3.10 Path properties of simple r.w.

$S_n = S_0 + X_1 + \ldots + X_n = S_0 + \mathcal{H}_n - \mathcal{T}_n$

$S_0, X_1, \ldots$ are independent, $P(X_i = 1) = p$, $P(X_i = -1) = q = 1-p$,

$\mathcal{H}_n = \#$ of $+1$ steps, $\mathcal{T}_n = \#$ of -1 steps: $\mathcal{H}_n + \mathcal{T}_n = n$.

Basic properties of paths

① Let $N_n(a,b)$ be number of paths from a to b in n steps (from $(0,a)$ to (n,b)):

$S_n = a + \mathcal{H}_n - \mathcal{T}_n = b$; since $\mathcal{T}_n = n - \mathcal{H}_n$,

$a + \mathcal{H}_n - (n - \mathcal{H}_n) = b$, $\quad 2\mathcal{H}_n = n + b - a$,

$\mathcal{H}_n = \dfrac{n+b-a}{2}$, $\quad \mathcal{T}_n = \dfrac{n+a-b}{2}$.

$$N_n(a,b) = \begin{cases} \dbinom{n}{\frac{n+b-a}{2}} & \text{if } \frac{n+b-a}{2} \in \{0,1,\ldots,n\} \\ 0, & \text{otherwise} \end{cases}$$

Note $P(S_n = b \mid S_0 = a) = \dbinom{n}{\frac{n+b-a}{2}} p^{\frac{n+b-a}{2}} q^{\frac{n+a-b}{2}}$.

$\left(= \dbinom{n}{\frac{n+b-a}{2}} 2^{-n} \quad \text{if } p = q = \tfrac{1}{2} \right)$

② Reflection principle

Consider $N_n^0(a,b) = \#$ of paths from a to b in n steps via 0.

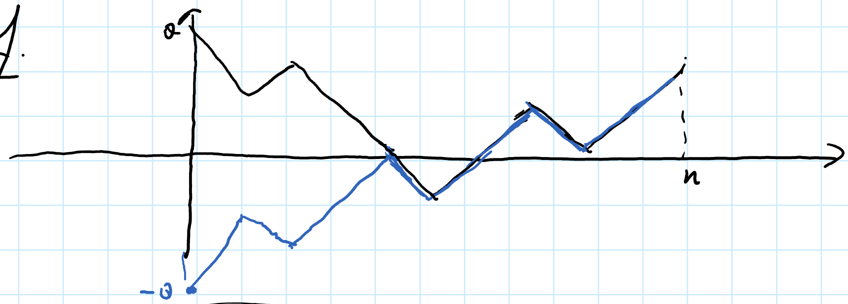
Note $N_n^0(a,b) = N_n(a,b)$ if $a < 0, b > 0$ or $a > 0, b < 0$.

Thm 1. Let $a > 0$, $b > 0$. Then

$$N_n^0(a,b) = N_n(-a,b)$$

Reflection principle.

Proof.



There is one-to-one correspondence between blue and black paths.

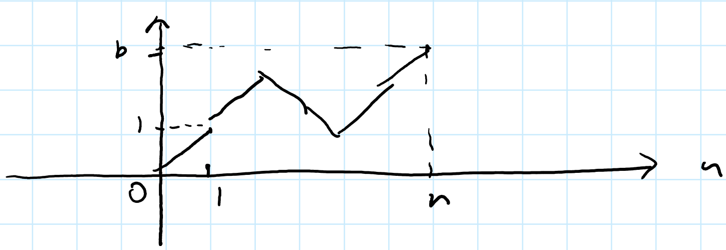
Consider $\overline{N_n^0}(a,b)$ = # of paths from a to b in n steps without visiting 0 inbetween.

Thm 2. Let $b > 0$. Then

$$\overline{N_n^0}(0,b) = \frac{b}{n} N_n(0,b) = \frac{b}{n}\binom{n}{\frac{n+b}{2}}$$

Note $b \leq n$

Why?



$$\overline{N_n^0}(0,b) = \overline{N_{n-1}^0}(1,b) = N_{n-1}(1,b) - N_{n-1}^0(1,b) =$$

$$= N_{n-1}(1,b) - N_{n-1}(-1,b) = \ldots$$

Remark 1. a) For $b \neq 0$, $\overline{N_n^0}(0,b) = \frac{|b|}{n} N_n(0,b)$.

b) "no zero visits in n steps" $= \{S_1 \neq 0, \ldots, S_n \neq 0\} = \{\underbrace{S_1 \cdots S_n}_{\text{product}} \neq 0\} =$

$= \{\tau_0 > n\}$, $\tau_0 = \min\{n \geq 1 : S_n = 0\}$.

