Day 1

Connection in Graphs*

In previous lecture we have seen 20 avatars(Basics types) of graph. Based on these terminology This week we will study some advance terminology of graph theory.

1.1 What is Path

There is many example of path in real life. Lets start with simple example. Suppose I am living at model town and want to go to airport(See Figure 1.1). The way I have searched on google map and it returned is called **path**

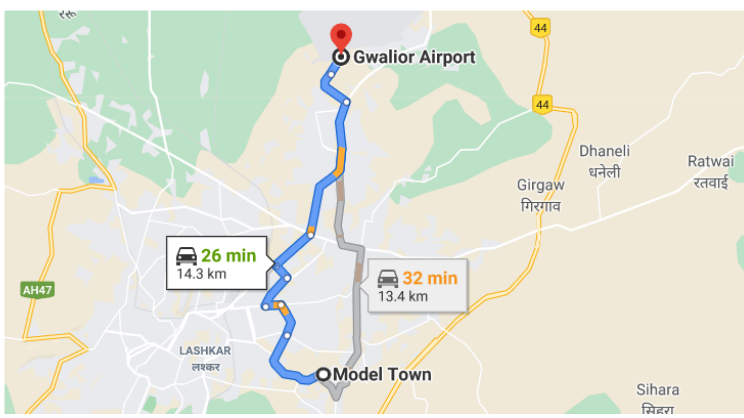

Figure 1.1: Simple example of path using google map

figure 1.1 showing two path from same source to destination. First path taking 26 minutes and second path is taking 32 minutes. So either I will follow first path or second path.

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Conclusion from Figure 1.1:

1. Source and destination in path is always different.
2. There may be more than one **path** from source to destination. In figure 1.1 there is two path.
3. People usually follow shortest path because to save time. But its up to them, they can also take long path according to there comfort.

Let's look the path more formally in terms of graph theory.

Definition:

1. In graph theory, a path in a graph $G(V, E)$ is a finite or infinite sequence of edges E which joins a sequence of vertices V .
2. According to Introduction to Graph Theory by D. West.
   An open walk in which no vertex appears more than once is called a path also called simple path or an elementary path
3. A path is a particularly simple example of a tree, and in fact the paths are exactly the trees in which no vertex has degree 3 or more.

Example 1:

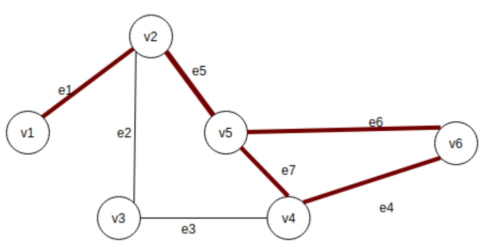


Figure 1.2: Shows a path (e1,e5, e7,e4,e6) or (v1,v2,v5,v4,v6,v5)

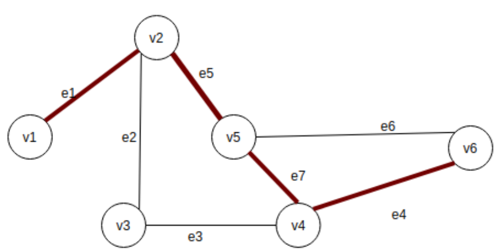
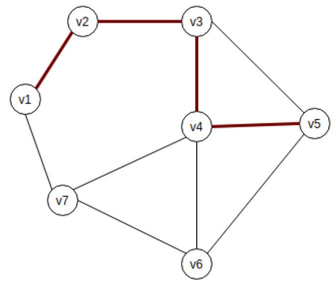


Figure 1.3: A simple path (e1, e5, e7, e4)

Example 2:

The following coloured walk is a path because the walk $v1, v2, v3, v4, v5$ does not repeat any edges.



Important fact about Path:

1. A path is a walk where all edges are distinct.

2. A simple path is a path where all vertices are distinct,
   In figure 1.2 example of path in which vertex v5 is repeated and in figure 1.3 no vertex is repeated.

3. Sometime path can be considered as simple path so on that case path is a trail in which neither vertices nor edges are repeated.

4. Let G be a graph, A trail is a walk $v_0, e_1, v_1, ..., v_k$ with no repeated edge. The length of a trail is its number of edges.

5. Let us take u and v as the endpoints then a u,v-trail is a trail with first vertex u and last vertex v.

6. A trail is said to be closed if its endpoints are the same. For a simple graph (which has no multiple edges), a trail may be specified completely by an ordered list of vertices

7. The number of edegs in a path is called the **length of a path**.

8. A **directed path** is a directed trail in which all vertices are distinct.

9. A **directed path** (sometimes called **dipath**) in a directed graph is a finite or infinite sequence of edges which joins a sequence of distinct vertices, but with the added restriction that the edges be all directed in the same direction.

10. A path such that no graph edges connect two nonconsecutive path vertices is called an **induced path**

11. A path that includes every vertex of the graph is known as a **Hamiltonian path**.

Finding paths:

To find shortest and longest paths in graphs there exist number of algorithms. Commonly used algorithms are as follows:

1. **Dijkstra's algorithm** produces a list of shortest paths from a source vertex to every other vertex in directed and undirected graphs with non-negative edge weights (or no edge weights).

2. **Bellman–Ford algorithm** can be applied to directed graphs with negative edge weights. The Floyd–Warshall algorithm can be used to find the shortest paths between all pairs of vertices in weighted directed graphs.

1.2 What is Walk

Definition:

1. A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk. In walk both edges and vertices can be repeated

2. Let G be a graph and let $v_0, v_n \in V(G)$.
   A walk here is a finite alternating sequence $W(v_0, v_n) = (v0, e1, v1, e2, v2, ..., e_t, v_n)$ of vertices and edges such that e_i is an edge incident with vertices v_{i-1} and v_i , i =1,2,...,t.

3. here v_0 is called the origin(starting vertex) and v_n is called the terminus(end vertex). Other vertices are called the internal vertices. Note that v_0 and v_n can also be internal vertices.

Example 1:

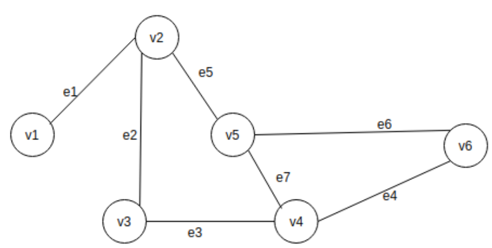


Figure 1.4

Figure 1.5 and Figure 1.6 are example of walk because walk can repeat edge.

