Estimation and Application of the Vasicek Interest Rate Model

Andrea Buffoli, Davide Rolfi

June 9, 2025

1 Introduction

The Vasicek model is one of the earliest and most fundamental models for describing the evolution of short-term interest rates. It belongs to the class of affine term structure models and is widely used in fixed income theory, quantitative finance, and interest rate derivatives pricing.

Figure 1 displays the historical evolution of the 3-month US Treasury Constant Maturity Rate (GS3M) over the period 1990–2025. The time series reflects the daily trajectory of short-term interest rates, expressed on an annualized basis. As a widely recognized benchmark, the GS3M rate is frequently employed in monetary policy assessments and in modeling the short end of the yield curve.

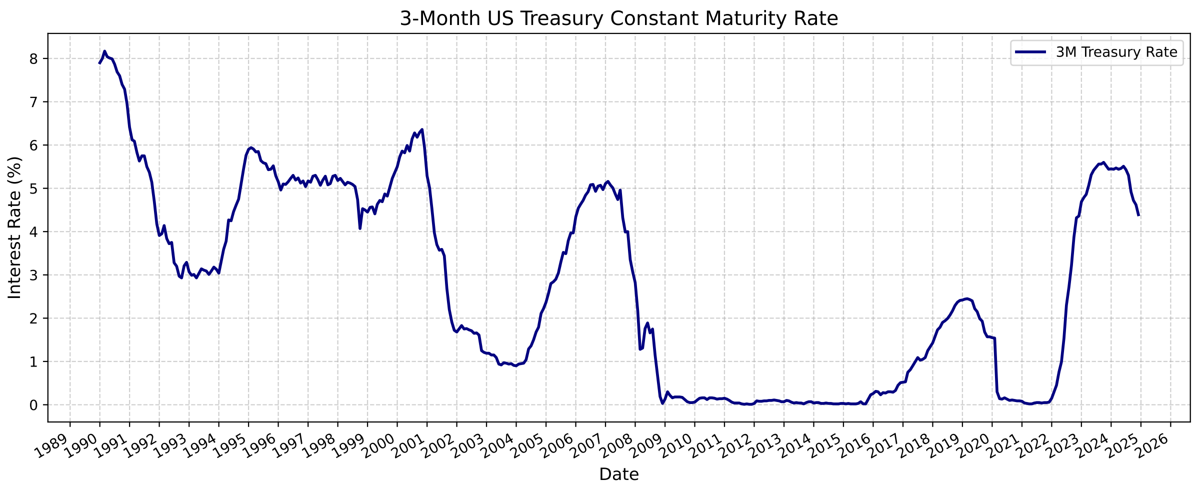

Figure 1: Historical evolution of the 3-month US Treasury Constant Maturity Rate (GS3M) from 1990 to 2025.

2 Mathematical Formulation of the Vasicek Model

The Vasicek model describes the evolution of the instantaneous short-term interest rate $r(t)$ through the following stochastic differential equation (SDE):

$$dr(t) = a(b - r(t))dt + \sigma dW(t), \tag{1}$$

where:

- $r(t)$ is the short-term interest rate,
- $a > 0$ is the speed of mean reversion,
- b is the long-term mean level,
- $\sigma > 0$ is the volatility of the interest rate,
- $W(t)$ is a standard Brownian motion.

This is a mean-reverting Ornstein-Uhlenbeck process. The rate $r(t)$ tends to revert toward the long-term mean b at a speed determined by a. The parameter σ controls the randomness of the rate's movement.

2.1 Solution to the SDE

The SDE in Equation (1) has a closed-form solution:

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dW(s), \tag{2}$$

which implies that:

$$\mathbb{E}[r(t)] = r(0)e^{-at} + b(1 - e^{-at}), \quad \mathbb{V}[r(t)] = \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{3}$$

3 Bond Pricing in the Vasicek Model

The price $P(t,T)$ at time t of a bond maturing at time T is given by:

$$P(t,T) = \exp\left\{-A(T-t) - B(T-t)r(t)\right\}, \tag{4}$$

where:

$$B(\tau) = \frac{1 - e^{-a\tau}}{a}, \tag{5}$$

$$A(\tau) = \left(b - \frac{\sigma^2}{2a^2}\right)(B(\tau) - \tau) - \frac{\sigma^2}{4a} B(\tau)^2, \tag{6}$$

with $\tau = T - t$. This property makes the Vasicek model very useful for pricing interest rate derivatives, bond portfolios, and managing interest rate risk.

