REBUTTAL OF THE "LETTER TO THE EDITOR" OF ANNALS OF APPLIED STATISTICS ON LAMBERT W × F DISTRIBUTIONS AND THE IGMM ALGORITHM

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I discuss comments and claims made in Stehlḱ and Hermann (2015) about skewed Lambert W × F random variables and the IGMM algorithm. I clarify misunderstandings about the definition and use of Lambert W × F distributions and show that most of their empirical results cannot be reproduced. I also introduce a variant of location-scale Lambert W × F distributions that are well-defined for random variables $X \sim F$ with non-finite mean and variance.

1. Introduction. In their "Letter to the Editor" Stehlḱ and Hermann (2015) relate work on the exact distribution of the likelihood ratio test statistic involving the Lambert W function (Stehlík, 2003) to the transformation-based approach of Lambert W × F distributions to model asymmetric data (Goerg, 2011). See recent work by Witkovsky, Wimmer and Duby (2015) on logarithmic Lambert W × χ^2 distributions that elegantly links these two areas of research.

Stehlík and Hermann present a convergence analysis of the IGMM algorithm when applied to data with non-existing mean or variance. They aim to show that using IGMM for the LATAM log-return series in Goerg (2011) was inappropriate since their statistical analysis implies that the LATAM series does not have a finite mean. See Figure 1 for a time series and density plot of this dataset.

I appreciate that Stehlík and Hermann took the time and effort to explore skewed Lambert W × F distributions and properties of the IGMM estimator. Here I show though that most of their findings are spurious: their methodological concerns foot on a misunderstanding of the definition of location-scale Lambert W × F distributions and their empirical findings are not reproducible. For the remainder of this work I closely follow the organization of Stehlḱ and Hermann (2015) and address comments and claims in similar order. When I use "they" or "their" without reference, I refer to their "Letter to the Editor".

2. Heavy Tails: On three regimes of the IGMM-algorithm. There seems to be a confusion about parameters μ_X and σ_X of location-scale Lambert W × F distributions: they *are* the expectation and standard deviation of the input random variable X, which means that they must exist. If they do not, then the Lambert W × F random variable is not of location-scale type from Definition 2.3 in Goerg (2011). For example, for a student-t input distribution the degrees of freedom parameter ν must be greater than 2. This $\nu > 2$ restriction
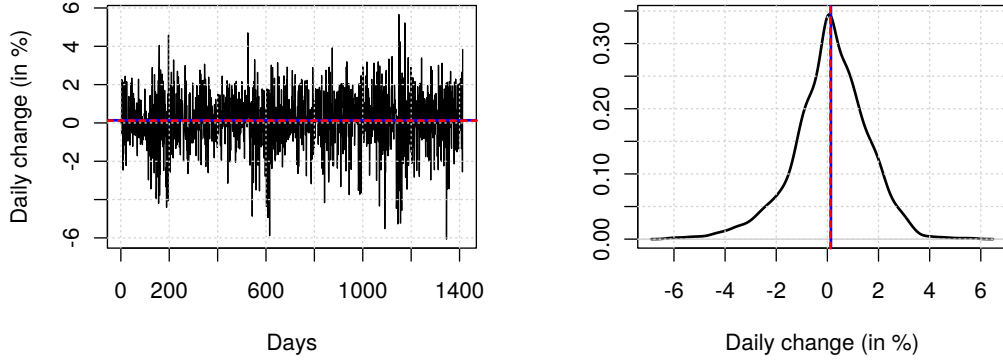

Fig 1: Time series plot and kernel density estimate of the LATAM daily log-return series (dataset `equityFunds` in R package **fEcofin**). Blue, solid line represents the median; red, dashed line the sample average.

is explicitly stated in Goerg (2011) (paragraphs above "Notation 2.4"). Also `beta2tau()` in the **LambertW** R package (Goerg, 2016) throws an error if $\nu \leq 2$.[1] Hence estimating μ_X and σ_X is only a well-defined task for input X with finite mean μ_X and variance σ_X^2. In this case the `IGMM` algorithm estimates μ_X and σ_X by sample mean and sample standard deviation in step 8 of Algorithm 3 of Goerg (2011). In practice, of course, researchers should first check if the dataset at hand satisfies this assumption (see also Section 2.1).

