§3.7 Hierarchical models

In Ex2: Number N of fish J catches is Poisson(λ).
Y is the number of H's in N tosses of a coin with $P(H)=p$.

Comment on Ex.2. In this example,

a) we can say Y is binomial(N,p): given $N=n$, Y is binomial(n,p).

b) alternatively to a), Y can be modeled as

(3) $\boxed{Y=\sum_{i=1}^{N} X_i}$, where X_i are independent Bernoulli(p), independent of N.

Again, given $N=n$, Y is the sum of n independen r.v.,
$Y=\sum_{i=1}^{n} X_i$ is binomial(n,p):

$$E(Y|N=n)=\sum_{i=1}^{n}E(X_i)=n\cdot p,\quad Var(Y|N=n)=Var\left(\sum_{i=1}^{n}X_i\right)$$
$$=\sum_{i=1}^{n}Var(X_i)=n\cdot pq.$$

Properties of conditional expectation

1. $E(Y)=E[E(Y|X)]$; $E[k(X)\,Y]=E[k(X)\,E(Y|X)]$

2. Recall $E(Y|X)=h(X)$, where $h(x)=E(Y|X=x)$.
$E(Y|X)$ is the best mean square estimate of Y by a function of X:

$$E[(Y-h(X))^2]\le E[(Y-g(X))^2] \text{ for any other } g(X).$$

Why?: $E[(Y-g(X))^2]=E[((Y-h(X))+(h(X)-g(X)))^2]$

Why?: $E[(Y-g(X))^2] = E[((Y-h(X)) + \underbrace{(h(X)-g(X))}_{k(X)})^2]$

$= E[(Y-h(X))^2] + E[k(X)^2] + 2\boxed{E[k(X)(Y-h(X))]}^{=0}$

$\geq E[(Y-h(X))^2]$, because $h(X) = E(Y|X)$, and

$E[k(X)(Y-h(X))] = E[k(X)Y] - E[k(X)h(X)] = 0.$

More properties

a) $E[g(X)|X] = g(X)$

$E[g(X)Y|X] = g(X)E[Y|X]$

Given X, $g(X)$ is a 'constant'.

Indeed, $E[g(X)Y|X=x] = E[g(x)Y|X=x] = g(x)E[Y|X=x]$.

b) Conditional expectation is expectation: it has all expectation properties.

c) If X, Y are independent, then $P(Y=y|X=x) = P(Y=y)$,

$E(Y|X=x) = E(Y)$, $E(Y|X) = E(Y)$.

3.8 Sums of r.v

Thm 1. Let $f(x,y)$ be joint pmf of X, Y. Then

$$f_{X+Y}(z) = \sum_x f(x, z-x) = \sum_y f(z-y, y), \quad -\infty < z < \infty.$$

Proof. $\{X+Y=z\} = \bigcup_x \{X+Y=z, X=x\}$ disjoint implies

$P(X+Y=z) = \sum_x P(X+Y=z, X=x) = \sum_x P(Y=z-x, X=x)$

$= \sum_x f(x, z-x).$

<u>Corollary</u>. If X, Y are independent, then

$$f_{X+Y}(z) = \sum_x f_X(x)\, f_Y(z-x) = \sum_y f_X(z-y) f_Y(y),$$

because $f(x,y) = f_X(x)\, f_Y(y)$.

<u>Ex 1</u>. Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be independent. Show that $X+Y \sim \text{Poisson}(\lambda+\mu)$.

<u>Answer</u>. Range of $X+Y$ is $\{0, 1, 2, \dots\}$.

$f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x \geq 0$, $f_Y(y) = e^{-\mu}\frac{\mu^y}{y!}$, $y \geq 0$. For $z \geq 0$,

$$f_{X+Y}(z) = \sum_x \underbrace{f_X(x)}_{x \geq 0}\, \underbrace{f_Y(z-x)}_{z-x \geq 0} = \underbrace{\sum_{x \geq 0,\, z-x \geq 0}}\, e^{-\lambda}\frac{\lambda^x}{x!}\, e^{-\mu}\frac{\mu^{z-x}}{(z-x)!} =$$

$$= \sum_{x=0}^{z} \frac{e^{-(\lambda+\mu)}}{z!} \frac{z!}{x!(z-x)!} \lambda^x \mu^{z-x} = \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x} =$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^z}{z!}, \quad z = 0, 1, \dots,$$

Poisson$(\lambda+\mu)$-probability.

because $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

<u>3.5</u>. Various discrete r.v.