4.1 Expectation, independence, examples

Continuous r. vectors

Def. (X,Y) is continuous r. vector (jointly continuous) if
$$P(a<X\le b,\ c<Y\le d)=\int_a^b\int_c^d f(x,y)\,dy\,dx \quad \text{for all } a<b,\ c<d.$$
f is called pdf of (X,Y).

Recall 1. If (X,Y) are jointly cont. with pdf $f(x,y)$, then X is continuous with pdf
$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy,\quad -\infty<x<\infty,\ \text{ because}$$
df of X is $F(x)=P(X\le x)=P(X\le x,\ -\infty<Y<\infty)$
$$=\int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f(u,y)\,dy\right)dx\,du.$$

2. If $f(x,y)$ is joint pdf of (X,Y), then
$$P((X,Y)\in B)=\iint_B f(x,y)\,dy\,dx = \text{volume under the surface } z=f(x,y)$$
above B, for any $B\subset\mathbb{R}^2$, $B\in\mathcal{B}$.

Independence.

Def. X,Y are independent if their joint df
$$F(x,y)=P(X\le x,\ Y\le y)=P(X\le x)P(Y\le y)=F_X(x)F_Y(y)$$

$F(x,y) = P(X \le x, Y \le y) = P(X \le x) P(Y \le y) = F_X(x) F_Y(y)$
for all x, y.

Remark. If X, Y are independent, then
$P(X \in B, Y \in D) = P(X \in B) P(Y \in D)$ for any $B, D \in \mathcal{B}$.

Claim. Jointly cont. (X,Y) has X and Y independent
if and only if their joint pdf
$$f(x,y) = f_X(x) f_Y(y) \quad \text{for all } x, y.$$

Why? joint df $F(x,y) = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f(u,v)\, dv \right) du =$
$= F_X(x) F_Y(y) = \int_{-\infty}^{x} f_X(u)\, du \int_{-\infty}^{y} f_Y(v)\, dv, \quad -\infty < x, y < \infty.$

$$\frac{\partial^2 F}{\partial x \partial y} = f(x,y) = f_X(x) f_Y(y).$$

Uniform r. vector

Def. (X,Y) is uniform in $D \subset \mathbb{R}^2$ $(D \in \mathcal{B})$,
if (X,Y) joint pdf
$$f(x,y) = \begin{cases} \frac{1}{|D|} & \text{if } (x,y) \in D \\ 0 & \text{otherwise.} \end{cases}$$

Note For any $B \subset D$, $P((X,Y) \in B) = \frac{|B|}{|D|}$.

Claim. Let (X,Y) be uniform in D. Then

a) X, Y are independent <u>if and only if</u> D is rectangle with sides parallel to coordinate axis.

b) (X, Y) is uniform in the rectangle $D = (a, b) \times (c, d)$ if and only if X, Y are independent uniform: X is uniform in (a, b), Y is uniform in (c, d).

<u>Proof of b)</u> Let (X, Y) be uniform in $(a, b) \times (c, d)$.

For any $(r, l) \subset (a, b)$, $(s, t) \subset (c, d)$

$$P(r < X \le l,\ s < Y \le t) = \frac{(l-r)(t-s)}{(b-a)(d-c)}.$$ Hence

$$P(r < X \le l) = \frac{l-r}{b-a}, \quad P(s < Y \le t) = \frac{t-s}{d-c}, \text{ and}$$

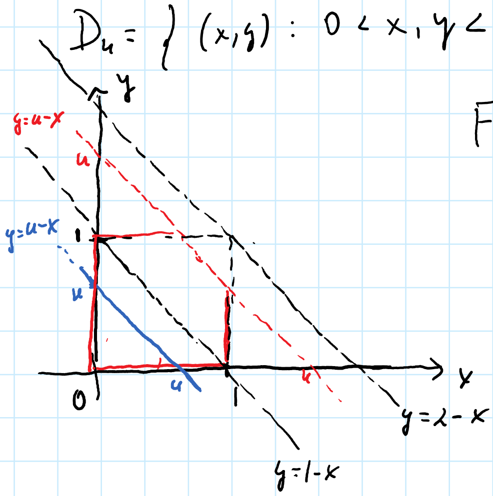
$$P(r < X \le l,\ s < Y \le t) = P(r < X \le l)\, P(s < Y \le t)$$

<u>Ex 2.</u> Let X, Y be independent uniform in $(0, 1)$. Find df and pdf of $X + Y$.

