5.9-10 Two limit theorems

Let $X_1, X_2, \ldots$ be i.i.d. We say $X_1, \ldots, X_n$ is an <u>independent sample of size n</u>.

$T = T_n = X_1 + \ldots + X_n$ is called sample sum (total)

$\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n} = \dfrac{T_n}{n}$ is called <u>sample mean</u>

<u>Example 1</u>. A coin with $P(H) = p$ is tossed repeatedly, $X_k = \#$ of H in kth toss, $k = 1, \ldots, n$, are i.i.d. Bernoulli(p),

$T = X_1 + \ldots + X_n = \#$ of H in n tosses $\sim$ binomial(n,p)

$\bar{X} = \dfrac{X_1 + \ldots + X_n}{n} = \frac{1}{n} \cdot T$ is <u>sample proportion</u> (relative frequency) of H in n tosses.

<u>Claim 1</u>. Let $X_1, \ldots, X_n$ be i.i.d. with $E(X) = \mu$, $Var(X) = \sigma^2$.

Then a) $E(T) = n\mu$, $Var(T) = n\sigma^2$

b) $E(\bar{X}) = \frac{1}{n} \cdot n\mu = \mu$, $Var(\bar{X}) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$, because $\bar{X} = \frac{1}{n} T$.

c) $\phi_T(t) = \phi_X(t)^n$,

$\phi_{\bar{X}}(t) = \phi_{\frac{1}{n}T}(t) = \phi_X\left(\frac{t}{n}\right)^n$

1. <u>LLN</u>.

<u>Thm 1</u>. Let $X_1, \ldots, X_n$ be i.i.d. with $E(X) = \mu$, $Var(X) = \sigma^2$.

Then $\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n} \longrightarrow \mu$ in probability: for each $\varepsilon > 0$,

$P(|\bar{X}_n - \mu| > \varepsilon) \longrightarrow 0$ as $n \to \infty$.

Comment: It can be proved that $X_n \to \mu$ with prob. 1: $\bar{X}_n \approx \mu$ for large n.

Proof. Since $E(|X|) < \infty$,

$$\phi_X(t) = 1 + it\mu + r(t), \text{ with } \frac{r(t)}{t} \to 0 \text{ as } t \to 0$$

Hence

$$\phi_{\bar{X}_n}(t) = \phi_X\left(\frac{t}{n}\right)^n = \left(1 + \frac{it\mu}{n} + \frac{nr(\frac{t}{n})}{n}\right)^n \to e^{it\mu} \text{ for all } t$$

$$\boxed{\text{Calculus: If } c_n \to c, \text{ then } \left(1 + \frac{c_n}{n}\right)^n \to e^c}$$

Thus $\bar{X}_n \xrightarrow{D} \mu \iff \bar{X}_n \to \mu$ in probability.

Example 1 If $Y = Y_n$ is binomial (n, p), then $\frac{Y_n}{n} \to p$ as $n \to \infty$, because $Y_n = X_1 + \dots + X_n$, where X_i are indep. Bernoulli(p), $\frac{Y_n}{n}$ is relative frequency of successes.

2 Let X_i be indep. uniform in (a, b), let h be a bounded function on $[a, b]$. Then

$$\frac{h(X_1) + \dots + h(X_n)}{n} \approx E[h(X)] = \frac{1}{b-a}\int_a^b h(x)\,dx$$

for large n.

2. Central limit theorem (CLT)

Question 1. How well $\bar{X}_n$ approximates μ?

Question 2. Let $X_1, \dots, X_n$ be i.i.d. What is the distribution of

$$T_n = X_1 + \dots + X_n \text{ and } \bar{X}_n = \frac{X_1 + \dots + X_n}{n} \text{ for large } n?$$

$T_n = X_1 + \dots + X_n$ and $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ for large n?

