Math 505A HW7 Neel Gupta

1) An urn contains balls #-ed $1, 2, \ldots, n$. We remove k balls & sum their number.

Let X = the sum of the i^{th} ball, $i \in [1, k]$.

$X = \sum_{i=1}^{k} X_i$, so the mean is the expected value.

$$\mu = E[X] = E\left[\sum_{i=1}^{k} X_i\right]$$

$E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E(X_i)$. X_i is the number on i^{th}

The range of the numbers is 1 to n, so the average number along $X_i = \frac{n+1}{2}$ or $\frac{\sum_{i=1}^{n} i}{n}$

so, $\sum_{i=1}^{n} i = \frac{(n+1)n}{2} \cdot \frac{1}{(n)} = \frac{(n+1)}{2}$, so

$$\Rightarrow E[X] = \sum_{i=1}^{k}\left(\frac{n+1}{2}\right) = \frac{k(n+1)}{2},$$

$$\boxed{\mu = \frac{k(n+1)}{2}}$$

$$Var(X) = E(X^2) - (E(X))^2.$$

$$E(X^2) = E\left(\left(\sum_{i=1}^{k} X_i\right)^2\right) = k E(X_1^2) + k(k-1) E(X_1 X_2).$$

$$k E(X_1^2) + k(k-1) E(X_1 X_2)$$

~~Let $E(X)$ be the $\sum_{i=1}^{n} p(x_i) x_i$, so there will be~~

$\cancel{\Rightarrow \frac{k}{n}\sum}$ ~~an n term in $E(X_i^2)$~~ :

~~$\sum_{i=1}^{n} p(x_i) = n p(x_i)$, so~~

$$E(X_1^2) = \sum_{i=1}^{n} x_i P(X_i = x_i^2) = P(x_1)(x_1) + P(x_2)(2) + P(x_3)(3)$$

$$E(X_1 X_2) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n-1} p(X_i = x_i) p(X_j = x_j)$$

$\underbrace{1 + 2 + 3 + \ldots}$ n terms, so dividing by n gives sum of P-s.

$$\Rightarrow E(X^2) = \frac{k}{n} \sum_{i=1}^{n} \cancel{P(x_i)} j + \frac{k(k-1)}{n(n-1)}$$

$\sum_{i>j}^{n/n-1} ij$ = By generalized solution the of the Binomial Thm,

$$\Rightarrow \frac{k}{n}\left\{\frac{(n+2)(n+1)(n)}{3} - \frac{1}{2}(n)(n+1)\right\} + \frac{k(k-1)}{n(n-1)} \sum_{j=1}^{n} j \left\{ \begin{matrix} \text{i expand} \\ = n(n+1) - j \end{matrix} \right\} (j-1).$$

Problem 1

$$\frac{k}{n}\left\{\frac{1}{3n}n(n+1)(n+2)-\frac{1}{2}(n)(n+1)\right\} +$$

$$\frac{k(k-1)}{n(n-1)}\sum_{j=1} j\left\{\cancel{(n)}(n+1)-j(j+1)\right\}$$

$$= \frac{1}{3}k(n+1)(n+2)-\frac{(n+1)}{2}$$

$$(n+1)\left(\frac{1}{3}k(n+2)-\frac{1}{2}\right)=\frac{1}{6}k(n+1)(2n+1)$$

$$\sum_{j=1}^{n} j\{n(n+1)-j(j+1)\} = \cancel{n(n(n+1)-n(n+1))}$$

$$\cancel{n(n+1)(n-1)(n+2)(4)}$$

$$\Rightarrow \cancel{\frac{1}{6}k(n+1)(2n+1)}\ \text{~~By Binomial Theorum~~}\ j$$

$$\cancel{+n(n+1)-2} \quad \cancel{(n-1)(n+1)(n)} = \cancel{(n)(n+1)(n+1)}$$

$$\sum_{j=1}^{n} j\{(n)(n+1)-j(j+1)\} = 1(n(n+1)-2)+2(n(n+1)-2(3))$$

$$+3(n(n+1)-(3)(4))+4(n(n+1)-(4)(5))\ldots$$

General term is $\frac{1}{12}(3n+2)(n+1)$, so

$$\cancel{E(X^2)=\frac{1}{6}k}$$

$$k(k-1)E(X_iX_j)=k(k-1)\left(\frac{1}{12}\right)(3n+2)(n+1).$$

$$\Rightarrow E(X^2)=\frac{1}{6}k(n+1)(2n+1)+\frac{1}{12}(k)(k-1)(3n+2)(n+1)$$

$Var(X)=E(X^2)-\mu^2$ (from earlier) $\rightarrow$

$$Var(X)=\frac{1}{6}k(n+1)(2n+1)+\frac{1}{12}k(k-1)(3n+2)(n+1)$$

$$-\left(\frac{k(n+1)}{2}\right)^2 = \frac{1}{6}k(n+1)(2n+1)+\frac{1}{12}k(k-1)(3n+2)\ldots$$

