Mathematical Methods III (B Course): Example Sheet 1 Easter 2021

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1. Let $\mathbf{e}_1$ represent the vector "go 1 mile East" and let $\mathbf{e}_2$ represent the vector "go 1 mile North". Let $\mathbf{x}$ be the vector

$$\mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2.$$

Give $\mathbf{x}$ relative to the basis

$$\mathbf{e}_1' = \mathbf{e}_1, \;\; \mathbf{e}_2' = \text{ “go 2 miles North-East”}.$$

2. Write the system of equations

$$\begin{aligned} p + z &= 2 \\ 2p + y + w &= 4 \\ y + 3w &= 3 \\ 2z + p + 3y &= 6\,, \end{aligned}$$

in matrix form $\mathbf{A}\,\mathbf{x} = \mathbf{d}$ in terms of the column vectors

$$\mathbf{x} = \begin{pmatrix} p \\ y \\ z \\ w \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 6 \end{pmatrix},$$

but do not bother to solve the equations.

3. Consider the column matrix $\mathbf{u}$ and the row matrix $\mathbf{v}$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 & 2 & b \end{pmatrix}.$$

Show that $\mathbf{v} \cdot \mathbf{u} = 2 + 2b$ and evaluate the 3×3 matrix $\mathbf{u}\,\mathbf{v}$.

4. The matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

State which of the following products do not exist (i.e. are inconsistent with the rules of matrix multiplication) and evaluate those that do: $\mathbf{A^2}, \mathbf{AB}, \mathbf{AC}, \mathbf{CA}, \mathbf{B^2}, \mathbf{BC}, \mathbf{CB}, \mathbf{C^2}$.

5. Let $\mathbf{D}$ be an $(N \times M)$ matrix. For what values of N and M do (i) $\mathbf{D}\mathbf{D}^T$ and (ii) $\mathbf{D^T D}$ exist?

6. Solve the system of equations

$$\begin{pmatrix} 6 & 7 & 3 & -1 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 19 \\ 4 \\ 8 \end{pmatrix}$$

for x, y, z, w.

Gaussian elimination aims to cast more complicated systems of equations into a form similar to the system seen here because they are easy to solve.

7. Find a 3×3 matrix $\mathbf{A}$ such that the equation

$$x^2 + 4y^2 + 9z^2 + 4xy - 6xz = 1$$

can be written in the matrix form

$$\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1.$$

8. Solve the following equations

$$3x + 2y = 9, \qquad x + 3y = 17,$$

by the method of Gaussian elimination, (*i.e.* divide the first one by three and then eliminate x from the second one).

9. Let

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Calculate $\mathbf{M}^2$, $\mathbf{M}\mathbf{M}^T$, $\mathbf{M}^T\mathbf{M}$. Express $\mathbf{M}$ as the sum of a symmetric and an antisymmetric matrix.

10. Define a new sort of multiplication of matrices *(bullet multiplication)* by

$$(\mathbf{A} \bullet \mathbf{B})_{ij} = \sum_k (\mathbf{A})_{ik} (\mathbf{B})_{jk}.$$

How can this be expressed in terms of ordinary matrix multiplication? Show using suffix notation that bullet multiplication is not associative i.e. $(\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C} \neq \mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C})$.

11. Prove the following results:

(i) The trace of the product of a symmetric and an antisymmetric matrix is zero.

(ii) If $\mathbf{A}$ is antisymmetric, then $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$ for any column vector $\mathbf{x}$.

Is the geometrical action of an antisymmetric matrix on an arbitrary vector a rotation?

12. The matrices $\mathbf{A}$ and $\mathbf{B}$ and the vector $\mathbf{x}$ have elements a_{ij} , b_{ij} and x_i. What are the elements of $\mathbf{A}^T$? Write down the matrices represented by

(i) $\sum_{j,k} a_{ij}b_{jk}x_k$,
(ii) $\sum_j x_j b_{ij}$,
(iii) $\sum_{i,j} a_{ij}b_{kj}x_i$,
(iv) $\sum_{j,k} a_{ij}a_{kj}a_{km}$.

13. Show, by producing specific examples, that:

(i) The product of two non-zero matrices can be zero;
(ii) $\mathbf{AB}$ can be zero even if $\mathbf{BA}$ is non-zero;
(iii) The product of two non-zero symmetric matrices can be anti-symmetric.

