1a.

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

$$g=a+b+c-2s=0$$

$$L=\sqrt{s(s-a)(s-b)(s-c)}-\lambda(a+b+c-2s)$$

$$\frac{\partial L}{\partial a}=-\frac{1}{2}\sqrt{\frac{s(s-b)(s-c)}{(s-a)}}-\lambda=0 \quad ①$$

$$\frac{\partial L}{\partial b}=-\frac{1}{2}\sqrt{\frac{s(s-a)(s-c)}{(s-b)}}-\lambda=0 \quad ②$$

only valid because 2s is considered as constant, if not s(a,b,c) so you would need to replace s before differentiation

$$\frac{\partial L}{\partial c}=-\frac{1}{2}\sqrt{\frac{s(s-a)(s-b)}{(s-c)}}-\lambda=0 \quad ③$$

$$\frac{\partial L}{\partial \lambda}=-(a+b+c-2s)=0$$

From ① and ②:

you can see a=b from 1 and 2 and that b=c from 2 and 3 !!!

$$\lambda^2=\frac{1}{4}\frac{s\cancel{(s-b)}(s-c)}{\cancel{(s-a)}}\frac{s\cancel{(s-a)}(s-c)}{\cancel{(s-b)}}=\frac{s^2(s-c)^2}{4}$$

$$\Rightarrow (s-c)^2=\frac{4\lambda^2}{s^2} \Rightarrow s-c=\frac{2\lambda}{s} \quad \text{as } s, s-c>0$$

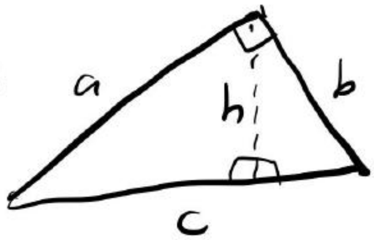
$$\Rightarrow c=s-\frac{2\lambda}{s}$$

Similarly, from ② and ③, $a=s-\frac{2\lambda}{s}$

and from ① and ③, $b=s-\frac{2\lambda}{s}$

$\therefore a=b=c$, so the triangle is equilateral.

Right-angled triangle



$c = \sqrt{a^2+b^2}$

$$A = \frac{1}{2}\sqrt{a^2+b^2}\,h \implies \sqrt{a^2+b^2} = \frac{2A}{h}$$

1/2 ab??? would be easier

with constraint

$$g = a+b+\sqrt{a^2+b^2} - P = 0$$

$\xleftarrow{}$ perimeter

$$L = A - \lambda g$$

$$L = \frac{1}{2}\sqrt{a^2+b^2}\,h - \lambda\left(a+b+\sqrt{a^2+b^2} - P\right)$$

$$\frac{\partial L}{\partial a} = \frac{a}{2\sqrt{a^2+b^2}}h - \lambda\left(1+\frac{a}{\sqrt{a^2+b^2}}\right) = 0$$

$$\frac{\partial L}{\partial b} = \frac{b}{2\sqrt{a^2+b^2}}h - \lambda\left(1+\frac{b}{\sqrt{a^2+b^2}}\right) = 0$$

$$\frac{\partial L}{\partial \lambda} = -\left(a+b+\sqrt{a^2+b^2} - P\right) = 0$$

$$a\left(\frac{1}{2}\sqrt{a^2+b^2}\,h\right) - \lambda\left(a^2+b^2+a\sqrt{a^2+b^2}\right) = b\left(\frac{1}{2}\sqrt{a^2+b^2}\,h\right) - \lambda\left(a^2+b^2+b\sqrt{a^2+b^2}\right) = 0$$

$\lambda\left(a^2+b^2+a\sqrt{a^2+b^2}\right) = aA$	$\lambda\left(a^2+b^2+b\sqrt{a^2+b^2}\right) = bA$
$\lambda\left(\left(\frac{2A}{h}\right)^2 + a\left[\frac{2A}{h}\right]\right) = aA$	$\lambda\left(\left(\frac{2A}{h}\right)^2 + b\left(\frac{2A}{h}\right)\right) = bA$
$\frac{4A^2}{h^2}\lambda + aA\left(\frac{2\lambda}{h} - 1\right) = 0$	$\frac{4A^2}{h^2}\lambda + bA\left(\frac{2\lambda}{h} - 1\right) = 0$

$$A(b-a)\left(\frac{2\lambda}{h} - 1\right) = 0$$

$A \neq 0 \implies a = b$ or $2\lambda = h$

CASE 1: $2\lambda = h$ [Substituting into $\frac{\partial L}{\partial a} = 0$:]

$$\frac{a}{2\sqrt{a^2+b^2}} h - \frac{1}{2} h \left(1 + \frac{a}{\sqrt{a^2+b^2}}\right) = 0$$

$h \neq 0$

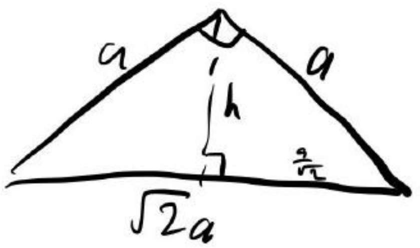
$$\therefore \frac{a}{\sqrt{a^2+b^2}} = 1 + \frac{a}{\sqrt{a^2+b^2}}$$

no solutions.

