# Symbolic Dynamics

## Entropy of (m,l)-Bernoulli transformations

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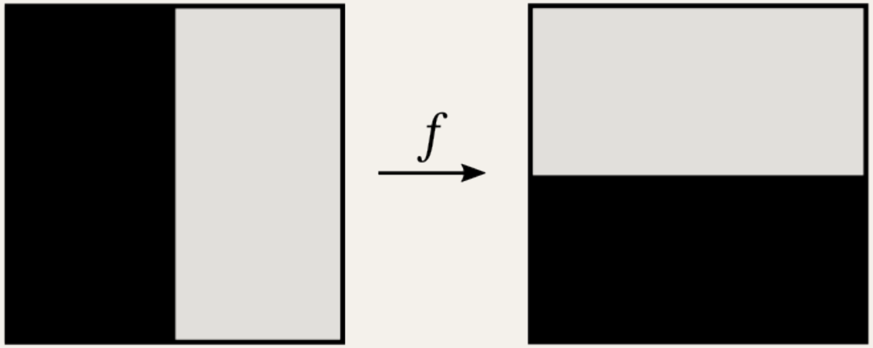
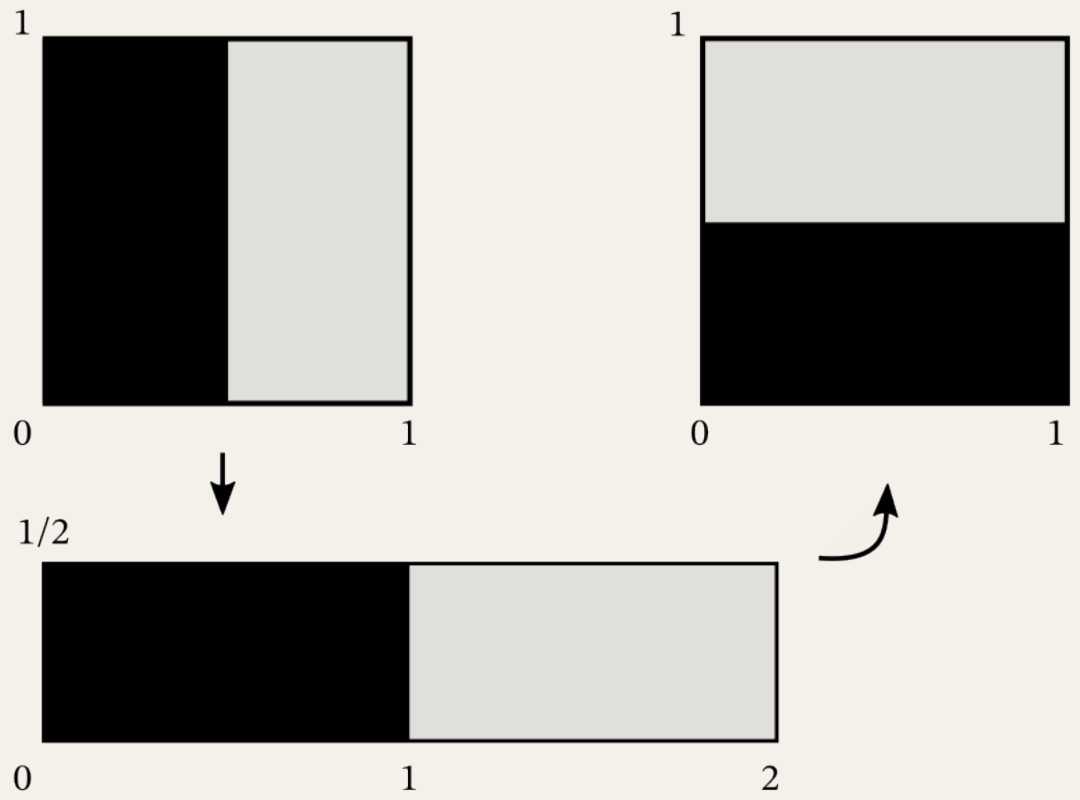
# Baker map

$$X = [0,1]^2,\ f(x,y) = \begin{cases} (2x, y/2) & \text{if } x \in \left[0, \frac{1}{2}\right) \\ (2x-1, (y+1)/2) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



