Math 505A HW10 Neel Gupta

1. A coin-making machine produces quarters s.t. prob that heads comes up is random variable distributed on $\mathcal{U}(0,1)$ and let U be this variable. A coin pops out and is flipped 10 times. Let X = # of heads in 10 tosses.

Let X = # of H in 10 tosses.

U = probability of heads $\sim U(0,1)$

a) Find $P(X=j \mid U=u)$, $j=0,\dots,10$ and $P(X=j \mid U)$, $j=0,\dots,10$.

There are 10 trials with $p=U$, so $X \sim Bin(10, U)$

Then $E(X \mid U=u) = 10u$ and

$E(X \mid U) = 10U \because X \sim Bin(10, U)$

$P(X=j \mid U=u)$ is pmf of $X \sim Bin(10,u)$ when $X=j$

Pmf of binomial r.v. $\sim Bin(n,p)$ when $Bin(n,p)=k$ is $\binom{n}{k} p^k (1-p)^{n-k}$ then

$P(X=j \mid U=u) = \binom{10}{j} u^j (1-u)^{10-j}$, $j=0,\dots,10$

then $P(X=j \mid U) = \binom{10}{j} U^j (10-U)^{10-j}$, $j=0,\dots,10$

where $U \sim U(0,1)$.

b) Find $P(X=j)$, $j=0,\dots,10$, the dist. of X.

Since $P(X=j) = E(I_{\{x=j\}})$ then

$P(X=j \mid U) = E(I_{\{x=j\}} \mid U)$

Then $P(X=j) = E(P(X=j \mid U)) = \int_0^1 P(X=j \mid U=u)\,du$, $j=0,10$

Given $P(X=j \mid U=u) = \binom{10}{j} u^j (1-u)^{10-j}$

$P(X=j) = \int_0^1 \binom{10}{j} u^j (1-u)^{10-j}\,du = \binom{10}{j} \int_0^1 u^j (1-u)^{10-j}\,du$

Given $\int_0^1 u^j (1-u)^{n-j}\,du = \frac{j!(n-j)!}{(n+1)!}$ then

$P(X=j) = \binom{10}{j}\left(\frac{j!(10-j)!}{(11!)}\right) = \frac{10!}{\cancel{(10-j)!j!}} \frac{\cancel{(j!)(10-j)!}}{11!} = \frac{\cancel{10!}}{11 \cdot \cancel{10!}}$

$P(X=j) = \frac{1}{11}$

1. c) Given ~~P(X~~ $X \sim Bin(10, U)$

then $X | U = u \sim Bin(10, u)$

Find $E(X)$ and $Var(X)$.

$E(X | U = u) = 10u$ and $E(X|U) = 10U$

then $E(X) = E(E(X | U)) = E(10U)$

$= \int_0^1 10u \, du = 10 \frac{u^2}{2}\Big|_0^1 = 10(\frac{1}{2})$

$E(X) = 5$

$Var(X) = E(V(X|U)) + V(E(X|U))$

$X|U \sim Binom(10, U)$ implies $E(X|U) = 10U$ and $V(X|U) = 10(U)(1-U)$

~~$\Rightarrow E(10U) + V($~~

$V(X) \Rightarrow E(10U(1-U)) + V(10U)$

~~$E(X)$~~ $= 10 \, E(U(1-U)) + 100 \, V(U)$

$V(U) = E(U^2) - (E(U))^2$

$= \int_0^1 x^2 dx - \left(\int_0^1 x \, dx\right)^2$

$= \frac{x^3}{3}\Big|_0^1 - \left(\frac{x^2}{2}\Big|_0^1\right)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$

$V(X)$ ~~$E(X)$~~ $= 10 \int_0^1 u(1-u) du + 100\left(\frac{1}{12}\right)$

$= 10\left(\int_0^1 u \, du - \int_0^1 u^2 du\right) + \frac{100}{12}$

$= 10\left(\frac{u^2}{2}\Big|_0^1 - \frac{u^3}{3}\Big|_0^1\right) + \frac{100}{12}$

$= 10\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{100}{12}$

$= \frac{10}{6} + \frac{100}{12} = \frac{120}{12} = 10$

$V(X) = 10.$ and $E(X) = 5.$

2) Let X, Y be indep. std. normal. What is joint pdf?