CASE 2: $a = b$

$$P = a+b+\sqrt{a^2+b^2} = 2a + \sqrt{2}a = a(2+\sqrt{2}) \Rightarrow a = \frac{P}{2+\sqrt{2}}$$

$$A = \frac{1}{2}\sqrt{a^2+b^2}\,h = \frac{1}{2}\sqrt{a^2+a^2}\,h = \frac{ah}{\sqrt{2}}$$



$$a^2 = h^2 + \frac{a^2}{2} \Rightarrow h = \frac{a}{\sqrt{2}}$$

$$\therefore A = \frac{a^2}{2} = \frac{P^2}{2(2+\sqrt{2})^2} = \frac{P^2(2-\sqrt{2})^2}{2(4-2)^2} = \frac{4-4\sqrt{2}+2}{8}P^2$$

$$= \frac{3-2\sqrt{2}}{4}P^2$$ where P is the perimeter. $P = 2s$

$$\therefore A = \frac{3-2\sqrt{2}}{4}(2s)^2 = (3-2\sqrt{2})s^2$$

ok

$\boxed{20.}$ $r=(x^2+y^2+z^2)^{1/2}$ with constraints

$$g=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$h=lx+my+nz=0$$

$$L=r-\lambda g-\mu h$$

$$L=(x^2+y^2+z^2)^{1/2}-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)-\mu(lx+my+nz)$$

$$\frac{\partial L}{\partial x}=x(x^2+y^2+z^2)^{-1/2}-\frac{2\lambda}{a^2}x-\mu l=\frac{x}{r}-\frac{2\lambda}{a^2}x-\mu l=0 \quad (1)$$

$$\frac{\partial L}{\partial y}=\frac{y}{r}-\frac{2\lambda}{b^2}y-\mu m=0 \quad (2)$$

$$\frac{\partial L}{\partial z}=\frac{z}{r}-\frac{2\lambda}{c^2}z-\mu n=0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda}=-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0 \quad (4)$$

$$\frac{\partial L}{\partial \mu}=-(lx+my+nz)=0 \quad (5)$$

From (1): $x\left(\frac{1}{r}-\frac{2\lambda}{a^2}\right)=\mu l$

ok

$$x\left(\frac{a^2-2r\lambda}{ra^2}\right)=\mu l$$

$$x=\frac{ra^2\mu l}{a^2-2r\lambda}$$

From (2): $y=\frac{rb^2\mu m}{b^2-2r\lambda}$

From (3): $z=\frac{rc^2\mu n}{c^2-2r\lambda}$

Substituting l, m, n into ⑤:

$$\frac{1}{\mu}\left[x^2\left(\frac{1}{r}-\frac{2\lambda}{a^2}\right)+y^2\left(\frac{1}{r}-\frac{2\lambda}{b^2}\right)+z^2\left(\frac{1}{r}-\frac{2\lambda}{c^2}\right)\right]=0$$

$$\frac{1}{r}(x^2+y^2+z^2)-2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad r^2=x^2+y^2+z^2$$

$$\therefore r-2\lambda=0 \Rightarrow 2\lambda=r$$

Substituting x, y, z into ⑤:

$$r\mu\left(\frac{a^2l^2}{a^2-2r\lambda}+\frac{b^2m^2}{b^2-2r\lambda}+\frac{c^2n^2}{c^2-2r\lambda}\right)=0$$

Substituting $2\lambda=r$:

ok, you could also had multiply the first eq by x, the second by y and the third by z then adding up

$$\frac{a^2l^2}{a^2-r^2}+\frac{b^2m^2}{b^2-r^2}+\frac{c^2n^2}{c^2-r^2}=0 \quad \text{as required.}$$

Geometric interpretation

g is an ellipsoid and h is a plane. The intersection of these is an ellipse. We are therefore finding stationary points of r on an ellipse. r is the Cartesian distance between two points whose difference in coordinates is (x, y, z).

The Lagrange multiplier $\lambda = \frac{1}{2r}$ so it is inversely proportional to the function r.

could you check what is happening in 2D and find and exact solution either with Lagrange or by directly solving the intersection point from the plane+ellipsoid eqs??