Homework 1

8/31/22 10:43 PM

1 hw1

1. Let $\{A_i, i \in I\}$ be a collection of sets. Prove De Morgan's Laws:

$$(\cup_i A_i)^c = \cap_i A_i^c, \ (\cap_i A_i)^c = \cup_i A_i^c.$$

Hint. The first one: $x \in (\cup_i A_i)^c \iff x \notin \cup_i A_i$ (it means there is no i such that $x \in A_i$) $\iff x \notin A_i$ for any $i \iff x \in A_i^c$ for any $i \iff x \in \cap_i A_i^c$.

2. *a)* Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of Ω. Show that $\mathcal{F}_1 \cap \mathcal{F}_2$, the collection of subsets of Ω that belong to both $\mathcal{F}_1$ and $\mathcal{F}_2$, is a σ-field.

Hint. Verify the definition of a σ-field. Since $\Omega \in \mathcal{F}_1$ and $\Omega \in \mathcal{F}_2$, then $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ etc...

b) Let $\mathcal{A}$ be a collection of subsets of Ω, and let $\mathcal{F}_i, i \in I$, be all σ-fields that contain $\mathcal{A}$. Show that $\mathcal{F} = \cap_i \mathcal{F}_i$ is a σ-field.

Comment. b) is of interest if $\mathcal{A}$ itself is not a σ-field. Note that $\mathcal{P}(\Omega)$, the σ-field of all subsets of Ω, is among $\mathcal{F}_i$. The collection $\mathcal{F} = \cap_i \mathcal{F}_i$ is called the smallest σ-field containing $\mathcal{A}$ (or σ-field generated by $\mathcal{A}$).

3. Suppose that eight distinct envelopes are placed at random in three distinct mailboxes.

a) In how many different ways this can be done?

b) What is the probability that every mailbox receives at least one envelope? *Hint. Consider finding probability of the complementary event A = "at least one mailbox is empty". Then one could use one of the following two options:*

(i) *Note A = "exactly one empty" or "exactly two empty"*

(ii) *$A = A_1 \cup A_2 \cup A_3$, where A_i = "ith mailbox empty", and by drawing Venn diagram (inclusion/exclusion):*

$$\#A = \#A_1 + \#A_2 + \#A_3 - \#(A_1 \cap A_2) - \#(A_1 \cap A_3) - \#(A_2 \cap A_3).$$

4. How many different letter arrangements of length 4 (four letter "words") can be made using the letters MOTTO?

5. (a) In how many ways can 3 boys and 3 girls sit in a row?

(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

(c) In how many ways if only the boys must sit together?

(d) In how many ways if no two people of the same sex are allowed to sit together?

Hint. It is like arranging books on a shelf (problem considered in class: check Content folder on blackboard).

1) Assume $x \in (\cup_i A_i)^c$, then x is not in $\cup_i A_i$.
$x \notin \cup_i A_i$. For every i, $x \notin A_i$ because x is not in the union.
By definition, $x \in A_i^c$ for every i.
Then $x \in \cap_i A_i^c$. $\square$ (1)

Assume $x \in \cap_i A_i^c$, then $x \in A_i^c$ for every i.
$x \in A_i^c$ means $x \notin A_i$ by def. of complement.
Since $x \notin A_i$ for every i, which means $x \notin \cup_i A_i$
which means $x \in (\cup_i A_i)^c$. $\square$ (2)
By (1) & (2), $(\cup_i A_i)^c \subseteq \cap_i A_i^c$ & $\cap_i A_i^c \subseteq (\cup_i A_i)^c$, so $(\cup_i A_i)^c = \cap_i A_i^c$

Assume $x \in (\cap_i A_i)^c$, then $x \notin \cap_i A_i$.
then $x \notin A_i$ for every i, then by definition of complementation,
$x \in A_i^c$ for every i, which is just the same as saying $x \in \cup_i A_i^c$. $\square$ (3)
Assume $x \in \cup_i A_i^c$, then $x \in A_i^c$ for all i.
By definition of complement, $x \notin A_i$ for all i, which is the same as saying x is not a part of the intersection.
$x \notin A_i^c$ for all i which means $x \notin (\cap_i A_i)$ which means $x \in (\cap_i A_i)^c$. $\square$ (4)
By (3) & (4), $(\cap_i A_i)^c \subseteq \cup_i A_i^c$ & $\cup_i A_i^c \subseteq (\cap_i A_i)^c$, so $(\cap_i A_i)^c = \cup_i A_i^c$

2) a) Let $\mathcal{F}_1$ & $\mathcal{F}_2$ be σ-fields of subsets of Ω.
Show that $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field.
1) $\emptyset \in \mathcal{F}_1$ & $\emptyset \in \mathcal{F}_2$ because $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-field, so $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$.
2) Let set $A \in \mathcal{F}_1 \cap \mathcal{F}_2$, then
$A \in \mathcal{F}_1$ & $A \in \mathcal{F}_2$.
Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-fields, $A \in \mathcal{F}_1$ implies $A^c \in \mathcal{F}_1$ and $A \in \mathcal{F}_2 \rightarrow A^c \in \mathcal{F}_2$, so
$A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$.
3) Let sets $A_1, A_2, \dots, A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$.
By union, $A_1 A_2 \dots A_n \in \mathcal{F}_1$ and
$A_1 A_2 \dots A_n \in \mathcal{F}_2$.
Since $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-fields, $A_1 A_2 \dots A_n \in \mathcal{F}_1 \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}_1$ where n can be infinite.
Similarly, $A_1 A_2 \dots A_n \in \mathcal{F}_2$ implies that $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_2$ where n can be countably infinite, so
$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$.
By (1), (2), and (3), $F_1 \cap F_2$ is a σ-field.

