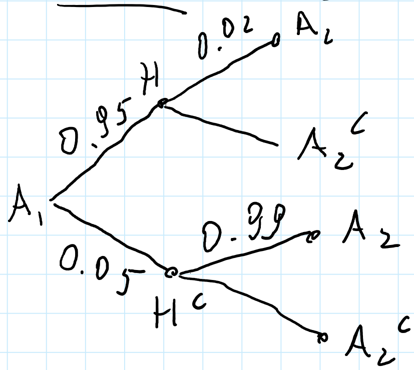
1.4 Conditional probability

Ex1 (false positive). A test for rare disease is positive for 99% of ill and 2% of healthy people.
J was tested positive. We found
$P(\text{J healthy} \mid \text{1st test positive}) = P(H \mid A_1) = 0.95.$
His 2nd test is positive again. Find prob. J is healthy.

Answer. A_2 = "2nd test positive", H^c = "ill", H = "healthy"



$$P(H \mid A_2) = \frac{0.02 \cdot 0.95}{0.02 \cdot 0.95 + 0.99 \cdot 0.05} = 0.28$$

Remark 1. Changes of probability:

(i) Before any test, prob. of a randomly selected person healthy,
$P(H) = 0.999$ ← "general population".

(ii) 1st test positive
$P(H \mid A_1) = 0.95$

(iii) 2nd test positive (J is in population of people with 1st positive test)
$P(H \mid A_1 A_2) = 0.28$

Ex2 (tv game) Award is behind one of 3 doors.
You choose 1st door. Presenter opens the 2nd door

and offers to switch to 3rd door. Would you do it?

<u>Answer</u>. L_k = "award behind k-th door", $k = 1, 2, 3$,
B = "2nd door opened".

<u>We need to compare</u> $P(L_1 | B)$, $P(L_3 | B)$, $P(L_2 | B) = 0$

We know (i) before B, $P(L_1) = P(L_2) = P(L_3) = 1/3$.

(ii) $P(B | L_1) = \frac{1}{2}$, $P(B | L_2) = 0$, $P(B | L_3) = 1$

Now,

$$P(L_1 | B) = \frac{P(B | L_1) P(L_1)}{P(B)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{1/2} = 1/3$$

$$P(B) = P(B | L_1) P(L_1) + P(B | L_2) P(L_2) + P(B | L_3) P(L_3) =$$

$$= \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{3}{2} = 1/2$$

Hence $P(L_3 | B) = 1 - \frac{1}{3} = \frac{2}{3}$ ✓ : better to switch.

<u>Comment</u>. Note $P(L_1 | B) = P(L_1) = \frac{1}{3}$
$P(B | L_1) = P(B) = \frac{1}{2}$

We say L_1 and B are <u>independent</u>.

<u>1.5 Independence</u>

<u>Def</u>. A, B are <u>independent</u> if

$$P(A | B) = P(A), \text{ equivalently,}$$

$$P(B | A) = P(B), \text{ equivalently,}$$

$$P(AB) = P(A) P(B)$$

<u>Equivalence in consequence of Multiplication Law</u>:

$$P(AB) = P(A | B) P(B)$$

$$= P(B | A) P(A)$$

Remark 1. If A, B are indep., then A^c and B, A and B^c, A^c and B^c are independent.

For instance, $P(AB^c) = P(A \setminus B) = P(A) - P(AB) =$
$= P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A)\,P(B^c)$.

Ex 1. Two fair dice are rolled. Consider
$A =$ "sum $= 7$", $B =$ "1st is 4", $C =$ "2nd is 3".
Show that any pair of those events is independent.

Answer. $\Omega = \{(i,j) : 1 \le i, j \le 6\}$, $\#\Omega = 6 \cdot 6 = 6^2 = 36$.
$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$, $\# A = 6$.
$P(A) = \frac{6}{36} = \frac{1}{6}$, $P(B) = \frac{6}{36} = \frac{1}{6}$, $P(C) = \frac{6}{36} = \frac{1}{6}$.

$P(AB) = \frac{1}{36} = P(A)P(B)$ | $P(BC) = \frac{1}{36} = P(B)P(C)$.
$P(AC) = \frac{1}{36} = P(A)\,P(C)$

Comment $P(ABC) = \frac{1}{36} \neq P(A)P(B)P(C)$

Def. A, B, C are indep. if
a) all pairs are independent,
b) $P(ABC) = P(A)P(B)P(C)$.

Independence of a family of events

Def. A_i, $i \in I$, are independent if for any finite $J \subset I$

$$P\Big(\bigcap_{j \in J} A_j\Big) = \prod_{j \in J} P(A_j) \leftarrow \text{product of probabilities}$$

$$P\Big(\bigcap_{j\in J} A_j\Big) = \prod_{j\in J} P(A_j)$$

Remark 2 a) If A, B, C are independent, then A is independent of any event made of B and C, like $B\cup C$, B^c, C^c, $B\cap C^c$, ...
Similarly with A^c.

b) If A_i, $i\in I$, are independent, $J, K \subset I$ are disjoint, then any event made of A_j, $j\in J$, and any event made of A_k, $k\in K$, are independent.

Remark 3. Events happening as a result of repeated experiments often are considered independent: coin is tossed, die is rolled ...