III Workshop Nordestino de Sistemas Dinâmicos - Salvador/BA

Ornstein isomorphism theorem for n-to-1 extended Bernoulli transformations

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Setup

- $\left(X_1, \mathcal{B}_, \mu\right)$ a probability space or a Lebesgue space.
- $T : X \to X$ a measure-preserving transformation.

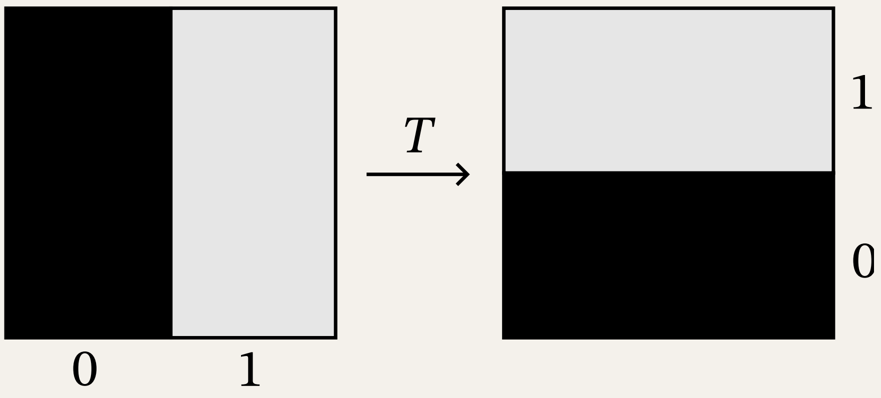
We say that $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ defined on $(X_1, \mathcal{B}_1, \mu)$ and $(X_2, \mathcal{B}_2, \nu)$, are *isomorphic* if there are $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ such that

- $\mu(A_1) = \nu(A_2) = 1$
- $T_1(A_1) \subset A_1,\ T_2(A_2) \subset A_2$
- $\exists\ \varphi : A_1 \to A_2$ invertible measure preserving map such that

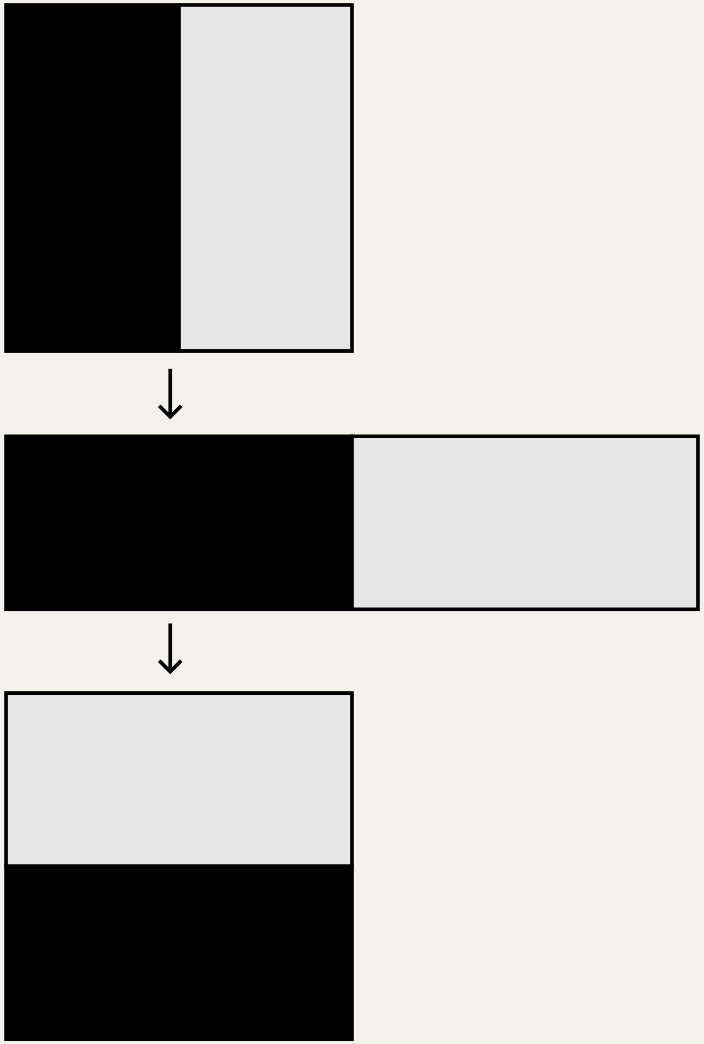
$$\varphi \circ T_1 = T_2 \circ \varphi.$$

