Math 505A Homework 8 Neel Gupta

1) Consider S: a simple r.w. over $\{0,1,\dots,N\}$ which goes right w/ prob. p and left w/ $q := 1-p$. Absorbing barriers are at 0 & N. Let $S_0 = k$, $k \in \{0, N\}$. Then show X, an r.v. of the # of positive steps before absorbtion has expectation,

$$E(X) = \frac{1}{2}[D_k - k + N(1-p_k)].$$

where D_k = "mean # of steps until absorption" &
p_k = "probability of absorbing @ 0"

Let Y = "number of negative steps til absorption"
Z_k = "number of steps til absorption"

Ans num pos steps + num neg steps = total steps

D_k & Z_k, which means that Z_k's expectation is $D_k \to E(Z_k) = D_k$.

Z_k is total steps, so $X + Y = Z_k$.

Since $E[Z_k] = D_k$, so

$E(X+Y) = D_k$.

The number of pos steps - the number neg steps gives us a system of how many steps we took given we know which barrier absorbed.

if absorbed by 0 then $-k$ steps.

which means $X - Y = -k$ @ 0.

& $X - Y = N - k$ @ N because $N-k$ positive steps to reach N.

$$\Rightarrow X - Y = \begin{cases} -k & \text{if absorbed @ } 0 \\ N-k & \text{if absorbed @ } N. \end{cases}$$

1) $X - Y = \begin{cases} -k & @ \, 0 \\ N \cdot k & @ \, N \end{cases}$

$P(\text{Walk ends at } 0) = P_k$

$P(\text{Walk ends at } N) = 1 - P_k$, so

$E(X-Y) = -k \cdot P_k + (N-k)(1-P_k)$

$E(X+Y) = D_k$, so

$E(X+Y) + E(X-Y) = E(2X) =$

$\Rightarrow D_k \cancel{-k \cdot P_k} + N - k - NP_k \cancel{+ kP_k}$

$E(2X) = D_k - k + N - NP_k$

$\rightarrow E(X) = \frac{1}{2}[D_k - k + N(1-P_k)$ ▨

Q.E.D.

Problem 2 Consider Simple Symmetric r.w. S_n with $S_0 = 0$, $p = q = \frac{1}{2}$. Let $T_0 = \min\{n \geq 1, S_n = 0\}$ or "the first return to start point."

Show that

$P(T_0 = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n} = \frac{1}{2n-1}P(S_{2n} = 0)$.

Ans First by conditioning on the first step

$P(T_0 = 2n) = p\,P(T = 2n \mid X_1 = 1) + q\,P(T = 2n \mid X_1 = -1)$

~~By Hitting time theorum,~~

$P(T_0 = 2n) = \frac{1}{2}P(T = 2n \mid X_1 = 1) + \frac{1}{2}P(T = 2n \mid X_1 = -1)$

which takes $2n-1$ steps to either hit -1 or 1 which means you returned to 0 after going up 1 then going down 1 in $2n-1$ steps returns us.

By Hitting Thm (3.10.14)

$P(T=2n) = \frac{1}{2} f_{-1}(2n-1) + \frac{1}{2} f_1(2n-1)$

the terms on the right hand side are equally distributed even if $p \neq \frac{1}{2}$, so the time to hit -1 or 1 in $2n-1$ steps is the same probability w/ equal distributions

$\rightarrow f_{-1}(2n-1) = f_1(2n-1) = \frac{|-1|}{2n-1} P(S_{2n-1} = 1)$

$\rightarrow f_{-1}^{(2n-1)} = \frac{1}{2n-1} P(S_{2n-1} = -1) = f_1^{(2n-1)} = \frac{1}{2n-1} P(S_{2n-1} = 1)$

Then $P(2n-1=1)$ is the chance to get 1 more positive result than neg result, so $p=\frac{1}{2}$ for everything, so chance to choose n heads and $n-1$ tails is indep., so By mult. prin.,

$\frac{1}{2n-1} P(S_{2n-1}=1) = \frac{1}{2n-1} \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n-1}$

chance to choose n ~~heads~~ $+1$'s,

so then $P(T_0 = 2n-1) = \frac{1}{2n-1} \binom{2n-1}{n} 2^{-(2n-1)}$ ▨

which by time homogeneity of r.w. can be shifted to show

$P(\tau_0 = 2n) = \left(\frac{1}{2n-1}\right) \binom{2n}{n} 2^{-2n}$

To then think of ending when equal number of ~~heads~~ $(+1)$ have been drawn to balance out the r.w. to end at 0.