As a consequence, Stehlík and Hermann's simulation study (p. 3, steps 1. - 4.) and the associated results in Table 1 are not set up correctly. In particular, the transformation in step 2. is *not* a location-scale Lambert W × F random variable as defined in Goerg (2011): U must have zero mean and unit variance, and μ and σ must be the mean and standard deviation of X. In Stehlík and Hermann's setup the parameters do not play that role: they use σ_X as the scale parameter, but s is the scale parameter of a student-t with a standard deviation of $\sigma_X = s \cdot \sqrt{\frac{\nu}{\nu-2}}$. For $\nu = 5$, both are well-defined with $\sigma_X = s \cdot \sqrt{\frac{5}{5-2}} \approx s \cdot 1.29$ (thus the ratio σ_X/σ in their Table 1 lies approximately at 1.25 not around 1.0); for $\nu \in \{1, 1.5\}$ σ_X is not finite, thus location-scale Lambert W × F inference – with `IGMM`, maximum likelihood estimation (MLE), or any other estimator – is not well defined.

2.0.1. *New variant of location-scale Lambert W × F distributions.* In hindsight it would have been more clear to refer to the original transformation with μ_X and σ_X in Goerg (2011) as *mean-variance* Lambert W × F distributions with *location-scale* input in order to avoid any such confusion that arose in Stehlík and Hermann's letter. I will take this opportunity to introduce a new variant of location-scale Lambert W × F distributions.

DEFINITION 2.1 (Unrestricted Location-scale Lambert W × F). *Let $X \sim F_X(x \mid \boldsymbol{\beta})$ be a continuous location-scale random variable with location and scale parameters c and s, and $\boldsymbol{\beta}$*

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[1]Prior to **LambertW** v0.5.0 `beta2tau` issued a warning to notify users about this improper application.

	Estimate	Std. Error	t value
c	0.197	0.037	5.269
s	1.241	0.042	29.816
ν	7.092	1.379	5.142
γ	-0.053	0.014	-3.944

TABLE 1

mean-variance Lambert $W \times t$

	Estimate	Std. Error	t value
c	0.197	0.037	5.269
s	1.241	0.042	29.814
ν	7.092	1.379	5.142
γ	-0.045	0.011	-3.999

TABLE 2

unrestricted location-scale Lambert $W \times t$

TABLE 3

Lambert $W \times t$ MLE for the LATAM data: unrestricted location-scale version has a slightly smaller $\widehat{\gamma}$ than the mean-variance version, since $\widehat{s} = 1.24 < 1.46 = \widehat{\sigma}_X$, and the non-linearity of transformation (1).

parametrizes the distribution. Let

$$(1) \qquad Y = \left(\frac{X - c}{s} \cdot e^{\gamma \frac{X-c}{s}} \right) \cdot s + c = \left(U \cdot e^{\gamma U} \right) \cdot s + c,$$

where $U = \frac{X-c}{s}$. Then Y has an unrestricted location-scale *Lambert $W \times F$ distribution*.

Using general location and scale, rather than mean and standard deviation, is more natural when viewed solely as a distribution as it does not require the existence of first and second moments; e.g., Eq. (1) is well defined for t distributions with $\nu \leq 2$. Viewed as a data-generating process, however, transformation (1) has the main disadvantage that the image y_i of a specific realization $x_i \in \mathbb{R}$ and fixed γ depends on the value of the scale parameter of F_X. Reversely, for a fixed observed y_i and given variance and skewness of Y, the skewness parameter will be closer to zero for $F_X^{(1)}$ than for an alternative distribution $F_X^{(2)}$ if $s^{(1)} < s^{(2)}$ – hence making γ estimates incomparable across distributions with different scale parametrization (see Table 3 for illustration of this property on the LATAM data). It is for this reason that I had originally defined location-scale Lambert W × F distributions only for location-scale input random variables with finite mean and variance.

For future reference, researchers should pick the version that is appropriate for their analysis and state whether they use the *mean-variance* (original *location-scale*) or the unrestricted *location-scale* from Definition 2.1. Of course, in the latter case using the IGMM algorithm in its original form only makes sense for distributions where location and scale parameters coincide with first and/or second moments, e.g., Normal or exponential.

2.1. *Robust testing for normality against Pareto tail.* Here Stehlík and Hermann aimed to find out empirically for what type of data IGMM yields proper statistical inference. This is indeed an important question that I had not addressed in the original paper. Before going into details of their analysis, I present a bootstrap approach to check if IGMM estimates have good properties for inference on *mean-variance* Lambert W × F distribution. This bootstrap convergence analysis is now readily available in the **LambertW** R package (Goerg, 2016).

