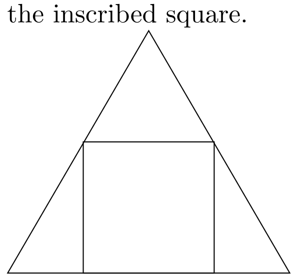
Maths Problems Set 1 (13 December 2019)

1. Each side length of the equilateral triangle below is 1. Find the area of the inscribed square.



2. Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

   (a) Remove an equal number of balls from each bag;

   (b) Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations?

   Operation (b) is now replaced with

   (b') Triple the number of balls in one bag.

   Is it now always possible to empty both bags after a finite sequence of operations?

3. Prove that there are infinitely many non-trivial Pythagorean triples, i.e. not a multiple of a different Pythagorean triple.

4. Straight lines are drawn on a infinite 2D plane. Show that the resulting regions can be coloured with two colors such that no adjacent regions have the same color.

5. For a four digit number n (using at least two different digits with leading zeros allowed), the digits of n are rearranged to form the largest possible four digit number a and the smallest possible four digit numbers b (with leading zeros if necessary). The difference between the a and b is found, and this process is repeated with n being this difference.

   Prove that the process eventually hits 6174 and remains there in at most 7 iterations for any four digit number n.

6. Gnomes have friendships. Friendship is commutative. A gnome is odd if it has an odd number of friends. Show that there is always an even number of odd gnomes.

7. Prove that if A and B are coprime where $A, B \in \mathbb{Z}$, then $\exists\ s, t \in \mathbb{Z}$ such that $As + Bt = 1$.

8. Evaluate $\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\ \mathrm{d}x$.

9. Find the smallest $a \geq 1$ such that $e^{y-x} \geq \frac{a+\sin x}{a+\sin y}$ for all $y \geq x$.

10. Evaluate $\lim\limits_{n\to\infty}(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n})$.

11. Find an expression for $\int_1^n (-1)^{\lfloor x \rfloor} \lfloor x \rfloor^{-1}\ \mathrm{d}x$ where $n \in \mathbb{N}$.

12. Prove that every prime has infinitely many multiples in the Fibonacci sequence. Prove that they have a constant frequency. Deduce that there are infinitely many Fibonacci numbers that are the product of only primes that had not previously occurred.

13. Find all solutions to the Diophantine equation $4x^2 = y^3 + 1$.

14. Let $A = \{0, 1, ..., 2^n - 1, 2^n\}$ and $B = \{0, ..., n\}$. How many functions g can be defined from A to B such that both of the following conditions hold:

    - for all $x \in B$ we have $g(2^x) = x$
    - for all $y, z \in A$ with $y \leq z$ we have $g(y) \leq g(z)$

15. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $a, b \in \mathbb{Z}$:

    (a) $f(a) + f(b) = f(f(a + b))$, or

    (b) $f(a) + f(2b) = f(f(a + b))$, or

    (c) $f(2a) + 2f(b) = f(f(a + b))$

16. There are n points on 2D plane. Show that it is always possible to select at least $\sqrt{n}$ of this points so that the points selected do not form any equilateral triangles.