# Extended Symbolic Dynamics

## Entropies and the Isomorphism problem

Neemias Martins joint with R. Varão, P. Mattos and P. Mehdipour

neemias.org

# Contents

# Extended Symbolic Dynamics

# Setup

- $(X, \mathcal{B}, \mu)$ a probability space
- $f : X \to X$ a measure-preserving map
- We say that $f : X \to X$ and $g : Y \to Y$ defined on $(X, \mathcal{B}_1, \mu)$ and $(Y, \mathcal{B}_2, \nu)$, are isomorphic if there are invariant measurable sets $X_1 \subset X$ and $Y_1 \subset Y$ of full measure and a invertible measure preserving map $\varphi : X_1 \to X_2$ such that

$$\varphi \circ f = g \circ \varphi.$$

# Bernoulli shifts



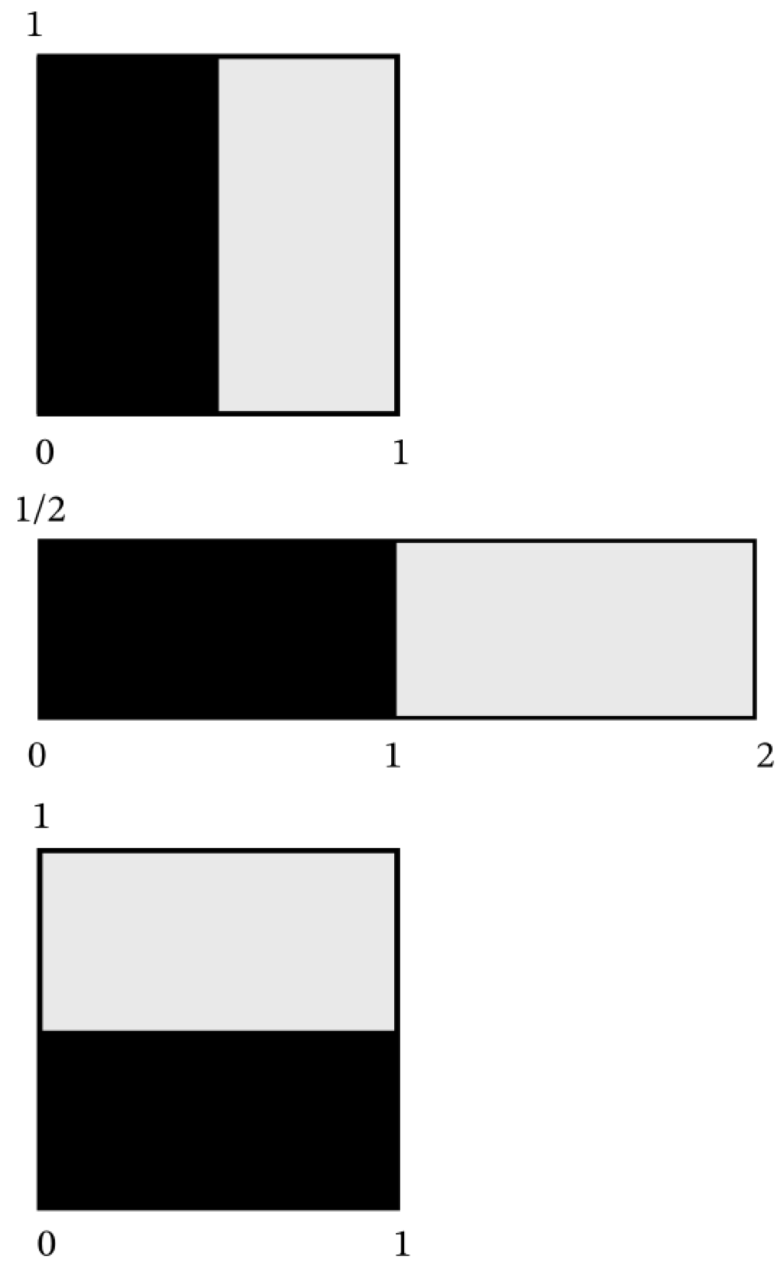
Let $\mathbb{A}$ an finite alphabet and consider $\Sigma_{\mathbb{A}} = \mathbb{A}^{\mathbb{Z}}$ the space of all sequences of letters of $\mathbb{A}$ :