14. For any square matrix $\mathbf{M}$, the matrix $\exp \mathbf{M}$ is defined by

$$\exp \mathbf{M} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{M}^n}{n!}.$$

For the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and any real number θ, show that

$$\exp \theta\mathbf{M} = \mathbf{I} + \mathbf{M}\sin\theta + \mathbf{M}^2(1 - \cos\theta),$$

and hence show that

$$\exp \theta_1\mathbf{M} \;\exp \theta_2\mathbf{M} = \exp(\theta_1 + \theta_2)\mathbf{M}.$$

Show also that $(\exp \theta\mathbf{M})(\exp \theta\mathbf{M})^T = \mathbf{I}$. Without detailed calculations, explain whether this result will hold for general $\mathbf{M}$, or whether $\mathbf{M}$ here is of special type (which you should specify).

15. Let $\mathbf{b} = (b_1, b_2, b_3)$ be a fixed vector. Find the (3×3) matrix $\mathbf{B}$ which satisfies $\mathbf{B}\,\mathbf{x} = \mathbf{b}\times\mathbf{x}$ for any vector $\mathbf{x}$ (i.e. find the elements of $\mathbf{B}$ in terms of the components of $\mathbf{b}$). By considering $\mathbf{B^2x}$, verify the formula $\mathbf{b} \times (\mathbf{b} \times \mathbf{x}) = (\mathbf{b}.\mathbf{x})\,\mathbf{b} - (\mathbf{b}.\mathbf{b})\,\mathbf{x}$. (I am using the notation $\mathbf{a} \times \mathbf{b}$ for the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$.)

16. Calculate $\det \mathbf{A}$, $\det \mathbf{B}$, $\det \mathbf{AB}$, and $\det(\mathbf{A}^{-1})$ for $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$, $\det \mathbf{A^{-1}} = 1/\det \mathbf{A}$.

17. Calculate the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

by

(i) row operations (Gaussian elimination),

(ii) expanding along the first row, i.e. by calculating $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$,

(iii) expanding down the first column, i.e. by calculating $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$. Here A_{ij} is the cofactor of a_{ij}.

18. What can be said about the columns of a matrix whose determinant is zero? For which value of a are the vectors $(1, 0, 1, 0)$, $(2, 1, 3, 2)$, $(4, 0, 1, 3)$ and $(2, 0, 3, a)$ linearly dependent?

19. In the theory of superconductivity, the energy E satisfies an equation of the form

$$\begin{vmatrix} a-E & -b & -b & -b & -b \\ -b & a-E & -b & -b & -b \\ -b & -b & a-E & -b & -b \\ -b & -b & -b & a-E & -b \\ -b & -b & -b & -b & a-E \end{vmatrix} = 0\,.$$

Show by solving this equation that there is one state with energy $E = a - 4b$ and all the others have energy $E = a + b$.

20. Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and hence solve $\mathbf{A}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

21. The two $(N \times N)$ matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{AB} = \mathbf{I}$. The N rows of $\mathbf{A}$ are denoted $\mathbf{a}_i$ for $1 \le i \le N$ and the N columns of $\mathbf{B}$ are denoted by $\mathbf{b}_j$ for $1 \le j \le N$. What can be said about the scalar products $\mathbf{a}_i \cdot \mathbf{b}_j$?

Show that in the case $N = 3$, these relationships are satisfied by the vectors $\mathbf{a}_i$ and

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]} \quad \mathbf{b}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]} \quad \mathbf{b}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]}$$

where $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ denotes the scalar triple product.

22. Show that if the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$ has a solution, then $\mathbf{y}$ must be a linear combination of the column vectors of $\mathbf{A}$.

23. Cramer's rule for writing down the solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$ is

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}}\,,$$

where $\mathbf{A}_i$ is the matrix obtained by replacing the *i-th* column of $\mathbf{A}$ by column vector $\mathbf{y}$. Use the cofactor expression for the inverse of a matrix to derive Cramer's rule in the 3×3 case, commenting on the cases where there is no solution and where the solution is not unique.

24. Find the rotation matrix $\mathbf{R}$ such that $\mathbf{y} = \mathbf{R}\,\mathbf{x}$ represents an anticlockwise rotation about the z-axis through an angle of $\pi/2$ radians.

