CSCI 270 - Spring 2023 - HW12 Solution

Due: 11:59 PM, April 23, 2023 (PST)

[Important] Although we release the solution early, you still need to submit your homework before the deadline to get the credit. Please fill out this [form] if you need feedback from the TAs. The feedback will be given after the exam.

1. [25 pts] Suppose you are designing a scheduling algorithm for a manufacturing plant that has a set of M machines and a set of N tasks to be completed. Each task can be assigned to only one machine, and each machine can only perform one task at a time. Let p_{ij} be the processing time of task i on machine j. The goal is to minimize the overall time it takes to complete all tasks. How would you formulate this problem as an ILP? What decision variables would you use, and what constraints and the objective function would you include?

Answer:

To formulate the scheduling problem as an ILP, we can use the following decision variables:

- Let x_{ij} be a binary variable indicating whether task i is assigned to machine j.
- Let C be the time at which the last task is completed, also known as "makespan".

Then we can formulate the problem as follows:

$$\text{minimize } C$$

subject to:

$$C \geq \sum_{i=1}^{N} p_{ij} x_{ij},\ \forall j \text{ (task assignment constraint)}$$

$$\sum_{j=1}^{M} x_{ij} = 1,\ \forall i \text{ (machine capacity constraint)}$$

$$x_{ij} \in \{0, 1\},\ \forall i, j \text{ (domain of the decision variables)}$$

Here, the objective function is to minimize the maximum completion time, and the constraints ensure that each task is assigned to exactly one machine and that the completion time of each task is greater than or equal to the total processing time on the machine it is assigned to.

Rubric:

Criteria	Points
Correct use of decision variables	5
Correct formulation of the objective function	5
Correct formulation of task assignment constraints	5
Correct formulation of machine capacity constraints	5
Correct domain of the decision variables	5

2. [25 pts] Formulate the problem of finding a Min-S-T- cut of a directed network with source s and sink t as an Integer Linear Program and explain your program

Answer:

minimize $\sum_{(u,v) \in E} c(u, v) \, . \, x_{(u,v)}$

subject to :

- $x_v - x_u + x_{(u,v)} \geq 0 \quad \forall \ (u, v) \in E$
- $x_u \in \{0, 1\} \ \forall u \in V : u \neq s, u \neq t$
- $x_{(u,v)} \in \{0, 1\} \ \forall (u, v) \in E$
- $x_s = 1$
- $x_t = 0$

- The variable x_u indicates if the vertex u is on the side of s in the cut. That is, $x_u = 1$ if and only if u is on the side of s.
- Setting $x_s = 1$ and $x_t = 0$ ensures that s and t are separated.
- Likewise, the variable $x_{(u,v)}$ indicates if the edge (u, v) crosses the cut.
- In the minimization objective, the notation $c(u, v)$ denotes the capacity of the edge (u,v).
- The first constraint ensures that if u is on the side of s and v is on the side of t, then the edge (u, v) must be included in the cut.
    - Possible scenarios of x_u and x_v:

i. If u is on the side of s and v is on the side of t : This implies $x_u = 1$ and $x_v = 0$. Thus, for the first constraint to be valid, $x_{(u,v)}$ must be set to 1, indicating that edge (u,v) is part of the cut.

ii. If both u and v are on the same side, then $x_u = x_v$. In this case, first constraint already becomes satisfied and it is not necessary for $x_{(u,v)}$ to be 1. Since the ILP tries to minimize $c(u, v) . x_{(u,v)}$, all the $x_{(u,v)}$ corresponding to this case, will eventually be set to 0.

- The formulated ILP will find a feasible configuration that yields the minimum sum of the capacities crossing the cut, thus finding the min-cut.

One can also visualize this program as initially starting with "s" node on one side, and all the other nodes on the T side. This would mean that the initial configuration would have all the edges from source node "s" as part of the cut, and as the iterations progress, the program checks if each node should be part of the S set or not (similar to what we did in class for finding min-cut) and finally reaches the minimal value, yielding the min-cut

Rubric:

Criteria	Points
Correct use of decision variables	5
Correct formulation of the objective function	5
Correct formulation of vertex constraints and proper initialization for source and sink nodes.	5
Correct formulation of edge inclusion constraint	5
Correct High-level Explanation	5

3. [25 pts] A set of n space stations need your help in building a radar system to track spaceships traveling between them. The ith space station is located in 3D space at coordinates (x_i, y_i, z_i). The space stations never move. Each space station "i" will have a radar with power r_i, where r_i is to be determined. You want to figure out how powerful to make each space station's radar transmitter is, so that whenever any spaceship travels in a straight line from one station to another, it will always be in the radar range of either the first space station (its origin) or the second space station (its destination). A radar with power r is capable of tracking space ships anywhere in the sphere with radius r centered at itself. Thus, a spaceship is within radar range through its strip from space station i to space station j if every point along the line from (x_i, y_i, z_i) to (x_j, y_j, z_j) falls within either the sphere of radius ri centered at (x_i, y_i, z_i) or the sphere of radius r_j centered at (x_j, y_j, z_j). The cost of each radar transmitter is proportional to its power, and you want to minimize the total cost of all of the radar transmitters. You are given all of the (x_1, y_1, z_1), ..., (x_n, y_n, z_n) values, and your job is to choose values for r_1, ..., r_n. Express this problem as a linear program.

(a) Describe your variables for the linear program (5 pts).

Answer:

r_i=the power of the i^{th} radar transmitter, i=1,2,...n

(b) Write out the objective function (8 pts).

Answer:

Minimize $r_1 + r_2 + ... + r_n$ or $\sum_{i=1}^{n} r_i$

(c) Describe the set of constraints for LP. You need to specify the number of constraints needed and describe what each constraint represents (12 pts)

Answer:

$r_i + r_j \geq \sqrt{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)}$

Or, $r_i + r_j \geq d_{i,j}$ for each pair of stations i and j, where $d_{i,j}$ is the distance from station i to station j

We need $\sum_{i=1}^{n-1} i = n(n-1)/2$ constraints of inequality (The number of constraints is due to the number of unique paths between each pair of space stations)

Rubric:

Criteria	Points
Correct use of variables	5
Correct formulation of the objective function	8

Correct formulation of constraint equation	8
Correct number of constraints	4

4. [25pts] Recall the maximum-bipartite-matching problem. Write a linear program that solves this problem given a bipartite graph $G = (V, E)$, where the set of vertices on the left is L, and the set on the right is R (i.e., $L \cup R = V$).

Answer:

We can solve the maximum-bipartite-matching problem by viewing it as a network flow problem, where we append a source s and sink t, each connected to every vertex in L and R respectively by an edge with capacity 1, and we give every edge already in the bipartite graph capacity 1. Let the set of vertices of the network flow be $V' = \{s\} \cup L \cup R \cup \{t\}$ The integral maximum flows are in correspondence with maximum bipartite matchings. In this setup, the linear programming problem to solve is as follows:

$$maximize \sum_{v \in L} f_{(s,v)}$$

subject to

$$f_{(u,v)} \leq 1 \ \forall u, v \in V' \text{ (flow constraints on edges)}$$

$$\sum_{v \in V'} f_{(v,u)} = \sum_{v \in V'} f_{(u,v)}, \ \forall u \in V \text{ (flow constraints on vertices)}$$

$$f_{(u,v)} \geq 0, \ \forall u, v \in V' \text{ (positive flow constraints)}$$

Rubric:

Criteria	**Points**
Correct use of variables	5
Correct formulation of the objective function	8
Correct formulation of constraints	12