1. One deck of cards (deck 1) has 5 red cards and 3 black cards, another deck (deck 2) has 2 red cards and 5 black cards. One card is randomly selected and removed from deck 1 and added to deck 2.

a) A card is then randomly selected from the augmented deck 2. It turns out that card is red. What is the probability that a black card was selected and removed from deck 1?

Answer. Let B_1 = "black card was removed from deck 1", R_2 = "red card was selected from deck 2". By Bayes formula,

$$\mathbf{P}(B_1|R_2) = \frac{\mathbf{P}(R_2|B_1)\,\mathbf{P}(B_1)}{\mathbf{P}(R_2|B_1)\,\mathbf{P}(B_1) + \mathbf{P}(R_2|R_1)\,\mathbf{P}(R_1)} = \frac{\frac{2}{8}\frac{3}{8}}{\frac{2}{8}\frac{3}{8} + \frac{3}{8}\frac{5}{8}} = \frac{6}{6+15} = \frac{6}{21} = \frac{2}{7}$$

b) Then a card is selected from the deck 1 again. What is the probability it is black?

Answer. Let B_3 = 'black is selected from deck 1". We need to compute, using $\mathbf{P}\left(B_1^c|R_2\right) = 1 - \frac{2}{7} = \frac{5}{7}$,

$$\begin{aligned}\mathbf{P}(B_3|R_2) &= \mathbf{P}(B_3|B_1)\,\mathbf{P}(B_1|R_2) + \mathbf{P}\left(B_3|B_1^c\right)\mathbf{P}\left(B_1^c|R_2\right) \\ &= \frac{2}{7}\frac{2}{7} + \frac{3}{7}\frac{5}{7} = \frac{19}{49}.\end{aligned}$$

2. We toss n coins, and each one shows heads with probability p, independently from the others. Each coin which shows heads is tossed again. Let X be the number of heads in the first toss of n coins, and Y be the number of heads in the second toss of X coins.

a) Find the generating function and moment generating function of Y.

Answer. We have $Y = \sum_{i=1}^{X} X_i$,where X_i is the number of heads in the ith toss: X_i are Bernoulli(p) independent of X. Hence by Theorem we know,

$$\begin{aligned}G_Y(s) &= G_X\left(G_{X_1}(s)\right) = (p(ps+q)+q)^n \\ &= \left(p^2 s + 1 - p^2\right)^n,\end{aligned}$$

that is Y is binomial$\left(n, p^2\right)$.

`Alternatively`, given $X = k$, Y is binomial(k, p), and

$$\mathbf{E}\left(s^Y|X=k\right) = (ps+q)^k,\, \mathbf{E}\left(s^Y|X\right) = (ps+q)^X.$$

Thus

$$\begin{aligned}G_Y(s) &= \mathbf{E}\left(s^Y\right) = \mathbf{E}\left[\mathbf{E}\left(s^Y|X\right)\right] = \mathbf{E}\left[(ps+q)^X\right] = G_X(ps+q) \\ &= (p(ps+q)+q)^n = \left(p^2 s + 1 - p^2\right)^n.\end{aligned}$$

b) What is the distribution of Y (its mass function)? What are $\mathbf{E}(X), \mathbf{E}\left(X^2\right)$ and $\operatorname{var}(X)$?

Answer. By part a), Y is binomial $\left(n, p^2\right)$.

Since X is binomial(n, p), we have

$$\begin{aligned}
\mathbf{E}(X) &= np, \operatorname{Var}(X) = np(1-p), \\
\mathbf{E}\left(X^2\right) &= np(1-p) + n^2p^2.
\end{aligned}$$

3. k distinct balls are placed into n distinct boxes at random with all n^k ways equally likely, $n > 7$ (equivalently, the balls are placed independently of one another and each ball is equally likely to land in any of the boxes). Let X be the number of non empty boxes.

a) Find $\mathbf{E}(X)$ and $\operatorname{Var}(X)$.

Answer. Let $A_i =$ "ith box is not empty", $X_i = I_{A_i}, i = 1, \ldots, n$. Then the number of non empty boxes

$$X = \sum_{i=1}^{n} X_i.$$

First $\mathbf{E}(X) = \sum_{i=1}^{n} \mathbf{E}(X_i)$,

$$\mathbf{E}(X_i) = \mathbf{P}(A_i) = 1 - \mathbf{P}\left(A_i^c\right) = 1 - \left(1 - \frac{1}{n}\right)^k, \mathbf{E}(X) = n\left[1 - \left(1 - \frac{1}{n}\right)^k\right].$$

