Encoding n-to-1 baker's maps

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Introduction

The baker's map 1-to-1

Let $X = [0,1] \times [0,1]$ and consider the Lebesgue space $(X, \overline{\mathcal{B}}, m)$.

The **baker's map** is the transformation $T : X \to X$ given by

$$T(x,y) = \begin{cases} (2x, \frac{1}{2}y); & 0 \leq x < \frac{1}{2},\ 0 \leq y \leq 1 \\ (2x-1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{2} \leq x \leq 1,\ 0 \leq y \leq 1. \end{cases}$$

» The baker's map preserves the Lebesgue measure.

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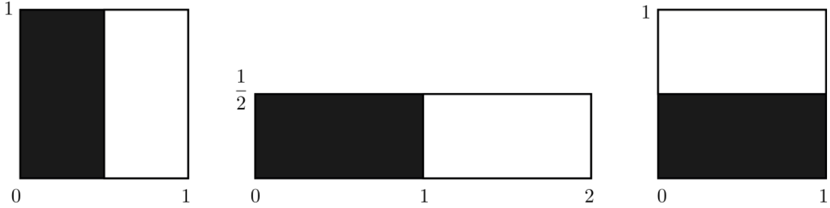
$m(T^{-1}(A)) = m(A)\ \forall A \in \overline{\mathcal{B}}$

The baker's map 1-to-1



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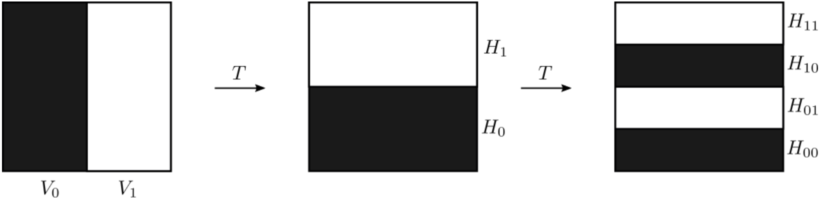
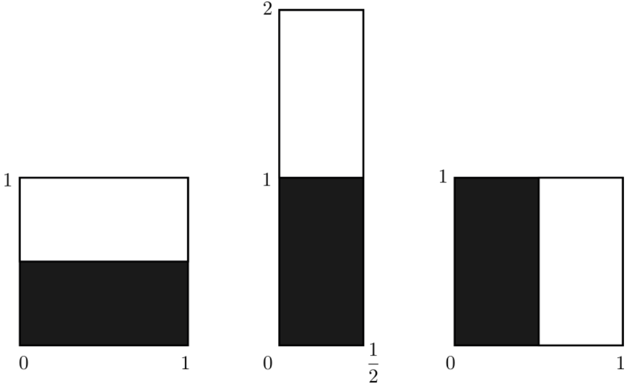


The baker's map 1-to-1

The baker's map is invertible and its inverse is given by

$$T^{-1}(x,y) = \begin{cases} (\frac{1}{2}x, 2y); & 0 \le x < 1,\ 0 \le y < \frac{1}{2} \\ (\frac{1}{2}x + \frac{1}{2}, 2y - 1) & 0 \le x \le 1,\ \frac{1}{2} \le y \le 1. \end{cases}$$

The baker's map 1-to-1



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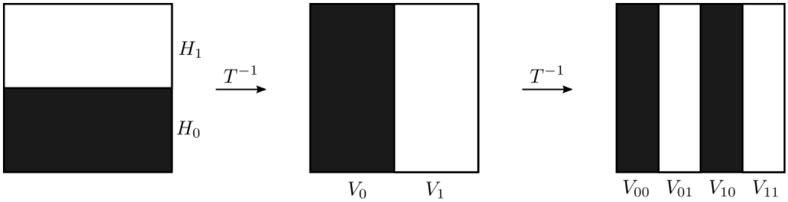


