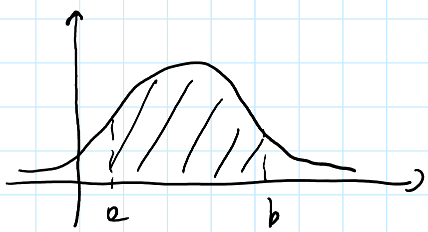
2.3. Continuous r.v. has df given by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty, \quad f \text{ is pdf of } X.$$

Note. 1. $F(x)$ is continuous, $P(X=a) = F(a) - F(a-) = 0$

2. $P(a < X \le b) = F(b) - F(a) = \int_a^b f(t)\,dt$ = shadowed area



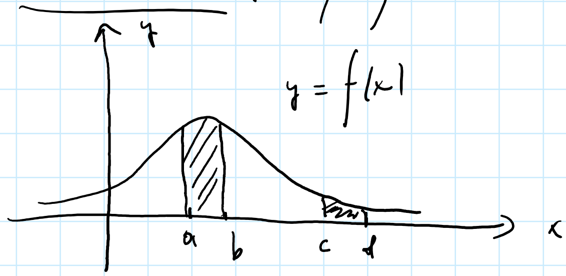
3. $P(a < X \le b) = \int_a^b f(t)\,dt \approx f(a)(b-a)$ if $a \approx b$.

4. $f(x) = F'(x)$ (if f is cont. at x). Calculus I:

$$f(x) = F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{P(x < X \le x+h)}{h},$$

and $\frac{P(x < X \le x+h)}{h}$ is average probability density in $(x, x+h]$.

Remark 1. pdf is more "visual" than df



Values of X between a and b more likely than between c and d.

Def. $f \ge 0$ on $\mathbb{R}$ is called pdf if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Ex1. a) For what C, $f(x) = Ce^{-\lambda x}$, $x > 0$ is a pdf? Here $\lambda > 0$, $f(x) = 0$, if $x \le 0$.

<u>Answer</u> $C > 0$,

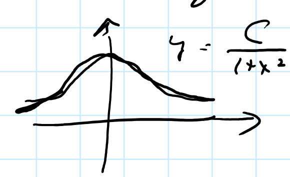
$$1 = \int_{-\infty}^{\infty} f(x)\,dx = C\int_0^{\infty} e^{-\lambda x}\,dx = C\,\frac{e^{-\lambda x}}{-\lambda}\Big|_0^{\infty} = \frac{C}{\lambda}$$

<u>$C = \lambda$.</u> Hence

$$(1)\quad \boxed{f(x) = \lambda e^{-\lambda x},\ x > 0}$$ is a pdf.

<u>Def.</u> A cont. r.v. X with pdf in (1) is called <u>exponential(λ)</u> - r.v.

X models lifetime of an electronic device, waiting time for an earthquake.

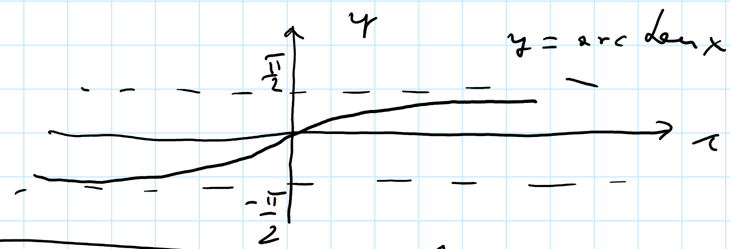


b) For what C, $f(x) = \frac{C}{1+x^2}$, $x \in \mathbb{R}$, is a pdf?

Answer. $C > 0$, $1 = \int_{-\infty}^{\infty} f(x)\,dx = C\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = C\arctan x\Big|_{-\infty}^{\infty}$

$= C\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi C$,

$C = \frac{1}{\pi}$



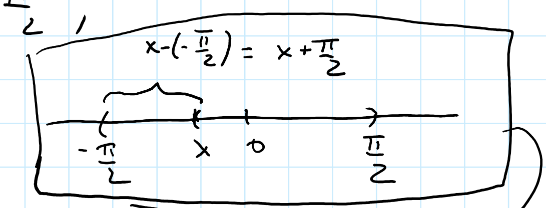
Hence $$\boxed{f(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2},\quad -\infty < x < \infty} \quad (2)$$

X with pdf in (2) is called <u>Cauchy</u> r.v.

