Contents

1 hw1

1. Let $\{A_i, i \in I\}$ be a collection of sets. Prove De Morgan's Laws:

$$(\cup_i A_i)^c = \cap_i A_i^c, \ (\cap_i A_i)^c = \cup_i A_i^c.$$

Hint. The first one: $x \in (\cup_i A_i)^c \iff x \notin \cup_i A_i$ (it means there is no i such that $x \in A_i$) $\iff x \notin A_i$ for any $i \iff x \in A_i^c$ for any $i \iff x \in \cap_i A_i^c$.

2. *a)* Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of Ω. Show that $\mathcal{F}_1 \cap \mathcal{F}_2$, the collection of subsets of Ω that belong to both $\mathcal{F}_1$ and $\mathcal{F}_2$, is σ-field.

Hint. Verify the definition of a σ-field. Since $\Omega \in \mathcal{F}_1$ and $\Omega \in \mathcal{F}_2$, then $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ etc...

b) Let $\mathcal{A}$ be a collection of subsets of Ω, and let $\mathcal{F}_i, i \in I$, be all σ-fields that contain $\mathcal{A}$. Show that $\mathcal{F} = \cap_i \mathcal{F}_i$ is a σ-field.

Comment. b) is of interest if $\mathcal{A}$ itself is not a σ-field. Note that $\mathcal{P}(\Omega)$, the σ-field of all subsets of Ω, is among $\mathcal{F}_i$. The collection $\mathcal{F} = \cap_i \mathcal{F}_i$ is called the smallest σ-field containing $\mathcal{A}$ (or σ-field generated by $\mathcal{A}$).

3. Suppose that eight distinct envelopes are placed at random in three distinct mailboxes.

a) In how many different ways this can be done?

b) What is the probability that every mailbox receives at least one envelope? *Hint. Consider finding probability of the complementary event $A =$"at least one mailbox is empty". Then one could use one of the following two options:*

(i) *Note $A =$"exactly one empty" or "exactly two empty"*

(ii) $A = A_1 \cup A_2 \cup A_3$, *where $A_i =$ "ith mailbox empty", and by drawing Venn diagram (inclusion/exclusion):*

$$\#A = \#A_1 + \#A_2 + \#A_3 - \#(A_1 \cap A_2) - \#(A_1 \cap A_3) - \#(A_2 \cap A_3).$$

4. How many different letter arrangements of length 4 (four letter "words") can be made using the letters MOTTO?

5. (a) In how many ways can 3 boys and 3 girls sit in a row?

(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

(c) In how many ways if only the boys must sit together?

(d) In how many ways if no two people of the same sex are allowed to sit together?

Hint. It is like arranging books on a shelf (problem considered in class: check Content folder on blackboard).

2 hw2

1. A woman has n keys, of which two will open her door. (a) If she tries the keys at random, discarding those that do not work,what is the probability that she will open the door on her kth try? (b) If she does not discard previously tried keys, what is the probability of no right key in k tries ($k \geq 1$)?

If $n = 9$, how many tries are needed to be 90% sure that the door is opened?

2. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife. *Hint.* Let $A_k =$ "kth couple sits together", $k = 1, 2, 3, 4$. Apply inclusion/exclusion principle and count thinking about the book arrangement on the shelf:

$$\begin{aligned} &\mathbf{P}\,(\text{at least one couple sits together}) \\ = \;& \mathbf{P}\,(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum_{i=1}^{4} \mathbf{P}\,(A_i) - \sum_{i<j} \mathbf{P}\left(A_i A_j\right) + \sum_{i<j<k} \mathbf{P}\left(A_i A_j A_k\right) - \mathbf{P}\,(A_1 A_2 A_3 A_4) \end{aligned}$$

For instance with $i < j$, $A_i A_j =$ "the ith couple and the jth couple go together, the remaining 4 people are arranged in any order".

3. *Answering a question take into consideration all the information revealed before it.*

A man has five coins, two of which are double-headed, one is double-tailed, and two are normal.

He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads? *Hint. In the previous question, you found the conditional probability that the discarded coin is double-headed.*

4. The hats of 10 people are mixed. Everybody picks up randomly a hat one by one. For any number n of people we found in class the probability $p_n = \mathbf{P}\,(\text{no matches})$.

a) Find probability that only people $1, 2, 3$ (the first three) get the right hats (no matches for the rest). *Hint. If $A_i =$ "ith person gets the right hat", then by multiplication law,*

$$\begin{aligned} \mathbf{P}\,(\text{only } 1,2,3 \text{ get the right hats}) &= \mathbf{P}\,(1,2,3 \text{ get the right hats and no matches for the rest}) \\ &= \mathbf{P}\,(\text{no matches for the rest} | A_1 A_2 A_3)\, \mathbf{P}\,(A_1 A_2 A_3)\,. \end{aligned}$$

Use p_n with an appropriate n in your answer.

b) Find probability of exactly three matches.

5. Jane has 3 children, each of which is equally likely to be a boy or a girl independently of the others. Consider the events:

$$\begin{aligned} A &= \text{"all the children are of the same sex"}, \\ B &= \text{"there is at most one boy"}, \\ C &= \text{"the family includes a boy and a girl"}. \end{aligned}$$

(a) Show that A is independent of B, and that B is independent of C.
(b) Is A independent of C?
(c) Do these results hold if Jane has four children?

3 hw3

1. In each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any given packet contains any specific Vice-Chancellor being 1/5, independently of all other packets. Show that the probability that each of the last three Vice-Chancellors is obtained in a bulk purchase of six packets is

$$1-3\left(\frac{4}{5}\right)^6+3\left(\frac{3}{5}\right)^6-\left(\frac{2}{5}\right)^6.$$

Hint. Let A_k="kth chancellor is in at least one of the 6 packets", $k=1,\ldots,5$. Then, by de Morgan law and inclusion-exclusion principle,

$$\begin{aligned}\mathbf{P}\left(A_3\cap A_4\cap A_5\right) &= 1-\mathbf{P}\left((A_3\cap A_4\cap A_5)^c\right)=1-\mathbf{P}\left(A_3^c\cup A_4^c\cup A_5^c\right),\\ \mathbf{P}\left(A_3^c\cup A_4^c\cup A_5^c\right) &= \sum_{k=3}^{5}\mathbf{P}\left(A_k^c\right)-\sum_{3\le i<j\le 5}\mathbf{P}\left(A_i^c\cap A_j^c\right)+\mathbf{P}\left(A_3^c\cap A_4^c\cap A_5^c\right).\end{aligned}$$

2. There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability p, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C.