Baker's map

$$X = [0,1]^2,\ T(x,y) = \begin{cases} (2x, y/2) & \text{if } x \in \left[0, \frac{1}{2}\right) \\ (2x-1, (y+1)/2) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



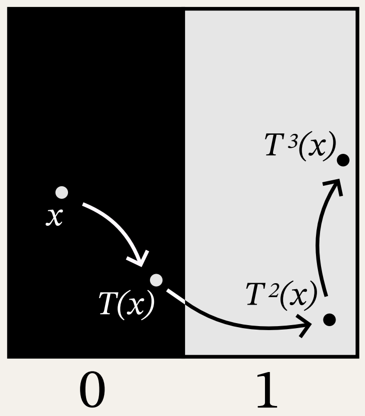
Baker's map



Coding the baker's map

To each orbit $\{\ldots, T^{-1}(x), x, T(x), \ldots\}$ we relate a sequence $(x_n)_n$ of zeros and ones:

- if $T^n x \in \left[0, \frac{1}{2}\right) \times [0,1]$, code $x_n = 0$
- if $T^n x \in \left[\frac{1}{2}, 1\right] \times [0,1]$, code $x_n = 1$



$$\mathcal{O}(x) = (\ \ldots\ ;\ 0011\ldots\ )$$
$$\mathcal{O}(T(x)) = (\ \ldots 0\ ;\ 011\ldots\ )$$

A is a finite alphabet

$\Sigma_{\mathrm{A}} = \mathrm{A}^{\mathbb{Z}} = \Big\{ (x_n)_{n\in\mathbb{Z}} : x_n \in \mathrm{A} \Big\}$

$$(x_n)_{n\in\mathbb{Z}} = (\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1\cdots\ ) \in \Sigma_{\mathrm{A}}$$

Σ_{A} is a compact metric space with

$$d((x_n),(y_n)) = 2^{-\inf\ \{|i|\ :\ x_i \neq y_i\}}$$

The *Bernoulli shift* $\sigma : \Sigma_{\mathrm{A}} \to \Sigma_{\mathrm{A}}$ is the map

$$\sigma(\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1\cdots\ ) = (\ \cdots x_{-1}x_0\ ;\ x_1x_2\cdots\ ).$$

Symbolic Dynamics

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] = \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] = \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

$$(\; \cdots x_{i-1} \; s_i s_{1+1} \cdots s_k \; x_{k+1} \cdots \;) \in C_i[s_i \ldots s_k]$$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define a probability measure by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \; \ldots \; \mu(C_k[s_k]) = p_{s_i} \; \ldots \; p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is a probability space.

A measurable map $T : X \to X$ is a *Bernoulli transformation* if it is isomorphic to a Bernoulli shift.

Kolmogorov-Sinai Entropy

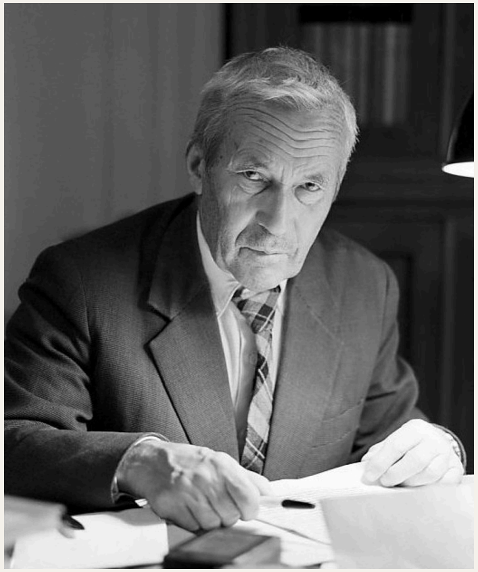
Entropy of a partition,

$$H_\mu(\mathcal{P}) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

Entropy of a transformation,

$$h_\mu(T) \stackrel{\text{def}}{=} \sup_{\mathcal{P}} \lim_{k \to \infty} \frac{1}{k} H_\mu\left(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{P} \right).$$

$T_1 \simeq T_2 \Rightarrow h_\mu(T_1) = h_\mu(T_2)$.



