Math 505A HW11 Neel Gupta

1. Suppose (X,Y) has joint density function
$f_{XY}(x,y) = g(\sqrt{x^2+y^2})$, for some func g.
Let $R = \sqrt{X^2+Y^2}$, and Θ be polar angle of (X,Y).
Find joint pdf of (R,Θ). Indep?

Move $f(x,y)$ to $g(p)$ given polar change:

$$\iint_D f(x,y)\,dx\,dy = \int_\alpha^\beta \int_a^b g(r)\,r\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

joint pdf $\longrightarrow P(R \le b, \Theta \le \beta)$

$$P(0 \le R \le b, 0 \le \Theta \le \beta) = F_{R,\Theta}(b,\beta)$$

$$F_{R,\Theta}(b,\beta) = P(0 \le R \le b, 0 < \Theta, \beta)$$
$$= \int_0^\beta \int_0^b g(r)\,r\,dr\,d\theta$$
$$= \int_0^b g(r)\,r\,dr \int_0^\beta 1\,d\theta$$
$$= \left(\int_0^b g(r)\,r\,dr\right)\beta$$

$$F_{R,\Theta}(b,\beta) = \beta \int_0^b g(r)\,r\,dr$$

$$f_{R,\Theta}(b,\beta) = \frac{\partial^2 F(b,\beta)}{\partial\beta\,\partial b} = g(b)(b) = bg(b)$$
$$= bg(b),\ b > 0,\ \beta \in [0, 2\pi]$$

Since b is the root of squares and $\Theta \in [0, 2\pi]$.
Θ is uniformly distributed on $(0, 2\pi)$, so
$f_\Theta(\beta) = \frac{1}{2\pi}$ $\therefore$ $\Theta \sim U(0,2\pi)$, then Θ and R are indep, since $f_{R,\Theta} = \underbrace{\left(\frac{b}{1}\right)(2\pi)g(b)}\left(\frac{1}{2\pi}\right)$
which multiplies pdfs to make joint pdf.

2) Let X_1, X_2, X_3 be indep. exponential r.v w/ $\lambda = 1$: $P(X_i > x) = e^{-x}$, $x > 0$, $i = 1,2,3$.

Let $Y_1 = \frac{X_1}{X_1 + X_2 + X_3}$, $Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$, $Y_3 = X_1 + X_2 + X_3$

a) Find joint pdf of (Y_1, Y_2, Y_3) and $E(Y_i)$, $i \in \{1,2,3\}$

$Y_1 = \frac{X_1}{Y_3}$, $Y_2 = \frac{X_1 + X_2}{Y_3}$, $Y_3 = X_1 + X_2 + X_3$
$= \cancel{Y_1 Y_3} + Y_2 Y_3 \cancel{- Y_1 Y_3} + X_3$

$X_1 = Y_1 Y_3$ $\quad Y_2 Y_3 = X_1 + X_2$
$= Y_1 Y_3 + X_2$ $\Rightarrow Y_3 = \cancel{Y_1 Y_3} + Y_2 Y_3 + X_3$

$X_2 = Y_2 Y_3 - Y_1 Y_3$

$X_3 = Y_3 - Y_2 Y_3$

$P(X_i > x) = e^{-x}$, $x > 0$, $i = 1,2,3$

$$|J| = \begin{vmatrix} x_{1y_1} & x_{1y_2} & x_{1y_3} \\ x_{2y_1} & x_{2y_2} & x_{2y_3} \\ x_{3y_1} & x_{3y_2} & x_{3y_3} \end{vmatrix} = \begin{vmatrix} Y_3 & 0 & Y_1 \\ -Y_3 & Y_3 & (Y_2 - Y_1) \\ 0 & -Y_3 & 1 - Y_2 \end{vmatrix}$$

$$\det |J| = Y_3 \left[Y_3 (1 - Y_2) + Y_3 (Y_2 - Y_1) \right] + Y_1 \left[(Y_3)^2 - 0 \right]$$
$$= Y_3 \left[Y_3 \cancel{- Y_2 Y_3} + \cancel{Y_2 Y_3} - Y_1 Y_3 \right] + Y_1 Y_3^2$$
$$= Y_3^2 \cancel{- Y_1 Y_3^2} \cancel{+ Y_1 Y_3^2} = Y_3^2$$

Joint PDF of X_1, X_2, X_3 is

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = e^{-x_2} e^{-x_3} e^{-x_1}$$
$$= \begin{cases} e^{-(x_1 + x_2 + x_3)} & \text{if } x_1, x_2, x_3 > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{y_1, y_2, y_3}(y_1, y_2, y_3) = F_{X_1, X_2, X_3}(y_1, y_2, y_3)$$
$$= e^{-(\cancel{Y_1 Y_3} + \cancel{Y_2 Y_3} - \cancel{Y_1 Y_3} + Y_3 - \cancel{Y_2 Y_3})} (|J|)$$
$$= \exp\{-(Y_3)\} / y_3^2 /$$
$$= \begin{cases} \exp\{-Y_3\} Y_3^2 & \text{if } 0 < Y_1, Y_2 < 1, \; 0 < Y_3 < \infty \\ 0 & \text{otherwise} \end{cases}$$

