Math 505A HW9 Neel Gupta

1. Let S_n be simple r.w. Assume $S_0 = 0$.

Let $\tau_0 = \min\{k > 0, S_k = 0\}$, the first return to 0.

Show: $\sum_{j=1}^{n} P(\tau_0 > j) P(S_{n-j} = 0) = 1$.

The moment of the last return given by $\sigma_n = \max\{k \leq n : S_k = 0\}$.

Then for any j $P(\sigma_n = j) = P(S_j = 0) P(\tau_0 > n-j)$

Summing over n, the probability of ~~the~~ $\sigma_n = 1$

$\sum_{j=0}^{n} P(\sigma_n = j) = 1$ then implying By path prop.

$= \sum_{j=0}^{n} P(S_j = 0) P(\tau_0 > n-j) = 1$

Let $n-j = k$ then $j = n-k$ when $j=0$, $k=n$; $j=n$, $k=0$

$\Rightarrow \sum_{k=0}^{n} P(S_{n-k} = 0) P(\tau_0 > k) = 1.$

Q. E. D.

2) Annual rainfall is indep and identical r.v's $\{X_r, r \geq 1\}$. Find prob. of

a) $X_1 < X_2 < X_3 < X_4$

$P(X_1 < X_2 < X_3 < X_4) = \dfrac{1}{\#\text{ of ways to order them}}$

All X_i's have same pdf $f(x)$, so

$$P(X_1 < X_2 < X_3 < X_4) = \iiiint_{x_1 < x_2 < x_3 < x_4} f(x_1) f(x_2) f(x_3) f(x_4)$$

So since they're iid with all X_i's, we can use counting to find the # of ways to order X_1 to X_4 is $4!$ ways, so $P(X_1 < X_2 < X_3 < X_4) = \dfrac{1}{4!}$

2. b) Find $P(X_1 > X_2 < X_3 < X_4)$

$X_2 < X_1 \quad X_2 < X_3 \quad X_3 < X_4$

Let I_A be the event indicator for which rainfall r.v.'s are strictly less than or greater.

$$I_{x_1 > x_2 < x_3 < x_4} = I_{x_1 > x_2} \cdot I_{x_2 < x_3} \cdot I_{x_3 < x_4}$$
$$= I_{x_1 > x_2} \cdot I_{x_2 < x_3 < x_4}$$
$$= (1 - I_{x_1 < x_2})(I_{x_2 < x_3 < x_4})$$

$$E(I_{x_1 > x_2 < x_3 < x_4}) = P(X_1 > X_2 < X_3 < X_4)$$

$$\Rightarrow P(X_1 > X_2 < X_3 < X_4) =$$
$$E(I_{x_2 < x_3 < x_4} - I_{x_1 < x_2 < x_3 < x_4})$$

By part (a) and counting,

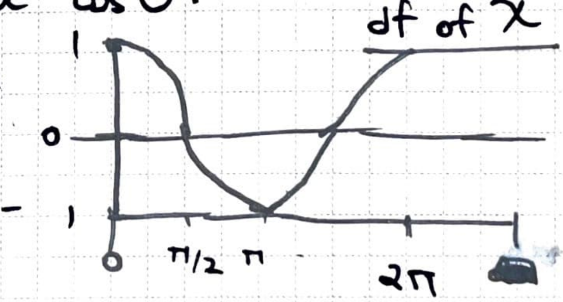
$$= \boxed{\frac{1}{3!} - \frac{1}{4!}}$$

3) Let Θ be uniform on $(0, 2\pi)$ and $a > 0$.

Find the pdf of $Y = a \cos\Theta$.

Let $X = \cos\Theta$. then

$P(X \le x) = \frac{\text{length}}{2\pi}$



where length = the difference between the 2 seperate x coords.

Small case x is percent of data between the points of arccos (x) and $2\pi -$ arccos (x), so

$$\text{length} = \frac{2\pi - \arccos(x) - \arccos(x)}{2\pi} = \boxed{\frac{\pi - \operatorname{acos}(x)}{\pi}}$$

$$\cancel{P(X \le x)} = \frac{\pi - \operatorname{acos}(x)}{\pi} = F_X(x)$$

$$P(Y \le y) = P\left(\frac{\cancel{a}\cos\Theta}{\cancel{a}} \le \frac{y}{a}\right) = P\left(\cos\Theta \le \frac{y}{a}\right)$$

