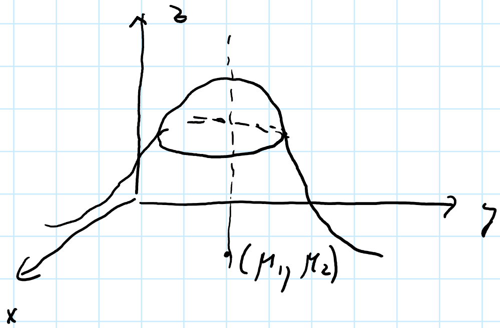
4.6-7 Normal bivariate r. vector

Recall. If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ are independent, then

1. (X, Y) is jointly cont. r. vector with pdf

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}, \quad (x,y) \in \mathbb{R}^2.$$

2. $aX + bY$ is normal r.v.



Question. Are there possibly dependent normal r.v. X, Y that are jointly continuous and 2. holds?

Answer. Yes

Examples. 1. X = height, Y = weight of a football player.

2. X = SAT score, Y = GPA of a freshman.

3. $Y = X + \varepsilon Z$, where $X \sim N(\mu, \sigma^2)$, $Z \sim N(0,1)$ are independent.

"General" model. Assume $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ and

(1) $\boxed{Y = kX + c + V, \text{ where } V \sim N(0, \sigma^2) \text{ and } X \text{ are independent.}}$

pendent.

Comments on (1). a) $kX + c$ is called "explanation of Y by X" and V explanation of Y by factors independent of X.

b) Taking expectation of both sides of (1) we get

$\mu_2 = k\mu_1 + c$, $c = \mu_2 - k\mu_1$:

$Y - \mu_2 = k(X - \mu_1) + V$,

c) X, V are uncorrelated: by class exercises,

$k = \rho \frac{\sigma_2}{\sigma_1}$ with $\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_1 \sigma_2}$,

$Var(V) = \sigma_2^2(1-\rho^2)$

Summarizing (1) becomes

$$\boxed{\begin{array}{l}(2) \quad Y = \mu_2 + \rho\sigma_2 \cdot \frac{X-\mu_1}{\sigma_1} + V = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X-\mu_1) + V, \\ \text{where } V \sim N(0, \sigma_2^2(1-\rho^2)) \text{ and } X \sim N(\mu_1, \sigma_1^2) \text{ are independent.}\end{array}}$$

Some consequences of (2) (Conditioning).

1. Given $X = x$,

$Y \overset{d}{=} \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1) + V$ with $V \sim N(0, \sigma_2^2(1-\rho^2))$

or $Y \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)\right)$. Hence

a) cond. pdf of Y given $X = x$ is the pdf of

$N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right).$

b) $E(Y \mid X=x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1)$

$E(Y \mid X) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X-\mu_1).$

$Var(Y \mid X=x) = \sigma_2^2(1-\rho^2).$

c) joint pdf of (X,Y) is

$f(x,y) = f(y \mid x) f_X(x)$

$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2(1-\rho^2)} Q(x,y)\right\}$, where

(3) $Q(x,y) = \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)$

<u>Def.</u> Jointly cont. r. vector (X,Y) is called <u>normal bivariate</u> with parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho$ ($\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$, $-1 < \rho < 1$) if their joint pdf is $f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2(1-\rho^2)} Q(x,y)\right\}$ with Q given by (3).

<u>Def.</u> $Z = (Z_1, Z_2)$ is called <u>standard</u> normal bivariate if Z_1, Z_2 are independent standard normal: $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = 0$.

<u>Properties of normal bivariate (X,Y) with $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$:</u>

1. $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, $\rho = \rho(X,Y)$.

2. X, Y are independent if and only if $\rho = 0$.

3. Roles of X, Y can be switched: not only

(i) $Y = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X - \mu_1) + V$, $V \sim N(0, \sigma_2^2(1-\rho^2))$ and X independent

but also

(ii) $X = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(Y - \mu_2) + \tilde{V}$, $\tilde{V} \sim N(0, \sigma_1^2(1-\rho^2))$ and Y independent.

4. $aX + bY$ is normal for any $(a, b) \neq 0$.

Why? Since $Y = kX + c + V$, we have

$aX + bY = (a + bk)X + bc + bV$, and X, V are independent.

5. Given $X = x$, $Y \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1-\rho^2)\right)$:

$E(Y \mid X = x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1)$, $\mathrm{Var}(Y \mid X) = \sigma_2^2(1-\rho^2)$.

6. If (X, Y) is normal bivariate $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then $(X, Y) = (Z_1, Z_2)A + (\mu_1, \mu_2)$, where (Z_1, Z_2) is standard normal bivariate, and A is 2×2 matrix with $\det A \neq 0$.

Why? $X = \sigma_1 \underbrace{\left(\frac{X - \mu_1}{\sigma_1}\right)}_{= Z_1} + \mu_1 = \sigma_1 Z_1 + \mu_1$

$$Y = \rho \sigma_2 \left(\frac{X - \mu_1}{\sigma_1}\right) + \mu_2 + \sigma_2\sqrt{1-\rho^2}\, \underbrace{\frac{V}{\sigma_2\sqrt{1-\rho^2}}}_{= Z_2},$$

$$(X, Y) = (Z_1, Z_2)\begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} + (\mu_1, \mu_2),\quad \det A = \sigma_1\sigma_2\sqrt{1-\rho^2} > 0.$$

7. Let (X,Y) be normal bivariate and A be 2×2 matrix with $\det A \neq 0$. Then $(U,V) = (X,Y)A + (b_1,b_2)$ is normal bivariate for any (b_1,b_2) (proved later).