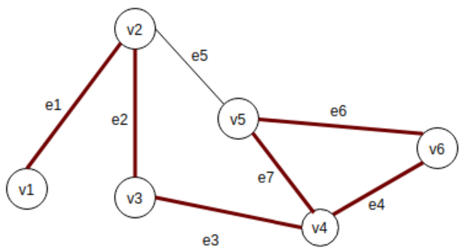


Figure 1.5: Walk without repeated edge (e1,e2,e3,e4,e6,e7)

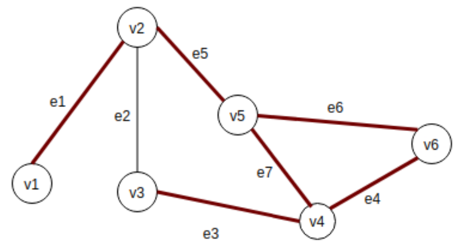


Figure 1.6: Walk with repeated edge (e1,e5,e6,e4,e7,e6,e4,e3)

Types of walk

Open walk: A walk is said to be an open walk if:

1. Length of the walk is greater than zero.
2. The starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.

Closed walk: A walk is said to be a closed walk if:

1. Length of the walk is greater than zero
2. The starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.

Important fact about walk

1. Walk can repeat anything (edges or vertices)
2. The length of walk can be calculated as the number of edges it contains where an edge is counted as many times as it occurs.
3. Walk is called a closed walk, if v_0(starting vertex) $= v_n$(ending vertex).
4. If there exists a (v_0, v_n)-walk, then there exists a (v_n, v_0)-walk.
5. If G is a simple graph, W is denoted as a sequence of vertices $(v_0, v_1, ..., v_n)$ with the understanding that $(v_i, v_i + 1)$ is an edge, for $i = 0, 1, \ . \ . \ . \ , t - 1$.
6. A walk in a graph is a sequence of edges such that each edge (except the first one) starts with a vertex where the previous edges ended.

1.3 What is Cycle

Definition:

1. A cycle is a path or a non-empty trail in which first and the last vertices are same.
2. Cycle is a closed trail in which the "first vertex = last vertex" is the only vertex that is repeated.

Example 1:

Figure 3 shows cycles with three and four vertices.

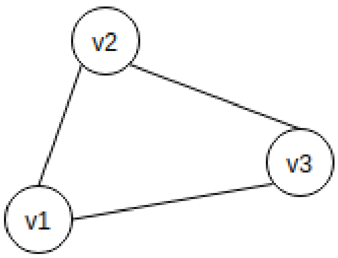


Figure 1.7: Cycle of size 3

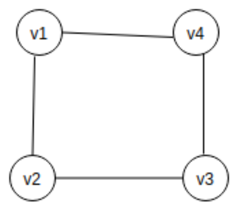


Figure 1.8: Cycle of size 4

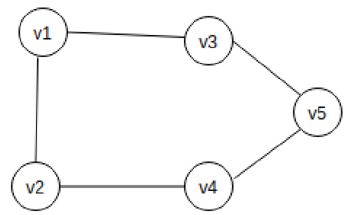


Figure 1.9: Cycle of size 5

Above figures are example of cycle with three node, four node and five node respectively.

Example 2:

Let's look at figure 1.9 and find the cycle of different size.

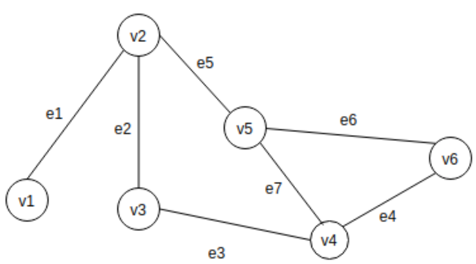


Figure 1.10: Graph G

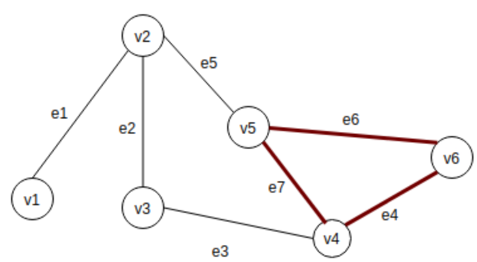


Figure 1.11: Cycle of size 3

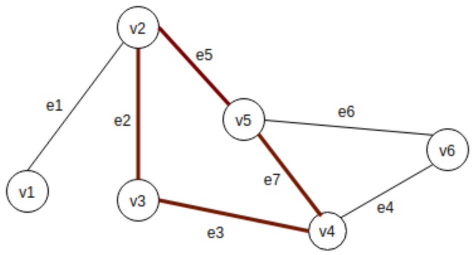


Figure 1.12: Cycle of size 4

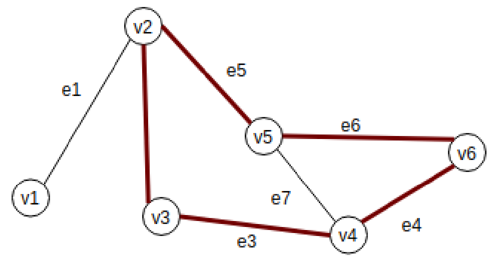


Figure 1.13: Cycle of size 5

Example 3:

In this example there are many cycle out of which in one colored cycle we can notice how no edges are repeated in the walk $v2, v3, v7, v6, v2$, which makes it definitely a trail, and that the start and end vertex v2 is the same which makes it closed.

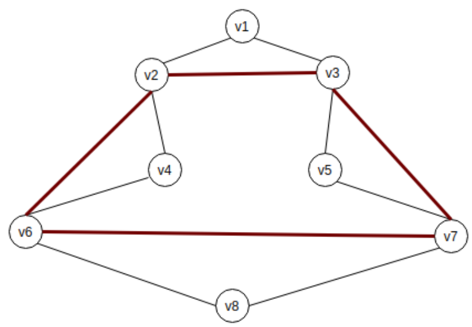


Figure 1.14

Theorem 1:

Statement: If G is simple and $\delta(G) \geqslant 2$, then there exists a cycle of length of at least $\delta(G) + 1$ in G.

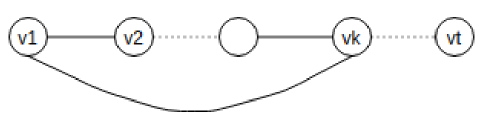


Figure 1.15

Proof. Let P be a path of maximum length in G. Let $P = (v_1, v_2, ..., v_n)$.
If v is a vertex adjacent with v_1, then
$v \in v_2, v_3, ..., v_n$; else $(v, v_1, v_2, ..., v_n)$ is a path of greater length, which is a contradiction to the maximality of P.
So, $N(v_1) \subseteq v_2, v_3, ..., v_n$. Let v_k be the last vertex in P to which v_1 is adjacent; see Figure 2.3. Then the subpath $Q = (v_1, v_2, .., v_k)$ contains at least deg $(v1) + 1 \geqslant \delta(G) + 1$. vertices, and so $(v_1, v_2, ..., v_k, v_1)$ is a cycle of length $\geqslant \delta(G) + 1$.

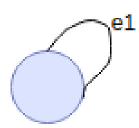
Theorem 2:

Statement: Every graph G with $m(G) \geqslant n(G)$ contains a cycle.

