Continuous r. vectors

Ex 2. Let X_1, X_2, X_3 be i.i.d. continuous with pdf $f(x)$.

a) Find $P(X_1 < X_2)$

Answer. $P(X_1 < X_2) = P(X_2 < X_1) = \frac{1}{2}$.

b) Find $P(X_1 < X_2 < X_3)$, $P(X_2 < X_3 < X_1)$.

Answer. Joint pdf of (X_1, X_2, X_3) is $f(x_1) f(x_2) f(x_3)$

$$P(X_1 < X_2 < X_3) = \iiint_{x_1 < x_2 < x_3} f(x_1) f(x_2) f(x_3)\, dx_1\, dx_2\, dx_3$$

As in a) we find

$$P(X_1 < X_2 < X_3) = P(X_2 < X_3 < X_1) = P(\text{any ordering}) = \frac{1}{3!}$$

because there are $3!$ ways to order 3 numbers,

$$\Omega = \bigcup_{(i_1, i_2, i_3)} \{X_{i_1} < X_{i_2} < X_{i_3}\},$$

$$1 = P(\Omega) = \sum_{i_1, i_2, i_3} \underbrace{P(X_{i_1} < X_{i_2} < X_{i_3})}_{a} = 3!\, a.$$

Functions of r.v.

Claim 1. a) Let X be cont. r.v. with pdf $f(x)$, $Y = g(X)$. Then

$$E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

b) Let (X, Y) be jointly cont. with pdf $f(x,y)$. and $V = h(X, Y)$. Then and $V = h(X,Y)$. Then

$$E(V) = E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy.$$

Properties of expectation

1. (Linearity) $E(aX + bY) = a E(X) + b E(Y)$.
2. If $X \le Y$, then $E(X) \le E(Y)$; 3. $E(c) = c$.
4. If X, Y are indep., then

$$E[g(X)\, h(Y)] = E[g(X)]\, E[h(Y)].$$

Why 4.? $E[g(X)\, h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) h(y) f_X(x) f_Y(y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \int_{-\infty}^{\infty} h(y) f_Y(y)\, dy.$$

Def. Covariance of X, Y is the number

$$\mathrm{Cov}(X,Y) = E[(X-\mu_1)(Y-\mu_2)], \text{ where } \mu_1 = E(X), \mu_2 = E(Y).$$

$$\mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_1)(y-\mu_2) f(x,y)\, dx\, dy = E(XY) - \mu_1\mu_2.$$

$$\mathrm{Var}(X) = \mathrm{Cov}(X,X)$$

Remark 1. $\mathrm{Cov}(X,Y)$ has all the properties listed for discrete r.v. X, Y: variance - covariance expansion, linearity in X and Y.

Def. Correlation of X, Y is the number

$$\rho = \rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_1 \sigma_2}, \quad \sigma_1 = \sqrt{\mathrm{Var}(X)}, \sigma_2 = \sqrt{\mathrm{Var}(Y)}.$$

$\rho = \rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_1 \sigma_2}$, $\sigma_1 = \sqrt{Var(X)}$, $\sigma_2 = \sqrt{Var(Y)}$.

Again $-1 \le \rho \le 1$, and the following statement holds

<u>Claim 2</u>. Let $\mu_1 = E(X)$, $\mu_2 = E(Y)$, $\sigma_1 = \sqrt{Var(X)}$, $\sigma_2 = \sqrt{Var(Y)}$, $\rho = \rho(X,Y)$.

Then

$Y - \mu_2 = \rho \dfrac{\sigma_2}{\sigma_1}(X - \mu_1) + V$, where X and V are uncorrelated ($Cov(V,X) = 0$), $E(V) = 0$, $Var(V) = \sigma_2^2(1-\rho^2)$

4.4. <u>Examples of r.v.</u>

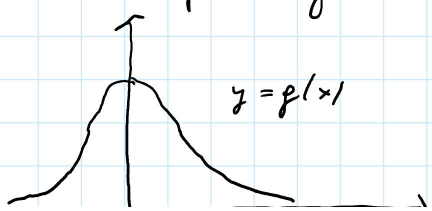
<u>Def.</u> a) Continuous r.v. X is called <u>normal</u> with parameters μ, σ^2 (we write $X \sim N(\mu, \sigma^2)$) if its pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Here $\sigma > 0$, $-\infty < \mu < \infty$.

b) We say Z is <u>standard normal</u> if $Z \sim N(0,1)$: Z is normal with parameters $\mu = 0$, $\sigma^2 = 1$. Its pdf

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$



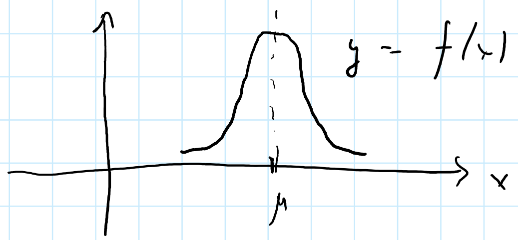
$E(Z) = 0$?

<u>Note</u> 1. $g(x) = g(-x)$: g is even. Therefore Z and $-Z$ are identically distributed, both $N(0,1)$

2. The pdf of $N(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sigma}\, g\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < x < \infty,$$

where $g(x)$ is pdf of $N(0,1)$.



$E(X) = \mu$?

<u>Basic facts</u>

1. $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$ (polar coordinates)

2. $E(Z) = 0$, $Var(Z) = 1$, $Z \sim N(0,1)$.

<u>Why?</u> $E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x\, e^{-x^2/2}}_{\text{odd}}\, dx = 0.$

$$Var(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx = 1.$$

3. a) If $Z \sim N(0,1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$: hence $E(X) = \mu$, $Var(X) = \sigma^2$, σ is standard deviation of X.
b) If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.