Figure 2 shows IGMM estimates of $\tau = (\mu_X, \sigma_X, \gamma)$ for the LATAM data and for a simulated Gaussian (Regime I) and Cauchy (Regime III) sample of the same length ($N = 1413$). I resampled the data with replacement and obtained $\widehat{\tau}_{IGMM}^{(n)}$ for varying sample sizes n. As Stehlík and Hermann show on p. 4, the IGMM estimate for σ_X (and μ_X) diverge if the data lies in Regime II (or Regime III). Figure 2 compares convergence properties for increasing sample
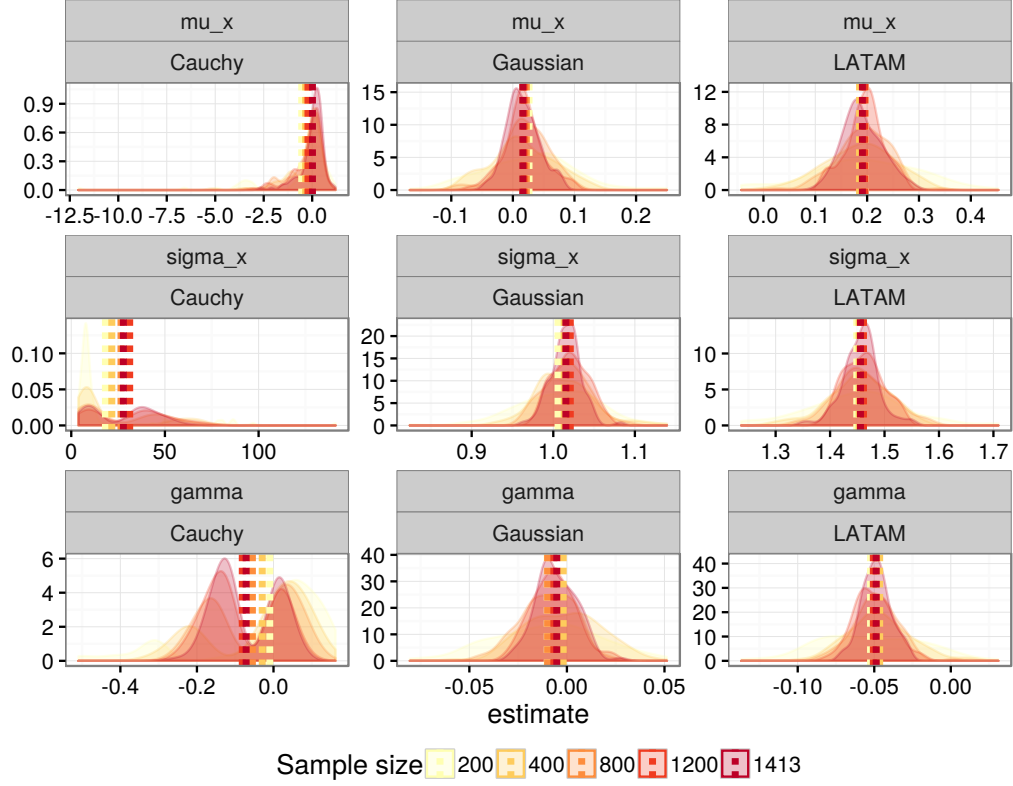
Fig 2: Bootstrap IGMM estimates with varying sample sizes (original size: $N = 1413$) for simulated Cauchy and Gaussian sample and the LATAM data (vertical lines represent averages).

size n: as expected, $\widehat{\mu}_X^{(n)}$ and $\widehat{\sigma}_X^{(n)}$ do not converge for the Cauchy sample; for the Gaussian and LATAM data they do. As n increases the uncertainty decreases in all estimates for the Gaussian and LATAM data, whereas the distribution of $\widehat{\mu}_X^{(n)}$ for the Cauchy sample does not change – a well-known characteristic of averages for Cauchy samples. Figure 3 shows that this uncertainty converges at the usual rate of $n^{-1/2}$ for a Normal sample and the LATAM data.

By contraposition of Stehlík and Hermann's arguments we can therefore conclude that the LATAM data does not lie in Regime II or III, and thus a skewed mean-variance Lambert W × F distribution is a viable option. One can then estimate $\tau = (\mu_X, \sigma_X, \gamma)$ with IGMM (or any other estimator for $\tau = \tau(\theta)$, where $\theta = (\boldsymbol{\beta}, \gamma)$ and $\boldsymbol{\beta}$ parametrizes $X \sim F(x \mid \boldsymbol{\beta})$).