Proof. If G contains a loop ($=$ *a* 1-cycle) or a multiple edge ($=$ *a* 2-cycle) we are through.
So, we prove the theorem for simple graphs. This we do by induction on n.
If $n \geqslant 3$, then there is only one graph with $m \geqslant n$, namely C_3.
So we proceed to the induction step.
If $\delta(G) \geqslant 2$, then G contains a cycle by above theorem.
Next assume that $\delta(G) \geqslant 1$, and let v be a vertex of degree $\geqslant 1$ in G.
Then $G - v$ is a graph with $m(G - v) \geqslant n(G - v)$.
Therefore, by induction hypothesis, $G - v$ contains a cycle. Hence G too contains a cycle.

Theorem 3:

Statement: Every closed odd walk contain an odd cycle.
Proof:



By using induction law
For l=1, obviously true.
Suppose it is true for $l < L$, then we have to show that this is also true for L.
Case1: If there is no repetition of vertex in walk, then a closed walk = a closed cycle.
Case2: If there is repetition in vertex in the walk, and let us suppose v is vertex which repeats. Break the walk into two v-v walks (say w1 and w2). Since $|w1| + |w2| = odd$, that means either w1 or w2 is odd walk. And surely they both are less than L. From the induction one of them (odd walk one) contain odd cycle.

Important fact about cycle

1. A graph without cycles is called an acyclic graph.

2. A directed graph without directed cycles is called a directed acyclic graph.

3. A connected graph without cycles is called a tree.

Relation between Path, Trail, Walk and Cycle

Let G be a graph and let $v_0, v_n \in V(G)$

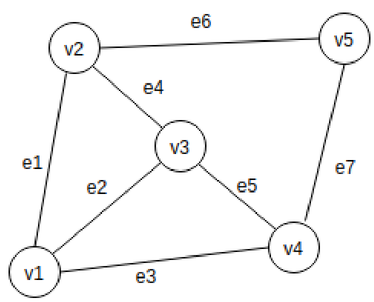
1. A (v_0, v_n)-**walk** is called a (v_0, v_n)-trail, if no edge is repeated (but vertices may get repeated). It is called a closed trail if $v_0 = v_n$.

2. A (v_0, v_n)-walk is called a (v_0, v_t)-path,if no vertex is repeated (and therefore no edge is repeated. By definition, every path is a trail and every trail is a walk. However, a walk need not be a trail and a trail need not be a path.

3. A closed walk $W(v_0, v_n)$ is called a cycle, if all its vertices are distinct except that $v_0 = v_n$

4. A cycle with k vertices is called a k-cycle and it is denoted by C_k.

5. A C_3 is also a K_3 and it is referred to as a triangle. A 1-cycle is a loop and a 2-cycle consists of two multiple edges.



We illustrate these concepts by taking a graph.

1. $W1(v1, v5) = (v1, e1, v2, e4, v3, e2, v1, e3, v4, e5, v3, e4, v2, e6, v5)$.

2. $W2(v1, v5) = (v1, e1, v2, e4, v3, e2, v1, e3, v4, e7, v5)$.

3. $W3(v1, v5) = (v1, e1, v2, e4, v3, e5, v4, e7, v5)$.

4. $W4(v1, v1) = (v1, e1, v2, e4, v3, e2, v1, e3, v4, e5, v3, e2, v1)$.

5. $W5(v1, v1) = (v1, e1, v2, e4, v3, e5, v4, e3, v1)$.

Here,
W1 is a (v_1, v_5)-walk of length 7. It is not a trail.
$W2$ is a trail of length 5 but it is not a path.
$W3$ is a path of length 4.
$W4$ is a closed walk of length 6 but it is not a cycle.
$W5$ is a cycle of length 4. As per our convention, W5 is also denoted by $(v_1, v_2, v_3, v_4, v_1)$.

1.4 Connected and disconnected graph

1. A graph is connected if there is path between any two pair of vertices in that graph. Otherwise it is disconnected.

2. A graph G is connected if there is a path in G between its every pair of vertices.

Connected Components

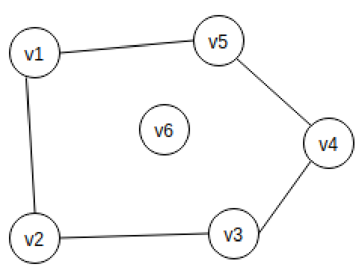
1. A graph H is a connected component("island") of G if:

    (a) H is a subgraph of G,

    (b) H is connected, and

    (c) H is maximal, i.e., adding any vertices to H will disconnect it.

2. In short, H is a connected component of G if H is a maximal subgraph of G that is connected. Those connected subgraphs which are not contained in larger connected subgraphs of G. The graph which is not connected will have two or more connected components.



Cut-edge or Cut-vertex

1. A cut-edge or cut-vertex of a graph is an edge or vertex whose deletion increases the number of components.

Example 1:

In below figure, colored edge is a cut edge because on removal of this the graph will be disconnected so this edge called cut edges.

In below figure, if you remove any of the colored vertex corresponding edges will also remove and graph G become disconnected. So these two vertex is cut vertex.

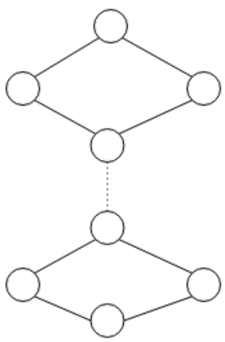


Figure 1.16: Graph G

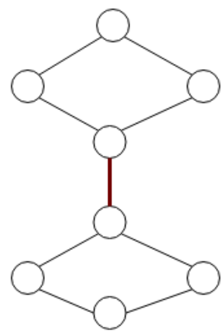


Figure 1.17: Cut Edges

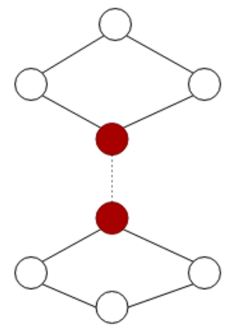


Figure 1.18: Cut Vertices

Important facts about Connectivity:

1. G has connectivity k if there is a cutset of size k but no smaller cutset.
2. If there is no cutset and G has at least two vertices, we say G has connectivity n - 1; if G has one vertex, its connectivity is undefined.
3. If G is not connected, we say it has connectivity 0.
4. G is k-connected if the connectivity of G is at least k.
5. The connectivity of G is denoted k(G)

Theorem 1:

Statement: Prove that every graph with n vertices and m edges has at least $n-m$ connected components.

Proof. We will prove this claim by doing induction on m.
Base Case: $m = 0$. A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. Hence the claim is true for $m = 0$.
Induction Hypothesis: Assume that for some $k \geqslant 0$, every graph with n vertices and k

edges has at least $n - k$ connected components.
Induction Step: We want to prove that a graph, G, with n vertices and k + 1 edges has at least $n - (k+1) = n - k - 1$ connected components.
Consider a subgraph $G^{'}$ of G obtained by removing any arbitrary edge, say $\{u, v\}$, from G. The graph $G^{'}$ has n vertices and k edges.
By induction hypothesis, G has at least $n - k$ connected components. Now add $\{u, v\}$ to $G^{'}$ to obtain the graph G. We consider the following two cases.