Corollary. Let $S_0 = 0$, $b \neq 0$. Then

$$P(S_1 \dots S_n \neq 0,\ S_n = b) = \frac{|b|}{n} P(S_n = b), \text{ equivalently,}$$

$$P(S_1 \dots S_n \neq 0 \mid S_n = b) = \frac{|b|}{n}$$

Proof. Let $b > 0$. Recall

$$P(\underbrace{S_1 \dots S_n \neq 0}_{\tau_0 > n},\ S_n = b) = \overline{N_n^0}(0,b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}} = \frac{b}{n} \overbrace{N_n(0,b)\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}}}^{\substack{P(S_n = b).\\ \shortparallel}}$$

Some exercises

Ex 1. A coin was tossed 20 times. Consider $S_n = \mathcal{H}_n - \mathcal{T}_n$, $1 \le n \le 20$, $S_0 = 0$. Given $S_{20} = 4$, find probability that $\mathcal{H}_1 > \mathcal{T}_1, \dots, \mathcal{H}_{20} > \mathcal{T}_{20}$.

Answer. $P(\mathcal{H}_1 > \mathcal{T}_1, \dots, \mathcal{H}_{20} > \mathcal{T}_{20} \mid \mathcal{H}_{20} - \mathcal{T}_{20} = 4) =$

$$= P(S_1 \dots S_{20} \neq 0 \mid S_{20} = 4) = \frac{4}{20} = \frac{1}{5}.$$

Ex 2. Consider simple r.w. S_n with $S_0 = k$, $0 < k < N$. Let $\tau_k = \min\{n \ge 1 : S_n = 0 \text{ or } S_n = N\}$: τ_k is time to reach the boundary from k.

a) Find $D_k = E(\tau_k)$

Answer. 1. 1st step analysis gives system of equs for D_k:

$$(1) \quad \begin{cases} D_k = p D_{k+1} + q D_{k-1} + 1, & k = 1, \dots, N-1. \\ D_0 = D_N = 0 \end{cases}$$

because $D_k = E(\tau_k) = E(\tau_k \mid X_1 = 1) \overbrace{P(X_1 = 1)}^{p} + E(\tau_k \mid X_1 = -1) \overbrace{P(X_1 = -1)}^{q}$

$= p\cdot(D_{k+1}+1) + q(D_{k-1}+1) = pD_{k+1} + qD_{k-1} + 1$, because

Given $X_1 = 1$, $\tau_k = 1 + \tau_{k+1}$: $E(\tau_k \mid X_1 = 1) = 1 + E(\tau_{k+1})$

Given $X_1 = -1$, $\tau_k = 1 + \tau_{k-1}$:

Now, (1) is non-homogeneous linear eqn:

general solution $D_k = A + B\left(\frac{q}{p}\right)^k + L_k$, where

L_k is particular solution to (1) and A, B to be found.

L_k is of the form

$$L_k = \begin{cases} ck & \text{if } q \neq p \\ ck^2 & \text{if } q = p = \frac{1}{2} \end{cases}$$ with c found from (1)

A, B are found using boundary conditions $D_0 = D_N = 1$.

Result:

$$D_k = \begin{cases} k(N-k), & p = q = \frac{1}{2}. \\ \frac{1}{q-p}\left(k - N\frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N}\right), & p \neq q. \end{cases}$$

b) Find $\overline{D}_k = \lim_{N\to\infty} D_k$

Answer. $$\overline{D}_k = \begin{cases} +\infty, & p \geq q \quad (\text{includes } p = q = \frac{1}{2}) \\ \frac{k}{q-p}, & q > p \end{cases}$$

Remark 1 Let $k > 0$. $S_0 = k$, $\overline{\tau}_k = \min\{n \geq 1 : S_n = 0\}$.

Then $\overline{D}_k = E(\overline{\tau}_k)$ is expected ruin time (number of games to be ruined). Recall for $p = q = \frac{1}{2}$, $P(\overline{\tau}_k < \infty) = 1$. but $E(\overline{\tau}_k) = +\infty$.