$$\ldots(n+1) - \frac{k^2(n+1)^2}{4} \Rightarrow$$

$$k(n+1)\left\{\frac{1}{6}(2n+1)+\frac{1}{12}(3n+2)(k-1)-\frac{1}{4}k(n+1)\right\}$$

Problem 2

a)

Let $Y = g(X) + W$ where X and W are indep. random variables.

Since X and W are independent, $P(X=x, W=w) = P(X=x) \cdot P(W=w)$

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

By conditional probability Law.

Since $Y = g(X) + W$

$$= \frac{P(X=x, g(X)+W=y)}{P(X=x)}$$

$$= \frac{P(X=x, W=y-g(X))}{P(X=x)}$$

Since W & X are indep:

$$\Rightarrow \frac{\cancel{P(X=x)} \cdot P(W=y-g(X))}{\cancel{P(X=x)}}$$

$$= P(W = y - g(X))$$

$$= P(W + g(X) = y) \quad \blacksquare \quad \text{QED.}$$

b) Let $X = Y + U$, where Y & U are independent.

Assume the pmf $f_Y(y)$ and $f_U(u)$.

Find joint pmf of X and Y, $f(x,y)$.

Let $V = Y$ & $X = Y + U \Rightarrow U = X - Y$

The pmf of X & Y in terms of U and V can help us because V and V are indep $\because$ U & Y are.

Move f_{XY} into UV world by doing a change of variables. since

$$f_{XY}(X,y) = f_{U,V}(X,Y) \times \det A$$

where $A = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}.$

2) $f_{XY}(x,y) = 1\, f_{U,V}(x,y)$ Since

$U_y = -1 \qquad U_x = 1$

$V_y = 1 \qquad V_x = 0$

$$\det \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (1)(1) - (-1)(0) = \boxed{1}.$$

$f_{XY}(x,y) = f_{U,V}(x,y)$. Since $V = Y$

$f_{XY}(x,y) = f_{U,Y}(x,y)$. Since U & Y are independent,

$\underline{f_{XY}(x,y) = f_U(x) \cdot f_Y(y)}$

b) $P(Y=y \mid X=x) = \dfrac{P(X=x, Y=y)}{P(X=x)}$

Given $X = Y + U$

$\Rightarrow P(Y=y \mid X=x) = \dfrac{P(Y+U=x, Y=y)}{P(Y+U=x)}$

$= \dfrac{P(U=x-y, Y=y)}{P(U=x-y)} \quad \because Y=y.$

Since U & Y are independent,

$\Rightarrow P(Y=y \mid X=x) = \dfrac{\cancel{P(U=x-y)} \cdot P(Y=y)}{\cancel{P(U=x-y)}}$

$\Rightarrow P(Y=y \mid X=x) = P(Y=y)$

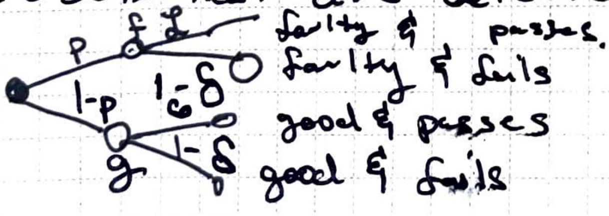
Since the pmf of Y is known as $f_Y(y)$

$\Rightarrow P(Y=y \mid X=x) = f_Y(y)$.

Problem 3

A factory produces n robots, with p chance of being faulty, the test detects fault if present with p δ.

Let X be # of faulty ~~robots~~ robots. Let Y be # of faulty robots that are detected as faulty



Problem 3

a) Since we can assume ~~normal~~ independence.

Passing has no correlation with being faulty, so

$$P(\text{faulty} \mid \text{passed}) = \frac{P(\text{faulty})\,(P(\text{passed} \mid \text{faulty}))}{P(\text{passed})}$$

Chance that it passes given faulty → $1-\delta$.