For $i \neq j$, by inclusion/exclusion,

$$\begin{aligned}
\mathbf{P}\left(A_i \cap A_j\right) &= 1 - \mathbf{P}\left(A_i^c \cup A_j^c\right) = 1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k, \\
\operatorname{Cov}\left(X_i, X_j\right) &= 1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k - 1 + 2\left(1 - \frac{1}{n}\right)^k - \left(1 - \frac{1}{n}\right)^{2k} \\
&= \left(1 - \frac{2}{n}\right)^k - \left(1 - \frac{1}{n}\right)^{2k},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Var}(X) &= \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{i<j} \operatorname{Cov}\left(X_i, X_j\right) \\
&= n\left[1 - \left(1 - \frac{1}{n}\right)^k\right]\left(1 - \frac{1}{n}\right)^k + 2\binom{n}{2}\left[\left(1 - \frac{2}{n}\right)^k - \left(1 - \frac{1}{n}\right)^{2k}\right] \\
&= n\left[1 - \left(1 - \frac{1}{n}\right)^k\right]\left(1 - \frac{1}{n}\right)^k + n(n-1)\left[\left(1 - \frac{2}{n}\right)^k - \left(1 - \frac{1}{n}\right)^{2k}\right].
\end{aligned}$$

b) Let A be the event that boxes 1 and 2 are both empty, B be the event that boxes 3, 4 are empty, and C be the event that boxes 5, 6, 7 are empty. Find $\mathbf{P}(A \cup B \cup C)$.

Answer. By inclusion/exclusion principle,

$$
\begin{aligned}
& \mathbf{P}(A \cup B \cup C) \\
= \; & \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) \\
& - \mathbf{P}(AB) - \mathbf{P}(BC) - \mathbf{P}(AC) + \mathbf{P}(ABC) \\
= \; & 2\left(1 - \frac{2}{n}\right)^k + \left(1 - \frac{3}{n}\right)^k - \left(1 - \frac{4}{n}\right)^k - 2\left(1 - \frac{5}{n}\right)^k \\
& + \left(1 - \frac{7}{n}\right)^k.
\end{aligned}
$$

4. Let (X, Y) be normal bivariate,

$$
\mathbf{E}(X) = \mathbf{E}(Y) = 0, \mathbf{E}\left(X^2\right) = \mathbf{E}\left(Y^2\right) = 1, \mathbf{E}(XY) = \rho.
$$

a) Find $\mathbf{E}(X|Y = y)$, $\mathbf{E}\left(X^2|Y = y\right)$, and $\mathbf{E}(X|Y)$, $\mathbf{E}\left(X^2|Y\right)$. Given $Y = 1$, what is the best mean square estimate of X?

Answer. (X, Y) is normal bivariate with parameters $\mu_1 = \mu_2 = 0, \sigma_1^2 = \sigma_2^2 = 1$, and ρ. By Theorem we know, given $Y = y$, $X \sim N\left(\rho y, 1 - \rho^2\right)$.

Hence

$$
\begin{aligned}
\mathbf{E}(X|Y = y) &= \rho y, \operatorname{Var}(X|Y = y) = 1 - \rho^2, \\
\mathbf{E}\left(X^2|Y = y\right) &= 1 - \rho^2 + (\rho y)^2 = 1 - \rho^2 + \rho^2 y^2, \\
\mathbf{E}(X|Y) &= \rho Y, \mathbf{E}\left(X^2|Y\right) = 1 - \rho^2 + \rho^2 Y^2.
\end{aligned}
$$

Given $Y = 1$, the best estimate of X is $\mathbf{E}(X|Y = 1) = \rho$.

b) Compute $\mathbf{E}\left(X^2 Y^2\right)$ and the correlation coefficient between X^2 and Y^2.

Answer. Using part a),

$$
\begin{aligned}
\mathbf{E}\left(X^2 Y^2\right) &= \mathbf{E}\left[Y^2 \mathbf{E}\left(X^2|Y\right)\right] = \mathbf{E}\left[Y^2\left(1 - \rho^2 + \rho^2 Y^2\right)\right] \\
&= \left(1 - \rho^2\right)\mathbf{E}\left[Y^2\right] + \rho^2 \mathbf{E}\left(Y^4\right) = 1 - \rho^2 + 3\rho^2 = 1 + 2\rho^2, \\
\operatorname{Cov}\left(X^2, Y^2\right) &= 1 + 2\rho^2 - 1 = 2\rho^2.
\end{aligned}
$$