Andrei Kolmogorov

Ornstein isomorphism theorem

Ornstein, 1970

Bernoulli shifts with the same entropy are isomorphic.

Donald Ornstein

Enconding n-to-1 baker’s maps

Mehdipour, P., Martins, N.
Archiv der Mathematik.
119, 199–211, (2022).

Kristina Armitage/Quanta Magazine

Zip shifts

- A and B be two alphabets with $|\mathrm{A}| \geq |\mathrm{B}|$
- $\kappa : \mathrm{A} \to \mathrm{B}$ a surjective map
- Σ the space of all sequence of letters

$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1} \ ; \ x_0x_1\cdots)$$

$$\text{with} \quad x_{-1}, x_{-2}, \ldots \in \mathrm{B} \quad \text{and} \ x_0, x_1, \ldots \in \mathrm{A}.$$

The (full) *zip shift* map is $\sigma_\kappa : \Sigma \to \Sigma$ with

$$\sigma_\kappa(\ \cdots x_{-1} \ ; \ x_0x_1\cdots \ ) = (\cdots x_{-1}\kappa(x_0) \ ; \ x_1x_2\cdots).$$

Zip shift space

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define $\left(p_\beta : \beta \in \mathrm{B}\right)$

$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \kappa^{-1}(\beta)} p_\alpha .$$

The measure μ is defined by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ \ldots \ \mu(C_k[s_k]) = p_{s_i} \ \ldots \ p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is the *zip shift space.*

Some properties of zip shifts

- *σ_κ is a local homeomorphism*
- *σ_κ preserves the measure μ*
- *σ_κ is mixing and ergodic*
- *σ_κ has density of periodic points*

LM-Bernoulli property

A map is a *LM-Bernoulli transformation* if is isomorphic to a zip shift map.

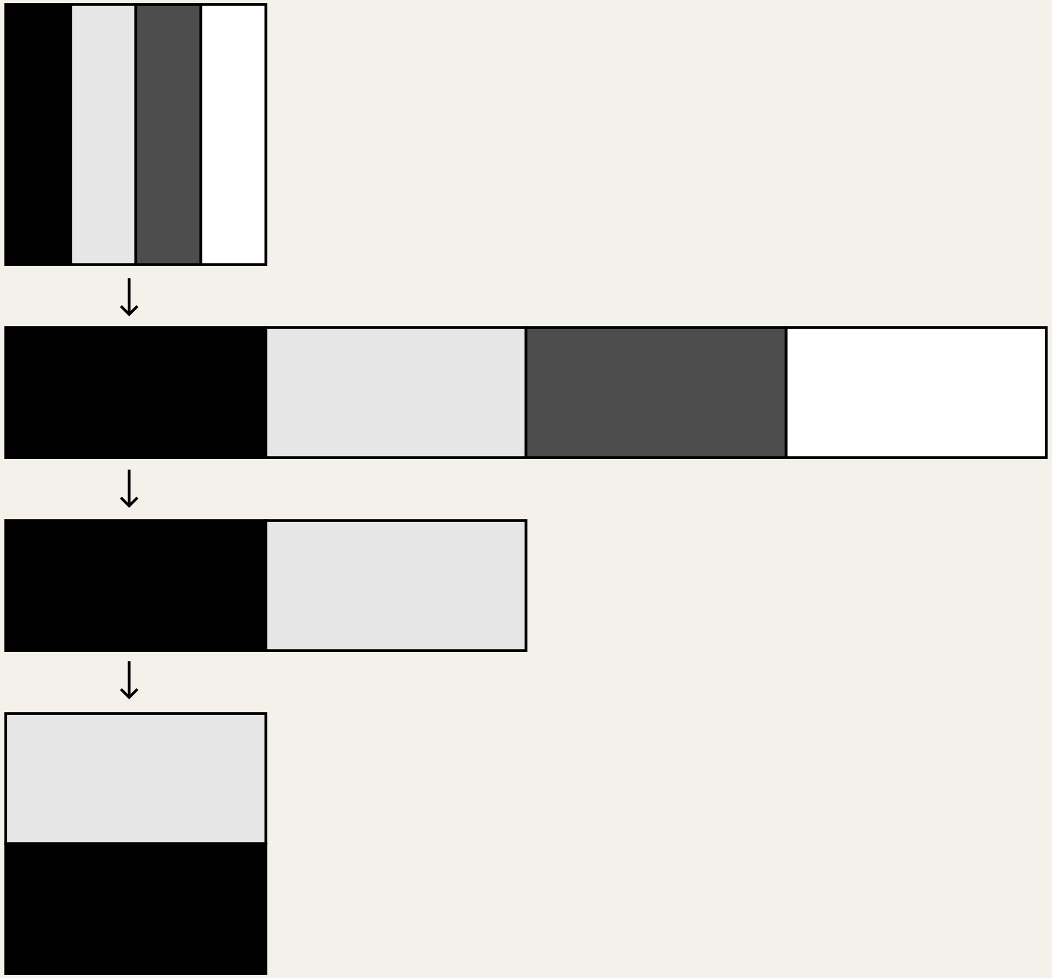
When we want to highlight or consider fixed cardinalities $m = |\mathrm{A}|, \ell = |\mathrm{B}|$, we refer to LM-Bernoulli by (m, ℓ)-Bernoulli transformation.