If, in addition, there is a direct road from A to C, this road being blocked with probability p independently of the others, find the required conditional probability.

3. Consider a gambler G ruin problem where he starts with k dollars, $0<k<N$. A fair coin is tossed repeatedly. G wins \$1 if H, and loses \$1 if T. The game stops in two cases: either G is ruined or G reaches the desired amount \$$N$. Show that the game stops with probability 1. Hint. Besides the ruin probability p_k consider the probability $\bar{p}_k$ to reach N. Besides the equation (6) on p.17 for p_k, write a similar equation for $\bar{p}_k$ and solve it (what are $\bar{p}_0, \bar{p}_N$?). Find $p_k+\bar{p}_k$.

4. A communication channel transmits a signal as sequence of digits 0 and 1. The probability of incorrect reception of each digit is p. To reduce the probability of error at reception, 0 is transmitted as 00000 (five zeroes) and 1, as 11111. Assume that the digits are received independently and the majority decoding is used. Compute the probability of receiving the signal incorrectly if the original signal is (a) 0; (b) 101. Evaluate the probabilities when $p=0.2$. Hint. Number of errors in a reception of 5 digits is binomial r.v.

5. A coin with $\mathbf{P}(H)=p, \mathbf{P}(T)=q=1-p$, is tossed repeatedly (indefinitely). Let $H_k=$ "H in the kth toss", $T_k=$ "T in the kth toss". Assume all tosses are independent.

(a) Find $\mathbf{P}$ (at least one H after n) $=\mathbf{P}\left(\cup_{m=n}^{\infty}H_m\right)=1-\mathbf{P}\left(\cap_{m=n}^{\infty}T_m\right)$.

Hint. Recall, by continuity of probability, $\mathbf{P}\left(\cap_{m=n}^{\infty}T_m\right)=\lim_{l\to\infty}\mathbf{P}\left(\cap_{m=n}^{n+l}T_m\right)$.

(b) Find probability of infinitely many H.

Hint. $\mathbf{P}$ (infinitely many H) $=\mathbf{P}\left(\cap_{n=1}^{\infty}\cup_{m=n}^{\infty}H_m\right)$: use the previous part (a) and continuity of probability.

4 hw4

1. Consider the following strategy playing the roulette. Bet \$1 on red. If red appears (which happens with probability 18/38), then take \$1 and stop playing for the day. If red does not appear, then bet additional \$1 on red each of the following two rounds, and then stop playing for the day no matter the outcome. Let X be the net gain (a negative gain means a loss).

(a) What are possible values of X? Find $\mathbf{P}(X = k)$ for all possible values k; sketch the distribution function (cdf) of X.Hint. To find all $\mathbf{P}(X = k)$ it is convenient to use probability tree.

(b) Compute $\mathbf{P}(X > 0)$, the probability of net win. Is it a good strategy? (c) You played 5 days. Find probability to win \$1 in at least three of 5 games.

2. a) Let U be a r.v. with distribution function

$$F_U(u) = \mathbf{P}(U \leq u) = \begin{cases} 0 & \text{if } u < 0, \\ u & \text{if } 0 \leq u \leq 1, \\ 1 & \text{if } u > 1. \end{cases}$$

We say U is uniformly distributed in the interval $[0, 1]$; note that

$$\mathbf{P}(a < U \leq b) = \frac{b-a}{1} = b - a, 0 \leq a < b \leq 1.$$

Let F be a distribution function which is continuous and strictly increasing (note that range of F is $(0, 1)$). Recall that in this case the inverse function $F^{-1} : (0, 1) \to \mathbf{R}$ is strictly increasing continuous and cancellation identities hold: $F\left(F^{-1}(x)\right) = x, x \in (0, 1)$, and $F^{-1}(F(x)) = x, x \in \mathbf{R}$.

a) Show that $X = F^{-1}(U)$ is a r.v. having distribution function F, i.e. $F = F_X$.

b) Let X be a r.v. with a continuous distribution function F. Find expression for the distribution functions of the following random variables:

(i) X^2; (ii) $|X|$ and $\sqrt{|X|}$; (iii) $F(X)$, assuming F is strictly increasing; (iv) $G^{-1}(F(X))$, assuming F and $G : \mathbf{R} \to (0, 1)$ are strictly increasing with $G^{-1} : (0, 1) \to \mathbf{R}$ as the inverse of G.

3. A coin is tossed repeatedly and heads turns up on each toss with probability p. Let H_n and T_n be the numbers of heads and tails in n tosses. Show that for each $\varepsilon > 0$,

$$\mathbf{P}\left(2p - 1 - \varepsilon \leq \frac{1}{n}(H_n - T_n) \leq 2p - 1 + \varepsilon\right) \to 1$$

as $n \to \infty$. Hint. $H_n + T_n = n$.

4. (Binomial distribution as an approximation) An urn contains N balls, m of which are red. A random sample of n balls is withdrawn without replacement from the urn. Assume $n < m < N$. The number X of red balls in this sample has the mass function

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, k = 0, \ldots, n.$$

This is called the hypergeometric distribution with parameters N, m, n, and X is a number of "successes" in n trials with initial "success" probability m/N but the trials are not independent.

Show further that if N, m approach ∞ in such a way that $m/N \approx p \in (0, 1)$, then for $k = 0, \ldots, n$,

$$\mathbf{P}(X = k) \approx \binom{n}{k} p^k (1-p)^{n-k}$$

Comment. It shows that for fixed n and large N, m, the distribution of X is approximately binomial$\left(n, p = \frac{m}{N}\right)$, the trials for large m, N with $m/N \approx p$ are approximately independent: the distribution in that case barely depends on whether or not the balls are replaced in the urn immediately after their withdrawal.

