Symbolic Dynamics

Entropy of (m,l)-Bernoulli transformations

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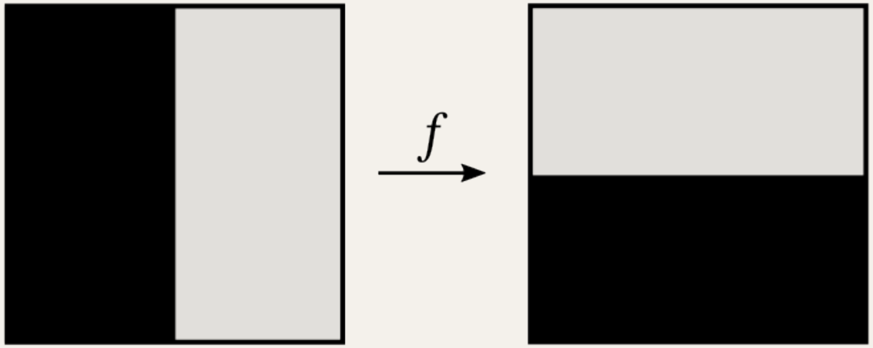
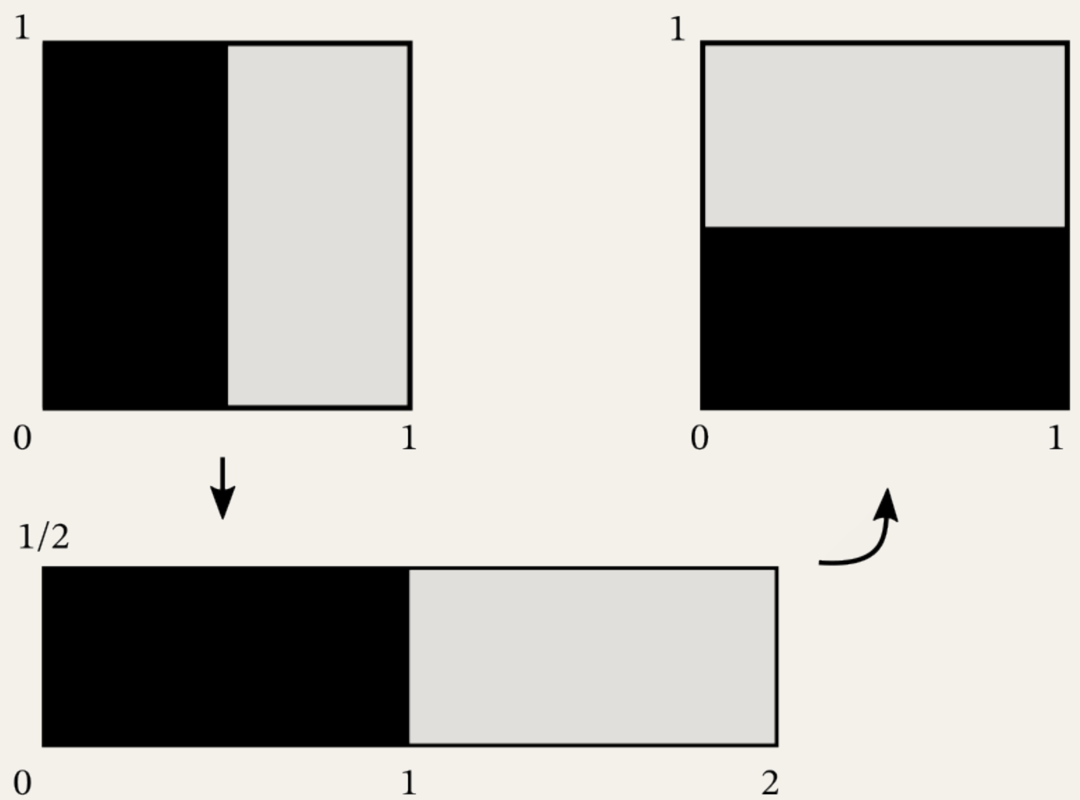
Baker map

$$X = [0,1]^2,\ f(x,y) = \begin{cases} (2x, y/2) & \text{if } x \in \left[0, \frac{1}{2}\right) \\ (2x-1, (y+1)/2) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



