<u>4.6-7</u> Normal bivariate r.v.

<u>Properties of normal bivariate (X,Y)</u> with $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho$.

1. $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, $\rho = \rho(X,Y)$.

4. $aX + bY$ is normal if $a \neq 0$ or $b \neq 0$.

<u>Why?</u> Since $Y = kX + V$, $aX + bY = (a+kb)X + bV$, and X, V are independent.

5. If A is 2×2 matrix with $\det A \neq 0$, then $(U, V) = (X, Y)A + (b_1, b_2)$ is normal bivariate for any $b_1, b_2 \in \mathbb{R}$.

<u>4.8</u> Multivariate normal

Le $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ and $B = (b_{ij})$ be $d\times d$ symmetric positive matrix: $b_{ij} = b_{ji}$ and

$$z B z' = \sum_{i,j=1}^{d} b_{ij} z_i z_j > 0 \text{ for any } z = (z_1, \dots, z_d) \neq 0.$$

<u>Note</u>: (i) $\det B > 0$; (ii) there is $d\times d$ matrix D so that $B = D'D$ ($\det B = (\det D)^2$

<u>Def</u>. $X = (X_1, \dots, X_d) \sim N(\mu, B)$ (X is <u>multivariate normal with parameters μ, B</u>) if joint pdf

$$f(x) = f(x_1, \dots, x_d) = \frac{1}{(2\pi)^{d/2}\sqrt{\det B}} \exp\left\{-\frac{1}{2}(x-\mu)B^{-1}(x-\mu)'\right\}, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Meaning of $\mu = (\mu_i)$, $B = (b_{ij})$:

$\mu_i = E(X_i)$, $b_{ij} = \mathrm{Cov}(X_i, X_j)$.

B is called covariance matrix of X.

<u>Remark1</u>. Normal bivariate (X_1, X_2) with parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho$ is $N(\mu, B)$ with $\mu = (\mu_1, \mu_2)$,

$$B = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \det B = \sigma_1^2\sigma_2^2(1-\rho^2) > 0.$$

<u>Thm</u>. Let $X \sim N(\mu, B)$.

a) If A is $d\times d$ matrix and $\det A \neq 0$, then $Y = XA + b \sim N(\tilde{\mu}, \tilde{B})$ with any $b = (b_1, \dots, b_d) \in \mathbb{R}^d$, where $\tilde{\mu} = E(Y) = \mu A + b$, $\tilde{B} = A' B A$.

b) If $B = D'D$, the $X = ZD + \mu$, where $Z = (Z_1, \dots, Z_d)$ is r. vector with independent $Z_i \sim N(0,1)$.

c) $a_1 X_1 + \dots + a_d X_d$ is normal for any $(a_1, \dots, a_d) \neq 0$.

<u>Why a)?</u> By definition,

$$f_X(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det B}} \exp\{-\tfrac{1}{2} Q_X(x)\}, \text{ where}$$

$$Q_X(x) = (x-\mu) B^{-1} (x-\mu)'.$$

For any function $h(Y)$,

$$E[h(Y)] = E[h(XA+b)] = \int_{\mathbb{R}^d} h(xA+b) f_X(x)\, dx =$$

changing variable of integration: $xA + b =: v$, $x = (v-b)A^{-1}$,
$dx = |\det A^{-1}|\, dv = \frac{1}{|\det A|} dv$

$$= \int_{\mathbb{R}^d} h(v) f_X((v-b)A^{-1}) \frac{1}{|\det A|} dv. \text{ Hence}$$

$$f_Y(v) = \frac{1}{|\det A|} f_X((v-b)A^{-1}) = \dots$$

b) Let $B = D'D$. By a)

$$Z = (X-\mu) D^{-1} \sim N(0, (D^{-1})' D' D D^{-1} = I) = N(0, I)$$

Hence $Z = (Z_1, \dots, Z_d)$, $Z_i \sim N(0,1)$ are independent,

$X - \mu = ZD$, $\underline{X = \mu + ZD}$.

c) Since $X = \mu + ZD$,

$a_1 X_1 + \dots + a_d X_d$ is linear combination of Z_i

c) Since $X = \mu + \Sigma D$,

$a_1 X_1 + \dots + a_d X_d$ is linear combination of Z_i which are indep. $N(0,1)$.