Hint. Write explicitly the binomials, divide numerator and denominator by N^n and group:

$$\begin{aligned} &\mathbf{P}(X = k) \\ = \quad & \frac{n!}{k!\,(n-k)!} \frac{\frac{m(m-1)\ldots(m-k+1)}{N^k} \frac{(N-m)(N-m-1)\ldots(N-m-(n-k)+1)}{N^{n-k}}}{\frac{N(N-1)\ldots(N-n+1)}{N^n}} \end{aligned}$$

5. Let X be binomial$(n, p = 1/2)$. It can be proved that the most likely values of X are

$$\begin{aligned} &\frac{n}{2} \text{ if } n \text{ is even,} \\ &\frac{n-1}{2} \text{ and } \frac{n+1}{2} \text{ if } n \text{ is odd.} \end{aligned}$$

Show that *for large n*,

$$\begin{aligned} \mathbf{P}\left(X = \frac{n}{2}\right) &= \binom{n}{n/2} 2^{-n} \approx \frac{1}{\sqrt{\pi \cdot \frac{n}{2}}} \text{ if } n \text{ is even,} \\ \mathbf{P}\left(X = \frac{n+1}{2}\right) &= \binom{n}{(n+1)/2} 2^{-n} \approx \frac{1}{\sqrt{\pi \cdot \frac{n}{2}}} \text{ if } n \text{ is odd.} \end{aligned}$$

Hint. Use Stirling formula according to which

$$k! \approx \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \text{ for large } k,$$

more precisely

$$\lim_{k \to \infty} \frac{k!}{\sqrt{2\pi k} \left(\frac{k}{e}\right)^k} = 1.$$

5 hw5

1. A box contains b blue and r red balls (total number of balls in the box is $n = b + r$).

All balls are removed at random one by one and arranged in a row. Let X_i be the number of red balls between the $(i-1)$th and ith blue ball drawn, $i = 2, \ldots, b$; Let X_1 be the number of red balls until the first blue ball shows up, and X_{b+1} be the number of red balls after the last blue ball drawn. Consider the random vector $X = (X_1, \ldots X_{b+1})$. The range of X are all the vectors $(k_1, \ldots, k_{b+1})$ with nonnegative integer components $k_1, \ldots, k_{b+1}$ such that $k_1 + \ldots + k_{b+1} = r$. For such a vector $(k_1, \ldots, k_{b+1})$ with $k_i \geq 0$, and $k_1 + \ldots + k_{b+1} = r$, find

$$\mathbf{P}(X = (k_1, \ldots, k_{b+1})) = \mathbf{P}(X_1 = k_1, \ldots, X_{b+1} = k_{b+1}).$$

Note $X_i = 0$ means there are no red balls between the $(i-1)$th and ith blue balls if $i = 2, \ldots, b$; $X_1 = 0$ means the row starts with blue ball; $X_{b+1} = 0$ means the last ball is blue.

Hint. The computation is simple. Do not overthink. The vector $(k_1, \ldots, k_{b+1})$ with nonnegative integer components determines specifically (uniquely) r red ball and b blue ball "seats" (positions). For instance, if $n = 5, b = 2, r = 3$, and $(k_1, k_2, k_3) = (0, 2, 1)$, then five "seat" color arrangement is $BRRBR$, and $\mathbf{P}(X = (0, 2, 1)) = ?$

2. a) Given 1000 married couples, compute the probability that, in at least three of them, both husband and wife were born on the same day.

b) Given 300000 married couples, compute the probability that, in at least three of them, both husband and wife were born on April 18.

Use binomial and its Poisson approximation. Is approximation accurate?

3. a) Let X and Y have joint df F. Show that for any, $a < b$ and $c < d$,

$$\begin{aligned} & \mathbf{P}(a < X \leq b, c < Y \leq d) \\ = \; & F(b, d) - F(a, d) - F(b, c) + F(a, c). \end{aligned}$$

Find (in terms of F) the probability

$$\mathbf{P}(X = b, Y = d).$$

Comment. This equality implies that if F is a joint distribution function, then for any $a < b$ and $c < d$,

$$F(b, d) - F(a, d) - F(b, c) + F(a, c) \geq 0 \tag{5.1}$$

If F is twice continuously differentiable around the rectangle $(a, b] \times (c, d]$, using Taylor formula, we have

$$\begin{aligned} 0 &\leq F(b, d) - F(a, d) - F(b, c) + F(a, c) \\ &= \int_0^1 \int_0^1 \frac{\partial^2 F(a + s(b-a), c + r(d-c))}{\partial x \partial y} dr ds \; (d-c)(b-a). \end{aligned}$$

Therefore,

$$\frac{\partial^2 F(a, c)}{\partial x \partial y} \geq 0 \tag{5.2}$$

and $\frac{\partial^2 F(x,y)}{\partial x \partial y} \geq 0$ in the rectangle above guaranties that (5.1) holds.

b) Is the function $F(x,y) = 1 - e^{-xy}, 0 \leq x, y < \infty$, the joint df of some pair of r.v.?

Comment. In order for F to be joint distribution function, besides the properties (a), (b), (c) of Lemma 5, p.39, the inequality (5.1) must hold as well, which translates into (5.2), and $\frac{\partial^2 F(x,y)}{\partial x \partial y} \geq 0$ if F is twice continuously differentiable around (x, y).