1. Case I: u and v belong to the same connected component of $G^{'}$. In this case, adding the edge $\{u, v\}$ to $G^{'}$ is not going to change any connected components of $G^{'}$. Hence, in this case the number of connected components of G is the same as the number of connected components of G which is at least $n - k > n - k - 1$.

2. CaseII: u and v belong to different connected components of $G^{'}$. In this case, the two connected components containing u and v become one connected component in G. All other connected components in $G^{'}$ remain unchanged.

Thus, G has one less connected component than $G^{'}$. Hence, G has at least $n-k-1$ connected components.

Theorem 2:

Statement: Prove that every connected graph with n vertices has at least $n - 1$ edges.

Proof: . We will prove the contrapositive, i.e., a graph G with $m \leq n - 2$ edges is disconnected. From the result of the previous problem, we know that the number of components of G is at least

$$n - m \geqslant n - (n - 2) = 2$$

which means that G is disconnected. This proves the claim.
One could also have proved the above claim directly by observing that a connected graph has exactly one connected component. Hence, $1 \geqslant n - m$. Rearranging the terms gives us $m \geqslant n$ - 1.

Theorem 3:

Statement: An edge is a cut-edge if and only if it belongs to no cycle.

Proof. Take any edge $e = \{u, v\}$. Remove this edge from our graph:
if the graph is still connected, then there is some path from u to v not involving e; consequently, if we add e to the end of this path, we get a cycle.
Thus, if e is not a cut-edge, it's involved in a cycle.
Conversely:suppose that $e = \{u, v\}$ lies in a cycle. Let P be the path from u to v that

doesn't use e (i.e. go the other way around the cycle.) Pick any x, y in G; because G is connected, there's a path from x to y in G.
Take this path, and edit it as follows:
whenever the edge e shows up, replace this with the path P (or P traced backwards, as needed.) This then creates a walk from x to y; by deleting cycles, this walk will always become a path, and thus G is connected. So if e is involved in a cycle, it's not a cut-edge.

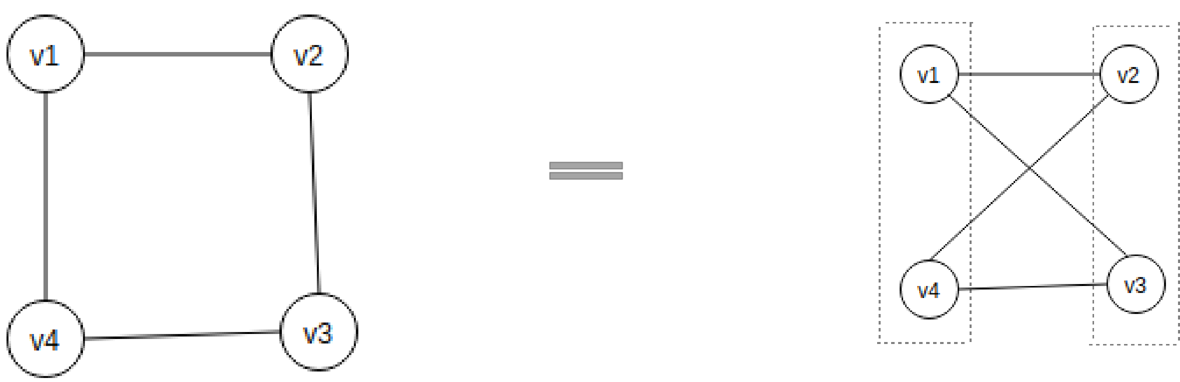
Day 2

Bipartite Graph, Eulerian Circuit[*]

2.1 Bipartite Graph

Definition:

1. A bipartite graph is a special kind of graph with the following properties:
   (a) It consists of two sets of vertices X and Y.
   (b) The vertices of set X join only with the vertices of set Y.
   (c) The vertices within the same set do not join.
2. A bipartition of a graph is a specification of two disjoint independent sets in G whose union become $V(G)$. The statement "G is a bipartite graph with bipartition X and Y" specifies one such partition.



Here, The vertices of the graph can be decomposed into two sets. The two sets are $X = \{A, C\}$ and $Y = \{B, D\}$. The vertices of set X join only with the vertices of set Y and vice-versa. The vertices within the same set do not join. Therefore, it is a bipartite graph.

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Bipartite Graph Properties

1. Bipartite graphs are 2-colorable.
2. Bipartite graphs contain no odd cycles.
3. Every sub graph of a bipartite graph is itself bipartite.
4. There does not exist a perfect matching for a bipartite graph with bipartition X and Y if $|X| \neq |Y|$.
5. In any bipartite graph with bipartition X and Y, Sum of degree of vertices of set X = Sum of degree of vertices of set Y
6. Number of complete matchings for $K_{n,n} = n!$

Theorem 1:

Statement: A graph is bipartite iff it has no odd cycle.

Proof: **Necessary condition:** Assume graph is bipartite and X and Y are two independent sets. To have a cycle, one has to traverse X to Y to X OR Y to X to Y one or more time. Therefore, a bipartite graph can not have odd cycle.

Sufficient condition: If G has no odd cycle => it does not contain a cycle OR it contain even cycle. It does not contain a cycle => take one vertex in X and next vertex in Y. It contain even cycle => Partition the graph such that each even length cycle is one subgraph. We know C_n is bipartite for even length cycle.

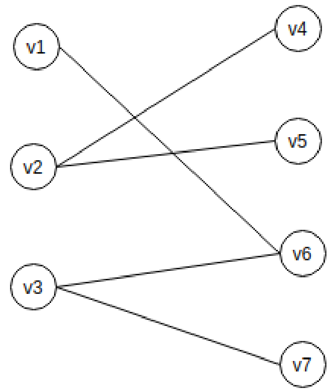
Or

One Other way is very easy: if G is bipartite with vertex sets V_1 and V_2, every step along a walk takes you either from V_1 to V_2 or from V_2 to V_1. To end up where you started, therefore, you must take an even number of steps.

Conversely, suppose that every cycle of G is even.
Let v_0 be any vertex. For each vertex v in the same component C_0 as v_0 let $d(v)$ be the length of the shortest path from v_0 to v. Color red every vertex in C_0 whose distance from v_0 is even, and color the other vertices of C_0 blue. Do the same for each component of G. Check that if G had any edge between two red vertices or between two blue vertices, it would have an odd cycle. Thus, G is bipartite, the red vertices and the blue vertices being the two parts.

Example 1:

The chromatic number of the following bipartite graph is 2.