The space of symbol sequences Σ_S

Let $S = \{0, 1, \ldots l-1\}$, $l \geq 2$ be a collection of symbols. Consider

$$\Sigma_S := \prod_{i=-\infty}^{\infty} S_i;\ S_i = S\ \forall i.$$

If $(s_n) \in \Sigma_S$, we write (s_n) as

$$(s_n) = (\cdots s_{-n} \cdots s_{-1} \cdot s_0 s_1 \cdots s_n \cdots);\ s_i \in S\, \forall i.$$

The space of symbol sequences Σ_S

Basic cylinder sets

$$C_i^j = \{(s_n) \in \Sigma_S \,|\, s_i = j \in S\}.$$

Cylinder sets

$$C_i^{j_0 \cdots j_k} = \{(s_n) \in \Sigma_S \,|\, s_i = j_0, \cdots, s_{i+k} = j_k\} = C_i^{j_0} \cap \cdots \cap C_{i+k}^{j_k}.$$

Let $\mathcal{C}$ the σ-algebra generated by all cylinder sets, thus $(\Sigma_S, \mathcal{C})$ is a measurable space.

The space of symbol sequences Σ_S

We define a probability measure on $(\Sigma_S, \mathcal{C})$ considering $P = (p_s | \, s \in S)$ a probability distribution such that

$$\mu(C_i^j) = p_j.$$

The measure of the cylinder set are defined by

$$\mu(C_i^{j_0 \cdots j_k}) = \mu(C_i^{j_0} \cap \cdots \cap C_{i+k}^{j_k}) = p_{j_0} \cdots p_{j_k}.$$

μ satisfies the axioms of a measure, then $(\Sigma_S, \mathcal{C}, \mu)$ is a measure space.

The shift map

The **shift map** is the map $\sigma : \Sigma_S \to \Sigma_S$ such that $\sigma((s_n)) = (s_{n+1})$, i.e

$$\sigma(\cdots s_{-n} \cdots s_{-1} \bullet s_0 s_1 \cdots s_n \cdots) = (\cdots s_{-n} \cdots s_{-1} s_0 \bullet s_1 s_2 \cdots s_n \cdots).$$

Isomorphism mod 0

Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ two probability spaces and

$$T_1 : X_1 \to X_1, \quad T_2 : X_2 \to X_2$$

measure-preserving transformations.

We say that T_1 is **isomorphic** to T_2 if there are $M_1 \in \mathcal{B}_1, M_2 \in \mathcal{B}_2$ with $\mu_1(M_1) = \mu(M_2) = 1$ such that

- **(i)** $T_1(M_1) \subset M_1$, $T_2(M_2) \subset M_2$;
- **(ii)** There is invertible measure-preserving transformation $\varphi : M_1 \to M_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$, i.e the following diagram commutes

$$\begin{array}{ccc} \Sigma_S & \xrightarrow{\sigma} & \Sigma_S \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[T]{} & X \end{array}$$

The Bernoulli transformations

Let $(X, \overline{\mathcal{B}}, m)$ a Lebesgue space. A transformation $T : X \to X$ has the **Bernoulli** property if T is isomorphic to a shift map.

The Bernoulli transformations

- A map having the Bernoulli property is mixing

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B) \quad \forall A, B \in \mathcal{B}$$

The Bernoulli transformations

- A map having the Bernoulli property is mixing
- **The baker's map has the Bernoulli property.**

Take $\sigma : \Sigma_S \to \Sigma_S; S = \{0, 1\}$

The n-to-1 baker’s map

The n-to-1 baker's map

Let $X = [0,1] \times [0,1]$. The following transformation $T : X \to X$ represents a **n-to-1 baker's map**:

$$T(x,y) = \begin{cases} (2nx, \frac{1}{2}y); & 0 \leq x < \frac{1}{2n},\ 0 \leq y \leq 1 \\ (2nx - 1, \frac{1}{2}y + \frac{1}{2}); & \frac{1}{2n} \leq x \leq \frac{2}{2n},\ 0 \leq y \leq 1 \\ (2nx - 2, \frac{1}{2}y); & \frac{2}{2n} \leq x \leq \frac{3}{2n},\ 0 \leq y \leq 1 \\ \vdots & \vdots \\ (2nx - (2n-1), \frac{1}{2}y + \frac{1}{2}); & \frac{2n-1}{2n} \leq x \leq 1,\ 0 \leq y \leq 1. \end{cases}$$