4. Let $X_1, \ldots, X_n$ be identically distributed continuous random variables, i.e. they have the same df and pdf f. Assume that for any $-\infty < x_1, x_2, \ldots, x_n < \infty$,

$$\mathbf{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \mathbf{P}(X_1 \leq x_1) \mathbf{P}(X_2 \leq x_2) \ldots \mathbf{P}(X_n \leq x_n).$$

Such a collection of r.v.'s is called a random sample: we have n independent "observations" of a random variable X with df F and pdf f.

a) Show that $(X_1, \ldots, X_n)$ is jointly continuous with joint df

$$G(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(u_1) f(u_2) \ldots f(u_n) \, du_n \ldots du_1,$$

and joint pdf

$$g(x_1, \ldots, x_n) = f(x_1) f(x_2) \ldots f(x_n), (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

Hint. Use independence to find joint df, then take mixed derivative in $x_1, \ldots, x_n$.

b) Explain why with probability 1 we have $X_i \neq X_j$ for any $i \neq j$ (which means that with probability 1, the sample values $X_1(\omega), \ldots, X_n(\omega), \omega \in \Omega$, are all distinct)?

Hint. For any pair $i \neq j$, the pair (X_i, X_j) is jointly continuous.

5. Let X and Y be independent random variables taking values in the positive integers and having the same mass function $f(x) = 2^{-x}$ for $x \in \{1, 2, \ldots\}$, that is they are geometric with $p = 1/2$. Find their joint probability mass function and:

(a) $\mathbf{P}(\min\{X, Y\} \leq x)$. Hint. Find $\mathbf{P}(\min\{X, Y\} > x)$.

(b) $\mathbf{P}(Y > X)$; (c) $\mathbf{P}(X = Y)$, (d) $\mathbf{P}(X \geq kY)$, for a given positive integer k;

(e) $\mathbf{P}(X \text{ divides } Y)$. Hint. X divides Y means $Y = lX$ for some $l \in \{1, 2, \ldots\}$. Answer is a series.

6 hw6

1. (Continuation of #1 of hw5) A box contains b blue and r red balls (total number of balls in the box is $n = b + r$). All balls are removed at random one by one and arranged in a row. Let X_i be the number of red balls between the $(i-1)$th and ith blue ball drawn, $i = 2, \ldots, b$; Let X_1 be the number of red balls until the first blue ball shows up, and X_{b+1} be the number of red balls after the last blue ball drawn. In #5 of hw4, we found the pmf of X :

$$\begin{aligned} \mathbf{P}\left(X = (k_1, \ldots, k_{b+1})\right) &= \mathbf{P}\left(X_1 = k_1, \ldots, X_{b+1} = k_{b+1}\right) \\ &= f\left(k_1, \ldots, k_{b+1}\right) = \frac{b!r!}{n!} = \frac{1}{\binom{n}{r}} \end{aligned} \tag{6.3}$$

for any nonnegative integers $k_1, \ldots, k_{b+1}$ so that $k_1 + \ldots + k_{b+1} = r$.

a) Find $\mathbf{E}(X_1), \ldots, \mathbf{E}(X_{b+1})$. Hint. The pmf f of X in (6.3) is symmetric in $(k_1, \ldots, k_{b+1})$: if we shuffle (rearrange) $k_1, \ldots, k_{b+1}$ the value of f does not change (it is still $f(k_1, \ldots, k_{b+1})$). Therefore all the components $X_1, \ldots, X_{b+1}$ are identically distributed (they have the same pmf, that is all marginal pmfs are identical).

b) Let Y_i be the number of balls needed to be removed until the ith blue ball shows up, $i = 1, \ldots, b$. Find $\mathbf{E}(Y_i), i = 1, \ldots, b$. Hint. $Y_1 = X_1 + 1$

c) Find the pmf of X_1 and Y_1. What are the pmf of $X_2, \ldots, X_{b+1}$?

2. A biased coin is tossed n times with probability p of H. A run is a sequence of throws which result in the same outcome. For example, HHTHTTH contains 5 runs. Show that the expected number of runs is $1 + 2(n-1)pq$, where $q = 1 - p$. Find the variance of the number of runs. Hint: use indicators of A_j=" jth and $j+1$ -th outcomes are different".

3. A building has 10 floors above the street level. 20 of people enter at street level and board an elevator. They choose floors independently, equally likely at random. Let T be the number of stops the elevator must make.

Find $\mathbf{E}(T)$and Var(T). Hint. Use the indicators of the event that ith floor selected. Note "ith floor selected" = "at least one person selects floor i". What is the probability that no one selects the ith floor?

4. *Coupons*. Every package of some intrinsically dull commodity includes a small and exciting plastic object. There are n different types of object, and each package is equally likely to contain any given type. You buy one package each day.

(a) Let X_j be number of days which elapse between the acquisitions of the jth new type of object and the $(j+1)$th new type. Recognize the distribution of X_j and find $\mathbf{E}(X_j)$. Hint. Since we have already j objects there are $n - j$ new left. After the acquisition of the jth new type, what is the probability that the next package will contain nothing new?

(b) Let X be the number of days needed for you to have a full set of objects. Find $\mathbf{E}(X)$. Express X using X_j.

(c) Let A_i be the event that none of the first k packages contain the ith object. Find $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup A_4)$.

5. You roll a fair die repeatedly. If it shows 1, you must stop, but you may choose to stop at any prior time. Your score is the number shown by the die on the final roll. Consider the following strategy $S(k)$: stop the first time that the die shows k or greater, $k = 4, 6$.

a) What is the probability that "k or greater" ($k = 4, 6$) shows up before 1? What is the probability that 1 shows up before "k or greater" ($k = 4, 6$)?

Hint. Consider A^k= "k or greater shows up before 1", and use first step analysis to write an equation for $\mathbf{P}(A_k)$.

b) Let X_k be the score when the strategy $S(k)$ was used. Find $\mathbf{E}(X_k), k = 4, 6$. Which strategy yields the higher expected score?

7 hw7

1. An urn contains n balls numbered 1, 2, ..., n. We remove k balls at random without replacement, and add up their numbers. Find the mean and an expression for the variance of the total. Recall

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

2. a) Let

$$Y = g(X) + W,$$

and W and X are independent discrete r.v. Show that

$$\mathbf{P}(Y = y|X = x) = \mathbf{P}(g(x) + W = y).$$

b) Let

$$X = Y + U,$$

and Y and U are independent. Assume the pmf $f_Y(y)$ and $f_U(u)$ are known. Find the joint pmf $f(x, y)$ of X and Y, and

$$\mathbf{P}(Y = y|X = x)$$

in terms of f_U and f_Y.