Example 2:

Given a bipartite graph G with bipartition X and Y, There does not exist a perfect matching for G if $|X| \neq |Y|$

or

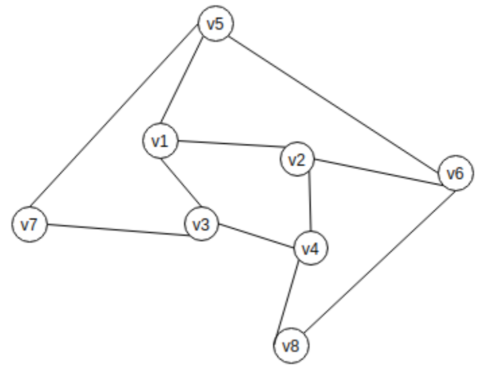
A perfect matching exists on a bipartite graph G with bipartition X and Y if and only if for all the subsets of X, the number of elements in the subset is less than or equal to the number of elements in the neighborhood of the subset.

Example 3:

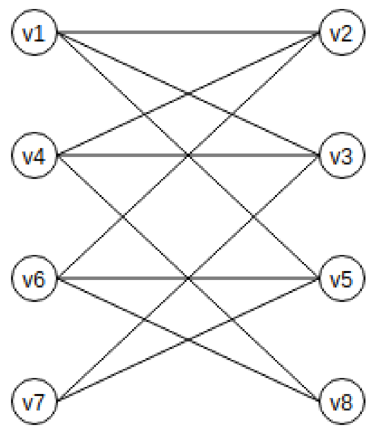
Maximum possible number of edges in a bipartite graph on 'n' vertices = (1/4) x n^2.
Suppose the bipartition of the graph is (V1, V2) where $|V1|$ = k and $|V2| = n - k$.
The number of edges between V1 and V2 can be at most $k(n - k)$
which is maximized at k = $n/2$.
Thus, maximum (1/4) x n^2 edges can be present.
Also, for any graph G with n vertices and more than (1/4) x n^2 edges, G will contain a triangle. This is not possible in a bipartite graph since bipartite graphs contain no odd cycles.

Example 4:

Is the following graph a bipartite graph?



Solution- The given graph may be redrawn as-



Here, This graph consists of two sets of vertices. The two sets are $X = \{v1, v4, v6, v7\}$ and $Y = \{v2, v3, v5, v8\}$. The vertices of set X are joined only with the vertices of set Y and vice-versa. Also, any two vertices within the same set are not joined. This satisfies the definition of a bipartite graph.
Therefore, Given graph is a bipartite graph.

Example 5:

The maximum number of edges in a bipartite graph on 12 vertices is? Solution- We know, Maximum possible number of edges in a bipartite graph on 'n' vertices = (1/4) x n2. Substituting n = 12, we get-
Maximum number of edges in a bipartite graph on 12 vertices =

$$(1/4) \text{ x } (12)2$$
$$= (1/4) \text{ x } 12 \text{ x } 12$$
$$= 36$$

Therefore, Maximum number of edges in a bipartite graph on 12 vertices = 36.

Trivial and Non Trivial

1. The trivial graph is the graph on one vertex
2. A non-trivial connected component is a connected component that isn't the trivial graph, which is another way of say that it isn't an isolated point.

2.2 Complete Bipartite Graph

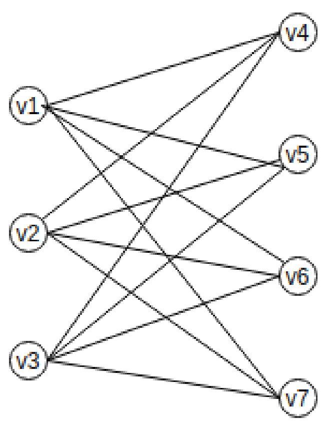
Definition 1:

A complete bipartite graph may be defined as follows

1. A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph.
2. Complete bipartite graph is a graph which is bipartite as well as complete.

Example 1:

The following graph is an example of a complete bipartite graph:



This graph is a bipartite graph as well as a complete graph.Therefore, it is a complete bipartite graph.

1. This graph is called as $K_{3,4}$.
2. Bipartite graphs are 2-colorable.
3. If graph is bipartite with no edges, then it is 1-colorable.

Properties of complete bipartite graph

1. The complete bipartite graph $K_{m,n}$ has a vertex covering number of $min_{m,n}$and an edge covering number of $max_{m,n}$.
2. The complete bipartite graph $K_{m,n}$ has a maximum independent set of size $max_{m,n}$.
3. A complete bipartite graph $K_{m,n}$ has $m^{(n-1)}\ n^{(m-1)}$ spanning trees.
4. A complete bipartite graph Km,n has a maximum matching of size $min_{m,n}$.

Example 2:

Give the Example to prove that A simple path is a bipartite graph.

solution: Yes simple path of any length is a bipartite graph because it can be partitioned into two disjoint subset X, Y .
See figure 2.1 is a clear explanation of path of length 3 and its equivalent bipartite graph.

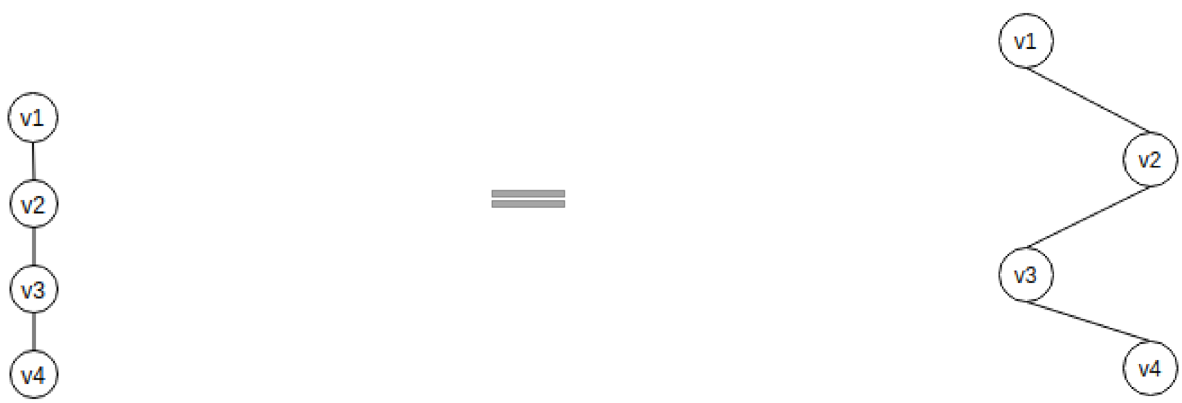


Figure 2.1: Example of simple path is a bipartite graph

Example 3:

Give the Example to prove that C_n is bipartite iff n is even.

Solution: You can take cycle of any size, here I am explaining using n=4.
figure 2.2 is a clear conversion of cycle into bipartite graph.