Ex 2. a) Let Θ be uniform in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Write pdf and cdf of Θ

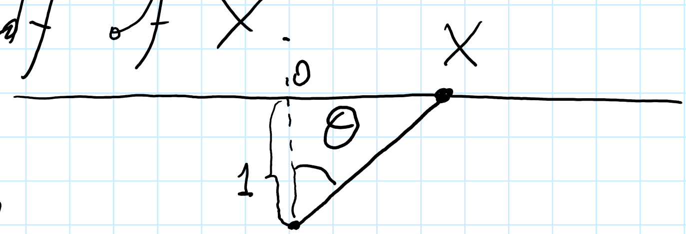
Answer. the pdf and cdf:

$$f_\Theta(x) = \begin{cases} \frac{1}{\pi}, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0, & \text{otherwise} \end{cases}$$



$$F_\Theta(x) = \begin{cases} 0, & x < -\frac{\pi}{2} \\ \frac{x+\pi/2}{\pi} = \frac{1}{\pi}x + \frac{1}{2}, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1, & x \geq \frac{\pi}{2}. \end{cases}$$

b) A robot 1 mile away from the front line shoots randomly along it. Let X be location of the hit. Find cdf and pdf of X.



Answer. The angle Θ is uniform in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\frac{X}{1} = \tan\Theta$,

$X = \tan\Theta$. Range of $X = \mathbb{R} = (-\infty, \infty)$

For $x \in \mathbb{R}$,

$$F(x) = P(X \leq x) = P(\tan\Theta \leq x) = P(\Theta \leq \arctan x)$$

$$= \frac{1}{\pi}\arctan x + \frac{1}{2}, \quad -\infty < x < \infty.$$

$$f(x) = F'(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

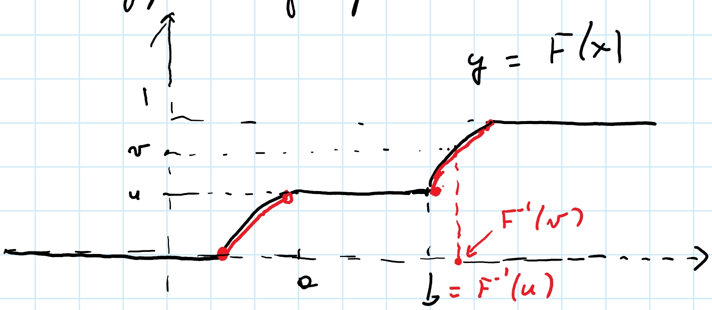
$f(x) = F'(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \; -\infty < x < \infty.$

Some functions of r.v.'s

Let F be continuous df.

The inverse of F. How do we find the inverse?

"Typical" graph:



Range of F = Domain of F^{-1}

a) Given v, $F^{-1}(v)$ is the unique solution to $v = F(x)$ for x.

b) Given u, any $x \in [a,b]$ solves $u = F(x)$: we define $F^{-1}(u) := b$.

Note $P(a \le X \le b) = F(b) - F(a) = 0$

c) $F(F^{-1}(y)) = y$ for all $y \in [0,1]$.

Claim 1. a) Let X be r.v. with continuous df F. Then $F(X)$ is uniform in $(0,1)$.

b) If F is continuous df and U is uniform in $(0,1)$, then F is df of $X = F^{-1}(U)$.

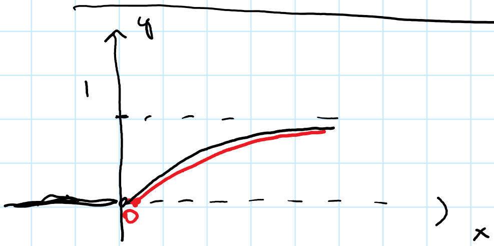
Why b)? $F_X(x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$

Ex 1. a) Let X be exponential (λ) - r.v.

Its pdf is $f(x) = \lambda e^{-\lambda x}, \; x > 0$;

$P(X > x) = e^{-\lambda x}, \; x > 0$; df $F(x) = 1 - e^{-\lambda x}, \; x > 0$

$P(X > x) = e^{-\lambda x}$, $x > 0$, df $F(x) = 1 - e^{-\lambda x}$, $x > 0$



Find $F^{-1}(v)$, $0 < v < 1$

__Answer__. Solving $v = F(x) = 1 - e^{-\lambda x}$ for x: $e^{-\lambda x} = 1 - v$, $-\lambda x = \ln(1-v)$, $x = -\frac{\ln(1-v)}{\lambda}$.

b) Let U be uniform in $(0,1)$. Find function of U whose df is F.

__Answer__ by Claim 1, $F^{-1}(U) = -\frac{\ln(1-U)}{\lambda}$ is r.v. whose df is $F(x) = 1 - e^{-\lambda x}$, $x > 0$