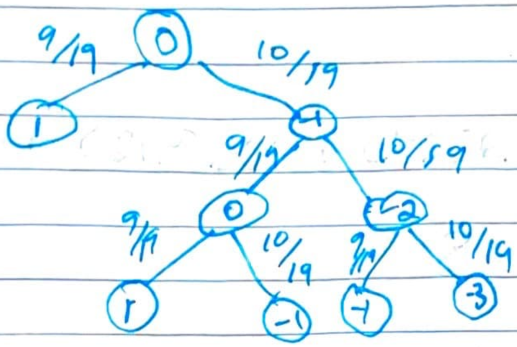
Math 505A HW4

1) Bet \$1 on red. $P(\text{red}) = \frac{18}{38} = \frac{9}{19}$

If red, stop w/ \$1.
If not red, play 2 more rounds & stop.



Let X be the net gain.

a) Values X can take on: $1, -1, -3$

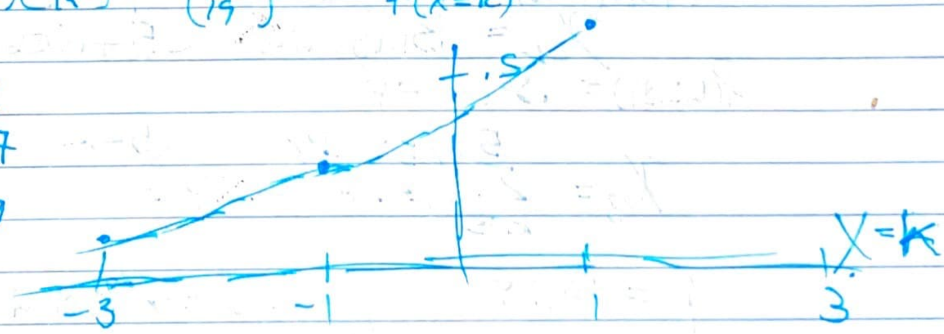
$P(X=k)$ for $-1, 1, -3$

$P(1) = 9/19 + \left(\frac{10}{19}\right)\left(\frac{9}{19}\right)\left(\frac{9}{19}\right)$

$P(-1) = \left(\frac{10}{19}\right)\left(\frac{9}{19}\right)\left(\frac{10}{19}\right) + \left(\frac{10}{19}\right)\left(\frac{10}{19}\right)\left(\frac{9}{19}\right) = \frac{2(10)^2 9}{19^3}$

$P(-3) = \left(\frac{10}{19}\right)\left(\frac{10}{19}\right)\left(\frac{10}{19}\right) = \left(\frac{10}{19}\right)^3$

$P(-1) = .26242$
$P(1) = .59177$
$P(-3) = .14579$



b) $P(X>0) = P(1) = .59177$

To determine if this is a good strategy, we calculate $E(X)$ of how much we expect to lose daily.

$$E(X) = \sum_x x P(X=x)$$

x	$P(X=x)$
-1	.2624
1	.5918
-3	.1458

$$E(x) = (-1)(.2624) + (1)(.5918) + (-3)(.1458)$$

$$E(X) = -0.108$$

On average, the game is expected to lose \$.11 daily, so this is not a good strategy.

c) You played 5 days. Find P("win \$1 in at least 3")

Let X_k be an r.v. that is binomial representing winning \$1 on the kth day.

From a) ~~[illegible]~~ P("3 wins of 5")

X_k = Binomial (5 tries, [illegible])

P("winning \$1")

$P(\text{win }\$1) = .5918 = p$

$$X_3 = \sum_{k=3}^{5} \binom{5}{3} p^k (1-p)^{5-k}$$

$p = .5918 \qquad q = 1-p = 0.4082$

$P(X \geq k=3)$

$$= \sum_{k=3}^{5} \binom{5}{k} (.5918)^k (.4082)^{5-k}$$

$$\boxed{P(X \geq 3) = .6683}$$

2) Let U be an r.v. with distribution function

$$F_U(u) = P(U \le u) = \begin{cases} 0 & \text{if } v < 0 \\ v & \text{if } 0 \le v \le 1 \\ 1 & \text{if } > 1 \end{cases}$$

U is uniformly distributed over $[0,1]$.