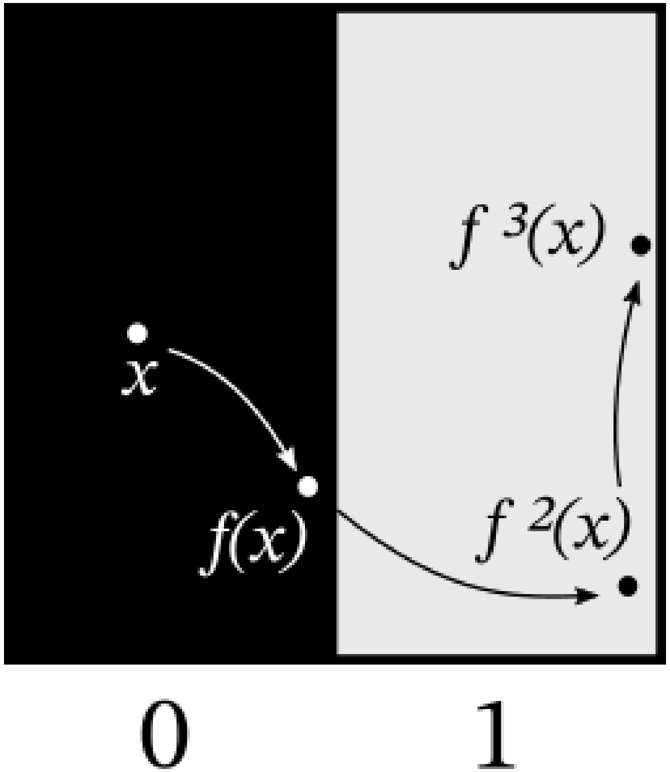
$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1}; x_0x_1\cdots) \in \Sigma_{\mathbb{A}}$$

- The **shift map** is the transformation $\sigma : \Sigma_{\mathbb{A}} \to \Sigma_{\mathbb{A}}$ tal que $\sigma(x_n) = (x_{n+1})$.
- A measurable map $f : X \to X$ is a **Bernoulli shift** if is isomorphic to a shift map.

# Baker map








- A órbita de $x$ é representada por $(\cdots\ ; 0011\cdots)$
- A órbita de $f(x)$ é representada por $(\cdots\ 0; 11\cdots)$

Let $\mathrm{A}$ and $\mathrm{B}$ be two finite alphabets with $\mathrm{A} \geq \mathrm{B}$ and $\varphi : \mathrm{A} \to \mathrm{B}$ a surjective map. Consider $\Sigma$ the space of all sequence of symbols

$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1}; x_0x_1 \cdots)$$

with $x_{-1}, x_{-2}, \ldots \in \mathrm{B}$ and $x_0, x_1, \ldots \in \mathrm{A}$.

- The **zip shift** [1] map is the transformation $\sigma_\varphi : \Sigma \to \Sigma$ such that

$$\sigma_\varphi(\cdots x_{-1}; x_0x_1 \cdots) = (\cdots x_{-1}\varphi(x_0); x_1x_2 \cdots)$$

- A measurable map $f : X \to X$ is a **extended Bernoulli** if is isomorphic to a zip map. We say that $f$ is a $(m, l)$-Bernoulli when $\#\mathrm{B} = m$ and $\#\mathrm{A} = l$.

Let $\mathcal{C}$ the $\sigma$-algebra generated by cylinders sets

- $C_i^s \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_{i\dots k}^{s_i \dots s_k} \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \dots, x_k = s_k\}$.