3. A factory has produced n robots, each of which is faulty with probability p. To each robot a test is applied which passes all good robots and detects the fault (if present) with probability δ. Let X be the number of faulty robots, and Y the number detected as faulty. Assume the usual independence.

(a) What is the probability that a robot passed is in fact faulty?

(b) Let Z be the number of passed faulty robots. Given $Y = k$, what is the distribution of Z? What is $\mathbf{E}(Z|Y)$?

(c) Show that

$$\mathbf{E}(X|Y) = \frac{np(1-\delta) + (1-p)Y}{1-p\delta}.$$

4. a) We define the conditional variance, Var$(Y|X)$, as a random variable

$$\mathrm{Var}(Y|X) = \mathbf{E}\left(Y^2|X\right) - (\mathbf{E}(Y|X))^2.$$

Show that

$$\mathrm{Var}(Y) = \mathbf{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathbf{E}(Y|X)).$$

b) (application of a)) We have a coin that shows heads with probability p. We tossed it repeatedly and counted how many tosses were needed for the first heads to show up. If that number is X, we roll the fair die X times. Let Y be the total score in X rolls of the die.

Find $\mathbf{E}(Y|X = n)$, $\mathbf{E}(Y|X)$ and $\mathbf{E}(Y)$. Find Var$(Y|X = n)$,Var$(Y|X)$, Var(Y) . Hint. $Y = \sum_{i=1}^{X} Y_i$, where Y_i is the score in the ith roll.

5. 51 passengers bought tickets on a 51-seat carriage. One seat was reserved for each passenger. The first 50 passengers took the seats at random so that all 51! possible seating arrangements (with one empty seat) are equally likely. The last passenger insisted on taking the assigned seat. If that seat is occupied, then the passenger in that seat has to move to the corresponding assigned seat, and so on.

a) Find the probability that the last passengers seat is not empty.

b) Compute the expected value of the number M of passengers who have to change their seats.

Hint. One way could be to denote E_n the expected value of the number of passengers who have to change their seats in n seat plane (instead of 51), and write a recursion for E_n in terms of E_{n-1}, and going "down" to get a general formula for E_n without knowing the distribution of $M = M_n$. You can obtain recursion by conditioning with respect to whether the last passengers seat is empty or not (1st step analysis). Another way is by simply finding the pmf of $M = M_n$ explicitly: directly or writing a recursion for $p_{n,k}$, probability that k passengers were moved in n seat plane, in terms of $p_{n-1,k-1}$.

8 hw8

1. Consider a simple random walk on the set $\{0, 1, \ldots, N\}$ in which each step is to the right with probability p or to the left with probability $q = 1 - p$. Absorbing barriers are placed at 0 and N. Let $S_0 = k, 0 < k < N$.

Show that the number X of positive steps of the walk before absorption satisfies

$$\mathbf{E}(X) = \frac{1}{2}\left[D_k - k + N(1 - p_k)\right],$$

where D_k is the mean number of steps until absorption and p_k is the probability of absorption at 0 (the gambler ruin probability). Hint. If Z_k is the number of steps until absorption, and Y is the number of negative steps until absorption, then $Z_k = X + Y$, $D_k = \mathbf{E}(Z_k) = \mathbf{E}(X + Y)$. What can you say about $k + X - Y$? How many values it takes?

2. Consider a symmetric simple random walk S_n with $S_0 = 0, p = q = 1/2$. Let $\tau_0 = \min\{n \geq 1 : S_n = 0\}$ be the time of the first return of the walk to its starting point.

Show that

$$\mathbf{P}(\tau_0 = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n} = \frac{1}{2n-1}\mathbf{P}(S_{2n} = 0).$$

Hint. Recall we found and computed $\mathbf{P}(\tau_0 > 2n) = \mathbf{P}(S_{2n} = 0)$.

3. Let S_n be symmetric simple r.w. ($p = q = 1/2$), and $S_0 = 0$, i.e.,

$$S_n = X_1 + \ldots + X_n, n \geq 1,$$

where X_i are independent identically distributed, $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$.

a) Show that $\bar{S}_n = -S_n, n \geq 0$, is symmetric r.w. as well, that is the sequences $\{S_n, n \geq 0\}$, and $\{-S_n, n \geq 0\}$ are identically distributed. Hint: X_i and $-X_i$ have identical mass functions, and $-X_i$ are independent.

b) For $b \neq 0$, set $\tau_b = \tau_b(S) = \min\{n > 0 : S_n = b\}$. Show that

$$\mathbf{P}(\tau_b < \tau_{-b}) = \mathbf{P}(\tau_{-b} < \tau_b) = 1/2.$$

Hint. Recall for any $a \neq 0$, $\mathbf{P}(\tau_a < \infty) = 1$. Since the sequences $\{S_n, n \geq 0\}$, and $\{-S_n, n \geq 0\}$ are identically distributed,

$$\mathbf{P}(\tau_b(S) < \tau_{-b}(S)) = \mathbf{P}(\tau_b(-S) < \tau_{-b}(-S)),$$

where

$$\begin{aligned}
\tau_b(S) &= \min\{n > 0 : S_n = b\}, \tau_b(-S) = \min\{n > 0 : -S_n = b\}, \\
\tau_{-b}(S) &= \min\{n > 0 : S_n = -b\}, \tau_{-b}(-S) = \min\{n > 0 : -S_n = -b\}.
\end{aligned}$$

c) Let $\sigma_k = \min\{n > 0 : S_n \notin (-k, k)\}$. Find $\mathbf{E}(S_{\sigma_k})$ and $\mathrm{var}(S_{\sigma_k})$. Hint: $\sigma_k = \min\{\tau_k, \tau_{-k}\}$ and $\mathbf{P}(\tau_k < \infty) = \mathbf{P}(\tau_{-k} < \infty) = 1$. What values S_{σ_k} and $S_{\sigma_k}^2$ take?

