1) Determine o campo gradiente de f e esboce-o:

$$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

Notação:

$$\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j} = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}$$

$$= \langle f_x(x,y), f_y(x,y) \rangle$$

$$= (f_x(x,y), f_y(x,y))$$

Sol:

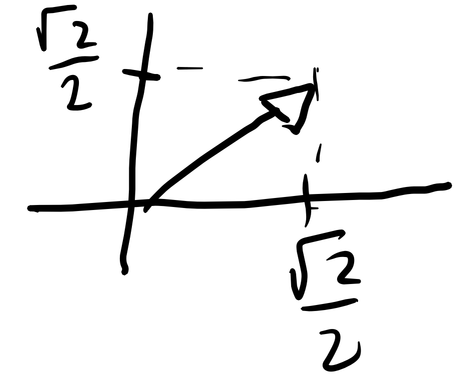
$$\nabla f(x,y) = \left\langle \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \; , \; \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y \right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}} , \frac{y}{\sqrt{x^2+y^2}} \right\rangle.$$

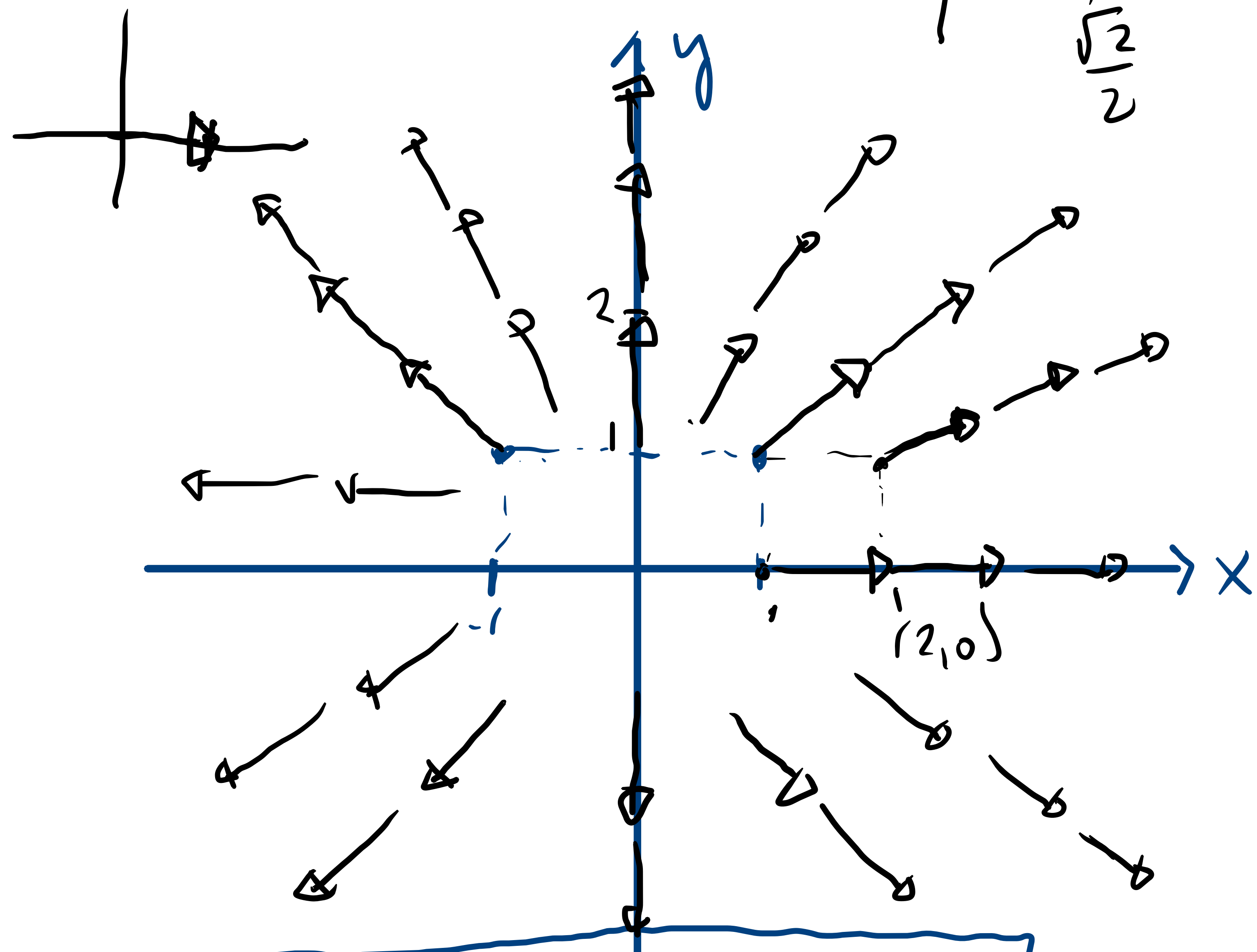
$$= \frac{x}{\sqrt{x^2+y^2}}\,\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\,\vec{j}.$$

