CSCI 270 - Spring 2023 - HW 11

Due: April 19, 2023

1. (15pts) Determine if the following statements are true or false. For each statement, briefly explain your reasoning.

   (a) If Ham-Cycle is polynomial time reducible to interval scheduling problem then P = NP.

   (b) The NP-Hard class of problems does not contain any decision problems.

   (c) If there exists an algorithm that solves problem X with pseudo-polynomial runtime, then X must be NP-Hard.

   (d) Suppose there is a 7-approxmiation algorithm for the general Traveling Salesman Problem. Then there exists a polynomial time solution for the 3-SAT problem.

   (e) A vertex that is part of a minimum vertex cover can never be a part of a maximum independent set.

   Solution:

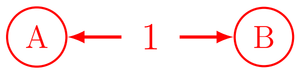
   (a) True, Interval scheduling is a polynomial time solvable problem whereas Ham-Cycle is an NP-Complete problem. If we can reduce Ham-Cycle to Interval scheduling problem then it means that we can solve Ham-Cycle in Polynomial time. Hence, P = NP

   (b) False, Checking whether the next move in chess is optimal or not is a decision problem that is part of the NP-Hard set.

   (c) False, Consider the problem of finding if the sum of a set of n integers is K. This problem can be solved by an algorithm with pseudo-polynomial runtime by using dynamic programming, where the runtime is $O(nS)$, where S is the sum of the input integers. However, this problem is not NP-Hard as it can be solved in polynomial time by simply adding up all the numbers and checking if it is equal to K.

   (d) True, Existence of the 7-approx algo for the General TSP would imply P=NP since it does not have a const. factor approximation unless P=NP. Hence, 3-SAT would also be solvable in poly-time.

(e) False, In the below graph, we can take A as both Maximal Independent Set and the Minimal Vertex Cover



Rubric (15 pts) For each question:

- 1 pts: Correct choice
- 2 pts: Valid reasoning for the answer

2. (15pts) Given an undirected graph with positive edge weights, the BIGHAM- CYCLE problem is to decide if it contains a Hamiltonian cycle C such that the sum of weights of edges in C is at least half of the total sum of weights of edges in the graph. Show that finding BIG-HAM-CYCLE in a graph is NP-Complete.

Solution:

The certifier takes an undirected graph (the BHC instance) and a sequence of edges as input. It verifies that the sequence of edges forms a Hamiltonian cycle and that the total weight of the cycle is at least half of the total weight of the edges in the graph. Thus BIG-HAM-CYCLE is in NP.

We claim that Hamiltonian Cycle is polynomial time reducible to BIG-HAMCYCLE. To see this, given an instance of problem Hamiltonian cycle in an undirected graph G = (V, E), pick an edge e, set its weight to $|E|$, and assign the rest of the edges weight of 1. When this weighted graph is fed into the BIG-HAM-CYCLE decider blackbox, it returns "yes" if and only if G has a Hamiltonian cycle containing the edge e. By repeating the above once for every edge e in the graph G, we can decide if the graph has a Hamiltonian cycle.

Rubric (15 pts)

- 5 pts: Prove the problem is in NP.
- 3 pts: Reduction from Hamiltonian Cycle
- 7 pts: Run BIG-HAM-CYCLE on the new weighted graph.

3. (15pts) Given an undirected connected graph G= (V, E) in which a certain number of tokens t(v) = 1 or 2 placed on each vertex v. You will now play the following game. You pick a vertex u that contains at least two tokens, remove two tokens from u and add one token to any one of adjacent vertices. The objective of the game is to perform a sequence of moves such that you are left with exactly one token in

the whole graph. You are not allowed to pick a vertex with 0 or 1 token. Prove that the problem of finding such a sequence of moves is NP-Hard by reduction from Hamiltonian Path.

Solution:

Construct n games. In each of them G' choose one different vertex to place 2 coins and then place 1 coin on the rest. Then, the given graph G has a HP if and only if at least one of these games is winnable.

Claim: G has a HP if at least one of the n constructed games has a winning sequence.

Since there is only one vertex with 2 tokens, we will start right there playing the game. Each next move is forced. By construction before the last move we will end up with a single vertex having two tokens on it. Making the last 2 move, we will have exactly one token on the board.

Proof:

- Suppose G has a Hamiltonian Path P. Suppose one of the endpoints of P is vertex u. Then the constructed game which has 2 coins on u, can have a winning sequence starting at u and tracing P.
- Suppose one of the constructed games with 2 coins on a vertex u, has a winning sequence. Note that every time we make a move at a node having 2 tokens, the next neighbour to add a coin to must be chosen as one which already has a coin (otherwise there is no new move and the game stops and cannot be won). Thus, the winning sequence must trace out a path starting at u. This path P passes through all the nodes by construction, and thus, gives a hamiltonian path of G. If all the edges don't have unit capacity, removing k edges from min-cut in the above mentioned way does not guarantee to have the smallest possible max-flow value.