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in $\mathrm{A}$, we define a probability distribution $(p_\beta : \beta \in \mathrm{B})$ where

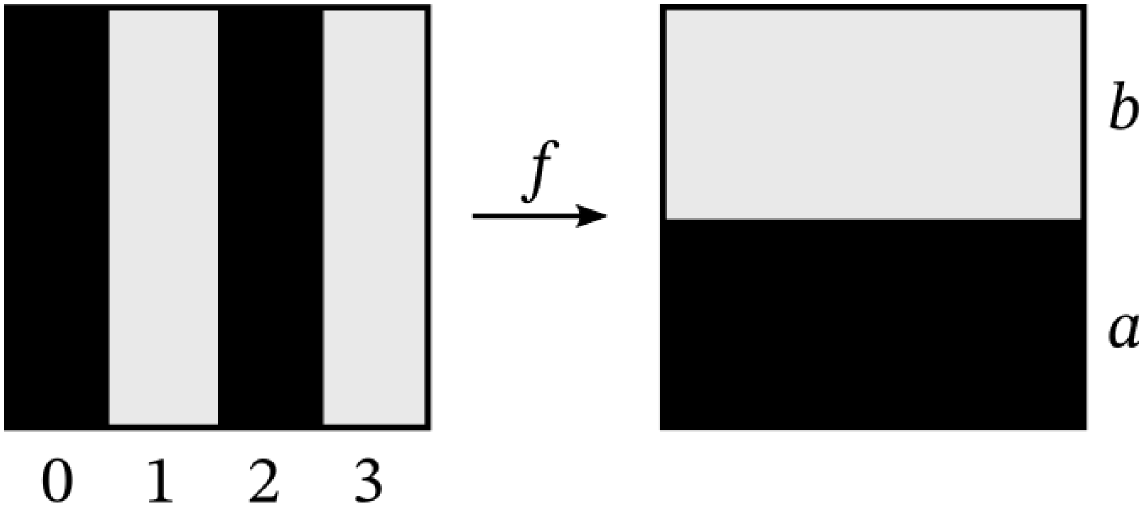
$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \varphi^{-1}(\beta)} p_\alpha.$$

The measure $\mu$ is defined by

- $\mu(C_i^s) = p_s$
- $\mu\left(C_{m\dots n}^{j_m \dots j_n}\right) = \mu(C_m^{s_m}) \dots \mu(C_n^{s_n}) = p_{s_m} \dots p_{s_n}$.

