

TALK - 29/08/2025

The Ornstein theory for n-to-1 symbolic dynamics

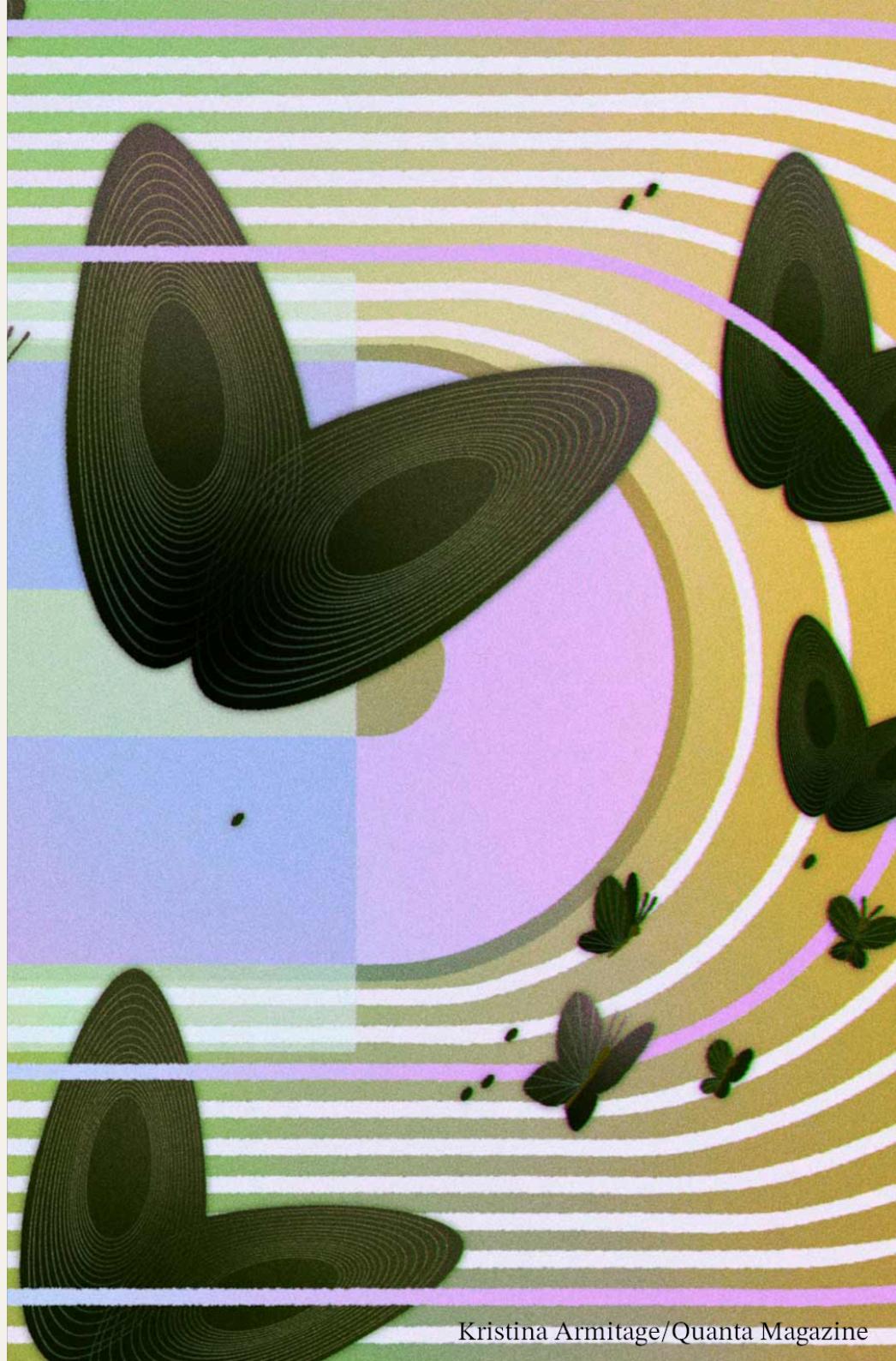
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Introduction



Kristina Armitage/Quanta Magazine

Setup

(X, \mathcal{B}, μ) a probability space or a Lebesgue space.

$T : X \rightarrow X$ a measure-preserving transformation.

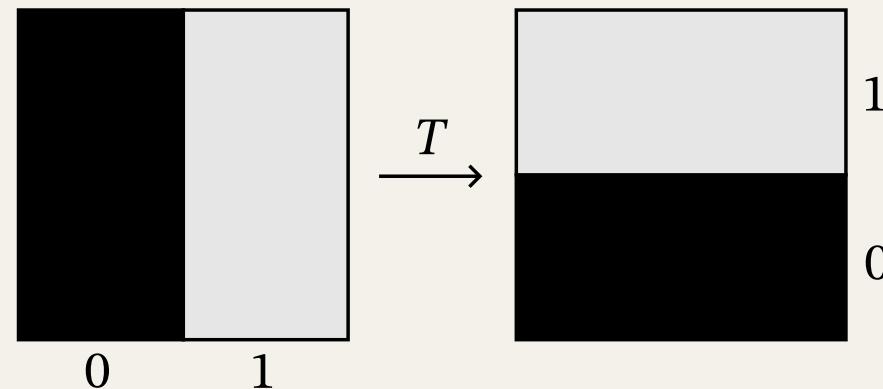
We say that $T_1 : X_1 \rightarrow X_1$ and $T_2 : X_2 \rightarrow X_2$ defined on $(X_1, \mathcal{B}_1, \mu)$ and $(X_2, \mathcal{B}_2, \nu)$, are *isomorphic* if there are $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ such that

- $\mu(A_1) = \nu(A_2) = 1$
- $T_1(A_1) \subset A_1, T_2(A_2) \subset A_2$
- $\exists \varphi : A_1 \rightarrow A_2$ invertible measure preserving map such that

$$\varphi \circ T_1 = T_2 \circ \varphi.$$

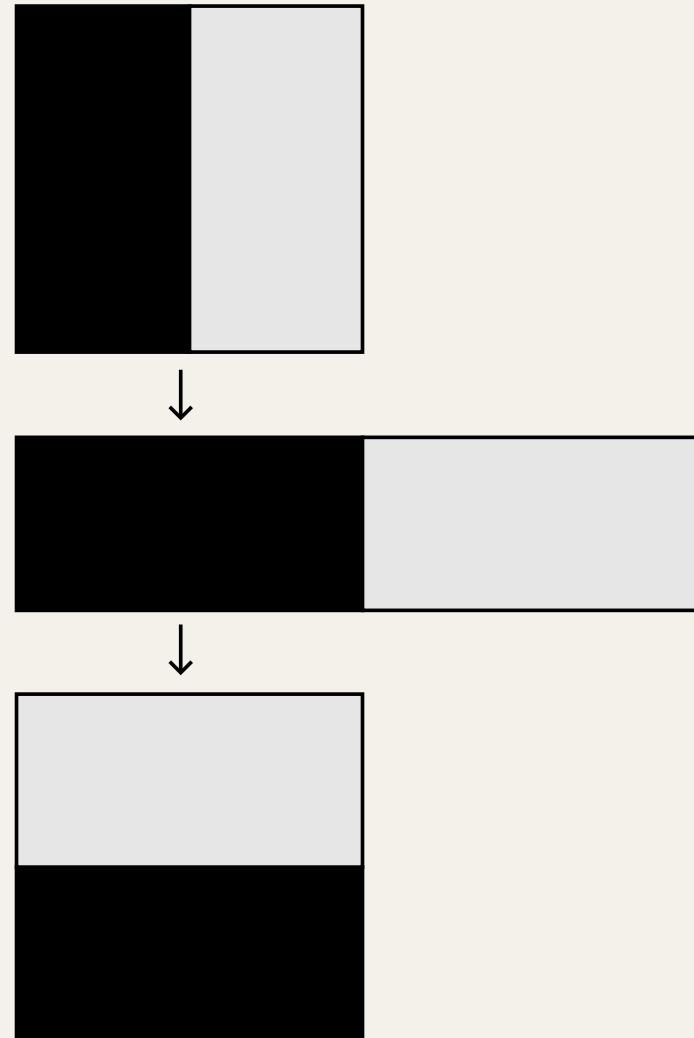
Baker's map

$$X = [0, 1]^2, \quad T(x, y) = \begin{cases} (2x, y/2) & \text{if } x \in \left[0, \frac{1}{2}\right) \\ (2x - 1, (y + 1)/2) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



