CS 215.

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ANSWERS : Follow from next page.

- Instructions for running the MATLAB code is specified in the INSTRUCIONS.txt file.

1.] $X_1, X_2, \dots, X_n.$ $n > 0.$ i.i.d.

cdf $F_X(x)$. pdf $f_X(x) = F'_X(x)$.

$Y_1 = \max(X_1, X_2, \dots, X_n)$.

$F_{Y_1}(y)$ = probability that $\max(X_1, X_2, \dots X_n) \leq y$

$= P(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y)$

$= P(X_1 \leq y) . P(X_2 \leq y) \dots P(X_n \leq y)$ {as X_i are independent}

$= F_X(y) . F_X(y) \dots F_X(y)$

$\boxed{F_{Y_1}(y) = (F_X(y))^n}$ ← cdf of Y_1 $\equiv$ $\boxed{F_{Y_1}(x) = (F_X(x))^n}$

$= n.$

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pdf of Y_1 → $f_{Y_1}(x) = F'_{Y_1}(x)$

$$= \left(F_X(x)^n \right)'$$

$$= \boxed{n \cdot F_X(x)^{n-1} \; F'_X(x)}$$

$$\boxed{f_{Y_1}(x) = n \cdot (F_X(x))^{n-1} \; f_X(x)}$$ ← pdf of Y_1.

$Y_2 = \min(x_1, x_2, \ldots, x_n)$.

cdf of Y_2 : $F_{Y_2}(x) = P(\min(x_1, x_2, \ldots, x_n) \leq x)$

$= P($ atleast one element of $x_1, x_2, \ldots, x_n$ is $\leq x)$

$= 1 - P($all among $x_1, x_2, \ldots, x_n > x)$

$= 1 - P(x_1 > x) \cdot P(x_2 > x) \cdots P(x_n > x)$

{all x_i are independent}

$= 1 - (1 - P(X_i \le x))^n$ (as X_i are identically distributed, all have same cdf)

$= 1 - (1 - F_X(x))^n$

$\Rightarrow \boxed{F_{Y_2}(x) = 1 - (1 - F_X(x))^n}$ ← cdf of Y_2.

$f_{Y_2}(x) = F_{Y_2}'(x)$

$= (1 - (1 - F_X(x))^n)'$

$= -n(1 - F_X(x))^{n-1} \cdot -F_X'(x) = \boxed{n(1-F_X(x))^{n-1} F_X'(x)}$

$\boxed{f_{Y_2}(x) = n(1 - F_X(x))^{n-1} f_X(x)}$ → pdf of Y_2.

2.] $X \sim \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$

$$\sum_{i=1}^{K} p_i = 1.$$

$$\Rightarrow \boxed{\phi_X(t) = \sum_{1}^{K} p_i \phi_{X_i}(t)} \quad \text{where } X_i \sim \mathcal{N}(\mu_i, \sigma_i^2).$$

↳ MGF of X.

$$\phi_{X_i}(t) = \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right).$$

$$\Rightarrow \boxed{\phi_X(t) = \sum_{i=1}^{K} p_i \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right)} \rightarrow \text{MGF of } X.$$

$$E(X) = \phi_X'(0)$$

$$= \sum_{1}^{K} p_i(\mu_i + \sigma_i^2 t) \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right) \Big|_{t=0}$$

$$\boxed{E(X) = \sum_{1}^{K} p_i \mu_i}$$

$$\text{Var}(X) = \phi_X''(0) - (\phi_X'(0))^2 = E(X^2) - (E(X))^2$$

$$\phi_X'(t) = \sum_{1}^{K} p_i(\mu_i + \sigma_i^2 t) \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right).$$

$$\phi_X''(t) = \sum_{1}^{K} p_i \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right)\left(\sigma_i^2 + (\mu_i + \sigma_i^2 t)^2\right)$$

$$\Rightarrow \text{Var}(X) = \phi_X''(0) - (\phi_X'(0))^2$$

$$\boxed{\text{Var}(X) = \sum_{1}^{K} p_i(\sigma_i^2 + \mu_i^2) - \left(\sum_{1}^{K} p_i \mu_i\right)^2}$$

For Z;

$$x_i \sim \mathcal{N}(\mu_i, \sigma_i^2).$$

$$Z = \sum_1^K p_i X_i.$$

$$E(z) = \cancel{\Sigma}\, E\left(\sum_1^K p_i x_i\right) = \sum_1^K E(p_i x_i)$$

$$= \sum_1^K p_i E(x_i) = \boxed{\sum_1^K p_i \mu_i = E(z)}.$$

$$\phi_z(t) = E\left(e^{t(\sum_1^K p_i x_i)}\right)$$

$$= \prod_{i=1}^K E\left(e^{t p_i X_i}\right) \qquad \{x_i \text{ are independent}\}$$

$$= \prod_1^K \exp\left(\mu_i p_i t + \frac{\sigma_i^2 t^2 p_i^2}{2}\right)$$

$$\boxed{\phi_Z(t) = \exp\left(t \sum_1^K \mu_i p_i + \frac{t^2}{2} \sum_1^K \sigma_i^2 p_i^2\right)} \rightarrow \text{MGF of } Z.$$