$(X, \mathcal{C}, \mu)$ is the **zip shift space**

# Baker's map n-to-1

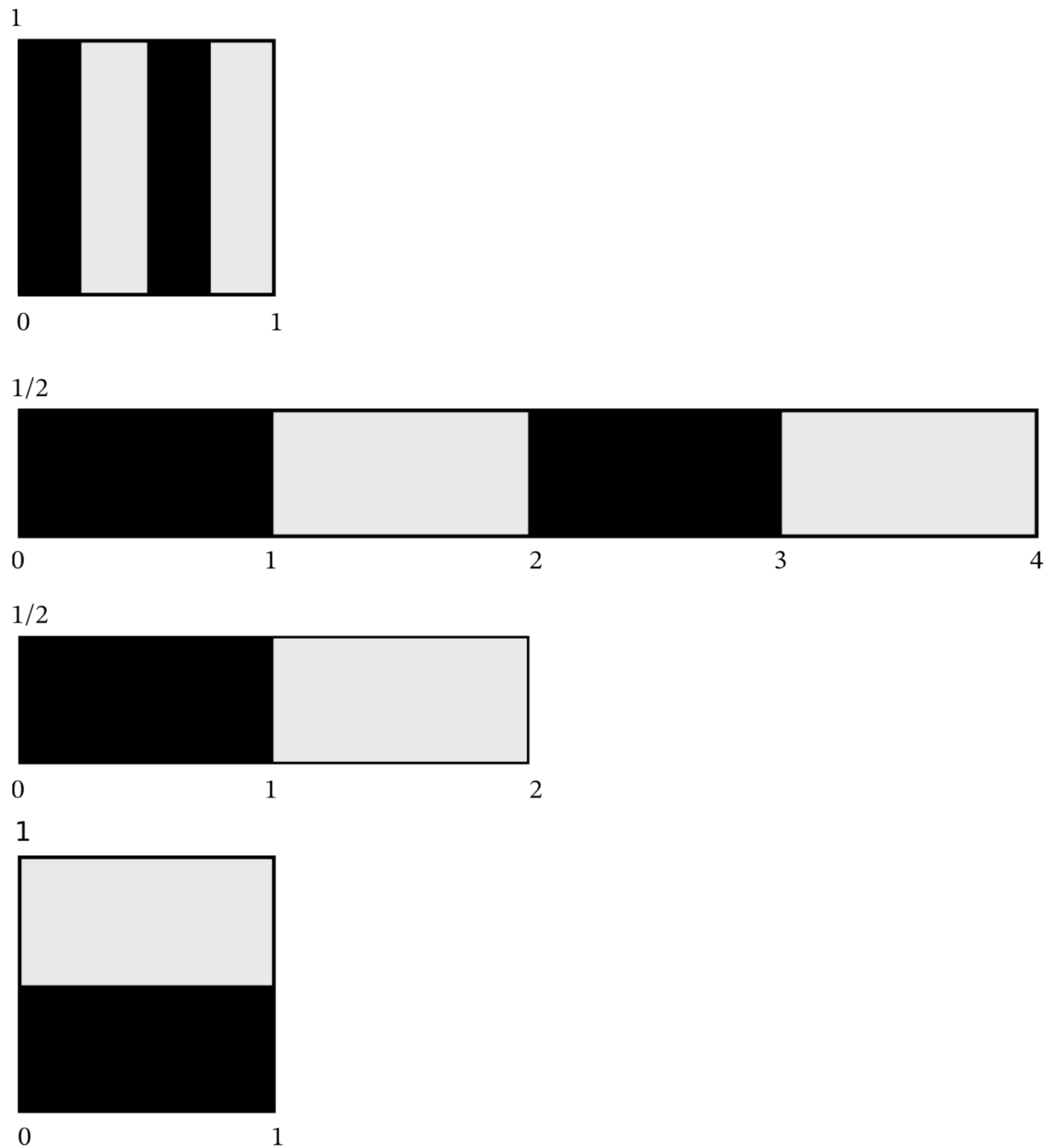



**Proposition (Mehdipour, - )**

The baker's n-to-1 is a (2,2n)-Bernoulli transformation. [2]

# Baker's map n-to-1





# Kolmogorov-Sinai Entropy

# Kolmogorov-Sinai Entropy



- Entropy of a finite partition $\mathcal{P}$:

$$H_\mu(\mathcal{P}) \stackrel{\text{def}}{=} \sum_{P\in\mathcal{P}} -\mu(P)\log\mu(P)$$

- Entropy of $f$ with respect to $\mathcal{P}$ :

$$h_\mu(f,\mathcal{P}) \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{k} H_\mu\left(\bigvee_{i=0}^{k-1} f^{-1}\mathcal{P}\right)$$

- Entropy of $f$ :

$$h_\mu(f) \stackrel{\text{def}}{=} \sup_{\mathcal{P}} h_\mu(f,\mathcal{P})$$

**Theorem (Kolmogorov, Sinai)**

Let $\mathcal{P}_1 \preceq \mathcal{P}_2 \preceq \ldots \preceq \mathcal{P}_n \preceq \ldots$ a non decreasing sequence of partitions with finite entropy such that $\cup_n \mathcal{P}_n$ generates the $\sigma$-algebra of measurable sets, up to measure zero. Then

$$h_\mu(f) = \lim_{n\to\infty} h_\mu(f, \mathcal{P}_n).$$

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# Entropy of a zip shift



We define:

- $\mathcal{C}_i$ the partition of $\Sigma$ into cylinders of index $i$ : $C_i^s$.
- $\mathcal{C}_{m\ldots n}$ the partition of $\Sigma$ into cylinders of index from $m$ to $n$.