3. a) ~~PDF of y $Y = a\cos\theta$~~

$\cancel{P\left(\cos\theta \le \frac{y}{a}\right) = \frac{\pi - a\cos\left(\frac{y}{a}\right)}{\pi \cdot a}}$

$F_X(x) = \frac{\pi - \cos^{-1}(x)}{\pi} \overset{d}{\Rightarrow} \frac{+1}{\pi\sqrt{1-x^2}}$ $\quad f_Y(y) = \cancel{P(X \le x)}$

$F_Y(y) = P(Y \le y) = P(a\cos\theta \le y)$

$P\left(\cos\theta \le \frac{y}{a}\right) = F_X\left(\frac{y}{a}\right) \quad \because F_X(x) = P(\cos\theta \le x)$

$F_Y(y) = F_X\left(\frac{y}{a}\right)$

Then the density function of Y given by $f_Y(y)$

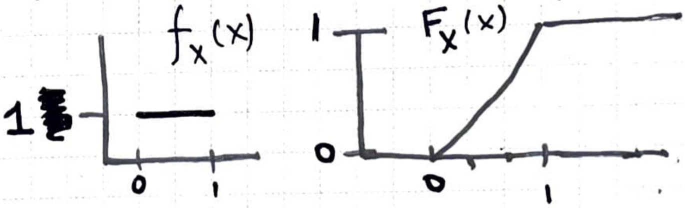
$f_Y(y) = F_Y'(y) = F_X'\left(\frac{y}{a}\right)$

$F_X'\left(\frac{y}{a}\right) = \frac{1}{a}\left(\cancel{\pi}\, 1 - \frac{\cos^{-1}\left(\frac{y}{a}\right)}{\pi}\right)'$

$= \frac{1}{a\pi\sqrt{1-\left(\frac{y}{a}\right)^2}}$

$\underline{f_Y(y) = \frac{1}{a\pi\sqrt{1-\left(\frac{y}{a}\right)^2}}}$

b) Let X be uniform$(0,1)$. Find the df and pdf of $U = 1 - X$.



$F_X(x) = P(X \le x)$

$P(U \le u) = P(1 - X \le u) = P(-X \le u - 1)$

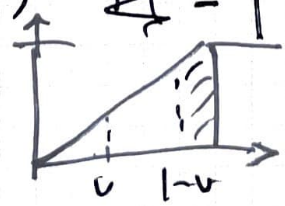
$= P(X \ge 1 - u)$

$F_X(x) = \int_{x=0}^{1} f_X(x) = \int_0^1 1\, dx = x\Big|_0^1 = 1$, so

$F_X(x) = x$ when $x \in [0,1]$

then $P(X \ge 1-u) = 1 - P(X \le 1-u)$

~~then~~ because



the shaded area is 1 - area up to $(1-u)$.

3. b) $P(X \ge 1-v) = 1 - P(X \le 1-v)$

Since $1-v \in (0,1)$,

$\cancel{F_X(1-v) = 1-v =}$

$F_V(u) = P(X \ge 1-v) = 1 - P(X \le 1-v) =$

$1 - F_X(1-v) =$

$1-(1-v) = 1-1+v = u$

then $F_V(u) = u$ when $v \in (0,1)$ so $1-v \in (0,1)$

then $f_V(u) = \frac{d}{du} F_V(u) = 1$

c) Let $X, Y \sim$ uniform$(0,1)$. Find df and pdf of $V = X - Y$.

Range of V is $(-1,1)$.

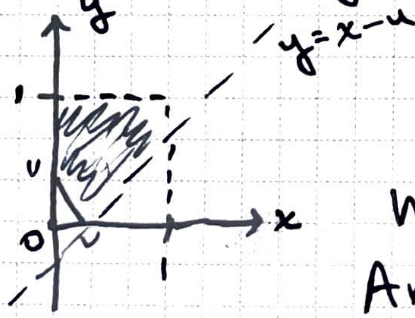
For $\cancel{0<u}$ $-1 < u < 1$, then

$F_V(u) = P(V \le u) = P(X - Y \le u)$

$\cancel{P(X-Y\le u) = P(X \le u+y) = \text{area under curve}}$ region R.

$\cancel{R = \{(x,y): 0<x,y<1,\ X<u+y\}}$

$P(X \le u+y) = |D_u|$ $D_u = \{(x,y): 0 \le x,y \le 1,\ X \le u+y\}$



$y \ge x - u$

$|D_u| = 1 -$ non shaded triangle $= F_V(u)$

When $X=1$, $y = 1-u$ and $\nabla X = 1-u$

Area of white $\Delta = \frac{1}{2}(1-u)^2$

$F_V(u) = 1 - \frac{1}{2}(1-u)^2$

4. Let $X_1, \dots, X_n$ be i.i.d. with common pdf f.

Arrange all $X_i(\omega)$'s in non-decr. order s.t.

