## Counting.

**Claim.** Assume we have $n_1$ copies of letter $L_1, \dots,$ $n_k$ copies of letter $L_k$, $n_1 + \dots + n_k = n$. Number of words of length $n$ using those letters is

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! \dots n_k!}$$

"seat assignment": divide a row of $n$ seats $\overbrace{\square \dots \square}^{n}$ into $k$ groups of size $n_1, \dots, n_k$.

**Ex1.** There are 8 white 2 red balls in the box. They are taken out one by one.

Find probability that a) 1st ball is red: $\frac{2}{10} = \frac{1}{5}$.

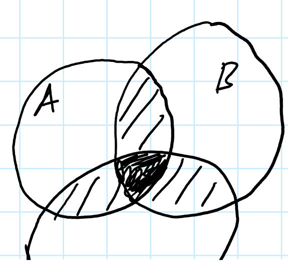
b) $k$-th ball is red: $\frac{2 \cdot 9!}{10!} = \frac{2}{10} = \frac{1}{5}$.

**This week:**

1. Inclusion/exclusion principle (#4 of 1.3, p. 8)
2. 1.4-5 Conditional probability and independence

## Inclusion/exclusion principle

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
$- P(AB) - P(AC) - P(BC)$
$+ P(ABC)$



$$-P(AB)-P(AC)-P(BC)$$
$$+P(ABC)$$


3. To find $P(A_1 \cup \ldots \cup A_n) = P(\text{at least one } A_i \text{ happens})$,

(i) <u>include</u> probabilities of all <u>odd</u> intersections

(ii) <u>exclude</u> " ———————— " <u>even</u> intersections

Result:

$$\boxed{\begin{aligned} P(A_1 \cup \ldots \cup A_n) &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) \\ (1) \qquad & - \ldots + (-1)^{n+1} P(A_1 \cdots A_n). \end{aligned}}$$

<u>Comment</u>. 1. Generic term in (1):

$$(-1)^{k+1} \sum_{i_1 < \ldots < i_k} P(A_{i_1} \cdots A_{i_k})$$

2. There are $\binom{n}{k}$ terms in the sum $\sum_{i_1 < \ldots < i_k} P(A_{i_1} \ldots A_{i_k})$

<u>Ex 1</u>. Hats of $n$ people are mixed up. Everybody picks up randomly a hat (one by one).

Find probability of no match.

<u>Answer</u>. $P(\text{no match}) = 1 - P(\text{at least one match})$

$= 1 - P(A_1 \cup \ldots \cup A_n)$, where $A_i$ = "$i$th person gets the right hat".

$$P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1 < \ldots < i_k} P(A_{i_1} \cdots A_{i_k})$$

<u>By counting</u>, ① $P(A_{i_1} \cdots A_{i_k}) = \dfrac{(n-k)!}{n!}$

By Counting, ① $P(A_{i_1} \dots A_{i_k}) = \dfrac{\phantom{(n-k)!}}{n!}$

② $\displaystyle\sum_{i_1 < \dots < i_k} P(A_{i_1} \dots A_{i_k}) = \binom{n}{k} \frac{(n-k)!}{n!} = \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} = \frac{1}{k!}$

Hence $\displaystyle P(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n+1} \frac{1}{n!}$

$P(\text{no match}) = 1 - P(A_1 \cup \dots \cup A_n) = \frac{1}{2!} - \frac{1}{3!} + \dots - (-1)^{n+1} \frac{1}{n!} \approx e^{-1} = \frac{1}{e} = \underline{0.37}$

## 1.4. Conditional probability

Example 1. Fair die is rolled: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
Consider $A$: "odd number" $= \{1, 3, 5\}$, $P(A) = \frac{3}{6} = \frac{1}{2}$
Someone tells "score is $< 4$" $= B = \{1, 2, 3\}$ happened.
Question: Given $B$, what are chances of $A$?

Answer. $B = \{1, 2, 3\}$ becomes sample space, and we find proportion of $A$ in $B$:

$\dfrac{\#(A \cap B)}{\# B} = \dfrac{2}{3} := P(A|B)$ because $\#(AB) = \#\{1, 3\} = 2$

Note $P(A|B) = \dfrac{\#(A \cap B)}{\# B} = \dfrac{\#(A \cap B)/\#\Omega}{\# B / \# \Omega} = \dfrac{P(AB)}{P(B)}$

is relative 'weight' of $A$ in $B$.

General definition. Cond. prob. of $A$ given $B$ is the number

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)}} \leftarrow (\text{relative 'weight' of } A \text{ in } B).$$

Comment. a) $P(A|B)$, $A \in \mathcal{F}$, is probability in $A$.

b) If $\Omega$ is finite and all outcomes are equally likely ("classical setting"), then

$$P(A|B) = \frac{\#(AB)}{\# B}.$$

Ex 2. 8 white and 2 red balls in the box. Balls are taken out one by one

Given three first are white, find $P(R_6)$, $R_6$ = "6th is red".

Answer. $P(R_6 \mid W_1 W_2 W_3) = \frac{2}{7}$.

Similarly, $P(R_6 \mid W_1 R_2 W_3) = \frac{1}{7}$, $P(R_6 \mid R_1 W_2 R_3) = 0$.