Stehlík and Hermann aimed to answer this question indirectly: they use stable distribution estimates to suggest that the LATAM data lies in Regime III, and hence can not be accurately analyzed with IGMM. From the above bootstrap analysis we already know that it can not lie in Regime III (or II). It is thus a natural question to ask how they arrived at conclusions supporting Regime III. Below I discuss some shortcomings of stable distributions as a model for financial data in general, and also show that most of their results using Hill estimates cannot be reproduced.
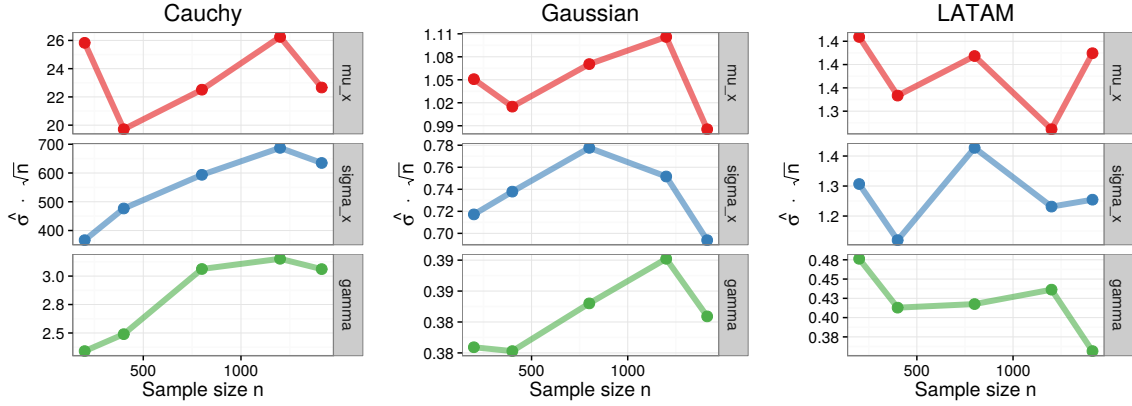
Fig 3: Sample standard deviation of IGMM bootstrap estimates times $\sqrt{n}$ with 100 replications for Caucy, Gaussian, and LATAM data.

Stehlík and Hermann estimate the tail index α of a Pareto distribution for the input of the LATAM series based on a modified Hill estimator for dependent data (Jordanova, Dusek and Stehlík, 2013). They conclude that the underlying input data falls in a regime ("Regime III") where neither mean nor variance exist. These findings are in contrast to the financial time series literature which is mainly concerned with the existence of finite fourth – not first – moments (Zadrozny, 2005; Mantegna and Stanley, 1998; Cont, 2001; Huisman et al., 2001).

To support their hypothesis about non-finite mean Stehlík and Hermann reference empirical findings based on stable distributions in Akgiray, Geoffrey Booth and Seifert (1988). The very same year though, Akgiray and Geoffrey Booth (1988) also published a meta analysis which rejects the stable-law model for a vast majority of 200 stock return series and they conclude that "[...] statistical inference should not be based on index α estimated from samples from stock returns". While stable distributions have good theoretical properties as a stochastic model for financial returns, empirical evidence of finite moments has researchers led to develop less restrictive distributions (Kim et al., 2009, 2006; Rosiński, 2007). Thus using stable distributions as an argument for non-existing mean is controversial – as exactly these restrictions on finite moments limit their aptness as a data-generating process for stock returns (Grabchak and Samorodnitsky, 2010; Lau, Lau and Wingender, 1990).

All parametric unconditional fits considered in Goerg (2011) (t, Lambert W × t, skew-t) can reject the hypothesis of a non-existing mean: the student-t MLE gives $\widehat{\nu} = 6.22$, with a 95% confidence interval (CI) of $(4.14, 8.3)$; the Lambert W × t fit has a similar 95% CI for $\widehat{\nu} = 7.09$: $(4.39, 9.79)$; similarly the skew-t fit gives $\widehat{\nu} = 7.16$ and skewness parameter $\widehat{\alpha} = -0.8$. To double-check their Hill estimates of the tail parameter I use the MLE for a continuous power law fit (Clauset, Shalizi and Newman, 2009) on the absolute values of the negative returns;[2] this yields an estimate of $\widehat{\alpha} = 3.99$ well outside Regime II or III (with optimal cutoff at $\widehat{x}_{\min} = 2.18$ – in absolute value). Since the LATAM series is not i.i.d., but exhibits dependence in the squared returns, I estimated a $GARCH$ model (Bollerslev, 1987) and showed that the standardized residuals also exhibit significant skewness (Goerg,

