Counting Principles

1. **Multiplication principle.** If k experiments result in $n_1, n_2, \dots, n_k$ outcomes, then there are

$n_1 \cdot n_2 \cdot \ldots \cdot n_k$ different outcomes of k experiments.

If $n_1 = \ldots = n_k = n$, then there are n^k different outcomes.

Ex1. In how many different ways 5 gifts can be awarded to 7 children?

Answer. We have 5 experiments with 7 outcomes (rolling a die with 7 faces 5 times):

$$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5 = 16,807$$

2. Permutations (orderings)

$$\boxed{\begin{array}{l}\text{Number of orderings of } k \text{ out of } n \text{ objects is} \\ n \cdot (n-1) \cdot \ldots \cdot (n-(k-1)) = n \cdot (n-1) \ldots (n-k+1) = \dfrac{n!}{(n-k)!} \\ \text{Number of orderings of } n \text{ objects is } n!\end{array}}$$

Recall $0! = 1$, $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

Ex2. There are 5 history, 3 chemistry and 2 physics books on a shelf.

a) How many orderings of those books are there?

$10! = 3,628,800$

b) How many orderings if the same subject books go together?

Answer. (1) Consider 3 "blocks": $\boxed{H}$, $\boxed{C}$, $\boxed{P}$

There are $3!$ block orderings

(2) There are $5!$ orderings in block $\boxed{H}$

$3!$ " ——— " $\boxed{C}$

$2!$ " ——— " $\boxed{P}$

4 experiments

(3) By multiplication principle:

$3! \cdot 5! \cdot 3! \cdot 2! = 8640$

c) How many orderings if history books go together?

Answer.

(1) 6 blocks, $\boxed{H}$ + 5 blocks containing 1 book each: $6!$

(2) Inside blocks: $5! \cdot 1! \ldots 1! = 5!$

(3) $6!\,5! = 86400$

3. Combinations

3a Binomial coefficient

Total number of distinct unordered groups of size k out of n is

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}} \quad (1)$$

, because by multiplication

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}} \quad (1),$$ because by multiplication principle, total number of orderings of k out of n,

$$\frac{n!}{(n-k)!} = \binom{n}{k}\cdot k!,$$ and solving for $\binom{n}{k}$ we get (1).

Ex3. 7 white and 3 red balls in the box.

a) Two balls are randomly selected (no order). Find $P(\geq 1 \text{ red})$.

Answer. $P(\geq 1 \text{ red}) = 1 - P(2 \text{ white balls selected}) =$

$$= 1 - \frac{\binom{7}{2}}{\binom{10}{2}} = 1 - \frac{\frac{7!}{5!\,2!}}{\frac{10!}{2!\,8!}} = 1 - \frac{\frac{6\cdot 7}{\cancel{2}}}{\frac{9\cdot 10}{\cancel{2}}} = 1 - \frac{7}{15} = \frac{8}{15}.$$

b) Two balls are selected one by one.

Find $P(\geq 1 \text{ red})$ the same answer

Answer. Like in a),

$$P(\geq 1 \text{ red}) = 1 - P(\text{both white}) =$$

$$= 1 - \frac{\binom{7}{2}2!}{\binom{10}{2}2!} = 1 - \frac{21}{45} = 1 - \frac{7}{15} = \frac{8}{15}$$

2nd answer quickly with ordering:

$$P(\text{both white}) = \frac{7\cdot 6}{10\cdot 9} = \frac{7}{15}, \quad 1 - \frac{7}{15} = \frac{8}{15}.$$

$10 \cdot 9$

b) Multinomial coefficient

$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! \dots n_k!}$ is the number of ways to divide n objects into k distinct groups of sizes $n_1, n_2, \dots, n_k$ $(n_1 + \dots + n_k = n)$

because that number equals

$$\binom{n}{n_1}\binom{n-n_1}{n_2} \dots \binom{n-n_1-\dots-n_{k-1}}{n_k} = \frac{n!}{n_1! \dots n_k!}$$

Note if $k=2$, $n = n_1 + n_2$, then

$\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$ is binomial coefficient

Ex4. Deck of card is dealt to 4 people, J is one of them.

Find P(J gets 4 aces).

1st answer. $\dfrac{\binom{48}{9,13,13,13}}{\binom{52}{13,13,13,13}} = 0.0026$

2nd answer. Equivalently, 13 cards are dealt to J.

Hence $P(\text{J gets 4 aces}) = \dfrac{\binom{48}{9}}{\binom{52}{13}} = 0.0026.$

Ex5. How many words can be formed from MOTTO?

Answer 5 letters: 2 T's, 2 O's, 1 M:

Think about seat assignment in a row of 5 "chairs":

$$\binom{5}{2,2,1} = \frac{5!}{2!2!} = 2\cdot 3\cdot 5 = 30.$$

Remark 1. If we have n_1 copies of letter $L_1, \dots, n_k$ copies of letter L_k, then number of words of length $n = n_1 + \dots + n_k$ with those letters is

$$\binom{n}{n_1, \dots, n_k}.$$

Ex 6. There are 8 white and 2 red balls in the box. All of them are taken out one by one. Find prob.

a) 1st ball is red: $\frac{2}{10} = \frac{1}{5}$

b) k-th ball is red:

Answer. $\Omega = \{$all orderings of 10 balls$\}$, $\#\Omega = 10!$

A_k: "kth ball is red".

$$P(A_k) = \frac{\# A_k}{\# \Omega} = \frac{2\cdot 9!}{10!} = \frac{2}{10} = \frac{1}{5}.$$

Ex 7. 20 people shake hands. How many distinct handshakes are there?

Answer. # of handshakes = # of distinct pairs

$$= \binom{20}{2} = \frac{20!}{2!\,18!} = \frac{20\cdot 19}{2} = 190.$$