4. a) Let $S_0 = a > 0$, $p = q = 1/2$. Let $\tau_0 = \min\{k \geq 0 : S_k = 0\}$, the hitting time of zero. For $a \geq 1$, $j \geq 1, n \geq 1$, express

$$\mathbf{P}\left(S_n = j, \tau_0 \leq n | S_0 = a\right) \text{ and } \mathbf{P}\left(S_n = j, \tau_0 > n | S_0 = a\right)$$

in terms of finitely many basic probabilities. By basic probabilities for S_n,$n \geq 0$, we mean probabilities of the form

$$\mathbf{P}(S_n = k | S_0 = 0), \mathbf{P}(S_n \geq k | S_0 = 0), \mathbf{P}(S_n \leq k | S_0 = 0)$$

Hint. Reflection principle, and space homogeneity.

b) For $a \geq 1$, $j \geq 1, n \geq 1$, show that

$$\mathbf{P}\left(\tau_0 > n | S_0 = a\right) = \sum_{j=1-a}^{a} \mathbf{P}\left(S_n = j | S_0 = 0\right).$$

Hint. Remove S_n in joint probabilities of a); telescopic sums?

5. In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assuming that all $\binom{m+n}{m} = \binom{m+n}{\frac{m+n+(n-m)-0}{2}}$ orderings (orders in which those votes were cast) are equally likely, show that probability that A is always ahead in the count of the votes is $\frac{n-m}{n+m}$.

Hint. $m + n$ votes were cast one by one. Let $\mathcal{H}_k$ be number of votes for A and $\mathcal{T}_k$ be number of votes for B after k people voted. Think about the differences $S_k = \mathcal{H}_k - \mathcal{T}_k, k \geq 1, S_0 = 0$.

9 hw9

1. Let S_n be a simple r.w. Assume $S_0 = 0$. Let $\tau_0 = \min\{k > 0 : S_k = 0\}$, the first return time back to zero. Show that

$$1 = \sum_{j=0}^{n} \mathbf{P}(\tau_0 > j)\, \mathbf{P}(S_{n-j} = 0).$$

Hint. $\mathbf{P}(\sigma_n = j)$, where $\sigma_n = \max\{k \le n : S_k = 0\}$?

2. The annual rainfall figures in Bandrika are independent identically distributed continuous random variables $\{X_r, r \ge 1\}$. Find the probability that:

(a) $X_1 < X_2 < X_3 < X_4$.

(b) $X_1 > X_2 < X_3 < X_4$. Hint. One way is to enumerate all possibilities and rewrite this event as a disjoint union of (a) type. The other way (without thinking): with probability 1,

$$I_{\{X_1 > X_2 < X_3 < X_4\}} = I_{\{X_1 > X_2\}} I_{\{X_2 < X_3 < X_4\}} = \left(1 - I_{\{X_1 < X_2\}}\right) I_{\{X_2 < X_3 < X_4\}}.$$

3. a) Let Θ be uniform on $(0, 2\pi)$, and $a > 0$. Find the pdf of $Y = a\cos\Theta$. Hint. Find pdf of $X = \cos\Theta$ first.

b) Let X be uniform in $(0,1)$. Find the df and pdf of $U = 1 - X$.

c) Let X, Y be independent uniform in $(0,1)$. Find the df and pdf of $V = X - Y$. Hint. Determine the areas under $x - y \le v$.

4. *Order statistics*. Let $X_1, \ldots, X_n$ be independent identically distributed variables with a common pdf f. Such a collection is called a random sample. For each $\omega \in \Omega$, arrange the sample values $X_1(\omega), \ldots, X_n(\omega)$ in non-decreasing order $X_{(1)}(\omega), \ldots, X_{(n)}(\omega)$, where (1), (2),..., (n) is a (random) permutation (arrangement) of 1, 2,..., n. The new variables $X_{(1)}, \ldots, X_{(n)}$ are called the order statistics.

a) Show, by a symmetry argument, that the joint distribution function of the order statistics satisfies

$$\begin{aligned}
&\mathbf{P}\left(X_{(1)} \le y_1, \ldots, X_{(n)} \le y_n\right) \\
=\ & n!\mathbf{P}(X_1 \le y_1, \ldots, X_n \le y_n, X_1 < \ldots < X_n) \\
=\ & n! \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} \chi(x_1, \ldots, x_n) f(x_1) \ldots f(x_n)\, dx_1 \ldots dx_n,
\end{aligned}$$

where

$$\chi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_1 < \ldots < x_n \\ 0 & \text{otherwise.} \end{cases}$$

Note $X_{(1)} = \min_{1 \le k \le n} X_k$, $X_{(n)} = \max_{1 \le k \le n} X_k$.

Hint. We have

$$\left\{X_{(1)} \le y_1, \ldots, X_{(n)} \le y_n\right\} = \cup_{j_1, \ldots, j_n} \left\{X_{j_1} \le y_1, \ldots, X_{j_n} \le y_n, X_{j_1} < \ldots < X_{j_n}\right\},$$

where the union is taken over all possible different orderings (permutations) $j, \ldots, j_n$ of $\{1, \ldots, n\}$. All the sets in the union are disjoint, and there are $n!$ of them. With probability 1,

$$\Omega = \cup_{j_1, \ldots, j_n} \left\{X_{j_1} < \ldots < X_{j_n}\right\}.$$

b) Find the marginal density function of the kth order statistic $X_{(k)}$ of a sample with size n directly (without using the joint df find in a)). Hint. First, show that the df of $X_{(k)}$ is

$$F_{X_{(k)}}(x) = \mathbf{P}\left(X_{(k)} \leq x\right) = \sum_{j=k}^{n} \binom{n}{j} F(x)^{j} \left[1 - F(x)\right]^{n-j} :$$

note $\sum_{j=1}^{n} I_{\{X_j \leq x\}}$ is a binomial r.v.

c) Find the joint density function of the order statistics of n independent uniform variables in $(0, T)$. Hint: use a)

5. Let $X_1, \ldots, X_n$ be positive independent identically distributed continuous random variables. Show that, if $m < n$, then $\mathbf{E}\left(\frac{S_m}{S_n}\right) = \frac{m}{n}$, where $S_k = X_1 + \ldots + X_k$.