b) Let A be a collection of subsets of Ω.
Let $F_i, i \in I$, be all σ-fields that cointain A.
Show $\mathcal{F} = \cap_i \mathcal{F}_i$ is a σ-field.
$\mathcal{F} = (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \mathcal{F}_i)$.
1) By definition of σ-field, $\emptyset \in \mathcal{F}_1$, $\emptyset \in \mathcal{F}_2, \dots, \emptyset \in \mathcal{F}_i$.
If $\emptyset \in \mathcal{F}_1, \emptyset \in \mathcal{F}_2, \dots, \emptyset \in \mathcal{F}_i$, then $\emptyset \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \cap \mathcal{F}_i)$, then
$\emptyset \in \cap_i \mathcal{F}_i$.
2) Let $A \in \mathcal{F}$, then $A \in \cap_i \mathcal{F}_i$, then
$A \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \cap \mathcal{F}_i)$, then $A \in \mathcal{F}_1$, $A \in \mathcal{F}_2, \dots A \in \mathcal{F}_i$.
Since $\mathcal{F}_1 \dots \mathcal{F}_i$ are σ-field, $A \in \mathcal{F}_1$ implies $A^c \in \mathcal{F}_1$. Similarly, $A \in \mathcal{F}_2 \rightarrow A^c \in \mathcal{F}_2$ ... ultimately $A \in \mathcal{F}_i$ implies $A^c \in \mathcal{F}_i$.
$A^c \in (\mathcal{F}_1 \cap \mathcal{F}_2 \dots \mathcal{F}_i) \Rightarrow A^c \in \cap_i \mathcal{F}_i$
3) Let $A_1, A_2, \dots A_n \in \mathcal{F}$. $\mathcal{F} = (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \dots \cap \mathcal{F}_i)$
By union, $A_1, A_2 \dots A_n \in \mathcal{F}_1$ and
$A_1, A_2 \dots A_n \in \mathcal{F}_2$ ...
$A_1, A_2 \dots A_n \in \mathcal{F}_i$.
$A_1, A_2 \dots A_n \in \mathcal{F}_1$ shows $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_1$
$A_1, A_2 \dots A_n \in \mathcal{F}_2$ shows $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_2$. Similarly, $A_1 A_2 \dots A_n \in \mathcal{F}_i = \bigcup_{i=1}^{n} A_i \in \mathcal{F}_i$.
Since $\bigcup_{i=1}^{n} A_i$ is in $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_i$, $\bigcup_{i=1}^{n} A_i \in \cap_i \mathcal{F}_i$.
By (1), (2), (3), $\cap_i \mathcal{F}_i$ is a σ-field.

3) 8 distinct envelopes are placed at random in 3 boxes.

a) There are 3 ways to arrange each envelope and 8 envelopes. 3^8 ways to place them. 6561 ways to arrange them.

b) P("every mail box recieves 1 mail")
For every mailbox to recieve 1 mail, no boxes should be empty.
A box is empty if
- exactly 1 empty, or
- exactly 2 empty.

P(2empty) = 3/6561 because 3 ways to choose which box gets all 8.
P(1empty) = (3 # ways to pick empty box)(# ways to arrange 8 over 2 boxes) $= (3)(2^8)/6561$.
P(at least 1 in all) = 1 − P(1empty) − P(2 empty)
$= 1 - \frac{3}{6561} - \frac{768}{6561} = \frac{5790}{6561} = \boxed{.8825}$

4) How many 4 letter word can be made of "MOTTO"?
By permutations there are $\frac{5!}{(5-4)!}$ permutations of length 4.
There are 120 4 letter permutations of MOTTO.
Because there are 2Os, the number of unique permutations is $\frac{120}{2!} = 60$.
Because there are 2Ts, the number of unique permutations is $60/2! = \boxed{30}$

5) a) There are 6 people. The number of ways to arrange them is $6! = \boxed{720}$
b) There are 2 blocks, $\boxed{B}$ & $\boxed{G}$. The number of ways to arrange the blocks is $2! = 2$. Inside each block there are 3 people, so 3! ways of ordering both the girls and boys. $(3!)(3!)(2$ ways to arrange blocks$) = (6)(6)(2) = \boxed{72}$
c) There are 4 blocks when only boys sit together, $\boxed{B}, G_1, G_2, G_3$. The number of ways to arrange the blocks is $4! = 24$. Inside the boys block, there are 3! ways of ordering them. $(3!)(24$ ways to arrange blocks$) = (6)(24) = \boxed{144}$
d) $\boxed{G}$ $\underline{B_1}$ $\boxed{G}$ $\underline{B_2}$ $\boxed{G}$ $\underline{B_3}$
3! ways to arrange girls
3! ways to arrange boys.
or Boys start.
$\boxed{B}$ $\underline{G_1}$ $\boxed{B}$ $\underline{G_2}$ $\boxed{B}$ $\underline{G_3}$
2! ways to arrange blocks.
Total number of ways to arrange them is $(3!)(3!)(2) = 6 \cdot 6 \cdot 2 = \boxed{72}$