Let $X \sim N$~~(μ_1, σ_1^2)~~$(0, 1)$ and $Y \sim N$~~(μ_2, σ_2^2)~~$(0, 1)$

then $f_X(x) = P(X = x) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-x^2}{2}\right\}$

and $f_Y(y) = P(Y = y) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-y^2}{2}\right\}$

2. a) Since X, Y are indep.,

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2+y^2)\right\} \cdot (1)$$

where (1) is the det $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is the jacobian of the transformation of $u(x,y) = (x,y)$.

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left\{\frac{-1}{2}(x^2+y^2)\right\}$$

Are $R^2 = X^2 + Y^2$ and $V = \frac{X}{\sqrt{X^2+Y^2}}$ indep?

Joint distribution function of R^2, V is $F_{R^2,V}(x,y)$

where x and y are the ~~values R^2 and V take on~~ input values.

$$F_{R^2,V}(x,y) = P(R^2 \le r, V \le v)$$

$$P(R^2 \le r, V \le v) = P\left(X^2+Y^2 \le r, \frac{X}{\sqrt{X^2+Y^2}} \le v\right)$$

Any combination of $X, Y \in \mathbb{R}^2$'s integral of density function over the region of $\mathbb{R}^2$ gives the probability over regions which are not all of $\mathbb{R}^2$.

$$\rightarrow F_{R^2,V}(x,y) = P\left(X^2+Y^2 \le r, \frac{X}{\sqrt{X^2+Y^2}} \le v\right)$$

$$= \iint_{\substack{x^2+y^2 \le r \\ \frac{x}{\sqrt{x^2+y^2}} \le v}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\substack{x^2+y^2 \le r \\ \frac{x}{\sqrt{x^2+y^2}} \le v}} \frac{1}{2\pi} \exp\left\{\frac{-1}{2}(x^2+y^2)\right\} dx\,dy$$

Let $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow \frac{1}{2\pi} \iint_{\substack{r^2 \le u \\ r \le \sqrt{u} \\ \cos\theta \le v}} e^{-\frac{r^2}{2}} (r)\,dr\,d\theta$$ where $r^2 \le u$ and $\frac{\cancel{r}\cos\theta}{\sqrt{r^2}} \le v$

value taken on by R^2 instead of r

2.a. $P(R^2 \le u, V \le v) = \frac{1}{2\pi} \iint_{\cos\theta \le v} \int_0^{\sqrt{u}} r e^{\frac{-r^2}{2}} \, dr \, d\theta$

When $\cos\theta \le v$, then θ can only take on values $\theta \in [0, 2\pi]$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{u}} r e^{\frac{-r^2}{2}} \, dr \, d\theta$$

Let $\underline{u} = -\frac{r^2}{2}$ (Let $\underline{u}$ be different then u)

$0 \to 0$

vsub $-1 \, d\underline{u} = -r \, dr$ and $\sqrt{u} \to \underline{u}$-sub

$\frac{(\sqrt{u})^2}{2} \to \frac{u}{2}$

$$= \frac{-1}{2\pi} \int_0^{2\pi} \int_0^{u/2} e^{\underline{u}} \, d\underline{u} \, d\theta$$

$$= \frac{-1}{2\pi} \int_0^{2\pi} e^{\underline{u}} \Big|_0^{u/2} d\theta = \frac{1}{2\pi} (1 - e^{u/2}) \int_0^{2\pi} d\theta$$

$$= \frac{\theta}{2\pi} (1 - e^{u/2})$$

Since ~~$\cos\theta \le v$ and and~~ V is a function of θ and u is the value R^2 takes on and v is the value V takes on, we see that the V dependent parts $\left(\frac{\theta}{2\pi}\right)$ and the r dependent parts multiply in the df of both R^2, V, so R^2 and V are independent.