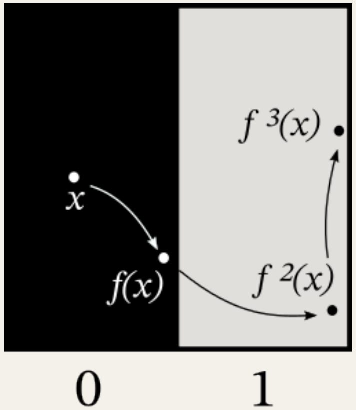
# Baker map



# Coding the Baker map

To each orbit $\{\dots, f^{-1}(x), x, f(x), \dots\}$ we relate a sequence $(x_n)_n$ of zeros and ones:

- if $f^n x \in \left[0, \frac{1}{2}\right)$, code $x_n = 0$
- if $f^n x \in \left[\frac{1}{2}, 1\right]$, code $x_n = 1$




Orbit of $x$ : ( … ; $0011\dots$ ). Orbit of $f(x)$ : ( …$0$ ; $011\dots$ ).

# Symbolic Dynamics

$\mathrm{A}$ is a finite alphabet

$\mathrm{A}^{\mathbb{Z}} = \left\{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathrm{A}\right\}$

$$(x_n)_{n\in\mathbb{Z}} = (\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1 \cdots\ )$$

$\mathrm{A}^{\mathbb{Z}}$ is a compact metric space with

$$d((x_n),(y_n)) = 2^{-\inf\{|i| : x_i \neq y_i\}}$$

$\sigma : \mathrm{A}^{\mathbb{Z}} \to \mathrm{A}^{\mathbb{Z}}$ is the (full) **shift map**

$$\sigma(x_n) = (x_{n+1}).$$

# Symbolic Dynamics

Let $\mathcal{C}$ the $\sigma$-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

$$( \;\cdots x_{i-1}\; s_i s_{1+1} \cdots s_k \; x_{k+1} \cdots \; ) \in C_i[s_i \ldots s_k]$$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in $\mathrm{A}$, we define a measure

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ldots \mu(C_k[s_k]) = p_{s_i} \ldots p_{s_k}.$

# Symbolic Dynamics

We say that $f : X \to X$ and $g : Y \to Y$, defined on $(X, \mathcal{B}_1, \mu)$ and $(Y, \mathcal{B}_2, \nu)$, are isomorphic if there are measurable sets $A \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$ such that

- $\mu(A) = \nu(B) = 1$
- $f(A) \subset A,\ g(B) \subset B$
- $\exists\, \varphi : A \to B$ invertible measure preserving map such that

$$\varphi \circ f = g \circ \varphi.$$

A map $f : X \to X$, defined on $(X, \mathcal{B}, \nu)$, is a Bernoulli shift if is isomorphic to a shift map on $(\Sigma, \mathcal{C}, \mu)$.

# Zip shifts

- $\mathrm{A}$ and $\mathrm{B}$ be two finite alphabets $\mathrm{A} \geq \mathrm{B}$
- $\varphi : \mathrm{A} \to \mathrm{B}$ a surjective map
- $\Sigma$ the space of all sequence of letters

$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1}\,;\, x_0x_1\cdots)$$
$$\text{with } x_{-1}, x_{-2}, \ldots \in \mathrm{B} \text{ and } x_0, x_1, \ldots \in \mathrm{A}.$$

The full zip shift map is $\sigma_\varphi : \Sigma \to \Sigma$ with

$$\sigma_\varphi(\ \cdots x_{-1}\,;\, x_0x_1\cdots\ ) = (\cdots x_{-1}\varphi(x_0)\,;\, x_1x_2\cdots).$$

$f : X \to X$ is a extended Bernoulli when is isomorphic to a zip map on $(\Sigma, \mathcal{C}, \mu)$. If $\#\mathrm{B} = m$ and $\#\mathrm{A} = l$, we say $f$ is a (m,l)-Bernoulli.

