3.9. Simple random walk

$$S_n = S_{n-1} + X_n = S_0 + X_1 + \dots + X_n, \quad n \geq 0.$$

$S_0, X_1, X_2, \dots$ are independent, $P(X_i = 1) = p$, $P(X_i = -1) = q = 1-p$.

Markov property. For any $n, m \geq 0$, $a, b, l_0, \dots, l_n \in \mathbb{Z}$,

$$P(\underbrace{S_{n+m} = b}_{\text{the future}} \mid \underbrace{S_0 = l_0, \dots, S_n = l_n}_{\text{the past}}) = P(S_{n+m} = b \mid \underbrace{S_n = l_n}_{\text{the present moment}}) \qquad (1)$$

Moreover, given $S_n = l_n$, the future and past are independent.

Why (1)? LHS of (1) $= \dfrac{P(\overset{l_n}{\cancel{S_n}} + \sum_{i=1}^{m} X_{n+i} = b, \cancel{S_0 = l_0, \dots, S_n = l_n})}{\cancel{P(S_0 = l_0, \dots, S_n = l_n)}} =$

$$= P(l_n + \sum_{i=1}^{m} X_{n+i} = b) = \text{RHS of (1)}.$$

R.w. with absorbing barriers is simple r.w. which stops when "boundary" is reached.

a) Consider S_n with $S_0 = k$, $0 < k < N$. Let $\tau = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = N\}$. It is time (= number of steps) needed to reach the boundary $\{0, N\}$.

Denoting $n \wedge \tau = \min\{n, \tau\}$, the sequence of r.v.

$R_n = S_{n \wedge \tau}$, $n \geq 0$, is r.w. with absorbing barrier $\{0, N\}$: motion stops if 0 or N are reached.

Example 1 S_n is G's wealth at time n, $S_0 = k$, $0 < k < N$.

G stops when 0 or N is reached.

b). Consider S_n with $S_0 = k$, $k > 0$, and motion stops when 0 is reached ($\{0\}$ is a single barrier).

Example 2. S_n is G's wealth, $S_0 = k > 0$, and G stops if ruined.

Ex1. Let $S_n, n \geq 0$, $S_0 = k$, $0 < k < N$. Consider event A_k = "0 is reached before N".

Find $p_k = P(A_k)$

Answer

$$(2) \quad p_k = \begin{cases} 1 - \frac{k}{N}, & p = q = \frac{1}{2} \\ 1 - \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}, & p \neq q \end{cases}$$

p_k is ruin probability for r.w. with barriers $\{0, N\}$

Some specific numbers. Let $p = P(H) = 0.495$,

$k = 50$, $N = 500$: $p_{50} = 0.9992$

$k = 50$, $N = 300$: $p_{50} = 0.996$

$k = 50$, $N = 200$: $p_{50} = 0.97\ldots$

Why (2)? 1st step analysis (system of eqns for p_k):

$$p_k = P(A_k) = P(A_k \mid X_1 = 1)\, P(X_1 = 1) + P(A_k \mid X_1 = -1)\, P(X_1 = -1)$$

$$= p_{k+1} \cdot p + p_{k-1} \cdot q \qquad \big(P(A_k \mid X_1 = 1) = P(A_{k+1}) \big)$$

System of eqns:

$$\begin{cases} p_k = p \cdot p_{k+1} + q \cdot p_{k-1}, & k = 1, \ldots, N \\ p_0 = 1, \quad p_N = 0 \end{cases}$$

Trial solution: $p_k = \theta^k$, θ to be found

$\theta^k = p \cdot \theta^{k+1} + q\, \theta^{k-1}$ | dividing by θ^k,

$1 = p\theta + \frac{q}{\theta}$, equivalently

$p_0 = 1, \quad p_N = 0$

$1 = p\theta + \frac{q}{\theta}$, equivalently

$p\theta^2 - \theta + q = 0$: $\quad \theta = \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm \sqrt{(2q-1)^2}}{2p}$;

$\theta = \frac{1 \pm |2q-1|}{2p}$: $\quad \theta_1 = 1, \quad \theta_2 = q/p$.

General solution $p_k = A + B\left(\frac{q}{p}\right)^k$, A, B to be found:

$$\begin{aligned} 1 &= p_0 = A + B \\ 0 &= p_N = A + B\left(\frac{q}{p}\right)^N \end{aligned} \quad \Bigg| \quad \begin{aligned} B &= \frac{1}{1-\left(\frac{q}{p}\right)^N} \\ A &= -B\left(\frac{q}{p}\right)^N \end{aligned}$$

Remark 1. Similarly we find

$$\bar{p}_k = P(N \text{ is reached before } 0) = \begin{cases} \frac{k}{N}, & p = q = \frac{1}{2} \\ \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N}, & p \neq q, \end{cases}$$

and $p_k + \bar{p}_k = 1$: 0 or N is reached with probability 1.

Ex 2. Let $k > 0$, $S_0 = k$ (single barrier $\{0\}$).

Find $\tilde{p}_k = P(0 \text{ is reached})$ ← ruin probability

Answer. (3) $\tilde{p}_k = \begin{cases} 1 & \text{if } p \le q \\ \left(\frac{q}{p}\right)^k & \text{if } p > q \end{cases}$ (G is ruined with any $k>0$)

Why (3)? $\{0 \text{ is reached}\} = \bigcup_{N=1}^{\infty} \{0 \text{ is reached before } N\}$ ← increasing

$\tilde{p}_k = P(0 \text{ is reached}) = \lim_{N \to \infty} p_k$

Some specific numbers. $p = 0.505$, $k = 50$: $\tilde{p}_{50} = 0.37$

$k = 100$: $\tilde{p}_{100} = 0.14$

$k = 100: \ \tilde{p}_{100} = 0.14$

3.10 Path properties of r.w.

Basic properties

① Any path in n steps from a to b (from $(0,a)$ to (n,b)) must have

$$r = \frac{n+b-a}{2} \text{ "+" steps and } l = \frac{n+a-b}{2} \text{ "−" steps.}$$

because $\begin{cases} r+l = n \\ a+r-l = b \end{cases}$

② If $\frac{1}{2}(n+b-a) \in \{0,1,\dots,n\}$, then there are

$$N_n(a,b) = \binom{n}{\frac{n+b-a}{2}} \text{ paths from } a \text{ to } b \text{ in } n \text{ steps}$$

If $\frac{1}{2}(n+b-a) \notin \{0,1,\dots,n\}$, then $N_n(a,b) = 0$

③ $$P(S_n = b \mid S_0 = a) = \binom{n}{\frac{n+b-a}{2}} p^{\frac{n+b-a}{2}} q^{\frac{n+a-b}{2}}$$

$$\left(= \binom{n}{\frac{n+b-a}{2}} 2^{-n} \text{ if } p = q = \frac{1}{2} \right).$$

Comment about ③ Let $S_0 = a$. Then $S_n = a + H_n - T_n$ $= a + H_n - (n - H_n) = a + 2H_n - n = b$ if and only if $H_n = \frac{n+b-a}{2}$. Since $H_n = \#$ of H's in n tosses is binomial(n,p), $P(S_n = b \mid S_0 = a) = P\left(H_n = \frac{n+b-a}{2}\right)$ is binomial probability