Hint. The random variables $\frac{X_i}{S_n} = \frac{X_i}{X_1 + \ldots + X_n}, i = 1. \ldots, n$, are identically distributed. First think what is $\mathbf{E}\left(\frac{X_i}{S_n}\right)$?

10 hw10

1. A coin-making machine produces quarters in such way that, for each coin, the probability U to turn up heads is uniform in $(0,1)$. A coin pops out (randomly) and is flipped 10 times. Let X be number of heads in those 10 tosses.

a) What are

$$\begin{aligned}\mathbf{P}(X=j|U=u), j &= 0,\ldots,10 \text{ and} \\ \mathbf{P}(X=j|U), j &= 0,\ldots,10?\end{aligned}$$

Hint. $X \sim$binomial$(10,U)$.

b) Find $\mathbf{P}(X=j), j=0,\ldots,10$, the distribution of X. Hint: Since

$$\mathbf{P}(X=j)=\mathbf{E}\left(I_{\{X=j\}}\right), \mathbf{E}\left(I_{\{X=j\}}|U\right)=\mathbf{P}(X=j|U),$$

we have

$$\mathbf{P}(X=j)=\mathbf{E}[\mathbf{P}(X=j|U)]=\int_0^1 \mathbf{P}(X=j|U=u)\,du, j=0,\ldots,10.$$

It is known that

$$\int_0^1 u^j(1-u)^{n-j}\,du=\frac{j!(n-j)!}{(n+1)!}.$$

c) Find $\mathbf{E}(X)$ and var(X).

2. a) Let X, Y be independent standard normal. What is joint pdf of (X,Y)? Are $R^2=X^2+Y^2$ and $V=\frac{X}{\sqrt{X^2+Y^2}}$ independent? Hint. Use polar coordinates to find the joint df of $\left(R^2,V\right)$.

b) Let X,Y,Z be independent standard normal. Show that

$$\frac{X+YZ}{\sqrt{1+Z^2}} \text{ and } Z$$

are independent standard normal. Hint: condition on Z.

3. Let (X_1,X_2) be normal bivariate with $\mu_1=\mu_2=0, \sigma_1^2=\sigma_2^2=1$, and $\rho=3/5$. Let (Y_1,Y_2) be the midterm and final exam scores of a randomly selected student. Assume

$$Y_1=80+3X_1, Y_2=75+2X_2.$$

Given a student got 90 in the midterm exam,

(a) What is the expectation and variance of her final exam score? Hint. It might be easier to reduce the question to (X_1,X_2) but also (Y_1,Y_2) is a normal bivariate.

(b) What is the probability that she got more than 75 in the final exam? Express the probability in terms of $\Phi(x)$, the df of a standard normal r.v.

4. Let X, Y be normal bivariate r.v. with $\mu_1 = \mu_2 = 0$, variances σ_1^2, σ_2^2 and correlation coefficient ρ.

a) Write what are $\mathbf{E}(X|Y)$,Var$(X|Y)$?

b) Show that

$$\begin{aligned}\mathbf{E}(X|X+Y=z) &= \frac{\sigma_1^2 + \rho\sigma_1\sigma_2}{\sigma_2^2 + 2\rho\sigma_1\sigma_2 + \sigma_1^2} z, \\ \operatorname{var}(X|X+Y=z) &= \frac{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}{\sigma_2^2 + 2\rho\sigma_1\sigma_2 + \sigma_1^2}.\end{aligned}$$

Hint. $(X, X+Y)$is normal bivariate.

5. Let $Y = X + \varepsilon Z$, where $X \sim N\left(0, \sigma_1^2\right), Z \sim N(0,1)$ are independent (X is a random "target", εZ is a "noise", Y is what we observe and register). Find the best mean square estimate of X based on Y (recall it is $\hat{X} = \mathbf{E}(X|Y)$). How is $X - \hat{X}$ distributed? Find the mean square error $\mathbf{E}\left[\left(X - \hat{X}\right)^2\right]$.

11 hw11

1. Suppose (X, Y) has joint density of the form $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$, for some function g. Let $R = \sqrt{X^2 + Y^2}$, and Θ be the polar angle of (X, Y). Find joint pdf of (R, Θ). Are R, Θ independent? Identify the distribution of Θ.

Hint. Use polar coordinates in $\mathbf{R}^2 : x = r\cos\theta, y = r\sin\theta$ with $r > 0, \theta = \arctan\frac{y}{x} \in (0, 2\pi)$. Note $r^2 = x^2 + y^2$. Recall Calculus: if D is a polar rectangle,

$$D = \{(x, y) : a < r < b, \alpha < \theta < \beta\} \text{ with } 0 < a < b, 0 < \alpha < \beta < 2\pi,$$

then

$$\int\int_D g(x, y)\, dxdy = \int_a^b \int_\alpha^\beta g(r\cos\theta, r\sin\theta)\, rd\theta dr.$$

Hence if $f(x, y)$ is joint pdf of (X, Y), then with D above we have

$$\mathbf{P}(a < R < b, \alpha < \Theta < \beta) = \int\int_D f(x, y)\, dxdy.$$

2. Let X_1, X_2, X_3 be independent exponential r.v. with parameter $\lambda = 1 : \mathbf{P}(X_i > x) = e^{-x}, x > 0, i = 1, 2, 3$. Let

$$Y_1 = \frac{X_1}{X_1 + X_2 + X_3}, Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, Y_3 = X_1 + X_2 + X_3.$$

a) Find the joint pdf of (Y_1, Y_2, Y_3) and $\mathbf{E}(Y_i), i = 1, 2, 3$. Hint: Find the inverse and Jacobian etc.

b) Show that (i) (Y_1, Y_2) is distributed as the order statistics of two uniform r.v in $(0, 1)$, see #4c) of hw9; Y_3 is gamma$(1, 3)$-distributed; (ii) Y_3 and (Y_1, Y_2) are independent.