$P(R^2 \le u, V \le v) = \frac{\theta}{2\pi}(1 - e^{u/2})$ implies R^2 and V are indep since they multiply and seperate.

b) Let X, Y, Z be independent $\sim N(0,1)$.

~~Given $Z = z$,~~ Show that $\frac{X + YZ}{\sqrt{1+Z^2}}$ and Z are indep. standard normal variables.

Joint df given by $P\left(\frac{X+YZ}{\sqrt{1+Z^2}} \le u, Z \le z\right)$

Using expectation of indicators to represent prob.

$$P\left(\frac{X+YZ}{\sqrt{1+Z^2}} \le u, Z \le z\right) = E\left[I_{\left\{\frac{X+YZ}{\sqrt{1+Z^2}} \le u\right\}} \cdot I_{\{Z \le z\}}\right]$$

Conditioning indicator on value of Z

Since $E(U g(V)) = E[E(U|V) g(V)]$

2.b. $P\left(\frac{X+YZ}{\sqrt{1+Z^2}} \le u, Z \le z\right) = E\left[E\left(I_{\left\{\frac{X+YZ}{\sqrt{1+Z^2}}\right\}} \mid Z\right) I_{\{Z \le z\}}\right]$

Given $Z = z$

$$E\left(I_{\left\{\frac{X+Yz}{\sqrt{1+z^2}}\right\}} \mid Z=z\right) = E\left(I_{\left\{\frac{1}{\sqrt{1+z^2}}X + \frac{z}{\sqrt{1+z^2}}Y \le u\right\}} \mid Z=z\right)$$

$E\left(I_{\left\{\frac{1}{\sqrt{1+z^2}}X + \frac{z}{\sqrt{1+z^2}}Y \le u\right\}} \mid Z=z\right)$ is the expectation of the ~~sum~~ indicator of std. normal, then the expectation of the indicator of std normal is given by the df of std normal variables. ~~$\Phi(u)$~~

~~$P\left(\frac{X+YZ}{\sqrt{1+Z^2}} \le u, Z \le z\right) = E\left[\Phi(u) I_{\{Z \le z\}}\right]$~~

~~because the expectation of the indicator of~~

Given $Z=z$, $\frac{X+Yz}{\sqrt{1+z^2}}$ is normally distributed by linearity of normal random variables.

$$X\left(\frac{1}{\sqrt{1+z^2}}\right) + \left(\frac{z}{\sqrt{1+z^2}}\right)Y$$

$$E\left(\frac{X+YZ}{\sqrt{1+Z^2}} \mid Z\right) = \frac{1}{\sqrt{1+z^2}} E(X \mid Z) + \frac{z}{\sqrt{1+z^2}} E(Y \mid Z)$$

$$= \frac{1}{\sqrt{1+z^2}} E(X) + \frac{z}{\sqrt{1+z^2}} E(Y)$$

Since $X \sim N(0,1)$ and $Y \sim N(0,1)$, $E(X)$ and $E(Y) = 0$

$$E\left(\frac{X+YZ}{\sqrt{1+Z^2}} \mid z\right) = 0 + 0 = 0$$

Then $V\left(\frac{X+YZ}{\sqrt{1+Z^2}} \mid z\right) = E\left[\left(\frac{X+YZ}{\sqrt{1+Z^2}}\right)^2 \mid Z\right] - (0)^2$

$$E\left[\left(\frac{X+YZ}{\sqrt{1+Z^2}}\right)^2 \mid z\right] = \frac{1}{(1+z^2)}\left(E(X^2 \mid Z) + 2E(X \mid Z)E(Y \mid Z) + z^2 E(Y^2 \mid Z)\right)$$

