3.9 Simple random walk (r.w.)

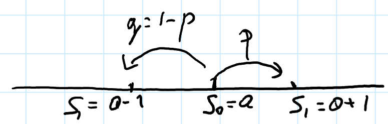
It is random walk (motion) of a point (particle) on the grid of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ in discrete time:

__Def.__ R.w. is a sequence of $\mathbb{Z}$-valued r.v.'s S_n, $n = 0, 1, \dots$, where S_n is the position of a moving point at time n:

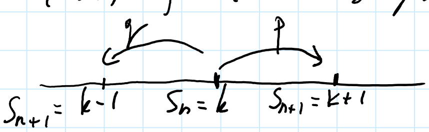
(i) $S_0 = a \in \mathbb{Z}$ is initial position

(ii) A coin with $P(H) = p$ is tossed:

the point moves to $\begin{cases} S_1 = a+1 & \text{if } H \\ S_1 = a-1 & \text{if } T \end{cases}$



(iii) If in n steps $S_n = k$, then the coin is tossed again:



$$S_{n+1} = k+1 \quad \text{if } H$$
$$S_{n+1} = k-1 \quad \text{if } T.$$

__Def.__ r.w. S_n, $n \geq 0$, is a sequence of $\mathbb{Z}$-valued r.w. so that

$$S_1 = S_0 + X_1 \qquad \underline{P(X_i = 1) = p, \; P(X_i = -1) = q = 1-p}$$

$$S_2 = S_1 + X_2 = S_0 + X_1 + X_2$$

$$\cdots$$

$$S_n = S_{n-1} + X_n = S_0 + X_1 + \dots + X_n$$

It is assumed that $S_0, X_1, X_2, \dots$ are independent

__Note__ a) $S_n = S_0 + X_1 + \dots + X_n$ is a sum of independent r.v.

b) (i) $S_{n+m} = S_m + \sum_{i=1}^{n} X_{m+i}$: $\tilde{S}_n = S_{n+m}$, $n = 0, 1, \dots$, is r.w. $\tilde{S}_0 = S_m$ (it starts at S_m)

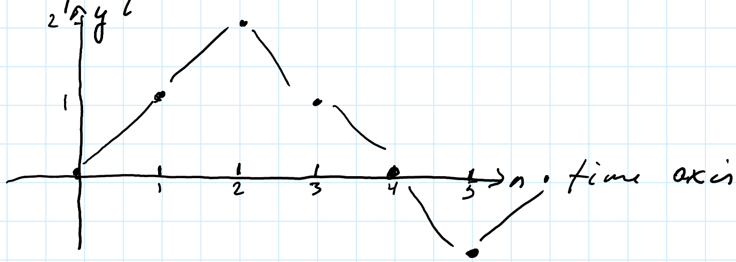
(ii) $S_{n+m} - S_m = \sum_{i=1}^{n} X_{m+i}$ and S_m are independent

(iii) for any $0 \leq j_1 < j_2 < \dots < j_\ell$,

$S_{j_2} - S_{j_1}, \; S_{j_3} - S_{j_2}, \; \dots, \; S_{j_\ell} - S_{j_{\ell-1}}$ are independent.

$S_{j_2} - S_{j_1}, S_{j_3} - S_{j_2}, \dots, S_{j_e} - S_{j_{e-1}}$ are independent.

<u>Def.</u> The path of r.w. (path of a particle) is collection of $\{(n, S_n) : n \geq 0\}$ which is linearly interpolated.



<u>Remark 1</u>. If $S_0 = 0$, $P(X_i = \pm \frac{1}{2})$, then $E(X_i) = 0$, $Var(X_i) = E(X_i^2) = 1$. By CLT,

$$\frac{X_1 + \dots + X_n}{\sqrt{n}} = \frac{S_n}{\sqrt{n}} \approx \text{standard normal r.v. } \underline{\text{for large } n.}$$

If we compress the path $\{(k, S_k) : 0 \leq k \leq n\}$ horizontally to $[0,1]$, and vertically by $\frac{1}{\sqrt{n}}$, for large n, we see approximately Wiener process:

$W_t^n := \frac{S_{[nt]}}{\sqrt{n}}$, $0 \leq t \leq 1$, is approximately Wiener process.