Since the random walk is finite because the probability of ending in $2n$ steps is 1 since $\sum_{n=1}^{\infty} P(T_0 = 2n) = 1$

because of the conditioning on the firststep which means the random walk ~~must~~ that starts with 1 can make it 0 in $2n-1$ steps and vice versa, so

$$P(\tau_0 = 2n) = \frac{1}{2n-1}\binom{2n}{n} 2^{-2n}$$

implies that the probability of choosing equal +1's and -1's is the same as the probability of ending on 0 in $2n$ steps which implies that ~~P(T_0=2n)~~

$$P(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$$

Because $P(T_0 < \infty) = P(\bigcup_{n=1}^{\infty} \{T_0 \leq 2n\})$,

so $P(\tau_0 = 2n) = \frac{1}{2n-1} P(S_{2n} = 0)$

Q.E.D

3) Let S_n be Symmetric simple r.w. $(p=q=\frac{1}{2})$, and $S_0 = 0$ s.t. $S_n = X_1 + \dots + X_n$, $n \geq 1$ and all X_i's are identical and independent, $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$.

3) a) $S_n = X_1 + \dots + X_n, n \geq 1$

Since all X_i's are independent and identical, all X_is can be replaced by $-X_i$'s $\therefore$ identical independence, So S_n and $-S_n$ are equally distributed, so $\underline{S_n} = -S_n, n \geq 0$ which is also a symmetric random walk.

Because $X_i = -X_i$ ~~each~~ and all $-X_i$'s are independent, $S_n = -S_n$ which $= -X_1 + -X_2 + \dots$

▨ Q.E.D

b) For $b \neq 0$, set $\tau_b = \tau_b(S) = \min\{n>0; S_n = b\}$

Show that: $P(\tau_b < \tau_{-b}) = P(\tau_{-b} < \tau_b) = \frac{1}{2}$.

Since $\{S_n, n \geq 0\}$ and $\{-S_n, n \geq 0\}$ are identically distributed, $\forall a \neq 0$,

$P(\tau_a < \infty) = 1$ which means there are finite amount of n's in the r.w. ending @ a.

$\longrightarrow$ for any $b \neq 0$, τ_b is reachable w/ $P=1$.

$P(\tau_b < \tau_{-b}) = P(\min\{S_n = b\} < \min\{S_n = -b\})$

$P(\min\{S_n = b, n>0\} < \min\{S_n = -b, n>0\}) =$

$P(\min\{S_n = -b, n>0\} < \min\{S_n = b, n>0\})$.

Since S_n & $-S_n$ are equally distributed by (a).

$P(\min\{S_n = b\} < \min\{S_n = -b\}) = P(\min\{S_n = b\} < \min\{-S_n = b\})$.

which then implies that

$P(\tau_b(S) < \tau_{-b}(S)) = P(\tau_b(-S) < \tau_{-b}(-S))$.

3) b) Since $P(\tau_b(S) < \tau_{-b}(S)) = P(\tau_b(-S) < \tau_{-b}(-S))$

Then $P(\tau_b = \tau_{-b}) = 1$ then

$P(\tau_b = -\tau_b) = 0$, so

$P(\{\tau_b < \tau_{-b}\} \cup \{\tau_{-b} < \tau_b\} \vee \{\tau_b = -\tau_b\})$

$= 1$, so which implies that

$P(\tau_b < \tau_{-b}) + P(\tau_{-b} < \tau_b) = 1$, so

$P(\tau_b < \tau_{-b}) = P(\tau_{-b} < \tau_b) = \frac{1}{2}$

Q.E.D.

c) Let $\sigma_k = \min\{n > 0 : S_n \notin (-k, k)\}$.

Find $E(S_{\sigma_k})$ and $Var(S_{\sigma_k})$.

Since S_n can't be in between k and $-k$

then $\sigma_k = \min\{n > 0 : S_n = k \text{ or } S_n = -k\}$

$= \min\{\min\{S_n = k\} \text{ or } \min\{S_n = -k\}\}$

moment at τ_k moment at τ_{-k}

$= \min\{\tau_k, \tau_{-k}\}$.

Since σ_k is defined as $\min\{\tau_k, \tau_{-k}\}$,

$P(\tau_k < \infty) = P(\tau_{-k} < \infty) = 1$

which means τ_k and τ_{-k} are finite

Then τ_k and τ_{-k} are r.v.'s

then $E(S_{\sigma_k}) = E(X_1) E(\sigma_k)$

By Wald's equation and

$Var(S_{\sigma_k}) = Var(X_1) E(\sigma_k) + E^2(X_1) Var(\sigma_k)$

and $E(X_1) = 0$ and $Var(X_1) = E(X_1^2) = 1$ by r.w.

