1. It is reported that 50% of all computer chips produced are defective. Inspection ensures that only 5% of the legally marketed chips are defective. However, some chips are illegally marketed without any inspection. 2% of all available chips are illegally marketed.

a) If you buy a chip and find that it is defective, what is the probability it was illegally marketed? If you buy a chip and find that it is not defective, what is the probability it was illegally marketed?

Answer. Denoting $B=$ "chip is illegally marketed", $D=$ "chip is defective", by Bayes formula,

$$\begin{aligned}\mathbf{P}(B|D) &= \frac{\mathbf{P}(D|B)\,\mathbf{P}(B)}{\mathbf{P}(D)} = \frac{\mathbf{P}(D|B)\,\mathbf{P}(B)}{\mathbf{P}(D|B)\,\mathbf{P}(B)+\mathbf{P}(D|B^c)\,\mathbf{P}(B^c)} \\ &= \frac{0.5\cdot 0.02}{0.5\cdot 0.02+0.05\cdot 0.98} = \frac{10}{10+49} = \frac{10}{59} =: p_1.\end{aligned}$$

Similarly,

$$\mathbf{P}(B|D^c) = \frac{\mathbf{P}(D^c|B)\,\mathbf{P}(B)}{\mathbf{P}(D^c)} = \frac{0.5\cdot 0.02}{0.5\cdot 0.02+0.95\cdot 0.98} = \frac{100}{100+95\cdot 98} = \frac{10}{941} =: p_2$$

b) Suppose we buy n chips. Let X be the number of illegally marketed chips among those n, and V be the number of chips among those n that are defective. Find $\mathbf{E}(X|V=k), \mathbf{E}(X|V)$.

Answer. We have $X = X_1 + X_2$, where X_1 is the number of illegally marketed chips that are defective, X_2 is the number of illegally marketed chips that are not defective. Then

$$\mathbf{E}(X|V=k) = \mathbf{E}(X_1|V=k) + \mathbf{E}(X_2|V=k).$$

Given $V=k$, X_1 is binomial(k,p_1), and X_2 is binomial$(n-k,p_2)$. Hence

$$\mathbf{E}(X|V=k) = kp_1 + (n-k)\,p_2, \mathbf{E}(X|V) = Vp_1 + (n-V)\,p_2.$$

c) If you buy three chips and all three are defective, what is the probability that at least two of them are illegally marketed?

Answer. Given three defective, the number of illegal chips is binomial with $n=3, p=p_1$. Hence

$$\begin{aligned}\mathbf{P}(\text{at least two illegal}) &= 1-\mathbf{P}(\text{at most one illegal}) \\ &= 1-(1-p_1)^3 - 3p_1(1-p_1)^2.\end{aligned}$$

Comment. #1 of our mt, hw7, faulty robots, problems.

2. The number N of cars arriving at a drive-up window in a given day is Poisson with parameter λ. The numbers X_i of passengers in these cars are independent binomial with parameters $n = 4, p = 1/2$.

a) Find the moment generating function of the total number Y of passengers passing by the drive-up window in a given day.

Answer. Given $N = k$, $Y = \sum_{i=1}^{k} X_i$, that is $Y = \sum_{i=1}^{N} X_i$. According to theorem we know,

$$G_Y(s) = G_N(G_X(s)).$$

Since N is Poisson(λ), and X is binomial($n = 4, p = 1/2$), we have

$$\begin{aligned} G_N(s) &= e^{\lambda(s-1)}, G_X(s) = (ps+q)^4, s \in \mathbf{R}, p = \frac{1}{2}, q = \frac{1}{2}, \\ G_Y(s) &= G_N(G_X(s)) = \exp\left\{\lambda\left[(ps+q)^4 - 1\right]\right\}, s \in \mathbf{R}, \\ M_Y(t) &= G_Y\left(e^t\right) = \exp\left\{\lambda\left[\left(pe^t+q\right)^4 - 1\right]\right\}, t \in \mathbf{R}. \end{aligned}$$

b) Find $\mathbf{E}(Y|N=n), \mathbf{E}(Y|N), \mathrm{Var}(Y|N), \mathbf{E}(Y^2|N)$.

