1.4 Conditional probability

Def. $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$, $A, B \in \mathcal{F}$, $P(B) > 0$.

Remark 1. a) $P(A|B)$ is probability in A.
b) If Ω is finite and outcomes equally likely, then

$$P(A|B) = \frac{\#(A\cap B)}{\# B}.$$

Ex 1. 8 white and 2 red balls in the box. They are taken out one by one. Find

a) $P(R_6) = \frac{2}{10} = \frac{1}{5}$ (R_6 = "6th is red").

b) $P(R_6 | W_1 W_2 W_3) = \frac{2}{7}$

c) $P(R_6 | W_1 R_2 W_3) = \frac{1}{7}$; d) $P(R_6 | W_1 R_2 R_3) = 0$.

Important formulas (based on $P(A|B) = \frac{P(AB)}{P(B)}$

① **Multiplication rule**

$$P(A\cap B) = P(A|B)\,P(B) = P(B|A)\,P(A).$$

In general,

$$P(A_1 \dots A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 A_2)\dots P(A_n|A_1\dots A_{n-1})$$

Ex 2. There are 7 white and 3 red balls in the box. Find $P(R_1 W_2 R_3)$

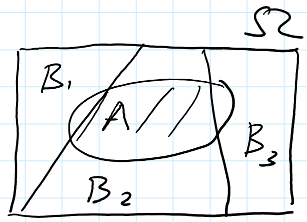
Answer. $P(R_1 W_2 R_3) = P(R_1)\,P(W_2|R_1)\,P(R_3|R_1 W_2)$

$= \frac{3}{}\ \frac{7}{}\ \frac{2}{}$

$= \frac{\ }{10} \cdot \frac{\ }{9} \cdot \frac{\ }{8}$

<u>Counting</u> $\frac{3 \cdot 2 \cdot 7}{10 \cdot 9 \cdot 8}$

(2) <u>Totaly probability law</u>



If $B_1, \dots, B_n$ are disjoint, and $\Omega = \bigcup_{i=1}^{n} B_i$, then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

<u>Note</u> $\Omega = B \cup B^c$ is the simplest case:

$$P(A) = P(A \mid B) P(B) + P(A \mid B^c) P(B^c).$$

<u>Why?</u> $A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (AB_i)$ $\leftarrow$ disjoint

$$P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_i P(A \mid B_i) P(B_i)$$

(3). <u>Bayes formula</u> (reversal of conditioning)

$$P(B \mid A) = \frac{P(A \mid B) P(B)}{P(A)} = \frac{P(A \mid B) P(B)}{P(A \mid B) P(B) + P(A \mid B^c) P(B^c)}.$$

<u>Why?</u> $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B) P(B)}{P(A)} = \dots$

<u>Ex2</u> (false positive) A rare disease affects 1 person in 1 000. The test of this disease is positive for 99% of ill and 2% of healthy people.

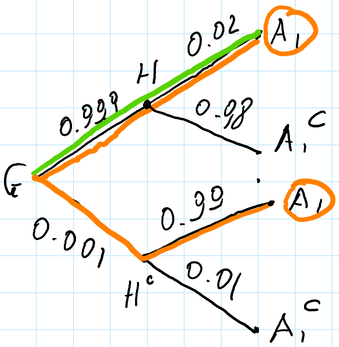
a) Find probability of positive test for a randomly selected person.

Answer. A_1 = "test positive", A_1^c = "test negative",
H = "healthy", H^c = "ill"

We know: $P(A_1|H) = 0.02$, $P(A_1|H^c) = 0.99$
$P(H) = 0.999$, $P(H^c) = 0.001$.

By total probability law,

$$P(A_1) = P(A_1|H)P(H) + P(A_1|H^c)P(H^c) =$$
$$= 0.02 \cdot 0.999 + 0.99 \cdot 0.001$$
$$= 0.02097$$



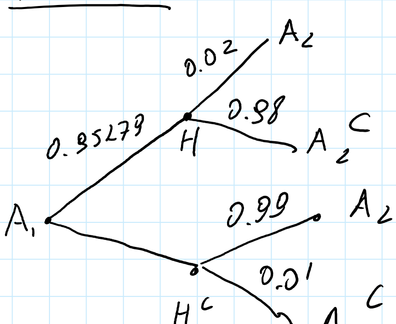
b) J is tested positive. Find prob. J is healthy.

Answer. $P(H|A_1) = \dfrac{P(A_1|H)\,P(H)}{P(A_1)}$

$$= \frac{0.02 \cdot 0.999}{0.02 \cdot 0.999 + 0.99 \cdot 0.001} = 0.95279$$

c) J decides to be tested again and test is positive. Find probability J is healthy.

Answer. $P(H|A_1 A_2) =$



$$= \frac{0.02 \cdot 0.95279}{0.02 \cdot 0.95279 + 0.99(1 - 0.95279)}$$

$$= 0.29$$

Ex4 (TV game). Award is placed behind 3 doors.

You choose 1st door. Presenter opens 2nd door and offers to switch to 3rd door.
Is it worth to switch?

<u>Answer</u>. L_k = "award behind kth door", $k = 1, 2, 3$

B = 2nd door opened.

We need to compare $P(L_1/B)$, $P(L_3/B)$, $P(L_2/B) = 0$

<u>We know</u> (before B happened) $P(L_1) = P(L_2) = P(L_3) = \frac{1}{3}$.

$P(B/L_1) = \frac{1}{2}$, $P(B/L_2) = 0$, $P(B/L_3) = 1$.

$$P(B) = P(B/L_1)\,P(L_1) + P(B/L_2)\,P(L_2) + P(B/L_3)\,P(L_3)$$
$$= \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}.$$

$$P(L_3/B) = \frac{P(B/L_3)\,P(L_3)}{P(B)} = \frac{1\cdot 1/3}{1/2} = 2/3$$

$P(L_1/B) = 1/3$.

$\Big\{$ <u>Comment</u>. We found $P(L_1/B) = P(L_1) = \frac{1}{3}$, $P(B/L_1) = P(B) = 1/2$ / We say L_1, B are indep.