$\boxed{9.}$ (a) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ with $y(0)=0$ and $y'(0)=1$.

Trial solution $y = e^{\lambda x}$.

$$\frac{dy}{dx} = \lambda e^{\lambda x}, \quad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^2 - 5\lambda + 6) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 2, 3$$

general solution: $y = Ae^{2x} + Be^{3x}$

$$\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x}$$

Substituting $y=0$ and $\frac{dy}{dx} = 1$ at $x=0$.

$$0 = Ae^0 + Be^0 \Rightarrow 0 = A + B$$

$$1 = 2Ae^0 + 3Be^0 \Rightarrow 1 = 2A + 3B$$

$$2A + 2B = 0$$

$$B = 1$$

$$\Rightarrow A = -1$$

$\therefore$ particular solution: $y = -e^{2x} + e^{3x}$

ⓑ $\left(\frac{d^2}{dx^2}+n^2\right)y=0$ with $y(0)=0$ and $y'(0)=1$.

$$\frac{d^2y}{dx^2}+n^2y=0$$

auxiliary equation: $\lambda^2+n^2\lambda=0$

$$\lambda(\lambda+n^2)=0$$

$$\lambda=0,-n^2$$

general solution: $y=A+Be^{-n^2x}$

$$\frac{dy}{dx}=-Bn^2e^{-n^2x}$$

substituting $y=0$ and $\frac{dy}{dx}=1$ at $x=0$

$$0=A+B$$

$$1=-Bn^2 \Rightarrow B=-n^{-2}$$

$$\therefore A=n^{-2}$$

particular solution: $y=n^{-2}-n^{-2}e^{-n^2x}$

(c) $\left(\frac{d^2}{dx^2}+2\frac{d}{dx}+4\right)y=0$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=0$$

auxiliary equation: $\lambda^2+2\lambda+4=0$

$$\lambda=\frac{-2\pm\sqrt{4-16}}{2}$$
$$=\frac{-2\pm2\sqrt{3}i}{2}$$
$$=-1\pm\sqrt{3}i$$

$$\lambda_1=\overline{\lambda_2}$$

$$e^{\lambda_1 x}=e^{(-1+\sqrt{3}i)x}=e^{-x}(\cos\sqrt{3}x+i\sin\sqrt{3}x)$$

$\therefore y=e^{-x}\cos\sqrt{3}x$ and $y=e^{-x}\sin\sqrt{3}x$ are both solutions.

general solution: $y=e^{-x}(A\cos\sqrt{3}x+B\sin\sqrt{3}x)$

$$\frac{dy}{dx}=-e^{-x}(A\cos\sqrt{3}x+B\sin\sqrt{3}x)+e^{-x}(\sqrt{3}B\cos\sqrt{3}x-\sqrt{3}A\sin\sqrt{3}x)$$

substituting $y=0$ and $\frac{dy}{dx}=1$ at $x=0$:

$$\cos\sqrt{3}x=\cos(0)=1$$
$$\sin\sqrt{3}x=\sin(0)=0$$

$$0=A$$
$$1=-A+\sqrt{3}B=\sqrt{3}B\Rightarrow B=\frac{1}{\sqrt{3}}$$

particular solution: $y=\dfrac{e^{-x}\sin\sqrt{3}x}{\sqrt{3}}$

(d) $\frac{d^2y}{dx^2} + 9y = 18$ with $y(0)=0,\ y'(0)=1$

Complementary Function

Auxiliary equation: $\lambda^2 + 9 = 0$

$$\lambda = \pm 3i$$

$$y_c = c_1 \cos(3x) + c_2 \sin(3x)$$

Particular Integral

Trial solution: $y = d_1$

$$\frac{d^2y}{dx^2} = 0$$

$$0 + 9d_1 = 18 \Rightarrow d_1 = 2$$

$$\therefore y_p = 2$$

General Solution

$$y = y_c + y_p$$

$$\therefore y = c_1 \cos(3x) + c_2 \sin(3x) + 2$$

Particular Solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + 2$$

$$\frac{dy}{dx} = -3c_1 \sin(3x) + 3c_2 \cos(3x)$$

$$0 = c_1 + 2 \Rightarrow c_1 = -2$$

$$1 = 3c_2 \Rightarrow c_2 = \frac{1}{3}$$

$$\therefore y = -2\cos(3x) + \frac{1}{3}\sin(3x) + 2$$

(e) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$ with $y(0)=0$ and $y'(0)=1$

Complementary Function

Auxiliary Equation: $\lambda^2 - 3\lambda + 2 = 0$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda = 1, 2$$