Let F be a distribution function which is continuous and strictly increasing, s.t. F^{-1} spans a similar range to F's domain and cancellation identities hold.

a) Show that $X = F^{-1}(U)$ is an r.v. having distribution function F, $F = F_X$.

F_X is the CDF of F, so
$F_X = P(X \le x)$

$X = F^{-1}(U)$, so we are trying to show that the ~~cdf~~ cdf of $X = F^{-1}(U)$ is F.

$P(U \le x) = x$ over $[0,1]$ is true of U.

$P(X \le x) = P(F^{-1}(U) \le x) = P(F(F^{-1}(U)) \le F(x))$
Since F's cancellation identity holds, $= P(U \le F(x))$

$P(U \le F(x)) = \boxed{F(x)}$

b) (i) $P(\sqrt{X^2} \le \sqrt{x}) = F(x)$

$P(X \le \pm\sqrt{x}) = P(X \le \sqrt{x})\, P(X$

$P(-\sqrt{x} \le X \le \sqrt{x})$

$P(X \le \sqrt{x}) - P(X \le -\sqrt{x})$

$\boxed{F(\sqrt{x}) - F(-\sqrt{x})}$

(ii) $P(|X| \le x) = F(x)$

$P(-x \le X \le x) = F(x)$

$P(X \le x) - P(X \le -x) =$

$\boxed{F(x) - F(-x)}$

$P((\sqrt{|X|})^2 \le x^2) = F(x)$

$P(|X| \le x^2)$

$P(-x^2 \le X \le x^2) =$

$P(X \le x^2) - P(X \le -x^2)$

$\boxed{F(x^2) - F(-x^2)}$

(iii) $P(F(X) \le x) = F(x)$

$P(F^{-1}(F(X)) \le F^{-1}(x)) = F(x)$

$P(X \le F^{-1}(x)) = F(X)$

$F(F^{-1}(X)) = \boxed{X}$

small case

(iiii) $P(G^{-1}F(X) \le x) = F(x)$

$P(G^{-1}F(X) \le G(x)) = P(X \le G(x))$

$\boxed{G(x)}$

$P(F(X) \le G(x)) \le P(X$

$P(X \le x) = G(x)$

3) A coin is tossed and $P(H)=p$. Let H_n and T_n be the # of heads and tails in n tosses.

So, $T_n + H_n = n$, so $T_n = n - H_n$ and $H_n = n - T_n$

H_n is distributed binomially with n tries and probability H shows up of p ~ Binom(n,p);

$E(H_n) = np$ and $\text{Std. dev.}(H_n) = \sqrt{np(1-p)}$

Now consider: $S_n = \frac{1}{n}(H_n - T_n)$

$= S_n = \frac{1}{n}(H_n - n + H_n) = S_n = \frac{1}{n}(2H_n - n)$

Then to calculate the mean, we get the expectation of the average standard deviation of the number of Hs.

$E(S_n) = \mu = E\left(\frac{1}{n}(2H_n - n)\right) =$

$= \frac{1}{n}(2E(H_n) - n)$

From binom(n,p), $E(H_n) = np$

$\Rightarrow \frac{1}{n}(2np - n) = \frac{2np - n}{n} = \boxed{2p-1}$

$\mu = 2p-1$

$\sigma^2 = Var(S_n) = var\left(\frac{1}{n}[2H_n - n]\right) = \frac{1}{n^2} var(2H_n - n)$

$\Rightarrow \frac{4}{n^2} Var(H_n) = \frac{4(\sqrt{np(1-p)})^2}{n^2} = \frac{4\cancel{n}p(1-p)}{n^{\cancel{2}}}$

$= \frac{4p(1-p)}{n}$

The given expression $\forall \varepsilon > 0$.