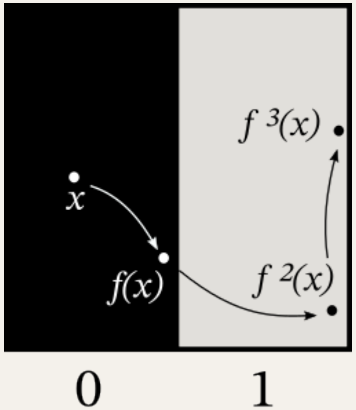
Baker map



Coding the Baker map

To each orbit $\{\ldots, f^{-1}(x), x, f(x), \ldots\}$ we relate a sequence $(x_n)_n$ of zeros and ones:

- if $f^n x \in \left[0, \frac{1}{2}\right)$, code $x_n = 0$
- if $f^n x \in \left[\frac{1}{2}, 1\right]$, code $x_n = 1$



Orbit of x : $(\ \ldots\ ;\ 0011\ldots\ )$. Orbit of $f(x)$: $(\ \ldots 0\ ;\ 011\ldots\ )$.

Symbolic Dynamics

A is a finite alphabet

$\mathrm{A}^{\mathbb{Z}} = \left\{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathrm{A}\right\}$

$$(x_n)_{n\in\mathbb{Z}} = (\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1\cdots\ )$$

$\mathrm{A}^{\mathbb{Z}}$ is a compact metric space with

$$d((x_n),(y_n)) = 2^{-\inf\{|i|\,:\,x_i\neq y_i\}}$$

$\sigma : \mathrm{A}^{\mathbb{Z}} \to \mathrm{A}^{\mathbb{Z}}$ is the (full) **shift map**

$$\sigma(x_n) = (x_{n+1}).$$

Symbolic Dynamics

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

$$(\;\cdots x_{i-1}\; s_i s_{1+1} \cdots s_k \; x_{k+1} \cdots \;) \in C_i[s_i \ldots s_k]$$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define a measure

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ldots \mu(C_k[s_k]) = p_{s_i} \ldots p_{s_k}$.

Symbolic Dynamics

We say that $f : X \to X$ and $g : Y \to Y$, defined on $(X, \mathcal{B}_1, \mu)$ and $(Y, \mathcal{B}_2, \nu)$, are isomorphic if there are measurable sets $A \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$ such that

- $\mu(A) = \nu(B) = 1$
- $f(A) \subset A, \; g(B) \subset B$
- $\exists\, \varphi : A \to B$ invertible measure preserving map such that

$$\varphi \circ f = g \circ \varphi.$$

A map $f : X \to X$, defined on $(X, \mathcal{B}, \nu)$, is a **Bernoulli shift** if is isomorphic to a shift map on $(\Sigma, \mathcal{C}, \mu)$.

Zip shifts

- A and B be two finite alphabets $\mathrm{A} \geq \mathrm{B}$
- $\varphi : \mathrm{A} \to \mathrm{B}$ a surjective map
- Σ the space of all sequence of letters

$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1}\,;\, x_0x_1\cdots)$$
$$\text{with } \; x_{-1}, x_{-2}, \ldots \in \mathrm{B} \;\text{ and }\; x_0, x_1, \ldots \in \mathrm{A}.$$

The full **zip shift** map is $\sigma_\varphi : \Sigma \to \Sigma$ with

$$\sigma_\varphi(\ \cdots x_{-1}\,;\, x_0x_1\cdots\ ) = (\cdots x_{-1}\varphi(x_0)\,;\, x_1x_2\cdots).$$

$f : X \to X$ is a **extended Bernoulli** when is isomorphic to a zip shift on $(\Sigma, \mathcal{C}, \mu)$. If $\#\mathrm{B} = m$ and $\#\mathrm{A} = l$, we say f is a **(m,l)-Bernoulli**.