Baker's map

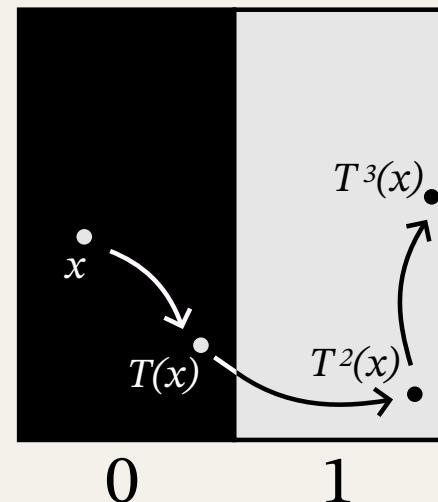
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Coding the baker's map

To each orbit $\{..., T^{-1}(x), x, T(x), ...\}$ we relate a sequence $(x_n)_n$ of zeros and ones:

- if $T^n x \in \left[0, \frac{1}{2}\right) \times [0, 1]$, code $x_n = 0$
- if $T^n x \in \left[\frac{1}{2}, 1\right] \times [0, 1]$, code $x_n = 1$



$$\mathcal{O}(x) = (\dots ; 0011\dots)$$

$$\mathcal{O}(T(x)) = (\dots 0 ; 011\dots)$$

Symbolic Dynamics

A is a finite alphabet

$$\Sigma_A = A^{\mathbb{Z}} = \{(x_n)_{n \in \mathbb{Z}} : x_n \in A\}$$

$$(x_n)_{n \in \mathbb{Z}} = (\dots x_{-2}x_{-1}; x_0x_1\dots) \in \Sigma_A$$

Σ_A is a compact metric space with

$$d((x_n), (y_n)) = 2^{-\inf \{|i| : x_i \neq y_i\}}$$

The *Bernoulli shift* $\sigma : \Sigma_A \rightarrow \Sigma_A$ is the map

$$\sigma(\dots x_{-2}x_{-1}; x_0x_1\dots) = (\dots x_{-1}x_0; x_1x_2\dots).$$

Symbolic Dynamics

Let \mathcal{C} the σ -algebra generated by the cylinder sets

- $C_i[s] = \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \dots s_k] = \{(x_n) \in \Sigma : x_i = s_i, \dots, x_k = s_k\}$
 $(\dots x_{i-1} [s_i s_{i+1} \dots s_k] x_{k+1} \dots) \in C_i[s_i \dots s_k]$

Given a probability distribution $(p_\alpha : \alpha \in A)$ in A , we define a probability measure by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \dots s_k]) = \mu(C_i[s_i]) \dots \mu(C_k[s_k]) = p_{s_i} \dots p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is a probability space.

Symbolic Dynamics

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A measurable map $T : X \rightarrow X$ is a *Bernoulli transformation* if it is isomorphic to a Bernoulli shift.

Isomorphism problem of Bernoullis shifts

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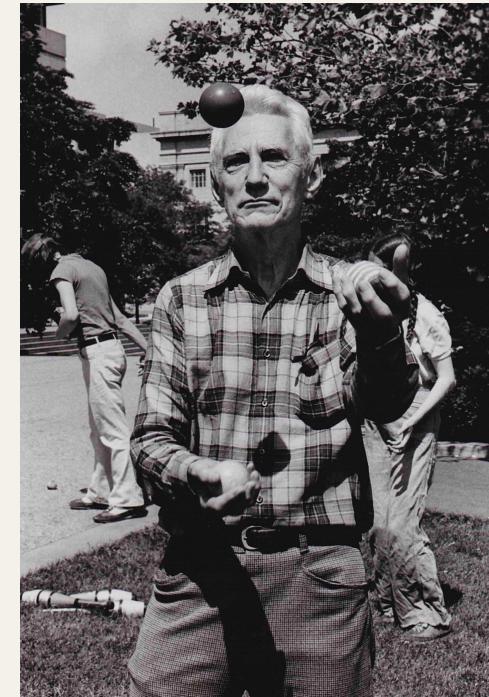
- Von Neumann: spectral isomorphism.
- Kolmogorov and Sinai entropy.
- Ornstein isomorphism theorem.

Shannon Entropy

For a probability distribution $\rho = (p_\alpha : \alpha \in A)$,

$$h(\rho) \stackrel{\text{def}}{=} \sum_{\alpha \in A} -p_\alpha \log p_\alpha.$$

Entropy is the measure of uncertainty.



Claude Shannon

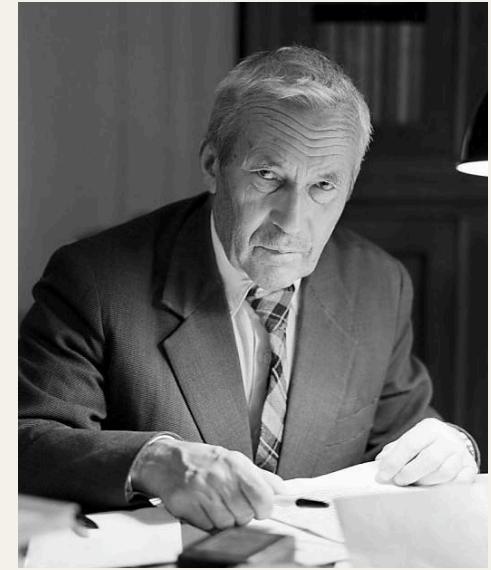
Kolmogorov-Sinai Entropy

Entropy of a partition,

$$H_\mu(\mathcal{P}) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

Entropy of a transformation with respect to a partition,

$$h_\mu(T, \mathcal{P}) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \frac{1}{k} H_\mu \left(\bigvee_{i=0}^{k-1} T^{-i} \mathcal{P} \right).$$



Andrei Kolmogorov

Entropy of a transformation,

$$h_\mu(T) \stackrel{\text{def}}{=} \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

$$T_1 \simeq T_2 \Rightarrow h_\mu(T_1) = h_\mu(T_2).$$

Kolmogorov-Sinai Entropy

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Kolmogorov-Sinai theorem

Let $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \dots$ to be a non-decreasing generating sequence of partitions with finite entropy. Then,

$$h_\mu(T) = \lim_k h_\mu(T, \mathcal{P}_k).$$



Yakov Sinai

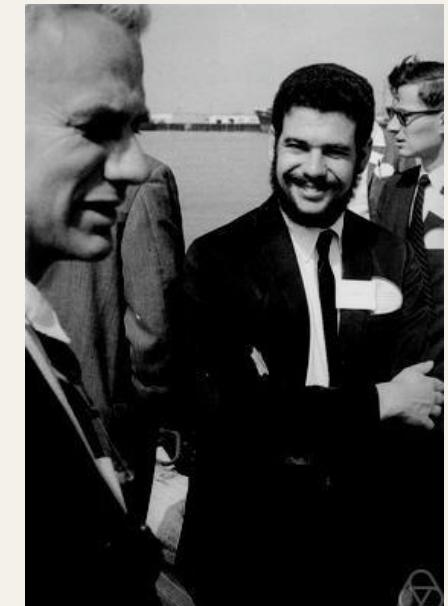
Ornstein isomorphism theorem

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Ornstein, 1970

Bernoulli shifts with the same entropy are isomorphic.

