Example 3. J departs for work "randomly" between 7 and 7:10 am: $\Omega = (0,10)$

We found that

for $C =$ "J departs between $0 < a < b < 10$" $= (a,b)$,

(1) $P(C) = \frac{b-a}{10} = \frac{|C|}{10}$, where $|C| =$ length of C.

Remark 1. (1) makes sense only for $C \subset (0,10)$ that are measurable (have length). The collection ~~of~~ of all measurable subsets of Ω is a σ-field.

Def. A collection $\mathcal{F}$ of Ω-subsets is called σ-field if a) $\emptyset, \Omega \in \mathcal{F}$

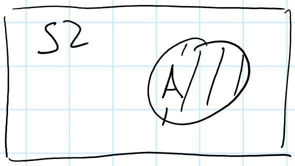
b) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,

c) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

Remark 2. For any sample space Ω, a), b), c) hold for $\mathcal{P}(\Omega)$, the collection of all subsets of Ω. In Example 3, the σ-field $\mathcal{F}$ of all measurable subsets of $\Omega = (0,10)$ is smaller than $\mathcal{P}(\Omega)$: $\mathcal{F} \subsetneq \mathcal{P}(\Omega)$. It coincides with the smallest σ-field that contains all subintervals of $\Omega = (0,10)$.

Venn diagrams and properties of $P(A)$ $\left(= \frac{\#A}{\#\Omega}\right.$ in "standard" setting$\left.\right)$

Venn diagrams and properties of $P(A)$ $\left(= \frac{\# A}{\# \Omega} \text{ in "standard setting"} \right)$

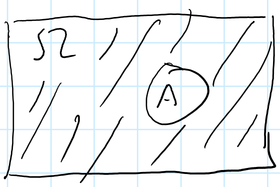
1. 

$P(A) = \frac{\# A}{\# \Omega}$ implies $P(\Omega) = 1$, $P(\emptyset) = 0$

$0 \le P(A) \le 1$.

$P(A)$ measures plausibility of A.

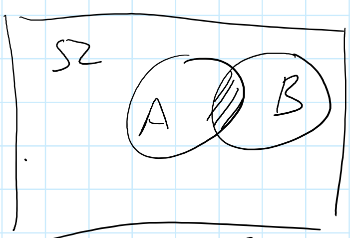
2. not A (complementary event) $= A^c = \Omega \setminus A$



$$P(A^c) = \frac{\# \Omega - \# A}{\# \Omega} = 1 - \frac{\# A}{\# \Omega} = 1 - P(A).$$

$$P(A) = 1 - P(A^c).$$

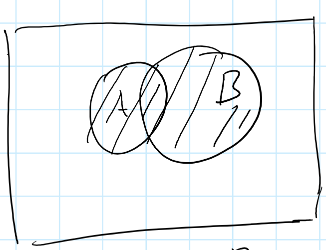
3. "A and B" $= A \cap B = AB$



$A \cap B$

$A_1 \cap \ldots \cap A_n =$ "all A_i happen".

4. "A or B" $= A \cup B =$ "A or B or both"



$A \cup B$

$A_1 \cup \ldots \cup A_n =$ "there is i so that A_i happens" = "at least one of A_i occurs".

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ inclusion exclusion principle

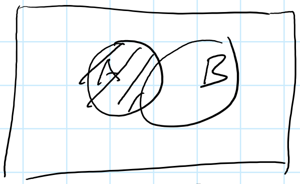
b) If A, B are disjoint (mutually exclusive), then

$P(A \cup B) = P(A) + P(B)$

c) Always $P(A \cup B) \le P(A) + P(B)$

$P(A_1 \cup \ldots \cup A_n) \le P(A_1) + \ldots + P(A_n)$.

5. "A not B" $= A \cap B^c = A \setminus B$

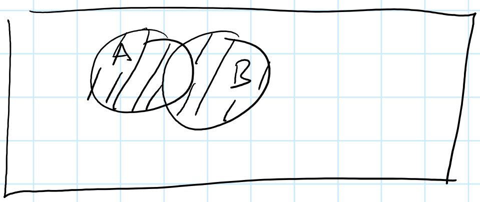


A \ B

a) $P(A \setminus B) = P(A) - P(A \cap B)$

b) If $B \subset A$, then $P(A \setminus B) = P(A) - P(B)$

c) $A \triangle B$ = "A or B but not both"



A △ B

$$P(A \triangle B) = P(A) + P(B) - 2P(AB)$$

Axioms of probability

Def. Probability space is a triplet $(\Omega, \mathcal{F}, P)$, where Ω is sample space, $\mathcal{F}$ is σ-field of events, and P is probability defined on $\mathcal{F}$.

Def. Given Ω with σ-field $\mathcal{F}$ of events, probability P assigns to each $A \in \mathcal{F}$ a number $P(A)$ so that

a) $0 \le P(A) \le 1$, $P(\Omega) = 1$, $P(\emptyset) = 0$.

b) If $A_1, A_2, \ldots$ are disjoint (mutually exclusive), then

then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) = \lim_{N\to\infty} \sum_{1}^{N} P(A_n) \leftarrow \sigma\text{-additivity.}$$

Recall $\bigcup_{n=1}^{\infty} A_n$ = "at least one A_n happens"

Note ① b) implies that if $A_1, \dots, A_n$ are disjoint, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1}^{n} P(A_i)$

② All properties we derived for "classical setting" hold in general case.

Ex 1. A fair coin is tossed until H shows up. Find probability that H shows up.

Answer. Let A_j = "1st H shows up in the j-th toss" $= \{\omega_j\}$. Then

$$A = \text{"H shows up"} = \bigcup_{j=1}^{\infty} A_j \quad \leftarrow \text{disjoint}$$

$$P(A) = \sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

Counting Principles