3.10 Path properties of r.w.

1. $N_n(a,b)$ = # of paths from a to b in n steps

$$= \binom{n}{\frac{n+b-a}{2}}$$

$$P(S_n = b \mid S_0 = a) = \binom{n}{\frac{n+b-a}{2}} p^{\frac{n+b-a}{2}} q^{\frac{n+a-b}{2}}.$$

2. For $a, b > 0$, $N_n^0(a,b)$ = # of paths from a to b in n steps via zero $= N_n(-a, b)$

3. For $b \neq 0$, $\overline{N_n^0}(0,b)$ = # of paths from 0 to b in n steps without zero visits

$$= \frac{|b|}{n} N_n(0,b)$$

Claim 1. Let $S_0 = 0$, $b \neq 0$. Then

$$P(S_1 \dots S_n \neq 0,\ S_n = b) = \frac{|b|}{n} P(S_n = b), \text{ or}$$

$$P(S_1 \dots S_n \neq 0 \mid S_n = b) = \frac{|b|}{n}.$$

Some exercises

Ex 2. Consider simple r.w., $S_0 = k$, $0 < k < N$.

Let $\tau_k = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = N\}$: τ_k is time needed to reach boundary from k.

a) Find $E(\tau_k)$

Answer. 1. 1st step analysis gives a system of eqns for $D_k = E(\tau_k)$:

$$(1) \quad \begin{cases} D_k = p D_{k+1} + q D_{k-1} + 1, & k = 1, 2, \dots, N-1 \\ D_0 = D_N = 0, \end{cases}$$

because $D_k = E(\tau_k \mid X_1 = 1) \overbrace{P(X_1 = 1)}^{p} + E(\tau_k \mid X_1 = -1) \overbrace{P(X_1 = -1)}^{q}$

$= (1 + D_{k+1}) p + (1 + D_{k-1}) q = p D_k + q D_k + 1.$

Given $X_1 = 1$, $\tau_k = 1 + \tau_{k+1}$: $E(\tau_k \mid X_1 = 1) = 1 + E(\tau_{k+1}) = 1 + D_{k+1}$

$X_1 = -1$, $\tau_k = 1 + \tau_{k-1}$: $E(\tau_k \mid X_1 = -1) = 1 + E(\tau_{k-1}) = 1 + D_{k-1}$

Now (1) is nonhomogeneous linear eqn. General solution: $D_k = A + B\left(\frac{q}{p}\right)^k + L_k$, where L_k is particular solution to (1), and A, B to be found

L_k is of the form

$$L_k = \begin{cases} c\, k & , q \neq p \\ c\, k^2 & , q = p \end{cases} \text{ with } c \text{ found from (1).}$$

A, B are found from $D_0 = D_N = 0$.

<u>Result</u>;

$$D_k = \begin{cases} k(N-k) & , p = q = \frac{1}{2} \\ (q-p)\left(k - N \cdot \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}\right) & , p \neq q . \end{cases}$$

b) Find $\overline{D_k} = \lim_{N \to \infty} D_k$

<u>Answer</u> $$\overline{D}_k = \begin{cases} +\infty & \text{if } p \geq q \text{ (includes } p = q = \tfrac{1}{2}\text{)}. \\ \frac{k}{q-p} & \text{if } q > p \end{cases}$$

<u>Remark 1</u>. Let $k > 0$, $S_0 = k$, $\overline{\tau}_k = \min\{n \geq 1 : S_n = 0\}$.

Then $\overline{D}_k = E(\overline{\tau}_k)$ is expected ruin time.

Recall for $p = q = \frac{1}{2}$, $P(\overline{\tau}_k < \infty) = 1$,

<u>but</u> $E(\overline{\tau}_k) = +\infty$.

<u>Ex 3</u>. Let $S_0 = 0$. We found for $b > 0$,

$\overline{N_n^0}(0, b) = \overline{N_{n-1}^0}(1, b) = N_{n-1}(1, b) - N_{n-1}^0(1, b) =$

$= N_{n-1}(1, b) - N_{n-1}(-1, b) = N_{n-1}(0, b-1) - N_{n-1}(0, b+1)$.

Show that

no returns to zero in $2n$ step

$R_{2n} = \#$ of paths in $\{S_1 \dots S_{2n} \neq 0\} = N_{2n}(0, 0)$.

