Q2. (i) Let the flow path be $s \to x_1 \to x_2 \to \cdots \to x_n \to t$

$$\sum_{v \neq s,t} \left[\sum_{w} f(v \to w) - \sum_{u} f(u \to v) \right]$$

$$= \sum_{v \neq s,t} \Big[f(v \to x_1) + f(v \to x_2) + \cdots + f(v \to x_n) - f(x_1 \to v) - f(x_2 \to v) - \cdots - f(x_n \to v) \Big]$$

$$\begin{aligned}
= &\cancel{f(x_1 \to x_1)} + \cancel{f(x_1 \to x_2)} + \cancel{f(x_1 \to x_3)} + \cdots + \cancel{f(x_1 \to x_n)} \\
&- \cancel{f(x_1 \to x_1)} - \cancel{f(x_2 \to x_1)} - \cancel{f(x_3 \to x_1)} - \cdots - \cancel{f(x_n \to x_1)} \\
&+ \cancel{f(x_2 \to x_1)} + \cancel{f(x_2 \to x_2)} + \cancel{f(x_2 \to x_3)} + \cdots + \cancel{f(x_2 \to x_n)} \\
&- \cancel{f(x_1 \to x_2)} - \cancel{f(x_2 \to x_2)} - \cancel{f(x_3 \to x_2)} - \cdots - \cancel{f(x_n \to x_2)} \\
&+ \cancel{f(x_3 \to x_1)} + \cancel{f(x_3 \to x_2)} + \cancel{f(x_3 \to x_3)} + \cdots + \cancel{f(x_3 \to x_n)} \\
&- \cancel{f(x_1 \to x_3)} - \cancel{f(x_2 \to x_3)} - \cancel{f(x_3 \to x_3)} - \cdots - \cancel{f(x_n \to x_3)} \\
&+ \cdots - \cdots \quad \text{and so on, cancelling out} \ldots \\
&+ \cancel{f(x_n \to x_1)} + \cancel{f(x_n \to x_2)} + \cancel{f(x_n \to x_3)} + \cdots + \cancel{f(x_n \to x_n)} \\
&- \cancel{f(x_1 \to x_n)} - \cancel{f(x_2 \to x_n)} - \cancel{f(x_3 \to x_n)} - \cdots - \cancel{f(x_n \to x_n)}
\end{aligned}$$

$= 0$ as required.

[redacted]

(ii) $$\sum_{v \neq t} \left[\sum_w f(v \to w) - \sum_u f(u \to v) \right]$$

$$= \sum_{v \neq s,t} \left[\sum_w f(v \to w) - \sum_u f(u \to v) \right] + \left[\sum_w f(s \to w) + \sum_u f(u \to s) \right]$$

$$= 0 + value(f)$$

$$= value(f)$$

$$\sum_{v} \left[\sum_w f(v \to w) - \sum_u f(u \to v) \right]$$

$$= \sum_{v \neq t} \left[\sum_w f(v \to w) - \sum_u f(u \to v) \right] + \left[\sum_w f(t \to w) - \sum_u f(u \to t) \right]$$

$$= value(f) + \sum_w f(t \to w) - \sum_u f(u \to t)$$

Total sum net flow in network $= 0$, as if flow comes from somewhere it needs to go somewhere.

$$\therefore \sum_{v} \left[\sum_w f(v \to w) - \sum_u f(u \to v) \right] = 0$$

$$\therefore 0 = value(f) + \sum_w f(t \to w) - \sum_u f(u \to t)$$

$$\sum_u f(u \to t) - \sum_w f(t \to w) = value(f)$$

net flow into t = total flow into t − total flow out of t

$$= \sum_u f(u \to t) - \sum_w f(t \to w)$$

$$= value(f)$$ as required. ∎