3.5 Various discrete r.v.

1. $X \sim$ binomial(n, p), $E(X) = np$, $Var(X) = npq$.

If $X \sim bin(n,p)$, $Y \sim bin(m,p)$ are independent, then $X+Y \sim bin(n+m, p)$.

2. $X \sim$ Poisson(λ). We know $E(X) = \lambda$.

Ex 1. Confirm that $\boxed{Var(X) = \lambda}$

Answer. $Var(X) = E(X^2) - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

We compute

$$E[X(X-1)] = E(X^2) - E(X) = E(X^2) - \lambda$$

$$= \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \lambda^2 \sum_{k=2}^{\infty} \frac{k(k-1)\lambda^{k-2}}{k!}$$

$$= e^{-\lambda} \lambda^2 \frac{d^2}{d\lambda^2} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = e^{-\lambda} \lambda^2 \frac{d^2}{d\lambda^2} (e^{\lambda}) = \lambda^2$$

3. $X \sim$ geometric(p), $P(X=k) = q^{k-1} p$, $k = 1, 2, \ldots$, $E(X) = \frac{1}{p}$,

$Var(X) = \frac{q}{p^2}$ (the same technique as Poisson)

$P(X > k) = q^k$, $k = 1, 2, \ldots$.

4. X is negative binomial (r, p). Example:

A coin with $P(H) = p$ is tossed repeatedly.

X = # of tosses needed for r H's to show up.

Note. 1. X is geometric(p) if $r = 1$.

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2. $X = X_1 + \dots + X_r$, where X_i are independent geometric(p):

$$E(X) = r \cdot \frac{1}{p}, \quad Var(X) = r \cdot \frac{q}{p^2}.$$

3. For $k \geq r$,

$$P(X=k) = \binom{k-1}{r-1} p^r q^{k-r}$$

$\{X=k\}$ = "all words of length k with r H's and $k-r$ T's with the last letter H".

Ex 2. Let $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ) be independent. Show that given $X+Y=n$,

$$X \sim \text{binomial}\left(n, p = \frac{\lambda}{\lambda+\mu}\right), \quad q = 1-p = \frac{\mu}{\lambda+\mu}:$$

$$P(X=k \mid X+Y=n) = \binom{n}{k} p^k q^{n-k}, \quad k=0,1,\dots,n.$$

Answer. Recall $X+Y \sim$ Poisson$(\lambda+\mu)$.

$$P(X=k \mid X+Y=n) = \frac{P(X=k, \cancel{X+Y}=n)}{P(X+Y=n)} = \frac{P(X=k)\,P(Y=n-k)}{P(X+Y=n)}$$

$$= \frac{\cancel{e^{-\lambda}} \frac{\lambda^k}{k!} \cancel{e^{-\mu}} \frac{\mu^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda+\mu)}} \frac{(\lambda+\mu)^n}{n!}} = \frac{n!}{k!(n-k)!} \frac{\lambda^k}{(\lambda+\mu)^k} \cdot \frac{\mu^{n-k}}{(\lambda+\mu)^{n-k}}$$

$$= \binom{n}{k} \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{n-k}, \quad p = \frac{\lambda}{\lambda+\mu}, \; q = \frac{\mu}{\lambda+\mu}.$$

Ex 3. Number of accidents per day at an intersection is Poisson(λ). There were 5 accidents last week. Find probability of at most one accident the 1st day of the week.

Answer. X = # of accidents 1st day $\sim$ Poisson(λ)
Y = # " —" following 6 days $\sim$ Poisson(6λ)

Given $X+Y=5$, $X \sim \text{binomial}\left(5, \frac{\lambda}{\lambda+6\lambda} = \frac{1}{7}\right)$

$$P(X \le 1 \mid X+Y=5) = \left(\frac{6}{7}\right)^5 + 5 \cdot \frac{1}{7}\left(\frac{6}{7}\right)^4 \approx 0.85.$$

Comment on Ex3. We know $n=5$ Poisson events happened in time interval $[0,7]$. In such case, it is like 5 time moments "hit" time interval $[0,7]$ independently and without any "preferences": $A=[0,1]$ is hit with probability $\frac{|A|}{7} = \frac{1}{7}$.