So $E(S_{\sigma_k}) = 0$ and $Var(S_{\sigma_k}) = \cancel{E(\tau_k)}\, E(\sigma_k)$

Since τ_k and τ_{-k} are identical

$= E(\tau_k) \cdot P(\tau_k < \tau_{-k}) + E(\tau_{-k}) P(\tau_{-k} < \tau_k)$

$= \frac{1}{2} E(\tau_k) + \frac{1}{2} E(\tau_{-k}) = \boxed{E(\tau_k)}$

Problem 4

Let $S_0 = a > 0$, $p = q = \frac{1}{2}$, Let $\tau_0 = \min\{k > 0, S_k = 0\}$

for $a \geq 1$, $j \geq 1$, $n \geq 1$,

express $P(S_n = j, \tau_0 \leq n \mid S_0 = a)$ and

$P(S_n = j, \tau_0 < n \mid S_0 = a)$.

in basic probabilities.

$P(S_n = j, \tau_0 \leq n \mid S_0 = a)$ is prob. to go from a to j in n steps via 0.

Since $p = \frac{1}{2}$, $P(S_n = j, T_0 \leq n \mid S_0 = a) =$ (# of paths that satisfy $a \to j$ in n steps via 0) $\cdot$ (probability of selecting specific path in n step) $\cdot$ or $(\frac{1}{2})^n$.

By reflection principle of r.w.,

number of steps to go from a to j in n step via 0 = number of steps to go from $-a$ to j in steps via 0.

$P(S_n = j, \tau_0 \leq n \mid S_0 = a) = P(S_n = j \mid S_0 = -a)$

By space homogeneity,

$P(S_n = j \mid S_0 = -a) = \boxed{P(S_n = j + a \mid S_0 = 0)}$

$\to P(S_n = j \mid S_0 = a) = P(S_n = j + a \mid S_0 = 0)$

~~Then~~ let $P(S_n = j \mid T_0 > n) = P(S_n = j \mid S_0 = a) - P(S_n = j \,\&\, \tau_0 \leq n \mid S_0 = a)$.

$\to P(S_n = j, T_0 > n) = \boxed{P(S_n = j - a \mid S_0 = 0) - P(S_n = j + a \mid S_0 = 0)}$

Problem 4

b) Show that $P(\tau_0 > n \mid S_0 = a) = \sum_{j=1-a}^{a} P(S_n = j \mid S_0 = 0)$

By part (a)

$P(S_n = j, T_0 > n \mid S_0 = a) = P(S_n = j-a \mid S_0 = 0) - P(S_n = j+a \mid$

$\to P(\tau_0 > n \mid S_0 = a) = \sum_{j \geq 1} P(S_n = j-a \mid S_0 = 0) - P(S_n = j+a \mid S_0 = 0)$

$\to$ Given $S_0 = 0$ for all terms $= \sum_{j \geq 1} P(S_n = j-a) - P(S_n = j+a) + \dots$

which is a telescoping sum from $j = 1-a$ to a because ~~j varies between~~

~~$j-a \leq j \leq j+a$, so~~

~~$P(\tau_0 > n \mid S_0 = a) = \sum$~~

the values S_n takes on is between $j-a$ to $j+a$, so the sum goes from $1-a$ to a, so j can vary from 1 to a, so

$$P(\tau_0 > n \mid S_0 = a) = \sum_{j=1-a}^{a} P(S_n = j \mid S_0 = 0)$$

Problem 5 A recieves n votes
B recieves m votes s.t. $n > m$

Assuming all $\binom{m+n}{m} = \binom{m+n}{\frac{m+n+(n-m)+0}{2}}$

orderings of voters is equally likely.
Show that P(A is always ahead)

$= \frac{n-m}{n+m}$.

$m+n$ total votes were cast. Let H_k = # of A votes
Let T_k = # of B votes after k people voted.
$S_k = H_k - T_k$, $k \geq 1$, $S_0 = 0$.
The number of ways candidate A can be in lead after first vote is 1. A must get first vote.
The # of ways where A is ahead the entire time with getting the first vote vs. # of ways A gets 1st vote but $m=n$ or $n>m$ at any time k before N.
Because of this we can compare going from 1 A vote to getting $m+n$ votes over $m-n$ votes.
By reflection we can also start when the first vote is for B, so S_k when $k=1 = -1$.
Then starting from 0 there are $\binom{m+n-1}{m-1} - \binom{m+n-1}{m}$ paths that have 1 ~~m votes~~ more A votes than B.

$$\#\text{paths} = \binom{m+n-1}{m-1} - \binom{m+n-1}{m}$$

$$= \frac{(m+n-1)!}{(m-1)!(n!)} - \frac{(m+n-1)!}{m!(n-1)!}$$

$$= \frac{(m-n)(n+m-1)!}{m!n!} = \left(\frac{m-n}{m+n}\right)\binom{m+n}{m}$$

5) ~~Prob Ada~~ P(A always in lead) =

$$\frac{(\#\text{ ways A is ahead})}{(\text{total ways})} = \frac{\frac{m-n}{m+n}\binom{m+n}{m}}{\binom{m+n}{m}}$$

$$\rightarrow P(\text{A always ahead}) = \boxed{\frac{m-n}{m+n}}$$

Q.E.D.