Some relations involving LambertW

$$W(z)\exp(W(z)) = z \iff e^{-W(z)} = \frac{W(z)}{z} \iff ze^{-W(z)} = W(z)$$

$$W_\delta(z) = \operatorname{sgn}(z)\sqrt{\frac{W(\delta z^2)}{\delta}}$$

$$W_\delta(z)^2 = \frac{W(\delta z^2)}{\delta} \iff W_\delta(z)^2 = z^2 e^{-W(\delta z^2)}$$

$$\frac{W_\delta(z)}{z} = \frac{1}{|z|}\sqrt{\frac{W(\delta z^2)}{\delta}} = \exp\{-\frac{1}{2}W(\delta z^2)\} = \exp\{-\frac{\delta}{2}W_\delta(z)^2\}$$

$$\log\frac{W_\delta(z)}{z} = -\frac{1}{2}W(\delta z^2)$$

Suppose $Y \sim \text{LambertW} \times F_X$

$$z = \frac{y-\mu}{\sigma}$$

$$g_Y(y \mid \beta, \delta) = f_X(W_\delta(z)\sigma + \mu \mid \beta) \cdot \frac{W_\delta(z)}{z \cdot [1 + W(\delta z^2)]}$$

$$\begin{aligned}
\log g_Y &= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) + \log\frac{W_\delta(z)}{z} - \log(1 + W(\delta z^2)) \\
&= \log f_X(W_\delta(z)\sigma + \mu \mid \beta) - \frac{1}{2}W(\delta z^2) - \log(1 + W(\delta z^2))
\end{aligned}$$

For $X \sim \mathcal{N}(\mu, \sigma)$

$$\begin{aligned}
\log f_X(W_\delta(z)\sigma + \mu \mid \beta) &= \log\frac{1}{\sqrt{2\pi\sigma^2}} + \log\exp\{-\frac{1}{2\sigma^2}(W_\delta(z)\sigma + \mu - \mu)^2)\} \\
&\propto -\log\sigma - \frac{1}{2}W_\delta(z)^2 \\
&\propto -\log\sigma - \frac{1}{2}\frac{W(\delta z^2)}{\delta}
\end{aligned}$$

$$\begin{aligned}
\log g_Y &\propto -\log\sigma - \frac{1}{2}\frac{W(\delta z^2)}{\delta} - \frac{1}{2}W(\delta z^2) - \log(1 + W(\delta z^2)) \\
&\propto -\log\sigma - \frac{1}{2}\left(1 + \frac{1}{\delta}\right)W(\delta z^2) - \log(1 + W(\delta z^2))
\end{aligned}$$

Now assume $Y_1, \ldots, Y_N$ i.i.d.

Then $g_Y(Y_1, \ldots, Y_N) = \prod_{i=1}^{N} g_y(y_i \mid \beta, \delta)$ and the likelihood is

$$\log \prod_{i=1}^{N} g_y(y_i \mid \beta, \delta) = -N \log \sigma - \frac{1}{2}\left(1 + \frac{1}{\delta}\right) \sum_{i=1}^{N} W(\delta z_i^2) - \sum_{i=1}^{N} \log(1 + W(\delta z_i^2))$$