1. A coin-making machine produces quarters in such a way that for each coin the probability U to turn up heads assumes equally likely one of nine values $\{p_1, \ldots, p_9\}$:

$$\mathbf{P}(U = p_i) = \frac{1}{9}, i = 1, \ldots, 9,$$

and $p_i = i/10, i = 1, \ldots, 9$. A coin pops out of the machine.

Compute the conditional pmf of U given that the coin is tossed once and turns up heads. Write an expression for the conditional expectation of U.

Answer. Let H = "the coin turns up heads". For $i = 1, \ldots, 9$, by Bayes,

$$\begin{aligned}\mathbf{P}(U = p_i|H) &= \frac{\mathbf{P}(U = p_i, H)}{\mathbf{P}(H)} = \frac{\mathbf{P}(H|U = p_i)\mathbf{P}(U = p_i)}{\sum_{j=1}^{9}\mathbf{P}\left(H|U = p_j\right)\mathbf{P}\left(U = p_j\right)} \\ &= \frac{p_i \cdot \frac{1}{9}}{\sum_{j=1}^{9} p_j \cdot \frac{1}{9}} = \frac{p_i}{\sum_{j=1}^{9} p_j} = \frac{\frac{i}{10}}{\sum_{j=1}^{9}\frac{j}{10}} = \frac{i}{\sum_{j=1}^{9} j} = \frac{i}{\frac{9\cdot 10}{2}} = \frac{i}{45}.\end{aligned}$$

Then, by definition,

$$\mathbf{E}(U|H) = \sum_{i=1}^{9} p_i \mathbf{P}(U = p_i|H) = \sum_{i=1}^{9}\frac{i}{45}\cdot\frac{i}{10} = \sum_{i=1}^{9}\frac{i^2}{450}.$$

Comment. The heads probability of the coin is unknown but the best mean square estimate of that probability (based on the single toss result) is the number $\mathbf{E}(U|H) = \sum_{i=1}^{9}\frac{i^2}{450} = \frac{19}{30}$.

2. Every package of some intrinsically dull commodity includes a small and exciting plastic object. There are 5 different types of object, and each package is equally likely to contain any given type independently of other packages. You bought n packages, $n > 5$.

(a) Find the probability that no one package has type 1 object; Find the probability no one package has type 1 or type 2 object.

Answer. First,

$$\begin{aligned}\mathbf{P}(\text{a package does not include type 1}) &= 1 - \frac{1}{5} = \frac{4}{5}, \\ \mathbf{P}(\text{a package does not include type 1 or type 2}) &= 1 - \frac{2}{5} = \frac{3}{5}.\end{aligned}$$

Using independence,

$$\begin{aligned}\mathbf{P}(\text{no one package has type 1}) &= \left(\frac{4}{5}\right)^n = \frac{4^n}{5^n} \\ \mathbf{P}(\text{no one package has type 1 or type 2}) &= \left(\frac{3}{5}\right)^n = \frac{3^n}{5^n}.\end{aligned}$$

(b) Let X be the number of different types of object found in n packages. For instance $X = 1$ if all n packages contained an object of the same type.

Find $\mathbf{E}(X)$ and $\mathrm{Var}(X)$.

Answer (our elevator problem, #3 of hw6: "packages"="people", "types" = "building floors"). Let A_i = "at least one of n packages has type i", $i = 1, 2, 3, 4, 5$. Then $X = \sum_{i=1}^{5} I_{A_i}$,

$$\mathbf{E}(X) = \sum_{i=1}^{5} \mathbf{P}(A_i)$$

and, using the variance covariance expansion,

$$\mathrm{Var}(X) = \sum_{i=1}^{5} \mathrm{Var}\left(I_{A_i}\right) + 2\sum_{i<j} \mathrm{Cov}\left(I_{A_i}, I_{A_j}\right).$$

