**1.** A box contains 5 fair coins and other 4 coins with heads probability 1/3.

a) One coin is randomly selected and tossed twice. We see heads in the first toss and tails in the second toss. Find probability that the coin is fair.

*Answer.* Consider the events: $H_1 =$ "heads in the 1st toss", $T_2 =$ "tails in the 2nd toss", $N =$ "selected coin is fair", $N^c =$ "selected coin is not fair". By Bayes formula

$$\begin{aligned}
\mathbf{P}(N|H_1T_2) &= \frac{\mathbf{P}(H_1T_2|N)\,\mathbf{P}(N)}{\mathbf{P}(H_1T_2|N)\,\mathbf{P}(N) + \mathbf{P}(H_1T_2|N^c)\,\mathbf{P}(N^c)} \\
&= \frac{\left(\frac{1}{2}\right)^2 \frac{5}{9}}{\left(\frac{1}{2}\right)^2 \frac{5}{9} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{9}} = \frac{45}{77}.
\end{aligned}$$

b) We throw away this coin and choose randomly another coin from the 8 coins in the box and toss it. What is the probability to see heads? $\frac{45}{77} = 0.584\,42$

*Answer.* Consider $H_3 =$ "heads in the toss of another coin coin". Then by total probability law 3 times,

$$\begin{aligned}
\mathbf{P}(H_3|H_1T_2) &= \mathbf{P}(H_3|N)\,\mathbf{P}(N|H_1T_2) + \mathbf{P}(H_3|N^c)\,\mathbf{P}(N^c|H_1T_2) \\
&= \left(\frac{4}{8}\frac{1}{2} + \frac{4}{8}\frac{1}{3}\right)\cdot\frac{45}{77} + \left(\frac{5}{8}\frac{1}{2} + \frac{3}{8}\frac{1}{3}\right)\left(1 - \frac{45}{77}\right) = \frac{131}{308} = 0.425\,32.
\end{aligned}$$

**2.** A fair die is rolled $n$ times. You are paid \$1 for any two consecutive identical scores. For instance, with $n = 7$ for 1222551 you get 3 dollars.

Let $X$ be the total payment.

a) Find $\mathbf{E}(X)$ and $\mathrm{Var}(X)$. Hint. Use indicators.

*Answer.* Let $A_i =$ " $i$th and $(i+1)$ scores are identical, $i = 1, \ldots, n-1$. Then

$$\begin{aligned} X &= \sum_{i=1}^{n-1} I_{A_i}, \mathbf{E}(X) = \sum_{i=1}^{n-1} \mathbf{E}(I_{A_i}) = \sum_{i=1}^{n-1} \mathbf{P}(A_i), \\ \mathrm{Var}(X) &= \sum_{i=1}^{n-1} \mathrm{Var}(I_{A_i}) + 2 \sum_{1 \le i < j \le n-1} \mathrm{Cov}\left(I_{A_i}, I_{A_j}\right). \end{aligned}$$

Assuming independence of the rolls (or assuming all $6^n$ scores equally likely and counting),

$$\begin{aligned} \mathbf{P}(A_i) &= \sum_{k=1}^{6} \mathbf{P}(\text{score } k \text{ in } i\text{th and } i+1) = 6 \cdot \frac{1}{6^2} = \frac{1}{6}, \\ \mathrm{Var}(I_{A_i}) &= \mathbf{P}(A_i)[1 - \mathbf{P}(A_i)] = \frac{1}{6}\left(1 - \frac{1}{6}\right) = \frac{5}{36}. \end{aligned}$$

Because of independence, $\mathrm{Cov}\Big(I_{A_i}, I_{A_j}\Big) = 0$ if $i + 1 < j$, and

$$\begin{aligned} \mathrm{Cov}\left(I_{A_i}, I_{A_{i+1}}\right) &= \mathbf{P}(A_i \cap A_{i+1}) - \mathbf{P}(A_i)\mathbf{P}(A_{i+1}), \\ \mathbf{P}(A_i \cap A_{i+1}) &= \mathbf{P}(i\text{th, } i+1 \text{ and } i+2 \text{ scores identical}) = 6 \cdot \frac{1}{6^3} = \frac{1}{36}. \end{aligned}$$

Hence $\mathrm{Cov}\Big(I_{A_i}, I_{A_{i+1}}\Big) = 0$, $A_i$ and $A_{i+1}$ are independent as well, and

$$\mathrm{Var}(X) = \sum_{i=1}^{n-1} \mathrm{Var}(I_{A_i}) = (n-1) \cdot \frac{5}{36} = \frac{5(n-1)}{36}.$$

b) Find probability that there are no identical consecutive scores in the first 4 rolls.

*Answer.* By b), all $A_i$ are pairise independent, and

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = 6\left(\frac{1}{6}\right)^4 = \frac{1}{6^3} = \mathbf{P}(A_1)\mathbf{P}(A_2)\mathbf{P}(A_3).$$

Thus $A_1, A_2, A_3$ are independent, and

$$\mathbf{P}(A_1^c \cap A_2^c \cap A_3^c) = \mathbf{P}(A_1^c)\mathbf{P}(A_2^c)\mathbf{P}(A_3^c) = \left(1 - \frac{1}{6}\right)^3 = \frac{5^3}{6^3}.$$

**3.** The number $X$ of electrons that hit the plate is Poisson with parameter $\lambda_1 = 2$. Every impact produces independently a number of secondary electrons that is Poisson with parameter $\lambda_2 = 1$.

Let $Y$ be the total number of secondary electrons.

a) Find $\mathbf{P}(Y = j | X = n), n \geq 0, j \geq 0$. Hint. A sum of independent Poisson is Poisson.

b) Find $\mathbf{E}(Y|X = n)$ and $\mathbf{E}(Y|X)$ and $\mathbf{E}(Y)$.

c) Find $\mathbf{E}(Y^2|X = n)$, $\mathbf{E}(Y^2|X)$ and $\mathrm{Var}(Y)$.

*Answer.* a) Given $X = n$, we have $Y = \sum_{i=1}^{n} Y_i$, where $Y_i$ are independent Poisson(1), $Y_i$ is the number of secondary electrons produced by the $i$th primary electron among $n$. Thus, given $X = n$, $Y$ is Poisson$(n)$ : for $n \geq 0$,

$$\mathbf{P}(Y = j | X = n) = e^{-n} \frac{n^j}{j!}, j \geq 0.$$

b) Since, given $X = n$, $Y$ is Poisson$(n)$,

$$\mathbf{E}(Y|X = n) = n, \mathbf{E}(Y|X) = X,$$

and

$$\mathbf{E}(Y) = \mathbf{E}[\mathbf{E}(Y|X)] = \mathbf{E}(X) = 2.$$

c) Since, given $X = n$, $Y$ is Poisson$(n)$,

$$\begin{aligned}
\mathbf{E}\left(Y^2|X = n\right) &= n + n^2, \mathbf{E}\left(Y^2|X\right) = X + X^2, \\
\mathbf{E}\left(Y^2\right) &= \mathbf{E}\left[\mathbf{E}\left(Y^2|X\right)\right] = \mathbf{E}(X) + \mathbf{E}\left(X^2\right) = 2 + 2 + 2^2 = 8, \\
\mathrm{Var}(Y) &= \mathbf{E}\left(Y^2\right) - (\mathbf{E}(Y))^2 = 8 - 2^2 = 4.
\end{aligned}$$