$$= \frac{1}{1+z^2}\left(E(X^2) + 2(0)(0) + z^2 E(Y^2)\right)$$

since $E(Y) = 0 = E(X)$

then $E(X^2) = 1 = E(Y^2)$

$$\mathrm{Var}\left(\frac{X+YZ}{\sqrt{1+Z^2}} \mid z\right) = \cancel{\frac{1}{1+z^2}}\left(\cancel{1+z^2}(+1)\right) = 1$$

$$\frac{X+YZ}{\sqrt{1+Z^2}} \mid Z \sim N(0,1)$$

2. b) Since $\frac{X+YZ}{\sqrt{1+Z^2}} \mid Z \sim N(0,1)$

$$P\left(\frac{X+YZ}{\sqrt{1+Z^2}} \le u, Z \le u\right) = E\left[E\left(I_{\left\{\frac{X+YZ}{\sqrt{1+Z^2}} \le u\right\}} \mid Z\right) I_{\{Z \le z\}}\right)$$
$$= E[\Phi(u) I_{\{Z \le z\}}]$$
$$= \Phi(u) E(I_{\{Z \le z\}}) = \Phi(u) P(Z \le z)$$
$$= \Phi(u)\Phi(z)$$

Since joint df of r.v.'s is multiplicative and seperated, the r.v.'s are independent $\square$.

3. a) Let (X_1, X_2) be normal bivariate w/ $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$, and $\rho = 3/5$. Let (Y_1, Y_2) be midterm and final exam scores. Assume $Y_1 = 80 + 3X_1$ and $Y_2 = 75 + 2X_2$

$$E(Y_1) = 80 + 3E(X_1) = 80$$
$$E(Y_2) = 75 + 2E(X_2) = 75$$

$$Var(Y_1) = 0 + 3^2 V(X_1) = 9 \quad \because V(X_1) = 1$$
$$Var(Y_2) = 0 + 2^2 V(X_2) = 4 \quad \because V(X_2) = 1$$

$$Cov(Y_1, Y_2) = \cancel{\sigma_1^2 \sigma_2^2} \sqrt{V(Y_1)} \cdot \sqrt{V(Y_2)} \cdot Cov(X_1, X_2)$$
$$= (3)(2) \cancel{\rho \quad 6} \tfrac{3}{5} \times Cov(X_1, X_2)$$
$$= 6(\rho_{X_1 X_2}) = 18/5$$

~~Cov(X_1, X_2)~~

Now, $\rho_{Y_1 Y_2} = \frac{Cov(Y_1, Y_2)}{\sqrt{Var(Y_1) \cdot Var(Y_2)}} = \frac{18/5}{\sqrt{9 \cdot 4}} = \frac{18/5}{6}$

$$= 3/5$$

$$\rho_{Y_1, Y_2} = 3/5$$

Now

$$E(Y_2 \mid Y_1 = 90) = E(Y_2) + \rho_{Y_1 Y_2} \frac{V(Y_2)}{V(Y_1)} [90 - E(Y_1)]$$
$$= \cancel{80}\,75 + \frac{\cancel{3}\,1}{5} \cdot \frac{4}{\cancel{9}\,3}(90 - 80)$$
$$= 75 + \frac{\cancel{40}\,8}{\cancel{15}\,3} = 75 + 8/3 = \boxed{77.667}$$

$$V(Y_2 \mid Y_1 = 90) = (1 - \rho_{Y_1 Y_2}^2)(V(Y_2))$$
$$= \left(1 - \left(\tfrac{3}{5}\right)^2\right)(4)$$
$$= \left(\tfrac{16}{25}\right)(4) = \boxed{2.56}$$

3. b) Given $Y_2 \mid Y_1 = 90$ is normally distributed

with mean = 77.667 and variance = 2.56

$$P(Y_2 > 75 \mid Y_1 = 90) = 1 - \phi\left(\frac{75 - 77.66}{\sqrt{2.56}}\right)$$

$$= 1 - \phi\left(\frac{-2.667}{\sqrt{2.56} = 1.6}\right)$$

$$= 1 - \phi(-1.667)$$

$$= \phi(1.667)$$

$$P(Y_2 > 75 \mid Y_1 = 90) = \phi(1.67) = \boxed{0.9525}$$

4. Let X, Y be normal bivariate r.v. w/ $\mu_1 = \mu_2 = 0$ and $\sigma_1^2, \sigma_2^2, \rho$.

a) What are $E(X|Y)$ and $V(X|Y)$?