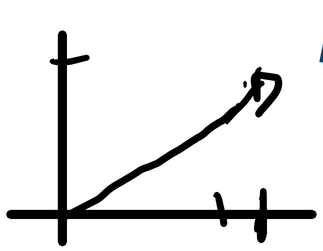
$$\nabla f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle.$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



(x,y)	$\nabla f(x,y)$
$(1,0)$	$\langle 1,0\rangle$
$(2,0)$	$\langle 1,0\rangle$
$(1,1)$	$\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
$(2,2)$	$\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
$(0,1)$	$\langle 0,1\rangle$
$(0,2)$	$\langle 0,1\rangle$
$(-1,1)$	$\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
$(2,1)$	$\left\langle \boxed{\frac{2}{\sqrt{5}}}, \frac{\sqrt{2}}{2}\right\rangle$





$\frac{2\sqrt{5}}{5}$

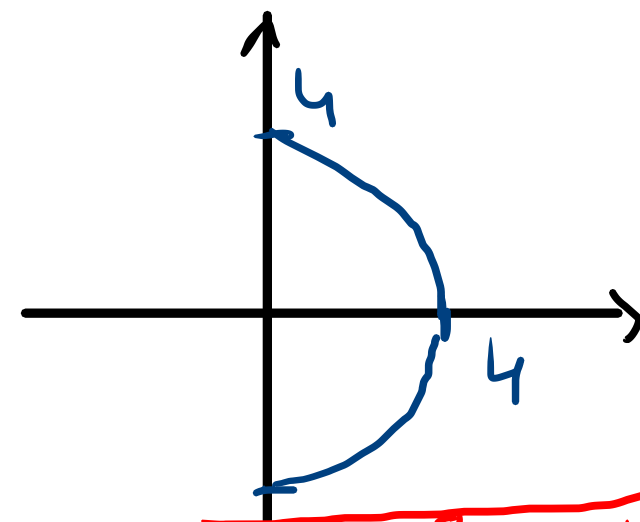
Note que

$$|\nabla f(x,y)| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1$$

2. Calcule a integral de linha, sendo C a curva dada:

$$\int_C xy^4\,ds\,, \quad C \text{ é a metade direita do círculo } x^2+y^2=\underline{16}$$



$x = x(t)$ $\quad a \leq t \leq b$
$y = y(t)$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

As equações paramétricas de C são:

$$x = 4\cos t \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$
$$y = 4\,\text{sen}\,t$$

$$\int_C x y^4 \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\,\text{sen}\,t)^4 \cdot \sqrt{(-4\,\text{sen}\,t)^2 + (4\cos t)^2} \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \cdot \text{sen}^4 t \cdot \sqrt{16\,(\underbrace{\text{sen}^2 t + \cos^2 t}_{1})} \, dt$$

$$= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \text{sen}^4 t \, dt$$

$$u = \text{sen}\,t$$

$$du = \cos t \, dt$$

$$\int u^4 \, du = \frac{u^5}{5} + C$$

$$= 4^6 \left. \frac{\text{sen}^5 t}{5} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{4^6}{5} \cdot (1 - (-1)) = \frac{4^6 \cdot 2}{5} = 1638{,}4.$$

