Examples of discrete r.v.

3. X is Poisson(λ): it models

(i) number of earthquakes
(ii) number of customer arrivals } in a given time interval.
(iii) number of typos on a printed page.
...

$$\boxed{\begin{gathered}\text{Range of } X = \{0,1,2,\dots\} \\ P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!},\quad k=0,1,\dots\end{gathered}}$$

Note for $\lambda = 1$, $P(X \geq 6) = \sum_{k=6}^{\infty} P(X=k) = 0.0006$

Where Poisson(λ) comes from?

Poisson X arises as approximation of binomial(n,p) - r.v. Y, when n is "large", p is "small", $\lambda = n\cdot p$ is "moderate":

$$(2)\quad P(Y=k) = \binom{n}{k} p^k q^{n-k} \approx e^{-\lambda}\frac{\lambda^k}{k!} \text{ with } \lambda = n\cdot p,\ k=0,1,\dots$$

It can be shown,

$$\boxed{|P(X\in A) - P(Y\in A)| \leq np\cdot p = np^2}$$

Why (2)? $P(Y=k) = \frac{n!}{(n-k)!}\cdot\frac{1}{k!}\cdot p^k \frac{(1-p)^n}{\underbrace{(1-p)^k}_{\approx 1}} = \frac{n\cdot(n-1)\dots(n-k+1)}{n^k}\cdot\frac{1}{k!}(np)^k\left(1-\frac{np}{n}\right)^n$

$\underset{\lambda=n\cdot p}{\approx} \frac{\lambda^k}{k!}e^{-\lambda}$ because $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$, $\frac{n\dots(n-k+1)}{n^k}\approx 1$ for large n.

Ex 1. Number N of fish J catches per day is Poisson (λ). J comes home, counts his fish and tosses that many times coin with $P(H)=p$, $P(T)=q=1-p$.

Y comes home, counts no fish ...

a coin with $P(H) = p$, $P(T) = q = 1-p$.

Let X be number of H's, Y be the number of T's in N tosses.

a) Find joint pmf of (X, Y).

Comments. 1. Range of X = Range of Y = $\{0, 1, 2, \dots\}$

2. $X+Y = N$, $Y = N - X$.

Answer. Joint pmf

$$f(x,y) = P(X=x, Y=y) = P(X=x, Y=y, N=x+y) = P(X=x, N=x+y) =$$

$$= \underbrace{P(X=x \mid N=x+y)}_{\text{binomial}(x+y,\, p)} P(N=x+y) = \binom{x+y}{x} p^x q^y \cdot e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^x}{x!} e^{-\lambda q} \frac{(\lambda q)^y}{y!}, \quad x, y = 0, 1, 2, \dots$$

b) Find marginal pmf's. Are X, Y independent?

Answer. $f_X(x) = \sum_y f(x,y) = e^{-\lambda p} \frac{(\lambda p)^x}{x!} \underbrace{\sum_y e^{-\lambda q} \frac{(\lambda q)^y}{y!}}_{=1} = e^{-\lambda p} \frac{(\lambda p)^x}{x!}$,

$x = 0, 1, 2, \dots$: X is Poisson(λp).

Similarly $f_Y(y)$ is Poisson(λq), $f(x,y) = f_X(x) f_Y(y)$:

X, Y are independent.

3.3 Expectation (mean value, average of X)

Let X be discrete r.v.

Def. Expectation of X is the number

$$E(X) = \sum_x x P(X=x) = \sum_x x f_X(x) \quad \text{assuming} \quad \sum_x |x| P(X=x) < \infty.$$

It is weighted sum of possible values of X;

$P(X=x)$ is the weight of x.

Note 1. If $X \geq 0$, then $E(X) \geq 0$

2. $E(c) = c$, here c is any constant.

Meaning of $E(X)$

Example. Let X be daily net gain of G:

$P(X=1) = 0.59 = p_1$; $P(X=-1) = 0.26 = p_2$, $P(X=-3) = 0.15 = p_3$:

$p_1+p_2+p_3=1$

G asks question: what is my average gain per day if I play n days?

Answer. Let X_i = net gain in ith day, $i=1,\dots,n$

Average gain per day is

$$\frac{X_1+\dots+X_n}{n} = (1)\underbrace{\frac{(\#\text{ of days \$1 won})}{n}}_{\approx p_1} + (-1)\underbrace{\frac{(\#\text{ of days \$1 lost})}{n}}_{\approx p_2} + (-3)\underbrace{\frac{(\#\dots\text{\$3 lost})}{n}}_{\approx p_3}$$

Bernoulli thm

$$\approx (1)\cdot p_1 + (-1)p_2 + (-3)p_3 =: E(X) = -0.11 \text{ (dollars)}.$$

Expectation of a function of r.v.

$$E\,g(X) = \sum_x g(x)\,P(X=x) = \sum_x g(x) f_X(x)$$

$$E\,g(X,Y) = \sum_x g(x,y)\,P(X=x, Y=y) = \sum_{x,y} g(x,y) f_{X,Y}(x,y).$$

Properties of $E(X)$

1. $E(X+Y) = E(X)+E(Y)$, $E(X-Y) = E(X)-E(Y)$.

   $E(aX+bY) = a\,E(X)+b\,E(Y)$.

2. $E(XY) = E(X)E(Y)$ if X, Y are independent.

Why 2.? $E(XY) = \sum_{x,y} xy\,P(X=x, Y=y) = \sum_{x,y} yx\,P(X=x)\,P(Y=y) =$

$$= \left(\sum_y y\,P(Y=y)\right)\left(\sum_x x\,P(X=x)\right) = E(X)E(Y).$$

Ex1. a) Let X be Bernoulli(p). Find $E(X) = 1\cdot p + 0\cdot q = p$.

b) Let X be binomial(n,p). Find $E(X)$.

Answer.

Ex2. A deck of cards was dealt to 4 people. J is one of them. Let X = # of aces J gets. Find $E(X)$.

Ex 3. Let X be Poisson(λ). Find $E(X)$