# The zip shift space

Let $\mathcal{C}$ the $\sigma$-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in $\mathrm{A}$, we define $\left(p_\beta : \beta \in \mathrm{B}\right)$

$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \varphi^{-1}(\beta)} p_\alpha.$$

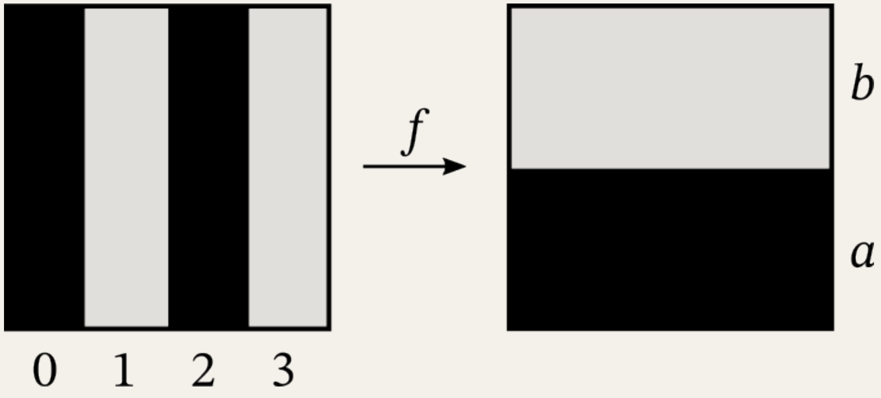
The measure $\mu$ is defined by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ldots \mu(C_k[s_k]) = p_{s_i} \ldots p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is the zip shift space

# Baker's map 2-to-1

# Baker's map n-to-1




**Mehdipour, ---**

The baker’s n-to-1 is a (2,2n)-Bernoulli transformation.

# Entropy

- Von Neumann: Isomorphism problem of Bernoulli shifts
- Shannon entropy

$$H(p_1, \dots, p_k) \stackrel{\text{def}}{=} -\sum_{i=1}^{k} p_i \log p_i$$

- Kolmogorov-Sinai entropy as a invariant of measure isomorphim:

$$H_\mu(\mathcal{P}) \stackrel{\text{def}}{=} H(\mu(P_1), \dots, \mu(P_k))$$

$$h_\mu(f) \stackrel{\text{def}}{=} \sup_{\mathcal{P}} \lim_{k \to \infty} \frac{1}{k} H_\mu\left(\bigvee_{i=0}^{k-1} f^{-i}\mathcal{P}\right)$$

- Ornstein theorem: Bernoulli shifts of same entropy are isomorphic.

# Folding entropy

- $\varepsilon := \{\{x\} : x \in X\}$ the partition of $X$ into single points
- $f^{-1}(\varepsilon) = \left\{f^{-1}(x) : x \in X\right\}$ the preimage partition of $\varepsilon$ by $f$.

The folding entropy of $f$ is defined by

$$\mathcal{F}_\mu(f) \stackrel{\text{def}}{=} H_\mu\big(\varepsilon \mid f^{-1}(\varepsilon)\big) = \int_X H_{\tilde{\mu}_x}(\varepsilon) d(f\mu)$$

where $\{\tilde{\mu}_x\}$ is a canonical family of conditional measures of $\mu$ disintegrated along the preimage sets $\left\{f^{-1}x\right\}$ for $\mu$-a.e. $x \in X$.

# Entropy of a zip shift

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- $h_\mu(\sigma_\varphi) = H\Big((p_\alpha)_{\alpha\in \mathrm{A}}\Big)$
- $\mathcal{F}_\mu(f) = H\Big((p_\alpha)_{\alpha\in A}\Big) - H\Big((p_\beta)_{\alpha\in \mathrm{B}}\Big).$

# Isomorphism theorem

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Let $f$ and $g$ be two n-to-1 $(m, l)$-Bernoulli transformations of same entropy. Then $f \cong g$.

# References


[1] N. Martins, P. G. Mattos, e R. Varão, «Folding Entropy for Extended Shifts». 2024.

[2] P. Mehdipour e N. Martins, «Encoding n-to-1 baker's transformations», *Arch. Math.*, vol. 119, pp. 199–211, 2022.

[3] S. Lamei e P. Mehdipour, «Zip shift space», 2022.

[4] W. Wu e Y. Zhu, «On preimage entropy, folding entropy and stable entropy», *Ergod. Th. & Dynam. Sys.*, vol. 41, pp. 1217–1249, 2021.

[5] G. Liao e S. Wang, «Continuity properties of folding entropy», *Israel Journal of Mathematics*, 2024.

[6] D. Ornstein, *Ergodic Theory, Randomness, and Dynamical Systems*. Yale Mathematical Monographs, 1974.

( . . . thank you . . . )