$P(\text{passed}) = 1 - P(\text{failed})$

$P(\text{failed}) = \text{Chance it was faulty } \& \ \underbrace{\text{test correct}}_{\delta}$

$P(\text{failed}) = p\delta$, so $p(\text{passed}) = 1 - p\delta$

$$\Rightarrow P(\text{faulty} \mid \text{passed}) = \frac{(P)(1-\delta)}{1-p\delta}$$

b) Let Z be the number of faulty robots passed.

Y be # detected as faulty.

Given $Y=k$, find $E(Z \mid Y)$ and dist. of Z

Total # of Faulty Robots $= Y + Z$

Let X be # of faulty robots.

X is binomial (n, p) and $X = Y + Z$.

Since $Z = X - Y$, Z is also binomially distributed.

$P(Z \mid Y=k) = Z = X - k$ since $X = x$ & $Y = k$.

Then $E(Z \mid Y) = E(X - k)$

$= E(X) - E(k)$

Given dist of X, $E(X) = np = \mu$.

$= \boxed{np - k}$

c) Assume $Y=y$ robots were found faulty, then

$$E[X \mid Y] = y + \frac{(n-y)(1-\delta)(p)}{1-\delta p} \quad \left(\text{by answer to (a)}\right)$$

The # of faulty is # faulty

\# faulty that were found passing.

Problem 3

c) $E(X|Y) = y + \dfrac{(n-y)(1-\delta)p}{(1-\delta p)}$

$$= \frac{(1-\delta p)y + (n-y)(1-\delta)p}{(1-\delta p)}$$

$$\cancel{= \frac{y - y\delta p + np - yp - n\delta p + y\delta p}{(1-\delta p)}}$$

$$\cancel{= \frac{y + np - yp - n\delta p}{1-\delta p}}$$

$$= \frac{y - y\delta p + n(1-\delta)p - y(1-\delta)p}{(1-\delta p)}$$

$$= \cancel{y} \; \frac{y - \cancel{y\delta p} + np - n\delta p - yp + \cancel{y\delta p}}{(1-\delta p)}$$

$$= \frac{y + np - n\delta p - yp}{(1-\delta p)}$$

$$= y - yp + np - n\delta p$$

$$E[X|Y] = \frac{y(1-p) + np(1-\delta)}{1-\delta p} \quad \blacksquare \quad \text{Q.E.D.}$$

Problem 4

a) We define conditional variance as:

$$Var(Y|X) := E(Y^2|X) - (E(Y|X))^2.$$

Taking its expectation

$$E(Var(Y|X)) = E[E(Y^2|X)] - E[E(Y|X)]^2]$$

The expected value of the conditional expectation of an r.v. is the expectation of the original var.

$$E[E(Y^2|X)] = E(Y^2)$$

$$\rightarrow E(Var(Y|X)) = E(Y^2) - E[(E(Y|X))^2].$$

Problem 4

a) (1) $E(Var(Y|X)) = E(Y^2) - E[(E(Y|X))^2]$

Now taking the varience of the conditional expectation, given $E(Y|X)$,

$$\Rightarrow Var(E(Y|X)) = E((E(Y|X))^2) - (E(E(Y|X)))^2$$

~~Given~~ Since $E(E(Y|X)) = E(Y)$,

$$Var(E(Y|X)) = E((E(Y|X))^2) - (E(Y))^2 \quad (2)$$

Combining (1) & (2), Cancels out

$$E(Var(Y|X)) = E(Y^2) - E[(E(Y|X))^2] \text{ \& }$$
$$Var(E(Y|X)) = E[(E(Y|X))^2] - (E(Y))^2$$
$$E(Var(Y|X)) + Var(E(Y|X)) = E(Y^2) - (E(Y))^2, \text{ so}$$
$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$$

QED

b) Coin with $p(H)=p$. Number of tosses until

$$E(Y) = \mu = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

$$Var(Y) = \sigma^2 = \sum_{k=1}^{6} k^2 \cdot \frac{1}{6} - \mu^2 = \frac{35}{12}.$$

$$E(Y|X=n) = \sum_{k=1}^{n} P(Y_i|X_i) y_i$$
$$= \underbrace{E(Y_1) + E(Y_2) + E(Y_3) + \dots}_{n \text{ times}}$$
$$= \boxed{\mu n.}$$

Given $X=x$,

$$E(Y|X) = \mu X = 3.5X$$
$$E(Y) = \mu E(X) \quad \because E(Y) = E(\mu X)$$
$$E(X) = \sum_{x} x P(X=k)$$

When $r = 1-p$, $E(x) = \frac{p}{1-p} \sum_{i=1} i r^i = \frac{p}{1-p} r \left(\frac{d}{dr}\left(\frac{1}{1-r}\right)\right)$