**Theorem ( - , Mattos, Varão)**

$$h_\mu(\sigma_\varphi) = H_\mu(\mathcal{C}_0).$$

[3]

# Sketch of proof

1. $(\mathcal{P}_n)_{n\in\mathbb{N}}$ with $\mathcal{P}_n = \mathcal{C}_{-n\ldots n-1}$is a non-decreasing *generating* sequence.
2. $\bigvee_{i=0}^{k-1} \sigma_\varphi^{-i}(\mathcal{P}_n) = \mathcal{C}_{-n\ldots n+k-2}$
3. $H_\mu(\mathcal{C}_{-n\ldots n+k-2}) = n \cdot H_\mu(\mathcal{C}_{-1}) + (n+k-1)\cdot H_\mu(\mathcal{C}_0)$
4. $h_\mu(f,\mathcal{P}_n) = \lim_{k\to\infty} \frac{1}{k}\big(H_\mu(\mathcal{C}_{-n\ldots n+k-2})\big) = H_\mu(\mathcal{C}_0)$
5. $h_\mu\big(\sigma_\varphi\big) = H_\mu(\mathcal{C}_0)$ by Kolmogorov-Sinai theorem.

# Folding Entropy

# Quotient measure



Let

- $(X, \mathcal{B}, \mu)$ be a probability space
- $\mathcal{P}$ be a measurable partition of $X$
- $\pi : X \to \mathcal{P}$ the canonical projection: $x \in X \mapsto \mathcal{P}(x) \in \mathcal{P}$

We define a $\sigma$-algebra in $\mathcal{P}$ by $\hat{\mathcal{B}} := \left\{\mathcal{Q} \subset \mathcal{P} : \pi^{-1}(\mathcal{Q}) \in \mathcal{B}\right\}$ and define the quotient measure $\hat{\mu} := \pi_* \mu$ :

$$\hat{\mu}(\mathcal{Q}) = \mu\left(\pi^{-1}\mathcal{Q}\right), \ \forall \mathcal{Q} \in \hat{\mathcal{B}}.$$

# Disintegration of a measure



A **disintegration** of $\mu$ with respect to a partition $\mathcal{P}$ is a family $\{\mu_P : P \in \mathcal{P}\}$ of measures in $X$ such that

- $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$
- $P \mapsto \mu_{P(A)}$ is measurable for every $A \in \mathcal{B}$
- $\mu_P(A) = \int_{P \in \mathcal{P}} \mu_P(A) d\hat{\mu}(P)$ for every $A \in \mathcal{B}$.

# Folding entropy

Let

- $\varepsilon := \{\{x\} : x \in X\}$ the partition of $X$ into single points
- $f^{-1}(\varepsilon) = \{f^{-1}(x) : x \in X\}$ the preimage partition of $\varepsilon$ by $f$.

The **folding entropy** ([4], [5], [6]) of $f$ is defined by

$$\mathcal{F}_\mu(f) = H_\mu\big(\varepsilon \mid f^{-1}(\varepsilon)\big)$$

where $H_\mu\big(\varepsilon \mid f^{-1}(\varepsilon)\big)$ is the conditional entropy.

# Properties of folding entropy



- $\mathcal{F}_\mu(f) \leq \log N$ if $f$ is $N$-to-1
- $\mathcal{F}_\mu(f) = \log N$ if all preimages are equally $\mu$-weighted
- $\mathcal{F}_\mu(f) = 0$ if $f$ is invertible (since $N = 1$).