(b) $f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = y_3^2 e^{-y_3}$

$= f_{Y_1}(y_1)\cdot f_{Y_2}(y_2)\cdot f_{Y_3}(y_3).$

then $f_{Y_1}(y_1) = \begin{cases} 1 & \text{if } 0<y_1<1 \\ 0 & \text{otherwise} \end{cases}$

$f_{Y_2}(y_2) = \begin{cases} 1 & \text{if } 0<y_2<1 \\ 0 & \text{otherwise} \end{cases}$ and

$f_{Y_3}(y_3) = \begin{cases} y_3^2 \exp(-y_3) & \text{if } y_3>0 \\ 0 & \text{otherwise} \end{cases}$

so the joint pdf is the product of marinal pdfs,

so Y_1, Y_2, Y_3 are independent r.v.s.

i) Since Y_1, Y_2 are indep, joint pdf (Y_1,Y_2)

$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)\cdot f_{Y_2}(y_2)$

$= 1$ if $0 < Y_2, Y_1 < 1$

$= \begin{cases} 1 & \text{if } (Y_1,Y_2) < 1 \;\&\; (Y_1,Y_2) > 0 \\ 0 & \text{otherwise} \end{cases}$

then (Y_1,Y_2) is the order statistic of 2 r.v.'s

that are distributed $\sim U(0,1)$. $\blacksquare$

then $f_{Y_3}(y_3) = e^{-Y_3} y_3^{3-1}$ if $y_3 > 0$

then pdf of Y_3 is eq. to pdf of $\Gamma(1,3)$

$\therefore Y_3 \sim \Gamma(1,3)$ $\blacksquare$

ii) $f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = f_{Y_3}(y_3)\cdot f_{Y_1,Y_2}(y_1,y_2)$

since Y_1, Y_2 are mutually independent.

$= e^{-y_3} y_3^2 (1)$

(Y_1,Y_2) and Y_3 are independent

$\therefore$ the joint pdf is the product of marginal pdfs.

$\blacksquare$

3) a) Let $X = (X_1, \dots, X_d)$ and $E(X_i^2) < \infty$ $(i=1,\dots,d)$.

Let $B = (b_{ij})$ with $b_{ij} = Cov(X_i, X_j)$ $(i,j=1,\dots,d)$

X is a r. vector $\in \mathbb{R}^d$

B is a $d \times d$ covariance matrix

Let z = a ~~random~~ vector $\in \mathbb{R}^d$

$$Var(zX) = \cancel{zBz}\ zV(X)z' = zBz'$$

$$= \sum_{i,j=1}^{d} \underbrace{Cov(X_i, X_j)}_{(b_{ij})} z_i z_j$$

$$= \sum_{i=1}^{d} z_i^2 Var(X_i) + 2\sum_{i<j}^{d} z_i z_j Cov(X_i, X_j)$$

$$= Var(z_1 X_1 + \dots + z_d X_d)$$

Since variance ≥ 0, $\therefore$ B is a nonnegative-definite

$\blacksquare$

b) Let $X = (X_1, \dots, X_d)$ be multivariate $\sim N(\mu, B)$

where $\mu \in \mathbb{R}^d = E(X_i), i=1,\dots,d$ and

B is a $d \times d$ matrix $= Cov(X_i, X_j) = (b_{ij})$, $j,i = 1,\dots,d$.

Let $c_1, \dots, c_d$ be constants.

Let $Y = c_1 X_1 + \dots + c_d X_d = cX$

~~(X, Y)~~ $= Y = (X_1, \dots, X_d) A \rightarrow \det A = (c_1 \cdot \dots \cdot c_d)$

det $A \neq 0$, so $Y \sim$ <u>multidimensional normal</u>

$$Y = X \begin{pmatrix} c_1 & 0 & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & c_d \end{pmatrix} \rightarrow \det A = (c_1 \cdot \dots \cdot c_d) \neq 0$$

given at least 1 is non 0

$$E(Y) = E(cX) = cE(X) = c\mu$$

$$V(Y) = V(cX) = cBc' = cV(X)c'$$

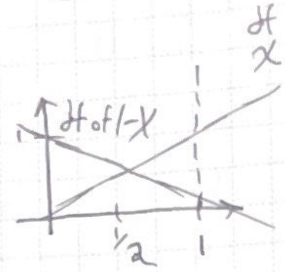
$Y \sim$ d-dimensionally $N(c\mu, cBc')$.

4) a) Let $X \sim U(0,1)$. Let $V = \min\{X, 1-X\}$.

Find df and pdf of V.