Answer. Given $N = n$, $Y = \sum_{i=1}^{n} X_i$ is binomial($4n, p = 1/2$). Hence,

$$\begin{aligned} \mathbf{E}(Y|N=n) &= 4np = 2n, \mathbf{E}(Y|N) = 2N, Var(Y|N) = 4Npq = N, \\ \mathbf{E}\left(Y^2|N\right) &= Var(Y|N) + (\mathbf{E}(Y|N))^2 = N + 4N^2. \end{aligned}$$

c) Suppose that the numbers of cars arriving at the drive-up window on different days are independent Poisson with the same parameter λ. Six cars arrived the last week at that window. What is the probability of at least two cars arriving the last day of that week.

Answer. Let U be the number of cars arriving the first six days, and V be the number of cars arriving the last day. Then $U \sim$ Poisson(6λ), $V \sim$ Poisson(λ) are independent. Given $U + V = 6$, V is binomial with $n = 6, p = \frac{\lambda}{7\lambda} = \frac{1}{7}$:

$$\begin{aligned} \mathbf{P}(V \geq 2|U+V=6) &= 1 - \mathbf{P}(V \leq 1|U+V=6) = 1 - \left(\frac{6}{7}\right)^6 - 6\left(\frac{1}{7}\right)\left(\frac{6}{7}\right)^5 \\ &= 1 - 2\left(\frac{6}{7}\right)^6. \end{aligned}$$

Comment. #3c) of our mt.

3. Suppose k balls are tossed into d boxes, with all d^k possibilities equally likely. Let D be the number of boxes that contain exactly two7 balls.

a) Compute probability that exactly two balls land in box 1.

Answer. Two balls that go to box 1 can be chosen in $\binom{k}{2}$ different ways, and the remaining $k-2$ equally likely into remaining $n-1$ equally likely into remaining $n-1$ box:

$$\mathbf{P}\left(\text{exactly two land into box 1}\right) = \frac{\binom{k}{2}\left(d-1\right)^{k-2}}{d^k} = \frac{k\left(k-1\right)\left(d-1\right)^{k-2}}{2!d^k} =: p_1.$$

Alternatively, number Y of balls that land into box 1 is binomial with $n = k, p = 1/n$:

$$\mathbf{P}\left(Y=2\right) = \binom{k}{2}\left(\frac{1}{d}\right)^2\left(1-\frac{1}{d}\right)^{k-2}.$$

b) Find $\mathbf{E}\left(D\right), \mathbf{E}\left(D^2\right)$ and $\mathrm{Var}(D)$.

Answer. Let $A_i =$ "exactly two balls land into ith box", $i = 1, \ldots, d$. Then for $i \neq j$,

$$\begin{aligned}
\mathbf{P}\left(A_i\right) &= p_1, \\
\mathbf{P}\left(A_iA_j\right) &= \frac{\binom{k}{2}\binom{k-2}{2}\left(d-2\right)^{k-4}}{d^k} \\
&= \frac{k\left(k-1\right)\left(k-2\right)\left(k-3\right)}{2!2!}\left(\frac{1}{d}\right)^4\left(1-\frac{2}{d}\right)^{k-4} =: p_2
\end{aligned}$$

Now,

$$\begin{aligned}
D &= \sum_{i=1}^{d} I_{A_i}, \mathbf{E}\left(D\right) = \sum_{i=1}^{d} p_1 = dp_1, \\
\mathbf{E}\left(D^2\right) &= \sum_{i=1}^{d}\mathbf{E}\left(I_{A_i}\right) + 2\sum_{i<j}\mathbf{E}\left(I_{A_iA_j}\right) = \sum_{i=1}^{d}\mathbf{P}\left(A_i\right) + 2\sum_{i<j}\mathbf{P}\left(A_iA_j\right) = dp_1 + 2\binom{d}{2}p_2 \\
&= dp_1 + d\left(d-1\right)p_2,
\end{aligned}$$

and

$$\mathrm{Var}\left(D\right) = \mathbf{E}\left(D^2\right) - \left(\mathbf{E}\left(D\right)\right)^2 = dp_1 + d\left(d-1\right)p_2 - d^2p_1^2.$$

c) Let $k \leq d$. Find probability that no box receives more than one ball.