Now, recall I_{A_i} is Bernoulli, and $\mathrm{Var}\big(I_{A_i}\big) = \mathbf{P}(A_i)\left[1 - \mathbf{P}(A_i)\right]$. Also,

$$\mathrm{Cov}(I_{A_i}, I_{A_j}) = \mathbf{P}(A_i \cap A_j) - \mathbf{P}(A_i)\,\mathbf{P}\left(A_j\right).$$

Note, by part (a),

$$\mathbf{P}(A_i) = 1 - \mathbf{P}\left(A_i^c\right) = 1 - \frac{4^n}{5^n}.$$

For $i \neq j$, by inclusion-exclusion and part (a),

$$\begin{aligned}
\mathbf{P}\left(A_i \cap A_j\right) &= 1 - \mathbf{P}\left(A_i^c \text{ or } A_j^c\right) = 1 - \mathbf{P}(A_i^c) - \mathbf{P}(A_j^c) + \mathbf{P}(A_i^c \cap A_j^c) \\
&= 1 - 2 \cdot \frac{4^n}{5^n} + \frac{3^n}{5^n}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{Var}\left(I_{A_i}\right) &= \frac{4^n}{5^n}\left(1 - \frac{4^n}{5^n}\right), \\
\mathrm{Cov}(I_{A_i}, I_{A_j}) &= 1 - 2 \cdot \frac{4^n}{5^n} + \frac{3^n}{5^n} - \left(1 - \frac{4^n}{5^n}\right)^2 \\
&= \frac{3^n}{5^n} - \left(\frac{4^n}{5^n}\right)^2 = \frac{3^n}{5^n} - \frac{4^{2n}}{5^{2n}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}(X) &= 5\left(1 - \frac{4^n}{5^n}\right), \\
\mathrm{Var}(X) &= 5 \cdot \frac{4^n}{5^n}\left(1 - \frac{4^n}{5^n}\right) + 20\left(\frac{3^n}{5^n} - \frac{4^{2n}}{5^{2n}}\right).
\end{aligned}$$

3. Suppose the number N of times a fair die is rolled is Poisson r.v. with $\lambda = 5$. Let Y be the total score in N rolls.

(a) Find $\mathbf{E}(Y|N=n)$, $\mathbf{E}(Y|N)$ and $\mathbf{E}(Y)$.

(b) Find $\mathbf{E}\left(Y^2|N=n\right)$, $\mathbf{E}\left(Y^2|N\right)$ and $\mathbf{E}\left(Y^2\right)$. Find $\mathrm{Var}(Y)$.

Answer. The die score X has

$$\begin{aligned}\mathbf{E}(X) &= \mu = \frac{1+2+3+4+5+6}{6} = \frac{7}{2},\\ \mathrm{Var}(X) &= \sigma^2 = \sum_{k=1}^{6} k^2 \cdot \frac{1}{6} - \mu^2 = \frac{35}{12}.\end{aligned}$$

Given $N = n$**,** we have $Y = X_1 + \ldots X_n$, where X_i are independent die scores (distributed like X). Hence,

$$\begin{aligned}\mathbf{E}(Y|N=n) &= \mathbf{E}(X_1) + \ldots + \mathbf{E}(X_n) = n\mu,\\ \mathbf{E}\left(Y^2|N=n\right) &= \mathrm{Var}(X_1 + \ldots + X_n) + (n\mu)^2 = n\sigma^2 + n^2\mu^2.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}(Y|N) &= \mu N = \frac{7}{2}N,\\ \mathbf{E}(Y) &= \mu\mathbf{E}(N) = 5\mu = \frac{35}{2},\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left(Y^2|N\right) &= \sigma^2 N + \mu^2 N^2\\ \mathbf{E}\left(Y^2\right) &= \sigma^2\mathbf{E}(N) + \mu^2\mathbf{E}\left(N^2\right) = 5\sigma^2 + \mu^2\left(5+5^2\right)\\ &= 5\sigma^2 + 30\mu^2.\end{aligned}$$

Finally, $\mathrm{Var}(Y) = 5\sigma^2 + 30\mu^2 - (5\mu)^2 = 5\sigma^2 + 5\mu^2 = \frac{455}{6}$.