4 Discretization and OLS Estimation

To estimate the model parameters from discrete data, we use Euler discretization:

$$r(t+1) = r(t) + a(b - r(t))\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon(t), \quad \varepsilon(t) \sim \mathcal{N}(0,1). \tag{7}$$

This can be rearranged as a linear regression:

$$r(t+1) = \alpha + \beta r(t) + \text{error}, \tag{8}$$

where $\beta = 1 - a\Delta t$ and $\alpha = ab\Delta t$.

4.1 Python Code for OLS Estimation with Real Data on Colab

```
import numpy as np
import pandas as pd
import statsmodels.api as sm
from pandas_datareader import data as pdr
import datetime
from google.colab import files

# Define time window
start = datetime.datetime(1990, 1, 1)
end = datetime.datetime(2024, 12, 31)

try:
    # Load GS3M data from FRED
    df = pdr.DataReader('GS3M', 'fred', start, end).dropna()
    r = df['GS3M'].values / 100  # convert from percent to decimal

    if len(r) < 2:
        raise ValueError("Not enough data points.")

    # OLS regression: r_{t+1} = alpha + beta * r_t + epsilon
    r_t = r[:-1]
    r_tp1 = r[1:]
    dt = 1 / 250

    X = sm.add_constant(r_t)
    model = sm.OLS(r_tp1, X).fit()
    alpha, beta = model.params
    residuals = model.resid
    r_squared = model.rsquared

    # Estimate Vasicek parameters
    if 0 < beta < 1:
        a = (1 - beta) / dt
        b = alpha / (1 - beta)
        sigma = np.std(residuals) / np.sqrt(dt)
    else:
        raise ValueError("Beta outside (0,1): invalid for Vasicek")

    # Save estimates as Colab variables
    alpha_ols = alpha
    beta_ols = beta
    R2_ols = r_squared
    a_vasicek = a
    b_vasicek = b
    sigma_vasicek = sigma

    # Display values
    print("Estimated parameters:")
    print(f"alpha_ols = {alpha_ols:.6f}")
    print(f"beta_ols  = {beta_ols:.6f}")
    print(f"R2_ols    = {R2_ols:.4f}")
    print(f"a_vasicek = {a_vasicek:.4f}")
    print(f"b_vasicek = {b_vasicek:.4f}")
    print(f"sigma     = {sigma_vasicek:.4f}")

    # Save to CSV
```

```
    estimates_df = pd.DataFrame({
        'alpha': [alpha_ols],
        'beta': [beta_ols],
        'R_squared': [R2_ols],
        'a': [a_vasicek],
        'b': [b_vasicek],
        'sigma': [sigma_vasicek]
    })
    csv_path = "/content/vasicek_estimates.csv"
    estimates_df.to_csv(csv_path, index=False)
    print(f"CSV saved to: {csv_path}")

    # Download CSV
    files.download(csv_path)

except Exception as e:
    print(f"Error: {e}")
```

Table 1: Estimated Vasicek Model Parameters (GS3M Series, 1990–2025)

Parameter	Description	Estimated Value
R^2	Coefficient of determination	0.9926
a	Speed of mean reversion $\left(\frac{1-\beta}{\Delta t}\right)$	2.2675
b	Long-term mean level $\left(\frac{\alpha}{1-\beta}\right)$	0.0182
σ	Volatility	0.0310

The estimates presented in Table 1 reflect the results of the model calibration performed on the levels of interest rates, which explains the high adjusted R^2 obtained. When the estimation is instead conducted on first differences, the inferred long-run level remains consistent; however, the regression on rate changes yields a substantially lower adjusted R^2, approximately 1% (see Table 2). As such, the predictive ability of the Vasicek model drops significantly when the autoregressive component is not explicitly considered, as illustrated in Figure 2.