Answer $R_{2n} = 2 \sum^{n} \{S_1 \dots S_{2n} \neq 0, S_{2n} = 2k\}$, telescoping sum

R_{2n}

<u>Answer</u> $R_{2n} = 2\sum_{k=1}^{n} \{S_1 \dots S_{2n} \neq 0,\ S_{2n} = 2k\}$ telescoping sum

$= 2\sum_{k=1}^{n} \overline{N_{2n}^{0}}(0, 2k) = 2\sum_{k=1}^{n} \left[N_{2n-1}(0, 2k-1) - N_{2n-1}(0, 2k+1)\right]$

$= 2\ N_{2n-1}(0,1) \cancel{- 2\cdot N_{2n-1}(0, 2n+1)} = N_{2n-1}(0,1) + N_{2n-1}(0,-1)$

$= N_{2n}(0,0).$

<u>Thm 4</u>. Let $S_0 = 0$, $p = q = \frac{1}{2}$. Then

$$P(S_1 \dots S_{2n} \neq 0) = P(\tau_0 > 2n) = P(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$$

where $\tau_0 = \min\{n \geq 1 : S_n = 0\}$

<u>Proof</u>. $P(S_1 \dots S_{2n} \neq 0) = R_{2n} \cdot 2^{-2n} = N_{2n}(0, 0)\, 2^{-2n}$

$= \binom{2n}{n} 2^{-2n}$

<u>Remark 1</u>. Recall our homework: $P(S_{2n} = 0 \mid S_0 = 0)$

$= \binom{2n}{n} 2^{-2n} \approx \frac{1}{\sqrt{\pi n}}$ as $n \to \infty$.

<u>Corollary</u>. Let $S_0 = 0$, $p = q = \frac{1}{2}$. Then

$P(\tau_0 < \infty) = 1.$

<u>Why?</u> $P(\tau_0 < \infty) = P\left(\bigcup_{n=1}^{\infty} \{\tau_0 \leq 2n\}\right) =$

$= \lim_{n\to\infty} P(\tau_0 \leq 2n) = \lim_{n\to\infty} \left[1 - P(\tau_0 > 2n)\right] = 1.$

<u>Remark 2</u>. For $a > 0$, $p = q = \frac{1}{2}$,

$$P(\tau_0 > 2n \mid S_0 = a) = \sum_{j=1-a}^{a} \underbrace{P(S_{2n} = j \mid S_0 = 0)}_{\frac{1}{\sqrt{\pi n}}}$$

$\approx \frac{a}{\sqrt{\pi n}}$

Hence $P(\tau_0 < \infty \mid S_0 = a) = 1.$

Moment of the last return

Example. Fair coin is tossed repeatedly,

$S_n = H_n - T_n$, $n \geq 1$, $S_0 = 0$.

H_n ← # of Hs, T_n ← # of Ts.

By Bernoulli thm, $\frac{S_n}{n} = \frac{H_n}{n} - \frac{T_n}{n} \approx 0$ for large n.

Question: How frequent are zero visits?

Time moment of the last return. Let $S_0 = 0$, $p = q = 1/2$.

At time moment n, consider $\sigma_n = \max\{k \leq n : S_k = 0\}$,

Def. r.v. σ_n is called moment of the last return (return because $S_0 = 0$)

Range of $\sigma_n = \{0, 1, \dots, n\}$.

Claim 1. Let $S_0 = 0$, $\tau_0 = \min\{n \geq 1 : S_n = 0\}$ ← moment of the first return.

Then for $j = 0, 1, \dots, n$,

$$P(\sigma_n = j) = P(S_j = 0)\, P(\tau_0 > n - j)$$

Proof. Case $j = n$:

$$P(\sigma_n = n) = P(S_n = 0)\, P(\tau_0 > 0) = P(S_n = 0)$$

Case $j < n$:

$$P(\sigma_n = j) = P(S_j = 0, S_{j+1} \dots S_{n-1} S_n \neq 0)$$

$$= P(S_{j+1} \cdots S_n \neq 0 \mid S_j = 0)\, P(S_j = 0)$$

$$= P(S_1 \cdots S_{n-j} \neq 0 \mid S_0 = 0)\, P(S_j = 0)$$

$$= P(\tau_0 > n-j \mid S_0 = 0)\, P(S_j = 0).$$

<u>Corollary</u>. Let $S_0 = 0$, $p = q = 1/2$. Then for $j = 0, 1, \dots, n$,

$$P(\sigma_{2n} = 2j) = P(S_{2j} = 0)\, P(\tau_0 > 2(n-j))$$

$$= P(S_{2j} = 0)\, P(S_{2(n-j)} = 0) \approx \frac{1}{\sqrt{\pi j}} \frac{1}{\sqrt{\pi (n-j)}} \quad \text{if}$$

<u>j and $n-j$ are large</u>.