$= \frac{p}{1-p} (r)\left(\frac{1}{(1-r)^2}\right)$ putting $r = 1-p$ back in

$= \frac{\cancel{p}\,(1-p)}{\cancel{(1-p)}\,(p)^2} = \frac{1}{p} = E(X)$, so

$$E(Y) = \mu\left(\frac{1}{p}\right) = \boxed{\frac{7}{2p}}$$

Problem 4

$$Var(Y \mid X=n) = E(Y^2 \mid X=n) - (3.5n)^2$$

$$Var(Y^2 \mid X=n) = (n)(\sigma_X^2) + (n\mu)^2$$

$$Var(Y \mid X=n) = n\sigma^2 + \cancel{(n\mu)^2} - \cancel{(n\mu)^2}, \text{ so}$$

$$Var(Y \mid X=n) = n\left(\frac{35}{12}\right)$$

from part(a) $Var(X)$ or $\sigma^2 = \frac{35}{12}$, so

$$Var(Y \mid X=n) = \boxed{\frac{35n}{12}}.$$

$$Var(Y \mid X) = E(Y^2 \mid X) - (3.5X)^2$$

$$E(Y^2 \mid X) = \sigma^2 X + \mu^2 X^2, \text{ given } X=x.$$

$$E(Y^2 \mid X) = \sigma^2 X + (\mu x)^2.$$

$$Var(Y \mid x) = \sigma^2 X + (\mu x)^2 - (\mu X)^2$$

$$Var(Y \mid X) = \boxed{\frac{35X}{12}}$$

$$Var(Y) = E(Var(Y \mid X)) + Var(E(Y \mid X))$$

$$= E\left(\frac{35}{12}X\right) + Var(3.5X)$$

$$= \frac{35}{12}E(X) + E((3.5X)^2) - (3.5x)^2$$

$$= \frac{35}{12p} + 3.5^2 E(x^2) - (3.5x)^2$$

$$= \frac{35}{12} \cdot \frac{1}{p} + 3.5^2\, Var(x)$$

$$Var(X) = q/p \quad \therefore \quad E(x^2) - E(x)^2$$

$$= \frac{p^2-p+1}{p^2} \quad -1(p-1)? \quad - \frac{1}{p^2} = \frac{(1-p)}{p} = \frac{q}{p}$$

$$=$$

$$E(x^2) - \frac{1}{p^2} = \frac{q}{p}$$

$$= \boxed{\frac{1-p}{p}}$$

$$Var(Y) = \frac{35}{12}p + \frac{q}{p}(3.5)^2$$

Problem 5

Let N be the # of passengers $= 51$
number of seats reserved for 1 person

Total number of seating arrangements $- 51!$

Given 50 are random & 51st needed last seat, using complementary probabilities, there are 50 non empty seats and 1 empty.

What is the probability the last seat is the empty one?

$P(\text{last seat is not empty}) = 1 - P(\text{last seat empty})$

$P(\text{last seat empty}) = \frac{1}{51}$.

$P(\text{last seat not empty}) = 1 - \frac{1}{51} = \frac{51-1}{51} = \frac{50}{51}$

$P(\text{last seat not empty}) = \boxed{.98039}$

b) Calculate the expected number of people who will have to move.

A displaced person will move to the assigned seat of the person corresponding to where they were supposed to sit.

Since seats being occupied has the same probab of the 50th person getting the last seat.

$P(\text{displaced seating w/ 51st wanting last}) = .9804$ for every single person. Let X be the r.v. of the number of people who have to move.

$$E(X) = \sum_{\forall x} x_i P(X_i) = \sum_{i=1}^{51} P(\text{displace}) = 51(P\text{displace})$$

Since the number of people who must move (X) is binomally distributed, $E(X) = np = 51 \cdot .9804 = 51\left(\frac{50}{51}\right) = 50$ $\boxed{E(x) = 50 \text{ people move}}$

b) By 1st step analysis

$P_{n,k} = P_{n-1,k-1} \cdot P(\text{last seat not empty})$

$= P_{n-1,k-1} \cdot \frac{n-1}{n}$

$= P_{n-2,k-2} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1}$

and so on.

$P(E_n$ = "num of people who have to change in n")

$= \frac{n-k}{n} = P_{nk}$, so $\left(\frac{n-k}{n}\right)^n = E(X)$

where X is #who have to move because k people want certain seats out of all n plane seats.

$E(X) = n-k.$

$P(X) = \frac{n-k}{n}.$