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[2]Estimates were obtained using the **poweRlaw** R package (Gillespie, 2015).
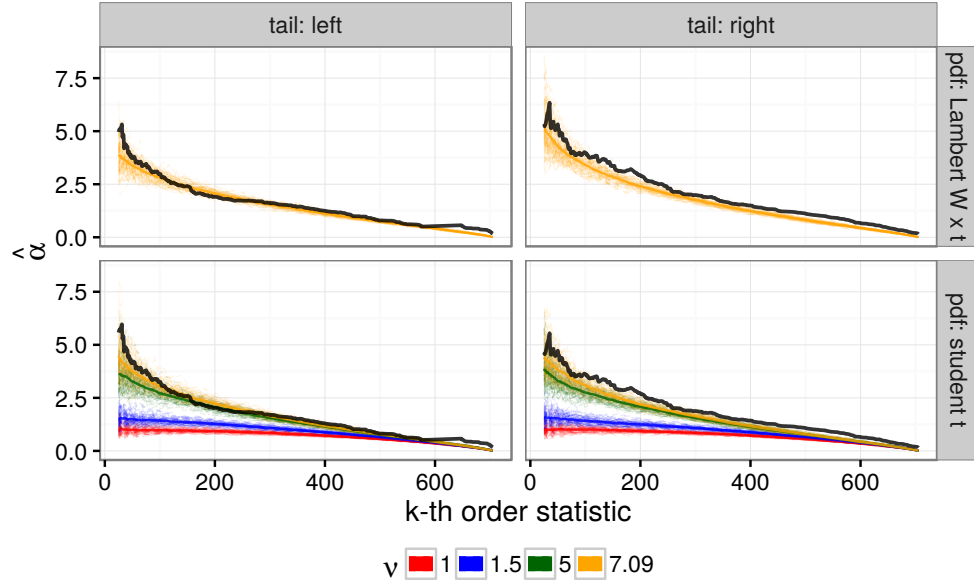
Fig 4: Hill curves for student t input and Lambert W × t output. Simulated values (red, blue, green, orange) plus observed and transformed LATAM data (black, solid); colored solid lines are pointwise averages at each k. Degrees of freedom $\nu = 1$, 1.5, and 5 correspond to Regime III, II, and I, respectively; $\nu = 7.09$ is the Lambert W × t MLE for ν.

2011). Also a $GARCH(1,1)$ model with heavy-tailed conditional skew-t innovations remains far outside Regime II or III with $\widehat{\nu} = 9.59$.

2.2. *A graphical screening between regimes of* `IGMM`.   One of the main premises of the Letter is that the input of the LATAM data falls in Regime III – as illustrated in Figure 2 of Stehlík and Hermann (2015). As shown above not only the data-driven bootstrap estimates, but also a comprehensive selection of statistical marginal and time series conditional models as well as the maximum likelihood Pareto tail estimates clearly reject their claim.

In this section I aim to replicate how Stehlík and Hermann arrived at their conclusion; however, their findings are not reproducible.

I follow their setup and draw i.i.d. samples from student-t distributions with $\nu = 1$, 2, and 5 degrees of freedom (Regime III, II, and I) and one with $\nu = 7.09$, which is the Lambert W × t MLE of ν for the LATAM data. Furthermore, I draw random samples from the estimated *true* Lambert W × t distribution with $\widehat{\theta}_{MLE} = (0.2, 1.24, 7.09, -0.05)$. Each simulation has the same number of samples as the LATAM data, $N = 1413$.[3] I then use the harmonic Hill estimator from Definition 1 of Stehlík and Hermann with $\beta = 2$ for the simulated data and with $\beta = 1.001$ for the LATAM data.[4] By taking absolute values Stehlík and Hermann

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[3]Stehlík and Hermann report a different sample size of $N = 1421$.

