Cauchy-Schwarz inequality: $|E\,UV| \le \sqrt{E(U^2)}\sqrt{E(V^2)}$

Proof. Let $F(t) = E[(U+tV)^2] = E(U^2) + 2$

$= c + 2bt + at^2 \ge 0$ for all t, $c = E(U^2)$, $b =$

Hence discriminant $= (2b)^2 - 4ac = 4(b^2 - ac) \ge 0$

$|b| \le ac$.

Ex 1. show that $E(|Y|) \le \sqrt{E(Y^2)}$

Answer. Cauchy-Schwarz with $U = |Y|$, $V = 1$.

More exercises

Ex 1. Let $\mu = E(V)$, $\sigma^2 = Var(V)$, $\sigma = \sqrt{Var(V)}$.

Confirm that

$$P(|V-\mu| \le 3\sigma) \ge \frac{8}{9} \approx 0.89$$ (If V is normal, then $P(|V-\mu| \le 3\sigma) = 0.997$)

Answer. Chebyshev:

$$P(|V-\mu| > 3\sigma) \le \frac{Var(V)}{9\sigma^2} = \frac{\sigma^2}{9\sigma^2} = \frac{1}{9}.$$

Ex 2. Let V, Z be independent, $Z \sim N(0,1)$,

$P(V=1) = P(V=-1) = \frac{1}{2}$.

a) Show that $X = VZ \sim N(0,1)$

Answer.

$$\phi_X(t) = E\left(e^{itVZ}\right) = E\left(e^{itVZ} 1_{\{V=1\}}\right) + E\left(e^{itVZ} 1_{\{V=-1\}}\right)$$

$$= P(V=1)\, E\left(e^{itZ}\right) + P(V=-1)\, E\left(e^{-itZ}\right) =$$

$$= \frac{1}{2} e^{-t^2/2} + \frac{1}{2} e^{-\frac{(-t)^2}{2}} = e^{-t^2/2} = \phi_Z(t).$$

b) Find $Cov(X, Z)$

<u>Answer</u>. $Cov(X, Z) = E(XZ) = E(VZ^2) = E(V)E(Z^2)$
$= 0$.

c) Are X, Z independent?

<u>Answer</u>. $P(X = Z) = P(VZ = Z) = P(V = 1) = \frac{1}{2} > 0$.
<u>No</u>.

d) Are $X = VZ$ and V independent?

<u>Answer</u>.

$$E\left(e^{isVZ} \mid V = 1\right) = e^{-s^2/2}$$

$$E\left(e^{isVZ} \mid V = -1\right) = e^{-s^2/2}$$

$$E\left(e^{isVZ} \mid V\right) = e^{-s^2/2}$$

<u>Joint cf</u>: $E\, e^{(isVZ + itV)} = E\left[e^{itV} E\left(e^{isVZ} \mid V\right)\right] =$

$= e^{-s^2/2} E\left(e^{itV}\right) = E\left(e^{isVZ}\right) E\left(e^{itV}\right)$. <u>Yes</u>.

<u>Ex 3</u>. Let $Y_n \sim$ geometric$\left(p = \frac{\lambda}{n}\right)$:

$P(Y_n = k) = q^{k-1} p, \quad k = 1, 2, \ldots$

$$G_{Y_n}(s) = \frac{ps}{1 - qs}$$

Show that $\frac{Y_n}{n} \xrightarrow{D} Y \sim$ exponential(λ)

<u>Answer</u>. $\phi_{Y_n}(t) = \dfrac{pe^{it}}{1 - (1-p)e^{it}} = \dfrac{p\, \cancel{e^{it}}}{\cancel{e^{it}}\left(p - (1 - e^{-it})\right)} = \dfrac{\frac{\lambda}{n}}{\frac{\lambda}{n} + (e^{-it} - 1)}$

$\phi_{\frac{Y_n}{n}}(t) = \phi_{Y_n}\left(\frac{t}{n}\right) = \dfrac{\lambda}{\lambda + n\left(e^{-i\frac{t}{n}} - 1\right)}$

$$= \frac{\lambda}{\lambda + n\left(e^{-it} - 1\right)}$$

$M_Y(t) = \dfrac{\lambda}{\lambda - t}, \quad \phi_Y(t) = \dfrac{\lambda}{\lambda - it}$

