3.10 Path Properties

Thm 4. Let $S_0 = 0$, $p = q = \frac{1}{2}$. Then

$$P(S_1 \dots S_{2n} \neq 0) = P(\tau_0 > 2n) = P(S_{2n} = 0) = \binom{2n}{n} 2^{-2n} \approx \frac{1}{\sqrt{\pi n}},$$

where $\tau_0 = \min\{n \geq 1 : S_n = 0\}$ is moment of 1st return.

Moment of the last return. Let $S_0 = 0$, $p = q = \frac{1}{2}$.
At time n consider the moment of the last return.

$\sigma_n = \max\{k \leq n : S_k = 0\}$

Range of $\sigma_n \subset \{0, 1, \dots, n\}$.

Claim 1. Let $S_0 = 0$, $p = q = \frac{1}{2}$, $\tau_0 = \min\{k \geq 1 \ S_k = 0\}$.
Then for $j = 0, 1, \dots, n$

$$P(\sigma_n = j) = P(S_j = 0)\, P(\tau_0 > n - j)$$

Proof. For $j < n$,

$$P(\sigma_n = j) = P(S_{j+1} \dots S_n \neq 0, S_j = 0)$$
$$= P(S_{j+1} \dots S_n \neq 0 \mid S_j = 0)\, P(S_j = 0) =$$
$$= P(S_1 S_2 \dots S_{n-j} \neq 0 \mid S_0 = 0)\, P(S_j = 0) = P(\tau_0 > n - j)\, P(S_j = 0).$$

Corollary. Let $S_0 = 0$, $p = q = \frac{1}{2}$. Then for $j = 0, 1, \dots, n$

$$P(\sigma_{2n} = 2j) = P(S_{2j} = 0) \cdot P(\tau_0 > 2(n-j))$$
$$\underset{\text{by Thm 4}}{=} P(S_{2j} = 0)\, P(S_{2(n-j)} = 0) \approx \frac{1}{\sqrt{\pi j}} \cdot \frac{1}{\sqrt{\pi (n-j)}} \quad \text{if } j \text{ and } n-j \text{ are large.}$$

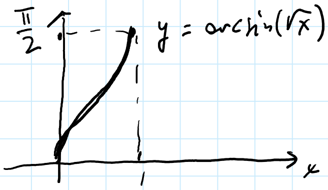
Remark. $\sigma_{2n} \in \{0, 1, \dots, 2n\}$, $\frac{\sigma_{2n}}{2n} \in [0, 1]$.

Limit thm. (Arcsin law) Let $S_0 = 0$, $p = q = \frac{1}{2}$,
$\sigma_{2n} = \max\{k \leq 2n : S_k = 0\}$. Then for $x \in (0, 1)$

$$P\left(\frac{\sigma_{2n}}{2n} \leq x\right) = P(\sigma_{2n} \leq 2nx) \approx \frac{2}{\pi} \arcsin(\sqrt{x})$$

$\sigma_{2n} = \max\{k \le 2n : S_k = 0\}$. Then for $x \in (0,1)$

$$P\left(\frac{\sigma_{2n}}{2n} \le x\right) = P(\sigma_{2n} \le 2nx) \approx \frac{2}{\pi}\arcsin(\sqrt{x})$$



Comment. 1. For $x = \frac{1}{2}$, $\arcsin\left(\sqrt{\frac{1}{2}}\right) = \frac{\pi}{4}$

$P(\sigma_{2n} \le n) \approx \frac{1}{2}$: for $\approx 50\%$ of paths last zero was n time units back ... 0 n 2n

2. There is no oscilations around zero: r.w. staying for long on positive or negative side

Proof. Recall by Corollary above,

$$P(\sigma_{2n} = 2j) = \frac{1}{\pi}\frac{1}{\sqrt{j(n-j)}} \quad \text{if } j, n-j \text{ are large}$$