Rubric (15 pts)

- 8 pts: Correct reduction from Hamiltonian Path to the game
- 7 pts: Prove that the reduction will give the correct answer.

4. (20 pts) In a certain town, there are many clubs, and every adult belongs to at least one club. The town's people would like to simplify their social life by disbanding as many clubs as possible, but they want to make sure that afterwards everyone will still belong to at least one club. Formally the Redundant Clubs problem has the following input and output. INPUT: List of people; list of clubs; list of members of each club; number K. OUTPUT: Yes if there exists a set of K clubs such that, after

disbanding all clubs in this set, each person still belongs to at least one club. No otherwise. Prove that the Redundant Clubs problem is NP -Complete.

Solution:

- We must show that Redundant Clubs is in NP, but this is easy: if we are given a set of K clubs, it is straightforward to check in polynomial time whether each person is a member of another club outside this set.
- We prove Redundant Clubs is in NP-Hard by reducing from a known NP-complete problem, Set Cover, e.g., Set Cover $\leq$p Redundant Clubs. We translate inputs of Set Cover to inputs of Redundant Clubs, so we need to specify how each Redundant Clubs input element is formed from the Set Cover instance.

We use the Set Cover's elements as our translated list of people, and make a list of clubs, one for each member of the Set Cover family. The members of each club are just the elements of the corresponding family. To finish specifying the Redundant Clubs input, we need to say what K is: we let K = F − KSC where F is the number of families in the Set Cover instance and KSC is the value K from the set cover instance. This translation can clearly be done in polynomial time (it just involves copying some lists and a single subtraction).

Finally, for the proof we need to show that we can remove K clubs if and only if we have a set cover of size KSC such that K = F − KSC . If we have a yes-instance of Set Cover, that is, an instance with a cover consisting of KSC subsets, the other K subsets form a solution to the translated Redundant Clubs problem, because each person belongs to a club in the cover. Conversely, if we have K redundant clubs, the remaining KSC clubs form a cover. So the answer to the Set Cover instance is yes if and only if the answer to the translated Redundant Clubs instance is yes. This completes the reduction, and we confirm that the given problem is NP-Hard.
Thus this problem is NP-Complete.

Rubric (20pt):

- 5 pts: Prove Redundant Clubs is in NP.
- 5 pts: Claim that Set Cover $\leq$p Redundant Clubs.
- 5 pts: Construction of the reduction.
- 4 pts: Reduction proof.
- 1 pt: Conclusion.

5. (15 pts) Given a graph G=(V, E) with an even number of vertices as the input, the

HALF-IS problem is to decide if G has an independent set of size $\frac{|V|}{2}$. Prove that HALF-IS is in NP-Complete.

Given a graph G(V, E) and a certifier S $\subset$ V, $|V| = \frac{|E|}{2}$, we can verify if no two nodes are adjacent in polynomial time (O($|S|^2$) = O($|V|^2$)). Therefore HALF-IS $\in$ NP. We prove the NP-Hardness using a reduction of the NP-complete problem Independent Set Problem (IS) to HALF-IS. Consider an instance of IS, which asks for an independent set A $\subset$ V, $|A| = k$, for a graph G(V, E), such that no two pair of vertices in A are adjacent to each other.

- If k = $\frac{|V|}{2}$, IS reduces to HALF-IS.
- If k < $\frac{|V|}{2}$, then add m new nodes such that $k + m = (|V| + m)/2$, i.e., $m = |V| - 2k$. Note that the modified set of nodes V' has even number of nodes. Since the additional nodes are all disconnected from each other, they form a subset of independent set. Therefore, the new graph G(V, E) where E = E has an independent-set of size $\frac{|V|}{2}$ if and only if G(V, E) has an independent set of size K.
- If k > $\frac{|V|}{2}$, then again add $m = |V| - 2$k new nodes to form the modified set of nodes V. Connect these new nodes to all the other $|V| + m1$ nodes. Since these m new nodes are connected to every other none of them should belong to an independent set. . Therefore, the new graph G(V, E') has an independent-set of size $\frac{|V|}{2}$ if and only if G(V, E) has an independent set of size .

Rubric (15 pts)

- 5 pts: Prove HALF-IS is in NP.
- 5 pts: Claim and construct the reduction: Independent Set $\leq$p HALF-IS.
- 5 pts: Reduction Proof and Conclusion