The n-to-1 baker's maps

$T : [0,1]^2 \to [0,1]^2$ given by

$$T(x,y) = \begin{cases} \left(2nx, \frac{1}{2}y\right) & \text{if } \ 0 \leq x < \frac{1}{2n} \\ \left(2nx - 1, \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \ \frac{1}{2n} \leq x < \frac{2}{2n} \\ \left(2nx - 2, \frac{1}{2}y\right) & \text{if } \ \frac{2}{2n} \leq x < \frac{3}{2n} \\ \qquad \vdots & \vdots \qquad \vdots \\ \left(2nx - (2n-1), \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \ \frac{2n-1}{2^n} \leq x \leq 1. \end{cases}$$

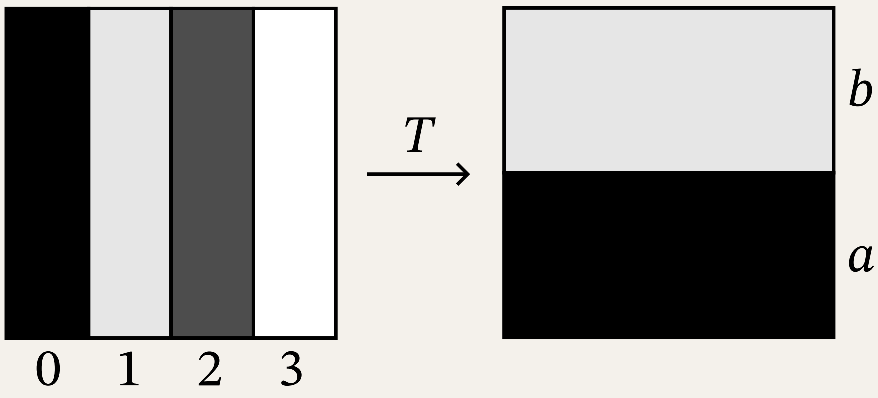
The 2-to-1 baker's maps



The n-to-1 baker's maps are LM-Bernoulli

Theorem

The n-to-1 baker's map is a $(2, 2n)$-Bernoulli transformation.



The n-to-1 baker's maps are chaotic

Theorem

The n-to-1 baker's map $\overline{T} : \overline{X} \to \overline{X}$ is chaotic in the sense of Devaney.