Moreover, the residuals from the level-based OLS estimation exhibit clear signs of heteroscedasticity (see Figure 4), thereby violating one of the core assumptions of the classical linear regression framework. This suggests that the standard OLS estimates may be inefficient and that robust standard errors or alternative estimation techniques should be considered. However, applying such corrections would implicitly assume a model with time-varying variance, which is inconsistent with the theoretical structure of the Vasicek model, where innovations are assumed to be homoscedastic and normally distributed. Therefore, inference measures such as standard errors, t-statistics, and p-values are not reported, as their interpretation would be invalid under these violated assumptions.

Overall, these findings indicate that the observed data are unlikely to be generated by a true Vasicek process. In particular, the estimation on levels reveals residuals that are heteroscedastic, non-Gaussian, and exhibit autoregressive patterns, features that contradict the model's theoretical assumptions of normally distributed, homoscedastic innovations and serially uncorrelated shocks. The level of the interest rate accounts for the vast majority of the explained variation, while the mean-reversion mechanism, central to the Vasicek formulation, appears to contribute relatively little. This raises doubts about the empirical adequacy of the Vasicek model when applied directly to real-world interest rate dynamics. It is also worth noting that the results are highly sensitive to the estimation window: in some subperiods, the mean-reverting component played a much more prominent role, while in others its contribution was negligible. This is consistent with the presence of structural breaks in the time series, which may affect the stability of the model parameters and the overall fit of the Vasicek framework.

Table 2: Estimated Vasicek Model Parameters on the Difference (GS3M Series, 1990–2025)

Parameter	Description	Estimated Value
R^2	Coefficient of determination	0.0112
a	Speed of mean reversion $\left(\frac{1-\beta}{\Delta t}\right)$	2.2675
b	Long-term mean level $\left(\frac{\alpha}{1-\beta}\right)$	0.0182
σ	Volatility	0.0310

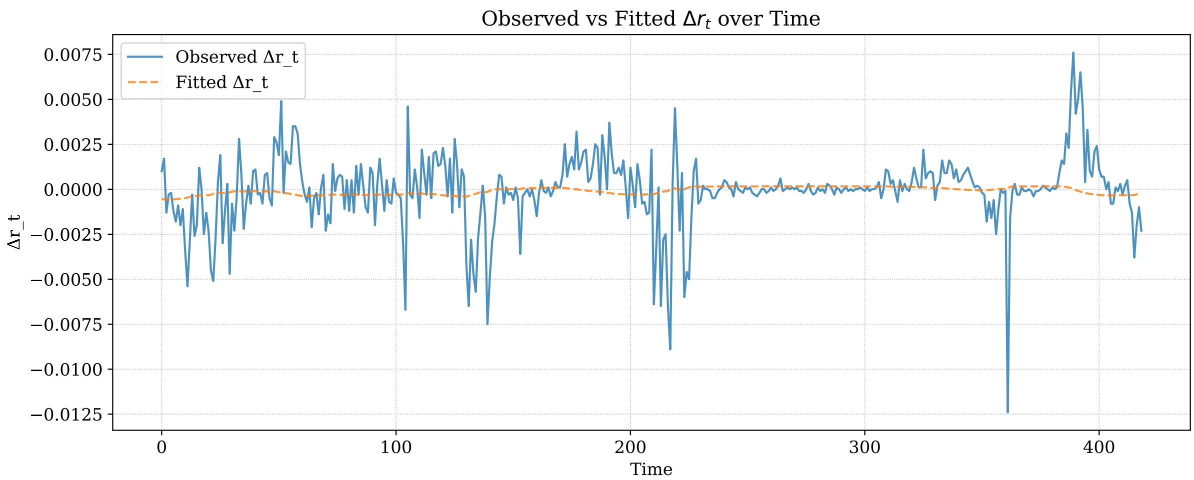

Figure 2: Observed versus fitted first differences Δr_t from the Vasicek model estimated via OLS.

Figure 3 illustrates the simulation of the Vasicek model using the parameters estimated via OLS. The simulated trajectories can be directly compared to the actual historical process, as represented by the observed time series. Figure 4 displays the time series of the model residuals. The graphical evidence suggests clear signs of heteroscedasticity, thereby violating one of the core assumptions underlying the Ordinary Least Squares (OLS) methodology.