$[nt]$ is integer part of $[nt]$: $[nt] = k$ if $k \leq nt < k+1$ or $\frac{k}{n} \leq t < \frac{k+1}{n}$

<u>Remark 2</u>. For any $m \geq 0$, $a, b \in \mathbb{Z}$,

$$\underline{P(S_{m+n} = b \mid S_m = a)} = \frac{P(\cancel{S_m}^{a} + \sum_{i=1}^{n} X_{m+i} = b, \cancel{S_m = a})}{\cancel{P(S_m = a)}},$$

$$= P(a + \sum_{i=1}^{n} X_{m+i} = b) = P(a + \sum_{i=1}^{n} X_i = b) = \underline{P(S_n = b \mid S_0 = a)}$$

$$= P(a + c + \sum_{i=1}^{n} X_i = b + c) = \underline{P(S_n = b + c \mid S_0 = a + c)}$$

<u>Properties of r.w. $S_n, n \geq 0$</u>

1. S_n is <u>time homogeneous</u>: for any $m \geq 0$, $a, b \in \mathbb{Z}$,

$P(S_n = b \mid S_0 = a) = P(S_{n+m} = b \mid S_m = a)$. (Remark 2)

2. S_n is space homogeneous: for any $a, b, c \in \mathbb{Z}$, $n \geq 0$,

$P(S_n = b \mid S_0 = a) = P(S_n = b + c \mid S_0 = a + c)$ (Remark 2)

3. Markov property: for any $n, m \geq 0$, $a, b, l_0, \dots, l_n \in \mathbb{Z}$,

$P(S_{n+m} = b \mid S_0 = l_0, \dots, S_n = l_n) = P(S_{n+m} = b \mid S_n = l_n)$ (1)

Moreover, given $S_n = l_n$, the future and past are independent.

$$\text{LHS of (1)} = \frac{P(\overset{l_n}{S_n} + \sum_{i=1}^{m} X_{n+i} = b, \cancel{S_0 = l_0, \dots, S_n = l_n})}{P(\cancel{S_0 = l_0, \dots, S_n = l_n})} =$$

$$= P(l_n + \sum_{i=1}^{m} X_{n+i} = b) = \text{RHS of (1)}.$$

Example 1 S_n is G's wealth at time n, unlimited borrowing allowed:

(i) $S_n = S_0 + H_n - T_n$, $n \geq 0$, where $H_n = \#$ of H in n tosses, $T_n = \#$ of T in n tosses.

(ii) $S_0 = 0$, $S_n = H_n - T_n$ is net gain in n games.

Note H_n, $T_n = n - H_n$ are binomial r.v.

R. w. with absorbing barriers is simple r.w. which stops when a "boundary" is reached.

a) Consider S_n with $S_0 = k$, $0 < k < N$. Let $\tau = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = N\}$. It is time (number of steps) to reach the boundary $\{0, N\}$. Then $R_n = S_{n \wedge \tau}$ with $n \wedge \tau := \min\{n, \tau\}$ is r.w. with absor

$R_n = S_{n\wedge\tau}$ with $n\wedge\tau := \min\{n,\tau\}$ is r.w. with absorbing barrier $\{0,N\}$: motion stops if 0 or N are reached.

Example 2. S_n is G's wealth at time n, $S_0 = k$, $0<k<N$, G stops when 0 or N is reached.

b) Consider S_n with $S_0 = k$, $k>0$, and motion stops when 0 is reached. There is a single barrier 0. (N is removed)

Example 3. S_n is G's wealth at time n, $S_0 = k$, $k>0$. G stops if ruined.