- Bôcher Memorial Prize
- Elect to American National Academy of Sciences
- Elect to American Academy of Arts and Sciences



Donald Ornstein, 1961

Non-invertible case: Extended Symbolic Dynamics

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Encoding n-to-1 baker's transformations

Folding entropy for extended shifts

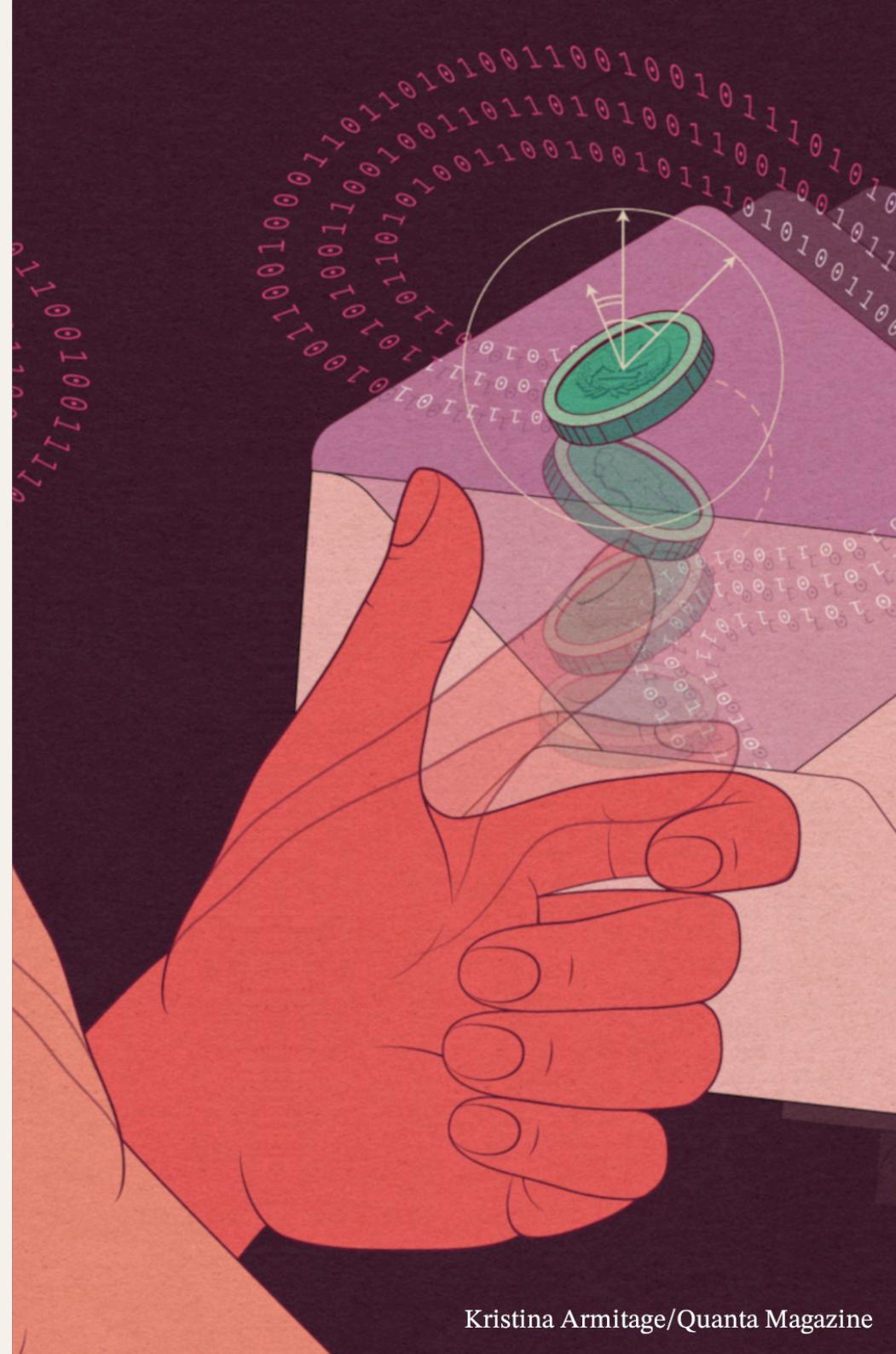
Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations

Encoding n-to-1 baker's maps

Mehdipour, P., Martins, N.

Archiv der Mathematik.

119, 199–211, (2022). 



Zip shifts

- A and B be two alphabets with $|A| \geq |B|$
- $\kappa : A \rightarrow B$ a surjective map
- Σ the space of all sequence of letters

$$(x_n)_{n \in \mathbb{Z}} = (\dots x_{-2} x_{-1}; x_0 x_1 \dots)$$

with $x_{-1}, x_{-2}, \dots \in B$ and $x_0, x_1, \dots \in A$.

The (full) *zip shift* map is $\sigma_\kappa : \Sigma \rightarrow \Sigma$ with

$$\sigma_\kappa(\dots x_{-1}; x_0 x_1 \dots) = (\dots x_{-1} \kappa(x_0); x_1 x_2 \dots).$$

Zip shift space

Let \mathcal{C} the σ -algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \dots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \dots, x_k = s_k\}$

Given a probability distribution $(p_\alpha : \alpha \in A)$ in A , we define
 $(p_\beta : \beta \in B)$

$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \kappa^{-1}(\beta)} p_\alpha.$$

The measure μ is defined by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \dots s_k]) = \mu(C_i[s_i]) \dots \mu(C_k[s_k]) = p_{s_i} \dots p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is the *zip shift space*.

A map is a *LM-Bernoulli transformation* if is isomorphic to a zip shift map. A LM-Bernoulli with $m = |A|, l = |B|$ is called a (m, l) -Bernoulli transformation.

Zip shifts

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- σ_κ is a local homeomorphism
- σ_κ preserves the measure μ
- σ_κ is mixing and ergodic
- σ_κ has density of periodic points

The n-to-1 baker's maps

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$T : [0, 1]^2 \rightarrow [0, 1]^2$ given by

$$T(x, y) = \begin{cases} \left(2nx, \frac{1}{2}y\right) & \text{if } 0 \leq x < \frac{1}{2n} \\ \left(2nx - 1, \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \frac{1}{2n} \leq x < \frac{2}{2n} \\ \left(2nx - 2, \frac{1}{2}y\right) & \text{if } \frac{2}{2n} \leq x < \frac{3}{2n} \\ \vdots & \vdots \\ \left(2nx - (2n-1), \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \frac{2n-1}{2n} \leq x \leq 1. \end{cases}$$

The n-to-1 baker's maps

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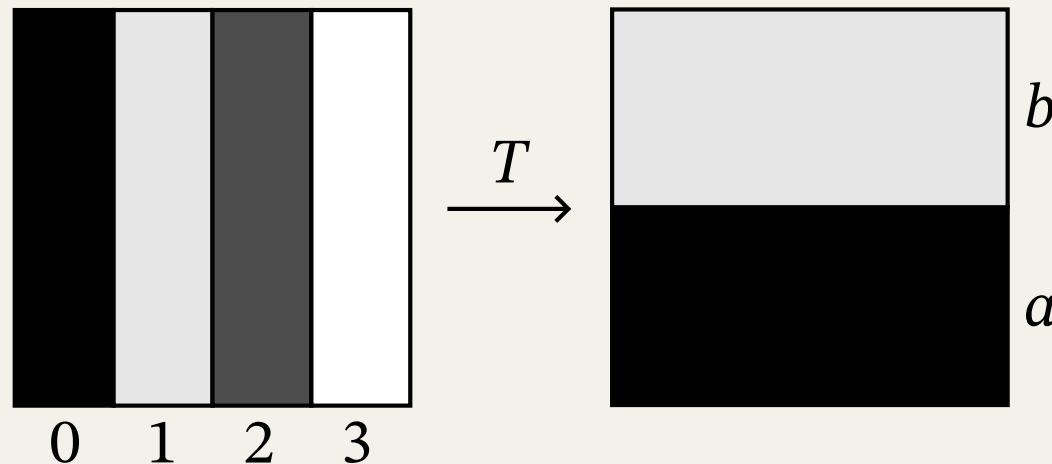


The n-to-1 baker's maps are LM-Bernoulli

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Theorem A

The n-to-1 baker's map is a $(2, 2n)$ -Bernoulli transformation.