$X_{(1)}(\omega), \dots, X_{(n)}(\omega)$ ~~are of~~ is a permutation of the order of $1, 2, \dots, n$ called the order statistics.

$P(X_{(1)} \le y_1, \dots, X_{(n)} \le y_n)$ is joint df.

Let $i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}$ be the index of the r.v. that has now been put into non-decr. order.

$\rightarrow P(X_{(1)} \le y_1, \dots, X_{(n)} \le y_n) =$

$$\bigcup_{i_1, i_2, \dots, i_n} (P(X_{i_1} \le y_1, X_{i_2} \le y_2, \dots, X_{i_n} \le y_n, X_{i_1} < \dots < X_{i_n}))$$

$$= \bigcup_{i_1, \dots, i_n} (P(X_{i_1} \le y_1, \dots, X_{i_n} \le y_n, X_{i_1} < \dots < X_{i_n}))$$

The union of disjoint event's probabilities is the sum of their individual probabilities, so

$$\Rightarrow = \sum_{i_1, i_2, \dots, i_n} P(X_{i_1} \le y_1, X_{i_2} \le y_2, X_{i_3} \le y_3, \dots, X_{i_n} \le y_n, X_{i_1} < \dots < X_{i_n})$$

Symmetric terms

There are $n!$ ways to arrange all by counting

$$\Rightarrow \sum_{i_1, \dots, i_n} P(X_{i_1} \le y_1, \dots, X_{i_n} \le y_n, X_{i_1} < \dots < X_{i_n})$$

$$= n!\, P(X_1 \le y_1, \dots, X_n \le y_n, X_1 < \dots < X_n)$$

~~Since all is are just reordering of indises,~~

$$= n!\, P(X_1 \le y_1, \dots, X_n \le y_n, X_1 < \dots < X_n)$$

The joint density fuction is the integral over all X_i's where $i \in (0, n)$.

4. a) $= n!\, P(X_1 \le y_1, \dots, X_n \le y_n, X_1 < \dots < X_n)$

then has joint pdf

$$= n! \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1) \dots f(x_n)\, dx_1 \dots dx_n$$

$x_1 \le y_1 \dots$
$x_n \le y_n,$
$x_1 < x_2 < \dots < x_n$

$$= n! \int_{-\infty}^{y_n} \cdots \int_{-\infty}^{y_1} f(x_1) \dots f(x_n)\, dx_1\, dx_2 \dots dx_n$$

$x_1 < x_2 < \dots < x_n$

Since we only want the order stats which has the non-decr. stipulation, we can add a flag / Bernoulli r.v. to check if we are in the non-decreasing case instead of having the boundary condition on the integral.

Let $\mathcal{X} = \begin{cases} 1 & \text{if } x_1 < x_2 < \dots < x_n \\ 0 & \text{otherwise} \end{cases}$

then

$$P(X_{(1)} \le y_1, \dots, X_{(n)} \le y_n) = n! \int_{-\infty}^{y_n} \cdots \int_{-\infty}^{y_1} \mathcal{X} f(x_1) \dots f(x_n)\, dx_1 \dots dx_n.$$

Q.E.D.

b) Find marginal density of $X_{(k)}$ of a sample size n.

Marginal density is derivative of marginal df.
Marginal df of k^{th} order statistic is

$$F_{X_{(k)}}(x) = P(X_{(k)} \le x)$$

4. b) Let $\mathbb{I}_k = \begin{cases} 1 & \text{if } X_{(k)} \leq x \\ 0 & \text{otherwise} \end{cases}$

then $\sum_{j=1}^{n} \mathbb{I}_j =:$ a simple r.v. S.

then S is distributed binomially

$S \sim \text{binom}(n, p = F(x) = P(X_k \leq x))$

because there are n indep. trials which all have the samepdf 'f' meaniq they all have Some df 'F' with the same probab of success. For the kth random variable to satisfy ordering condition, $X_{(k)}$ must be the kth smallest random variable, so the stipulation does not exist from $k+1 \to n$.