Answer. As before, there are d^k of equally likely possibilities. Then each of d boxes could contain at most one ball: there are $\binom{d}{k}k!$ ways for that:

there are $\binom{d}{k}$ distinct ways to choose k boxes that will contain one ball, and $k!$ different ways to arrange k balls in those k boxes. So,

$$\begin{aligned}
& \mathbf{P}\,(\text{no box receives more than one}) \\
= & \frac{\binom{d}{k}k!}{d^k} = \frac{d\,(d-1)\ldots(d-k+1)}{d^k} = \left(1-\frac{1}{d}\right)\left(1-\frac{2}{d}\right)\ldots\left(1-\frac{k-1}{d}\right).
\end{aligned}$$

4. Suppose X and Y are independent uniform in $(0,1)$.

a) Find the joint probability density function of $X-Y, X+Y$.

Answer. Consider $U = X-Y, V = X+Y$. We look at (U,V) as a function of (X,Y). Let $f(x)$ be the pdf of uniform in $(0,1)$. First, the joint pdf of (X,Y) is

$$f(x,y) = f(x)\,f(y) = 1, (x,y) \in D = \{(x,y) : 1 > x > 0, 1 > y > 0\},$$

and (U,V) is the function of (X,Y) defined by

$$u = x - y, v = x + y. \tag{1}$$

We find its inverse by solving (1) for x, y :

$$x = (u+v)/2, y = (v-u)/2, \tag{2}$$

Note that according to (2), $D = \{(x,y) : 1 > x > 0, 1 > y > 0\}$ is mapped onto

$$\begin{aligned}
S &= \{(u,v) : 1 > x = (u+v)/2 > 0, 1 > y = (v-u)/2 > 0\} \\
&= \{(u,v) : 2 > u+v > 0, 2 > v-u > 0\} = \{(u,v) : 2-u > v > -u, 2+u > v > u\}
\end{aligned}$$

The Jacobian of the inverse

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

By theorem in the class note of 11/9 (see Theorem in 4.7 of textbook), the joint pdf of (U,V) is

$$\begin{aligned}
g(u,v) &= f\left(\frac{u+v}{2}, \frac{v-u}{2}\right) |J(u,v)|\, I_S(u,v) = \frac{1}{2} I_{S(u,v)} \\
&= \frac{1}{2}, 2-u > v > -u, 2+u > v > u.
\end{aligned}$$

b) Compute $\mathrm{Cov}(X-Y, X+Y)$.

Answer. By the properties of the covariance, using independence,

$$\begin{aligned} Cov\left(X-Y,X+Y\right) &= Cov\left(X,X\right)-Cov\left(Y,Y\right) \\ &= \mathrm{Var}\left(X\right)-\mathrm{Var}\left(Y\right)=0, \end{aligned}$$

because X, Y are identically distributed.

c) Find the probability density function of $X+Y$.

Answer. There are various ways to find it. One of them is geometric by draw a square and computing areas. The other one could be to use joint pdf found in a):

$$\begin{aligned} f_V\left(v\right) &= \int_{-\infty}^{\infty} g\left(u,v\right)du = \int_{-v}^{v}\frac{1}{2}du = v, 0<v<1, \\ f_V\left(v\right) &= \int_{-\infty}^{\infty} g\left(u,v\right)du = \int_{v-2}^{2-v}\frac{1}{2}du = 2-v, 1<v<2. \end{aligned}$$

d) Find $\mathbf{E}\left(\frac{X}{X+Y}\right)$ and $\mathbf{E}\left(\frac{X-Y}{X+Y}\right)$.

Answer. Since $\frac{X}{X+Y}$ and $\frac{Y}{X+Y}$ are identically distributed (see #5 of hw9), by symmetry,

$$\begin{aligned} \mathbf{E}\left(\frac{X}{X+Y}\right) &= \mathbf{E}\left(\frac{Y}{X+Y}\right)=\frac{1}{2}, \\ \mathbf{E}\left(\frac{X-Y}{X+Y}\right) &= \mathbf{E}\left(\frac{X}{X+Y}\right)-\mathbf{E}\left(\frac{Y}{X+Y}\right)=\frac{1}{2}-\frac{1}{2}=0. \end{aligned}$$

5. Let X and Y be independent standard normal. Let $W=X+Y$ and $V=X-Y$.

a) Are W and V independent? Write the probability density function of V.