The n-to-1 baker's maps are chaotic

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Theorem B

The n-to-1 baker's map $\bar{T} : \bar{X} \rightarrow \bar{X}$ is chaotic in the sense of Devaney.

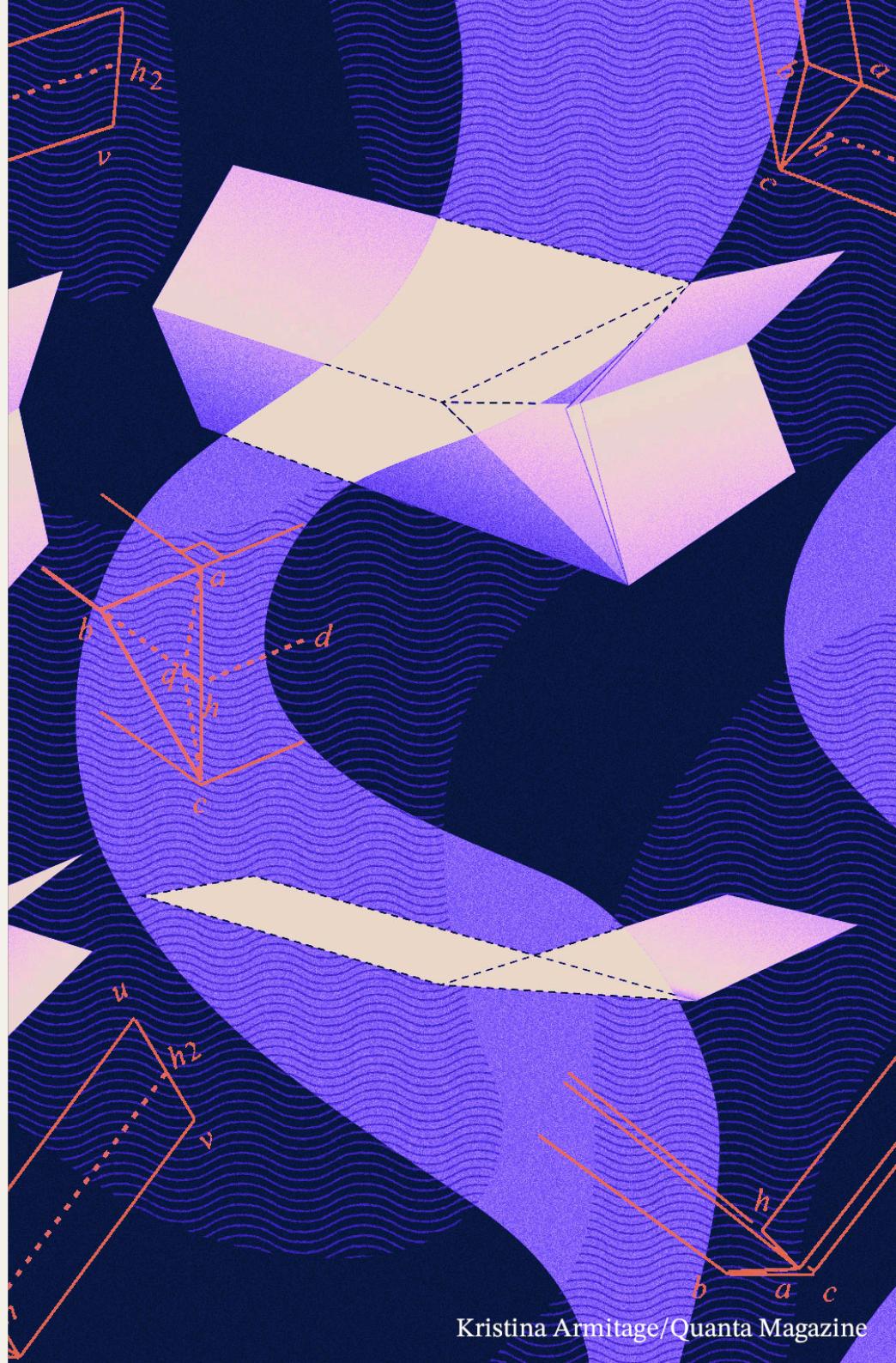
Devaney's chaos:

- Topologically transitive
- Density of periodic points
- Sensitive dependence on initial conditions.

Folding and Metric Entropies for Extended Shifts

Martins, N., Mattos, P.G., Varão, R.

arXiv:2407.01828 (2024). 



Partitions by cylinders

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$$\mathcal{C}_i = \{\square_i\} = \begin{cases} \{C_i[\alpha] : \alpha \in A\} & \text{if } i \geq 0 \\ \{C_i[\beta] : \beta \in B\} & \text{if } i < 0 \end{cases}$$

$$\mathcal{C}_{k_0 \dots k_1} = \{\square_{k_0} \dots \square_{k_1}\} = \{C_{k_0 \dots k_1}[s_{k_0} \dots s_{k_1}] : s_{k_0}, \dots, s_{k_1} \in A \cup B\} = \bigvee_{i=k_0}^{k_1} \mathcal{C}_i.$$

Partitions by cylinders

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- \mathcal{C}_i and \mathcal{C}_j are independent partitions, $\forall i, j$
- $\sigma_\kappa^i(\mathcal{C}_0) = \mathcal{C}_{-i}, \quad \forall i \geq 0$
- $\bigvee_{i=0}^{k-1} \sigma_\kappa^{-i}(\mathcal{C}_0) = \mathcal{C}_{0\dots k-1}$
- $\bigvee_{i=-k}^{k-1} \sigma_\kappa^{-i}(\mathcal{C}_0) = \mathcal{C}_{-k\dots k-1}$

Kolmogorov-Sinai entropy of zip shifts

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$$H_\mu(\mathcal{C}_0) = \sum_{\alpha \in A} -p_\alpha \log p_\alpha = h(\rho_A)$$

$$H_\mu(\mathcal{C}_{-1}) = \sum_{\beta \in B} -p_\beta \log p_\beta = h(\rho_B)$$

Kolmogorov-Sinai entropy of zip shifts

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$$H_\mu(\mathcal{C}_{-k_0 \dots k_1}) = k_0 H_\mu(\mathcal{C}_{-1}) + k_1 H_\mu(\mathcal{C}_0), \quad \forall k_0, k_1 \in \mathbb{N}$$

Kolmogorov-Sinai entropy of zip shifts

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$$\mathcal{P}_k := \mathcal{C}_{-k \dots k-1} \Rightarrow \bigvee_{i=0}^{n-1} \sigma_k^{-i} \mathcal{P}_k = \mathcal{C}_{-k \dots k+n-2}.$$

Kolmogorov-Sinai entropy of zip shifts

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$$\begin{aligned} h_\mu(\sigma_\kappa, \mathcal{P}_k) &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} \sigma_\kappa^{-i} \mathcal{P}_k \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} (n H_\mu(\mathcal{C}_{-1}) + (k+n-1)H_\mu(\mathcal{C}_0)) \\ &= H_\mu(\mathcal{C}_0) \end{aligned}$$

Theorem C

$$h_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0).$$

In fact, $\{\mathcal{P}_k\}$ is a generating sequence and is non-decreasing. Then, by the Kolmogorov-Sinai theorem,

$$h_\mu(\sigma_\kappa) = \lim_{k \rightarrow \infty} h_\mu(\sigma_\kappa, \mathcal{P}_k) = H_\mu(\mathcal{C}_0).$$

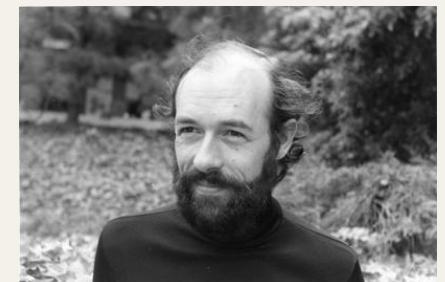
Folding entropy

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$$\mathcal{F}_\mu(T) := H_\mu(\mathcal{E}/T^{-1}\mathcal{E}).$$

$$H_\mu(\mathcal{P}/\mathcal{R}) := \int_{R \in \mathcal{R}} H_{\mu_R}(\mathcal{P}/R) \mu_R(dR),$$

where $\{\mu_R\}_{R \in \mathcal{R}}$ is a disintegration of μ with respect to \mathcal{R} .