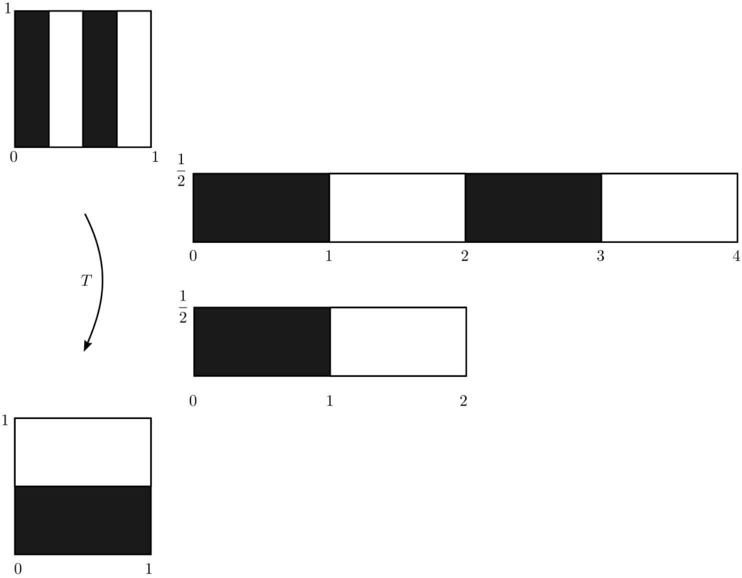
» The n-to-1 baker's map preserves the Lebesgue measure.

Example

The 2-to-1 baker's map is given by

$$T(x,y) = \begin{cases} (4x, \frac{1}{2}y); & 0 \leq x < \frac{1}{4},\ 0 \leq y \leq 1 \\ (4x-1, \frac{1}{2}y + \frac{1}{2}); & \frac{1}{4} \leq x \leq \frac{1}{2},\ 0 \leq y \leq 1 \\ (4x-2, \frac{1}{2}y); & \frac{1}{2} \leq x \leq \frac{3}{4},\ 0 \leq y \leq 1 \\ (4x-3, \frac{1}{2}y + \frac{1}{2}); & \frac{3}{4} \leq x \leq 1,\ 0 \leq y \leq 1. \end{cases}$$

Example



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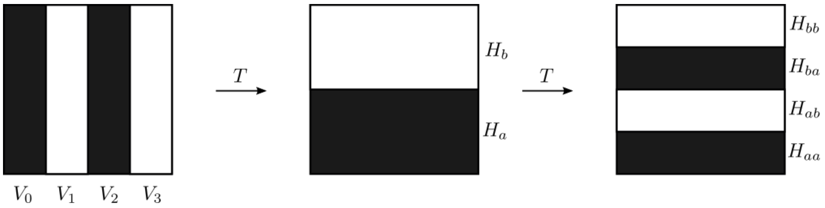

An extended two-sided shift map

A new symbol sequence space

Let $Z = \{a_1, a_2, \ldots, a_m\}$ and $S = \{0, 1, \ldots, l-1\}$ be two collections of symbols with $l \geq m$ and $\kappa : S \to Z$ be a surjective map.

Define $\Sigma = \Sigma_Z \times \Sigma_S$, where

$$\Sigma_Z = \prod_{i=-\infty}^{-1} Z_i \text{ with } Z_i = Z \text{ and } \Sigma_S = \prod_{i=0}^{\infty} S_i \text{ with } S_i = S.$$

Any $(s_n) \in \Sigma$ can be represented in the form

$$(\cdots s_{-2} s_{-1} \cdot s_0 s_1 \cdots),$$

where $(\cdots s_{-2} s_{-1}) \in \Sigma_Z$ and $(s_0 s_1 \cdots) \in \Sigma_S$.

