1. A personnel director has two lists of applicants for jobs. List 1 contains the names of four women and six men, whereas list 2 contains the names of five women and four men. A name is randomly selected from list 1 and added to list 2.

a) A name is then randomly selected from the augmented list 2. What is the probability that the name is that of a woman?

Answer. Let $M =$ "man's name is selected from list 1 and added to list 2", $W =$ "woman's name is selected from list 1 and added to list 2".

Let $W_1 =$ "woman's name selected from augmented list 2". By total probability law,

$$\begin{aligned} \mathbf{P}(W_1) &= \mathbf{P}(W_1|M)\,\mathbf{P}(M) + \mathbf{P}(W_1|W)\,\mathbf{P}(W) \\ &= \frac{5}{10}\frac{6}{10} + \frac{6}{10}\frac{4}{10} = 0.54 \end{aligned}$$

b) It turns out that the name selected from the augmented list 2 is that of a man. What is the probability that a woman's name was originally selected from list 1?

Answer. Let $M_1 =$ "man's name selected from augmented list 2". By Bayes formula,

$$\mathbf{P}(W|M_1) = \frac{\mathbf{P}(M_1|W)\,\mathbf{P}(W)}{\mathbf{P}(M_1)} = \frac{\mathbf{P}(M_1|W)\,\mathbf{P}(W)}{1-0.54} = \frac{\frac{4}{10}\frac{4}{10}}{\frac{46}{100}} = \frac{16}{46} = \frac{8}{23}.$$

c) Then a name is selected randomly from list 1 again. What is the probability it is a woman's name.

Answer. Let $W_2 =$ "woman's name selected from list 1 again". Taking M_1 into consideration (look at b)), by total probability law,

$$\begin{aligned} \mathbf{P}(W_2|M_1) &= \mathbf{P}(W_2|M)\,\mathbf{P}(M|M_1) + \mathbf{P}(W_2|W)\,\mathbf{P}(W|M_1) \\ &= \frac{4}{9}\left(1-\frac{8}{23}\right) + \frac{3}{9}\frac{8}{23} = \frac{28}{69}. \end{aligned}$$

2nd answer. By definition and total probability law,

$$\begin{aligned} \mathbf{P}(W_2|M_1) &= \frac{\mathbf{P}(M_1W_2)}{\mathbf{P}(M_1)} = \frac{\mathbf{P}(M_1W_2|M)\,\mathbf{P}(M) + \mathbf{P}(M_1W_2|W)\,\mathbf{P}(W)}{\mathbf{P}(M_1)} \\ &= \frac{\frac{4}{9}\frac{5}{10}\frac{6}{10} + \frac{3}{9}\frac{4}{10}\frac{4}{10}}{\frac{4}{10}\frac{4}{10} + \frac{5}{10}\frac{6}{10}} = \frac{1}{9}\frac{4\cdot 42}{46} = \frac{1}{3}\frac{2\cdot 14}{23} = \frac{28}{69}. \end{aligned}$$

2. Suppose that each of 5 jobs is assigned at random to one of three servers A, B and C. [For example, one possible outcome would be that job 1 goes to server B, job 2 goes to server C, job 3 goes to server C, job 4 goes to server B and job 5 goes to server A. "At random" here means that there are 3^5 equally likely outcomes.

(a) Find the probability that server C gets all 5 jobs. *Answer.* $\frac{1}{3^5}$.

(b) Let S be the number of servers that get exactly one job. Find $\mathbf{E}(S)$.

Answer. Let $A_i =$ "ith server gets exactly one job", $i = 1, 2, 3$. Then

$$S = I_{A_1} + I_{A_2} + I_{A_3}, \mathbf{E}(S) = 3 \cdot \frac{5 \cdot 2^4}{3^5} = 5\left(\frac{2}{3}\right)^4 = \frac{80}{81}.$$

because, by counting, $\mathbf{P}(A_i) = \frac{5 \cdot 2^4}{3^5}, i = 1, 2, 3$.