David Ruelle

Theorem D

$$\mathcal{F}_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).$$

Disintegration of the measure

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- $\hat{x}(\alpha) := (\cdots x_{-2} ; \textcolor{blue}{\alpha} x_0 \cdots), \quad \forall \alpha \in \kappa^{-1}(x_{-1})$
- $\hat{x} := \sigma_\kappa^{-1}(x) = \{\hat{x}(\alpha) : \alpha \in \kappa^{-1}(x_{-1})\}$
- $\hat{X} := \{\hat{x} : x \in X\} \subset \sigma_\kappa^{-1}(\mathcal{E}), X \subset \Sigma.$
- The *quotient measure* $\hat{\mu}$ is given by
$$\hat{\mu}(\hat{X}) = \mu(\pi^{-1}(\hat{X})) = \mu(\sigma_\kappa^{-1}(X)) = \mu(X).$$

Disintegration of the measure

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- For every $\beta \in B$, we define the following probability distribution

$$\left(q_\alpha^\beta : \alpha \in \kappa^{-1}(\beta) \right), \text{ where } q_\alpha^\beta := \frac{p_\alpha}{p_\beta}.$$

- The *conditional measure* on $\hat{x} \in \sigma_\kappa^{-1}(\varepsilon)$ is given by

$$\mu_{\hat{x}}(\{\hat{x}(\alpha)\}) := q_s^{x_{-1}}, \quad \forall \alpha \in \kappa^{-1}(x_{-1}).$$

The family $\{\mu_{\hat{x}}\}_{\hat{x} \in \sigma_\kappa^{-1}(\varepsilon)}$ is a *disintegration* of μ with respect to $\sigma_\kappa^{-1}(\mathcal{E})$.

Folding entropy of zip shifts

The folding entropy of σ_κ is given by

$$\mathcal{F}_\mu(\sigma_\kappa) \stackrel{\text{def}}{=} H_\mu(\mathcal{E}/\sigma_\kappa^{-1}(\mathcal{E})) = \int_{\hat{x} \in \sigma_\kappa^{-1}(\mathcal{E})} H_{\mu_{\hat{x}}}(\mathcal{E}/\hat{x}) d\hat{\mu}(\hat{x})$$

Folding entropy of zip shifts

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$$\begin{aligned}\mathcal{F}_\mu(\sigma_\kappa) &= \sum_{\beta \in B} \int_{\hat{x} \in \hat{C}_{-1}[\beta]} H_{\mu_{\hat{x}}}(\mathcal{E}/\hat{x}) d\hat{\mu}(\hat{x}) \\ &= \sum_{\beta \in B} \sum_{\alpha \in \kappa^{-1}(\beta)} (-q_\alpha^\beta \log q_\alpha^\beta) \hat{\mu}(\hat{C}_{-1}[\beta]) \\ &= \sum_{\beta \in B} \sum_{\alpha \in \kappa^{-1}(\beta)} (-q_\alpha^\beta \log q_\alpha^\beta) p_\beta \\ &= \sum_{\beta \in B} \sum_{\alpha \in \kappa^{-1}(\beta)} -p_\alpha (\log p_\alpha - \log p_\beta), \text{ since } q_\alpha^\beta \cdot p_\beta = p_\alpha \\ &= \sum_{\alpha \in A} -p_\alpha \log p_\alpha - \sum_{\beta \in B} -p_\beta \log p_\beta = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).\end{aligned}$$

Folding entropy of zip shifts

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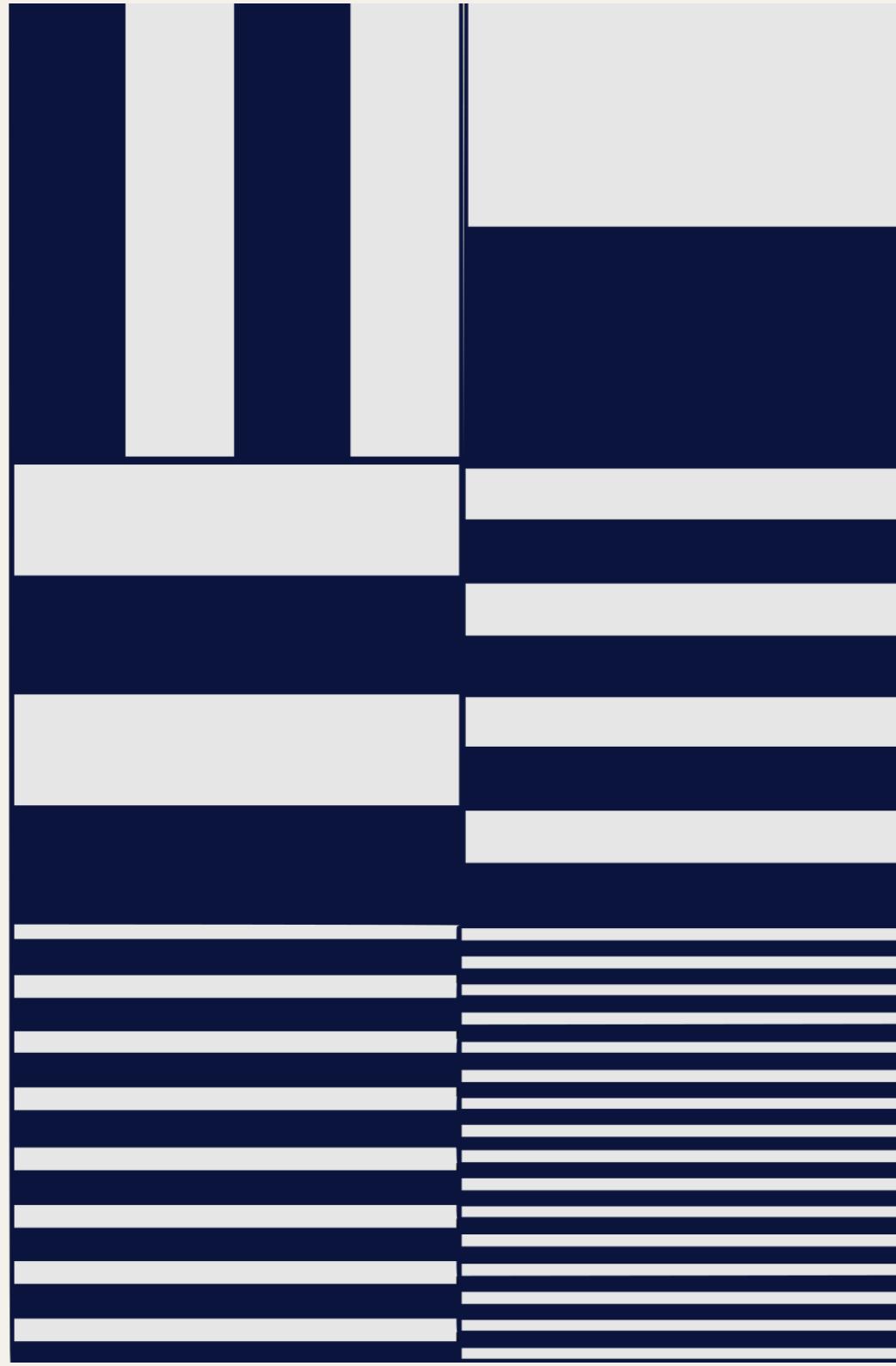
In particular,

$$h_\mu(\sigma_\kappa) = \mathcal{F}_\mu(\sigma_\kappa) + h(\rho_B).$$

Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations

Martins, N., Mehdipour, P., Varão, R.

Preprint (2025). 



Theorem E

Two n-to-1 LM-Bernoulli transformations of same entropy are isomorphic.

