Exponential, gamma r.v. and Poisson process

Def. $X \sim \Gamma(\lambda, n)$ if its pdf

$$f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x}, \quad x > 0$$

Remark 1. $\Gamma(\lambda, 1)$ is exponential (λ)

Claim 1. If $X \sim \Gamma(\lambda, n)$, $Y \sim \Gamma(\lambda, m)$ are independent, then $X+Y \sim \Gamma(\lambda, n+m)$.

Remark 2. a) If $X \sim \Gamma(\lambda, n)$, then

$$P(X > t) = \int_t^\infty \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x} dx = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

If $X \sim \Gamma(\lambda, 1) =$ exponential (λ), then $P(X > t) = e^{-\lambda t}$, $t > 0$.

Some exercises

Ex 1. (Memoryless property) Let X be exponential (λ), lifetime of a device. Then for any $s, t \geq 0$,

$$P(X > s) = P(X > t+s \mid X > t) = e^{-\lambda s}$$

Answer. $P(X > t+s \mid X > t) = \dfrac{P(X > t+s, X > t)}{P(X > t)} =$

$$= \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

Example. X = waiting time for next earthquake after 1994: $t_1 = 30$, $t_0 = 28$

$$P(X > t_1) = e^{-30\lambda}$$

$$P(X > t_1 \mid X > t_0) = P(X > t_1 - t_0 + t_0 \mid X > t_0) = e^{-\lambda(t_1 - t_0)} = e^{-2\lambda}$$

Poisson process

Ex 2. The light bulb in a room is replaced immediately after it dies. Let X_i, lifetime of ith light bulb, be exponential(λ). Let T_n be the time moment of nth replacement:

$T_1 = X_1$, $T_2 = X_1 + X_2$, ..., $T_n = X_1 + \dots + X_n$.

Note $T_n \sim \Gamma(\lambda, n)$. (as a sum of exponential r.v.'s)

Let $N(t)$ be the number of lightbulb replacements in time interval $[0, T]$.

Find the pmf $N(t)$

Answer. Range of $N(t) = \{0, 1, 2, \dots\}$

$$P(N(t) \le n) = P(T_{n+1} \geq t) = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$P(N(t) = n) = P(N(t) \le n) - P(N(t) \le n-1) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \dots$$

$N(t) \sim$ Poisson(λt).

Remark 3. a) $N(t)$, $t \geq 0$, is continuous time stochastic process counting "replacements":

$N(t)$ = # of replacements in $[0, t]$;

b) By memoryless property, $N(t)-N(s)$ = # of replacements in $(s,t]$ is Poisson$(\lambda(t-s))$.

c) It can be shown, again using memoryless property, that counts of replacements in non-overlapping time intervals are independent: for any $s < t < u$, $N(u)-N(t)$, $N(t)-N(s)$ are independent.

4.6 Conditional distribution and expectation

Let (X,Y) have joint pdf $f(x,y)$.

Question. Given X, what is the probability that $a \le Y \le b$.

Answer. $P(a < Y \le b \mid X) = \int_a^b f(y \mid X)\, dy$, where

$f(y|x) := \dfrac{f(x,y)}{f_X(x)}$; $f(y|x)$ is called cond. pdf of Y given $X = x$.

Note 1. $f(x,y) = f(y|x)\, f_X(x)$

2. $P(a < Y \le b \mid X = x) = \int_a^b f(y|x)\, dy$.

Why? Idea: $P(a < Y \le b \mid X = x) = \lim_{\varepsilon \to 0} P(a < Y \le b \mid x < X \le x+\varepsilon)$

$$= \lim_{\varepsilon \to 0} \frac{P(a < Y \le b,\ x < X < x+\varepsilon)}{P(x < X < x+\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}\int_x^{x+\varepsilon}\left(\int_a^b f(u,y)\,dy\right)du}{\frac{1}{\varepsilon}\int_x^{x+\varepsilon} f_X(u)\,du} = \int_a^b \frac{f(x,y)}{f_X(x)}\,dy$$

Def. (Cond. expectation).

a) $E(Y|X=x) = \int_{-\infty}^{\infty} y f(y|x)\,dy =: h(x)$

b) $E(Y|X) = \int_{-\infty}^{\infty} y f(y|X)\,dy = h(X)$.

<u>Note</u> $E(k(Y)|X) = \int_{-\infty}^{\infty} k(y) f(y|X)\,dy$.

All properties of $E(Y|X)$ listed in discrete case hold:

1. $E(Y) = E[E(Y|X)]$ can be read as

a) $E(Y) = \int_{-\infty}^{\infty} E(Y|X=x) f_X(x)\,dx$

b) $E(Y) = E[h(X)]$ if $h(x) = E(Y|X=x)$ is known.

Also $\boxed{P(A) = \int_{-\infty}^{\infty} P(A|X=x) f_X(x)\,dx}$

<u>Ex 1</u> (<u>hierarchical model</u>) A bacteria has lifetime $Y \sim$ exponential(X), where X is uniform in $(2,3)$, meaning that given $X=x$, $Y \sim$ exponential$(\lambda = x)$:

$f(y|x) = x e^{-xy}$, $y>0$, $2<x<3$. <u>Find</u>

a) joint pdf of (X,Y).

b) $E(Y|X=x)$, $Var(Y|X=x)$, $E(Y|X)$, $E(Y)$.

<u>Answer</u>. a) joint pdf

$f(x,y) = f(y|x) f_X(x) = x e^{-xy}$, $y>0$, $2<x<3$.

b) $E(Y|X=x) = \frac{1}{x}$, $Var(Y|X=x) = \frac{1}{x^2}$,

$$E(Y \mid X) = \frac{1}{X}, \quad Var(Y \mid X) = \frac{1}{X^2}.$$

$$E(Y) = E\left(\frac{1}{X}\right) = \int_2^3 \frac{1}{x}\,dx = \ln x \Big|_2^3 = \ln 3 - \ln 2 = \ln \frac{3}{2}.$$

c) $$Var(Y) = E(Var(Y/X)) + Var(E(Y/X)) =$$

$$= E\left(\frac{1}{X^2}\right) + Var\left(\frac{1}{X}\right) = 2E\left(\frac{1}{X^2}\right) - \left(E\left(\frac{1}{X}\right)\right)^2$$

$$= 2 \cdot \frac{1}{6} - (\ln 1.5)^2 = \underline{\frac{1}{3} - (\ln 1.5)^2};$$

because $$E\left(\frac{1}{X^2}\right) = \int_2^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_2^3 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$