$$\phi_Z'(t) = \left(\sum_1^K \mu_i p_i + t \sum_1^K \sigma_i^2 p_i^2\right) \exp\left(t \sum_1^K \mu_i p_i + \frac{t^2}{2} \sum_1^K \sigma_i^2 p_i^2\right)$$

$$\phi_Z''(t) = \exp\left(t \sum_1^K \mu_i p_i + \frac{t^2}{2} \sum_1^K \sigma_i^2 p_i^2\right) \left(\left(\sum_1^K \mu_i p_i + t \sum_1^K \sigma_i^2 p_i^2\right)^2 + \sum_1^K \sigma_i^2 p_i^2\right)$$

$$\phi_Z''(0) = \left(\sum_1^K \mu_i p_i\right)^2 + \sum_1^K \sigma_i^2 p_i^2 = E(Z^2)$$

$$Var(Z) = \cancel{\phi}\; E(Z^2) - E(Z)^2$$

$$= \sum_1^K \sigma_i^2 p_i^2 + \cancel{\left(\sum_1^K \mu_i p_i\right)^2} - \cancel{\left(\sum_1^K \mu_i p_i\right)^2}$$

$$\boxed{Var(Z) = \sum_1^K \sigma_i^2 p_i^2}$$

PDF of Z.

As derived, MGF of Z

$$\phi_Z(t) = \exp\left(t \sum_1^K \mu_i p_i + \frac{t^2}{2} \sum \sigma_i^2 p_i^2 \right).$$

$$= \exp\left(t\mu + \frac{t^2}{2}\sigma^2 \right). \qquad \left\{ \begin{aligned} \text{where } \mu &= \sum_1^{K} \mu_i p_i \\ \sigma^2 &= \sum_1^K \sigma_i^2 p_i^2 \end{aligned} \right\}.$$

Since MGF uniquely determines PDF

~~& PDF of X $Y\sim$~~

& MGF of $Y \sim \mathcal{N}(\mu, \sigma^2)$

is $\exp\left(t\mu + \frac{t^2}{2}\sigma^2 \right)$

we conclude PDF of Z & Y is same as MGF uniquely determines PDF.

$$\Rightarrow \boxed{ f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(z-\mu)^2}{2\sigma^2} \right) }$$

where $\mu = \sum_1^K \mu_i p_i$

$\sigma^2 = \sum_1^K \sigma_i^2 p_i^2$.

Question -3

Let $Y = X - \mu$

For $\tau > 0$, $P(X-\mu \geq \tau) = P(Y \geq \tau) = P(Y+t \geq \tau+t)$

$\therefore P(Y+t \geq \tau+t) \leq P((Y+t)^2 \geq (\tau+t)^2)$

Using Markov's inequality.

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$\Rightarrow P((Y+t)^2 \geq (\tau+t)^2) \leq \frac{E((Y+t)^2)}{(\tau+t)^2}$$

$$E((Y+t)^2) = E(Y^2 + 2tY + t^2)$$

$$= E(Y^2) + 2t\,E(Y) + \underbrace{E(t^2)}_{\to t^2}$$

Since $Y = X - \mu$

$E(Y^2) = \sigma^2$ & $E(Y) = 0$

$$\Rightarrow \frac{E((Y+t)^2)}{(\tau+t)^2} = \frac{E(Y^2+t^2)}{(\tau+t)^2} = \sigma^2+t^2$$

$$\therefore P(X-\mu \geq \tau) \leq P((Y+t)^2 \geq (\tau+t)^2) \leq \frac{\sigma^2+t^2}{(\tau+t)^2}$$

Setting $\frac{d}{dt}\left(\frac{\sigma^2+t^2}{(\tau+t)^2}\right) = 0$ (For getting minima)

$$\Rightarrow 2t(\tau+t)^2 = 2(\tau+t)(\sigma^2+t^2)$$

$$\Rightarrow t = \sigma^2/\tau$$

$\Rightarrow P(x-\mu \geq \tau) \leq \frac{\sigma^2+t^2}{(\tau+t)^2} = \frac{\sigma^2}{\sigma^2+\tau^2}$ (Putting $t = \sigma^2/\tau$)

For $\tau < 0$, put $k = -\tau$ & $Y = X - \mu$

$$P(X-\mu \leq \tau) = P(Y \leq -k) = P(-Y \geq k) = P(-Y+t \geq k+t)$$

$$\Rightarrow P(-Y+t \geq k+t) \leq P((-Y+t)^2 \geq (k+t)^2) \leq \frac{E((-Y+t)^2)}{(k+t)^2}$$

$$\therefore E((-Y+t)^2) = E(Y^2) - 2t\cancelto{0}{E(Y)} + E(t)^2 = \sigma^2+t^2$$

So, $P(X-\mu \leq \tau) \leq \frac{\sigma^2+t^2}{(k+t)^2}$ (Same as before)