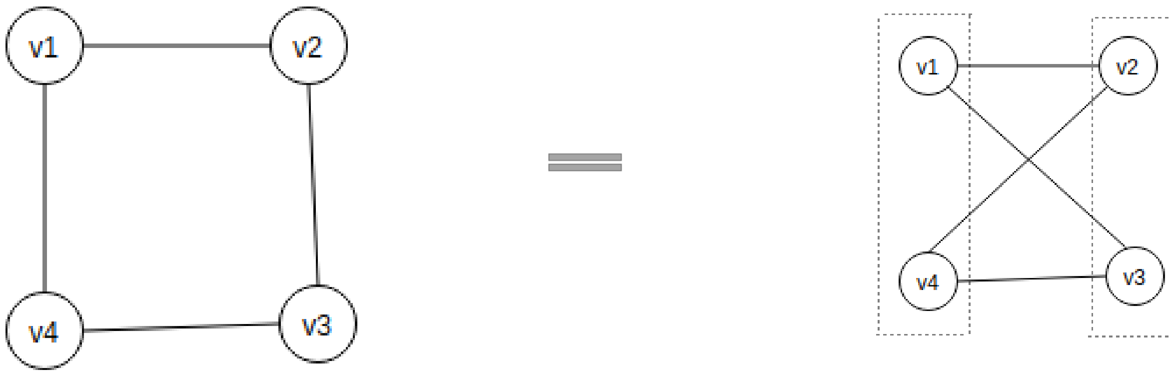


Figure 2.2: Cycle of size 4 can be converted into biparpite graph

Figure 2.3: Cycle of size 3 and 5 means odd cycle cannot convert into bipartite graph

2.3 Eulerian Circuit

Konigsberg bridge problem

1. In the early 18th century, the citizens of Königsberg spent their days walking on the intricate arrangement of bridges across the waters of the Pregel (Pregolya) River. for a total of seven bridges.

2. According to folklore, the question arose of whether a citizen could take a walk through the town in such a way that each bridge would be crossed exactly once

3. Therefore, each landmass, with the possible exception of the initial and terminal ones if they are not identical, must serve as an endpoint of an even number of bridges

4. However, for the landmasses of Königsberg, A is an endpoint of five bridges, and B, C, and D are endpoints of three bridges. The walk is therefore impossible.

5. Figure 2.6 is a Königsberg bridge problem as a graph consisting of nodes (vertices) representing the landmasses and arcs (edges) representing the bridges. The degree of a vertex of a graph specifies the number of edges incident to it.

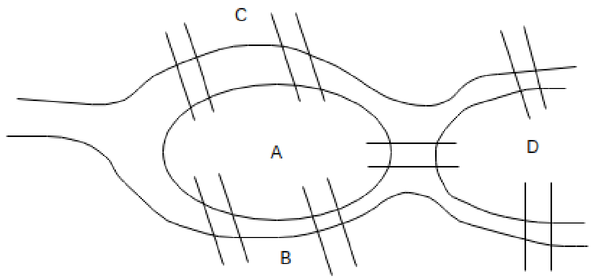


Figure 2.4: G_1

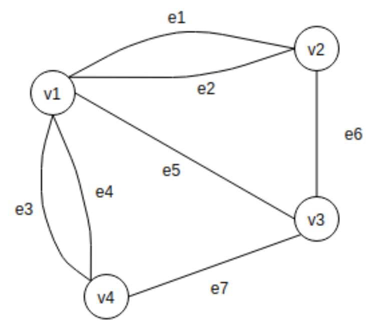


Figure 2.5: G_2

Eulerian path traverses each edge of a graph once and only once. Thus, Euler's assertion that a graph possessing such a path has **at most two vertices of odd degree** was the first theorem in graph theory

Definition of Eulerian Circuit

1. In simple term a graph in which need to traverse every edge exactly once and return to starting vertex.
2. We call a closed trail a circuit when we do not specify the first vertex but keep the list in cyclic order.
3. An Eulerian circuit in a graph is a circuit containing all the edges.
4. More formally a graph is Eulerian if it has a closed trail containing all the edges.

Theorem 1:

Statement: A graph G is Eulerian iff it has at most one non-trivial component and all its vertices have even degree.

Necessary condition:

Assume a graph G is Eulerian. => It has closed trail containing all the edges. =¿ There will be an incoming and outgoing edge for all the vertices. => All its vertices have even degree. (It has to have one non-trivial component, as more than one components closed trail might not be possible).

Sufficient condition:

Assume a graph G has at most one non-trivial component and all its vertices have even degree. If number of edges $m = 0$ then G is Eulerian. Assume this is true for all graphs

having the above properties and less than M edges.
Then consider a graph containing one non-trivial component has M edges and its each vertex has even degree. => Each vertex has at least two degree. (Why it can't be 0?)
Consider a graph containing one non-trivial component has M edges and its each vertex has even degree. => Each vertex has at least two degree.(Why it can't be 0?) => G contains cycle lets say it cycle C ,
Remove $E(C)$ from G to construct G'.
G' has less than M edges and Each vertex of G' has even degree. It can have more than one component. By the induction hypo. Each component of G' contain Eulerian Cycle. We combine these Eulerian cycles with C to construct an Eulerian circuit as follows: Traverse C until a component of G's appear, then traverse Eulerian cycle of that component, come back to C and repeat this. see below figure for more understanding.

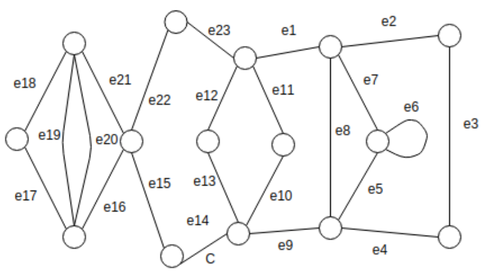


Figure 2.6

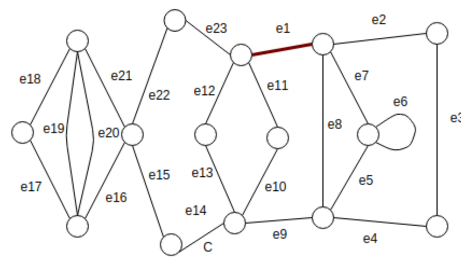


Figure 2.7

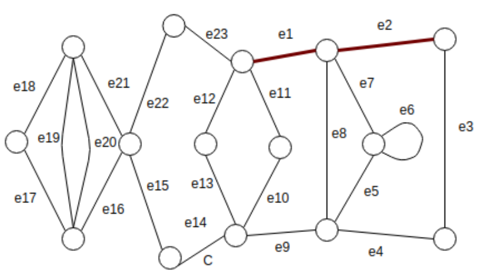


Figure 2.8

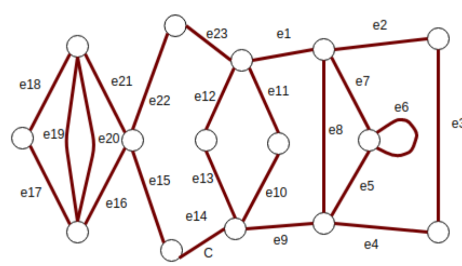


Figure 2.9

Day 3

Union of Graph, Miscellaneous Exercise*

3.1 Union of graphs

Definition 1:

The union of graphs $G1, G2, ...G_k$ written as $\bigcup_{i=1}^{k} G^i$ is the graph with vertex set $\bigcup_{i=1}^{k} E(G^i)$ and edge set $\bigcup_{i=1}^{k} V(G^i)$

Definition 2:

In simple way union of graph is represented as $G1 \cup G2$.
There are two definitions.