$P(2p-1 - \varepsilon \le \frac{1}{n}(H_n - T_n) \le 2p-1+\varepsilon) \longrightarrow 1$

as $n \longrightarrow \infty$.

Shown $\mu = 2p-1$ & $S_n = \frac{1}{n}(H_n - T_n)$

$= P(-\varepsilon \le \frac{1}{n}(H_n - T_n) - (2p-1) \le \varepsilon)$

$= P(-\varepsilon \le S_n - \mu \le \varepsilon) = P(|S_n - \mu| \le \varepsilon)$

$P(|S_n - \mu| \le \varepsilon) = 1 - P(|S_n - \mu| > \varepsilon)$

$1 - P(|S_n - \mu| > \varepsilon) \ge 1 - P(|S_n - \mu| \ge \varepsilon) =$

$1 - P(|S_n - \mu| \ge \frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\sqrt{\frac{p(1-p)}{n}}) \ge$

$1 - \left(\frac{1}{\varepsilon}\sqrt{\frac{p(1-p)}{n}}\right)^2$ By Chebyshev's Ineq.

If $1 - \left(\frac{1}{\varepsilon}\frac{\sqrt{p(1-p)}}{n}\right)^2$ approaches 1 as $n \to \infty$, then $P(2p-1-\varepsilon \le \frac{1}{n}(H_n - T_n) \le 2p-1+\varepsilon)$ $\to 1$.

$\lim_{n\to\infty}\left(1 - \left(\frac{1}{\varepsilon}\frac{\sqrt{p(1-p)}}{n}\right)^2\right) = 1$ $\because$

$1 - \left(\frac{1}{\varepsilon}\frac{\sqrt{p(1-p)}}{n}\right)^2 \le P(2p-1-\varepsilon \le \frac{1}{n}(H_n - T_n) \le 2p-1+\varepsilon)$

$\longrightarrow 1$.

4) $P(X=k) = \dfrac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \quad k \in [0, n]$

$$= \frac{\dfrac{m!}{k!(m-k)!} \cdot \dfrac{(N-m)!}{(n-k)!(N-m-n+k)!} \cdot \dfrac{n!}{k!(n-k)!}}{\dfrac{N!}{n!(N-n)!} \cdot \cancel{n!}}$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\dfrac{m!}{(m-k)!} \cdot \dfrac{(N-m)!}{(N+k-m-n)!}}{\dfrac{N(N-1)(N-2)\cdots(1)}{(N-n)(N-n-1)(N-n-2)\cdots}}\right)$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\dfrac{(m-k)!}{N^k} \cdot \dfrac{(N+k-m-n)!}{N^{n-k}}}{1}\right)$$

As $\dfrac{(N-n)!}{N^n}$ approaches 1 when N is large.

$$\lim_{m,N\to\infty} \frac{(m-k)!}{N^k} = \frac{M^k}{N^k} = p^k \quad \underbrace{\cdots}_{N-k \text{ times}}$$

m is the number of red balls, so m/N is p since its the probab (Red).

$$\lim_{m,N\to\infty} \frac{(N+k-m-n)(N+k-m-n-1)\cdots}{N^{n-k}} = \frac{(N-m)^{n-k}}{(N)^{n-k}} = q^{n-k}$$

$$= \boxed{\frac{n!}{(n-k)!\,k!}\, p^k q^{n-k}}$$

5. Let X be $\text{binom}(n, p = 1/2)$,
It can be proven that most likely values are

$$\begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} \text{ and } \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Show that $P(X = n/2) = \binom{n}{n/2} 2^{-n}$

$$\binom{n}{n/2} 2^{-n} = \frac{n!}{((n/2)!)^2} (2)^{-n}$$

By Stirling formula $\Rightarrow \dfrac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (2)^{-n}}{((n/2)!)^2}$

$$= \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n 2^{-n}}{\left(\sqrt{2\pi \frac{n}{2}}\left(\frac{n/2}{e}\right)^{n/2}\right)^2}$$

