4.7 Functions of r. vectors

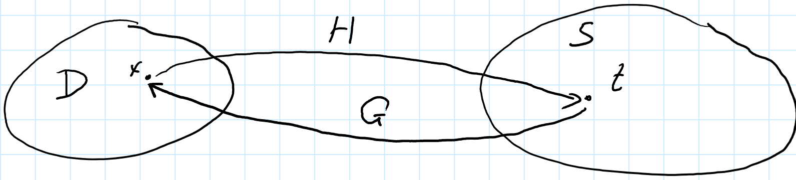
Assume $X = (X_1, \dots, X_n)$ is jointly continuous with joint pdf $f(x) = f(x_1, \dots, x_n)$ and range of $X = D \subset \mathbb{R}^n$. Let $T = (T_1, \dots, T_n) = H(X)$:

$$T_1 = H_1(X) \quad \Big| \quad (T_1, \dots, T_n) = (H_1(X), \dots, H_n(X))$$
$$T_n = H_n(X)$$

Question. Is T jointly continuous? If so, find joint pdf of $(T_1, \dots, T_n)$.

Assume $t = H(x), x \in D$, be one-to-one and continuously differentiable with the inverse $x = G(t)$: $G(H(x)) = x, x \in D$, $H(G(t)) = t$, $t \in S = \{t \in \mathbb{R}^d : G(t) \in D\}$.

$$\left.\begin{matrix} t_1 = H_1(x) \\ \vdots \\ t_n = H_n(x) \end{matrix}\right\} t = H(x)$$



$$\left\{\begin{matrix} x_1 = G_1(t) \\ \cdots \\ x_n = G_n(t) \end{matrix}\right| \ x = G(t)$$

Thm1. Under assumptions above, $T = H(X)$ is jointly continuous with joint pdf

$$(1) \quad f_T(t) = f(G(t))\,|J(t)|\,I_S(t), \text{ where } S = \{t \in \mathbb{R}^n : G(t) \in D\}$$

$$J(t) = \begin{vmatrix} \frac{\partial G_1}{\partial t_1} & \cdots & \frac{\partial G_1}{\partial t_n} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial t_1} & \cdots & \frac{\partial G_n}{\partial t_n} \end{vmatrix}$$ is determinant of $n \times n$ matrix of partial derivatives.

Comment. We can write (1) as

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$$f_T(t) = \begin{cases} f(G(t))\,|J(t)|, & t \in S = \{t \in \mathbb{R}^n : G(t) \in D\} \\ 0, & \text{otherwise.} \end{cases}$$

Why (1)? For any function $g(T)$,

$$E[g(T)] = E[g(H(X))] = \int_D g(H(x))\, f(x)\, dx =$$ calculus

$$\boxed{\text{integration variable change: } x = G(t),\ dx = |J(t)|\,dt}$$

$$= \int_S g(t)\, f(G(t))\,|J(t)|\,dt = \int_{\mathbb{R}^d} g(t)\, \underbrace{f(G(t))\,|J(t)|\, 1_S(t)}\, dt$$

Procedure. 1. We find the inverse $x = G(t)$ by solving for x the equation $t = H(x)$ with a given t.

2. Find $J(t)$

3. By Thm 1,

$$f_T(t) = f(G(t))\,|J(t)|\, 1_S(t), \text{ where } S = \{t : G(t) \in D\},$$

equivalently,

$$f_T(t) = \begin{cases} f(G(t))\,|J(t)|, & t \in S \\ 0, & \text{otherwise.} \end{cases}$$

Ex 1. Let X_1, X_2, X_3 be <u>indep.</u> exponential$(\lambda = 1)$

a) Find joint pdf of $T_1 = X_1$, $T_2 = X_1 + X_2$, $T_3 = X_1 + X_2 + X_3$

Answer. Range of $X = (X_1, X_2, X_3) = D = \{(x_1, x_2, x_3) : x_1 > 0, x_2 > 0, x_3 > 0\}$