Answer. First (X,Y) is bivariate standard normal. Since

$$(W,V)=(X,Y)\,A=(X,Y)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \det\left(A\right)=-2\neq 0,$$

(W,V) is normal bivariate. By properties of covariance, using independence of X,Y,

$$\begin{aligned} \mathrm{Cov}\left(W,V\right) &= \mathrm{Cov}\left(X+Y,X-Y\right)=\mathrm{Cov}\left(X,X\right)-\mathrm{Cov}\left(Y,Y\right) \\ &= \mathrm{Var}\left(X\right)-\mathrm{Var}\left(Y\right)=1-1=0. \end{aligned}$$

Hence W, V are independent. As a linear combination of independent standard normal, $V \sim N(0,2)$, because $\mathbf{E}(V) = 0$,Var$(V) = 2$.

b) Find $\mathbf{E}(X + 3Y|V)$,Var$(X + 3Y|V)$. What is the distribution of $X + 3Y$ given $V = v$? What is the best mean square estimate of $X + 3Y$ given $V = 1$?

Answer. First,

$$(X + 3Y, V) = (X + 3Y, X - Y) = (X, Y)\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} = (X, Y) D,$$

with $\det D = -4 \neq 0$. Hence $(X + 3Y, V) = (X + 3Y, X - Y)$ is normal bivariate. We find its parameters:

$$\begin{aligned}
\mu_1 &= \mu_2 = \mathbf{E}(X + 3Y) = \mathbf{E}(X - Y) = 0, \\
\sigma_1^2 &= \mathrm{Var}(X + 3Y) = \mathrm{Var}(X) + 3^2\mathrm{Var}(Y) = 10, \\
\sigma_2^2 &= \mathrm{Var}(X - Y) = 2, \\
\mathrm{Cov}(X + 3Y, X - Y) &= \mathrm{Cov}(X, X) - 3\mathrm{Cov}(Y, Y) = 1 - 3 = -2, \\
\rho &= \frac{\mathrm{Cov}(X + 3Y, X - Y)}{\sigma_1\sigma_2} = -\frac{2}{\sqrt{20}} = -\frac{1}{\sqrt{5}}
\end{aligned}$$

By theorem that we know, given $V = X - Y = v$, $X + 3Y$ is $N\left(\rho\frac{\sigma_1}{\sigma_2}V, \sigma_1^2(1 - \rho^2)\right) = N\left(-V, 10\left(1 - \frac{1}{5}\right) = 8\right)$. Hence

$$\mathbf{E}(X + 3Y|V) = \rho\frac{\sigma_1}{\sigma_2}V = -V, \mathrm{Var}(X + 3Y|V) = 8.$$

The best mean square estimate of $X + 3Y$ given $V = 1$ is -1.

6. a) Let X_k be independent Poisson with parameter $\lambda = 1, T_n = \sum_{k=1}^{n} X_k$. Find

$$\lim_{n\to\infty} \mathbf{P}(0 \leq T_n \leq n).$$

Answer. All X_k are i.i.d. with $\lambda = \mathbf{E}(X)$ =Var$(X) = 1$: $\mathbf{E}(T_n) = n$ =Var$(T_n) = n$. By CLT, for large n, $T_n = \sum_{k=1}^{n} X_k$ is approximately $N(n, n)$. Hence

$$\mathbf{P}(0 \leq T_n \leq n) = \mathbf{P}(T_n \leq n) = \mathbf{P}\left(\frac{T_n - n}{\sqrt{n}} \leq 0\right) \to \mathbf{P}(Z \leq 0) = \frac{1}{2} \text{ as } n \to \infty,$$

where $Z \sim N(0,1)$.

b) Let X_n be binomial$(n, p = \lambda/n)$ with $\lambda > 0$, and let X be Poisson with parameter λ. Show that X_n converges to X in distribution as $n \to \infty$. Recall $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$.

Answer. Since X is Poisson(λ), and X_n is binomial$(n, p = \lambda/n)$,

$$
\begin{aligned}
\Phi_X(t) &= \exp\left\{\lambda\left(e^{it} - 1\right)\right\}, t \in \mathbf{R}, \\
\Phi_{X_n}(t) &= \left(pe^{it} + q\right)^n = \left(\frac{\lambda}{n}e^{it} + 1 - \frac{\lambda}{n}\right)^n = \left(1 + \frac{\lambda\left(e^{it} - 1\right)}{n}\right)^n \\
&\to \exp\left\{\lambda\left(e^{it} - 1\right)\right\} = \Phi_X(t) \text{ for all } t.
\end{aligned}
$$

Hence X_n converges to X in distribution by continuity theorem.