1. The disjoint union of graphs, the union is assumed to be disjoint
2. The union of two graphs is defined as the graph $(V1 \cup V2, E1 \cup E2)$.

.

Example 1:

In figures G1 and G2 are the two graphs and figure 3.3 is showing the union of G1 and G2

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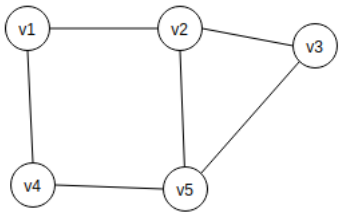


Figure 3.1: G_1

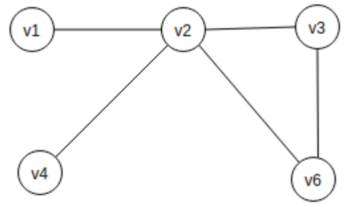


Figure 3.2: G_2

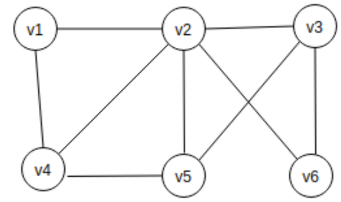
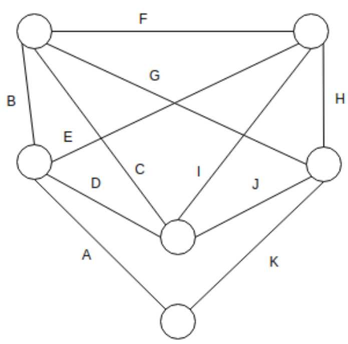
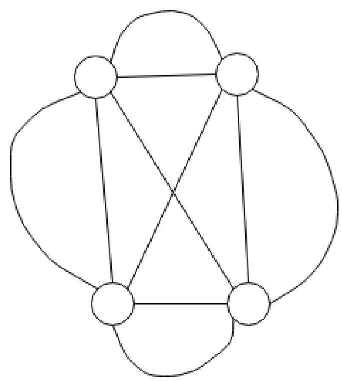


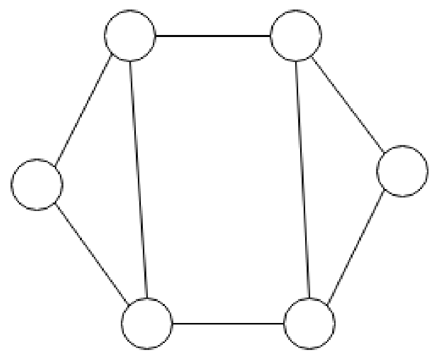
Figure 3.3: $G_1 \cup G_2$

Example 1:

Are these graph Eulerian?



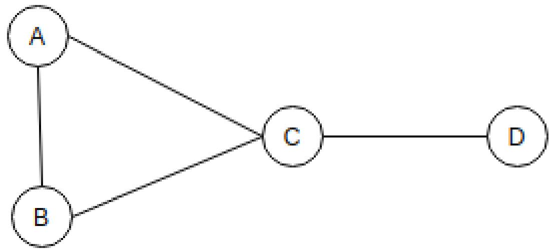




Solution: 1. True, First figure is Eulerian graph because according to lemma that if every vertex of graph has even degree then graph is Eulerian graph.
2. False, For figure 2 and 3 if we check the degree of each vertex then we can find that some vertex have odd degree. so both graph is not Eulerian Graph.

Example 2:

calculate the following using given figure:
Maximal paths: Maximal cliques:
Maximal independent sets:
And
Maximum paths:
Maximum cliques:
Maximum Independent sets:



In the below example of graph this structure include vertex A,B,C,D is called paw and Maximal path in a graph is the path where if you add an another edge its not going to increase the length of a path.

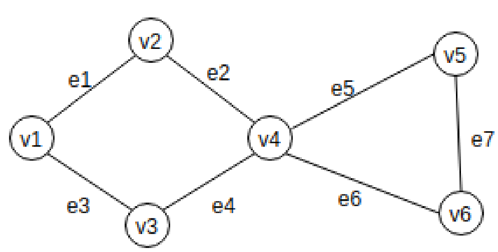
So here is the solution
Maximal paths: ACB, ABCD, BACD
Maximal cliques: ABC, CD
Maximal independent sets: C, BD, AD
Maximum paths: ABCD, BACD
Maximum cliques: ABC
Maximum Independent sets: BD, AD

Example 3:

Prove or disprove:

1. Every Eulerian bipartite graph has an even number of edges.
   True. Since graph is Eulerian, it can be decomposed into cycles. Since it is bipartite, all cycles are of even length. Hence, the edges comprise of some number of even-length cycles. hence number of edges is even.

2. Every Eulerian simple graph with an even number of vertices has an even number of edges.
   False. Consider a cycle of length 4 and a cycle of length 3 and connect them at one vertex only. The resulting graph is Eulerian (two cycles and connected), has 6 vertices (even), but 7 (odd) edges.



Day 4

Vertex Degrees and Counting*

4.1 Degree of a vertex

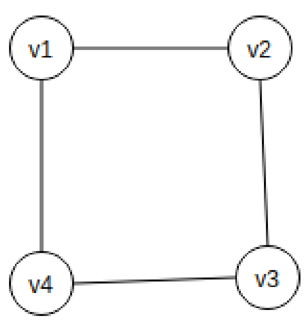
Definition:

1. Degree of a vertex v in a graph G written as d(v) is number of edges incidents to v, except that each loop at v counts twice. The maximum and minimum number of degrees are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

2. To analyze a graph it is important to look at the degree of a vertex. One way to find the degree is to count the number of edges which has that vertex as an endpoint. An easy way to do this is to draw a circle around the vertex and count the number of edges that cross the circle.

