Ex 5. Let $X \sim N(0, \sigma^2)$, $Y \sim N(aX, \tilde{\sigma}^2) \leftarrow$ given $X = x$, $Y \sim N(ax, \tilde{\sigma}^2)$.

a) Find $E(e^{tY} | X)$, $Cov(X, Y)$.

Answer. $E(e^{tY}|X) = e^{taX} e^{\tilde{\sigma}^2 t^2/2}$

$Cov(X, Y) = a\sigma^2$

b) Find joint mgf of (X, Y), and $M_Y(t)$

Answer. $M(s,t) = E(e^{sX+tY}) = E[e^{sX} E(e^{tY}|X)]$

$= e^{-\tilde{\sigma}^2 t^2/2} E(e^{(s+ta)X}) = e^{\tilde{\sigma}^2 t^2/2} e^{(s+ta)^2 \sigma^2/2} =$

$= \exp\{ \frac{1}{2} [t^2(\tilde{\sigma}^2 + a^2\sigma^2) + 2st a\sigma^2 + s^2\sigma^2] \}$

$M_Y(t) = M(0, t) = e^{(\tilde{\sigma}^2 + a^2\sigma^2)t^2/2}$: $Y \sim N(0, \tilde{\sigma}^2 + a^2\sigma^2)$

c) Is X, Y normal bivariate?

Answer. Covariance matrix $B = \begin{pmatrix} \sigma^2 & a\sigma^2 \\ a\sigma^2 & \tilde{\sigma}^2 + a^2\sigma^2 \end{pmatrix}$ Yes

$\det B = \sigma^2(\tilde{\sigma}^2 + a^2\sigma^2) - a^2\sigma^4 = \sigma^2\tilde{\sigma}^2 > 0$.

Ex 6. Let X, Y be indep. uniform in $(0,1)$.

a) Find pdf of $L = XY$

Answer. Range of $L = (0,1)$ (X, Y) is uniform in $[0,1]\times[0,1]$

$y = z/x$ For $z \in (0, 1)$

Answer: Range of L: $(0,1)$ (X,Y) is uniform in $[0,1]\times[0,1]$

$y = 1 = \frac{z}{x}$, $\underline{x = z}$

For $z \in (0,1)$,

$$P(L \le z) = P(XY \le z) = P\left(Y \le \frac{z}{X}\right)$$
$$= |A| = z\cdot 1 + \int_z^1 \frac{z}{x}\,dx$$
$$= z + z\ln x\Big|_z^1 = \underline{z - z\ln z}$$

$$f_L(z) = \begin{cases} 1 - 1 - \ln z = \ln z, & 0 < z < 1 \\ 0 & \text{otherwise.} \end{cases}$$

b) Find joint pdf of $(X, X+Y)$

Answer. $U = X$, $V = X+Y$: (U,V) is a function of (X,Y)

$\begin{cases} U = X \\ V = X+Y \end{cases}$ | $\begin{cases} u = x \\ v = x+y \end{cases}$ defines 1-to-1 mapping of $\mathbb{R}^2$

$\begin{cases} x = u \\ y = v - x = v - u \end{cases}$ | $J(u,v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$

$S = \{(u,v) : 0 < u < 1,\ 0 < v - u < 1\} = \{(u,v) : 0 < u < 1,\ u < v < 1+u\}$

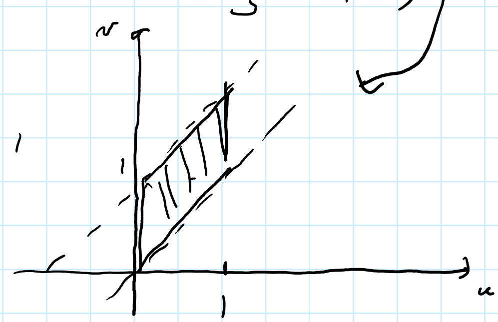
$\doteq$ shadowed region

Joint pdf of (U,V) is

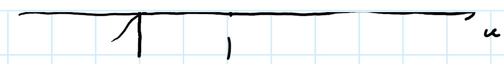
$g(u,v) = f(x(u,v), y(u,v))\,|J(u,v)|\,1_S(u,v) = 1\cdot 1\cdot I_S(u,v)$