~~E(X|Y) and~~ given $Y = y$, $X = \rho \frac{\sigma_1}{\sigma_2} y + V$, $V \sim N(0, \sigma_1^2(1-\rho^2))$ V is indep of X, Y

$$E(X \mid Y = y) = \rho \frac{\sigma_1}{\sigma_2} y + E(V)$$

$$= \rho \frac{\sigma_1}{\sigma_2} y + 0 \Rightarrow E(X|Y) = \rho \frac{\sigma_1}{\sigma_2} Y$$

$$Var(X|Y) = 0 + Var(V) = \sigma_1^2(1-\rho^2)$$

b) Now $z = X + Y$, find $E(X \mid X+Y = z)$ and $V(X \mid X+Y=z)$

$$X + Y \sim N(0, \sigma_1^2 + \sigma_2^2)$$

$(X, X+Y) = (X, Y) A + b$, where A is a matrix & b is a ~~vector~~ tensor

$$(X, X+Y) = (X, Y)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0$$

$\det A = 1 - 0 = 1 \neq 0 \rightarrow (X, X+Y)$ is normal bivariate.

$$E(X \mid X + Y = z) = \mu_x + \rho \frac{\sigma_x}{\sigma_{x+y}}((X+Y) - \mu_{X+Y})$$

$$= 0 + \rho \frac{\sigma_x}{\sigma_{x+y}}(z)$$

$$V(X+Y) = V(X) + V(Y) + 2Cov(X, Y)$$

$$= \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$$

We don't know ρ between (X) and (X+Y) : ρ_{X+Y}

$$\rho_{X+Y} = \frac{Cov(X, X+Y)}{\sigma_1 \sigma_{x+y}} = \frac{Cov(X,X) + Cov(X,Y)}{\left(\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\right)\sigma_1}$$

$$= \frac{V(X) + Cov(X,Y)}{\sigma_1\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} = \frac{\sigma_1^2 + \rho\sigma_1\sigma_2}{\sigma_1\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}$$

4. b) $\rho_{X+Y} = \dfrac{\sigma_1^2 + \rho\sigma_1\sigma_2}{\sigma_1\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}$

$\rightarrow E(X \mid X+Y=z) = \rho_{X+Y} \dfrac{\sigma_X}{\sigma_{X+Y}} z$

$= \dfrac{\sigma_1^2+\rho\sigma_1\sigma_2}{\cancel{\sigma_1}\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}} \cdot \dfrac{\cancel{\sigma_1}}{\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}} (z)$

$= \dfrac{\sigma_1^2+\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2} (z)$

$V(X \mid X+Y=z) = \sigma_1^2(1-\rho_{X+Y}^2)$

$= \sigma_1^2\left(1-\left(\dfrac{\sigma_1^2+\rho\sigma_1\sigma_2}{\sigma_1\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}\right)^2\right)$

$= \cancel{\sigma_1^2}\left(\dfrac{(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)\sigma_1^2 - ((\sigma_1^2)+\rho\sigma_1\sigma_2)^2}{\cancel{\sigma_1^2}(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)}\right)$

$\underline{\text{Numerator}}$

$(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)\sigma_1^2 - ((\sigma_1^2)^2+2\rho\sigma_1^3\sigma_2+\rho^2\sigma_1^2\sigma_2^2)$

$= \cancel{\sigma_1^4} + \sigma_1^2\sigma_2^2 + \cancel{2\rho\sigma_1^3\sigma_2} - \cancel{\sigma_1^4} - \cancel{2\rho\sigma_1^3\sigma_2} - \rho^2\sigma_1^2\sigma_2^2$

$= \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$

$V(X \mid X+Y=z) = \boxed{\dfrac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}$ ■

5. Let $Y = X + \varepsilon Z$, where $X \sim N(0, \sigma_1^2)$, $Z \sim N(0,1)$ are indep. Find $\hat{X} = E(X \mid Y)$.