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

Particular Integral

Trial solution: $y_p = d_1 e^{5x}$

$$\frac{dy}{dx} = 5d_1 e^{5x}$$

$$\frac{d^2y}{dx^2} = 25 d_1 e^{5x}$$

$$25d_1e^{5x} - 15d_1e^{5x} + 2d_1e^{5x} = e^{5x}$$

$$12d_1e^{5x} = e^{5x}$$

$$\therefore d_1 = \frac{1}{12}$$

$$\therefore y_p = \frac{1}{12}e^{5x}$$

General Solution

$$y = y_c + y_p = c_1e^{x} + c_2e^{2x} + \frac{1}{12}e^{5x}$$

Particular Solution

$$\frac{dy}{dx} = c_1e^{x} + 2c_2e^{2x} + \frac{5}{12}e^{5x}$$

$0 = c_1 + c_2 + \frac{1}{12}$ $\qquad$ $1 = c_1 + 2c_2 + \frac{5}{12}$

$c_2 = \frac{2}{3}$, $c_1 = -\frac{3}{4}$

$$\therefore y = -\frac{3}{4}e^{x} + \frac{2}{3}e^{2x} + \frac{1}{12}e^{5x}$$

(f) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

auxiliary equation: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

general solution: $y = (A + Bx)e^x$

(g) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x} + e^x$

Complementary Function

Auxiliary equation: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$y_c = (c_1 + c_2 x)e^x$

Particular Integral

Trial solution: $y_{p1} = d_1 e^{2x}$

$\frac{dy}{dx} = 2d_1 e^{2x}$

$\frac{d^2y}{dx^2} = 4d_1 e^{2x}$

Substituting into ODE:

$4d_1 e^{2x} - 2(2d_1 e^{2x}) + d_1 e^{2x} = e^{2x}$

$d_1 e^{2x} = e^{2x}$

$d_1 = 1$

$\therefore y_{p1} = e^{2x}$

Trial solution: $y_{p2} = d_2 x^2 e^x$

$$\frac{dy}{dx} = d_2 x^2 e^x + 2d_2 x e^x = d_2 e^x (x^2 + 2x)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2d_2 \frac{d}{dx}(xe^x)$$

$$= d_2 e^x (x^2 + 2x) + 2d_2 (e^x + xe^x)$$

$$= d_2 e^x (x^2 + 2x + 2x + 2)$$

$$= d_2 e^x (x^2 + 4x + 2)$$

Substituting into ODE:

$$d_2 e^x (x^2 + 4x + 2) - 2d_2 e^x (x^2 + 2x) + d_2 e^x x^2 = e^x$$

$$d_2 e^x \left(x^2(1 - 2 + 1) + x(4 - 4) + 2\right) = e^x$$

$$2d_2 e^x = e^x$$

$$d_2 = \frac{1}{2}$$

$$\therefore y_{p2} = \frac{1}{2} x^2 e^x$$

General Solution

$$y = y_c + y_{p1} + y_{p2}$$

$$= (c_1 + c_2 x) e^x + e^{2x} + \frac{1}{2} x^2 e^x$$

$$\therefore y = \left(c_1 + c_2 x + \frac{1}{2} x^2\right) e^x + e^{2x}$$

10. $-iR = \frac{q}{C} + V(t)$

$$-L\frac{dj}{dt} = \frac{q}{C} + V(t)$$

$$-R\frac{di}{dt} = \frac{1}{C}\frac{dq}{dt} + \frac{dV}{dt}$$

$$\frac{dj}{dt} = -\frac{1}{LC}q - \frac{1}{L}V$$

$$\frac{di}{dt} = -\frac{1}{RC}\frac{dq}{dt} - \frac{1}{R}\frac{dV}{dt}$$

$$\frac{dq}{dt} = i + j$$

$$\frac{d^2q}{dt^2} = \frac{di}{dt} + \frac{dj}{dt}$$

$$\frac{d^2q}{dt^2} = -\frac{1}{RC}\frac{dq}{dt} - \frac{1}{R}\frac{dV}{dt} - \frac{1}{LC}q - \frac{1}{L}V$$

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC}q = -\frac{1}{R}\frac{dV}{dt} - \frac{1}{L}V$$

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC}q = -\frac{1}{R}\frac{dV}{dt} - \frac{1}{L}V$$

$V(t) = 0$ and $\frac{dV}{dt}(t) = 0$ when $t > 0$

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

auxiliary equation: $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$

$$LRC\lambda^2 + L\lambda + R = 0$$

$$\lambda = \frac{-L \pm \sqrt{L^2 - 4LR^2C}}{2LRC}$$

Let $L = kR^2C$

$$\lambda = \frac{-kR^2C \pm \sqrt{k^2R^4C^2 - 4kR^4C^2}}{2kR^3C^2}$$

$$= \frac{-kR^2C \pm R^2C\sqrt{k^2 - 4k}}{2kR^3C^2}$$

$$= \frac{-k \pm \sqrt{k^2 - 4k}}{2kRC}$$

$$= -\frac{1}{2RC} \pm \frac{1}{RC}\sqrt{\frac{1}{4} - \frac{1}{k}}$$

$$= \frac{1}{RC}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{k}}\right)$$