So $P(X-\mu \leq \tau) \leq \frac{\sigma^2}{k^2+\sigma^2}$

$\therefore k = -\tau$

$\Rightarrow k^2 = \tau^2$

$\therefore P(X-\mu \leq \tau) \leq \dfrac{\sigma^2}{\tau^2+\sigma^2}$

Taking the compliment

$1- P(X-\mu \geq \tau) \leq \dfrac{\sigma^2}{\tau^2+\sigma^2}$

$\Rightarrow P(X-\mu \geq \tau) \geq 1- \dfrac{\sigma^2}{\tau^2+\sigma^2}$

Question - 4

Using markov's inequality

$P(X \geq a) \leq \frac{E(X)}{a}$

If $t > 0$, $P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$

$\Rightarrow P(X \geq x) \leq \frac{E(e^{tX})}{e^{tx}} = e^{-tx} \phi_X(t)$

If $t < 0$, $P(X \geq x) = P(e^{tX} \leq e^{tx})$

or $P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}} = e^{-tx} \phi_X(t)$

$\Rightarrow P(X \leq x) \leq e^{-tx} \phi_X(t)$

$\therefore P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_X(t)$

This was obtained by putting $x = (1+\delta)\mu$ in the previously proved inequality

$\Rightarrow P(X > (1+\delta)\mu) \leq \frac{\phi_X(t)}{e^{t\mu(1+\delta)}} = \frac{E(e^{tX})}{e^{t\mu(1+\delta)}}$

Since $X = \sum_i X_i$, Also all X_i are independent (given independent Bernoulli trials)

Also $E(X_i) = p_i$

$\therefore E(e^{tX}) = E(e^{t\sum X_i}) = E(\Pi e^{tX_i}) \underset{\text{(independent)}}{\xrightarrow{\quad}} = \Pi E(e^{tX_i})$

$\Pi E(e^{tX_i}) = \Pi(1 - p_i + p_i e^t) = \Pi(1 + p_i(e^t - 1))$

$\therefore 1 + x \leq e^x$. Put $x = p_i(e^t - 1)$

$\Rightarrow \Pi E(e^{tX_i}) \leq \Pi e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)} = e^{\mu(e^t-1)}$

Since $\sum p_i = \mu$

$\therefore P(X > (1+\delta)\mu) \leq \frac{\Pi E(e^{tX_i})}{e^{t\mu(1+\delta)}} \leq \frac{e^{\mu(e^t-1)}}{e^{t\mu(1+\delta)}}$

$\Rightarrow P(X > (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{\mu\cdot(1+\delta)t}}$ for any $t \geq 0$, $\delta \geq 0$

$\therefore e^{\mu e^t - \mu - \mu t - \mu t \delta}$ should be minimized

$\Rightarrow f(t) = \mu e^t - \mu - \mu t - \mu t \delta$ should be minimized

$\therefore f'(t) = \mu e^t - \mu - \mu\delta = 0$ (for minima)