Since $\operatorname{Var}\left(X^2\right) = \operatorname{Var}\left(Y^2\right) = \mathbf{E}\left(X^4\right) - \left(\mathbf{E}\left(X^2\right)\right)^2 = 3 - 1 = 2$, we have

$$
\rho\left(X^2, Y^2\right) = \frac{2\rho^2}{2} = \rho^2.
$$

5. Let X, Y be independent exponential with parameter 1.
a) Find the joint probability density function of X and Y.
Answer. Because of independence, the joint pdf

$$f(x, y) = f_X(x) f_Y(y) = e^{-x} e^{-y} = e^{-(x+y)}, x > 0, y > 0.$$

b) Find the joint probability density function of (X, V), and confirm that the probability density function of $V = X + Y$ is

$$f_V(v) = ve^{-v}, v > 0.$$

Answer. Consider $U = X, V = X + Y$. Since

$$u = x, v = x + y, x, y \in \mathbf{R},$$

defines one-to-one mapping of $\mathbf{R}^2$ onto $\mathbf{R}^2$, with the inverse

$$x = u, y = v - u,$$

the joint pdf of (U, V) is

$$\begin{aligned} g(u, v) &= f(u, v - u) |J(u, v)| = f(u, v - u) \\ &= e^{-(u+v-u)} = e^{-v}, v>u > 0, \end{aligned}$$

because

$$J(u, v) = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1.$$

Relabeling U as X, the joint pdf of (X, V) is $g(x, v) = e^{-v}, 0 < x < v$.
Then

$$f_V(v) = \int_{-\infty}^{\infty} g(x, v)\, dx = \int_0^v e^{-v} dx = ve^{-v}, v > 0.$$

c) Find the conditional probability density function of X given $V = v$. Find $\mathbf{E}(X|V = v)$, and $\mathbf{P}(X \leq 1/4|V = 1)$.
Answer. The conditional pdf of X given $V = v$ is

$$g(x|v) = \frac{g(x, v)}{f_V(v)} = \frac{e^{-v}}{ve^{-v}} = \frac{1}{v}, 0 < x < v, v > 0.$$

Hence given $V = v > 0$, X is uniform in $(0, v)$:

$$\mathbf{E}(X|V = v) = \frac{v}{2}, \mathbf{P}(X \leq 1/4|V = 1) = \frac{1}{4}.$$

6. a) Let X_n be a random variable with values in $\{0, 1, 2, \ldots, n\}$ such that

$$\mathbf{P}\left(X_n = k\right) = \frac{1}{n+1} \text{ for } k = 0, 1, \ldots, n,$$

with characteristic function

$$\Phi_{X_n}(t) = \frac{1}{n+1} \frac{e^{it(n+1)} - 1}{e^{it} - 1}.$$

Show that

$$\frac{X_n}{n} \overset{D}{\to} U$$

as $n \to \infty$, where U is uniform in $(0, 1)$.

Answer. We will show that $\frac{X_n}{n} \overset{D}{\to} U$ using continuity theorem. The characteristic function

$$\Phi_U(t) = \mathbf{E}\left(e^{itU}\right) = \int_0^1 e^{itu} du = \frac{e^{itu}}{it}\Big|_0^1 = \frac{e^{it} - 1}{it},$$

and, by definition of the derivative,

$$\begin{aligned} \Phi_{\frac{X_n}{n}}(t) &= \Phi_{X_n}\left(\frac{t}{n}\right) = \frac{1}{n+1} \frac{e^{it\frac{n+1}{n}} - 1}{e^{\frac{it}{n}} - 1} = \frac{n}{n+1} \frac{e^{it\frac{n+1}{n}} - 1}{it \frac{e^{\frac{it}{n}} - 1}{\frac{it}{n}}} \\ &\to \frac{e^{it} - 1}{it} \text{ as } n \to \infty. \end{aligned}$$

b) Let $U_1, U_2, \ldots$ be independent uniform in $(0, 1)$. What is the limit in probability (as $n \to \infty$) of

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} I_{\{U_k \leq \frac{1}{2}\}}?$$

Find

$$\lim_{n \to \infty} \mathbf{P}\left(Y_n > 1/2\right).$$

Answer. $X_k = I_{\{U_k \leq 1/2\}}$ are independent Bernoulli$(p = 1/2)$, $k = 1, \ldots, n$. By LLN,

$$Y_n = \bar{X}_n = \frac{X_1 + \ldots + X_n}{n} \to p = \frac{1}{2}$$

in probability, and by CLT,

$$\mathbf{P}\left(\bar{X}_n > 1/2\right) = \mathbf{P}\left(\bar{X}_n - 1/2 > 0\right) = \mathbf{P}\left(\frac{\bar{X}_n - \frac{1}{2}}{\frac{\sqrt{\frac{1}{2}\cdot\frac{1}{2}}}{\sqrt{n}}} > 0\right) \to \mathbf{P}\left(Z > 0\right) = \frac{1}{2}$$

as $n \to \infty$, where $Z \sim N\left(0, 1\right)$.