First check whether X, Y are norm. bivar.

$(X, X+\varepsilon Z) = (X, Z)A + b$, $A \in \mathbb{R}^2$ & b is tensor

$= (X,Z)\begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix}$ $\det A = \varepsilon \neq 0$

which means X, Y are norm. bivariate

By linearity, $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1^2 = \sigma_1^2$, $\sigma_2^2 = \varepsilon^2 + \sigma_1^2$

$\rho = \dfrac{Cov(X,Y)}{(\sigma_1)(\sqrt{\varepsilon^2+\sigma_1^2})} = \dfrac{Cov(X, X+\varepsilon Z)}{\sigma_1\sqrt{\varepsilon^2+\sigma_1^2}} = \cancel{Var(X)}$

$= \dfrac{Cov(X,X) + Cov(X, \varepsilon Z)}{\sigma_1\sqrt{\varepsilon^2+\sigma_1^2}}$

5. $$\rho = \frac{Cov(X,X) + Cov(X, \mathcal{E}Z)}{\sigma_1 \sqrt{\mathcal{E}^2 + \sigma_1^2}}$$

$$= \frac{V(X) + (\mathcal{E})(Cov(X,Z))}{\sigma_1 \sqrt{\mathcal{E}^2 + \sigma_1^2}} \quad \because Cov(X,Z) = 0 \ \because \text{indep.}$$

$$= \frac{V(X)}{\sigma_1 \sqrt{\mathcal{E}^2+\sigma_1^2}} = \frac{\sigma_1^2}{\cancel{\sigma_1}\sqrt{\mathcal{E}^2+\sigma_1^2}} = \frac{\sigma_1}{\sqrt{\mathcal{E}^2+\sigma_1^2}}$$

$$E(X/Y=y) = \rho \frac{\sigma_X}{\sigma_Y}(y)$$

$$= \frac{\sigma_1}{\sqrt{\mathcal{E}^2+\sigma_1^2}} \cdot \frac{\sigma_1 (y)}{\sqrt{\mathcal{E}^2+\sigma_1^2}} \quad \because Var(Y) = \mathcal{E}^2+\sigma_1^2$$

$$= \frac{\sigma_1^2}{\mathcal{E}^2+\sigma_1^2}(y)$$

$$E(X|Y) = \frac{\sigma_1^2}{\mathcal{E}^2+\sigma_1^2}(Y) = \hat{X}$$

$X - \hat{X}$ is distributed normal bivariately by lin comb of 2 ~~norm b~~ ~~norm bivar~~ normal variables.

$$E((X-\hat{X})^2) = V(X-\hat{X})$$

$$V(X-\hat{X}) = V(X) + V(\hat{X}) - 2\,cov(X,\hat{X})$$

Since $\hat{X} = E(X|Y) = \underbrace{\frac{\sigma_1^2}{\mathcal{E}^2+\sigma_1^2}}_{\text{a constant} := k}(Y)$

$$Cov(X,\hat{X}) = cov(X, kY)$$

$$Var(X-\hat{X}) = Var(X) + Var(\hat{X}) - 2k\,Cov(X,Y)$$

$$\sigma_1^2 \quad = V(E(X|Y)+0)$$

$$= k^2 V(Y)$$

$$Cov(X,Y) = \rho\, \sigma_x \sigma_y = \frac{\sigma_1}{\mathcal{E}^2+\sigma_1^2} \times \sigma_1 \times \sqrt{\mathcal{E}^2+\sigma_1^2}$$

$$V(X-\hat{X}) = \sigma_1^2 + \sigma_1^2 - 2\frac{\sigma_1^2}{\sigma_1^2+\mathcal{E}^2} = \boxed{\frac{2\sigma_1^2\mathcal{E}^2}{\sigma^2+\mathcal{E}^2}}$$