5.1 Generating and moment generating functions (gf and mgf)

Assume X takes values in $\{0,1,2,\dots\}$, $p_n = P(X=n)$, $n \geq 0$

__Def.__ gf of X is the function

(1) $G(s) = G_X(s) = \sum_{n=0}^{\infty} p_n s^n = p_0 + p_1 s + \dots$

__Note__ a) $G(0) = p_0 = P(X=0)$, $G(1) = \sum_{n=0}^{\infty} P(X=n) = 1$ ∴ (1) has convergence radius $R \geq 1$.

b) $G(s) = \sum_{n=0}^{\infty} s^n P(X=n) = E(s^X)$, $-R < s < R$,

__Properties of G_X__

1. p_n are Taylor coefficients: $p_n = \frac{G^{(n)}(0)}{n!}$, $G^{(n)} = \frac{d^n}{ds^n} G$
2. If $G_X(s) = G_Y(s)$, $-\varepsilon < s < \varepsilon$ for some $\varepsilon > 0$, then X, Y are identically distributed.
3. If X, Y are __independent__, then $G_{X+Y}(s) = G_X(s)\, G_Y(s)$

__Ex 1.__ Find $G_X(s) = E(s^X)$.

a) $X \sim \text{Bernoulli}(p)$: $G(s) = q + ps$, $-\infty < s < \infty$

b) $X \sim \text{Binomial}(n,p)$: $G(s) = (q+ps)^n$ ($X = X_1 + \dots + X_n$ ← indep. Bernoulli(p))

c) $X \sim \text{geometric}(p)$: $G(s) = \frac{ps}{1-qs}$, $-\frac{1}{q} < s < \frac{1}{q}$, because

$P(X=k) = q^{k-1} p$, $k = 1, 2, \dots$, and

$E(s^X) = \sum_{k=1}^{\infty} s^k q^{k-1} p = \frac{p}{q} \sum_{k=1}^{\infty} s^k q^k = \frac{p}{q} \frac{sq}{1-sq}$ if $|sq| < 1$.

$$E(s^X) = \sum_{k=1} s^k q^{k-1} p = \frac{p}{q} \sum_{k=1} s^k q^k = \frac{p}{q} \frac{sq}{1-sq}$$

d) $X \sim \text{Poisson}(\lambda)$: $G(s) = e^{\lambda(s-1)}$, $-\infty < s < \infty$, because

$$E(s^X) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda s} e^{-\lambda}$$

Ex2. Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be indep. Use gf to show that $X+Y \sim \text{Poisson}(\lambda+\mu)$.

Answer. $G_{X+Y}(s) = G_X(s)\, G_Y(s) = e^{\lambda(s-1)} e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)}$

Properties of G_X

4. $G'(1) = E(X)$

$G''(1) = E[X(X-1)]$

$G^{(k)}(1) = E[X(X-1)\ldots(X-k+1)]$

$$G'(s) = \sum_{n=1}^{\infty} n s^{n-1} p_n$$
$$G'(1) = \sum_{n=1}^{\infty} n p_n = E(X).$$

Mgfs

Def. mgf of X is the function

$$M(t) = M_X(t) = E(e^{tX}), \quad -\varepsilon < t < \varepsilon, \text{ for some } \varepsilon > 0$$

provided $E(e^{tX})$ exists.

Note 1. If X takes values in $\{0, 1, 2, \ldots\}$, then

$$M_X(t) = E(e^{tX}) = E[(e^t)^X] = G_X(e^t)$$

Examples: 1. $X \sim \text{Binomial}(n,p)$: $M_X(t) = G_X(e^t) = (q+pe^t)^n$

2. $X \sim \text{geometric}(p)$: $M_X(t) = \dfrac{pe^t}{1-qe^t}$, $\quad t < -\ln q$

3. $X \sim \text{Poisson}(\lambda)$: $M_X(t) = e^{\lambda(e^t-1)}$, $-\infty < t < \infty$.

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$\boxed{\text{Thm 1. If } M_X(t) \text{ exists, then } E(X^n) = M_X^{(n)}(0).}$

Why? $e^{tX} = \sum_{n=0}^{\infty} \frac{X^n}{n!} t^n$,

$M(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n$: $\frac{M^{(n)}(0)}{n!} = \frac{E(X^n)}{n!}$

Ex 3. Let $Z \sim N(0,1)$. Confirm that $M_Z(t) = e^{\frac{t^2}{2}}$ and find $E(Z^n)$ for all n.

Answer. $M(t) = E(e^{tZ}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} dx =$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(x^2 - 2tx + t^2) + \frac{1}{2}t^2\} dx = e^{\frac{t^2}{2}} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx$

$= e^{t^2/2}$. The moments:

$M_Z(t) = e^{\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}$ | $\frac{M^{(2n)}(0)}{(2n)!} = \frac{1}{2^n n!}$ implies

$E(Z^{2n}) = \frac{(2n)!}{n! \, 2^n}$ $\quad (E(Z^4) = 3)$.

$E(Z^{2n+1}) = 0$.

Joint gf's and mgf's

Def. a) For X, Y with values in $\{0, 1, 2, \dots\}$, their joint gf is

$G_{X,Y}(s,t) = E(s^X t^Y) = \sum_{k,j=0}^{\infty} p_{kj} s^k t^j$, where

$p_{kj} = P(X=k, Y=j)$.

Note: $G_{X,Y}(s, 1) = G_X(s)$, $G_{X,Y}(1,t) = G_Y(t)$

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b) joint mgf of X, Y is the function

$M_{X,Y}(s,t) = E[e^{sX+tY}]$ for $-\varepsilon < s, t < \varepsilon$

provided it exists. Note $M_{X,Y}(s,0) = M_X(s)$, $M_{X,Y}(0,t) = M_Y(t)$.

Thm 2. a) X, Y are independent if and only if

$G_{X,Y}(s,t) = G_X(s)\, G_Y(t)$

b) X, Y are indep. if and only if $M_{X,Y}(s,t) = M_X(s)\, M_Y(t)$