3. a) A $d \times d$ symmetric matrix $A = (a_{ij})$ is called nonnegative if for any $z = (z_1, \ldots, z_d) \in \mathbf{R}^d$,

$$zAz' = \sum_{i,j=1}^{d} a_{ij} z_i z_j = \sum_{i=1}^{d} a_{ii} z_i^2 + 2\sum_{i<j} a_{ij} z_i z_j \geq 0.$$

where $z = (z_1, \ldots, z_d)$ is a row vector, z' is the transpose of z.

Let $X = (X_1, \ldots, X_d)$ and $\mathbf{E}\left(X_i^2\right) < \infty, i = 1, \ldots, d$. Let $B = (b_{ij})$ with $b_{ij} = \text{Cov}(X_i, X_j), i, j = 1, \ldots, d$. Show that B is nonnegative definite.

b) Let $X = (X_1, \ldots, X_d)$ be multivariate normal $N(\mu, B)$, where μ is the vector of expected values and $B = (b_{ij})$ is the covariance matrix: $b_{ij} = \text{Cov}(X_i, X_j)$.

Let $c_1, \ldots, c_d$ be constants, and at least one of them $\neq 0$. What is the distribution of $Y = c_1 X_1 + \ldots + c_d X_d$? Determine the parameters of that distribution.

4. a) Let X be uniform in $(0, 1)$. Find the df and pdf of $V = \min\{1 - X, X\}$. Find the df and pdf of V.

b) Let X, Y be independent uniform in $(0, 1)$, and $V = \frac{Y}{X}$. What is the set of possible values of V? Find the df and pdf of $V = \frac{Y}{X}$. Hint. Draw a picture and determine the areas under $\frac{y}{x} \leq v$.

5. (St. Petersburg paradox) You pay $\$2^8 = \$256$ to enter and play the following game: A fair coin is tossed until H shows up, and if X is the number of tosses that was needed, you are paid $\$2^X$. For instance, if $X = 10$, then you are paid $\$2^{10}$).

Your win/loss $W = 2^X - 256$. What is $\mathbf{E}(W)$? What is the probability that $W \geq 0$?

12 hw12

1. The joint mgf of a random vector $X = (X_1, \dots, X_d)$ is defined as

$$M_X(t) = \mathbf{E}\left[e^{t_1 X_1 + \dots t_d X_d}\right] = \mathbf{E}\left[e^{tX'}\right], t = (t_1, \dots, t_d) \in \mathbf{R}^d,$$

where tX' is the product of the row vector $t = (t_1, \dots, t_d)$ and the column vector X', the transpose of random row vector $X = (X_1, \dots, X_d)$.

(i) Let $Z = (Z_1, \dots, Z_d)$ with independent $Z_i \sim N(0, 1)$. Show that

$$M_Z(t) = \exp\left\{\frac{1}{2}tt'\right\} = \exp\left\{\frac{1}{2}\left(t_1^2 + \dots + t_d^2\right)\right\}, t = (t_1, \dots, t_d) \in \mathbf{R}^d.$$

(ii) Let $X = (X_1, \dots, X_d)$ be multivariate normal $N(\mu, B)$. Show that

$$\begin{aligned} M_X(t) &= \exp\left\{\mu t' + \frac{1}{2}tBt'\right\} \\ &= \exp\left\{\sum_{j=1}^{d} t_j \mu_j + \frac{1}{2}\sum_{k,j=1}^{d} b_{kj} t_k t_j\right\}, t = (t_1, \dots, t_d) \in \mathbf{R}^d. \end{aligned}$$

Hint. We have $B = D'D$ for some $d \times d$ -matrix D, and $X = ZD + \mu$, where $Z = (Z_1, \dots, Z_d)$ is r. vector of independent $Z_i \sim N(0, 1)$: use definition of M_X and part (i).

2. Consider a branching process with immigration: each generation is supplemented by an "immigrant" with probability p. This means that the size Z_n of the n-th generation satisfies

$$Z_n = I_n + \sum_{k=1}^{Z_{n-1}} X_k,$$

where $I_n = 1$ with probability p and $I_n = 0$ otherwise; the number of children X_k of the kth person in the generation $n-1$ are independent identically distributed with generating function $G(s)$ and mean μ. We assume that Z_{n-1}, I_n and X_k are independent. Let $G_n(s) = G_{Z_n}(s)$ and $\mu_n = \mathbf{E}(Z_n)$.

(a) Show that $G_n(s) = [ps + (1-p)]\, G_{n-1}(G(s))$.

(b) Show that $\mu_n = p + \mu_{n-1}\mu$;

(c) Find μ_n and $\lim_{n\to\infty} \mu_n$.

3. a) A sequence of biased coins is flipped; the chance that the kth coin shows a head is U_k, where U_k is a random variable taking values in $(0, 1)$. Let X_n be the number of heads after n flips. Does X_n obey the central limit theorem when the U_k are independent and identically distributed?

b) Let X_n be binomial(n, U), where U is uniform in $(0, 1)$ (look at #1 of hw10). Show that

$$\frac{X_n}{n+1} \xrightarrow{D} U,$$

that is CLT does not hold for X_n.

4. a) Let H_n be number of H in n independent tosses of a p-coin. Apply CLT to approximate $\mathbf{P}\left(a < \frac{H_n}{n} < b\right), 0 < a < b < 1$, for large n.

b) Let Y be Poisson(λ). Apply continuity theorem to show that

$$\frac{Y-\lambda}{\sqrt{\lambda}} \xrightarrow{D} Z \sim N\left(0,1\right)$$

as $\lambda \to \infty$. Hint. Recall $\mathbf{E}\left(Y\right) =Var(Y) = \lambda$. Find characteristic function of $V = \frac{Y-\lambda}{\sqrt{\lambda}}$.

Comment. For large λ, Y is approximately $N\left(\lambda,\lambda\right)$.

5. Let $X_k, k \geq 1$, be i.i.d. random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2}$$

exists in an appropriate sense, and identify the limit.