25. Write the following equations in the form $\mathbf{A}\mathbf{x} = \mathbf{y}$:

$$\begin{aligned} x + y + az &= 1 \\ 3x + 4y + (2+3a)z &= 5 \\ -x + y + z &= b\,. \end{aligned}$$

By using row operations reduce $\mathbf{A}$ to upper triangular form and calculate $|\mathbf{A}|$, distinguishing between the cases $|\mathbf{A}| = 0$ and $|\mathbf{A}| \neq 0$.

Find the values of a and b which give rise to unique solutions, no solution or many solutions. (*The method to be used for this question is given in section 1.5.2 item (3i) of the notes.*)

26. Use row operations (Gaussian elimination) to solve the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\,.$$

Let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ be the solutions of the above equation corresponding to the three cases $\mathbf{y} = (1,0,0)$, $\mathbf{y} = (0,1,0)$, $\mathbf{y} = (0,0,1)$ respectively. What is the relationship between $\mathbf{A}$ and the matrix whose columns are $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$?

27. The effect of a *reflection matrix* is to reflect vectors in a fixed plane; in other words, the component of the vector normal to the plane is reversed and the component parallel to the plane is unchanged. What do you think the eigenvalues of a reflection matrix are? What is the geometrical significance of the corresponding eigenvectors. Are they orthogonal?

28. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{M} = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}\,.$$

Verify that: (i) the eigenvalues are real; (ii) the sum of the eigenvalues is equal to trace $\mathbf{M}$; (iii) the product of the eigenvalues is equal to det $\mathbf{M}$; (iv) the eigenvectors are orthogonal.

29. Find the eigenvalues and eigenvectors ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$), of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}\,.$$

Let $\mathbf{y} = (6, -3, 0)^T$. By taking vector dot products with each of the eigenvectors in turn, find p_1, p_2 and p_3 such that $\mathbf{y} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$.

By using this decomposition, writing also $\mathbf{x} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$, solve the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$. Is the solution unique?

30. The column vector $\mathbf{e}$ is n-dimensional and has unit length. The $n \times n$ matrix $\mathbf{A}$ is given by $\mathbf{A} = \lambda \mathbf{e}\mathbf{e}^T$, where λ is a real number. Show that $\mathbf{A}$ has eigenvector $\mathbf{e}$ with eigenvalue λ and that all other $n-1$ eigenvalues of $\mathbf{A}$ are zero.

The real symmetric $n \times n$ matrix $\mathbf{B}$ has orthonormal eigenvectors $\mathbf{e}_a,\ a = 1 \ldots n$ with eigenvalues $\lambda_a,\ a = 1 \ldots n$. Show that $\mathbf{B}$ can be written as

$$\mathbf{B} \ = \ \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T + \ldots + \lambda_n \mathbf{e}_n \mathbf{e}_n^T.$$

(*This is how examiners construct examples of matrices for exams so that the answers are nice numbers.*)

31. Write the conic $x^2 + 16xy - 11y^2 = 1$ in the form $\mathbf{x}^T \mathbf{A} \mathbf{x} = 1$, where $\mathbf{A}$ is a 2×2 matrix. Find the eigenvalues and eigenvectors of $\mathbf{A}$. Write down the equation of the conic referred to new axes along the directions of the eigenvectors.

32. A stationary point of the function $f(x, y)$ is a minimum if the Hessian matrix (i.e. the matrix of second derivatives) is positive definite (i.e. its eigenvalues are both positive) and a saddle if the eigenvalues have opposite signs. Investigate the stationary points of $f(x, y) = x^3 + xy + y^2$.

33*. Verify that the following matrix is orthogonal:

$$\mathbf{O} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \quad .$$

Given that it represents a rotation, what can be said about one of its eigenvalues? Find the axis of rotation (i.e., the axis about which the rotation takes place).

Harder part: By considering geometrically the effect of the matrix on vectors that lie in the 2D subspace orthogonal to the axis, deduce that such vectors rotate according to the 2D orthogonal matrix (defined to act in the 2D subspace)

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ ,$$

where θ is the angle of rotation. Deduce that two of the eigenvalues of $\mathbf{O}$ are $e^{i\theta}, e^{-i\theta}$ and that trace $\mathbf{O} = 1 + 2\cos\theta$. Hence, find the angle of rotation in this case. (Section 1.6 of the notes will help.)