1 Symmetric heavy-tailed transform

Suppose $Y \sim \text{LambertW} \times F_X$ and $X \sim f_X(\cdot \mid \beta)$, $\mu, \sigma \in \mathbb{R}$.

$$z := \frac{y-\mu}{\sigma}$$

By Goerg (2014)

$$g_Y(y \mid \beta, \delta) = f_X(W_\delta(z)\sigma + \mu \mid \beta) \cdot \frac{W_\delta(z)}{z \cdot [1 + W(\delta z^2)]} \qquad \delta \geq 0$$

where

$$W_\delta(z) := \text{sgn}(z)\sqrt{\frac{W(\delta z^2)}{\delta}} \quad \text{and} \quad \left|\frac{d}{dz}W_\delta(z)\right| = \frac{W_\delta(z)}{z \cdot [1 + W(\delta z^2)]}$$

The log-likelihood is

$$\begin{aligned}\log g_Y &= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) + \log \frac{W_\delta(z)}{z} - \log(1 + W(\delta z^2)) \\ &= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) + \frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log\delta - \log|z| - \log(1 + W(\delta z^2))\end{aligned}$$

because

$$W_\delta(z)^2 = \frac{W(\delta z^2)}{\delta} \quad \text{and} \quad \frac{W_\delta(z)}{z} = \sqrt{\frac{W_\delta(z)^2}{z^2}} = \frac{1}{|z|}\sqrt{\frac{W(\delta z^2)}{\delta}}$$

1.1 Gaussian model

For $X \sim \mathcal{N}(\mu, \sigma)$, we have $\mu = \mathbb{E}\,X, \sigma^2 = \text{Var}\ X$ and $z = \frac{y-\mu}{\sigma}$.

$$\begin{aligned}\log f_X(W_\delta(z)\sigma + \mu \mid \beta) &= \log \frac{1}{\sqrt{2\pi\sigma^2}} + \log\exp\{-\frac{1}{2\sigma^2}(W_\delta(z)\sigma + \mu - \mu)^2)\} \\ &\propto -\log\sigma - \frac{1}{2}W_\delta(z)^2 \\ &\propto -\log\sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)}\end{aligned}$$

because

$$W(z)\exp(W(z)) = z \iff \frac{W(z)}{z} = e^{-W(z)} \quad \text{and} \quad W_\delta(z)^2 = z^2\frac{W(\delta z^2)}{z^2\delta} = z^2 e^{-W(\delta z^2)}$$

Then the log-likelihood of a single observation is

$$\begin{aligned}\log g_Y &\propto -\log\sigma - \frac{1}{2}\frac{W(\delta z^2)}{\delta} - \frac{1}{2}W(\delta z^2) - \log(1 + W(\delta z^2)) \\ &\propto -\log\sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)} + \frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log\delta - \log|z| - \log(1 + W(\delta z^2))\end{aligned}$$

For $Y_1, \ldots, Y_N$ i.i.d, the log-likelihood is

$$\prod_{i=1}^{N} g_y(y_i \mid \beta, \delta) = \left(-N\log\sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)}\right) + \left(\frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log\delta - \log|z| - \log(1 + W(\delta z^2))\right)$$

1.2 Exponential model

For $X \sim \text{Exp}(\lambda)$, we have $\mu = 0, \sigma = \frac{1}{\lambda}$ and $z = \lambda y$.

$$\begin{aligned}
\log f_X(W_\delta(z)\sigma \mid \beta) &= \log \lambda + \log \exp\left(-\lambda W_\delta(z)\frac{1}{\lambda}\right) \\
&= \log \lambda - W_\delta(z) \\
&= \log \lambda - z e^{-\frac{1}{2}W(\delta z^2)} \qquad\qquad W_\delta(z)^2 = z^2 e^{-W(\delta z^2)}
\end{aligned}$$