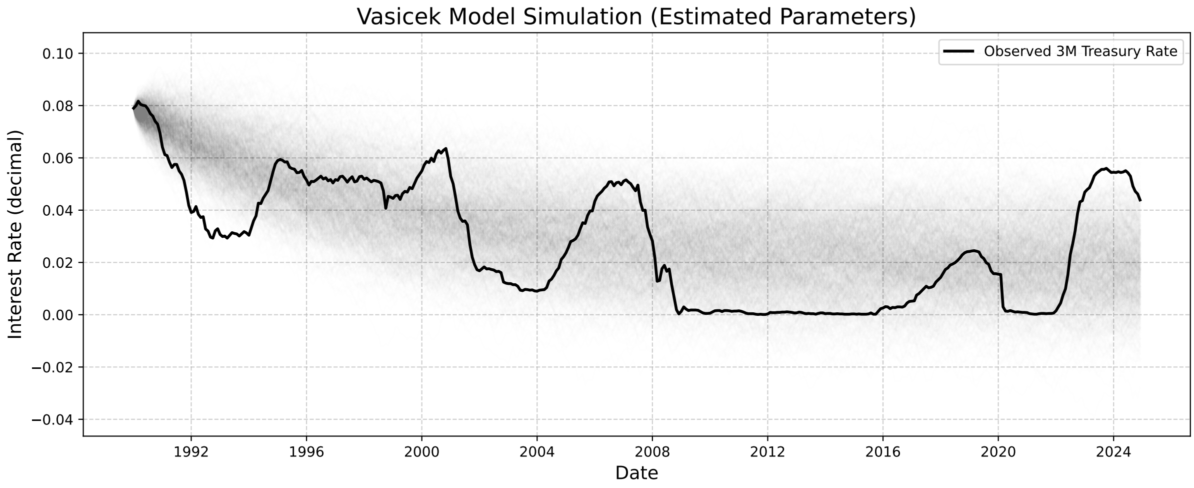

Figure 3: Simulation of the Vasicek model using OLS-estimated parameters, compared with the historical GS3M interest rate series.

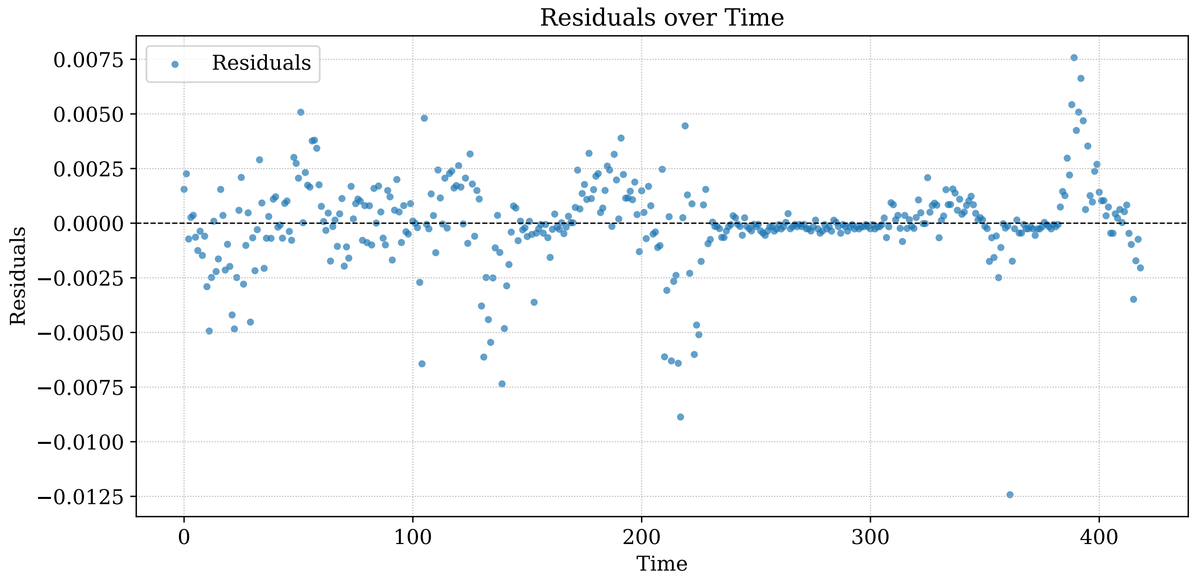

Figure 4: Time series of residuals from the Vasicek model estimated via OLS. Evidence of heteroscedasticity is visible.