3) Calcule

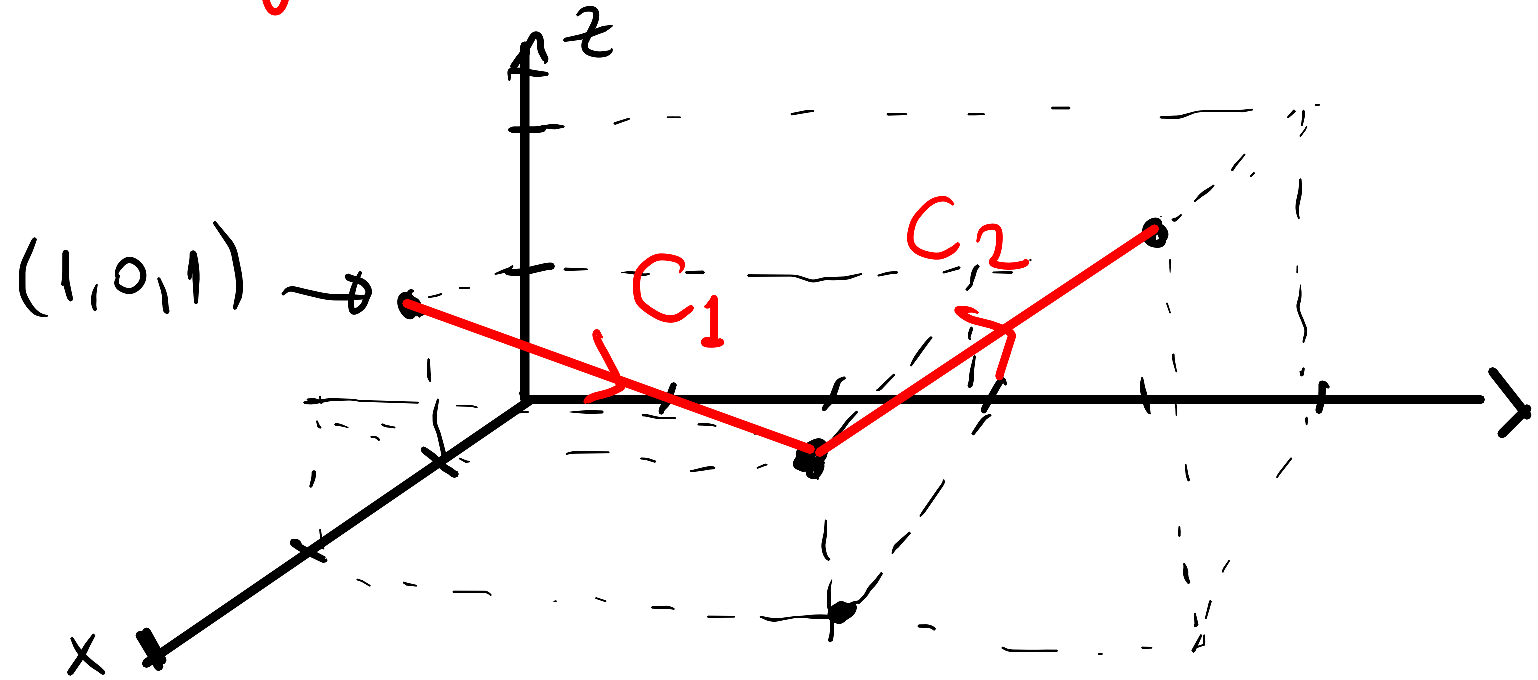
$$\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$$

C consiste nos segmentos de reta de $(1,0,1)$ a $(2,3,1)$ e de $(2,3,1)$ a $(2,5,2)$.

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$$

$$F = P\vec{i} + Q\vec{j} + R\vec{k} = \langle P, Q, R \rangle$$

$C = C_1 \cup C_2$



C_1: $\quad r(t) = (1-t)\, r_0 + t\, r_1 \qquad 0 \leq t \leq 1$

$$r(t) = (1-t)(1,0,1) + t(2,3,1)$$

$$r(t) = (1-t,\ 0,\ 1-t) + (2t,\ 3t,\ t)$$

$$x = 1-t+2t = 1+t \Rightarrow dx = dt$$

$$y = 3t \Rightarrow dy = 3\,dt$$

$$z = 1-t+t = 1 \Rightarrow dz = 0 \cdot dt$$

C_2 :

$$\alpha(t) = (1-t)\,r_1 + t\,r_2 \qquad 0 \le t \le 1$$

$$\alpha(t) = (1-t)(2,3,1) + t(2,5,2)$$

$$= \left(2-2t+2t,\ 3-3t+5t,\ 1-t+2t\right)$$

$$= (2,\ 3+2t,\ 1+t)$$

$x = 2 \qquad dx = 0 . dt$

$y = 3+2t \qquad dy = 2dt$

$z = 1+t \qquad dz = dt$

Portanto, $\qquad\qquad C = C_1 \cup C_2$

$$\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$$

$$= \int_{C_1} (x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2} (x+yz)\,dx + 2x\,dy + xyz\,dz$$

$$= \int_0^1 (1+t+3t)\,dt + 2(1+t)3\,dt + 0 + \int_0^1 0 + 2\cdot 2\cdot 2\,dt + \underbrace{2(3+2t)}_{6+4t}(1+t)\,dt$$

$$= \int_0^1 (1+4t+6+6t)\,dt + \int_0^1 (8+6+6t+4t+4t^2)\,dt$$

$$= \int_0^1 (10t+7)\,dt + \int_0^1 (4t^2+10t+14)\,dt$$

$$= \left[5t^2+7t\right]_0^1 + \left[\frac{4t^3}{3} + 5t^2 + 14t\right]_0^1 = 12 + \frac{4}{3} + 19 = \frac{97}{3}.$$