$\Rightarrow \mu e^t = \mu(1+\delta)$

$\Rightarrow e^t = 1+\delta$

$\Rightarrow t = \ln(1+\delta)$

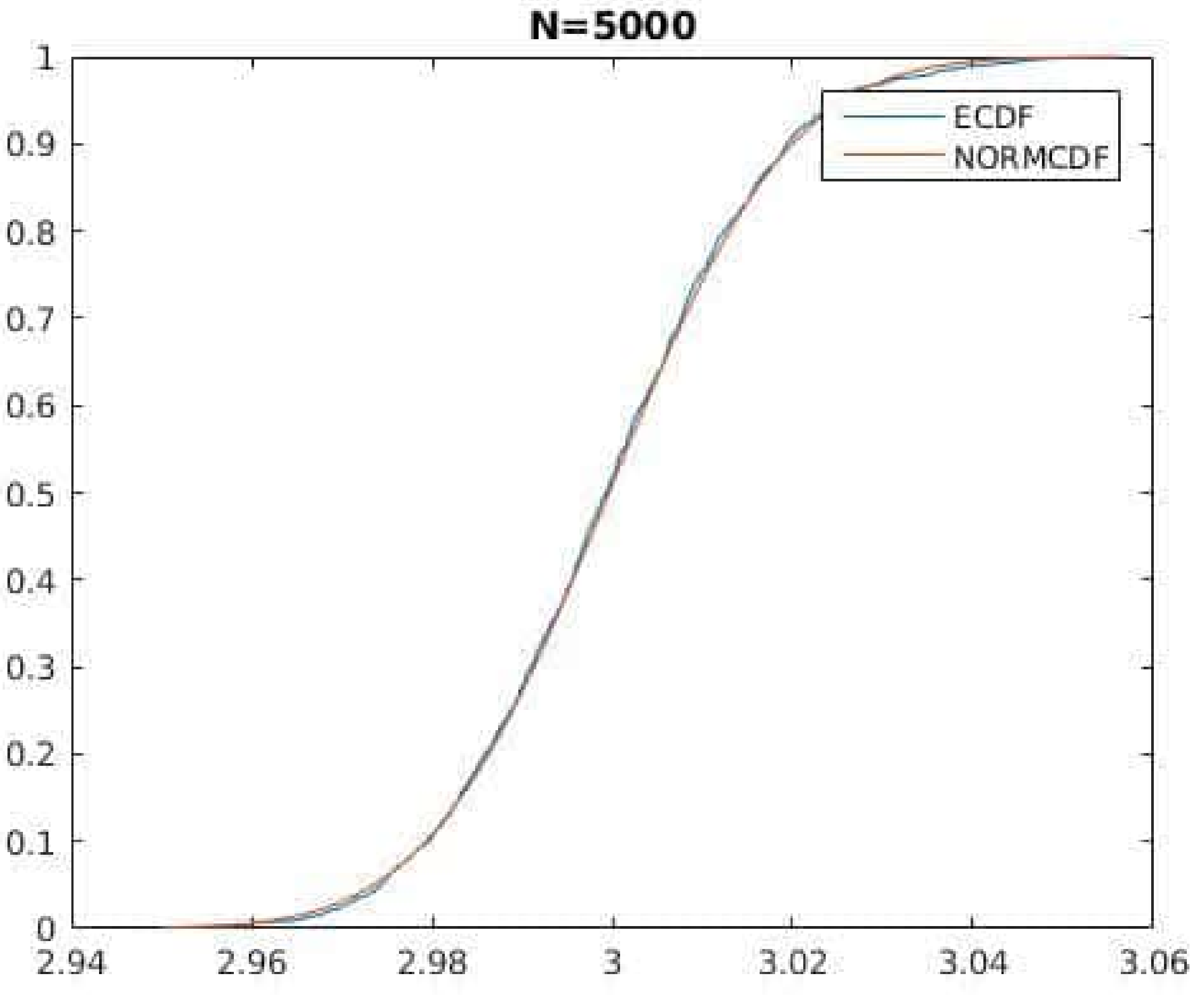
Also $f''(t) = \mu e^t > 0 \ \forall t$

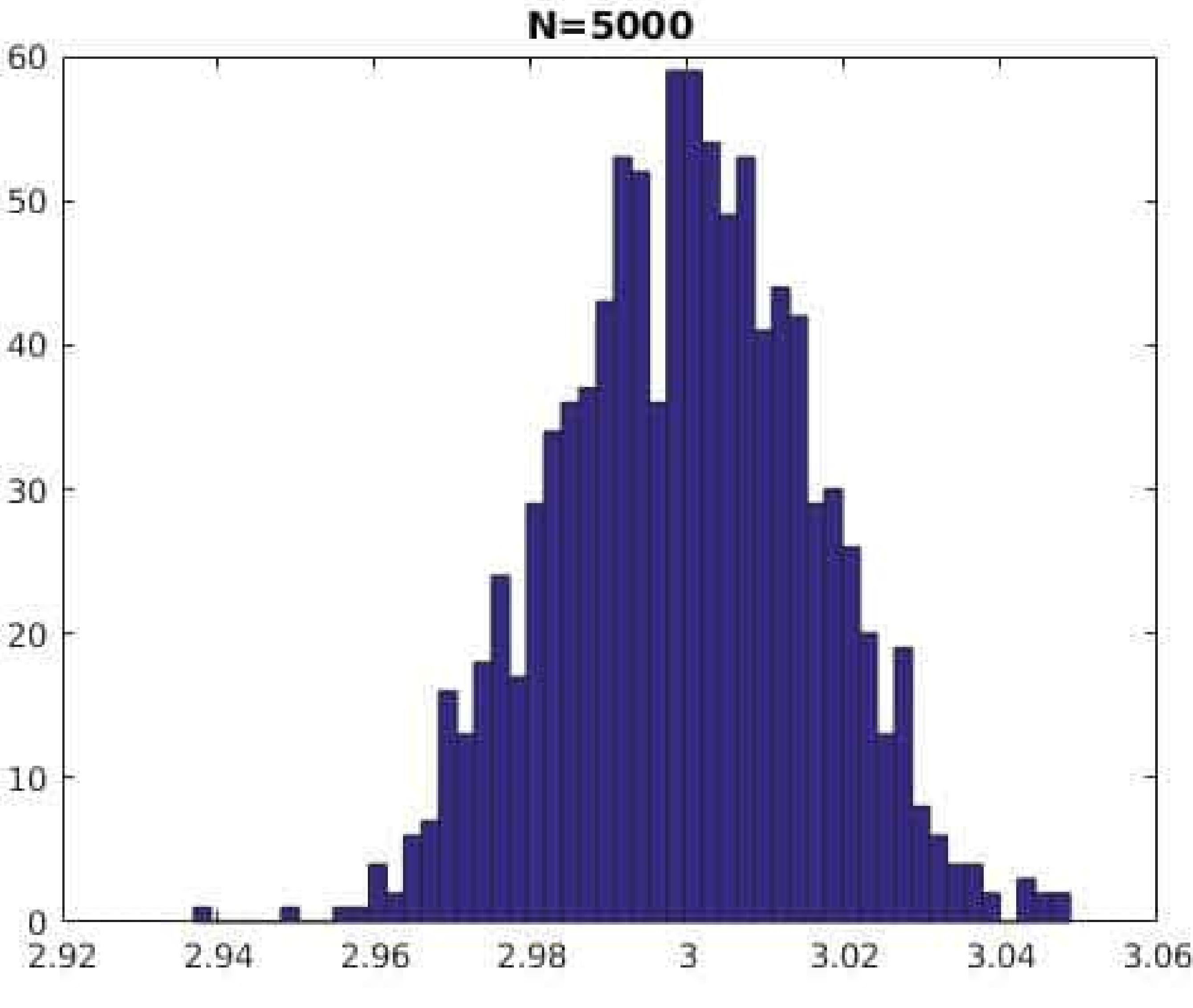
$\Rightarrow t = \ln(1+\delta)$ is a point of minima

