$\boxed{15.}$ (i) S_1: $x^2+y^2+z^2=a^2$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dS = r^2\sin\theta\, d\theta\, d\phi$

$0 \le \theta \le \pi$

$0 \le \phi < 2\pi$

$r = a$

$\underset{\sim}{\hat{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\underset{\sim}{F} = (\alpha x^3, \beta y^3, \gamma z^3)$

$= r^3(\alpha\sin^3\theta\cos^3\phi, \beta\sin^3\theta\sin^3\phi, \gamma\cos^3\theta)$

$\underset{\sim}{F}\cdot d\underset{\sim}{S} = \underset{\sim}{F}\cdot\underset{\sim}{\hat{n}}\, dS$

$= r^5\sin\theta\,(\alpha\sin^4\theta\sin^4\phi + \beta\sin^4\theta\cos^4\phi + \gamma\cos^4\theta)\,d\theta\, d\phi$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\cos^4 x = \frac{1}{4}(1+2\cos 2x+\cos^2 2x)$

$= \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right)$

$= \frac{1}{8}(2+4\cos 2x+1+\cos 4x)$

$= \frac{1}{8}(3+4\cos 2x+\cos 4x)$

$\int \cos^4 x\,dx = \frac{1}{8}\int(3+4\cos 2x+\cos 4x)\,dx$

$= \frac{1}{8}\left(3x+2\sin 2x+\frac{1}{4}\sin 4x\right)+C$

$\cos 2x = 1-2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\sin^4 x = \frac{1}{4}(1-2\cos 2x+\cos^2 2x)$

$= \frac{1}{4}\left(1-2\cos 2x+\frac{1}{2}(1+\cos 4x)\right)$

$= \frac{1}{8}(2-4\cos 2x+1+\cos 4x)$

$= \frac{1}{8}(3-4\cos 2x+\cos 4x)$

$\int \sin^4 x\,dx = \frac{1}{8}\int(3-4\cos 2x+\cos 4x)\,dx$

$= \frac{1}{8}\left(3x-2\sin 2x+\frac{1}{4}\sin 4x\right)+C$

$$\int_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{\theta=0}^{\pi} r^5 \sin\theta \left[\alpha \sin^4\theta \int_{\phi=0}^{\pi} \sin^4\phi \, d\phi + \beta \sin^4\theta \int_{\phi=0}^{\pi} \cos^4\phi \, d\phi + \gamma \cos^4\theta \int_{\phi=0}^{\pi} d\phi \right] d\theta$$

$$= r^5 \int_0^{\pi} \sin\theta \left[\frac{\alpha}{8} \sin^4\theta \left[3\phi - 2\sin 2\phi + \frac{1}{4}\sin 4\phi \right]_0^{\pi} + \frac{\beta}{8}\sin^4\theta \left[3\phi + 2\sin 2\phi - \frac{1}{4}\sin 4\phi \right]_0^{\pi} + \gamma\pi\cos^4\theta \right] d\theta$$

$$= \frac{\pi}{8} r^5 \int_0^{\pi} \sin\theta \left[3\alpha\sin^4\theta + 3\beta\sin^4\theta + 8\gamma\cos^4\theta \right] d\theta$$

$$= \frac{\pi}{8} r^5 \left[(3\alpha + 3\beta) \int_0^{\pi} \sin^5\theta \, d\theta + 8\gamma \int_0^{\pi} \sin\theta\cos^4\theta \, d\theta \right]$$

$$\int_0^{\pi} \sin\theta\cos^4\theta \, d\theta = -\int_1^{-1} u^4 \, du = \left[\frac{1}{5}u^5 \right]_{-1}^{1} = \frac{2}{5}$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$
$\theta = 0 \Rightarrow u = 1$
$\theta = \pi \Rightarrow u = -1$

$$\int_0^{\pi} \sin^5\theta \, d\theta = \int_0^{\pi} (1-\cos^2\theta)^2 \sin\theta \, d\theta$$

$$= \int_1^{-1} -(1-u^2)^2 \, du$$

$$= \int_{-1}^{1} (1 - 2u^2 + u^4) \, du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_{-1}^{1}$$

$$= 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$
$\theta = 0 \Rightarrow u = 1$
$\theta = \pi \Rightarrow u = -1$

$$\int_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \frac{\pi}{8} r^5 \left[(3\alpha + 3\beta) \times \frac{16}{15} + 8\gamma \times \frac{2}{5} \right]$$

$$= \frac{\pi}{5} r^5 \left(2\alpha + 2\beta + 2\gamma \right)$$

$$= \frac{2\pi}{5} r^5 \left(\alpha + \beta + \gamma \right)$$

$r = a$

$$\therefore \int_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \frac{2\pi}{5} a^5 \left(\alpha + \beta + \gamma \right)$$