Example 1:

What will be $\Delta(G)$ and $\delta(G)$ for k-regular graph?
Solution: The minimum and maximum degree of k-regular graph is k For example if a graph



2-regular graph

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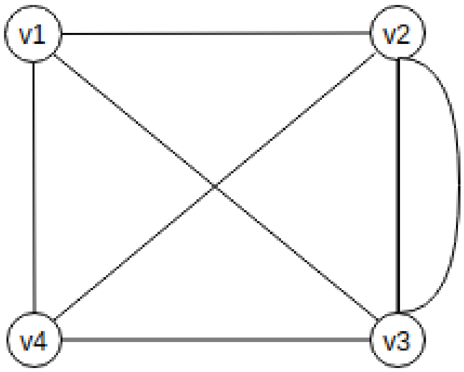
is 2 regular graph then degree of all vertex will we 2

Example 1:

Find degree of each node:



Vertex Name	Degree of vertex
V1	3
V2	4
V3	4
V4	3

Figure 4.1: Cycle of size 3 and 5 means odd cycle cannot convert into bipartite graph

Order and size of a graph :

The order of a graph G, $n(G)$ is number of vertices in G
The size of a graph G, $e(G)$ is number of edges in G.

Handshaking Lemma (1st Theorem of Graph Theory)

(Degree-sum formula) If G is a graph then.

$$\sum_{v \in V(G)} d(v) = 2\ e(G)$$

The formula implies that in any undirected graph, the number of vertices with odd degree is even. This formula called as the degree sum formula.

Important facts about degree of vertex:

1. A vertex with degree 0 is called an isolated vertex.
2. A vertex with degree 1 is called a leaf vertex or end vertex, and the edge incident with that vertex is called a pendant edge.
3. A vertex with degree $n - 1$ in a graph on n vertices is called a dominating vertex.

4. If each vertex of the graph has the same degree k the graph is called a k-regular graph and the graph itself is said to have degree k.

5. Similarly, a bipartite graph in which every two vertices on the same side of the bipartition as each other have the same degree is called a biregular graph.

6. An undirected, connected graph has an Eulerian path if and only if it has either 0 or 2 vertices of odd degree. If it has 0 vertices of odd degree, the Eulerian path is an Eulerian circuit.

4.2 Examples:

Example 1

There are 5 nodes in a graph. Each node has labelled using $\{1, 2, 3, 4, 5\}$. Two nodes are connected if their difference is a prime number. How many connected components this graph will have?
$(A)1$
$(B)2$
$(C)3$
$(D)4$

Solution: When you draw this graph you will find a graph given below. So clearly can

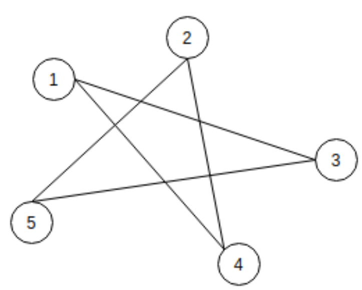


Figure 4.2

see this is a single connected component. So correct answer is 1 that is option A.

Example 2

If a node v is deleted in loopless graph G. Further, suppose adjacency (A) and incidence (M) matrix of G has size m x m and m x n, respectively. What will be the size of these matrices for $G - v$?.
(I) A will have size (m-1) x m
(II) M will have size (m-1) x n

Which among above are true?
(A) Only I is true (B) Only II is true (C) Both are true (D) Both are false.

Solution: Correct option is D that means both the statement are false because size of adjacency matrix will be $m-1$ x $m-1$ and size of incidence matrix will be $m-1$ x $(n - deg(v))$

Example 3

In a class with 9 students, each students sends valentine cards to three others. Determine whether it is possible that each student receives cards from the same three students to who he or she sends the card.

Solution: number of students = 9
Each students Send card to three others.
Hence total Card Distributed = $9 * 3 = 27$
it is not possible that each student receives cards from the same three students to whom he or she sent cards.
Similar transaction need to be in pair to happen this like
a sent b => b sent a
Hence total transaction need to be in even number but transactions are in odd number hence not possible if it have been 8 students then it would have been possible

Sender	Receiver
1	2 3 4
2	1 3 4
3	1 2 4
4	1 2 3
5	6 7 8
6	5 7 8
7	5 6 8
8	5 6 7

Example 4:

Suppose a simple graph has 15 edges, 3 vertices of degree 4, and all others of degree 3. How many vertices does the graph have?

Solution: 3 * 4 + (x-3) * 3 = 30 (use degree of sum formula)

Example 5:

A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

Solution: Given-
Number of edges = 24
Degree of each vertex = 4
Let number of vertices in the graph = n.
Using Handshaking Theorem, we have-
Sum of degree of all vertices = 2 x Number of edges
Substituting the values, we get-

$$\text{n x } 4 = 2 \text{ x } 24$$
$$\text{n} = 2 \text{ x } 6$$
$$\text{n} = 12$$

Thus, Number of vertices in the graph = 12.

Example 6

A graph contains 21 edges, 3 vertices of degree 4 and all other vertices of degree 2. Find total number of vertices.

Solution: Given-
Number of edges = 21 Number of degree 4 vertices = 3 All other vertices are of degree 2 Let number of vertices in the graph = n. Using Handshaking Theorem, we have- Sum of degree of all vertices = 2 x Number of edges Substituting the values, we get-

$$3 \text{ x } 4 + (\text{n} - 3) \text{ x } 2 = 2 \text{ x } 21\ 12 + 2\text{n} - 6 = 42\ 2\text{n} = 42 - 6\ 2\text{n} = 36$$

$\therefore n = 18$
Thus, Total number of vertices in the graph = 18.

Example 7:

A simple graph contains 35 edges, four vertices of degree 5, five vertices of degree 4 and four vertices of degree 3. Find the number of vertices with degree 2.

Solution- Given-

- Number of edges = 35
- Number of degree 5 vertices = 4
- Number of degree 4 vertices = 5
- Number of degree 3 vertices = 4

Let number of degree 2 vertices in the graph = n
Using Handshaking Theorem, we have-
Sum of degree of all vertices = 2 x Number of edges Substituting the values, we get-

$$4 \text{ x } 5 + 5 \text{ x } 4 + 4 \text{ x } 3 + \text{n x } 2 = 2 \text{ x } 35$$
$$20 + 20 + 12 + 2\text{n} = 70$$
$$52 + 2\text{n} = 70$$
$$2\text{n} = 70 - 52$$
$$2\text{n} = 18$$

$\therefore n = 9$
Thus, Number of degree 2 vertices in the graph = 9.

Example 8

A graph has 24 edges and degree of each vertex is k, then which of the following is possible number of vertices?

(A) 20 (B) 15 (C) 10 (D) 8

Solution: Given-
Number of edges = 24 Degree of each vertex = k Let number of vertices in the graph = n.
Using Handshaking Theorem, we have-
Sum of degree of all vertices = 2 x Number of edges
Substituting the values, we get-

$$\text{n x k} = 2 \text{ x } 24 \text{ k} = 48 \text{ / n}$$

Now, It is obvious that the degree of any vertex must be a whole number. So in the above equation, only those values of 'n' are permissible which gives the whole value of 'k'.
Now, let us check all the options one by one-

- For n = 20, k = 2.4 which is not allowed.

- For n = 15, k = 3.2 which is not allowed.
- For n = 10, k = 4.8 which is not allowed.
- For n = 8, k = 6 which is allowed.

Thus, Option (D) is correct.

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