Markov, Chebyshev inequalities and LLN (7.2, 7.3)

Recall $\boxed{X \le Y \text{ implies } E(X) \le E(Y)}$

1. Markov inequality. For any $a > 0$,

$$P(|V| \geq a) \leq \frac{E(|V|)}{a}.$$

Proof. For $a > 0$,

$$a\, I_{\{a \le |V|\}} \le |V|. \quad \text{Hence}$$

$$a\, E(1_{\{|V| \geq a\}}) = a P(|V| \geq a) \leq E(|V|).$$

2. Chebyshev inequality. Let $E(V) = \mu$, $Var(V) = \sigma^2$.
Then for each $\varepsilon > 0$,

$$P(|V - \mu| \geq \varepsilon) \leq \frac{Var(V)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}.$$

Proof. $P(|V - \mu| \geq \varepsilon) = P((V-\mu)^2 \geq \varepsilon^2) \overset{\text{Markov}}{\leq} \frac{E[(V-\mu)^2]}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}.$

Chebyshev inequality and weak LLN

Let $X_1, \ldots, X_n$ be i.i.d. with $E(X) = \mu$, $Var(X) = \sigma^2$.
Then for any $\varepsilon > 0$, by Chebyshev, $V = \bar{X}_n$),

$$(1) \quad \boxed{P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n \to \infty} 0} \leftarrow \text{weak LLN}$$

Comment on (1): $E[(\bar{X}_n - \mu)^2] = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$. We say

$\bar{X}_n \to \mu$ in mean square sense.

Def. $V_n \to V$ in mean square if $E[(V_n - V)^2] \xrightarrow{n\to\infty} 0$.

Note. mean square convergence implies convergence in probability:

$$P(|V_n - V| \geq \varepsilon) = P((V_n - V)^2 \geq \varepsilon^2) \leq \frac{E[(V_n - V)^2]}{\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

Some useful facts

Claim 1. If $V_n \to V$ in probability and g is continuous, then $g(V_n) \to g(V)$

Claim 2. If $V_n \xrightarrow{D} V$ and g is continuous, then $g(V_n) \xrightarrow{D} g(V)$.

Ex 1. Let $X_1, X_2, \dots X_n$ be i.i.d., $E(X) = \mu$, $Var(X) = \sigma^2$.

Show that sample variance

$$\tilde{S}_n^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{n} \longrightarrow \sigma^2 \text{ in probability}$$

(and $\tilde{S}_n = \sqrt{\tilde{S}_n^2} \to \sqrt{\sigma^2} = \sigma$)

Answer. Expanding numerator and using Claim 1,

$$\tilde{S}_n^2 = \sum_{i=1}^{n} (X_i^2 - 2X_i \bar{X}_n + \bar{X}_n^2)/n$$

$$= \frac{\sum_{i=1}^{n} X_i^2}{n} - 2\frac{\sum_{i=1}^{n} X_i}{n}\bar{X}_n + \frac{n\bar{X}_n^2}{n} = \frac{\sum X_i^2}{n} - \bar{X}_n^2$$

$$\to E(X^2) - \mu^2 = Var(X) = \sigma^2.$$

Slutsky thm. Let $Y_n \xrightarrow{D} Y$ and $V_n \to 1$ in probability. Then $V_n Y_n \xrightarrow{D} Y$.

probability. Then $V_n Y_n \xrightarrow{D} Y$.

Ex 2. Let $X_1, \dots, X_n$ be i.i.d., $E(X) = \mu$, $Var(X) = \sigma^2$.
Show that $\dfrac{\bar{X}_n - \mu}{\tilde{S}_n / \sqrt{n}} \xrightarrow{D} Z \sim N(0,1)$

Answer. $\dfrac{\bar{X}_n - \mu}{\frac{\tilde{S}_n}{\sqrt{n}}} = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \cdot \overbrace{\dfrac{\sigma}{\tilde{S}_n}}^{\to 1} \xrightarrow{D} Z \sim N(0,1)$.

Comment on Ex 2.: note

$$P\left(\frac{|\bar{X}_n - \mu|}{\tilde{S}_n/\sqrt{n}} \le 3 \right) \approx P(|Z| \le 3) = 0.997$$

$$\underbrace{P\left(\bar{X}_n - \frac{3\tilde{S}_n}{\sqrt{n}} \le \mu \le \bar{X}_n + 3\tilde{S}_n/\sqrt{n}\right)}_{0.997 \text{ confidence interval for } \mu.} = 0.997$$

Cauchy-Schwarz inequality: $|E\,UV| \le \sqrt{E(U^2)}\sqrt{E(V^2)}$.

Proof. Let $F(t) = E[(U + tV)^2] = E(U^2) + 2t\,E(UV) + t^2 E(V^2)$
$= c + 2bt + a t^2 \ge 0$ for all t, $c = E(U^2)$, $b = E(UV)$, $a = E(V^2)$
Hence discriminant $= (2b)^2 - 4ac = 4(b^2 - ac) \ge 0$ or
$|b| \le ac$.

Ex 3. Show that $E(|Y|) \le \sqrt{E(Y^2)}$

Answer. Cauchy-Schwarz with $U = |Y|$, $V = 1$.