Natural Sciences Tripos, Part IA
Mathematical Methods II, Course B
Answers to Example Sheet 2
Functions of More than One Variable

Dr S. M. Colwell

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Please communicate any errors to `smc1@cam.ac.uk`.

Skills section

S1. (Solutions listed in the order f_x, f_y, f_{xx}, f_{yy}, f_{xy}.)

(a) $3x^2 - 4xy + 3y^2$, $-2x^2 + 6xy - 12y^2$, $6x - 4y$, $6x - 24y$, $-4x + 6y$

(b) $-2xy^2\,\exp(-x^2y^2)$, $-2x^2y\,\exp(-x^2y^2)$, $2y^2(2x^2y^2 - 1)\exp(-x^2y^2)$, $2x^2(2x^2y^2 - 1)\exp(-x^2y^2)$, $4xy(x^2y^2 - 1)\exp(-x^2y^2)$

(c) $-(1/y)\exp(-x/y)$, $(x/y^2)\exp(-x/y)$, $(1/y^2)\exp(-x/y)$, $[x(x-2y)/y^4]\exp(-x/y)$, $[(y - x)/y^3]\exp(-x/y)$

(d) $\cos(x + y)$, $\cos(x + y)$, $-\sin(x + y)$, $-\sin(x + y)$, $-\sin(x + y)$

(e) $-(2x + y)/(x^2 + xy + 2y^2)^2$, $-(x + 4y)/(x^2 + xy + 2y^2)^2$, $2(3x^2 + 3xy - y^2)/(x^2 + xy + 2y^2)^3$, $2(-x^2 + 6xy + 12y^2)/(x^2 + xy + 2y^2)^3$, $(3x^2 + 17xy + 6y^2)/(x^2 + xy + 2y^2)^3$

S2. (a) $\mathrm{d}f = (x + y)^{-2}\exp[-1/(x + y)](\mathrm{d}x + \mathrm{d}y)$

(b) $\mathrm{d}f = (\cosh x/\sinh y)\,\mathrm{d}x - (\sinh x\cosh y/\sinh^2 y)\,\mathrm{d}y$

(c) $\mathrm{d}f = (x^2 + y^2)^{-1/2}(x\,\mathrm{d}x + y\,\mathrm{d}y)$

(d) $\mathrm{d}f = (x^2 + y^2)^{-1}(-y\,\mathrm{d}x + x\,\mathrm{d}y)$

(e) $\mathrm{d}f = x^{y-1}y\;\mathrm{d}x + x^y \ln x\;\mathrm{d}y$

S3. (a) $(0, 0)$

(b) $x = 0$ or $y = 0$ (two lines of stationary points)

(c) none

(d) $x + y = (n + \frac{1}{2})\pi$ for any integer n (infinitely many lines of stationary points)

(e) $(0, 0)$

Standard questions

4. $(-1, 2)$, $(-1, 2)$, $(2, -4)$, $(2, 4)$

5. (a) 0.1 %
   (b) 0.2 %

6.
$$\left(\frac{\partial f}{\partial r}\right)_\phi = -2r\cos\phi\sin\phi\,\exp(-r^2\cos\phi\sin\phi) = -\frac{2xy}{r}\,\exp(-xy)$$
$$\left(\frac{\partial f}{\partial \phi}\right)_r = r^2(\sin^2\phi - \cos^2\phi)\exp(-r^2\cos\phi\sin\phi) = (y^2 - x^2)\exp(-xy)$$

7.
$$-\left(\frac{xz + 4y^3}{yz + 3x^2}\right), \qquad -\left(\frac{xy + 5z^4}{xz + 4y^3}\right), \qquad -\left(\frac{yz + 3x^2}{xy + 5z^4}\right)$$

8.
$$-\left[\frac{pV^3 - a(V - 2b)}{V^3(V - b)}\right], \qquad \frac{RV^3}{pV^3 - a(V - 2b)}, \qquad \frac{V - b}{R}$$

10. (a) exact, $f = xy$, $y = c/x$
    (b) not exact, $\mu = x^{-2}\exp(-1/x)$, $y = c\,\exp(1/x)$
    (c) exact, $f = \frac{1}{2}(x^2 - y^2) + xy$, $y = x \pm (2x^2 + c)^{1/2}$
    (d) exact, $f = \sinh x\cos y + \cosh y\sin x$, $\sinh x\cos y + \cosh y\sin x = c$
    (e) not exact, $\mu = (\sin x + \cos x)^{-1}$, $y = c - \ln(\sin x + \cos x)$
    (f) exact, $f = \arctan(y/x)$, $y = cx$

12. $G = U - TS + pV$

13. (i)
$$\left(\frac{\partial S}{\partial V}\right)_T \Big/ \left(\frac{\partial S}{\partial p}\right)_V$$
    (iii)
$$\gamma = 1 + \frac{R}{C_\mathrm{v}}$$
    (iv)
$$\frac{5}{3}$$

14. $h = 1/\sqrt{2}$ at $(a/\sqrt{2}, a/\sqrt{2})$, $h = -1/\sqrt{2}$ at $(-a/\sqrt{2}, -a/\sqrt{2})$
    $h = 1/2$ at $(a, 0)$ and $(0, a)$, $h = -1/2$ at $(-a, 0)$ and $(0, -a)$

15. Saddle point $(0, 0)$, maxima $(1, 0)$ and $(-1, 0)$, minima $(0, 1)$ and $(0, -1)$

16. (a) Maximum $(0, 0)$

    (b) Maximum $(\pi/2, \pi/2)$

    (c) Saddle point $(1, 0)$, maxima $(1 + 1/\sqrt{2}, 1/\sqrt{2})$ and $(1 - 1/\sqrt{2}, -1/\sqrt{2})$, minima $(1 - 1/\sqrt{2}, 1/\sqrt{2})$ and $(1 + 1/\sqrt{2}, -1/\sqrt{2})$

17. (a) 0 at $(\pm 1, 0)$, $2/(3\sqrt{3})$ at $(1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$, $-2/(3\sqrt{3})$ at $(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$

    (b) $\mathrm{e}^{1/2}$ at $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$, $\mathrm{e}^{-1/2}$ at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$

19. $(3 - 2\sqrt{2})s^2$

21. $$n_s = g_s \left[\mathrm{e}^{\beta(E_s - \mu)} + 1\right]^{-1}$$

22. $$\frac{1}{2\beta}, \qquad \frac{3}{2\beta}$$