Given $X \sim U(0,1)$, df of X

and df of $1-X$



then $\min\{X, 1-X\} \in [0, .5]$, so

$0 \leq V \leq 1/2$.

df of V - $P(V \leq v)$

$P(V \leq v) = P((X \leq v) \cup (1-X \leq v))$

$= P(X \leq v) + P(1-X \leq v) - \cancel{P(X \leq v \cup 1-X \leq v)}$

disjoint $\therefore$ $P(X \leq v \cup 1-X \leq v) = 0$

$= P(X \leq v) + P(-X \leq v-1)$

$= P(X \leq v) + P(X \geq 1-v)$

$= P(X \leq v) + 1 - (1-v)$

$= v + 1 - (1-v) = v + 1 - 1 + v = 2v$

$F_V(v) = 2v, \quad 0 \leq v \leq \frac{1}{2}$ - df of V

$f_V(v) = 2, \quad 0 \leq v \leq \frac{1}{2}$ - pdf of V

4) b) Let $X, Y \sim U(0,1)$ and $V = \frac{X}{Y}$. Find df and pdf of V.

Let us transform $(X,Y) \to (U,V)$

Let $U = X$ and $V = Y/X$ then

$X = U$ and $Y = (X = U)(V)$

$= UV$

$0 \le X, Y \le 1 \to 0 \le U, U \cdot V, \le 1$

$0 \le U \le 1$

$\frac{0}{u} \le \frac{\cancel{u} \cdot v}{\cancel{u}} \le \frac{1}{u} \to 0 \le V \le \frac{1}{u}$

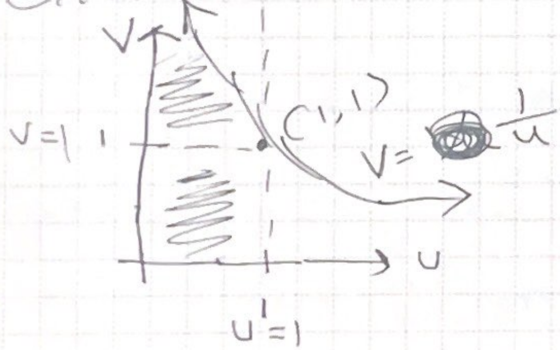
Jacobian:

$$|J| = \begin{vmatrix} \frac{dX}{du} & \frac{dX}{dV} \\ \frac{dY}{du} & \frac{dY}{dV} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$$

then

$$f_{u,v}(u,v) = f_X(x) \cdot f_Y(y) \cdot |J|$$
$$= (1) \cdot (1) \cdot u = u$$
$$= u, \quad 0 \le V \le \frac{1}{u}$$
$$0 \le u \le 1$$
$$\to 0 \le V \le \infty$$

Given $0 \le u \le 1$ and $0 \le v \le \frac{1}{u}$:



Region 1: $0 \le v \le 1$

Region 2: $1 \le v \le \frac{1}{u} (\infty)$

Then $0 \le v \le 1$ and $0 \le u \le 1$: Region 1

$$\therefore f_V(v) = \int_0^1 f_{U,V}(u,v)\,du = \int_0^1 u\,du = \frac{1}{2}$$

Also $1 \le v \le \frac{1}{u} \to 1 \le uv \to 0 \le u \le \frac{1}{v}$

$$\therefore f_V(v) = \int_0^{1/v} u\,du = \frac{u^2}{2}\Big|_0^{1/v} = \frac{1}{2}\left(\frac{1}{v}\right)^2$$
$$= \frac{1}{2v^2} \quad v > 1$$

4) b) $f_V(v) = \frac{1}{2}$ when $0 \le v \le 1$

$f_V(v) = \frac{1}{2v^2}$ when $v > 1$

Then:

$$f_V(v) = \begin{cases} \frac{1}{2}, & 0 \le v \le 1 \\ \frac{1}{2v^2}, & v > 1 \end{cases}$$

5) You pay \$256 to play. You win \$$2^X$ where X = # of tosses until fair coin shows H.

Your win/loss amount is given by

$W = 2^X - 256$

Then $W = 2^X - 2^8$

Find $E(W)$.

$$E(W) = E(2^X - 2^8) = E(2^X) - 2^8$$

$$E(2^X) = \sum_{k=1}^{\infty} 2^k P(X=k) = \sum_{k=1}^{\infty} 2^k \left(\frac{1}{2^k}\right)$$

$$\rightarrow E(W) = \sum_{k=1}^{\infty} 1 - 2^8 = +\infty$$

Find $P(W \ge 0)$.

$$\begin{aligned} P(W \ge 0) &= P(2^X - 2^8 \ge 0) \\ &= P(2^X \ge 2^8) \\ &= P(X \ge 8) \\ &= \sum_{k=8}^{\infty} 2^{-k} = \sum_{k=8}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2^8}}{1 - \frac{1}{2}} \\ &= \frac{1}{2^7} \end{aligned}$$

Paradox Found,

$$\infty \ne \frac{1}{2^7}$$