5.1 More on g.f.

Def. For X taking values in $\{0,1,2,\dots\}$,

$$G(s) = G_X(s) = E(s^X) = \sum_{n=0}^{\infty} s^n p_n, \quad p_n = P(X=n), \ n \geq 0.$$

Recall $G(0) = P(X=0)$, $G(1) = 1$.

Thm 1. Let X_i be i.i.d. with common $G_X(s)$. Let N be $\{0,1,2,\dots\}$-valued indep. of X_i, and $Y = \sum_{i=1}^{N} X_i$.

Then $G_Y(s) = G_N(G_X(s)) = G_N \circ G_X(s)$.

Proof. Given $N=n$, $Y = \sum_1^n X_i$ (X_i are i.i.d.):

$E(s^Y \mid N=n) = G_X(s)^n$, $E(s^Y \mid N) = G_X(s)^N$. Hence

$$G_Y(s) = E(s^Y) = E\left[G_X(s)^N\right] = G_N(G_X(s)).$$

5.4. Branching processes

Consider evolution in time of a population.

1 time unit = 1 generation. Denote Z_n = n-th generation size.

Assume $Z_0 = 1$. If X_k = # of children of k-th individual in generation $n-1$, then

$$Z_n = \sum_{k=1}^{Z_{n-1}} X_k,$$

X_k are i.i.d. with $E(X_k) = \mu$, $Var(X_k) = \sigma^2$ and common $G(s) = G_X(s)$.

Questions of interest:

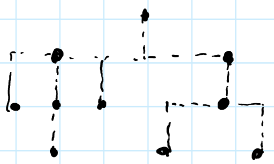
1. $E(Z_n)$, $Var(Z_n) = ?$
2. What is probability of extinction?

Note (i) $P(\text{extinction}) = \lim_{n\to\infty} P(Z_n = 0)$ because

$$\text{"extinction"} = \bigcup_{n=1}^{\infty} \{Z_n = 0\} \leftarrow \text{increasing.}$$

(ii) $P(Z_n = 0) = G_n(0)$, where $G_n(s) = G_{Z_n}(s)$

$G_1(s) = G_X(s)$.



$Z_0 = 1$
$Z_1 = 2$
$Z_3 = 4$ $\leftarrow$ branching process.
$Z_4 = 3$

Thm 2. a) $G_n(s) = G_{n-1}(G(s)) = G_{n-1}\circ G(s)$, $n \geq 2$.

b) $G_{n+m}(s) = G_n(G_m(s)) = G_n \circ G_m(s)$, $n, m \geq 1$.

Proof. a) By Thm 1, $G_n(s) = G_{n-1}(G(s)) =$

$= \dots = \underbrace{G\circ G\circ\dots\circ G}_{n \text{ times}}(s)$. For instance $G_2(s) = G_1(G(s)) =$
$= G(G(s)) = G\circ G(s)$.

b) $G_{n+m}(s) = \underbrace{G\circ\dots\circ G}_{n}\circ\underbrace{G\circ\dots\circ G}_{m} = G_n(G_m(s))$.

Thm 3. If $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, then

a) $E(Z_n) = \mu^n$, $Var(Z_n) = \begin{cases} n\sigma^2, & \mu = 1 \\ \sigma^2 \dfrac{\mu^{n-1} - \mu^{2n-1}}{1-\mu}, & \mu \neq 1. \end{cases}$

b) $\lim_{n\to\infty} E(Z_n) = \begin{cases} 0, & \mu < 1 \\ 1, & \mu = 1 \\ \infty, & \mu > 1 \end{cases}$ | $\lim_{n\to\infty} Var(Z_n) = \begin{cases} +\infty, & \mu \geq 1 \\ 0, & \mu < 1. \end{cases}$

Proof. Given $Z_{n-1} = k$, $Z_n = \sum_{j=1}^{k} X_j$:

$E(Z_n \mid Z_{n-1} = k) = k\mu$, $Var(Z_n \mid Z_{n-1} = k) = k\sigma^2$. Then

$E(Z_n \mid Z_{n-1}) = \mu Z_{n-1}$, $Var(Z_n \mid Z_{n-1}) = \sigma^2 Z_{n-1}$, and

$\mu_n := E(Z_n) = \mu E(Z_{n-1}) = \mu\cdot\mu_{n-1} = \dots = \mu^n$.

$Var(Z_n) = E(\sigma^2 Z_{n-1}) + \mu^2 Var(Z_{n-1}) \overset{\mu=1}{=} \sigma^2 + Var(Z_{n-1}) = \dots$

$\mathrm{Var}(Z_n) = E(\sigma^2 Z_{n-1}) + \mu^2 \mathrm{Var}(Z_{n-1}) \overset{\mu=1}{=} \sigma^2 + \mathrm{Var}(Z_{n-1}) = \dots$

$= \dots \; n\sigma^2, \; \dots$

<u>Thm 4</u>. Let $\eta_n = P(Z_n = 0)$. Then
$\eta = \lim_{n\to\infty} \eta_n = P(\text{extinction})$ is the smallest solution to
eqn $\boxed{\eta = G(\eta)}$, where $G = G_X$.

<u>Proof</u>. By Thm 2., $\eta_n = G_n(0) = G(G_{n-1}(0)) = G(\eta_{n-1})$.
Taking limit as $n \to \infty$, we get $\boxed{\eta = G(\eta)}$.

<u>Geometric branching</u> : Assume family size X has pmf
$P(X = n) = p^n q, \quad n = 0, 1, 2, \dots \quad q = 1 - p$

<u>Note</u> (i) $P(X = 0) = q \quad (= P(\text{no children}))$
$P(X \geq 1) = p \quad (= P(\geq 1 \text{ child}))$

(ii) $V = X + 1$ is geometric(q), $X = V - 1$,
$G(s) = \dfrac{q}{1 - ps}$.

Then $\eta = P(\text{extinction})$ is the smallest root of $s = G(s)$:

$s = \dfrac{q}{1 - ps}$, $\underbrace{ps^2 - s + q = 0}$:

$$\eta = \begin{cases} 1, & p \leq q \quad (\text{includes } p = q = \tfrac{1}{2}) \\ \dfrac{q}{p}, & q < p \end{cases}$$

5.7-9 Characteristic function

Recall 1. gf : $G(s) = E(s^X)$, $X \in \{0, 1, 2, \dots\}$

2. mgf : $M_X(t) = E(e^{tX})$, $-\varepsilon < t < \varepsilon$.

C... ... M... DNE: For instance,

M_X DNE.

1. X is Cauchy (has pdf $= \frac{1}{\pi} \frac{1}{1+x^2}$, $-\infty < x < \infty$.

2. $P(X = k) = \frac{3}{\pi^2} \frac{1}{k^2}$, $k = \pm 1, \pm 2, \ldots$

Def. Characteristic function of X is the function $\phi_X(t) = E(e^{itX})$, $-\infty < t < \infty$, where $i^2 = -1$, $e^{ia} = \cos(a) + i \sin(a)$.

Note $E(e^{itX}) = E[\cos(tX)] + i E[\sin(tX)]$ is well defined for all $-\infty < t < \infty$.