The df $F_{X_{(k)}}(x) = P(X_{(k)} \leq x)$

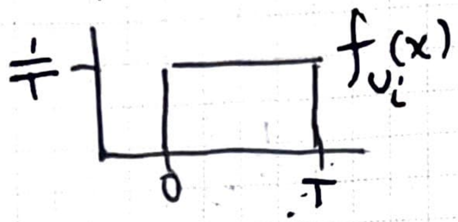
$$= \sum_{j=k}^{n} \binom{n}{j} F(x)^j \cdot (1-F(x))^{n-j}$$

then the pdf $f_{X_{(k)}}(x) = F'_{X_{(k)}}(x)$

$$\sum_{j=k}^{n} f(x) \binom{n}{j} \left(j F(x)^{j-1} (1-F(x))^{n-j} - F(x)^j \cdot (n-j)(1-F(x))^{n-j-1} \right)$$

$$= k \binom{n}{k} f(x) (F(x))^{k-1} (1-F(x))^{n-k}$$

4. c) Find joint density function of n independent uniform $(0,T)$ r.v.s.



Let U_i be a uniform $(0,T)$ where $i \in (0, n]$ and represents all uniforms from $U_1, \dots, U_n$.

By part (a), joint df is given by

$$F(x) = n! \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} \chi f(x_1) \dots f(x_n) dx_1 \dots dx_n$$

then $f(x) = n! \, \chi f(x_1) \dots f(x_n)$

where $\chi = \begin{cases} 1 & \text{if } x_1 < x_2 \dots < x_n \\ 0 & \text{otherwise} \end{cases}$ for a)

Let $\overline{\chi} = \begin{cases} 1 & \text{if } x_1 < x_2 < \dots < x_n \text{ and } 0 \le x_1, x_2, \dots, x_n \le T \\ 0 & \text{otherwise} \end{cases}$

then $F_U(x) = n! \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} \overline{\chi} f(x_1) \dots f(x_n) dx_1 \dots dx_n$

then $f_U(x) = n! \, \overline{\chi} f(x_1) \dots f(x_n)$

where $f(x_i) = \frac{1}{T} \; \forall i$ since Uni $(0,T)$

then $f_U(x) = n! \, \overline{\chi} \left(\frac{1}{T}\right)^n$

5. Let $X_1, \dots, X_n$ be pos. i.i.d. cont. r.v.
Given $m < n$, find $E\left(\frac{S_m}{S_n}\right)$.
Let $S_m = X_1 + \dots + X_m$ and $S_n = X_1 + \dots + X_n$

then $\frac{S_m}{S_n} = \frac{X_1}{S_n} + \dots + \frac{X_m}{S_n}$

5. then $\frac{x_1}{S_n} + \dots + \frac{x_m}{S_n} = 1$

$\frac{x_i}{S_n}$ are all identical and independent

Let $F_{\frac{x_i}{S_n}}(r)$ be the df of $\frac{x_i}{S_n}$

$$F_{\frac{x_i}{S_n}}(r) = P\left(\frac{x_i}{S_n} \le r\right) = \int \cdots \int_{\frac{x_i}{x_1+\dots+x_n} \le r} f(x_1)\dots f(x_n)\,dx_1 \dots dx_n$$

after doing a change of variables

Let $y_1 = x_i$ and $y_i = x_1$

then $P\left(\frac{x_i}{S_n} \le r\right) = \int \cdots \int_{\frac{y_1}{y_1+y_2+\dots+y_n}} f(y_1)\dots f(y_n)\,dy_1 \dots dy_n$

then all $\frac{x_i}{S_n}$'s are ~~independent and~~ identical because switching the indices shows that the probability is not dependent on i, so all $\frac{x_i}{S_n}$ are identical to $\frac{x_1}{S_n} = \frac{x_m}{S_n}$

then let $E\left(\frac{x_1}{S_n}\right) = a$

$\frac{x_1}{S_n} + \dots + \frac{x_m}{S_n} = 1 \rightarrow$ taking expectation of both sides

$E\left(\frac{S_n}{S_n}\right) = E\left(\frac{x_1}{S_n}\right) + \dots + E\left(\frac{x_m}{S_n}\right)$

~~$E(x_1$~~ then $a = \frac{1}{n}$ since there are n items with identical distributions that sum to 1, so $a = \frac{1}{n}$

$$E\left(\frac{S_m}{S_n}\right) = \underbrace{E\left(\frac{x_1}{S_n}\right) + \dots + E\left(\frac{x_m}{S_n}\right)}_{m \text{ terms}} = \boxed{\frac{m}{n}}$$

$\quad\; = (1/n)$