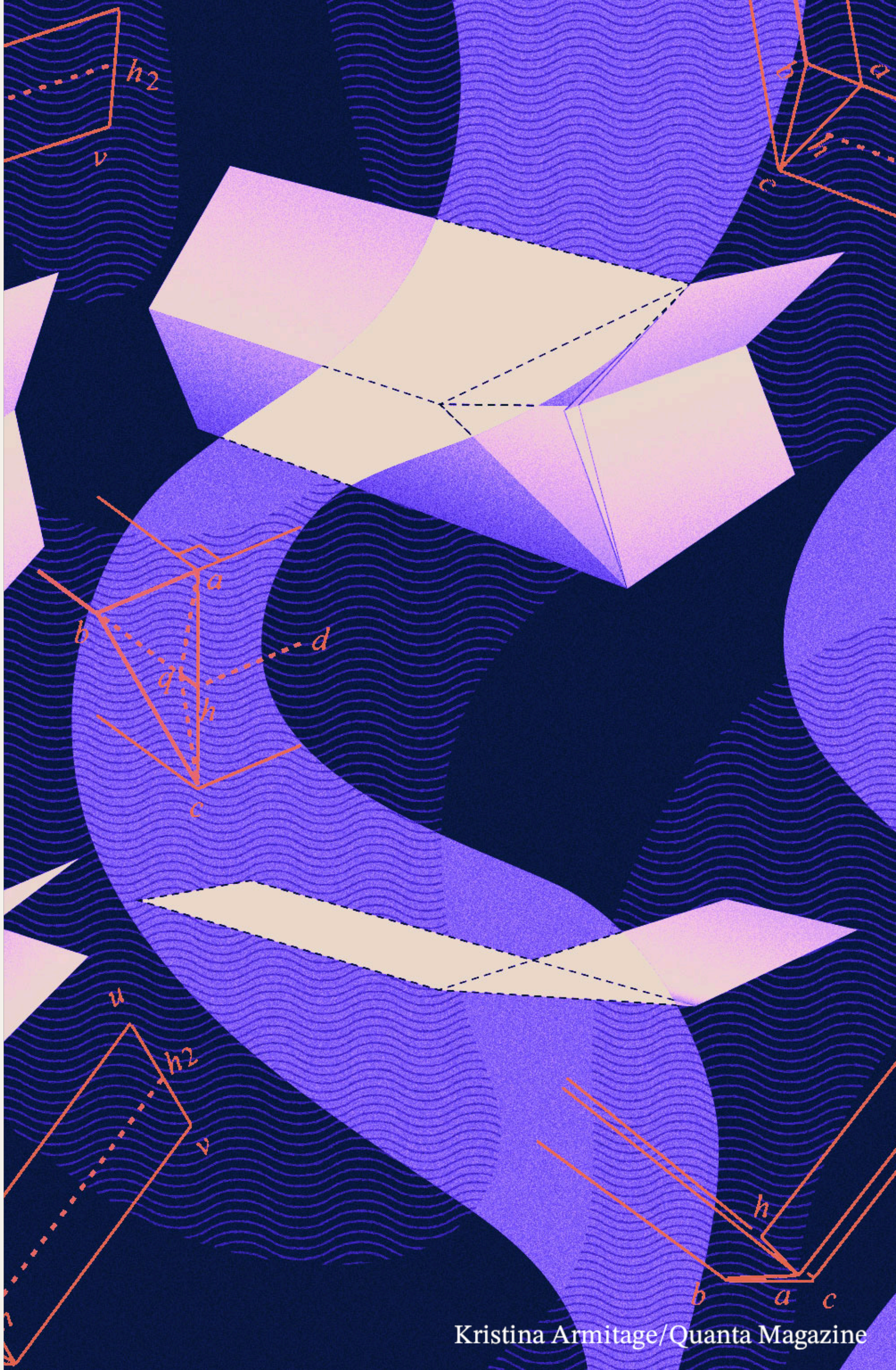
Devaney's chaos

- Topologically transitive
- Density of periodic points
- Sensitive dependence on initial conditions.

Folding and Metric Entropies for Extended Shifts

Martins, N., Mattos, P.G., Varão, R.

Journal of Dynamical Systems and Differential Equations, (2026).



Kolmogorov-Sinai entropy of zip shifts

$$\boxed{h_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0).}$$

Folding entropy

$$\mathcal{F}_\mu(T) := H_\mu(\mathcal{E}/T^{-1}\mathcal{E}).$$

$$H_\mu(\mathcal{P}/\mathcal{R}) := \int_{R\in\mathcal{R}} H_{\mu_R}(\mathcal{P}/R)\ \mu_R(dR),$$

where $\{\mu_R\}_{R\in\mathcal{R}}$ is a disintegration of μ with respect to $\mathcal{R}$.

