Homework 3

[illegible] - [illegible]

1) P("centers in figure") = 1/5
Bulk purchase pdf for m=n
Binomial(n, p) = Binomial(6, 1/5)

P("figure found in last 3") = 1 - P("figure not found in last 3")

Let A_1, A_2, A_3 be the events that the figure is not found in the 4th, 5th, or 6th boxes of cereal.

$P(\text{"not found in last 3"}) = 1 - P(\bigcup_{i=1}^{3} A_i)$

$= 1 - P(\bigcup_{i=1}^{3} A_i) = 1 - 3P(A_1) + 3P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$

$P(A_i) = 1 - p = 4/5$

$P(A_1 \cap A_2) = P(\text{"chance the } 4^{th} \text{ and } 5^{th} \text{ joint have it"})$ $\quad P(A_1^c) = 1/5 \quad P(A_2^c) = 1/5 \Rightarrow P(A_1^c \cap A_2^c) = 2/5 \Rightarrow P(A_1 \cap A_2) = 3/5$

$P(A_1 \cap A_2 \cap A_3) = 1 - 3p = 2/5$

With n=6 tries and multiplication principle, $P(A)$ happens $= (P(A))^n$

$P(\text{"figure found"}) = 1 - 3\left(\frac{4}{5}\right)^6 + 3\left(\frac{3}{5}\right)^6 - \left(\frac{2}{5}\right)^6$

2)

$P(\text{blocked}) = p$

$P(\text{"clear"}) = q = 1-p$

Find $P(\text{"A} \leftrightarrow \text{B" } | \text{ "A} \not\leftrightarrow \text{C"}) = \dfrac{P(A \leftrightarrow B \cap A \not\leftrightarrow C)}{P(A \not\leftrightarrow C)}$

$P(A \not\leftrightarrow B) = P(\text{"show blocking the crossing from A to B"}) = P(\text{r1 blocked}) \cdot P(\text{r2 blocked}) = p^2$

$P(A \leftrightarrow B) = 1 - P(A \not\leftrightarrow B \text{ is blocked}) = 1 - p^2$

Similarly, $P(B \not\leftrightarrow C) = 1 - p^2$ and $P(B \not\leftrightarrow C) = p^2$

$P(A \leftrightarrow C) = P(A \leftrightarrow B) \cdot P(B \leftrightarrow C) = (1-p^2)^2$

$P(A \not\leftrightarrow C) = 1 - P(A \leftrightarrow C) = 1 - (1-p^2)^2$

$P(A \leftrightarrow B \cap A \leftrightarrow C) = P(\text{"a to b is not blocked but b to c is blocked"}) =$

$P(A \leftrightarrow B \cap B \not\leftrightarrow C)$

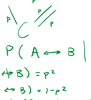
By Mult. principle of disjoint events,

$P(A \leftrightarrow B \cap B \not\leftrightarrow C) = P(A \leftrightarrow B) \cdot P(B \not\leftrightarrow C) =$

$(1-p^2)(p^2)$

$\dfrac{P(A \leftrightarrow B) \cap P(B \not\leftrightarrow C)}{P(A \not\leftrightarrow C)} = \dfrac{(1-p^2)p^2}{1-(1-p^2)^2}$

Now consider:



Find $P(A \leftrightarrow B \mid A \not\leftrightarrow C)$:

$P(A \not\leftrightarrow B) = p^2$

$P(A \leftrightarrow B) = 1-p^2$

$P(B \not\leftrightarrow C) = p^2 \Rightarrow P(B \leftrightarrow C) = 1-p^2$

$P(A \not\leftrightarrow C) = 1 - P(A \leftrightarrow C)$

Let $A' \leftrightarrow C'$ be the path from the A node to the C node w/o going through B.

$P(A \leftrightarrow C) = P(A \leftrightarrow B) \cdot P(B \leftrightarrow C) + P(A' \leftrightarrow C')$

The probability there is a path from A to C is given by $P(A' \leftrightarrow C') + P(A' \not\leftrightarrow C')\left(P(A \leftrightarrow B) \cdot P(B \leftrightarrow C)\right)$

The only reason to go through B is if $A' \leftrightarrow C'$ is closed.

$P(A \not\leftrightarrow C') = p$

$P(A' \leftrightarrow C') = 1-p$

$P(A \leftrightarrow C) = (1-p) + p(1-p^2)^2$

$P(A \not\leftrightarrow C) = 1 - P(A \leftrightarrow C)$

$P(A \not\leftrightarrow C) = 1 - (1-p) - p(1-p^2)^2$

$P(A \leftrightarrow B \mid A \not\leftrightarrow C) = \dfrac{P(A \leftrightarrow B \cap A \not\leftrightarrow C)}{P(A \not\leftrightarrow C)}$

$P(A \leftrightarrow B \cap A \not\leftrightarrow C) = p^2$

Disjoint events

$P(A \leftrightarrow B \cap A \not\leftrightarrow C) = P(A \leftrightarrow B \cap A' \not\leftrightarrow C' \cap B \not\leftrightarrow C)$

$P(A \leftrightarrow B) = 1-p^2$

$P(B \not\leftrightarrow C) = p^2$

$P(A' \not\leftrightarrow C') = p$

$P(A \rightarrow B \cap A' \not\leftrightarrow C' \cap B \not\leftrightarrow C) = (1-p^2)p^2 p$, so $P(A \leftrightarrow B \mid A \not\leftrightarrow C) = \dfrac{(1-p^2)p^3}{1-(1-p)-p(1-p^2)^2} \Big/ p = \dfrac{(1-p^2)p^2}{1-(1-p^2)^2}$

3) G either wins \$1 or loses \$1.
He starts with \$k where $0 < k < N$.
The game ends when he has either attained \$N or \$0.
It takes k goes to reach 0 and N-k goes to reach N.
With a fair coin, $P(H) = 1/2 \rightarrow P(+\$1) = 1/2$
$P(T) = 1/2 \rightarrow P(-\$1) = 1/2$
Let X be the random variable representing \$ we win.
Let A_N represent ending with \$N and the game stopping
Let A_0 represent ending with \$0 and the game stopping
$P(A_N)$ means k games which lose money & $P(A_0)$ is N-k games which add money.