Uniform zip shifts

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- σ_κ n-to-1 $\Rightarrow \sigma_\kappa$ is a (k, kn) -zip shift
- $\sigma_{\kappa_1}, \sigma_{\kappa_2}$ uniform n-to-1 (k, kn) -zip shifts $\Rightarrow \sigma_{\kappa_1} \simeq \sigma_{\kappa_2}$
- Let $\sigma_{\kappa_1}, \sigma_{\kappa_2}$ uniform n-to-1 zip shifts. Then

$$\sigma_{\kappa_1} \simeq \sigma_{\kappa_2} \Leftrightarrow h_\mu(\sigma_{\kappa_1}) = h_\mu(\sigma_{\kappa_2})$$

Ornstein characterization of Bernoulli shifts

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Ornstein, 1974

An automorphism $T : X \rightarrow X$ is isomorphic to a Bernoulli shift $\sigma : \Sigma_A \rightarrow \Sigma_A$ with distribution $\rho_A = (p_\alpha : \alpha \in A)$ if, and only if, there is a partition \mathcal{P} such that

- a) $\text{dist}(\mathcal{P}) = \rho_A$
- b) \mathcal{P} is a generating for T
- c) $\{T^k \mathcal{P}\}_{k \in \mathbb{N}}$ is a independent sequence.

Ornstein characterization of Bernoulli shifts

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Ornstein, 1974

Two Bernoulli transformations are isomorphic if, and only if, there are partitions \mathcal{P} and \mathcal{R} such that

$$\text{dist}\left(\bigvee_{i=0}^k T_1^{-i} \mathcal{P}\right) = \text{dist}\left(\bigvee_{i=0}^k T_2^{-i} \mathcal{R}\right), \quad \forall k \in \mathbb{N}.$$

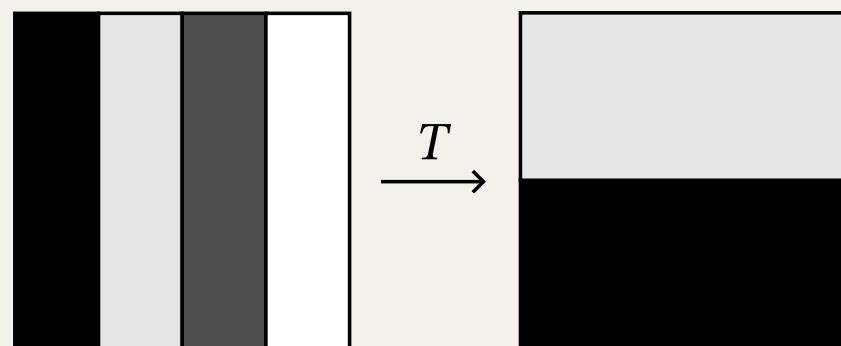
Domain and image partitions

- A image partition $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ of a n-to-1 local isomorphism a partition such that for all $P_i \in T^{-1}Q_j$, the map

$$T|_{P_i} : P_i \rightarrow X$$

is an automorphism.

- The collection \mathcal{P} of all P_i is a domain partition



Characterization of n-to-1 LM Bernoulli

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An n-to-1 local isomorphism $T : X \rightarrow X$ is a LM-Bernoulli transformation with distribution $\rho_A = (p_\alpha : \alpha \in A)$ if, and only if, there is a domain partition \mathcal{P} such that

- a) $\text{dist}(\mathcal{P}) = \rho_A$
- b) \mathcal{P} is a generating for T
- c) The sequences $\{T^k \mathcal{P}\}_{k \in \mathbb{N}}$ and $\{T^{-k} \mathcal{P}\}_{k \in \mathbb{N}}$ are independent.

The copying condition

Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and \mathcal{P} and \mathcal{R} be partitions of X_1 and X_2 , respectively.

The *process* (T_1, \mathcal{P}) is a *copy* of the process (T_2, \mathcal{R}) , and we denote by

$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R})$$

when, for all $k \geq 0$,

$$\text{dist}\left(\bigvee_{-k}^k T_1^{-i} \mathcal{P}\right) = \text{dist}\left(\bigvee_{-k}^k T_2^{-i} \mathcal{R}\right).$$

The copying condition

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Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and \mathcal{P} and \mathcal{R} to be generating partitions, respectively. Then,

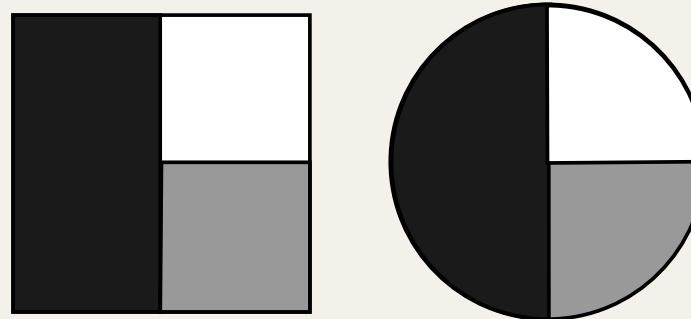
$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R}) \Leftrightarrow T_1 \simeq T_2.$$

Metrics on partitions and processes

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- Distance between the distributions of two partitions of same cardinality:

$$|\text{dist}(\mathcal{P}) - \text{dist}(\mathcal{R})| = \sum_{i=1}^k |\mu(P_i) - \mu(R_i)|.$$

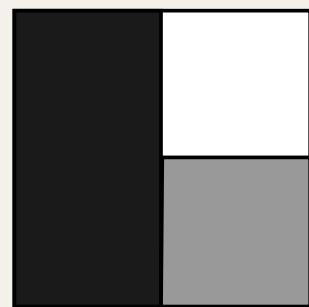


$$\text{dist}(\mathcal{P}) = \text{dist}(\mathcal{R}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right)$$

Metrics on partitions and processes

- Distance between partitions of the same space and same cardinality:

$$|\mathcal{P} - \mathcal{R}| = \sum_{i=1}^k \mu(P_i \Delta R_i)$$



$$\text{dist}(\mathcal{P}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right)$$

$$\text{dist}(\mathcal{R}) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8} \right)$$

$$|\mathcal{P} - \mathcal{R}| = \frac{1}{2}.$$

- $\mathcal{P} \mapsto h_\mu(T, \mathcal{P})$ is a continuous function in the partition metric.
- The space of all partitions is connected in the partition metric.

Metrics on partitions and processes

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- Distance between sequences of partitions

$$d\left(\{\mathcal{P}_i\}_1^k, \{\mathcal{R}_i\}_1^k\right) = \inf \frac{1}{k} \sum_{i=1}^k |\overline{\mathcal{P}}_i - \overline{\mathcal{R}}_i|,$$

where the infimum is taken over all sequences of partitions $\{\overline{\mathcal{P}}_i\}_1^k, \{\overline{\mathcal{R}}_i\}_1^k$ of a same Lebesgue space such that

$$\text{dist}\left(\bigvee_{i=0}^k \overline{\mathcal{P}}_i\right) = \text{dist}\left(\bigvee_{i=0}^k \mathcal{P}_i\right), \quad \text{dist}\left(\bigvee_{i=0}^k \overline{\mathcal{R}}_i\right) = \text{dist}\left(\bigvee_{i=0}^k \mathcal{R}_i\right)$$

Metrics on partitions and processes

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- Distance between processes

$$d((T_1, \mathcal{P}), (T_2, \mathcal{R})) = \sup_k d\left(\left\{T_1^{-i} \mathcal{P}\right\}_1^k, \left\{T_2^{-i} \mathcal{R}\right\}_1^k\right)$$

Finitely determined processes

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An n-to-1 LM-Bernoulli process (T_1, \mathcal{P}) is *finitely determined* if for every $\varepsilon > 0$, there are $\delta > 0$ and $k \in \mathbb{N}$ such that if an n-to-1 LM-Bernoulli process (T_2, \mathcal{R}) satisfies the conditions

- a) $|\mathcal{P}| = |\mathcal{R}|$
- b) $|h_\mu(T_1, \mathcal{P}) - h_\mu(T_2, \mathcal{R})| < \delta$
- c) $\left| \text{dist}\left(\bigvee_{i=0}^k T_1^{-1} \mathcal{P}\right) - \text{dist}\left(\bigvee_{i=0}^k T_2^{-1} \mathcal{R}\right) \right| < \delta,$

then

$$d((T_1, \mathcal{P}), (T_2, \mathcal{R})) < \varepsilon.$$

Finitely determined processes

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If (T, \mathcal{P}) is an n-to-1 LM-Bernoulli process and $\{T^{-i}\mathcal{P}\}_{i \in \mathbb{N}}$ is an independent sequence, then (T, \mathcal{P}) is finitely determined.