David Ruelle

Folding entropy of zip shifts

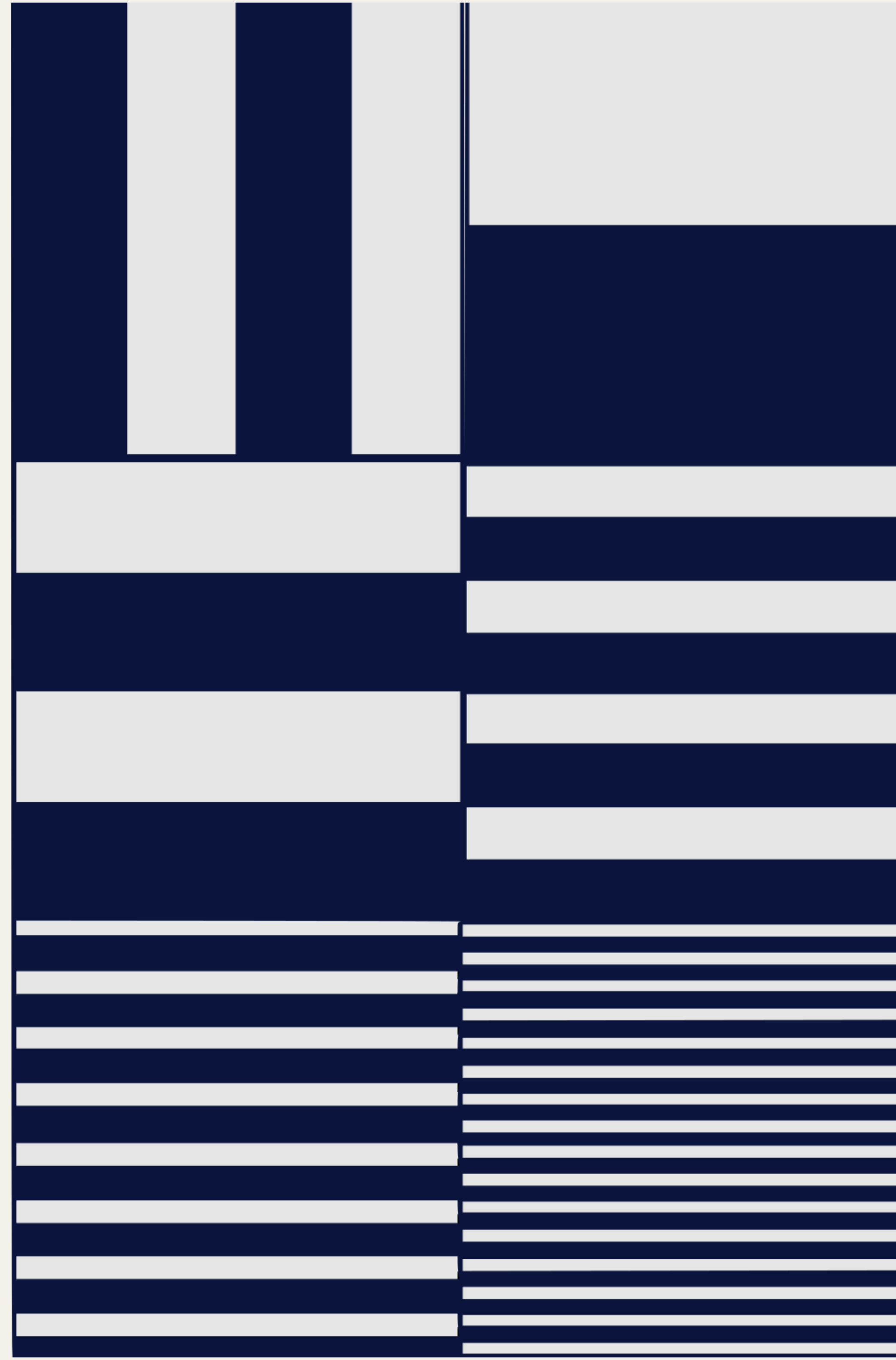
Theorem

$$\mathcal{F}_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).$$

Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations

Martins, N.,Mehdipour, P, Varão, R.

Preprint (2026) .



Isomorphism theorem

Main Theorem

Two n-to-1 LM-Bernoulli transformations of same entropy are isomorphic.