4.7 Functions of r.v's.

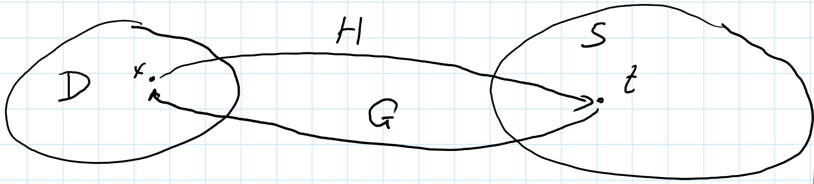
Assume $X = (X_1, \dots, X_n)$ is jointly continuous with joint pdf $f(x) = f(x_1, \dots, x_n)$, and range of $X = D \subset \mathbb{R}^d$. Let $T = (T_1, \dots, T_n) = H(X)$:

$$\begin{cases} T_1 = H_1(X) \\ \dots \\ T_n = H_n(X) \end{cases} \qquad (T_1, \dots, T_n) = (H_1(X), \dots, H_n(X))$$

Question. Is T jointly continuous? If so, find joint pdf of $(T_1, \dots, T_n)$.

Assume $t = H(x)$, $x \in D$, be one-to-one and continuously differentiable with the inverse $x = G(t)$: $G(H(x)) = x$, $x \in D$, $H(G(t)) = t$, $t \in S = \{t \in \mathbb{R}^d : G(t) \in D\}$.

$$\left.\begin{matrix} t_1 = H_1(x) \\ \vdots \\ t_n = H_n(x) \end{matrix}\right\} \quad t = H(x)$$



$$\left\{\begin{matrix} x_1 = G_1(t) \\ \dots \\ x_n = G_n(t) \end{matrix}\right. \quad x = G(t)$$

Thm. Under assumptions above, $T = H(X)$ is jointly continuous with joint pdf

$$f_T(t) = f(G(t))\, |J(t)|\, I_S(t), \text{ where } S = \{t \in \mathbb{R}^n : G(t) \in D\}$$

$$J(t) = \begin{vmatrix} \frac{\partial G_1}{\partial t_1} & \dots & \frac{\partial G_1}{\partial t_n} \\ \dots & & \dots \\ \frac{\partial G_n}{\partial t_1} & \dots & \frac{\partial G_n}{\partial t_n} \end{vmatrix}$$ is determinant of $n \times n$ matrix of partial derivatives.

Procedure: 1. We find the inverse $x = G(t)$ by solving for x the equation $t = H(x)$ with a given t.

2. Find $J(t)$

3. Write the answer: by Thm 1,

$$f_T(t) = f(G(t))\,|J(t)|\,1_S(t), \text{ where } S = \{t \in \mathbb{R}^d : G(t) \in D\}$$

equivalently,

$$f_T(t) = \begin{cases} f(G(t))\,|J(t)|, & t \in S = \{t : G(t) \in D\} \\ 0, & \text{otherwise} \end{cases}$$

Ex 1. Let X_1, X_2, X_3 be indep. exponential($\lambda = 1$)

a) Find joint pdf of $T_1 = X_1$, $T_2 = X_1 + X_2$, $T_3 = X_1 + X_2 + X_3$

Answer. Range of $X = (X_1, X_2, X_3) = D = \{(x_1, x_2, x_3) : x_1 > 0, x_2 > 0, x_3 > 0\}$

(i) We solve for x_1, x_2, x_3 the equations

$$\begin{cases} t_1 = x_1 \\ t_2 = x_1 + x_2 \\ t_3 = x_1 + x_2 + x_3 \end{cases} \quad \begin{vmatrix} x_1 = t_1 \\ x_2 = t_2 - t_1 \\ x_3 = t_3 - t_2 \end{vmatrix} \quad G(t) = G(t_1, t_2, t_3) = (t_1,\ t_2 - t_1,\ t_3 - t_2)$$

(ii) Computing $J(t) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$

(iii) Application of Thm 1:

joint pdf of X is $f(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} e^{-x_3} = e^{-(x_1 + x_2 + x_3)}$, $x_1 > 0$, $x_2 > 0$, $x_3 > 0$.

$$S = \left\{(t_1, t_2, t_3) : \begin{matrix} t_1 > 0 \\ t_2 - t_1 > 0 \\ t_3 - t_2 > 0 \end{matrix}\right\} = \{0 < t_1 < t_2 < t_3 < \infty\}$$

$$f_T(t) = e^{-t_3} \text{ if } \{0 < t_1 < t_2 < t_3 < \infty\}$$

b) Find $f_T(t_1, t_2 \mid t_3)$

Answer. $f_T(t_1, t_2 \mid t_3) = \dfrac{f_T(t_1, t_2, t_3)}{f_{T_3}(t_3)} =$

Answer: $f_T(t_1, t_2 \mid t_3) \quad f_{T_3}(t_3)$

$$= \frac{e^{-t_3}}{\frac{t_3^2}{2!} e^{-t_3}} = \frac{2!}{t_3^2} \quad \text{if} \quad 0 < t_1 < t_2 < t_3,$$

Comment Given $T_3 = t_3$, (T_1, T_2) is distributed as order statistic of two uniform in $(0, t_3)$.