Also, another option, $\mathbf{P}(A_i)$ is binomial probability: $X_i =$ number of jobs ith server gets is binomial$\left(n = 5, p = \frac{1}{3}\right)$:

$$\mathbf{P}(A_i) = \mathbf{P}(X_i = 1) = \binom{5}{1} \frac{1}{3} \left(\frac{2}{3}\right)^4.$$

2nd answer. Range of $X = \{0, 1, 2\}$, and, by counting, $\mathbf{P}(S = 1) = 3 \cdot \frac{5\left(2^4 - 4 - 4\right)}{3^5}, \mathbf{P}(S = 2) = 3 \cdot \frac{5 \cdot 4}{3^5}$,

$$\mathbf{E}(S) = 3 \cdot \frac{5\left(2^4 - 8\right)}{3^5} + 2 \cdot 3 \cdot \frac{5 \cdot 4}{3^5} = \frac{80}{81}.$$

(c) Find the probability that no server gets more than 2 jobs.

Answer. Counting directly (2 servers necessarily get exactly 2 and one server exactly 1 job),

$$\mathbf{P}(\text{no server gets } \geq 3 \text{ jobs}) = \frac{3 \cdot \frac{5!}{2!2!1!}}{3^5} = \frac{10}{27}.$$

2nd answer. Using complementary event,

$$\begin{aligned} \mathbf{P}(\text{no server gets } \geq 3 \text{ jobs}) &= 1 - \mathbf{P}(\text{at least one gets } \geq 3) \\ &= 1 - 3 \cdot \left[\binom{5}{3}\frac{2^2}{3^5} + 5 \cdot \frac{2}{3^5} + \frac{1}{3^5}\right] \end{aligned}$$

because

$$\begin{aligned} \mathbf{P}(\text{at least one gets } \geq 3) &= \mathbf{P}(\text{A or B or C gets } \geq 3) \\ &= 3\mathbf{P}(\text{A gets } \geq 3) = 3 \cdot \left[\binom{5}{3}\frac{2^2}{3^5} + 5 \cdot \frac{2}{3^5} + \frac{1}{3^5}\right]. \end{aligned}$$

$\mathbf{P}$ (A gets ≥ 3) can be found as binomial probability as well.

(d) Assume there are m jobs and n servers. Suppose that each of m jobs is assigned at random to one of n servers. Let S be the number of servers that get exactly one job. Find Var(S) in terms of m and n.

Answer. Let A_i = "ith server gets exactly one job", $i = 1, \ldots, n$. By counting,

$$\mathbf{P}(A_i) = \frac{m(n-1)^{m-1}}{n^m} = \frac{m}{n}\left(1 - \frac{1}{n}\right)^{m-1} =: p.$$

So,

$$S = \sum_{i=1}^{n} I_{A_i}, \mathbf{E}(S) = \sum_{i=1}^{n} \mathbf{P}(A_i) = \frac{m \cdot (n-1)^{m-1}}{n^{m-1}} = m\left(1 - \frac{1}{n}\right)^{m-1} = np$$

Now,

$$\operatorname{Var}(S) = \sum_{i=1}^{n} \operatorname{Var}(I_{A_i}) + 2\sum_{i<j} \operatorname{Cov}\left(I_{A_i}, I_{A_j}\right).$$

We find for $i \neq j$, by counting,

$$\begin{aligned} \mathbf{P}(A_i A_j) &= \frac{m(m-1)(n-2)^{m-2}}{n^m} = m(m-1)\left(\frac{1}{n}\right)^2\left(1 - \frac{2}{n}\right)^{m-2} =: a, \\ \operatorname{Cov}\left(I_{A_i}, I_{A_j}\right) &= \mathbf{P}(A_i A_j) - \mathbf{P}(A_i)\mathbf{P}(A_j) = a - p^2. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Var}(S) &= \sum_{i=1}^{n} \operatorname{Var}(I_{A_i}) + 2\sum_{i<j} \operatorname{Cov}\left(I_{A_i}, I_{A_j}\right) = n\left(p - p^2\right) + n(n-1)\left(a - p^2\right) \\ &= n(p - a) + n^2\left(a - p^2\right). \end{aligned}$$