Ornstein characterization of Bernoulli shifts

Ornstein, 1974

An automorphism $T : X \to X$ *is isomorphic to a Bernoulli shift* $\sigma : \Sigma_{\mathrm{A}} \to \Sigma_{\mathrm{A}}$ *with distribution* $\rho_{\mathrm{A}} = (p_\alpha : \alpha \in \mathrm{A})$ *if, and only if, there is a partition* $\mathcal{P}$ *such that*

a) $\mathrm{dist}(\mathcal{P}) = \rho_{\mathrm{A}}$

b) $\mathcal{P}$ *is a generating for* T

c) $\{T^k \mathcal{P}\}_{k \in \mathbb{N}}$ *is a independent sequence.*

Ornstein characterization of Bernoulli shifts

Ornstein, 1974

Two Bernoulli transformations are isomorphic if, and only if, there are partitions $\mathcal{P}$ and $\mathcal{R}$ such that

$$\operatorname{dist}\left(\bigvee_{i=0}^{k} T_1^{-i}\mathcal{P}\right) = \operatorname{dist}\left(\bigvee_{i=0}^{k} T_2^{-i}\mathcal{R}\right), \quad \forall k \in \mathbb{N}.$$

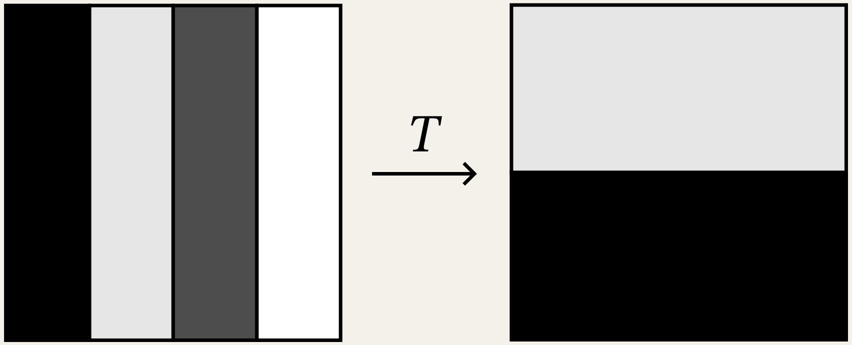
Domain and image partitions

- A image partition $\mathcal{Q} = \{Q_1, ..., Q_m\}$ of a n-to-1 local isomorphism a partition such that for all $P_i \in T^{-1}Q_j$, the map

$$T_{|_{P_i}} : P_i \to X$$

  is an automorphism.

- The collection $\mathcal{P}$ of all P_i is a domain partition



Characterization of n-to-1 LM Bernoulli

An n-to-1 local isomorphism $T : X \to X$ *is a LM-Bernoulli transformation with distribution* $\rho_{\mathrm{A}} = (p_\alpha : \alpha \in \mathrm{A})$ *if, and only if, there is a domain partition* $\mathcal{P}$ *such that*

a) $\mathrm{dist}(\mathcal{P}) = \rho_{\mathrm{A}}$

b) $\mathcal{P}$ *is a generating for* T

c) *The sequences* $\{T^k \mathcal{P}\}_{k \in \mathbb{N}}$ *and* $\{T^{-k} \mathcal{P}\}_{k \in \mathbb{N}}$ *are independent.*

The copying condition

Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and $\mathcal{P}$ and $\mathcal{R}$ be partitions of X_1 and X_2, respectively.

The *process* $(T_1, \mathcal{P})$ is a *copy* of the process $(T_2, \mathcal{R})$, and we denote by

$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R})$$

when, for all $k \geq 0$,

$$\operatorname{dist}\left(\bigvee_{-k}^{k} T_1^{-i}\mathcal{P}\right) = \operatorname{dist}\left(\bigvee_{-k}^{k} T_2^{-i}\mathcal{R}\right).$$

The copying condition

Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and $\mathcal{P}$ and $\mathcal{R}$ to be the domain generating partitions, respectively. Then,

$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R}) \Leftrightarrow T_1 \simeq T_2.$$

References

1. Mehdipour P, Martins N: **Encoding n-to-1 baker's transformations.** *Archiv der Mathematik* 2022, **109**:199–211.
2. Martins N, Mattos P, Varão R: **Folding and metric entropies of extended shifts.** *Journal of Dynamics and Differential Equations* 2026,
3. Martins N, Mehdipour P, Varão R: **Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations.** *preprint* 2026,
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5. Mehdipour P, Lamei S: **An n-to-1 Smale Horseshoe.** *arXiv:250214441* 2025,
6. Martins N: **Ornstein theory for extended symbolic dynamics.** *PhD Thesis, Unicamp* 2025,

Thank you!

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