$\boxed{5.}$ $(x+y+1)^2 \frac{dy}{dx} + (x+y+1)^2 + x^3 = 0$

$$\frac{dy}{dx} = -1 - \frac{x^3}{(x+y+1)^2}$$

Let $u = x+y+1$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$= 1 - 1 - \frac{x^3}{(x+y+1)^2}$$

$$\frac{du}{dx} = -\frac{x^3}{u^2}$$

$$\int u^2 du = -\int x^3 dx$$

$$\frac{1}{3} u^3 = -\frac{1}{4} x^4 + c$$

$$u^3 = -\frac{3}{4} x^4 + c'$$

$$u = \sqrt[3]{-\frac{3}{4} x^4 + c'}$$

$$x+y+1 = \sqrt[3]{-\frac{3}{4} x^4 + c'}$$

$$y = -x - 1 + \sqrt[3]{-\frac{3}{4} x^4 + c'}$$

$\boxed{6.}$ ⓐ $\dfrac{dy}{dx} - y = xy^5$

Bernoulli DE with $n=5$

Let $z = y^{-4}$

$$\frac{dz}{dx} = \frac{dy}{dx}(-4y^{-5})$$

$$= (y+xy^5)(-4y^{-5})$$

$$= -4x - 4y^{-4}$$

$$= -4x - 4z$$

$$\frac{dz}{dx} + 4z = -4x \qquad p(x) = 4$$

integrating factor: $\mu(x) = e^{\int p(x)dx} = e^{\int 4dx} = e^{4x}$

$$e^{4x}\left(\frac{dz}{dx} + 4z\right) = -4xe^{4x}$$

$$\frac{d}{dx}(ze^{4x}) = -4xe^{4x}$$

$$ze^{4x} = -4\int xe^{4x}dx \qquad u = x \quad v = \tfrac{1}{4}e^{4x}$$

$$u' = 1 \quad v' = e^{4x}$$

$$= -4\left(\frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}dx\right)$$

$$= \frac{1}{4}e^{4x} - xe^{4x} + C$$

$$z = \frac{1}{4} - x + Ce^{-4x}$$

$$y = z^4$$

$$\therefore y = \left(\frac{1}{4} - x + Ce^{-4x}\right)^4$$

(b) $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

Bernoulli DE where $n=1$

Let $z = y^{-1}$

$$\frac{dz}{dx} = \frac{dy}{dx}(-y^{-2})$$

$$= \left(y^2(\cos x - \sin x) - y\right)(-y^{-2})$$

$$= \cos x - \sin x + y^{-1}$$

$$= \cos x - \sin x + z$$

$$\frac{dz}{dx} - z = \cos x - \sin x \qquad p(x) = -1$$

integrating factor: $u(x) = e^{\int p(x)dx} = e^{-\int dx} = e^{-x}$

$$e^{-x}\left(\frac{dz}{dx} - z\right) = e^{-x}(\cos x - \sin x)$$

$$\frac{d}{dx}(ze^{-x}) = e^{-x}(\cos x - \sin x)$$

$$ze^{-x} = \int e^{-x}(\cos x - \sin x)dx$$

$u = \cos x - \sin x \quad v = -e^{-x}$

$u' = \sin x - \cos x \quad v' = e^{-x}$

$$= -e^{-x}(\cos x - \sin x) + \int e^{-x}(\sin x - \cos x)dx$$

$$= -e^{-x}(\cos x - \sin x) - ze^{-x} + C$$

$$2ze^{-x} = -e^{-x}(\cos x - \sin x) + C$$

$$z = \frac{\sin x - \cos x + C'e^{x}}{2}$$

$y = \frac{1}{z} \Rightarrow y = \frac{2}{\sin x - \cos x + C'e^{x}}$

$\boxed{7.}$ $(y-x)\dfrac{dy}{dx}+(2x+3y)=0$

$$\frac{dy}{dx}=\frac{2x+3y}{x-y}$$

Multiplying $x$ and $y$ by $\alpha$ does not change the equation.

$\therefore$ Homogeneous

Let $y=ux$

$$\frac{dy}{dx}=u+x\frac{du}{dx}=\frac{2x+3ux}{x-ux}$$

$$u+x\frac{du}{dx}=\frac{2+3u}{1-u}$$

$$x\frac{du}{dx}=\frac{2+3u}{1-u}-u$$

$$x\frac{du}{dx}=\frac{2+3u-u+u^2}{1-u}$$

$$\int\frac{1-u}{u^2+2u+2}\,du=\int\frac{1}{x}\,dx$$

$$\ln|x|=-\int\frac{u-1}{u^2+2u+2}\,du$$

$$=-\frac{1}{2}\int\frac{2u-2}{u^2+2u+2}\,du$$

$$=-\frac{1}{2}\left(\int\frac{2u+2}{u^2+2u+2}\,du-\int\frac{4}{u^2+2u+2}\,du\right)$$

$$=2\int\frac{1}{u^2+2u+2}\,du-\frac{1}{2}\int\frac{2u+2}{u^2+2u+2}\,du$$

$$\ln|x| = 2\int \frac{1}{1+(u+1)^2}du - \frac{1}{2}\int \frac{2u+2}{u^2+2u+2}du$$

Let $v=u+1$
$dv=du$
Let $w=u^2+2u+2$
$dw=(2u+2)du$

$$= 2\int \frac{1}{1+v^2}dv - \frac{1}{2}\int \frac{dw}{w}$$

$$= 2\tan^{-1}(v) - \frac{1}{2}\ln|w| + c$$

$$= 2\tan^{-1}(u+1) - \frac{1}{2}\ln|u^2+2u+2| + c$$

$$u = \frac{y}{x}$$

$$\therefore 2\tan^{-1}\left(\frac{y}{x}+1\right) - \frac{1}{2}\ln\left(\frac{y^2}{x^2}+\frac{2y}{x}+2\right) = \ln x + c'$$