$P(A_N) = \dfrac{\sum_{i=0}^{k} (\frac{1}{2})}{\sum_{i=0}^{N} (1/2)} = \dfrac{\overbrace{.5 + \dots + .5}^{k}}{\underbrace{.5 + \dots + .5}_{N \text{ times}}} = \dfrac{(N-k) \cdot .5}{N \cdot .5} = \dfrac{N-k}{N}$

$P(A_0) = \dfrac{\sum_{i=0}^{k} (\frac{1}{2})}{\sum_{i=0}^{N} (\frac{1}{2})} = \dfrac{\overbrace{.5 + \dots + .5}^{k}}{\underbrace{.5 + \dots + .5}_{N}} = \dfrac{k}{N}$

$P(\text{stop}) = P(A_0) + P(A_N)$

$P(\text{stop}) = \dfrac{k}{N} + \dfrac{N-k}{N} = \dfrac{N-k+k}{N} = \dfrac{N}{N} = \boxed{1}$

reaching \$0 $\quad$ reaching \$N

The game will not go on infinitely and will end for sure b/c $P_k + \overline{P_k} = 1$.

4) 5 signals in $\longrightarrow$ 1 signal out.

00000 $\rightarrow$ 0

Each has p(wrong) = .2

5 tries with p(wrong) = .2

Let X be a binom r.v. with n=5, p=.2 that a signal is wrong.

a) 0 got passed in as 00000

but if it were wrong it should be 1 $\rightarrow$ 00 111

$P(X_1 X_2 X_3) = \binom{5}{3} p^3 q^2 + \binom{5}{4}(.2)^4(.8) + \binom{5}{5}(.2)^5$

$P(0 \text{ was decoded wrong}) = \boxed{.0579}$

b) 101 was decoded wrong.

1 can be wrong. or
0 can be wrong. or
1 can be wrong.

Consider each bit being wrong is independent of every other bit and by part (a) $P(\text{"a bit was emitted wrong"}) = .0579$

$P(101 \text{ is decoded wrong}) = P((1 \text{ is wrong}) \cup (0 \text{ is wrong}) \cup (1 \text{ is wrong}))$

Let A_1, A_2, A_3 represent the ith bit is wrong.

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - \sum_{i<j}^{3} P(A_i A_j) + (P($

$= \sum_{i=1}^{3} P(A_i) - \sum_{i<j}^{3} P(A_i A_j) - P(A_1 A_2 A_3)$

$P(A_i) = .0579 \quad P(A_i^c) = 1 - .0579$

$\sum_{i=1}^{3} P(A_i^c) = 3(.0579)^1(1-.0579)^2$

$\sum_{i<j}^{3} P(A_i A_j) = 3(.0579)^2(1-.0579)$

$P(A_1 A_2 A_3) = (.0579)^3$

$P(101 \text{ is wrong}) = 3(.0579)(1-.0579)^2 - 3(.0579)^2(1-.0579) + (.0579)^3$

$P(101 \text{ is wrong}) = \boxed{.1449}$

5) $P(H) = p$, $P(T) = q = 1-p$

Let H_k = "H in the kth toss"
Let T_k = "T in the kth toss"

a) Find P(at least 1 H after n)

$P(\bigcup_{m=n}^{\infty} H_m) = 1 - \left(\bigcap_{m=n}^{\infty} T_m\right)$

$1 - P\left(\bigcap_{m=n}^{\infty} T_m\right) = 1 - \lim_{k\to\infty} P\left(\bigcap_{m=n}^{k+n} T_m\right)$

$P(T) = q$, so the infinite intersection = 0.

$= 1 - \lim_{k\to\infty} q^{k-1}$

$= 1 - \lim_{k\to\infty} (\text{a number} < 1)^{k+1} = 1 - 0 = \boxed{1}$

b) Find probability of infinite many H.

$P(\text{infinite H}) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} H_m\right)$.

$P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} H_m\right) = \lim_{k\to\infty} P\left(\bigcap_{n=1}^{k} \bigcup_{m=n}^{\infty} H_m\right)$

From part (a)

$P\left(\bigcup_{m=n}^{\infty} H_m\right)$ means at least 1 H happens after n.

$P\left(\bigcup_{m=n}^{\infty} H_m\right) = 1$.

$\lim_{k\to\infty} \bigcap_{n=1}^{k}$ (at least 1 H after n) = at least 1 H happens after 1.

So, $P(\text{Infinite H}) = \lim_{k\to\infty} P(\text{at least 1 H after 1}) = \lim_{k\to\infty} (\text{ans from part (a)}) = \lim_{k\to\infty} (1) = \boxed{1}$