5 Maximum Likelihood Estimation (MLE) of the Vasicek Model

5.1 Theory and Likelihood Formulation

Another method which one could use is the Maximum Likelihood Estimation (MLE), based on the exact transition density of the Vasicek process.

Recall the solution to the continuous-time Vasicek model over a time step Δt:

$$r(t+\Delta t) = r(t)e^{-a\Delta t} + b(1-e^{-a\Delta t}) + \eta(t), \quad \eta(t) \sim \mathcal{N}(0, Q), \tag{9}$$

with variance of the innovation given by:

$$Q = \frac{\sigma^2}{2a}(1-e^{-2a\Delta t}). \tag{10}$$

This leads to the conditional density:

$$r(t+1) \mid r(t) \sim \mathcal{N}(\mu(t), Q), \quad \mu(t) = r(t)e^{-a\Delta t} + b(1-e^{-a\Delta t}). \tag{11}$$

The log-likelihood for a sample of size n is:

$$\log L(a,b,\sigma) = -\frac{1}{2}\sum_{t=1}^{n-1}\left[\log(2\pi Q) + \frac{(r(t+1)-\mu(t))^2}{Q}\right], \tag{12}$$

which can be maximized numerically to obtain parameter estimates.

5.2 Python Code for MLE Estimation

```python
import numpy as np
import pandas as pd
from pandas_datareader import data as pdr
import datetime
from scipy.optimize import minimize
from google.colab import files

# Time window
start = datetime.datetime(1990, 1, 1)
end = datetime.datetime(2024, 12, 31)

try:
    # Load GS3M data from FRED and convert to decimal
    df = pdr.DataReader('GS3M', 'fred', start, end).dropna()
    r = df['GS3M'].values / 100  # percent to decimal

    if len(r) < 2:
        raise ValueError("Not enough data points.")

    # Define r_t and r_{t+1}
    r_t = r[:-1]
    r_tp1 = r[1:]
    dt = 1 / 250  # daily frequency

    # Define negative log-likelihood function for Vasicek model
    def vasicek_neg_log_likelihood(params):
        a, b, sigma = params
        if a <= 0 or sigma <= 0:
            return np.inf
        phi = np.exp(-a * dt)
        mu = r_t * phi + b * (1 - phi)
        Q = (sigma ** 2) * (1 - phi ** 2) / (2 * a)
        ll = -0.5 * np.sum(np.log(2 * np.pi * Q) + ((r_tp1 - mu) ** 2)
            / Q)
```

```
        return -ll  # negative log-likelihood for minimization

    # Initial guess and bounds
    initial_guess = [0.1, 0.02, 0.01]
    bounds = [(1e-5, 5), (-0.1, 0.2), (1e-6, 1)]

    # Minimize negative log-likelihood
    result = minimize(vasicek_neg_log_likelihood, initial_guess,
        bounds=bounds)

    if not result.success:
        raise RuntimeError("MLE optimization failed: " +
            result.message)

    # Extract estimated parameters
    a_mle, b_mle, sigma_mle = result.x

    # Display results
    print("MLE Estimated Vasicek Parameters:")
    print(f"a (speed of mean reversion)      = {a_mle:.6f}")
    print(f"b (long-term mean rate)          = {b_mle:.6f}")
    print(f"sigma (instantaneous volatility) = {sigma_mle:.6f}")

    # Save estimates to CSV
    estimates_df = pd.DataFrame({
        'a': [a_mle],
        'b': [b_mle],
        'sigma': [sigma_mle]
    })
    csv_path = "/content/vasicek_mle_estimates.csv"
    estimates_df.to_csv(csv_path, index=False)
    print(f"CSV saved to: {csv_path}")
    files.download(csv_path)

    # Save parameters to .npz for simulation use
    npz_path = "/content/vasicek_mle_params.npz"
    np.savez(npz_path, a=a_mle, b=b_mle, sigma=sigma_mle)
    print(f"NumPy parameters saved to: {npz_path}")
    files.download(npz_path)

except Exception as e:
    print(f"Error: {e}")
```