2nd answer. Directly,

$$\begin{aligned} S^2 &= \left(\sum_{i=1}^{n} I_{A_i}\right)^2 = \sum_{i=1}^{n} I_{A_i} + 2\sum_{i<j} I_{A_i} I_{A_j}, \\ \mathbf{E}\left(S^2\right) &= np + 2\binom{n}{2} a = np + n(n-1)a, \end{aligned}$$

and $\operatorname{Var}(S) = \mathbf{E}(S^2) - (\mathbf{E}(S))^2 = np + n(n-1)a - n^2p^2$.

Comment. The problem can be restated in our "elevator" hw setting: "jobs" = "people", "servers"= "floors", S = number of floors

3. The number X of electrons that hit the plate is Poisson with parameter $\lambda_1 = 2$. Every impact produces independently a number of secondary electrons that is Poisson with parameter $\lambda_2 = 1$. Let Y be the total number of secondary electrons.

a) Find $\mathbf{P}(Y = j|X = n), n \geq 0, j \geq 0$. Hint. A sum of independent Poisson is Poisson.

Answer. Let Y_i be number of secondary electrons produced by ith impact: Y_i are independent Poisson$(\lambda_2 = 1)$. Given $X = n$, $Y = Y_1 + \ldots + Y_n$ is Poisson$(n\lambda_2 = n)$ as a sum of independent Poisson. So,

$$\mathbf{P}(Y = j|X = n) = e^{-n\lambda_2}\frac{(n\lambda_2)^j}{j!} = e^{-n}\frac{n^j}{j!}, n, j \geq 0.$$

b) Find $\mathbf{E}(Y|X = n)$ and $\mathbf{E}(Y|X)$ and $\mathbf{E}(Y)$.
Answer. Again, given $X = n$, Y is Poisson(n) :

$$\mathbf{E}(Y|X = n) = n, \mathbf{E}(Y|X) = X, \mathbf{E}(Y) = \mathbf{E}(X) = \lambda_1 = 2.$$

c) Find $\mathbf{E}(Y^2|X = n)$, $\mathbf{E}(Y^2|X)$ and $\mathrm{Var}(Y)$.
Answer. Again, given $X = n$, Y is Poisson$(n\lambda_2 = n)$:

$$\begin{aligned}\mathbf{E}\left(Y^2|X = n\right) &= n + n^2, \mathbf{E}\left(Y^2|X\right) = X + X^2, \mathbf{E}\left(Y^2\right) = \mathbf{E}\left(X + X^2\right) = 2 + 2 + 2^2,\\ \mathrm{Var}(Y) &= \mathbf{E}\left(Y^2\right) - (\mathbf{E}(Y))^2 = 2 + 2 + 2^2 - 2^2 = 4.\end{aligned}$$

2nd answer. Let $Y_1.Y_2, \ldots$ be independent Poisson(λ_2), independent of X. We model Y as $Y = \sum_{i=1}^{X} Y_i$. Since given $X = n, Y$ is the sum of n independent Poisson: $Y = \sum_{i=1}^{n} Y_i$, using formula of #3 of hw7, we find

$$\begin{aligned}\mathrm{Var}(Y|X = n) &= n, \mathrm{Var}(Y|X) = X,\\ \mathbf{E}\left(Y^2|X = n\right) &= n + n^2, \mathbf{E}\left(Y^2|X\right) = X + X^2,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= \mathbf{E}[\mathrm{Var}(Y|X)] + \mathrm{Var}(\mathbf{E}(Y|X))\\ &= \mathbf{E}(X) + \mathrm{Var}(X) = 2 + 2 = 4.\end{aligned}$$