$\boxed{51.}$ ⓐ $f(x,y) = x^3 - 2x^2y + 3xy^2 - 4y^3$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2 \qquad \frac{\partial f}{\partial y} = -2x^2 + 6xy - 12y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 4y \qquad \frac{\partial^2 f}{\partial y^2} = 6x - 24y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 - 4xy + 3y^2\right) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2x^2 + 6xy - 12y^2\right)$$

$$= -4x + 6y \qquad = -4x + 6y$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

ⓑ $f(x,y) = e^{-x^2y^2}$

$$\frac{\partial f}{\partial x} = -2xy^2 e^{-x^2y^2} \qquad \frac{\partial f}{\partial y} = -2x^2y e^{-x^2y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-x^2y^2}\left(-2y^2 - 2xy^2(-2xy^2)\right) \qquad \frac{\partial^2 f}{\partial y^2} = e^{-x^2y^2}\left(-2x^2 - 2x^2y(-2x^2y)\right)$$

$$= 2y^2 e^{-x^2y^2}\left(2x^2y^2 - 1\right) \qquad = 2x^2 e^{-x^2y^2}\left(2x^2y^2 - 1\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-2xy^2 e^{-x^2y^2}\right)$$

$$= e^{-x^2y^2}\left(-4xy - 2xy^2(-2x^2y)\right)$$

$$= 4xy e^{-x^2y^2}\left(x^2y^2 - 1\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2x^2y e^{-x^2y^2}\right)$$

$$= e^{-x^2y^2}\left(-4xy - 2x^2y(-2xy^2)\right)$$

$$= 4xy e^{-x^2y^2}\left(x^2y^2 - 1\right)$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

(c) $f(x,y) = e^{-x/y}$

$$\frac{\partial f}{\partial x} = -\frac{1}{y} e^{-x/y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{y^2} e^{-x/y}$$

$$\frac{\partial f}{\partial y} = \frac{x e^{-x/y}}{y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{y^2 \cdot \frac{x^2 e^{-x/y}}{y^2} - 2xy e^{-x/y}}{y^4}$$

$$= \frac{x^2 e^{-x/y} - 2xy e^{-x/y}}{y^4}$$

$$= \frac{x e^{-x/y}}{y^4} (x - 2y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{e^{-x/y}}{y}\right)$$

$$= -\frac{y\left(\frac{x e^{-x/y}}{y^2}\right) - e^{-x/y}}{y^2}$$

$$= \frac{e^{-x/y}}{y^2}\left(1 - \frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x e^{-x/y}}{y^2}\right)$$

$$= \frac{e^{-x/y}}{y^2}\left(1 - \frac{x}{y}\right)$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

(d) $f(x,y)=\sin(x+y)$

$$\frac{\partial f}{\partial x}=\cos(x+y) \qquad \frac{\partial f}{\partial y}=\cos(x+y)$$

$$\frac{\partial^2 f}{\partial x^2}=-\sin(x+y) \qquad \frac{\partial^2 f}{\partial y^2}=-\sin(x+y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(\cos(x+y)\right)=-\sin(x+y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(\cos(x+y)\right)=-\sin(x+y)$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

(e) $f(x,y)=(x^2+xy+2y^2)^{-1}$

$$\frac{\partial f}{\partial x}=(-1)(2x+y)(x^2+xy+2y^2)^{-2}=-\frac{2x+y}{(x^2+xy+2y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2}=-\frac{2(x^2+xy+2y^2)^2-2(2x+y)(2x+y)(x^2+xy+2y^2)}{(x^2+xy+2y^2)^4}$$

$$=\frac{2(4x^2+4xy+y^2-x^2-xy-2y^2)}{(x^2+xy+2y^2)^3}$$

$$=\frac{2(3x^2+3xy-y^2)}{(x^2+xy+2y^2)^3}$$

$$\frac{\partial f}{\partial y}=(-1)(x+4y)(x^2+xy+2y^2)^{-2}=-\frac{x+4y}{(x^2+xy+2y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}=-\frac{4(x^2+xy+2y^2)^2-2(x+4y)^2(x^2+xy+2y^2)}{(x^2+xy+2y^2)^4}$$