4. Calcule a integral de linha $\int_C F \cdot dr$, sendo C dada pela função vetorial $r(t)$.

$$F(x,y) = xy\vec{i} + 3y^2\vec{j} = (xy, 3y^2)$$

$$r(t) = 11t^4\vec{i} + t^3\vec{j} = (\underbrace{11t^4}_{x}, \underbrace{t^3}_{y}) \; ; \; 0 \leq t \leq 1.$$

$$W = \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

$$r'(t) = (44t^3, 3t^2)$$

$$F(r(t)) \cdot r'(t) = (11t^4 \cdot t^3, 3(t^3)^2) \cdot (44t^3, 3t^2)$$

$$= (11t^7, 3t^6) \cdot (44t^3, 3t^2) = 484t^{10} + 9t^8.$$

$$\int_C F \cdot dr = \int_0^1 484t^{10} + 9t^8 \, dt$$

$$= 484 \frac{t^{11}}{11} + t^9 \Big|_0^1 = \frac{484}{11} + 1$$

$$= \frac{484 + 11}{11} = \frac{495}{11} .$$

5. Determine se F é ou não um campo vetorial conservativo. Se for, determine uma função f tq

$$F = \nabla f.$$

(f função potencial)

$$F(x,y) = (2x-3y)\vec{i} + (-3x+4y-8)\vec{j}$$

$$F = P\,\vec{i} + Q\,\vec{j}$$

D aberto simplesmente conexo, vale que:

Se $F = P\vec{i} + Q\vec{j}$ e P, Q tem derivadas parciais de 1ª ordem contínuas com $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$,

então F é conservativo

Note que $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

$$P(x,y) = 2x - 3y$$

$$Q(x,y) = -3x + 4y - 8$$

$$\frac{\partial P}{\partial y} = -3 \quad , \quad \frac{\partial Q}{\partial x} = -3$$

e o domínio de $\vec{F}$ é $\mathbb{R}^2$ que é simplesmente conexo e aberto, então $\vec{F}$ é conservativo.

Queremos f tq

$$\nabla f(x,y) = (2x - 3y,\ -3x + 4y - 8)$$

$$f_x = 2x - 3y$$

$$\Rightarrow f(x,y) = x^2 - 3xy + g(y)$$

$$\Rightarrow f_y = -3x + g'(y) = -3x + 4y - 8$$

$$\Rightarrow g'(y) = 4y - 8$$

$$\Rightarrow g(y) = \frac{4y^2}{2} - 8y + K$$

$$g(y) = 2y^2 - 8y + K$$

Tome $K = 0$.

$$\Rightarrow f(x,y) = x^2 - 3xy + 2y^2 - 8y$$

$$\nabla f = (2x - 3y,\ -3x + 4y - 8) = F$$

6. Mostre que a integral de linha é independente do caminho e calcule a integral.

$$\int_C (1-ye^{-x})\,dx + e^{-x}\,dy$$

C é qualquer caminho de $(0,1)$ a $(1,2)$

$\int_C F\,dr$ indep. do caminho sempre que

$$\int_{C_1} F\,dr = \int_{C_2} F\,dr \quad \text{p/ qq } C_1, C_2$$

caminhos suaves com mesmo ponto inicial e final

Resultado: $\displaystyle\int_{C_1} \nabla f\,dr = \int_{C_2} \nabla f\,dr$ sempre que ∇f é cont

$\Rightarrow$ F conservativo indep. do caminho

Vejamos que o campo é conservativo:

$$F(x,y) = \left(1 - ye^{-x}, \underbrace{e^{-x}}\right)$$

I: f; $\nabla f = F$

$$f_x = 1 - ye^{-x} \Rightarrow f(x,y) = x + ye^{-x} + g(y)$$

$$\Rightarrow f_y = e^{-x} + g'(y) = e^{-x}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K$$

Tome $K = 0$.

$$\Rightarrow f(x,y) = x + ye^{-x}.$$

Logo F é conservativo, daí a integral de linha independe do caminho.

Agora,

$$F = \nabla f$$

$$\int_C (1 - ye^{-x})\,dx + e^{-x} = \int_C F\,dr = \int_C \nabla f\,dr$$

$$(f(x,y) = x + ye^{-x})$$

$$= f(1,2) - f(0,1)$$

$$= 1 + 2e^{-1} - 1e^{-0}$$

$$= 2e^{-1} = \frac{2}{e}.$$

T.F p/ int. de linha

C suave dada por $r(t)$ $a \leq t \leq b$

f diferenciável com ∇f contínuo

$$\Rightarrow \int_C \nabla f\,dr = f(r(b)) - f(r(a))$$