Rokhlin lemma for LM-Bernoulli

Let $T : X \rightarrow X$ be a LM-Bernoulli transformation, $k \geq 0$ and $\varepsilon > 0$.

There is a disjoint measurable sequence

$$F, TF, \dots, T^{k-1}F,$$

such that $\mu\left(\bigcup_{i=0}^{k-1} T^i F\right) > 1 - \varepsilon$.

Stacks and Gadgets

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The sequence $\{T^i F\}_0^{k-1}$ is a *stack* of base F and length k .



Strong Rokhlin lemma for LM-Bernoulli

Let (T, \mathcal{P}) to be a LM-Bernoulli process, $k \geq 0$ and $\varepsilon > 0$. There is a stack

$$F, TF, \dots, T^{k-1}F,$$

such that $\mu\left(\bigcup_{i=0}^{k-1} T^i F\right) > 1 - \varepsilon$ and $\text{dist}(\mathcal{P}/F) = \text{dist}(\mathcal{P})$.

Stacks and Gadgets

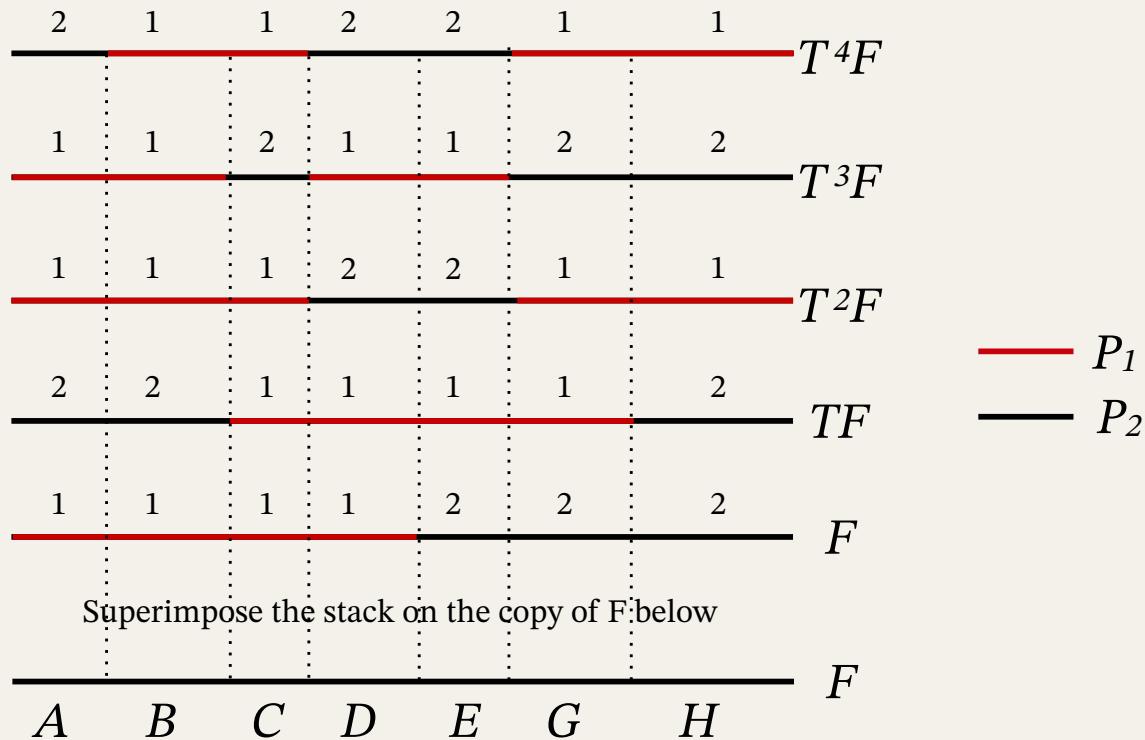
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The induced distribution of the base of the stack is the same as \mathcal{P} .

Stacks and Gadgets

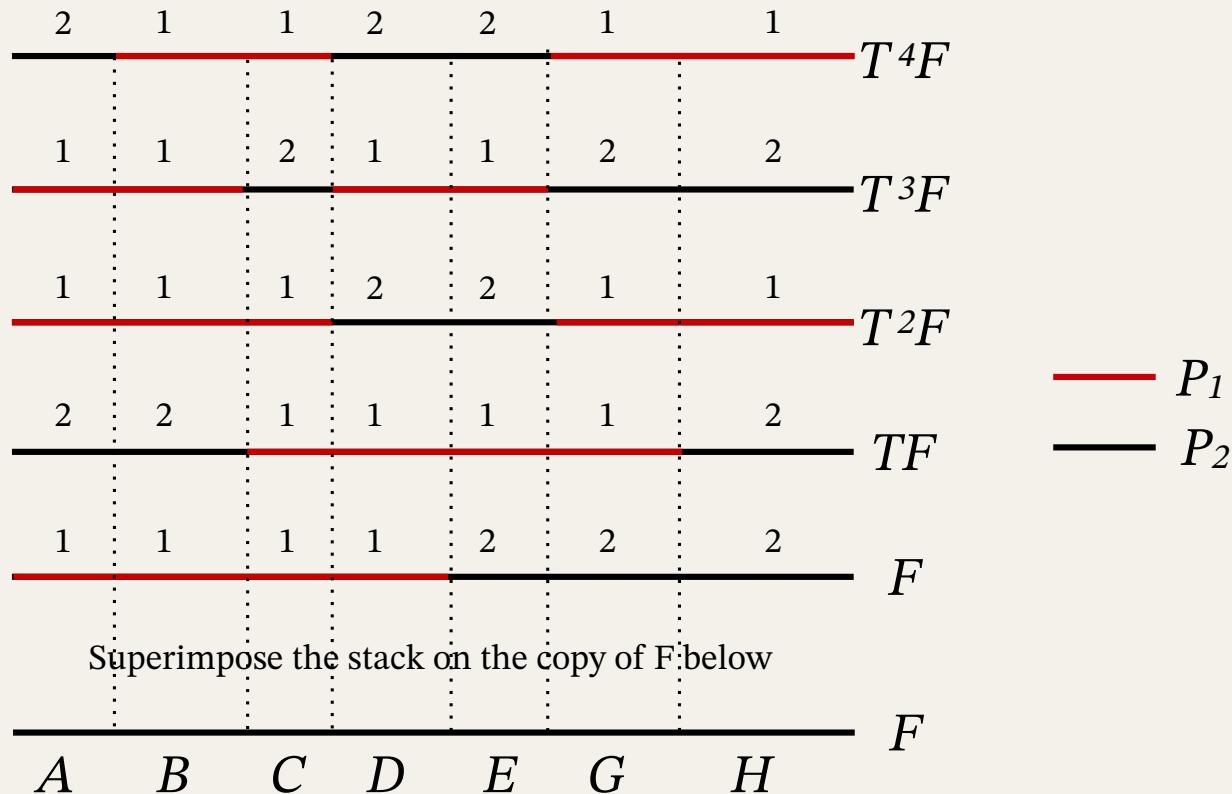
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$$\bigvee_{i=0}^{5-1} T^{-i}(\mathcal{P}/T^i F)/F = \bigvee_{i=0}^{5-1} T^{-i}\mathcal{P}/F.$$