The Zip-Shift Map

Let Σ and $\kappa : S \to Z$ be as stated previously. The map $\sigma_\kappa : \Sigma \to \Sigma$ given by

$$\sigma_\kappa(s_n) = (s_{n+1}) = (\cdots s_{-n} \cdots s_{-1}\kappa(s_0) \cdot s_1 \cdots s_n \cdots)$$

is called the **Zip-Shift map** with (m, l) symbols or simply a (m, l)-Zip-Shift map

The pair (Σ, σ_κ) is called the **Zip-Shift space** with (m, l) symbols.

Shift map is a Zip-Shift

Let $Z = S = \{0, 1, \dots l - 1\}$ with factor map $\kappa : S \to Z$ defined as $\kappa(s) = s$, then $\sigma_\kappa = \sigma : \Sigma \to \Sigma$ is a zip-shift.

Examples

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A (2,4) Zip-Shift

Let $Z = \{a, b\}$, $S = \{0, 1, 2, 3\}$ with $\kappa : S \to Z$ given by $\kappa(0) = \kappa(2) = a$ and $\kappa(1) = \kappa(3) = b$. Then, for example,

$$\sigma_\kappa(\cdots aba \bullet 103 \cdots) = (\cdots abab \bullet 03 \cdots)$$
$$\sigma_\kappa^2(\cdots aba \bullet 103 \cdots) = (\cdots ababa \bullet 3 \cdots)$$
$$\vdots$$

The topological framework

Let $\overline{d} : \Sigma \times \Sigma \to [0,1]$ given by

$$\overline{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{2^{M(x,y)}} & \text{if } x \neq y \end{cases} \qquad M(x,y) = \min\{|i|; x_i \neq y_i\}.$$

$(\Sigma, \overline{d})$ is a metric space and induces a topology on Σ.

The topological framework

Basic cylinder sets

$$C_i^j = \{(s_n) \in \Sigma | s_i = j\}; \quad j \in Z \text{ if } i < 0 \text{ and } j \in S \text{ if } i \geq 0.$$

Cylinder sets

$$C_i^{j_0 \cdots j_k} = \{(s_n) \in \Sigma | s_i = j_0, \ldots, s_{i+k} = j_k\} = C_i^{j_0} \cap C_{i+1}^{j_1} \cap \cdots \cap C_{i+k}^{j_k}.$$

The collection of all cylinder sets generates a basis for the topology induced by $(\Sigma, \overline{d})$.

Properties of the Zip-Shift

» $\sigma_\kappa(\Sigma) = \Sigma$ **and** σ_κ **is a local homeomorphism.**

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- » $\sigma_\kappa(\Sigma) = \Sigma$ and σ_κ is a local homeomorphism.
- » **Per(σ_κ)is dense in** Σ.

$$p \in Per(\sigma_\kappa) \Leftrightarrow p = (\overline{\kappa(s_0)\cdots\kappa(s_{n-1})} \cdot \overline{s_0 s_1 \cdots s_{n-1}})$$

Properties of the Zip-Shift

- $\sigma_\kappa(\Sigma) = \Sigma$ and σ_κ is a local homeomorphism.
- Per(σ_κ)is dense in Σ.
- σ_κ **is topologically transitive.**

$\forall A, B \subset \Sigma$ open and disjoint, $\exists m \in \mathbb{N}; \quad \sigma_\kappa^m(A) \cap B \neq \emptyset.$

Properties of the Zip-Shift

- $\sigma_\kappa(\Sigma) = \Sigma$ and σ_κ is a local homeomorphism.
- Per(σ_κ)is dense in Σ.
- σ_κ is topologically transitive.
- σ_κ **has a sensitive dependence of initial conditions.**

$\exists \delta > 0; \forall x \in \Sigma\, \forall N \ni x : \; \exists y \in N, n \in \mathbb{N}$ such that $\overline{d}(\sigma_\kappa^n(x), \sigma_\kappa^n(y)) > \delta.$

The measurable framework

Let $Z = \{a_1, a_2, \ldots, a_m\}$ and $S = \{0, 1, \ldots, l - 1\}$ be two collection of symbols such that $m \leq l$ and $\kappa : S \to Z$ a surjective map.