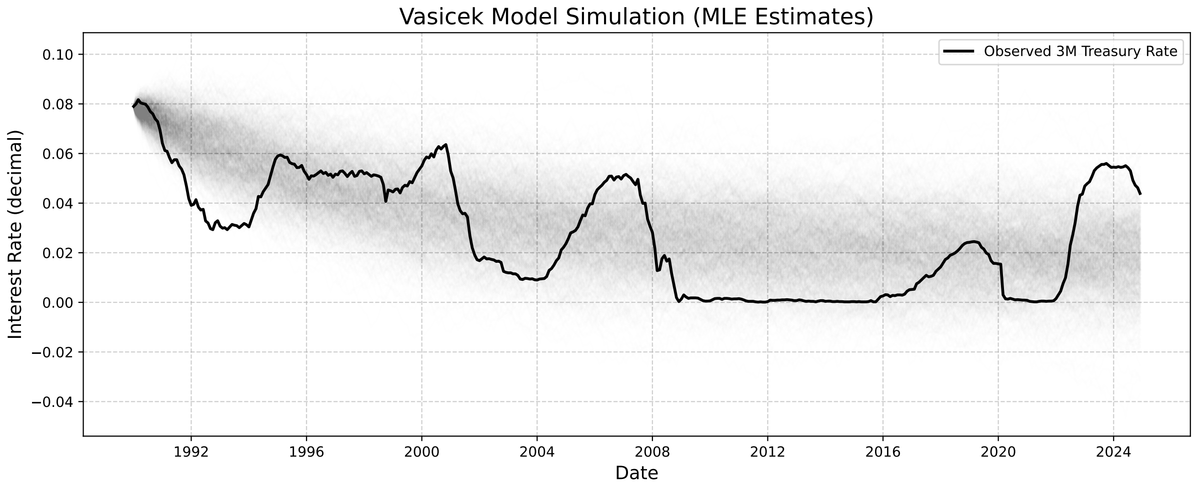

Figure 5: Simulation of the Vasicek model using MLE-estimated parameters, compared with the historical GS3M interest rate series.

Table 3: Estimated Vasicek Model Parameters (MLE, GS3M Series, 1990–2025)

Parameter	Description	Estimated Value
a	Speed of mean reversion	2.277746
b	Long-term mean level	0.018218
σ	Instantaneous volatility	0.031150

6 Conclusion

The Vasicek model remains a cornerstone in the modeling of short-term interest rates, providing a tractable framework for capturing mean-reverting dynamics and enabling closed-form solutions for bond pricing (depending on the risk neutral probability). Its analytical structure and intuitive economic interpretation make it a valuable tool in both academic research and practical applications. From an empirical perspective, the model can be estimated using a variety of methods, ranging from simple Ordinary Least Squares (OLS) based on discretized approximations to the famous Maximum Likelihood Estimation (MLE).

While the Vasicek model captures key features of interest rate behavior, such as mean reversion and stochastic volatility, its empirical performance is limited when the autoregressive structure is weak or when market frictions and non-linearities are present. Future research may consider extensions incorporating time-varying parameters, non-Gaussian shocks, or regime-switching dynamics to enhance the model's realism and predictive accuracy in modern financial environments.

Empirical evidence suggests that, over the analysis period 1990–2025, the persistence component of the Vasicek model accounts for a significantly larger portion of the variance in interest rates than the mean-reverting component. The historical time series exhibits multiple structural breaks, particularly around major financial crises and the subsequent monetary policy interventions by central banks. These events have markedly altered the dynamics of interest rates, often diminishing or even temporarily eliminating the mean-reverting behaviour that is central to the Vasicek framework.

In certain sub-periods, the mean reversion component is more pronounced, while in others, the data do not support its presence at all. This heterogeneity implies that the choice of the observation window is critical: the effectiveness and interpretability of the Vasicek model hinge upon the temporal context in which it is applied. Consequently, any analysis employing this model should explicitly account for structural shifts and consider tailoring the model's assumptions and calibration to the specific characteristics of the period under examination.

References

[1] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.