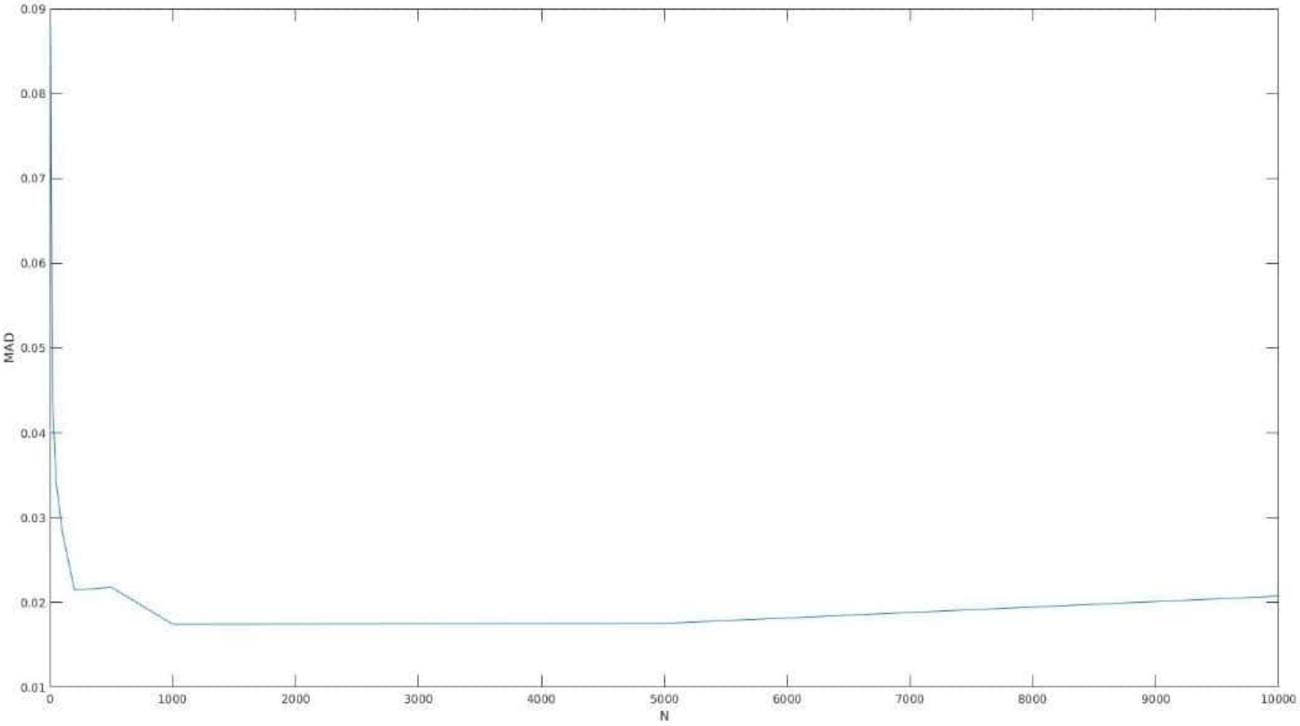
5.] Histogram & ECDF, NORMCDF plots are attached for N = 5000.

MAD vs N plot is attached below.

Instruction to run code : Just execute the 'a2q5.m' file.


N=5000
ECDF
NORMCDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.94
2.96
2.98
3
3.02
3.04
3.06