The zip shift space

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \dots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \dots, x_k = s_k\}$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define $\left(p_\beta : \beta \in \mathrm{B}\right)$

$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \varphi^{-1}(\beta)} p_\alpha.$$

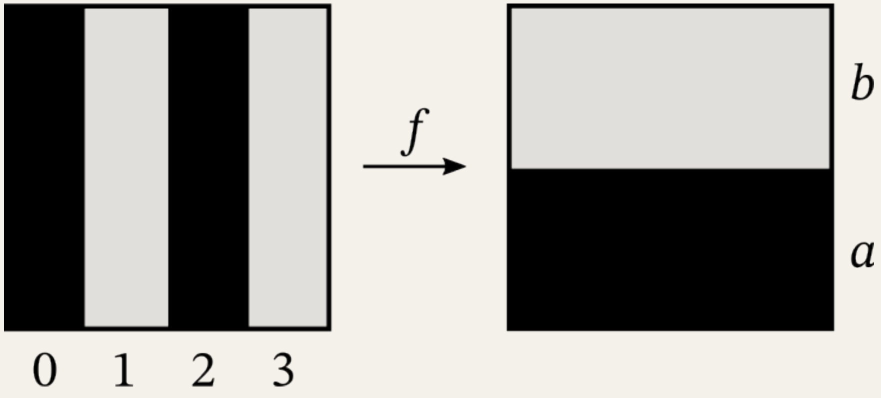
The measure μ is defined by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \dots s_k]) = \mu(C_i[s_i]) \dots \mu(C_k[s_k]) = p_{s_i} \dots p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is the zip shift space

Baker's map 2-to-1

Baker's map n-to-1



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The baker’s n-to-1 is a (2,2n)-Bernoulli transformation.

Entropy

- Von Neumann: Isomorphism problem of Bernoulli shifts
- Shannon entropy

$$H(p_1, \dots, p_k) \stackrel{\text{def}}{=} -\sum_{i=1}^{k} p_i \log p_i$$

- Kolmogorov-Sinai entropy as a invariant of measure isomorphim:

$$H_\mu(\mathcal{P}) \stackrel{\text{def}}{=} H(\mu(P_1), \dots, \mu(P_k))$$

$$h_\mu(f) \stackrel{\text{def}}{=} \sup_{\mathcal{P}} \lim_{k\to\infty} \frac{1}{k} H_\mu\left(\bigvee_{i=0}^{k-1} f^{-i}\mathcal{P}\right)$$

- Ornstein theorem: Bernoulli shifts of same entropy are isomorphic.

Folding entropy

- $\varepsilon := \{\{x\} : x \in X\}$ the partition of X into single points
- $f^{-1}(\varepsilon) = \{f^{-1}(x) : x \in X\}$ the preimage partition of ε by f.

The folding entropy of f is defined by

$$\mathcal{F}_\mu(f) \stackrel{\text{def}}{=} H_\mu(\varepsilon \mid f^{-1}(\varepsilon)) = \int_X H_{\tilde{\mu}_x}(\varepsilon) d(f\mu)$$

where $\{\tilde{\mu}_x\}$ is a canonical family of conditional measures of μ disintegrated along the preimage sets $\{f^{-1}x\}$ for μ-a.e. $x \in X$.

Entropy of a zip shift

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- $h_\mu(\sigma_\varphi) = H\Big((p_\alpha)_{\alpha\in \mathrm{A}}\Big)$
- $\mathcal{F}_\mu(\sigma_\varphi) = H\Big((p_\alpha)_{\alpha\in A}\Big) - H\Big((p_\beta)_{\alpha\in \mathrm{B}}\Big).$

Isomorphism theorem

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Let f and g be two n-to-1 (m, l)-Bernoulli transformations of same entropy. Then $f \cong g$.

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(. . . thank you . . .)