1 Symmetric heavy-tailed transform

Suppose $Y \sim \text{LambertW} \times F_X$ and $X \sim f_X(\cdot \mid \beta)$, $\mu, \sigma \in \mathbb{R}$.

$$z := \frac{y - \mu}{\sigma}$$

By Goerg (2014)

$$g_Y(y \mid \beta, \delta) = f_X(W_\delta(z)\sigma + \mu \mid \beta) \cdot \frac{W_\delta(z)}{z \cdot [1 + W(\delta z^2)]} \qquad \delta \geq 0$$

where

$$W_\delta(z) := \operatorname{sgn}(z)\sqrt{\frac{W(\delta z^2)}{\delta}} \quad \text{and} \quad \left|\frac{d}{dz}W_\delta(z)\right| = \frac{W_\delta(z)}{z \cdot [1 + W(\delta z^2)]}$$

The log-likelihood is

$$\begin{aligned}\log g_Y &= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) + \log \frac{W_\delta(z)}{z} - \log(1 + W(\delta z^2)) \\ &= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) + \frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log \delta - \log|z| - \log(1 + W(\delta z^2))\end{aligned}$$

because

$$W_\delta(z)^2 = \frac{W(\delta z^2)}{\delta} \quad \text{and} \quad \frac{W_\delta(z)}{z} = \sqrt{\frac{W_\delta(z)^2}{z^2}} = \frac{1}{|z|}\sqrt{\frac{W(\delta z^2)}{\delta}}$$

1.1 Gaussian model

For $X \sim \mathcal{N}(\mu, \sigma)$, we have $\mu = \mathbb{E}\,X, \sigma^2 = \text{Var}\ X$ and $z = \frac{y-\mu}{\sigma}$.

$$\begin{aligned}\log f_X(W_\delta(z)\sigma + \mu \mid \beta) &= \log \frac{1}{\sqrt{2\pi\sigma^2}} + \log \exp\{-\frac{1}{2\sigma^2}(W_\delta(z)\sigma + \mu - \mu)^2)\} \\ &\propto -\log \sigma - \frac{1}{2}W_\delta(z)^2 \\ &\propto -\log \sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)}\end{aligned}$$

because

$$W(z)\exp(W(z)) = z \iff \frac{W(z)}{z} = e^{-W(z)} \quad \text{and} \quad W_\delta(z)^2 = z^2\frac{W(\delta z^2)}{z^2\delta} = z^2 e^{-W(\delta z^2)}$$

Then the log-likelihood of a single observation is

$$\begin{aligned}\log g_Y &\propto -\log \sigma - \frac{1}{2}\frac{W(\delta z^2)}{\delta} - \frac{1}{2}W(\delta z^2) - \log(1 + W(\delta z^2)) \\ &\propto -\log \sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)} + \frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log \delta - \log|z| - \log(1 + W(\delta z^2))\end{aligned}$$

For $Y_1, \dots, Y_N$ i.i.d, the log-likelihood is

$$\prod_{i=1}^{N} g_y(y_i \mid \beta, \delta) = \left(-N\log \sigma - \frac{1}{2}z^2 e^{-W(\delta z^2)}\right) + \left(\frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log \delta - \log|z| - \log(1 + W(\delta z^2))\right)$$

1.2 Exponential model

For $X \sim \text{Exp}(\lambda)$, we have $\mu = 0, \sigma = \frac{1}{\lambda}$ and $z = \lambda y$.

$$\begin{aligned}
\log f_X(W_\delta(z)\sigma \mid \beta) &= \log \lambda + \log \exp\left(-\lambda W_\delta(z)\frac{1}{\lambda}\right) \\
&= \log \lambda - W_\delta(z) \\
&= \log \lambda - z e^{-\frac{1}{2}W(\delta z^2)} \qquad\qquad W_\delta(z)^2 = z^2 e^{-W(\delta z^2)}
\end{aligned}$$

For $Y_1, \ldots, Y_N$ i.i.d, the log-likelihood is

$$\prod_{i=1}^{N} g_y(y_i \mid \beta, \delta) = \left(N \log \lambda - \sqrt{\frac{W(\delta z^2)}{\delta}}\right) + \left(\frac{1}{2}\log W(\delta z^2) - \frac{1}{2}\log \delta - \log|z| - \log(1 + W(\delta z^2))\right)$$

1.3 General model

For X location-scale and/or scale family distribution, $\log f_X$ need not be calculated explicitly. It will suffice to make the following sampling statement

$$\sigma W(\delta z^2) \sim f_{X-\mu}(\cdot \mid \beta')$$

where β' are the parameters of the random variable $X - \mu$, given β are the parameters of X.