This week: 3.3, 3.6 Expectation, Variance, Covariance
3.4 Indicator method

3.3 Expectation, Variance

Def. $E(X) = \sum_x x\, P(X=x)$

Note a) If $X \geq 0$, then $E(X) = \sum_{x \geq 0} x\, P(X=x) \geq 0$;

b) If $X \geq 0$ and $E(X) = 0$, then $X = 0$ with probability 1:

$$\sum_{x \geq 0} x\, P(X=x) = 0 \Rightarrow x\, P(X=x) = 0 \text{ for all } x$$

Expectation of a function of r.v.:

$$\boxed{\begin{aligned} E\, g(X) &= \sum_x g(x)\, P(X=x) \\ E\, h(X,Y) &= \sum_{x,y} h(x,y)\, P(X=x, Y=y). \end{aligned}} \qquad (1)$$

Properties of $E(X)$:

1. $E(X+Y) = E(X) + E(Y)$, $E(aX+bY) = aE(X) + bE(Y)$

2. $E(XY) = E(X)E(Y)$ if X, Y are independent.

Why 2.? By (1) with $h(x,y) = xy$,

$$E(XY) = \sum_{x,y} xy\, P(X=x, Y=y) = \sum_{x,y} x\, P(X=x)\, y\, P(Y=y)$$

$$= \left(\sum_x x\, P(X=x)\right)\left(\sum_y y\, P(Y=y)\right) = E(X)\, E(Y).$$

Ex1 a) Let X be Bernoulli(p). Find

$$E(X) = 1\cdot p + 0\cdot(1-p) = p$$

b) Let X be binomial(n,p). Find

$$E(X) = n\cdot p$$

Answer. Recall $X = X_1 + \dots + X_n$, where X_i are Bernoulli(p):

$$E(X) = E(X_1) + \dots + E(X_n) = n\cdot p.$$

Ex2. a) Let X be geometric(p): $P(X=k) = q^{k-1}p$, $k = 1, 2, \dots$

Confirm $E(X) = \frac{1}{p}$

Answer. $E(X) = \sum_{k=1}^{\infty} k\, P(X=k) = \sum_{k=1}^{\infty} k\, q^{k-1} p$

$$= \frac{d}{dq}\left(\sum_{k=0}^{\infty} q^k p\right) = \frac{d}{dq}\left(\frac{p}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

b) Let X be Poisson(λ): $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, \dots$

Confirm $E(X) = \lambda$

Answer. $E(X) = \sum_{k=1}^{\infty} k\, P(X=k) = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} =$

$$= e^{-\lambda}\cdot\lambda \sum_{k=1}^{\infty} \frac{k\,\lambda^{k-1}}{k!} = e^{-\lambda}\cdot\lambda\, \frac{d}{d\lambda}\Big(\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}_{=e^{\lambda}}\Big) = e^{-\lambda}\,\lambda\, e^{\lambda} = \lambda.$$

Ex3. A deck of cards is dealt to 4 people. J is one of them. What is expected number of aces J gets?

Answer. X_i = # of aces ith player gets, $i = 1, 2, 3, 4$.

Now, $P(X_i = k)$, $k = 0, 1, 2, 3, 4$, does not depend on i:

$E(X_1) = E(X_2) = E(X_3) = E(X_4) =: a \leftarrow$ to be determined.

On the other hand, $X_1 + X_2 + X_3 + X_3 = 4$. Hence

$E(X_1) + E(X_2) + E(X_3) + E(X_4) = E(4) = 4$;

$$4 \cdot a = 4, \quad a = E(X_i) = 1.$$

Moments of X

Def. k-th moment of X is the number

$$m_k = E(X^k) = \sum_x x^k P(X=x), \quad k = 1, 2, \ldots$$

Note 1. $m_1 = E(X)$, $m_2 = E(X^2)$

2. $E(X - m_1) = E(X) - m_1 = 0$.

$X - m_1$ is called "centered X".

Def. kth central moment of X is the number

$$\sigma_k = E[(X - m_1)^k], \quad k = 1, 2, \ldots$$

Def. a) $\sigma_2 = E[(X - m_1)^2]$ is called variance of X (denoted $Var(X)$).

b) $\sigma = \sqrt{\sigma_2} = \sqrt{Var(X)}$ is called standard deviation of X

Meaning of $Var(X)$. It is mean deviation of X from its mean $m_1 = E(X)$:

$$E[\underbrace{(X - m_1)}_{\text{deviation}}{}^2] = Var(X) = \sigma_2.$$

Basic properties of $Var(X)$

Basic properties of $Var(X)$

1. $Var(X) = E[(X-m_1)^2] \geq 0$ (always).

   $Var(X) = 0$ if and only if X is constant with prob. 1.

2. $Var(X) = E[(X-m_1)^2] = E(X^2) - m_1^2 = m_2 - m_1$

3. $Var(aX+b) = a^2 Var(X)$

Why 2.? $E[(X-m_1)^2] = E(X^2) - 2m_1E(X) + m_1^2 = m_2 - 2m_1^2 + m_1^2 = m_2 - m_1^2$.

3? $E(aX+b) = aE(X) + b = am_1 + b$

$aX + b - E(aX+b) = a(X-m_1)$

$Var(aX+b) = E[a^2(X-m_1)^2] = a^2 Var(X)$.