This week. 2.5 (random vectors)

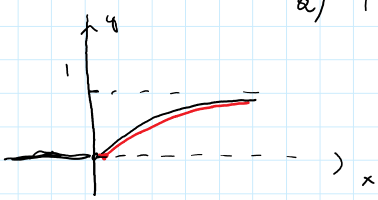
3.1, 3.5 (examples of discrete r.v.)

3.2 Independence; 3.3 Expectation.

Ex 1. Let X be exponential (λ) - r.v.

Its pdf is $f(x) = \lambda e^{-\lambda x}$, $x > 0$;

a) Find $P(X > x)$, $x > 0$, and df.



Answer. $P(X > x) = \lambda \int_x^\infty e^{-\lambda t} dt = e^{-\lambda t} \Big|_x^\infty = e^{-\lambda x}$, $x > 0$.

$F(x) = P(X \le x) = 1 - e^{-\lambda x}$, $x > 0$.

b) Find the inverse.

Answer. Given $0 < v < 1$, solve the equation $v = F(x)$ for x:

$v = 1 - e^{-\lambda x}$, $e^{-\lambda x} = 1 - v$, $-\lambda x = \ln(1-v)$, $x = -\frac{\ln(1-v)}{\lambda}$

So, $F^{-1}(v) = -\frac{\ln(1-v)}{\lambda}$, $0 < v < 1$

$F^{-1}(0) = 0$

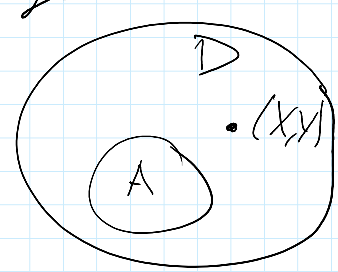
2.5. Random vectors

Often we need to regard r.v.'s $X_1, \dots, X_d$ as components of a random vector

$X = (X_1, \dots, X_d)$

Example 1. A dart is flung at a target $D \subset \mathbb{R}^2$.

Let (X, Y) be coordinates of the hit.



Assume any point in D can be hit "equally likely". In this case,

$P((X,Y) \in A) = \frac{|A|}{|D|}$, $|A|$ is area of $A \subset D$,

$|D|$

$A, D \in \mathcal{B}$, σ-field of all $A \subset D$ that have area.

Note $\frac{P((X,Y) \in A)}{|A|} = \frac{1}{|D|}$ for all $A \in \mathcal{B}$:

$\frac{1}{|D|}$ is probability density.

Def. We say (X, Y) is uniform r. vector in $D \subset \mathbb{R}^2$.

Example 2. A dice with $P(1) = p_1, \dots, P(6) = p_6$, $p_1 + \dots + p_6 = 1$, is rolled n times. Consider $X = (X_1, \dots, X_6)$, where X_i is the number of i's in n rolls. X is called multinomial r. vector:

a) $$\boxed{\begin{gathered} P(X_1 = k_1, \dots, X_6 = k_6) = \binom{n}{k_1, k_2, \dots, k_6} p_1^{k_1} \cdots p_6^{k_6} \\ k_1 + \dots + k_6 = n \end{gathered}}$$

b) $X_1 + \dots + X_6 = n$

c) If $p_1 = \dots = p_6 = 1/6$, then $p_1^{k_1} \cdots p_6^{k_6} = \left(\frac{1}{6}\right)^n = \frac{1}{6^n} = 6^{-n}$.

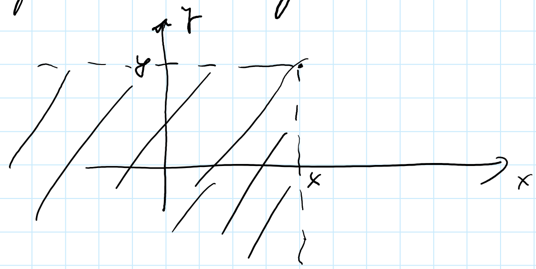
Def. A r. vector (or r.v.) X taking values in a finite set $\{a_1, \dots, a_N\}$ is called uniform if $P(X = a_1) = \dots = P(X = a_N) = \frac{1}{N}$.

Def. Joint df of $X = (X_1, \dots, X_d)$ is the function
$F(x_1, \dots, x_d) = P(X_1 \le x_1, \dots, X_d \le x_d)$, $-\infty < x_1, \dots, x_d < \infty$.