<u>Answer.</u> Range of $V := X + Y = (0, 2)$. For $0 < u < 2$,

$$F(u) = P(X + Y \le u) = P(Y \le u - X) = |D_u|, \text{ where}$$

$$D_u = \{(x, y) : 0 < x, y < 1,\ y \le u - x\}.$$ <u>Draw a picture:</u>



For $0 < u < 1$, $\leftarrow$ ("blue zone"),

$$F(u) = |D_u| = \frac{1}{2} u^2 = \text{area of blue triangle}$$

For $1 < u < 2$,

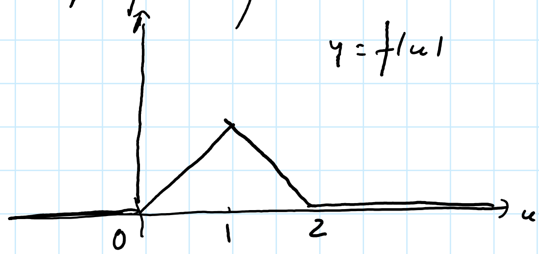
$F(u)$ = area of red trapezoid =

= 1 − area of triangle =

$$= 1 - \frac{1}{2}(2-u)^2$$

$F(u) = 1$ if $u > 2$.

The pdf $f(u) = F'(u) = \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 \le u < 2 \\ 0, & \text{otherwise} \end{cases}$



<u>Symmetry</u>

<u>Ex 2</u>. Let X_1, X_2, X_3 be indep identically distributed continuous with pdf $f(x)$.

a) Find $P(X_1 < X_2)$

<u>Answer</u>. $P(X_1 < X_2) = P(X_2 > X_1) = \frac{1}{2}$, because:

joint df is $f(x_1) f(x_2)$, $\Omega = \{X_1 < X_2\} \cup \{X_2 < X_1\}$

$1 = P(\Omega) = P(X_1 < X_2) + P(X_2 < X_1)$

Now $P(X_1 < X_2) = \iint\limits_{x_1 < x_2} f(x_1) f(x_2)\, dx_1\, dx_2 \underset{=}{\overset{y_1 = x_2,\ y_2 = x_1}{}}$

$= \iint\limits_{y_2 < y_1} f(y_2) f(y_1)\, dy_1\, dy_2 = P(X_1 > X_2)$.

b) Find $P(X_1 < X_2 < X_3)$, $P(X_2 < X_3 < X_1)$

<u>Answer</u>. X_1, X_2, X_3 has joint pdf $f(x_1) f(x_2) f(x_3)$

As in a) we find that

$P(X_1 < X_2 < X_3) = P(X_2 < X_3 < X_1) = P(\text{any ordering}) = \frac{1}{3!}$

because there are $3!$ ways to order 3 numbers, and

$\Omega = \bigcup\limits_{i_1, i_2, i_3} \{X_{i_1} < X_{i_2} < X_{i_3}\}$ disjoint

$$1 = P(\Omega) = \sum_{i_1, i_2 i_3} \underbrace{P(X_{i_1} < X_{i_2} < X_{i_3})}_{a} = 3!\, a .$$

Functions of r.v. and r. vectors

Claim 2. a) let X be cont. with pdf $f(x)$, $Y = g(X)$. Then

$$E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

b) Let (X,Y) have joint pdf $f(x,y)$. Let $V = h(X,Y)$. Then

$$E(V) = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy .$$

Properties of expectation

1. Linearity $E(aX + bY) = aE(X) + bE(Y)$;
2. If $X \le Y$, then $E(X) \le E(Y)$; 3. $E(c) = c$.
4. If X, Y are indep., then $g(X)$, $h(Y)$ are indep. for any functions g, h, and

$$E[g(X)\, h(Y)] = E(g(X))\, E(h(Y)).$$

For instance, $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dx\, dy = \int_{-\infty}^{\infty} x f_X(x) dx \int_{-\infty}^{\infty} y f_Y(y) dy.$