[4]Using the classic Hill estimator, e.g., `hillPlot()` in the **fExtremes** R package (Wuertz et al., 2013), is not only just marginally different in the parameter space ($\beta = 1$ vs. $\beta = 1.001$), but also the resulting Hill plots are essentially indistinguishable for the analyzed data and simulations.
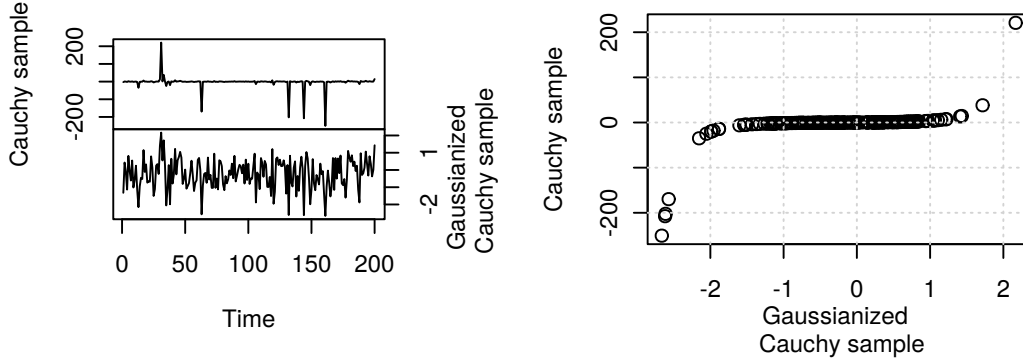
Fig 5: Gaussianizing a Cauchy: (left) random Cauchy sample $\mathbf{y}$ and its Gaussianized version $\mathbf{x}_{\widehat{\theta}_{IGMM}}$ using a heavy-tail Lambert W $\times$ Gaussian fit with $\widehat{\theta}_{IGMM} = (\mu_X = -0.23, \sigma_X = 0.88, \delta = 1.21)$ and fixed $\alpha \equiv 1$; (right) bijective mapping $\mathbf{y} \leftrightarrow \mathbf{x}_{\widehat{\theta}_{IGMM}}$.
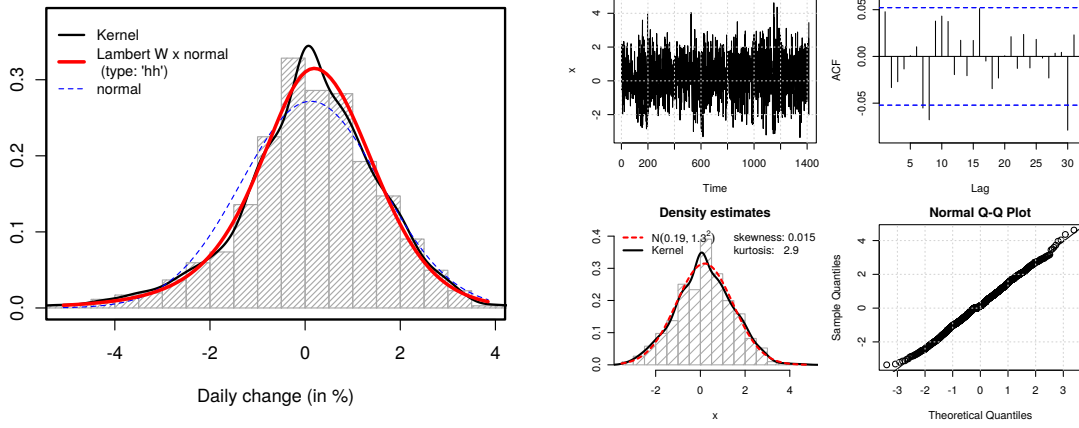
implicitly assume the series is centered around zero and left and right tail share the same properties. Since neither assumption applies to the LATAM data, I rather estimate two Hill curves for positive and negative values separately.[5] Each series was also centered by its median to ensure that Hill curves from different simulations all end at the same maximum order statistic.

Results from 100 replications are shown in Figure 4. The lower panel replicates Regimes I, II, and III from Figure 2 in Stehlík and Hermann (2015). However, the Hill estimates for the LATAM data *cannot* be reproduced: the black, solid line does not appear below – or even close to – Regime III (red) as Stehlík and Hermann show in their plots, but falls mostly on the orange Lambert W $\times$ t samples with $\nu = 7.09$, clearly *above* Regime III (red). Overall the Hill estimates even support the left-skewed Lambert W $\times$ t distribution as a feasible marginal model for the LATAM series.

2.3. *On Regime III of IGMM.* Stehlík and Hermann give suggestions for a robustification of the IGMM algorithm when μ_X and/or σ_X do not exist. However, in such a regime the original location-scale Lambert W $\times$ F distributions are not defined; thus estimating μ_X or σ_X with any algorithm is not a well-posed objective after all.

For such extremely heavy-tailed data I extended skewed to heavy-tailed Lambert W $\times$ F distributions (Goerg, 2015). As an illustration, consider a random i.i.d. sample from a standard Cauchy distribution – upper-left panel in Figure 5. After estimating the parameters of the heavy-tail Lambert W $\times$ Gaussian distribution (using IGMM with type = "h") the Cauchy data can be transformed to a Gaussianized version of itself (lower-left) using a bijective mapping (right). See Goerg (2015) for details and empirical performance of IGMM and MLE in presence of heavy tails (also covering Regime II and III cases).