(ii) $S_2 = S_c + S_T + S_B$

S_c : curved surface

$$x = a\cos\phi \qquad 0 \le \phi < 2\pi$$
$$y = a\sin\phi \qquad -h \le z \le h$$
$$z = z$$

$$dS = a\, d\phi\, dz$$

$$\underset{\sim}{\hat{n}} = (\cos\phi, \sin\phi, 0)$$

$$\underset{\sim}{F} = (\alpha x^3, \beta y^3, \gamma z^3)$$

$$= (\alpha a^3 \cos^3\phi, \beta y^3 \sin^3\phi, \gamma z^3)$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{S} = \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS$$

$$= (\alpha a^3 \cos^3\phi + \beta y^3 \sin^3\phi + 0)\, a\, d\phi\, dz$$

$$= a^4 (\alpha \cos^3\phi + \beta \sin^3\phi)\, d\phi\, dz$$

$$\int_{S_c} \underset{\sim}{F} \cdot d\underset{\sim}{S} = a^4 \int_{\phi=0}^{2\pi} \int_{z=-h}^{h} (\alpha\cos^3\phi + \beta\sin^3\phi)\, dz\, d\phi = 0$$

S_T : top

$$x = r\cos\phi \qquad 0 \le \phi < 2\pi$$
$$y = r\sin\phi \qquad 0 \le r \le a$$
$$z = h$$

$$dS = r\,d\phi\,dr$$
$$\underset{\sim}{\hat{n}} = (0,0,1)$$
$$\underset{\sim}{F} = (\alpha r^3\cos^3\phi,\ \beta r^3\sin^3\phi,\ \gamma h^3)$$
$$\underset{\sim}{F}\cdot d\underset{\sim}{S} = \underset{\sim}{F}\cdot\underset{\sim}{\hat{n}}\,dS = \gamma h^3 r\,d\phi\,dr$$
$$\int_{S_T}\underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\gamma h^3 r\,d\phi\,dr = \int_0^a 2\pi\gamma h^3 r\,dr = 2\pi\gamma a h^3$$

S_B : bottom

$$x = r\cos\phi,\ y = r\sin\phi,\ z = -h,\quad 0 \le \phi < 2\pi,\ 0 \le r \le a$$
$$dS = r\,d\phi\,dr$$
$$\underset{\sim}{\hat{n}} = (0,0,-1)$$
$$\underset{\sim}{F} = (\alpha r^3\cos^3\phi,\ \beta r^3\sin^3\phi,\ -\gamma h^3)$$
$$\underset{\sim}{F}\cdot d\underset{\sim}{S} = \underset{\sim}{F}\cdot\underset{\sim}{\hat{n}}\,dS = \gamma h^3 r\,d\phi\,dr$$
$$\int_{S_B}\underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_T}\underset{\sim}{F}\cdot d\underset{\sim}{S} = 2\pi\gamma a h^3$$

$$S_2 : \int_{S_2}\underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_c}\underset{\sim}{F}\cdot d\underset{\sim}{S} + \int_{S_T}\underset{\sim}{F}\cdot d\underset{\sim}{S} + \int_{S_B}\underset{\sim}{F}\cdot d\underset{\sim}{S} = 4\pi\gamma a h^3.$$

16. (i) The cube S has 6 faces: $\underset{\sim}{F} = (x^2+ay^2, 3xy, 6z)$

$\underline{S_1}$: top $\underset{\sim}{x} = (s,t,1)$, $\underset{\sim}{n} = (0,0,1)$ $0 \leq s \leq 1$, $0 \leq t \leq 1$

$dS = ds\,dt$

$$\int_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{S_1} \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS = \int_{t=0}^{1} \int_{s=0}^{1} 6\, ds\,dt = \int_0^1 6\,dt = 6$$

$\underline{S_2}$: bottom $\underset{\sim}{x} = (s,t,0)$, $\underset{\sim}{\hat{n}} = (0,0,-1)$, $0 \leq s \leq 1$, $0 \leq t \leq 1$

$$\int_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{t=0}^{1} \int_{s=0}^{1} 0\, ds\,dt = 0$$

$\underline{S_3}$: left $\underset{\sim}{x} = (0,s,t)$, $\underset{\sim}{\hat{n}} = (-1,0,0)$ $0 \leq s \leq 1$, $0 \leq t \leq 1$

$$\int_{S_3} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{t=0}^{1} \int_{s=0}^{1} (-as^2)\, ds\,dt = -\frac{1}{3}a \int_0^1 dt = -\frac{1}{3}a$$

$\underline{S_4}$: right $\underset{\sim}{x} = (1,s,t)$, $\underset{\sim}{\hat{n}} = (1,0,0)$ $0 \leq s \leq 1$, $0 \leq t \leq 1$

$$\int_{S_4} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{t=0}^{1} \int_{s=0}^{1} (1+as^2)\, ds\,dt = \int_0^1 \left(1+\frac{a}{3}\right) dt = 1 + \frac{1}{3}a$$