Answers are given by

<u>Thm 2</u> (CLT) Let X_k be i.i.d., $E(X) = \mu$, $Var(X) = \sigma^2$

Then for large n,

a) $T_n = X_1 + \dots + X_n$ is approximately $N(n\mu, n\sigma^2)$

b) $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ " ——— " $N(\mu, \frac{\sigma^2}{n})$

c) $\frac{T_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ " ——— " $Z \sim N(0,1)$

a), b), c) are <u>equivalent</u>

<u>Ex 3</u>. Estimate the error of μ approximation by $\bar{X}_n$.

<u>Answer</u>. By c), for large n,

$$P\left(\frac{|\bar{X}_n - \mu|}{\sigma/\sqrt{n}} \le 3\right) \approx P(|Z| \le 3) = P(-3 \le Z \le 3) = 0.9973$$

Hence $|\bar{X}_n - \mu| \le \frac{3\sigma}{\sqrt{n}}$ with certainty of $> 99\%$

<u>Proof of c)</u> in the case $\mu = 0, \sigma = 1$: $\frac{T}{\sqrt{n}} = \frac{1}{\sqrt{n}} T \xrightarrow{D} Z \sim N(0,1)$.

$$\boxed{\begin{array}{l}\text{Fact: If } E(V^2) < \infty, \text{ then} \\ \phi_V(t) = 1 + it\,E(V) - \frac{t^2}{2} E(V^2) + r(t), \text{ and } \frac{r(t)}{t^2} \xrightarrow{t \to 0} 0.\end{array}}$$

Since $\mu = 0, \sigma = 1$, we have $E(X^2) = Var(X^2) = 1$,

$\phi_X(t) = 1 - \frac{t^2}{2} + r(t)$ and $\frac{r(t)}{t^2} \to 0$. Then

$$\phi_T(t) = \phi_X(t)^n, \quad \phi_{\frac{1}{\sqrt{n}}T}(t) = \phi_X\left(\frac{t}{\sqrt{n}}\right)^n =$$

$$= \left(1 - \frac{t^2}{2n} + \frac{r\left(\frac{t}{\sqrt{n}}\right) \cdot n}{n}\right)^n \longrightarrow e^{-\frac{t^2}{2}} = \phi_Z(t)$$

CLT for binomial. Let $Y_n \sim$ binomial(n, p), then approximately

CLT for binomial. Let $Y_n \sim \text{binomial}(n,p)$, then approximately

a) $Y_n \sim N(np, np(1-p))$

b) $\hat{p}_n = \frac{Y_n}{n} \sim N\left(p, \frac{p(1-p)}{n}\right)$

c) $\frac{Y_n - np}{\sqrt{np(1-p)}} = \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$.

why? $Y_n = X_1 + \dots + X_n$, $X_i \sim \text{Bernoulli}(p)$ independent.

CLT for Poisson. Let $X \sim \text{Poisson}(\lambda)$. Then

$$Y = \frac{X-\lambda}{\sqrt{\lambda}} \xrightarrow{D} Z \sim N(0,1) \quad \text{as } \lambda \to \infty.$$

$$\phi_Y(t) = e^{-i\sqrt{\lambda}t} \exp\{\lambda(e^{i\frac{t}{\sqrt{\lambda}}} - 1)\,] = \exp\{\lambda[\underbrace{e^{i\frac{t}{\sqrt{\lambda}}} - 1 - \frac{it}{\sqrt{\lambda}}}_{(\frac{it}{\sqrt{\lambda}})^2}]\}$$

$$\approx e^{-t^2/2} = \phi_Z(t) \text{ for large } \lambda.$$

Joint cf, mgf, pf

Def. a) Joint cf of X, Y is

$$\phi(s,t) = E\left[e^{i(sX+tY)}\right], \quad s,t \in \mathbb{R}.$$

b) Joint mgf of X, Y is

$$M(s,t) = E\left(e^{sX+tY}\right), \quad -\varepsilon < s,t < \varepsilon \text{ for some } \varepsilon > 0$$

c) Joint pf of X, Y is

$$G(s,t) = E\left(s^X t^Y\right)$$

Note $\phi_X(s) = \phi(s,0)$, $\phi_Y(t) = \phi(0,t)$,

$M_X(s) = M(s,0)$, $M_Y(t) = M(0,t)$,