# 4.4. Normal r.v.

**Def.** a) $X \sim N(\mu, \sigma^2)$ if pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty. \quad \Big\} f(x) = \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)$$

b) $Z \sim N(0,1)$ is called standard normal: $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in \mathbb{R}$.

**Basic facts**

2. If $Z \sim N(0,1)$, then

3. a) If $Z \sim N(0,1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$:

$E(X) = \mu$, $Var(X) = \sigma^2$ ($\sigma$ is standard deviation of $X$)

b) If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.

Why a)? If $Z \sim N(0,1)$ and $X = \mu + \sigma Z$, then

$$F_X(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right),$$

$$f_X(x) = F_X'(x) = F_Z'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \varphi\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}.$$

4. If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.

5. Concentration around $\mu$: for $X \sim N(\mu, \sigma^2)$,

$$P(-3\sigma \le X - \mu \le 3\sigma) = 0.997$$

$$P(-2\sigma \le X - \mu \le 2\sigma) = 0.9545$$

$$P(-\sigma \le X - \mu \le \sigma) = 0.6827$$

## 4.8 Sums of cont. r.v.

**Thm 1.** Assume $(X,Y)$ has joint pdf $f(x,y)$. Then

a) $V = X+Y$ is continuous with pdf

$$f_V(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx = \int_{-\infty}^{\infty} f(z-y, y)\, dy, \quad -\infty < z < \infty.$$

b) If $X, Y$ are continuous independent, then joint pdf $f(x,y) = f_X(x) f_Y(y)$, and

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy.$$

Proof of a) df of $V = X+Y$ is

$$F(z) = P(X+Y \le z) = \iint_{x+y \le z} f(x,y)\, dx\, dy =$$

$$\boxed{\{(x,y) : x+y \le z\} = \{(x,y) : -\infty < x < \infty,\ y \le z-x\}}$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{z-x} f(x,y)\, dy \right) dx, \qquad -\infty < z < \infty.$$

$$F'(z) = f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx, \quad -\infty < z < \infty.$$

Applications of Thm 1

**Claim 1.** Let $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ be independent, then $aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

## Sum of exponential r.v.

**Ex 1.** Let $X_1, X_2$ be independen exponential $(\lambda)$. Find pdf of $X_1 + X_2$.

**Answer.** $X_1, X_2$ have the same pdf $f(x) = \lambda e^{-\lambda x}$, $x>0$.

Range of $X_1 + X_2 = (0, \infty)$. For $z > 0$,

$$f_{X_1+X_2}(z) = \int_{-\infty}^{\infty} \underbrace{\underbrace{f(z-y)}_{z-y>0}\, \underbrace{f(y)}_{y>0}}_{0<y<z}\, dy = \int_0^z \lambda e^{-\lambda(z-y)} \lambda e^{-\lambda y}\, dy$$

$$= \lambda^2 \int_0^z e^{-\lambda z}\, dy = \lambda^2 e^{-\lambda z} z = \underline{(\lambda z)\lambda e^{-\lambda z}, \quad z > 0.}$$

**Comment on Ex 1.** $X_1 + X_2$ with pdf $\underline{f(x) = (\lambda x)\lambda e^{-\lambda x}, x>0,}$ is gamma r.v. with parameters $\lambda$, $n = 2$:

we write $X_1 + X_2 \sim \Gamma(\lambda, n=2)$.

**Def.** Continuous r.v. $X$ is gamma $(\lambda, n)$ if its pdf

$$f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x}, \quad x > 0.$$

**Note:** a) $\Gamma(\lambda, 1)$ is exponential $(\lambda)$.

b) If $X, Y$ are independent $\Gamma(\lambda, 1)$, then $X+Y \sim \Gamma(\lambda, 2)$.

**Claim 2.** If $X \sim \Gamma(\lambda, n)$, $Y \sim \Gamma(\lambda, m)$ are independent, then $X + Y \sim \Gamma(\lambda, n+m)$.

**Remark 1.** a) If $X$ is exponential $\lambda$, then $E(X) = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$.

b) df of $X \sim \Gamma(\lambda, n)$ is known

$$P(X > t) = \int_t^\infty \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x} dx = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

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If $X \sim$ exponential $(\lambda)$, then $P(X > t) = e^{-\lambda t}$, $t > 0$.

c) If $X \sim \Gamma(\lambda, n)$, then $X = X_1 + \ldots + X_n$, where $X_i$ are indep. exponential $(\lambda)$: $E(X) = \frac{n}{\lambda}$, $Var(X) = \frac{n}{\lambda^2}$.

<u>Some exercises</u>.

<u>Ex 1</u>.