Math 395 Homework 1

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January 17, 2023

Problem 1. Prove that $\sqrt{2}$ is irrational.

Answer:
Assume that $\sqrt{2}$ is rational in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and share no common factors.

$$\begin{aligned}(\frac{p}{q})^2 &= (\sqrt{2})^2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2\end{aligned}$$

which implies that p^2 is even. If p^2 is even, then p is even, which implies that p^2 is divisible by 4. Hence q^2 is even, so q is even. Since p, q are even, they share a common factor which is a contradiction. Thus, $\sqrt{2}$ is irrational. $\square$

Problem 7. Prove that $\forall n \in \mathbb{N}$ and $\forall x \in \mathbb{R}, |\sin nx| \leq n|\sin x|$.

Answer:
Let $P(n) : |\sin nx| \leq n|\sin x|$.
For $P(1) : |\sin x| = |\sin x| \therefore P(1)$ is true.
Let I_0H_0 : Assume $P(k)$ for some $k \in \mathbb{N}$.
For $P(k+1)$:

$$\begin{aligned}|\sin((k+1)x)| &= |\sin(kx+x)| \\ &= |\sin kx \cos x + \cos kx \sin x| \\ &\leq |\sin kx||\cos x| + |\cos kx||\sin x| \\ &\leq |k|\sin x|| * |\cos x| + |\cos kx||\sin x| \quad &&\text{(by } I_0H_0\text{)} \\ &\leq k|\sin x| + |\sin x| &&(\because |\cos x| \leq 1 \text{ and } |\cos kx| \leq 1) \\ &= (k+1)|\sin x| \\ &\therefore P(k+1) \text{is true.}\end{aligned}$$

By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$. $\square$

Problem 8. Prove that $\forall n \in \mathbb{N}, n^n \leq (n!)^2$.

Answer:

$$n^n = \prod_{i=1}^{n} n$$

Similarly,

$$\begin{aligned}
(n!)^2 &= (1 * 2 * ... * n)(1 * 2 * ... * n) \\
&= (1 * 1)(2 * 2)...(n * n) \\
&= (1 * n)(2 * (n-1))(3 * (n-2))...((n-1) * 2)(n * 1) \\
&= \prod_{i=1}^{n} i(n+1-i) && (\because \text{ Gauss' Trick})
\end{aligned}$$

$$\prod_{i=1}^{n} n \leq \prod_{i=1}^{n} i(n+1-i)$$

Problem 9. Prove that $\forall n \in \mathbb{N}$

$$1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

Answer:
Let $P(n) : 1 + 2 + ... + n = \frac{n(n+1)}{2}$.
For $P(1) : \frac{1(1+1)}{2} = 1 \therefore P(1)$ is true.
Let I_0H_0 : Assume $P(k)$ is true for some $k \in \mathbb{N}$.
For $P(k+1)$:

$$\begin{aligned}
1 + 2 + ... + k + (k+1) &= \frac{k(k+1)}{2} + k + 1 && (\text{by } I_0H_0) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{k^2 + k + 2k + 2}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

$\therefore P(k+1)$ is true.

By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$.

Problem 10. Prove that $\forall n \in \mathbb{N}$

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Answer:
Let $P(n) : 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.
For $P(1) : \frac{1(2)(2+1)}{6} = 1 = 1^2 = 1 \therefore P(1)$ is true.
Let I_0H_0 : Assume $P(k)$ is true for some $k \in \mathbb{N}$.
For $P(k+1)$:

$$\begin{aligned} 1^2 + 2^2 + ... + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by } I_0H_0\text{)} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k^2+k)(2k+1) + 6(k^2+2k+1)}{6} \\ &= \frac{2k^3 + 3k^2 + k + 6k^2 + 12k + 6}{6} \\ &= \frac{2k^3 + 9k^2 + 13k + 6}{6} \end{aligned}$$

$\pm\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 6$ are the possible zeros of $2k^3 + 9k^2 + 13k + 6$ by Rational Root Theorem.

$$\begin{aligned} &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \\ &\therefore P(k+1) \text{is true.} \end{aligned}$$

By mathematical induction, $P(n)$ is true $\forall n \in \mathbb{N}$.