Let $\mathcal{C}$ be the σ-algebra generated by the family of all cylinder sets. Then $(\Sigma, \mathcal{C})$ is a measurable space.

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Now we define a probability distribution $P_Z = (p_z; z \in Z)$ taking $p_z = \sum_{s \in \kappa^{-1}(z)} p_s$ for every $z \in Z$ and define $\mu(C_{-i}^z) = p_z$, where $i > 0$.

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For every cylinder set $C_i^{j_0 \cdots j_k}$,

$$\mu(C_i^{j_0 \cdots j_k}) = \mu(C_i^{j_0} \cap \cdots \cap C_{i+k}^{j_k}) = p_{j_0} \cdots p_{j_k}.$$

μ satisfies the axioms of a measure, then $(\Sigma_S, \mathcal{C}, \mu)$ is a measure space and also a probability space.

Properties of the Zip-Shift

» σ_κ **preserves the probability measure** μ.

Properties of the Zip-Shift

- σ_κ preserves the probability measure μ.
- σ_κ **is mixing and ergodic.**

if $A \in \mathcal{C}; \sigma_\kappa^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A) = 1$.

The (m,l) Bernoulli transformations

A locally invertible measure preserving transformation $T : X \to X$ defined on a Lebesgue space $(X, \overline{\mathcal{B}}, \mu)$ is a **(m,l) Bernoulli transformation** if it isomorphic (mod 0) to an (m,l) Zip-Shift map.

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- The measure theoretical conjugation preserves the mixing property. Then, the (m,l) Bernoulli transformations are **mixing** and **ergodic**.

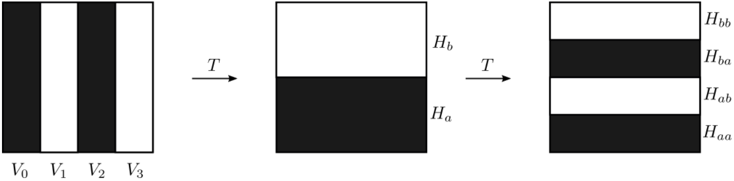
Enconding n-to-1 baker’s maps

Enconding n-to-1 baker's maps

Theorem (Mehdipour, Martins '22)

The n-to-1 baker's map is a (2,2n)-Bernoulli Transformation.

Encoding n-to-1 baker's maps



With this encoding, to each point p we associate a sequence of symbols $(s_n) \in \Sigma$ such that

$$s_i = s \in S \text{ iff } T^i(p) \in V_s; i \geq 0$$

and

$$s_{-i} = z \in Z \text{ iff } T^{-i}(x) \subset H_j;\ i > 0.$$

Sketch of proof

Let (Σ, σ_κ) the Zip Shift space such that

$$Z = \{a, b\} \text{ and } S = \{0, 1, \ldots, 2n-1\}$$

with $\kappa : S \to Z$ given by $\kappa(2k) = a$ and $\kappa(2k+1) = b$, for $k = 0, \ldots, n-1$.

We represent the symbols **a** and **b** numerically by **0** and **1**, respectively.

Sketch of proof

Consider the following map $\rho : \Sigma \to X$ given by

$$\rho((s_i)) = \left(\sum_{i=1}^{\infty} \frac{s_{i-1}}{(2n)^i}, \sum_{i=1}^{\infty} \frac{s_{-1}}{2^i} \right).$$

The orbits of the following lines have more than one representation:

$$\Omega_1 := \bigcup_i \{\frac{i}{2n}\} \times [0,1];\ i = 1, 2, \ldots, 2n-1$$

$$\Omega_2 := [0,1] \times \{\frac{1}{2}\}$$

Sketch of proof

Let $\Omega := \Omega_1 \cup \Omega_2$, $X_\Omega := \bigcup_{n=-\infty}^{\infty} T^i \Omega$ and $\Sigma_\Omega := \rho^{-1}(X_\Omega)$.

The sets X_Ω and Σ_Ω are invariant and have Lebesgue measure zero.