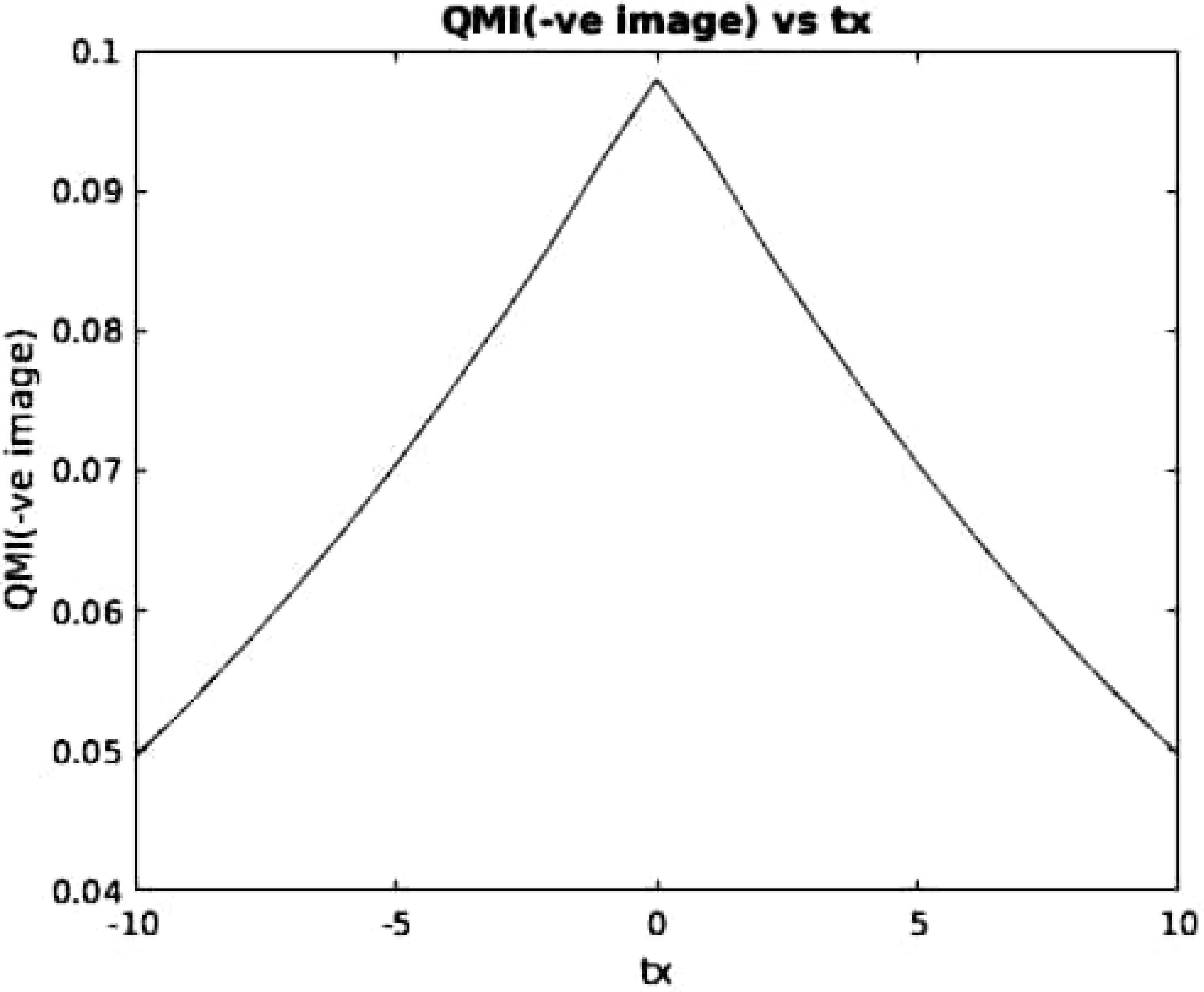

N=5000
60
50
40
30
20
10
0
2.92
2.94
2.96
2.98
3
3.02
3.04
3.06

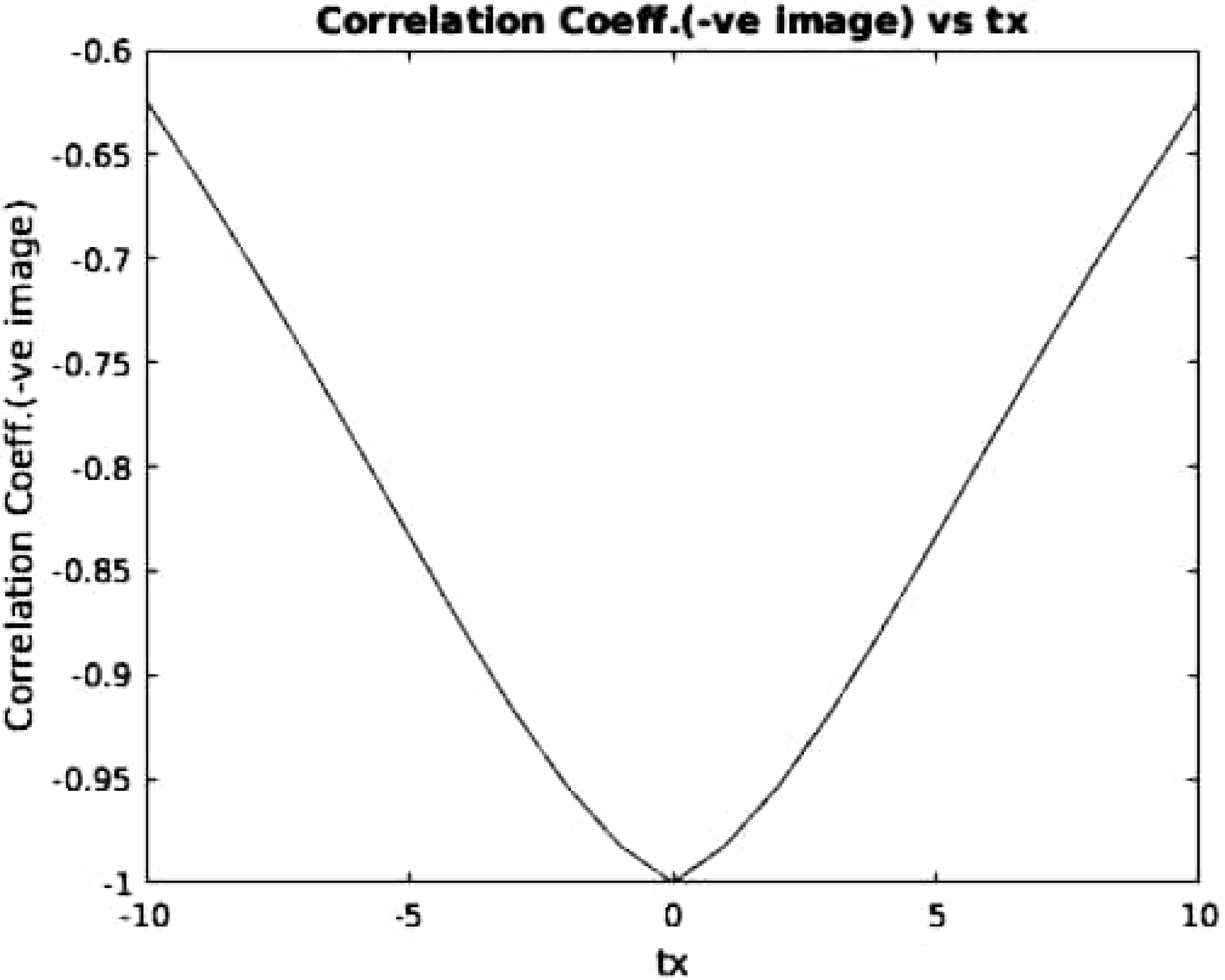

0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
MAD
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
N