For $Y_1, \ldots, Y_N$ i.i.d, the log-likelihood is

$$\prod_{i=1}^{N} g_y(y_i \mid \beta, \delta) = \left(N \log \lambda - \sqrt{\frac{W(\delta z^2)}{\delta}}\right) + \left(\frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log \delta - \log|z| - \log(1 + W(\delta z^2))\right)$$

1.3 General model

For X location-scale and/or scale family distribution, $\log f_X$ need not be calculated explicitly. It will suffice to call the following sampling statement

$$\sigma W(\delta z^2) + \mu \sim f_X(\cdot \mid \beta)$$

where β are the parameters of random variable X.

2 Asymmetric heavy-tailed transform

In the previous section, we generate $X \sim f_X(\cdot \mid \beta)$ and apply a transform to $U = \frac{X-\mu}{\sigma}$

$$Y = U \exp\left(\frac{\delta}{2}U^2\right) \quad \text{and} \quad \delta \geq 0$$

This transform spreads the outlying samples of further apart.

The inverse of this transform for $Z = \frac{Y-\mu}{\sigma}$ is the modified LambertW function

$$X = W_\delta\left(\frac{Y-\mu}{\sigma}\right)\sigma + \mu \iff U = \text{sgn}(Z)\sqrt{\frac{W(\delta Z^2)}{\delta}}$$

If we only expect to observe U-outliers on one-side of 0, then we can apply the asymmetric heavy-tail LambertW transform defined as

$$Y = \begin{cases} U \exp\left(\frac{\delta_l}{2}U^2\right) & U \leq 0 \\ U \exp\left(\frac{\delta_r}{2}U^2\right) & U > 0 \end{cases}$$

where $\delta_l, \delta_r \geq 0$.

Given x, we fix $\delta \in \{\delta_l, \delta_r\}$ and use the same likelihood as in the symmetric heavy-tail case. How do we choose between the left- and right-tail if we only observe Y? Because $W(z)$ is continuous-bijective and $W(0) = 0$, we can conclude $\text{sgn}(Z) = \text{sgn}(U)$. Using the sign of observed $z := \frac{y-\mu}{\sigma}$, we update the target using either δ_l or δ_r as the tail parameter.

3 Skewed non-negative transform

Suppose $Y \sim \text{LambertW} \times F_X$ and we want to apply a skew-transform to $X \sim f_X(\cdot \mid \beta),\ X \in \mathbb{R}_{\geq 0}$. For $\gamma \geq 0$

$$Y = U \exp\left(\gamma U\right) \sigma \quad \text{where} \quad U = \frac{X}{\sigma}$$

$$X = W_\gamma(Z)\sigma = \frac{W\left(\gamma Z\right)}{\gamma}\sigma \quad \text{where} \quad Z = \frac{Y}{\sigma}$$

The derivative of $\frac{d}{dz}W(z) = \frac{W(z)}{z(1+W(z))}$ thus

$$\frac{d}{dz}W_\gamma(z) = \frac{1}{\gamma}\frac{d}{dz}W(\gamma z) = \frac{W(\gamma z)}{\gamma z(1+W(\gamma z))}$$

By Goerg (2011)

$$h_Y(y|\beta,\gamma) = f_X\left(\frac{W(\gamma z)}{\gamma}\sigma \mid \beta\right)\frac{W(\gamma z)}{\gamma z(1+W(\gamma z))}$$

Using the same approach as in the first section

$$\log h_Y \propto \log f_X\left(\frac{W(\gamma z)}{\gamma}\sigma \mid \beta\right) + \log W(\gamma z) - \log \gamma z - \log(1+W(\gamma z))$$

where $\log f_X\left(\frac{W(\gamma z)}{\gamma}\right)$ can be replaced by

$$\frac{W(\gamma z)}{\gamma}\sigma \sim f_X(\cdot \mid \beta)$$

For non-negative X (or negative values of γ), the density also depends on the non-principal branch $W_{-1}(z)$.