Let $X_0 := X \setminus X_\Omega$ and $\Sigma_0 := \Sigma \setminus \Sigma_\Omega$.

Thus $\rho : \Sigma_0 \to X_0$ is a isomorphism.

Sketch of proof

Consider $(X, \overline{\mathcal{B}}, \mu)$ the Lebesgue probability space on X. One can show that ρ is a measurable map.

In order to obtain a measurable space $(\Sigma, \mathcal{C}, \mu)$, one uses the push-forward of the Lebesgue measure on X (for ρ^{-1}):

$$\mu = (\rho_*^{-1} m).$$

This measure is bi-invariant.

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$$\mu = (\rho_*^{-1} m).$$

This measure is bi-invariant.

$\rho^{-1} = g : (X, \overline{\mathcal{B}}, m) \to (\Sigma, \mathcal{C}) \Rightarrow \mu(A) = g_* m(A) = m(g^{-1}(A)), \forall A \in \mathcal{C}$

Sketch of proof

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{\sigma_\kappa} & \Sigma_0 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ X_0 & \xrightarrow[T]{} & X_0 \end{array}$$

To show that the above diagram commutes, one rewrites the n-to-1 baker's map in the form

$$T(x, y) = (2nx \mod 1, \frac{1}{2}(y + \kappa(s_0)))$$

where $\kappa(s_0)$ is considered by numeric representation of $a, b \in Z$.

Sketch of proof

Then,

$$
\begin{aligned}
T(\rho((s_i))) &= T\left(\sum_{i=1}^{\infty} \frac{s_{i-1}}{(2n)^i}, \sum_{i=1}^{\infty} \frac{s_{-i}}{2^i}\right) \\
&= \left(2n \sum_{i=1}^{\infty} \frac{s_{i-1}}{(2n)^i}, \frac{\kappa(s_0)}{2} + \sum_{i=1}^{\infty} \frac{s_{-i}}{2^{i+1}}\right) \\
&= \left(\sum_{i=1}^{\infty} \frac{s_i}{(2n)^i}, \sum_{i=1}^{\infty} \frac{s_{-i+1}}{2^i}\right) \\
&= \rho(\sigma_\kappa((s_i))).
\end{aligned}
$$

Mixing

Corollary

Since n-to-1 baker's maps are (2,2n)-Bernoulli transformations, they are mixing and ergodic.

Chaos

Theorem (Mehdipour, Martins '22)

The n-to-1 baker's map $T : X_0 \to X_0$ is chaotic

Sketch of proof

» X_Ω **contains all the discontinuity points, thus** $T : X_0 \to X_0$ **is continuous.**

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- **The measure theoretical conjugacy preserves the density of periodic points.**

Sketch of proof

- X_Ω contains all the discontinuity points, thus $T : X_0 \to X_0$ is continuous.
- The measure theoretical conjugacy preserves the density of periodic points.
- **Mixing and continuity on a Lebesgue space implies topological mixing, which implies topological transitivity.**

$$\forall A, B \text{ open}, \exists N > 0; \quad T^n(A) \cap B \neq \emptyset \quad \forall n \geq N$$

Sketch of proof

- X_Ω contains all the discontinuity points, thus $T : X_0 \to X_0$ is continuous.
- The measure theoretical conjugacy preserves the density of periodic points.
- Mixing and continuity on a Lebesgue space implies topological mixing, which implies topological transitivity.
- **Dense periodic points and topological transitivity implies sensitive dependence on initial conditions (Banks et al. '92).**

J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey.
On devaney's definition of chaos.
The American Mathematical Monthly, 99(4):332–334, 1992.

S. Lamei and P. Mehdipour.
An n-to-1 smale horseshoe.
preprint, 2020.

P. Mehdipour and N. Martins.
Encoding n-to-1 baker's transformations.
Arch. Math., 119:199–211, 2022.

R. Sturman, J. Ottino, and S. Wiggins.
***The Mathematical Foundations of Mixing*.**
Cambridge Univ. Press, Cambridge, 2006.

Thank you!