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[5]Using their absolute value approach does not change the results qualitatively.

(a) Model comparison: histogram, non-parametric ker-
nel density estimate (KDE), a fitted Normal distribution
baseline, and the parametric fit to the data.

(b) Properties of Gaussianized data: time series
plot, ACF estimates, fitted Normal distribution
compared to KDE, and a Normal QQ-plot.

Fig 6: Heavy double-tail Lambert W × Gaussian distribution fit to the LATAM data.

As yet another model to test the Regime II or III hypothesis for the LATAM data I fit a
two-sided heavy-tail Lambert W × Gaussian distribution via MLE: $\widehat{\delta}_l = 0.14$ and $\widehat{\delta}_r = 0.03$.
As expected the left tail is heavier than the right $(\widehat{\delta}_l > \widehat{\delta}_r)$, and $1/\widehat{\delta}_l = 6.92$ suggests that
moments up to order 6 exist (confidence intervals do not cover Regime II or III).[6] Figure 6a
shows that the model fits extremely well and neither parametric and non-parametric density
estimates nor QQ-plots indicate major deviations from Normality for the back-transformed
data (Fig. 6b).

3. Skewness: On Asset returns and t-distribution. Stehlík and Hermann discuss
a more subtle point about "symmetrization" since symmetry is often an inherent (physical)
property of a system or object. They opine that symmetry cannot be simply achieved by a
mere variable (data) transformation. As a popular counterexample consider the log-normal
distribution. It can either be seen as a useful distribution for non-negative right skewed data
or as a transformation that achieves symmetry in the random variable / data.

Yet, exactly this symmetry of a system gave rise to skewed Lambert W × F distributions in
the first place. The core idea originated from observing the "system" of a financial market and
how it processes new information into the price of a stock, currency, etc. In principle there was
no obvious reason why "bad" news should be more common or extreme than "good" news.
Put in other words, we would expect a symmetry in positive vs. negative returns. However,
as many empirical studies have shown (see references in Goerg, 2011), financial returns are
usually negatively skewed – seemingly in conflict with the symmetry of good vs. bad news.
One way to accomodate the observed asymmetry is to realize that in a financial market we
do not observe news *per se*, but people's reaction to them. As often is the case, people react

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[6]For a heavy tail Lambert W × Gaussian with $\alpha \equiv 1$ and tail parameter δ moments up to order $1/\delta$ exist.

more extremely to bad news than they do to good ones (see Beber and Brandt, 2010; Eil and Rao, 2011, for related studies).

The Lambert W × F framework integrates the symmetry of news $X \sim F$ (where F is a symmetric distribution with finite mean and variance), with the empirical evidence of negative skewness in financial returns, Y, by modeling the news processing as an asymmetric function (assuming $\mu_X = 0$ and $\sigma_X = 1$ for simplicity)

$$X \mapsto Y = X \cdot e^{\gamma X}, \tag{2}$$

where $\gamma \in \mathbb{R}$ encodes how people react to news: for $\gamma < 0$ bad news are exaggerated, for $\gamma = 0$ the quality of news has no effect, and for $\gamma > 0$ negative news affect prices less intensely.[7] Clearly this is only a model and I do not claim that (2) is the true data-generating process of financial markets. Yet, it conveniently embeds symmetry of news, asymmetric information processing, and empirical evidence of negative skewness in financial data in a statistical model that can be estimated from observed data.

If one cares about symmetry of a system, then one must decide if transforming random variables within the skewed Lambert W × F framework is appropriate given domain-specific context. If inherent symmetry is not important, then one can ignore the random variable (or data) transformation and only view Lambert W × F as yet another asymmetric distribution.

In Section 4 they use skewed input U, transform it via

$$Y = (U \cdot \exp(-bU)) \, c + a, \tag{3}$$

use `IGMM` to obtain an estimate of U, and finally test for symmetry to show that the method fails to recognize the skewness in the input.

Again, this is based on a misunderstanding of the Lambert W × F distributions in the location-scale setting and what IGMM aims to estimate. First, U must have zero mean and unit variance, which is in general not true for the skew-t distribution. Thus (3) describes a non-central, non-scale Lambert W × skew-t distribution (Definition 2.1 Goerg, 2011), which is furthermore scaled (by c) and shifted (by a). This is not a location-scale Lambert W × F distribution; so there are no μ_X or σ_X to be estimated.[8] Secondly, the authors assume that the estimate of U from `IGMM` should be close to their simulated skew-t U. This is also not the case. By default, `IGMM` sets the target skewness to 0, thus aiming to recover symmetric input. For a fair comparison one should try to obtain the true U, in which case the target skewness must be set to the theoretical skewness of a skew-t (or skew normal) distribution given the respective skewness parameter. However, this is quite an unusual route to take as there is little value in transforming skewed data to yet another skewed dataset.

