$\boxed{26.}$ $A\underline{x} = \underline{y}$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a+b \\ a+c \end{pmatrix} \begin{matrix} R_1 \\ R_2 \to R_1 + R_2 \\ R_3 \to R_1 + R_3 \end{matrix}$$

$$2z = a+c \Rightarrow z = \frac{a+c}{2}$$

$$2y + z = a+b$$

$$\Rightarrow y = \frac{1}{2}(a+b-z) = \frac{1}{2}\left(a+b-\frac{a+c}{2}\right) = \frac{a+2b-c}{4}$$

$$x+y+z = a$$

$$\Rightarrow x = a-y-z = a - \frac{a+2b-c}{4} - \frac{a+c}{2}$$

$$= \frac{4a-a-2b+c-2a-2c}{4}$$

$$= \frac{a-2b-c}{4}$$

$$\underline{e}_1 = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/2 \end{pmatrix} \quad \underline{e}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \quad \underline{e}_3 = \begin{pmatrix} -1/4 \\ -1/4 \\ 1/2 \end{pmatrix}$$

Let $B = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & -1/4 \\ 1/4 & 1/2 & -1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \Rightarrow B = A^{-1}$

$\therefore$ This matrix is the inverse of A.

$\boxed{29.}$

$$A\underline{v} = \lambda \underline{v}$$

$$(A - \lambda I)\underline{v} = 0$$

$$|A - \lambda I| = 0$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{array}{r} R_1 \\ R_2 \to R_2 + \frac{2}{4-\lambda} R_1 \\ R_3 \end{array} \begin{vmatrix} 4-\lambda & -2 & 0 \\ 0 & 3-\lambda-\frac{4}{4-\lambda} & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = 0$$

$$4-\lambda \begin{vmatrix} 3-\lambda-\frac{4}{4-\lambda} & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left((2-\lambda)\left(\frac{(3-\lambda)(4-\lambda)-4}{4-\lambda}\right) - 4\right) = 0$$

$$(2-\lambda)(\lambda^2 - 7\lambda + 8) - 16 + 4\lambda = 0$$

$$\cancel{2\lambda^2} - 14\lambda + \cancel{16} - \cancel{\lambda^3} + \cancel{7\lambda^2} - \cancel{8\lambda} - \cancel{16} + \cancel{4\lambda} = 0$$

$$-\lambda^3 + 9\lambda^2 - 18\lambda = 0$$

$$\lambda(\lambda^2 - 9\lambda^2 + 18) = 0$$

$$\lambda(\lambda-3)(\lambda-6) = 0$$

$$\lambda = 0, 3, 6$$

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

Let $\left(\begin{array}{ccc|c} a & b & c & j \\ d & e & f & k \\ g & h & i & l \end{array}\right)$ mean $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} j \\ k \\ l \end{pmatrix}$

CASE 1: $\lambda = 0$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 4-0 & -2 & 0 & 0 \\ -2 & 3-0 & -2 & 0 \\ 0 & -2 & 2-0 & 0 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \Rightarrow R_2 + \frac{1}{2}R_1 \\ R_3 \end{array}$$

$-2y + z = 0 \Rightarrow z = 2y$

$4x - 2y = 0 \Rightarrow 4x - z = 0 \Rightarrow z = 4x$

Let $z = 4$. $x = 1$, $y = 2$

$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Normalising: $\underline{e}_1 = \frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

CASE 2: $\lambda = 3$

$$\left(\begin{array}{ccc|c} 4-3 & -2 & 0 & 0 \\ -2 & 3-3 & -2 & 0 \\ 0 & -2 & 2-3 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \to R_2+2R_1 \\ R_3 \end{array}$$

$$-2y-z=0 \Rightarrow 2y=-z$$

$$x-2y=0 \Rightarrow 2y=x$$

Let $y=1$. $x=2,\ z=-2$

$$\underline{v}_2=\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Normalising: $\underline{e}_2=\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

<u>CASE 3</u> : $\lambda=6$

$$\left(\begin{array}{ccc|c} 4-6 & -2 & 0 & 0 \\ -2 & 3-6 & -2 & 0 \\ 0 & -2 & 2-6 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \to R_2-R_1 \\ R_3 \end{array}$$

$$-y-2z=0 \Rightarrow y=-2z$$

$$-2x-2y=0 \Rightarrow x=y=-2z$$

$$\underline{v}_3=\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Normalising: $\underline{e}_3=\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

<u>SUMMARY</u>

$\lambda_1=0$	$\lambda_2=3$	$\lambda_3=6$
$\underline{e}_1=\frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$	$\underline{e}_2=\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$	$\underline{e}_3=\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\underline{y} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{e}_1 = \frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \qquad \underline{e}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \underline{e}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$p_1 = \underline{y} \cdot \underline{e}_1 = \frac{3}{\sqrt{21}}(2-2) = 0$$

$$p_2 = \underline{y} \cdot \underline{e}_2 = 1(4-1) = 3$$

$$p_3 = \underline{y} \cdot \underline{e}_3 = \frac{3}{\sqrt{6}}(2-1) = \sqrt{\frac{3}{2}}$$

$$\underline{y} = 3\underline{e}_2 + \sqrt{\frac{3}{2}}\,\underline{e}_3$$

$$A\underline{x} = \underline{y}$$

Let $X = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \Rightarrow \quad X^T = \begin{pmatrix} \underline{e}_1 & \rightarrow \\ \underline{e}_2 & \rightarrow \\ \underline{e}_3 & \rightarrow \end{pmatrix}$

$$A'\underline{x}' = \underline{y}'$$

$$A' = X^T A X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\underline{y}' = X^T \underline{y} = \begin{pmatrix} 0 \\ 3 \\ \sqrt{3/2} \end{pmatrix}$$

$$A'\underline{x}' = y'$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ \sqrt{3/2} \end{pmatrix}$$

$$3q_2 = 3 \Rightarrow q_2 = 1$$

$$6q_3 = \sqrt{\frac{3}{2}} \Rightarrow q_3 = \sqrt{\frac{1}{24}}$$

$$\therefore \underline{x} = \underline{e}_2 + \sqrt{\frac{1}{24}}\,\underline{e}_3$$

This solution is not unique. $\underline{x} = k\underline{e}_1 + \underline{e}_2 + \sqrt{\frac{1}{24}}\,\underline{e}_3$ for all $k \in \mathbb{R}$.