$\underline{S_5}$: front $\underset{\sim}{x} = (s,1,t)$, $\underset{\sim}{\hat{n}} = (0,1,0)$ $0 \leq s \leq 1$, $0 \leq t \leq 1$

$$\int_{S_5} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{t=0}^{1} \int_{s=0}^{1} 3s\, ds\,dt = \frac{3}{2} \int_0^1 dt = \frac{3}{2}$$

$\underline{S_6}$: back $\underset{\sim}{x} = (s,0,t)$, $\underset{\sim}{\hat{n}} = (0,-1,0)$ $0 \leq s \leq 1$, $0 \leq t \leq 1$

$$\int_{S_6} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{t=0}^{1} \int_{s=0}^{1} 0\, ds\,dt = 0$$

$$\int_{S} \underset{\sim}{F} \cdot d\underset{\sim}{S} = 6 + 0 - \frac{1}{3}a + 1 + \frac{1}{3}a + \frac{3}{2} + 0 = \frac{17}{2}$$

(ii) $$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(bx+6)\,dx\,dy\,dz=\int_{z=0}^{1}\int_{y=0}^{1}\left(\frac{1}{2}b+6\right)dy\,dz=\frac{1}{2}b+6$$

$$\frac{1}{2}b+6=\frac{17}{2}$$

$$\frac{1}{2}b=\frac{5}{2}$$

$$b=5$$

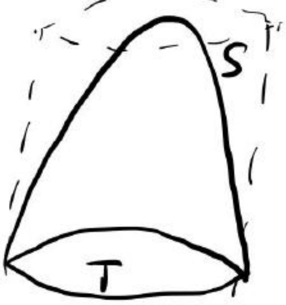
$\therefore$ These integrals have the same value for $b=5$ and all a.

19. Divergence theorem

$$\int_V (\nabla \cdot \underset{\sim}{F})\, dV = \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

, where S is a closed surface bounding V, and $d\underset{\sim}{S} = \underset{\sim}{\hat{n}}\, dS$, where $\underset{\sim}{\hat{n}}$ is an outward-pointing unit normal.

$S: x^2 + y^2 = 1 - z$



$$\underset{\sim}{x} = (r\cos\phi, r\sin\phi, z)$$

$$0 \le \phi < 2\pi, \quad 0 \le r \le 1 - z, \quad 0 \le z \le 1$$

$$dV = r\,dr\,d\phi\,dz$$

$$\underset{\sim}{F} = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$$

$$\nabla \cdot \underset{\sim}{F} = 3x^2 + 3y^2 + 6z = 3r^2\cos^2\phi + 3r^2\sin^2\phi + 6z = 3r^2 + 6z$$

$$\int_V (\nabla \cdot \underset{\sim}{F})\, dV = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \int_{r=0}^{1-z} (3r^2 + 6z)\, dr\, d\phi\, dz$$

$$= \int_0^{2\pi} d\phi \int_0^1 \left[r^3 + 6rz \right]_0^{1-z} dz$$

$$= 2\pi \int_0^1 \left[(1-z)^3 + 6(z - z^2) \right] dz$$

$$= 2\pi \left[-\frac{1}{4}(1-z)^4 + 6\left(\frac{1}{2}z^2 - \frac{1}{3}z^3 \right) \right]_0^1$$

$$= 2\pi \left(1 + \frac{1}{4} \right)$$

$$= \frac{5}{2}\pi$$

$\underline{T}$: $\hat{\underset{\sim}{n}} = (0, 0, -1)$

$\underset{\sim}{x} = (r\cos\phi, r\sin\phi, 0)$ $\qquad 0 \leq r \leq 1$

$dS = r\,dr\,d\phi$ $\qquad 0 \leq \phi < 2\pi$

$\underset{\sim}{F} \cdot \hat{\underset{\sim}{n}} = -x^2 - y^2 - 3z^2 = -r^2$

$$\int_T \underset{\sim}{F} \cdot d\underset{\sim}{S} = -\int_0^1 \int_0^{2\pi} r^3 \, d\phi \, dr$$

$$= -\int_0^1 r^3 dr \int_0^{2\pi} d\phi$$

$$= -\frac{1}{4} \times 2\pi$$

$$= -\frac{\pi}{2}$$

By the divergence theorem,

$$\int_V (\nabla \cdot \underset{\sim}{F}) \, dV = \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} + \int_T \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$\int_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{n}} \, dS = \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$= \int_V (\nabla \cdot \underset{\sim}{F}) \, dV - \int_T \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$= \frac{5}{2}\pi - \left(-\frac{\pi}{2}\right)$$

$$= 3\pi$$

$$\therefore \int_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{n}} \, dS = 3\pi$$