We solve for x_1, x_2, x_3 the equations

$$\begin{aligned} t_1 &= x_1 \\ t_2 &= x_1 + x_2 \\ t_3 &= x_1 + x_2 + x_3 \end{aligned} \quad\Bigg|\quad \begin{aligned} x_1 &= t_1 \\ x_2 &= t_2 - t_1 \\ x_3 &= t_3 - t_2 \end{aligned} \quad\Bigg|\quad \begin{aligned} G(t) &= G(t_1, t_2, t_3) \\ &= (t_1,\ t_2 - t_1,\ t_3 - t_2) \end{aligned}$$

$t_2 = x_1 + x_2$ | $x_2 = t_2 - t_1$ | $= (t_1,\ t_2 - t_1,\ t_3 - t_2)$

$t_3 = x_1 + x_2 + x_3$ | $x_3 = t_3 - t_2$ |

ii/ Computing $J(t) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$

iii) Application of Thm 1:

joint pdf of X is $f(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} e^{-x_3} = e^{-(x_1+x_2+x_3)}$, $x_1 > 0,\ x_2 > 0,\ x_3 > 0.$

$S = \left\{ (t_1, t_2, t_3) : \begin{matrix} t_1 > 0 \\ t_2 - t_1 > 0 \\ t_3 - t_2 > 0 \end{matrix} \right\} = \{ 0 < t_1 < t_2 < t_3 < \infty \}$

$f_T(t) = e^{-t_3}$ if $\{ 0 < t_1 < t_2 < t_3 < \infty \}$

b) Find $f_T(t_1, t_2 | t_3)$.

Answer. $f_T(t_1, t_2 | t_3) = \dfrac{f_T(t_1, t_2, t_3)}{f_{T_3}(t_3)} =$

$$= \frac{e^{-t_3}}{\frac{t_3^2}{2!} e^{-t_3}} = \frac{2!}{t_3^2}, \quad 0 < t_1 < t_2 < t_3 < \infty,$$

because $T_3 \sim \Gamma(\lambda = 1, n = 3)$

Comment. Given $T_3 = t_3$, (T_1, T_2) is the order statistic of two uniform in $(0, t_3)$ (#4c) hw9)

5.1 Generating and moment generating functions (gf and mgf).

Assume X takes values in $\{0, 1, 2, \ldots\}$, $p_n = P(X = n)$, $n = 0, 1, 2, \ldots$ Note $\sum_{n=0}^{\infty} p_n = 1$.

Def. gf of X is the function

$G(0) = p_0$

(1) $G(s) = G_X(s) = \sum^{\infty} p_n s^n = p_0 + p_1 s + \ldots$

$$(1) \quad G(s) = G_X(s) = \sum_{n=0}^{\infty} p_n s^n = p_0 + p_1 s + \dots$$

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<u>Note</u> (1) has convergence radius $R \geq 1$ and

$$G(s) = \sum_{n=0}^{\infty} s^n P(X=n) = E(s^X) \text{ if } -R < s < R, \; s \neq 0.$$

<u>Example</u>. If $X \sim \text{Poisson}(\lambda)$, then

$$G(s) = \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda s} \cdot e^{-\lambda} = e^{\lambda(s-1)}, \quad -\infty < s < \infty.$$

<u>Properties of $G = G_X$</u>

1. p_n are Taylor coefficients of G:

$$p_n = \frac{G^{(n)}(0)}{n!}, \quad \text{where } G^{(n)} = \frac{d^n}{ds^n} G,$$

$$p_0 = G(0) = P(X=0), \quad P(X > 0) = 1 - G(0).$$

2. If $G_X(s) = G_Y(s)$ for $-\varepsilon < s < \varepsilon$ for some $\varepsilon > 0$, then X and Y are identically distributed.

3. If X, Y are independent, then $G_{X+Y}(s) = G_X(s) G_Y(s)$.