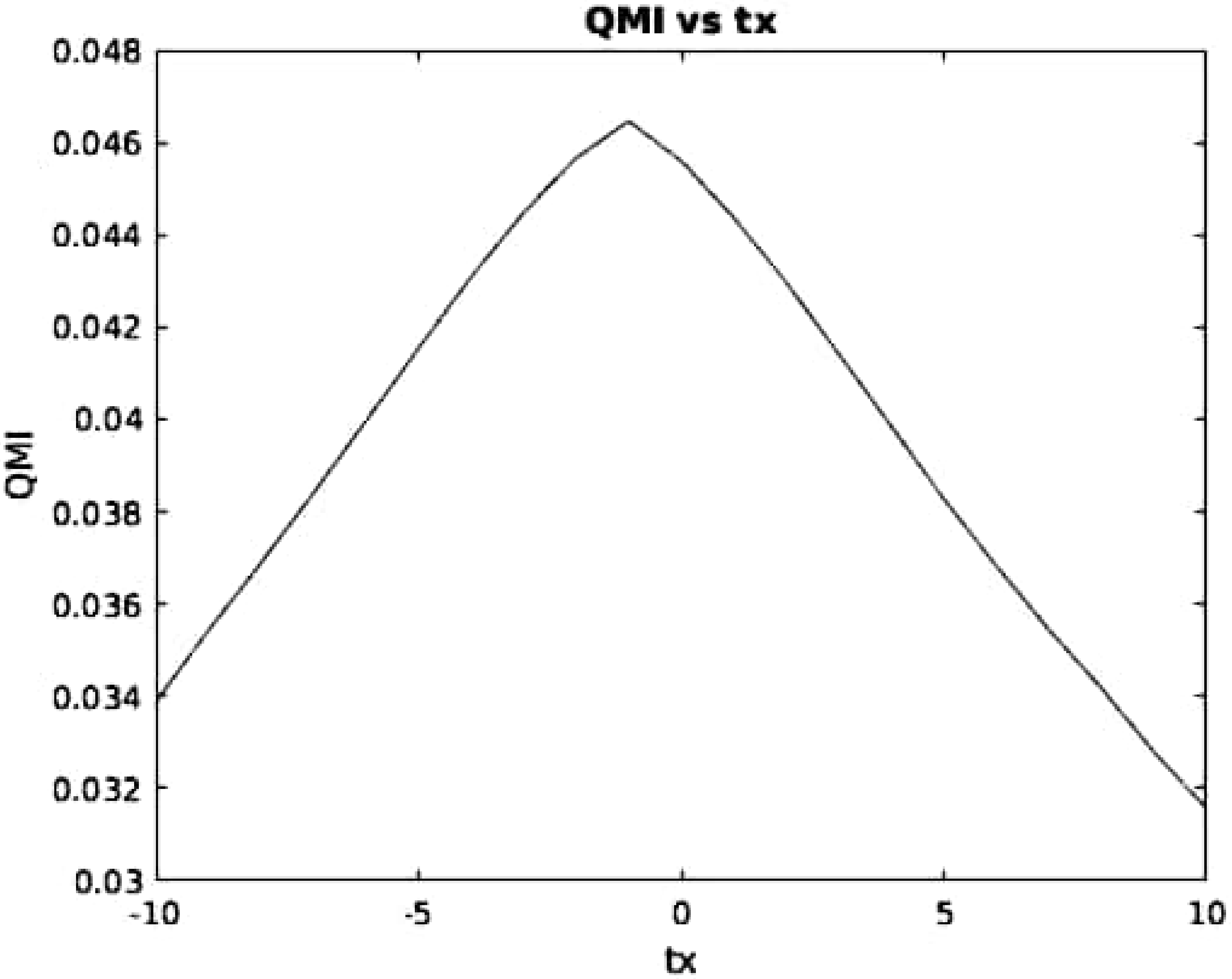
Question - 6

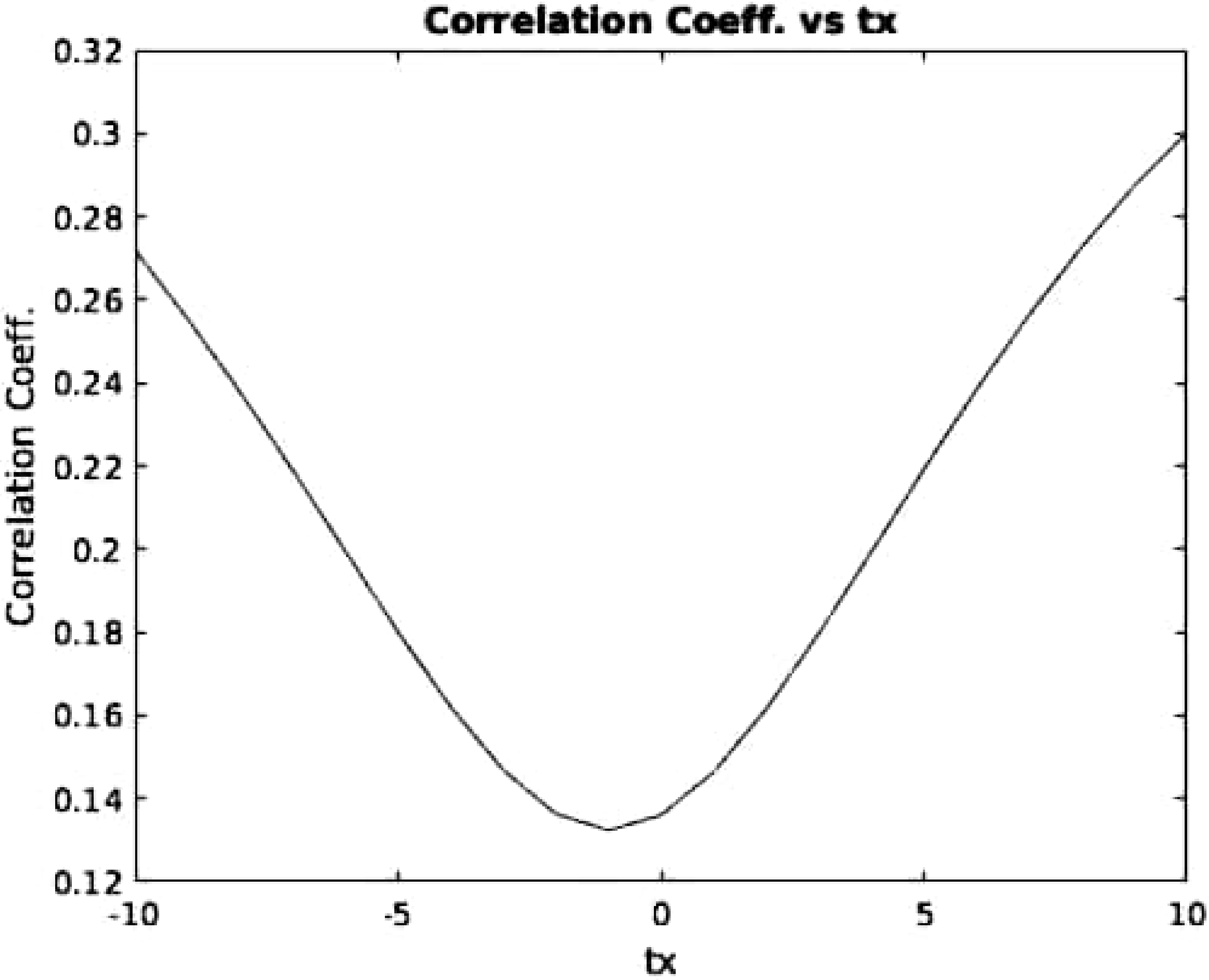
When two images are aligned (the shift along the x-axis) properly, the intensities $I_1(x,y)$ and $I_2(x,y)$ are highly correlated (or dependent).
Here, the proper alignment occurs when $t_x = 0$ ie. no shift in image 2.
We would expect the maxima to occur at $t_x = 0$ in the plots in case of QMI, which is as expected. This shows they are highly dependent when $t_x = 0$.
In I case, the correlation coeffecient does not follow the expectations because it does not make the full use of the complete PMF of intensities rather it uses only the mean & variance.
In II case, the negative of first image $(255 - I_1)$ and the first image (I_1) follow a linear relationship so correlation coeffecient $= -1$ at $t_x = 0$ when they are exactly opposites of each other.

The MATLAB code gives its output 4 graphs.
R1 & QMI1 correspond to image 1 & image 2.
R2 & QMI2 correspond to image 1 & image 3.
$\hookrightarrow 255 - I_1$


QMI(-ve image) vs tx
QMI(-ve image)
0.1
0.09
0.08
0.07
0.06
0.05
0.04
-10
-5
0
5
10
tx


Correlation Coeff.(-ve image) vs tx
Correlation Coeff.(-ve image)
tx
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-10
-5
0
5
10


QMI vs tx
0.048
0.046
0.044
0.042
0.04
0.038
0.036
0.034
0.032
0.03
QMI
-10
-5
0
5
10
tx



Correlation Coeff. vs tx
Correlation Coeff.
tx
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
-10
-5
0
5
10