$$= \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n 2^{-n}}{\pi n\left(\frac{n/2}{e}\right)^n} = \frac{\frac{\sqrt{2\pi n}(n^n)2^{-n}}{\cancel{e^n}} \cdot \cancel{e^n}}{\frac{\pi n (n/2)^n}{\cancel{e^n}} \cdot \cancel{e^n}}$$

$$= \frac{\sqrt{2\pi n}(n^n) 2^{-n}}{\pi n (n/2)^n} = \frac{\sqrt{2\pi n}\,(\cancel{n^n})\, 2^{-n}\,(2^n)}{\pi n\,(\cancel{n^n})\,\cancel{2^n}}$$

$$= \frac{\sqrt{2\pi n}\,(2^0)}{\pi n} = \boxed{\frac{\sqrt{2\pi n}}{\pi n} = \frac{\sqrt{2}\,(\pi n)^{1/2}}{\pi n} = \frac{1}{\sqrt{\pi (n/2)}}}$$

Also show:

$$\binom{n}{\frac{(n+1)}{2}} 2^{-n} \approx \frac{1}{\sqrt{\pi \cdot \frac{n}{2}}}$$

$$= \frac{n! \quad 2^{-n}}{\left(n-\frac{n+1}{2}\right)! \cdot \left(\frac{n+1}{2}\right)!} = \frac{n! \quad 2^{-n}}{\left(\frac{2n-1-n}{2}\right)! \left(\frac{n+1}{2}\right)!}$$

$$= \frac{n! \quad 2^{-n}}{\left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)!} = \frac{n! \cdot 2^{-n}}{\left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)!}$$

By Stirling's Formula, $\Rightarrow \dfrac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (2^{-n})}{\sqrt{2\pi\left(\frac{n+1}{2}\right)}\left(\frac{2n+2}{e}\right)^{\left(\frac{n+1}{2}\right)} \sqrt{2\pi\left(\frac{n-1}{2}\right)}\left(\frac{2n-2}{e}\right)^{\left(\frac{n-1}{2}\right)}}$

$$\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n (2^{-n})}{\sqrt{\pi(n+1)}\left(\frac{2n+2}{e}\right)^{\frac{n+1}{2}} \cdot \sqrt{\pi(n-1)}\left(\frac{2n-2}{e}\right)^{\left(\frac{n-1}{2}\right)}}$$

$$= \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}(2^{-n})}{\pi\sqrt{(n+1)(n-1)}\,\dfrac{(2n+2)^{\wedge}(n+1/2) \cdot (2n-2)^{\wedge}(n-1/2)}{e^{\frac{n+1}{2}} \cdot e^{\frac{n-1}{2}}}}$$

$$= \frac{\dfrac{\sqrt{2\pi n} \cdot n^n (2^{-n})}{\cancel{e^n}} \cdot \cancel{e^n}}{\dfrac{\pi\sqrt{(n-1)(n+1)}(2n+2)^{\left(\frac{n+1}{2}\right)}(2n-2)^{\left(\frac{n-1}{2}\right)}}{\cancel{e^n}} \cdot \cancel{e^n}}$$

$$= \frac{\sqrt{2\pi n}\,(n^n)(2^{-n})}{\pi\sqrt{(n-1)(n+1)}\,(2n+2)^{\left(\frac{n+1}{2}\right)}(2n-2)^{\left(\frac{n-1}{2}\right)}(2^n)}$$

$$\lim_{n\to\infty} \frac{n^n}{(2n+2)^{\left(\frac{n+1}{2}\right)}(2n-2)^{\frac{n-1}{2}}} = 1, \text{ so } \boxed{\frac{\sqrt{2\pi n}}{\pi n} = \frac{1}{\sqrt{\pi\left(\frac{n}{2}\right)}}}$$

$$\Rightarrow \frac{\sqrt{2\pi n}}{\pi\sqrt{(n+1)(n+1)}(2^n)} \approx$$