(a) $L = 8R^2C$

$\therefore k = 8$

$$\lambda = \frac{1}{RC}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{8}}\right)$$

$$= \frac{1}{RC}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{8}}\right)$$

$$= \frac{1}{RC}\left(-\frac{1}{2} \pm \frac{1}{2\sqrt{2}}\right)$$

$$= \frac{1}{RC}\left(-\frac{2 \pm \sqrt{2}}{4}\right)$$

$$= \frac{-(2 \pm \sqrt{2})}{4RC}$$

<u>General Solution</u>

$$q = Ae^{\frac{-(2+\sqrt{2})}{4RC}t} + Be^{\frac{-(2-\sqrt{2})}{4RC}t}$$

Particular Solution

When $t=0$, $q=Q$ and $\frac{dq}{dt} = -\frac{Q}{RC}$

$$q = Ae^{\frac{-(2+\sqrt{2})}{4RC}t} + Be^{\frac{-(2-\sqrt{2})}{4RC}t}$$

$$\frac{dq}{dt} = -\frac{(2+\sqrt{2})}{4RC}Ae^{\frac{-(2+\sqrt{2})}{4RC}t} - \frac{(2-\sqrt{2})}{4RC}Be^{\frac{-(2-\sqrt{2})}{4RC}t}$$

$$Q = A + B$$

$$-\frac{Q}{RC} = -\frac{2+\sqrt{2}}{4RC}A - \frac{2-\sqrt{2}}{4RC}B$$

$$4Q = (2+\sqrt{2})A + (2-\sqrt{2})B$$

$$4Q = (2+\sqrt{2})A + (2-\sqrt{2})(Q-A)$$

$$4Q = (2+\sqrt{2}-2+\sqrt{2})A + (2-\sqrt{2})Q$$

$$2\sqrt{2}A = Q(2+\sqrt{2})$$

$$A = \frac{2+\sqrt{2}}{2\sqrt{2}}Q$$

$$A = \frac{1+\sqrt{2}}{2}Q$$

$$B = \left(1 - \frac{\sqrt{2}+1}{2}\right)Q$$

$$B = \frac{2-\sqrt{2}-1}{2}Q$$

$$B = \frac{1-\sqrt{2}}{2}Q$$

$$\therefore q = \frac{1+\sqrt{2}}{2}Qe^{\frac{-(2+\sqrt{2})}{4RC}t} + \frac{1-\sqrt{2}}{2}Qe^{\frac{-(2-\sqrt{2})}{4RC}t}$$

ⓑ $\lambda = \frac{1}{RC}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{k}}\right)$

$L = 4R^2C$

$\therefore k = 4$

$\therefore \lambda = \frac{1}{RC}\left(-\frac{1}{2} \pm 0\right)$

$\lambda = -\frac{1}{2RC}$

<u>General Solution</u>

$q = (A + Bt)e^{-\frac{1}{2RC}t}$

<u>Particular Solution</u>

When $t = 0$, $q = Q$ and $\frac{dq}{dt} = -\frac{Q}{RC}$

$\frac{dq}{dt} = e^{-\frac{1}{2RC}t}\left(B - \frac{1}{2RC}(A + Bt)\right)$

$Q = A$

$-\frac{Q}{RC} = B - \frac{A}{2RC}$

$-\frac{A}{RC} = \frac{2BRC - A}{2RC}$

$-2A = 2BRC - A$

$2BRC = -A$

$B = -\frac{A}{2RC} = -\frac{Q}{2RC}$

$q = \left(Q - \frac{Q}{2RC}t\right)e^{-\frac{1}{2RC}t}$

$\therefore q = Q\left(1 - \frac{1}{2RC}t\right)e^{-\frac{1}{2RC}t}$

(c) $\lambda = \frac{1}{RC}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{4}-\frac{1}{k}}\right)$

$L = 2R^2C$

$\therefore k = 2$

$\lambda = \frac{1}{RC}\left(-\frac{1}{2} \pm \frac{1}{2}i\right)$

$\lambda = \frac{1}{2RC}(-1 \pm i)$

General Solution

$$q = e^{-\frac{1}{2RC}t}\left(A\cos\left(\frac{1}{2RC}t\right) + B\sin\left(\frac{1}{2RC}t\right)\right)$$

Particular Solution

When $t=0$, $q=Q$ and $\frac{dq}{dt} = -\frac{Q}{RC}$

$$\frac{dq}{dt} = e^{-\frac{1}{2RC}t}\left(-\frac{1}{2RC}\left(A\cos\left(\frac{1}{2RC}t\right) + B\sin\left(\frac{1}{2RC}t\right)\right) + \frac{1}{2RC}\left(B\cos\left(\frac{1}{2RC}t\right) - A\sin\left(\frac{1}{2RC}t\right)\right)\right)$$

$$\frac{dq}{dt} = \frac{e^{-\frac{1}{2RC}t}}{2RC}\left((B-A)\cos\left(\frac{1}{2RC}t\right) - (B+A)\sin\left(\frac{1}{2RC}t\right)\right)$$

$Q = A$

$-\frac{Q}{RC} = \frac{1}{2RC}(B-A)$

$-\frac{Q}{RC} = \frac{1}{2RC}(B-Q)$

$-2Q = B - Q$

$B = -Q$

$$\therefore q = Qe^{-\frac{1}{2RC}t}\left(\cos\left(\frac{1}{2RC}t\right) - \sin\left(\frac{1}{2RC}t\right)\right)$$