$\boxed{1a.}$ $A = \sqrt{s(s-a)(s-b)(s-c)}$

$g = a+b+c-2s = 0$

$$L = \sqrt{s(s-a)(s-b)(s-c)} - \lambda(a+b+c-2s)$$

$$\frac{\partial L}{\partial a} = -\frac{1}{2}\sqrt{\frac{s(s-b)(s-c)}{(s-a)}} - \lambda = 0 \quad ①$$

$$\frac{\partial L}{\partial b} = -\frac{1}{2}\sqrt{\frac{s(s-a)(s-c)}{(s-b)}} - \lambda = 0 \quad ②$$

$$\frac{\partial L}{\partial c} = -\frac{1}{2}\sqrt{\frac{s(s-a)(s-b)}{(s-c)}} - \lambda = 0 \quad ③$$

$$\frac{\partial L}{\partial \lambda} = -(a+b+c-2s) = 0$$

From ① and ②:

$$\lambda^2 = \frac{1}{4}\frac{s\cancel{(s-b)}(s-c)}{\cancel{(s-a)}}\frac{s\cancel{(s-a)}(s-c)}{\cancel{(s-b)}} = \frac{s^2(s-c)^2}{4}$$

$$\Rightarrow (s-c)^2 = \frac{4\lambda^2}{s^2} \Rightarrow s-c = \frac{2\lambda}{s} \quad \text{as } s, s-c > 0$$

$$\Rightarrow c = s - \frac{2\lambda}{s}$$

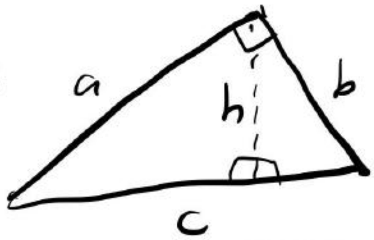
Similarly, from ② and ③, $a = s - \frac{2\lambda}{s}$

and from ① and ③, $b = s - \frac{2\lambda}{s}$

$\therefore a = b = c$, so the triangle is equilateral.

Right-angled triangle

$$A = \frac{1}{2}\sqrt{a^2+b^2}\,h \Rightarrow \sqrt{a^2+b^2} = \frac{2A}{h}$$



$$c = \sqrt{a^2+b^2}$$

with constraint

$$g = a+b+\sqrt{a^2+b^2} - P = 0$$

$\leftarrow$ perimeter

$$L = A - \lambda g$$

$$L = \frac{1}{2}\sqrt{a^2+b^2}\,h - \lambda\left(a+b+\sqrt{a^2+b^2} - P\right)$$

$$\frac{\partial L}{\partial a} = \frac{a}{2\sqrt{a^2+b^2}}h - \lambda\left(1+\frac{a}{\sqrt{a^2+b^2}}\right) = 0$$

$$\frac{\partial L}{\partial b} = \frac{b}{2\sqrt{a^2+b^2}}h - \lambda\left(1+\frac{b}{\sqrt{a^2+b^2}}\right) = 0$$

$$\frac{\partial L}{\partial \lambda} = -\left(a+b+\sqrt{a^2+b^2} - P\right) = 0$$

$$a\left(\frac{1}{2}\sqrt{a^2+b^2}\,h\right) - \lambda\left(a^2+b^2+a\sqrt{a^2+b^2}\right) = b\left(\frac{1}{2}\sqrt{a^2+b^2}\,h\right) - \lambda\left(a^2+b^2+b\sqrt{a^2+b^2}\right) = 0$$

$$\lambda\left(a^2+b^2+a\sqrt{a^2+b^2}\right) = aA \qquad \lambda\left(a^2+b^2+b\sqrt{a^2+b^2}\right) = bA$$

$$\lambda\left(\left(\frac{2A}{h}\right)^2 + a\left[\frac{2A}{h}\right]\right) = aA \qquad \lambda\left(\left(\frac{2A}{h}\right)^2 + b\left(\frac{2A}{h}\right)\right) = bA$$

$$\frac{4A^2}{h^2}\lambda + aA\left(\frac{2\lambda}{h} - 1\right) = 0 \qquad \frac{4A^2}{h^2}\lambda + bA\left(\frac{2\lambda}{h} - 1\right) = 0$$

$$A(b-a)\left(\frac{2\lambda}{h} - 1\right) = 0$$

$$A \neq 0 \Rightarrow a = b \text{ or } 2\lambda = h$$

CASE 1: $2\lambda = h$ [Substituting into $\frac{\partial L}{\partial a} = 0$:]

$$\frac{a}{2\sqrt{a^2+b^2}}h - \frac{1}{2}h\left(1 + \frac{a}{\sqrt{a^2+b^2}}\right) = 0$$

$$h \neq 0$$

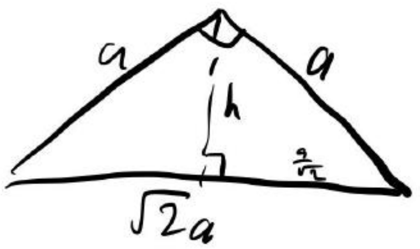
$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}} = 1 + \frac{a}{\sqrt{a^2+b^2}}$$

no solutions.