Stacks and Gadgets

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$$A = F \cap P_1 \cap T^{-1}(P_2) \cap T^{-2}(P_1) \cap T^{-3}(P_1) \cap T^{-4}(P_2).$$

\mathcal{P} -5-name of A : $(1, 2, 1, 1, 2)$

Getting a copy

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Let (T_1, \mathcal{P}) and (T_2, \mathcal{R}) to be two n-to-1 LM-Bernoulli processes with (T_2, \mathcal{R}) f.d. Given $\varepsilon > 0$, there are $\delta > 0$ and $k \in \mathbb{N}$ such that if:

- a) $h_\mu(T_1) \geq h_\mu(T_2, \mathcal{R})$
- b) $|\mathcal{P}| = |\mathcal{R}|$
- c) $\left| \text{dist}\left(\bigvee_{i=0}^{k-1} T_1^{-i} \mathcal{P}\right) - \text{dist}\left(\bigvee_{i=0}^{k-1} T_2^{-i} \mathcal{R}\right) \right| < \delta$
- d) $|h_\mu(T_1, \mathcal{P}) - h_\mu(T_2, \mathcal{R})| < \delta$

then there is $\overline{\mathcal{P}}$ such that $|\overline{\mathcal{P}} - \mathcal{P}| < \varepsilon$ and $(T_1, \overline{\mathcal{P}}) \sim (T_2, \mathcal{R})$.

Getting a better copy

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\mathcal{P} is a generating partition for T iff for each \mathcal{R} and $\varepsilon > 0$, there is k such that

$$\mathcal{R} \underset{\varepsilon}{\prec} \bigvee_{-k}^k T^i \mathcal{P}.$$

Getting a better copy

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Let \mathcal{P} be a generating partition for a n-to-1 LM-Bernoulli T , and suppose

$$h_\mu(T, \mathcal{P}) = h_\mu(T, \mathcal{R})$$

with $(T, \mathcal{P}), (T, \mathcal{R})$ both f.d. Given $\varepsilon > 0$, there is $\overline{\mathcal{R}}$ such that

- a) $(T, \overline{\mathcal{R}}) \sim (T, \mathcal{R})$
- b) $|\overline{\mathcal{R}} - \mathcal{R}| < \varepsilon$
- c) $\mathcal{P} \underset{\varepsilon}{\prec} \bigvee_{-k}^k T^i \overline{\mathcal{R}}, k \in \mathbb{N}$

Getting a better copy

Let \mathcal{P} be a generating partition for a n-to-1 LM-Bernoulli T , and suppose

$$h_\mu(T, \mathcal{P}) = h_\mu(T, \mathcal{R})$$

with $(T, \mathcal{P}), (T, \mathcal{R})$ both f.d. Given $\varepsilon > 0$, there is $\overline{\mathcal{R}}$ such that

- a) $(T, \overline{\mathcal{R}}) \sim (T, \mathcal{R})$
- b) $|\overline{\mathcal{R}} - \mathcal{R}| < \varepsilon$
- c) $\overline{\mathcal{R}}$ is generating for T .

Getting a better copy

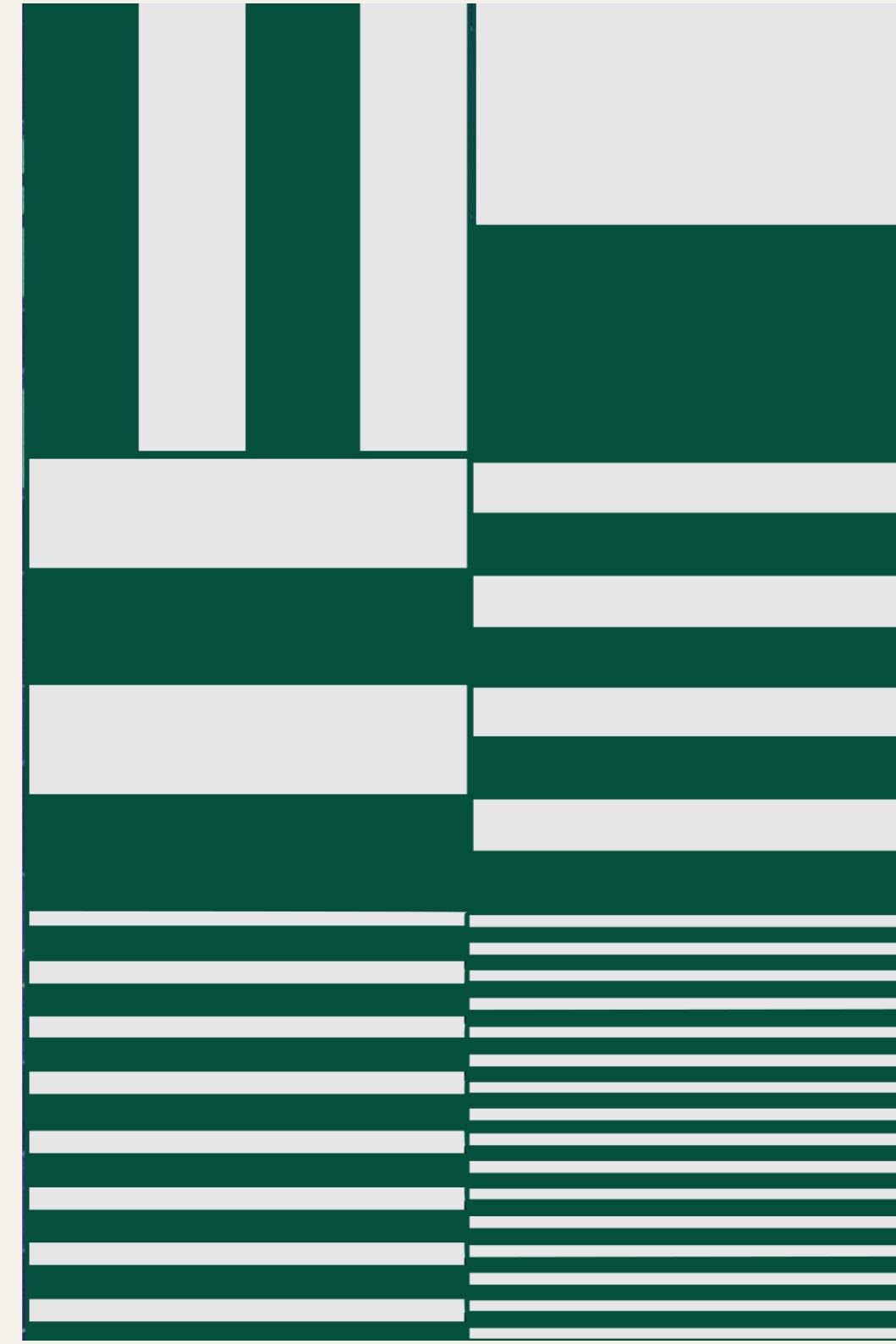
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Let (T_1, \mathcal{P}) and (T_2, \mathcal{R}) be two n-to-1 LM-Bernoulli processes f.d, \mathcal{P} and \mathcal{R} generating partitions, such that $h_\mu(T_1) = h_\mu(T_2)$. Then T_1 and T_2 are isomorphic.

- (T_2, \mathcal{R}) f.d \rightarrow choose \mathcal{P}' near to \mathcal{P} such that $(T_1, \mathcal{P}') \sim (T_2, \mathcal{R})$.
- $h_\mu(T, \mathcal{P}) = h_\mu(T, \mathcal{R})$, \mathcal{P} generating \rightarrow choose a generating $\overline{\mathcal{P}}$ near to \mathcal{P}' such that

$$(T_1, \overline{\mathcal{P}}) \sim (T_2, \mathcal{R}).$$

Conclusion



Thm A The n-to-1 baker's map are $(2, 2n)$ -Bernoulli.

Thm B The n-to-1 baker's map \bar{T} is chaotic.

Thm C $h_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0)$.

Thm D $\mathcal{F}_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1})$.

Thm E Two n-to-1 LM-Bernoulli maps of same entropy are isomorphic.

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Thank you!