3.0.1. *Autocorrelation.* Their findings that the "autocorrelation function of the back-transformed data of LATAM series has observed to be significant (e.g., lags 2, 7, 8, 13, and 30)" are debatable: only few lags (e.g., 1, 7, 8, 16, and 30) of the autocorrelation estimates of the LATAM

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[7]For extremely large (in absolute value) X, the mapped output Y is again closer to zero. As every model, also (2) is an approximation and for practical purposes such extreme values can be ignored as they usually occur with miniscule probability. For details see comments on the non-principal branch probability $p_{(-1)}$ in Goerg (2011).

[8]According to the new Definition 2.1 from Section 2.0.1 this can be now identified as an *unrestricted* location-scale Lambert W × t random variable.
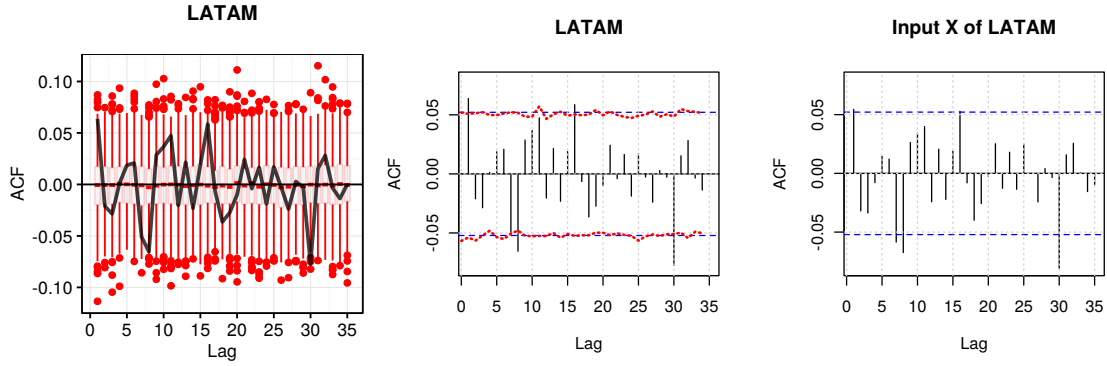
Fig 7: Estimated autocorrelation function (ACF) of LATAM data: (left) bootstrapped (red, dotted)); (center) standard 95% confidence intervals (blue, dashed); (right) ACF of back-transformed data.

series and its symmetrized input fall barely outside the 95% confidence intervals (Figure 7). Using a multiple lag test, e.g., Box-Ljung test (Ljung and Box, 1978), is inconclusive at best on rejecting white noise: testing the transformed data for all lags up to 30 none of the 30 p-values are below $\alpha = 0.01$; for $\alpha = 0.05$ 18 reject the null (note multiple hypothesis testing though). In any case, the ACF plot indicates already that this fund would not be the first choice for a successful trading strategy.

It is not clear from their Letter why these autocorrelation findings – even if they showed relevant deviations from white noise – matter for the main points of the paper. In Section 7.2, p. 29 of Goerg (2011) I only considered the unconditional distribution for the sake of illustrating the method. Lambert W × F time series models – including ARMA, GARCH, and SV models – were far beyond the scope of the paper and would not have added any further insights on the new methodology to model and symmetrize skewed data. I thus did not elaborate on this time series model fitting exercise.

4. Discussion. Based on claims in Stehlḱ and Hermann (2015) I clarify definitions and properties of location-scale Lambert W × F distributions, and explain when estimating μ_X and σ_X is a well-defined task. Their methodological concerns about IGMM are mostly spurious since they result from using incorrect definitions of skewed Lambert W × F random varibales. As a consequence I also introduce a new variant of location-scale Lambert W × F distributions which does not rely on the existence of first and second moments in $X \sim F$.

On the applied side, Stehlík and Hermann point out that a method of moments estimators should not be used for distributions whose moments do not exist, and perform an extensive simulation study to claim that the financial log-returns analyzed in Goerg (2011) lie in a regime of non-finite mean. Using data-driven bootstrap estimates and several parametric distribution and time series models I show that this is not the case. I also tried to replicate Stehlík and Hermann's findings based on Hill estimates, but failed to reproduce most of their results.

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