If $d=2$, $V=(X,Y)$, the df

$$F(x,y) = P(X \le x, Y \le y) = P((X,Y) \in (-\infty, x] \times (-\infty, y])$$

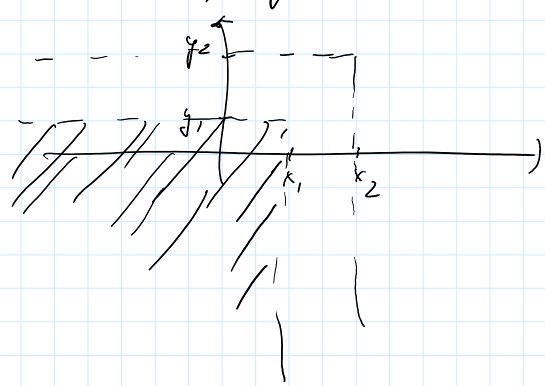
is the probability that (X,Y) falls into shadowed region:



<u>Properties of $F(x,y)$</u>

1. $0 \le F(x,y) \le 1$; If $x_1 \le x_2$, $y_1 \le y_2$, then

$F(x_1,y_1) \le F(x_2,y_2)$



2. $\lim\limits_{\substack{x\to\infty \\ y\to\infty}} F(x,y) = 1$,

$\lim\limits_{\substack{x\to-\infty \\ \text{or } y\to-\infty}} F(x,y) = 0$

3. $\lim\limits_{u,v \downarrow 0} F(x+u, y+v) = F(x,y)$.

<u>Remark 1.</u> $F(x,y)$ contains everything about probabilities related to (X,Y) including marginal df's:

$$\boxed{F_X(x) = \lim_{y\to\infty} F(x,y), \quad F_Y(y) = \lim_{x\to\infty} F(x,y)}$$ because

$\{X \le x\} = \{X \le x, Y < \infty\} = \bigcup\limits_{n=1}^{\infty} \{X \le x, Y \le n\}$ implies
$\uparrow$ increasing

$$F_X(x) = P(X \le x) = \lim_{n\to\infty} F(x,n) = \lim_{y\to\infty} F(x,y).$$

[illegible]

Jointly cont. r. vectors

Def. (X,Y) is a continuous r. vector if there is $f \geq 0$ so that its df

$$F(x,y) = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f(u,v)\, dv \right) du, \quad -\infty < x,y < \infty.$$

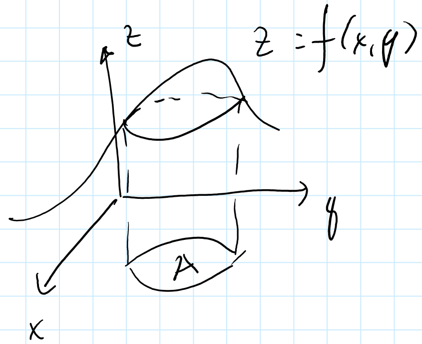
$f(x,y)$ is called joint pdf of (X,Y).

Note if (X,Y) is continuous, then

(i) $F(x,y)$ is continuous;

(ii) joint pdf

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$



(iii) For $A \subset \mathbb{R}^2$,

$$P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy, \quad A \in \mathcal{B}.$$

Geometrically, it is the volume under $z = f(x,y)$ above A.

Ex 1. Let (X,Y) be continuous. Find

a) $P(X=Y) = 0$; b) $P(Y = h(X)) = 0$

Answer. a) $P(X=Y) = P((X,Y) \in \Delta) = \iint_\Delta f(x,y)\, dx\, dy = 0$

$\Delta = \{(x,y) : x = y\}$ is the line: there is no volume above Δ.

b) Similarly, $A = \{ (x,y) : y = h(x) \}$ is a curve... .

Also, $P((X,Y) = (a,b)) = 0$.

Ex 2. Let (X,Y) be continuous with joint cdf $F(x,y)$ and joint pdf $f(x,y)$.

Then X is continuous with

$$F_X(x) = \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f(u,y)\,dy\right)du, \quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

Answer. For any x, y

$$F(x,y) = \int_{-\infty}^{x}\left(\int_{-\infty}^{y} f(u,v)\,dv\right)du$$

We found that $F_X(x) = \lim_{y\to\infty} F(x,y) = \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f(u,v)\,dv\right)du$,

$$f_X(x) = F_X'(x) = \int_{-\infty}^{\infty} f(x,v)\,dv, \quad -\infty < x < \infty$$

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx, \quad -\infty < y < \infty.$