CASE 2: $a = b$

$$P = a+b+\sqrt{a^2+b^2} = 2a+\sqrt{2}\,a = a(2+\sqrt{2}) \Rightarrow a = \frac{P}{2+\sqrt{2}}$$

$$A = \frac{1}{2}\sqrt{a^2+b^2}\,h = \frac{1}{2}\sqrt{a^2+a^2}\,h = \frac{ah}{\sqrt{2}}$$



$$a^2 = h^2 + \frac{a^2}{2} \Rightarrow h = \frac{a}{\sqrt{2}}$$

$$\therefore A = \frac{a^2}{2} = \frac{P^2}{2(2+\sqrt{2})^2} = \frac{P^2(2-\sqrt{2})^2}{2(4-2)^2} = \frac{4-4\sqrt{2}+2}{8}P^2$$

$$= \frac{3-2\sqrt{2}}{4}P^2$$ where P is the perimeter. $P = 2s$

$$\therefore A = \frac{3-2\sqrt{2}}{4}(2s)^2 = (3-2\sqrt{2})s^2$$

20. $r = (x^2+y^2+z^2)^{1/2}$ with constraints

$$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$h = lx + my + nz = 0$$

$$L = r - \lambda g - \mu h$$

$$L = (x^2+y^2+z^2)^{1/2} - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) - \mu(lx+my+nz)$$

$$\frac{\partial L}{\partial x} = x(x^2+y^2+z^2)^{-1/2} - \frac{2\lambda}{a^2}x - \mu l = \frac{x}{r} - \frac{2\lambda}{a^2}x - \mu l = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = \frac{y}{r} - \frac{2\lambda}{b^2}y - \mu m = 0 \quad (2)$$

$$\frac{\partial L}{\partial z} = \frac{z}{r} - \frac{2\lambda}{c^2}z - \mu n = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda} = -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \quad (4)$$

$$\frac{\partial L}{\partial \mu} = -(lx+my+nz) = 0 \quad (5)$$

From (1): $x\left(\frac{1}{r} - \frac{2\lambda}{a^2}\right) = \mu l$

$$x\left(\frac{a^2-2r\lambda}{ra^2}\right) = \mu l$$

$$x = \frac{ra^2\mu l}{a^2-2r\lambda}$$

From (2): $$y = \frac{rb^2\mu m}{b^2-2r\lambda}$$

From (3): $$z = \frac{rc^2\mu n}{c^2-2r\lambda}$$

Substituting l, m, n into ⑤:

$$\frac{1}{\mu}\left[x^2\left(\frac{1}{r}-\frac{2\lambda}{a^2}\right)+y^2\left(\frac{1}{r}-\frac{2\lambda}{b^2}\right)+z^2\left(\frac{1}{r}-\frac{2\lambda}{c^2}\right)\right]=0$$

$$\frac{1}{r}(x^2+y^2+z^2)-2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad r^2=x^2+y^2+z^2$$

$$\therefore r-2\lambda=0 \Rightarrow 2\lambda=r$$

Substituting x, y, z into ⑤:

$$r\mu\left(\frac{a^2l^2}{a^2-2r\lambda}+\frac{b^2m^2}{b^2-2r\lambda}+\frac{c^2n^2}{c^2-2r\lambda}\right)=0$$

Substituting $2\lambda=r$:

$$\frac{a^2l^2}{a^2-r^2}+\frac{b^2m^2}{b^2-r^2}+\frac{c^2n^2}{c^2-r^2}=0 \quad \text{as required.}$$

Geometric interpretation

g is an ellipsoid and h is a plane. The intersection of these is an ellipse. We are therefore finding stationary points of r on an ellipse. r is the Cartesian distance between two points whose difference in coordinates is (x, y, z).

The Lagrange multiplier $\lambda=\frac{1}{2r}$ so it is inversely proportional to the function r.