$$=\frac{2(x^2+8xy+16y^2-2x^2-2xy-4y^2)}{(x^2+xy+2y^2)^3}$$

$$=\frac{2(-x^2+6xy+12y^2)}{(x^2+xy+2y^2)^3}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{2x+y}{(x^2+xy+2y^2)^2}\right)=-\frac{(x^2+xy+2y^2)^2-2(x^2+xy+2y^2)(2x+y)(x+4y)}{(x^2+xy+2y^2)^4}$$

$$=\frac{4x^2+18xy+8y^2-x^2-xy-2y^2}{(x^2+xy+2y^2)^3}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{3x^2+17xy+6y^2}{(x^2+xy+2y^2)^3}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{x+4y}{(x^2+xy+2y^2)^2}\right)$$

$$=-\frac{(x^2+xy+2y^2)^2-2(x^2+xy+2y^2)(x+4y)(2x+y)}{(x^2+xy+2y^2)^4}$$

$$=\frac{4x^2+18xy+8y^2-x^2-xy-2y^2}{(x^2+xy+2y^2)^3}$$

$$=\frac{3x^2+17xy+6y^2}{(x^2+xy+2y^2)^3}$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$\boxed{S3.}$ (a) $f(x,y) = x^3 - 2x^2y + 3xy^2 - 4y^3$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2 \qquad \frac{\partial f}{\partial y} = -2x^2 + 6xy - 12y^2$$

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$3x^2 - 4xy + 3y^2 = 0 \Rightarrow 6x^2 - 8xy + 6y^2 = 0$$

$$-2x^2 + 6xy - 12y^2 = 0 \Rightarrow -6x^2 + 18xy - 36y^2 = 0$$

$$10xy + 30y^2 = 0$$

$$y(x + 3y) = 0$$

$y = 0$ or $x = -3y$

$3x^2 = 0$

$\Rightarrow x = 0$

$27y^2 + 12y^2 + 3y^2 = 0$

$y = 0$

$\Rightarrow x = 0$

$\therefore$ stationary point at $(0,0)$ with $f(0,0) = 0$.

(b) $f(x,y) = e^{-x^2y^2}$

$$\frac{\partial f}{\partial x} = -2xy^2e^{-x^2y^2} \qquad \frac{\partial f}{\partial y} = -2x^2ye^{-x^2y^2}$$

$$-2xy^2e^{-x^2y^2} = 0 \qquad -2x^2ye^{-x^2y^2} = 0$$

$$\Rightarrow -2xy^2 = 0 \qquad \Rightarrow -2x^2y = 0$$

$$xy^2 = 0 \qquad x^2y = 0$$

$\therefore$ $x = 0$ or $y = 0$

y can have any value | x can have any value

$\therefore$ stationary points:

$(0, y)$ for all y and $(x, 0)$ for all x

$f = 1$ at all of these points.

(c) $f(x,y) = e^{-x/y}$

$$\frac{\partial f}{\partial x} = -\frac{1}{y}e^{-x/y} \qquad \frac{\partial f}{\partial y} = \frac{xe^{-x/y}}{y^2}$$

$$-\frac{1}{y}e^{-x/y} = 0 \qquad \frac{xe^{-x/y}}{y^2} = 0$$

no solutions $\qquad x = 0$

$\therefore$ no stationary points.

ⓓ $f(x,y) = \sin(x+y)$

$$\frac{\partial f}{\partial x} = \cos(x+y) \qquad \frac{\partial f}{\partial y} = \cos(x+y)$$

$$\cos(x+y) = 0$$

$$x+y = \arccos(0)$$

$$x+y = n\pi + \frac{\pi}{2} \text{ for } \forall n \in \mathbb{Z}$$

stationary points:

$(x, n\pi + \frac{\pi}{2} - x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and

$(n\pi + \frac{\pi}{2} - y, y)$ for all $y \in \mathbb{R}$, $n \in \mathbb{Z}$.

At these points, $f = 1$ or -1.

ⓔ $f(x,y) = \frac{1}{x^2+xy+2y^2}$

$$\frac{\partial f}{\partial x} = -\frac{2x+y}{(x^2+xy+2y^2)^2} \qquad \frac{\partial f}{\partial y} = -\frac{x+4y}{(x^2+xy+2y^2)^2}$$

$2x+y = 0$ and $x+4y=0$ with $x^2+xy+2y^2 \neq 0$

$8x+4y=0$

$\therefore x=0, y=0$

$x^2+xy+2y^2 \neq 0$, so not a point on f.

$\therefore$ no stationary points.