For $0 < a < b < 1$,

$$P\left(a < \frac{\sigma_{2n}}{2n} \le b\right) = \sum_{a < \frac{2j}{2n} \le b} P(\sigma_{2n} = 2j) \approx \sum_{a < \frac{j}{n} \le b} \frac{1}{\pi\sqrt{j(n-j)}}$$

$$= \frac{1}{\pi}\sum_{a < \frac{j}{n} \le b} \frac{1}{\sqrt{\frac{j}{n}\left(1 - \frac{j}{n}\right)}} \cdot \frac{1}{n} \approx \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{x(1-x)}}\,dx = \frac{2}{\pi}\arcsin(\sqrt{x})\Big|_a^b$$

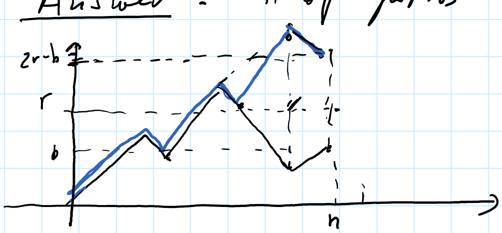
Riemann sum

Note $a < \frac{j}{n} \le b \iff na < j \le nb \iff n(1-b) \le n-j \le n(1-a)$;
For large n, both j and n-j are large.

Distribution of maximum

Ex 1. Let $S_0 = 0$, $M_n = \max\{S_k : 0 \le k \le n\}$.
Reduce $P(M_n \ge r,\ S_n = b)$ with $b < r$ to basic probabilities

Answer. # of paths in $\{M_n \ge r,\ S_n = b\} = N_n(0, 2r-b)$



Black paths = "$S_n = b$, $M_n \ge r$"
Blue paths = "$S_n = 2r-b$".

Hence $P(M_n \ge r,\ S_n = b) = N_n(0, 2r-b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \left(\frac{q}{p}\right)^{r-b} P(S_n = 2r-b)$

Remark. If $p = q = \frac{1}{2}$, $S_0 = 0$, then it can be shown,

$$P(M_n \ge r) = 2P(S_n > r) + P(S_n = r).$$

Chapter 4 . Continuous r.v.

4.1-3 pdf, independence, expectation

Recall: X is cont. r.v. if its df

$$F(x) = \int_{-\infty}^{x} f(y)\,dy, \quad -\infty < x < \infty,$$

for some $f \geq 0$; f is called pdf of X.

Note: $f(x) = F'(x)$ if F is cont. at x.

Def. $f \geq 0$ is called pdf if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Ex 1. Show that $f(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$, $0 < x < 1$, is pdf.

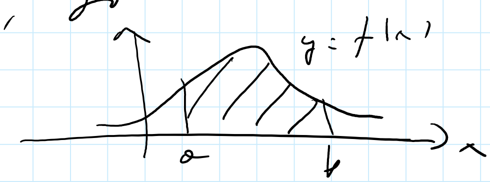
Answer. $f \geq 0$, $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = \frac{2}{\pi}\arcsin(\sqrt{x})\Big|_0^1$

$= \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$.

Properties of cont. r.v. X

1. $F(x) = P(X \leq x)$ is continuous in x, $P(X = x) = 0$ for all x.

2. $P(a < X \leq b) = \int_a^b f(x)\,dx$, for all $a < b$



$P(X \in B) = \int_B f(x)\,dx$, for all $B \in \mathcal{B}$

3. $P(a < X \leq b) = \int_a^b f(x)\,dx \approx f(a)(b-a)$ if $b \approx a$ and f cont. at a.

cont. r.v.

Expectation. For a cont. r.v. X with pdf $f(x)$, expectation of X is the number

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \lim_{n\to\infty} \sum_k \frac{k}{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} =$$

$$= \lim_{n\to\infty} \sum_k \underset{\text{value of } X}{\frac{k}{n}} \underbrace{P\left(\frac{k}{n} < X \le \frac{k+1}{n}\right)}_{P(\text{"}X=\frac{k}{n}\text{"})}$$

Meaning the same as in discrete case