$= 1$ if $0 < u < 1$, $u < v < 1+u$,

0 otherwise



Relabeling, pdf of X, $V = X+Y$ is



Relabeling, pdf of X, $V = X+Y$ is

$f(x,v) = 1$ if $0<x<1$, $x<v<1+x$

c) Find $f(v|x)$, $E(V|X=x)$, $E(V|X)$

Answer. $f(v|x) = \dfrac{f(x,v)}{f_X(x)} = \dfrac{1}{1} = 1$, $\underline{x<v<1+x}$, $0<x<1$.

Given $X=x$, V is uniform in $(x, 1+x)$: $E(V|X) = X+\frac{1}{2}$.

Remark. Given $V=v$, X is uniform in $\begin{cases} (0,v), & 0<v<1 \\ (v-1,1), & 1<v<2 \end{cases}$

Ex 7. There are m men and n women in a group. Let W = # of man-woman pairs with the same birthday. Find $E(W)$, $Var(W)$.

Answer. A_{ij} = "i-th man and j-th woman have the same birthday."

$$W = \sum_{i=1}^{m}\sum_{j=1}^{n} 1_{ij} \quad \Big| \quad E(W) = \sum_{i,j} P(A_{ij}) = \sum_{i,j} \frac{365}{365^2} = \frac{m\cdot n}{365}$$

$$Var(W) = \sum_{i,j} Var(1_{A_{ij}}) + \sum_{(i,j)\neq(k,l)} Cov(1_{A_{ij}}, 1_{A_{kl}})$$

For $(i,j)\neq(k,l)$,

$$Cov(1_{A_{ij}}, 1_{A_{kl}}) = P(A_{ij}\cap A_{kl}) - \left(\frac{1}{365}\right)^2 = 0.$$

$$P(A_{ij}\cap A_{kl}) = \frac{365\cdot 365}{365^4} = \left(\frac{1}{365}\right)^2$$

Case 1: $i \neq k,\ j \neq l$: $P(A_{ij} \cap A_{kl}) = 365^{\cdot}$

Case 2: $i = k,\ j \neq l$: $\frac{365}{(365)^3} = \left(\frac{1}{365}\right)^2$

Case 3: $i \neq k,\ j = l$: $\left(\frac{1}{365}\right)^2$

$$Var(W) = \sum_{i,j} \left(\frac{1}{365} - \frac{1}{365^2}\right) = mn \cdot \frac{364}{365^2}.$$

Comment. A_{ij}, A_{kl} are independent for any $(i,j) \neq (k,l)$.
but W is not binomial.

Ex 8. Number N of cars arriving at window per day is Poisson(λ). The numbers X_i of passengers in the cars are independent binomial ($n = 4$, $p = \frac{1}{4}$).
Find mgf of the total number Y of passengers in a given day.

Answer. $Y = \sum_{i=1}^{N} X_i$ $\qquad G_X(s) = (ps+q)^n$

$$G_N(s) = \exp\{\lambda(s-1)\}$$

$$G_Y(s) = G_N(G_X(s)) = \exp\{\lambda((ps+q)^n - 1)\}.$$

$$M_Y(t) = G_Y(e^t) = \exp\{\lambda((pe^t+q)^n - 1)\}.$$

Ex 9. Coin with $P(H) = p$ is tossed repeatedly. Let X_n be number of H in n tosses.
Express $p_n = P(X_n \text{ is odd})$ in terms of p_{n-1}.

Answer. H_1 = "H in 1st toss", $T_1 = H_1^c$.

$p_n = P(X_n \text{ is odd}) = P(X_n \text{ is odd} \mid H_1)\, P(H_1) +$

$+ P(X_n \text{ is odd} / T_1)\, P(T_1) = p \cdot (1 - p_{n-1}) + (1-p)\, p_{n-1}.$