This week: 1.1 - 1.3; Counting Principles (multiplication principle, permutations, combinations etc.)

Ch 1. Events and probabilities

Def. Event is something that happens as a result of random circumstances (as a result of an experiment, trial with uncertain outcome)

Def. Result of an experiment is called outcome.

Def. The set of all outcomes is called sample space, denoted Ω.

Events can be regarded as subsets of Ω.

Examples. 1. Fair die is rolled
2. Fair coin is tossed until the 1st H shows up.
3. J leaves for work "randomly" between 7 and 7:10am

Consider two versions of Example 1:

Example 1.a) Fair die is rolled: $\Omega = \{1, 2, 3, 4, 5, 6\}$

some events: $A = \{3\}$, $B =$ "multiple of 3" $= \{3, 6\}$.

All outcomes are equally likely:

$$P(1) = \dots = P(6) = \frac{1}{6}, \quad P(B) = \frac{\# B}{\# \Omega} = \frac{2}{6} = \frac{1}{3}$$

Meaning of $P(B) = \frac{1}{3}$: If die is rolled many times, Then in <u>about</u> 1/3 of all scores will be 3 or 6.

<u>Example 1.b)</u> I leave the room, roll a fair die, come back and tell you that B happened (score is multiple of 3): for you, $\Omega = B = \{3, 6\}$, $P(3) = P(6) = \frac{1}{2} = P(A)$.

Both, 1a) and 1.b) are cases of

<u>"Standard", classical setting:</u>

$\boxed{\begin{array}{l} \text{1. } \Omega = \{\omega_1, \dots, \omega_N\} \text{ is finite and all outcomes equally likely: } P(\omega_1) = \dots = P(\omega_N) = \frac{1}{N}; \\ \text{2. For any event } A \subset \Omega, \\ \quad P(A) = \frac{\# A}{\# \Omega}. \end{array}}$

<u>Infinite sample space</u>

<u>Example 2</u>. Fair coin is tossed until H shows up:

$\Omega = \{\omega_1, \omega_2, \dots\}$, where $\omega_1 = H_1$, $\omega_2 = T_1 H_2, \dots,$

$\omega_k = T_1 \dots T_{k-1} H_k, \dots$, and maybe $\omega_0 = T_1 T_2 \dots$.

$P(\omega_1) = P(H_1) = \frac{1}{2}$, $P(\omega_2) = P(T_1 H_2) = \frac{1}{4} = \frac{1}{2^2}$ ⟵ because in two tosses, $H_1 T_2$, $H_1 H_2$, $T_1 H_2$, $T_1 T_2$ are equally likely, ..., $P(\omega_k) = \frac{1}{2^k} = 2^{-k}$, $k = 1, 2, \dots$

In this case, any $A \subset \Omega$ is an event,

$P(A) = \sum_{\omega_i \in A} P(\omega_i)$. We will see $P(\omega_0) = 0$.

Example 3. J leaves for work randomly between 7 and 7:10 am ("equally likely" at any time moment between 7 and 7:10): $\Omega = (0, 10)$.

Some events:

$A =$ "J leaves between 7:05 and 7:10" $= (5, 10)$,

$$P(A) = \frac{1}{2} = \frac{5}{10} = \frac{|A|}{|\Omega|} = \frac{|A|}{10}$$

$B =$ "J leaves between 7:06 and 7:09" $= (6, 9)$,

$$P(B) = \frac{9-6}{10} = \frac{3}{10} = \frac{|B|}{|\Omega|} = \frac{|B|}{10},$$ $|A|, |B|$ are lengths of A, B. In general,

$$\boxed{\text{if } C = \text{"J departs between } 0 < a < b < 10\text{"} = (a, b), \quad P(C) = \frac{b-a}{10} = \frac{|C|}{|\Omega|} \longleftarrow (1)}$$

Question Does (1) make sense for any $C \in \mathcal{P}(\Omega)$, the set of all subsets of Ω? Recall $\emptyset \subset \Omega$, $\Omega \subset \Omega$

Answer. NOT all $C \subset \Omega = (0, 10)$ have length:

$P(C) = \frac{|C|}{10}$ makes sense only for $C \in \mathcal{F} \subset \mathcal{P}(\Omega)$, where $\mathcal{F}$ is a certain σ-field of Ω-subsets.

It is known $\mathcal{F} \neq \mathcal{P}(\Omega)$, $\mathcal{F}$ consists only of those Ω-subsets whose length can be measured.