**Theorem ( - , Mattos, Varão)**

Consider the probability space $(\Sigma, \mathcal{C}, \mu)$ and $\sigma_\varphi$ a zip shift map. Then

$$\mathcal{F}_\mu(f) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).$$

[3]

# Sketch of proof



- $\hat{x} \stackrel{\text{def}}{=} \sigma_\varphi^{-1}(x) = \big\{(\ldots x_{-2}; s x_0 \ldots) : s \in \varphi^{-1}(x_{-1})\big\} \in \sigma_\varphi^{-1}(\varepsilon)$
- $\hat{E} \stackrel{\text{def}}{=} \{\hat{x} : x \in E\} \subset \sigma_\varphi^{-1}(\varepsilon),\ E \in \mathcal{C}.$
- The **quotient measure** $\hat{\mu}$ is given by

$$\hat{\mu}\big(\hat{E}\big) = \mu\big(\pi^{-1}\big(\hat{E}\big)\big) \underbrace{=}_{\text{lemma}} \mu\big(\sigma_\varphi^{-1}(C)\big) = \mu(C).$$

# Sketch of proof



- For every $\beta \in \mathrm{B}$, we define the following probability distribution

$$\left(q^{\beta}_{\alpha} : \alpha \in \varphi^{-1}(\beta)\right), \text{where } q^{\beta}_{\alpha} = \frac{p_\alpha}{p_\beta}.$$

- The **conditional measure** on $\hat{x} \in \sigma_\varphi^{-1}(\varepsilon)$ is given by

$$\mu_{\hat{x}}(\{\,(\ldots x_{-2}; s x_1 \ldots)\,\}) := q_s^{x_{-1}}, \ \forall s \in \varphi^{-1}(x_{-1}).$$

- The family $\{\mu_{\hat{x}}\}$ is a **disintegration** of $\mu$ with respect to $\sigma_\varphi^{-1}(\varepsilon)$.

## Sketch of proof



The folding entropy of $\sigma_\varphi$ is given by

$$
\begin{aligned}
\mathcal{F}_\mu(\sigma_\varphi) &\overset{\text{def}}{=} H_\mu(\varepsilon \mid \sigma_\varphi^{-1}(\varepsilon)) \\
&= \int_{\hat{x}\in\sigma_\varphi^{-1}(\varepsilon)} H_{\mu_{\hat{x}}}(\varepsilon\,|_{\hat{x}})d\hat{\mu}(\hat{x}) \\
&= \sum_{\beta\in B} \int_{\hat{x}\in\hat{C}^{\beta}_{-1}} H_{\mu_{\hat{x}}}(\varepsilon\,|_{\hat{x}})d\hat{\mu}(\hat{x}) \\
&= \sum_{\beta\in B} \Big[ \sum_{\alpha\in\varphi^{-1}(\beta)} -q^{\beta}_{\alpha}\log q^{\beta}_{\alpha} \Big]\, \hat{\mu}\left(\hat{C}^{\beta}_{-1}\right) \\
&= \sum_{\beta\in B} \Big[ \sum_{\alpha\in\varphi^{-1}(\beta)} -q^{\beta}_{\alpha}\log q^{\beta}_{\alpha} \Big]\, {\color{red} p_\beta} \\
&= \sum_{\beta\in B} \sum_{\alpha\in\varphi^{-1}(\beta)} -{\color{blue} p_\alpha}(\log {\color{blue} p_\alpha} - \log {\color{red} p_\beta}), \text{since } q^{\beta}_{\alpha}\cdot {\color{red} p_\beta} = {\color{blue} p_\alpha} \\
&= {\color{blue} \sum_{\alpha\in A} -p_\alpha\log p_\alpha} - {\color{red} \sum_{\beta\in B} -p_\beta\log p_\beta} \\
&= {\color{blue} H_\mu(\mathcal{C}_0)} - {\color{red} H_\mu(\mathcal{C}_{-1})}.
\end{aligned}
$$

In particular,

$$h_\mu(\sigma_\varphi) = \mathcal{F}_\mu(\sigma_\varphi) + H_\mu(\mathcal{C}_{-1})$$

# Ornstein theory

# Isomorphism theorem of Bernoulli shifts



**Theorem (Ornstein)**

Bernoulli shifts of same entropy are isomorphic.

