5.7-9 Characteristic functions (cf)

Def. cf of X is the function

$$\phi(t) = \phi_X(t) = E(e^{itX}), \quad -\infty < t < \infty,$$

where $i^2 = -1$, $e^{ia} = \cos(a) + i\sin(a)$.

Note 1. $E[e^{itX}] = E[\cos(tX)] + iE[\sin(tX)]$, $|e^{itX}| \le 1$,

$|\phi_X(t)| \le 1$, $\phi_X(0) = 1$.

2. Recall

$$M_X(t) = E(e^{\boxed{t}X})$$

$$\phi_X(t) = E(e^{\boxed{it}X}) \overset{\text{formally}}{=} \begin{cases} M_X(it) \\ G_X(e^{it}) \end{cases}$$

$$G_X(s) = E(s^X)$$

Some cf

1. $X \sim \text{binomial}(n,p)$: $\phi(t) = (pe^{it}+q)^n$ | $G(s) = (ps+q)^n$
2. $X \sim \text{exponential}(\lambda)$: $\phi(t) = \frac{\lambda}{\lambda - it}$, $t \in \mathbb{R}$ | $M(t) = \frac{\lambda}{\lambda - t}$, $t < \lambda$
3. $Z \sim N(0,1)$: $\phi(t) = e^{\frac{(it)^2}{2}} = e^{-t^2/2}$ | $M(t) = e^{t^2/2}$
4. $X \sim \text{Cauchy}$: $\phi(t) = e^{-|t|}$: X has pdf $= \frac{1}{\pi} \frac{1}{1+x^2}$, $x \in \mathbb{R}$.
5. $X \sim \text{Poisson}(\lambda)$: $\phi(t) = \exp\{\lambda(e^{it}-1)\}$ | $G(s) = e^{\lambda(s-1)}$

Inversion thm Let F be df of X. Then

$$F(b) - F(a) = P(a < X \le b) = \lim_{N\to\infty} \int_{-N}^{N} \frac{e^{-iat} - e^{-ibt}}{2\pi i t} \phi_X(t)\,dt$$

for any a, b at which F is continuous.

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Claim 1. If X, Y are independent, then

$\Phi_{X+Y}(t) = \Phi_X(t)\, \Phi_Y(t)$ because

$E[e^{it(X+Y)}] = E[e^{itX} e^{itY}] = \dots$

Ex 1. a) Show that $\Phi_{aX+b}(t) = e^{ibt} \phi_X(at)$.

Answer. $E \exp\{it(aX+b)\} = E\left(e^{itaX} e^{itb}\right) =$

$= e^{itb} E(e^{i\boxed{ta}X}) = e^{itb} \phi_X(ta)$.

b) For $X \sim N(\mu, \sigma^2)$, $\phi_X(t) = e^{it\mu} e^{-\sigma^2 t^2/2}$

Answer. $X = \mu + \sigma Z$, $Z \sim N(0,1)$: by a),

$\phi_X(t) = e^{i\mu t} e^{-\frac{\sigma^2 t^2}{2}}$.

5.3. Continuity thm

Convergence of df's

Def. a) Let F_n, F be df. We say $F_n \to F$ if

$F_n(x) \longrightarrow F(x)$ for all x at which F is continuous

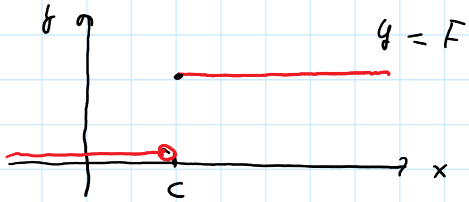
b) We say $X_n \xrightarrow{D} X$ if $F_{X_n} \longrightarrow F_X$ as $n \to \infty$.

Note b) means that for $a < b$

$F_{X_n}(b) - F_{X_n}(a) = P(a < X_n \le b) \to F(b) - F(a) = P(a < X \le b)$

(if F_X is cont. at a, b).

<u>Ex 1</u> Consider $X = c$ (constant with probability 1)



$y = F(x) = P(X \le x)$, $-\infty < x < \infty$

Show that $X_n \xrightarrow{D} c$ if and only if for each $\varepsilon > 0$, $P(|X_n - c| \le \varepsilon) = P(c-\varepsilon \le X_n \le c+\varepsilon) \to 1$, equivalently, $P(|X_n - c| > \varepsilon) \xrightarrow{n\to\infty} 0$.

<u>Answer.</u> $1 \ge P(c-\varepsilon \le X_n \le c+\varepsilon) \ge P(c-\varepsilon < X_n \le c+\varepsilon) = F_n(c+\varepsilon) - F_n(c-\varepsilon)$

$\longrightarrow F(c+\varepsilon) - F(c-\varepsilon) = 1$.

<u>Def.</u> a) $X_n \to c$ in probability if for each $\varepsilon > 0$,

$P(|X_n - c| > \varepsilon) \xrightarrow{n\to\infty} 0$;

b) $X_n \to X$ in probab. if for each $\varepsilon > 0$,

$P(|X_n - X| > \varepsilon) \xrightarrow{n\to\infty} 0$.

<u>Continuity thm.</u> $X_n \xrightarrow{D} X$ if and only if $\phi_{X_n}(t) \to \phi_X(t)$ for all t.

Proof is based on inversion thm.

<u>Corollary 1.</u> $X_n \to c$ in probability if and only if $\phi_{X_n}(t) \to e^{ict}$ for all t

<u>Why?</u> If $X = c$ (with probability 1), then $\phi_X(t) = e^{itc}$, and claim follows by Ex 1.