We find limit of $n\left(e^{-\frac{it}{n}} - 1\right) = \dfrac{e^{-\frac{it}{n}} - 1}{\frac{it}{n}} \cdot it =$

$\left(z = \frac{it}{n}\right)$ $e^{-z} - 1$ ... L'Hospital ... $-it$

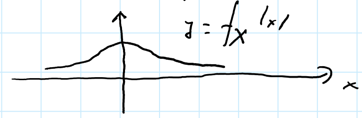
We find limit of $n\left(e^{\cdots}-1\right) = \frac{\quad}{\frac{it}{n}} \cdot it =$

$\overset{z=\frac{it}{n}}{=} \frac{e^{-z}-1}{z} \cdot it \xrightarrow{\text{L'Hospital}} -it$ as $z \to 0$

Hence $\phi_{\frac{Y_n}{n}}(t) \longrightarrow \phi_Y(t)$ as $n \to \infty$.

Ex 4. Let $X_1, \dots, X_n$ be independent Cauchy

Recall $f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty$



$\phi_X(t) = e^{-|t|}, \quad t \in \mathbb{R}.$

a) Find $\phi_{\bar{X}_n}(t)$

Answer. Recall $\phi_{\bar{X}_n}(t) = \phi_X\left(\frac{t}{n}\right)^n = \left(e^{-\left|\frac{t}{n}\right|}\right)^n = e^{-\frac{|t|}{n} \cdot n} =$

$= e^{-|t|} = \phi_X(t).$

b) Is there a constant c so that $\bar{X}_n \to c$ in probability?

Answer. No: $\bar{X}_n \to c$ in prob. $\iff \phi_{\bar{X}_n}(t) \to e^{itc}$.

Ex 5. Let $X \sim N(0, \sigma^2)$, $Y \sim N(aX, \tilde{\sigma}^2) \Leftarrow$ (given $X = x$, $Y \sim N(ax, \tilde{\sigma}^2)$).

a) Find $E(e^{itY} \mid X)$, $\operatorname{Cov}(X, Y)$

Answer. $E(e^{itY} \mid X = x) = e^{itax} e^{-\tilde{\sigma}^2 t^2/2}$,

$E(e^{itY} \mid X) = e^{itaX} e^{-\tilde{\sigma}^2 t^2/2}$

$\operatorname{Cov}(X, Y) = E(XY)$

$= E[X E(Y \mid X)] =$

$= E(aX^2) = \underline{a\sigma^2}$.

b) Find $\phi_Y(t)$. How is Y distributed

Answer. $\phi_Y(t) = E[E(e^{itY} \mid X)] = e^{-\tilde{\sigma}^2 t^2/2} E(e^{itaX})$

$= \exp\left\{-\frac{t^2}{2}\left(\tilde{\sigma}^2 + a^2\sigma^2\right)\right\}$: $\quad Y \sim N(0, \tilde{\sigma}^2 + a^2\sigma^2)$

c) Is X, Y normal bivariate?

Answer. joint cf:

$$\Phi(s,t) = E\left(e^{isX+itY}\right) = E\left[e^{isX} E(e^{itY} | X)\right] =$$

$$= e^{-\tilde{\sigma} t^2/2} E\left(e^{i(s+ta)X}\right) = e^{-\tilde{\sigma} t^2/2} e^{-(s+ta)^2 \sigma^2/2} =$$

$$= \exp\left\{-\frac{1}{2}\left[t^2(\tilde{\sigma}^2 + a^2\sigma^2) - 2sta\sigma^2 + \frac{s^2\sigma^2}{2}\right]\right\}$$

Yes: $B = \begin{pmatrix} \sigma^2 & a\sigma^2 \\ a\sigma^2 & \tilde{\sigma}^2 + a^2\sigma^2 \end{pmatrix}$ is covariance matrix.

In general $X = (X_1, X_2) \sim N(\mu, B)$ if

$$\Phi_X(t) = \exp\left\{it\mu' - \frac{1}{2} t B t'\right\} \quad , t = (t_1, t_2) \in \mathbb{R}^2$$

$$B = \left(\mathrm{Cov}(X_i, X_j)\right)_{1 \le i, j \le 2}.$$