[7]

**- , Varão, Mehdipour**

Let $f$ and $g$ be two n-to-1 $(m,l)$-Bernoulli transformations of same entropy. Then $f \cong g$.

.

# Sketch of proof



If $f : X \to X$ is n-to-1 extended Bernoulli with distribution $(p_\alpha : \alpha \in \mathrm{A})$, there is a **« domain partition »** $\mathcal{P}$ of $X$ such that

- $d(\mathcal{P}) = (p_\alpha)_{\alpha \in \mathrm{A}}$
- $\mathcal{P}$ is a generating partition for $f$
- $(f^{-n}(\mathcal{P}))_{n \in \mathbb{N}}$ and $(f^{n}(\mathcal{P}))_{n \in \mathbb{N}}$ are independent sequences

# Sketch of proof



We say that the process $(f, \mathcal{P})$ is a **copy** of $(g, \mathcal{R})$, and write $(f, \mathcal{P}) \sim (g, \mathcal{R})$, if for every $n \in \mathbb{N}$ :

$$d\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\right) = d\left(\bigvee_{i=0}^{n-1} g^{-i}\mathcal{R}\right).$$

Let $f$ and $g$ be two n-to-1 $(k, kn)$-Bernoulli transformations with $\mathcal{P}$ and $\mathcal{R}$ domain partitions which are generating for $f$ and $g$, respective. If $(f, \mathcal{P}) \sim (g, \mathcal{R})$, then $f$ and $g$ are isomorphic.

**Getting a good copy . . .**

If $f$ is a n-to-1 $(k, kn)$-Bernoulli with $\mathcal{P}$ a domain generating partition for $f$, and $\mathcal{P}'$ is a **« good »** partition with $H_\mu(f,\mathcal{P}) = H_\mu(f,\mathcal{P}')$, then given $\delta > 0$ there is a partition $\overline{\mathcal{P}}$ with

- $|\mathcal{P}' - \overline{\mathcal{P}}| < \delta$
- $(f,\mathcal{P}') \sim \left(f,\overline{\mathcal{P}}\right)$
- $\overline{\mathcal{P}}$ is generating for $f$.

## . . . to get a better copy

Let $f$ and $g$ be two n-to-1 $(k, kn)$-Bernoulli maps with $\mathcal{P}$ and $\mathcal{R}$ « good » generating partitions for $f$ and $g$, respectively. If

$$H_\mu(f, \mathcal{P}) = H_\mu(g, \mathcal{R}),$$

then there is a partition $\overline{\mathcal{P}}$

- $\left(f, \overline{\mathcal{P}}\right) \sim (g, \mathcal{R})$
- $\overline{\mathcal{P}}$ is generating for $f$.

# References


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[3] N. Martins, P. G. Martins, and R. Varão, “Folding Entropy for Extended Shifts.” 2024.

[4] D. Ruelle, “Positivity of entropy production in nonequilibrium statistical mechanics,” *J Stat Phys*, vol. 85, pp. 1–23, 1996.

[5] W. Wu and Y. Zhu, “On preimage entropy, folding entropy and stable entropy,” *Ergod. Th. & Dynam. Sys.*, vol. 41, pp. 1217–1249, 2021.

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[7] D. Ornstein, “Bernoulli shifts with the same entropy are isomorphic,” *Advances in Mathematics*, vol. 4, pp. 337–352, 1970.

(...obrigado!...)