Unit-1 class-11

1. A lamina is bounded by the curves $y=x^2-3x$ and $y=2x$. If the density at any point is given by λxy, find by double integration, mass of the lamina. Ans: $182\frac{7}{24}\lambda$

$y = x^2 - 3x$

$y = 2x$

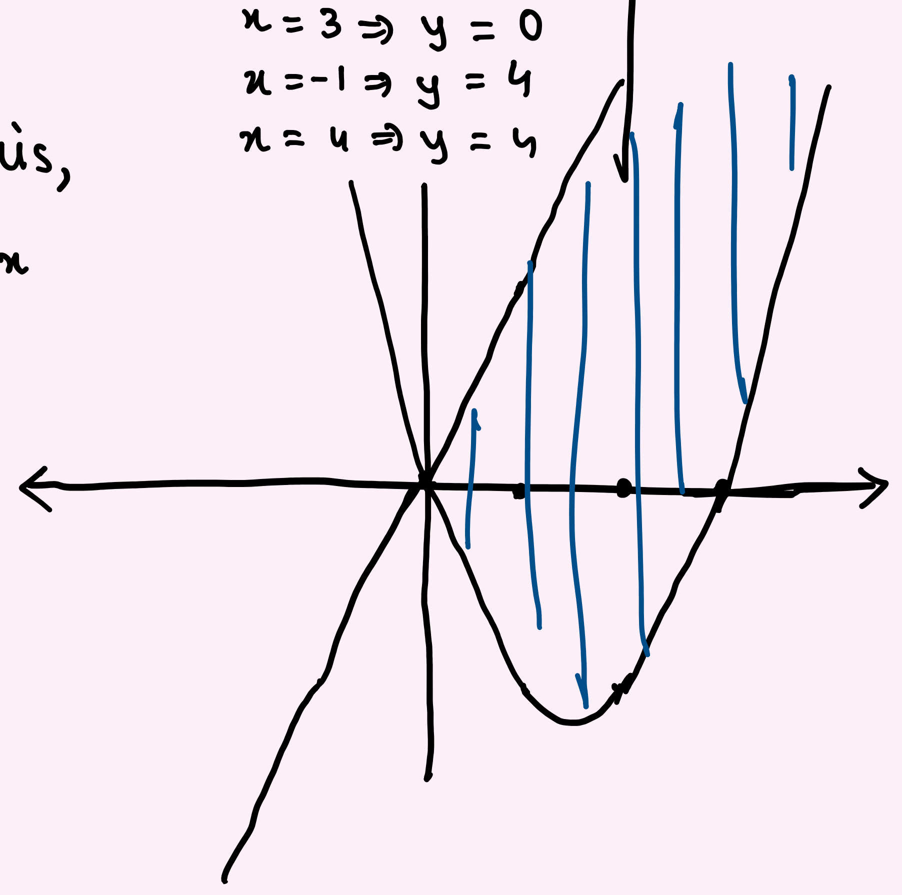
common point $\Rightarrow x = 0, 5$

$y=x^2-3x$	$y=2x$
$x=0 \Rightarrow y=0$	$x=0 \Rightarrow y=0$
$x=1 \Rightarrow y=-2$	$x=1 \Rightarrow y=2$
$x=2 \Rightarrow y=-2$	$x=2 \Rightarrow y=4$
$x=3 \Rightarrow y=0$	
$x=-1 \Rightarrow y=4$	
$x=4 \Rightarrow y=4$	

Line $\parallel^{el}$ to y-axis,

$y \Rightarrow x^2-3x$ to $2x$

$x \Rightarrow 0$ to 5



$$\int_0^5\int_{x^2-3x}^{2x} \lambda xy\,dy\,dx$$

$$\int_0^5\left[\frac{\lambda xy^2}{2}\right]_{x^2-3x}^{2x} dx$$

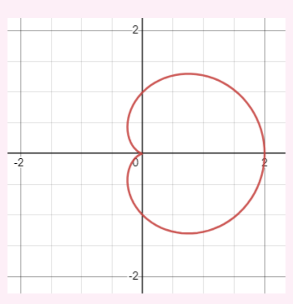
$$= \int_0^5 \frac{\lambda x}{2}\left((2x)^2-(x^2-3x)^2\right)dx = \int_0^5 \frac{\lambda x}{2}\left(4x^2-(x^4-6x^3+9x^2)\right)dx$$

$$= \frac{\lambda}{2}\int_0^5\left(-x^5+6x^4-5x^3\right)dx = \frac{\lambda}{2}\left[-\frac{x^6}{6}+\frac{6x^5}{5}-\frac{5x^4}{4}\right]_0^5$$

$$= \frac{4375\lambda}{24} = 182\frac{7}{24}\lambda$$

2. Find the mass of the lamina in the form of the cardioid $r=a(1+\cos\theta)$ whose density at any point varies as the square of its distance from the initial line. ans: $\frac{21\pi a\mu^4}{32}$

$r = a(1+\cos\theta)$



$\rho \propto r^2\sin^2\theta \Rightarrow \rho = \mu r^2\sin^2\theta$

$r^2 = x^2+y^2$ where $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r \Rightarrow 0$ to $a(1+\cos\theta)$

$\theta \Rightarrow 0$ to 2π

$$\iint \rho\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} \mu r^3\sin^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\mu r^4}{4}\right]_0^{a(1+\cos\theta)}\sin^2\theta\,d\theta$$

$$= \int_0^{2\pi}\frac{\mu a^4}{4}(1+\cos\theta)^4\sin^2\theta\,d\theta$$

$$= \frac{2\times\mu a^4}{4}\int_0^{\pi}(1+\cos\theta)^4\sin^2\theta\,d\theta$$

$$= \frac{\mu a^4}{2}\int_0^{\pi}\left[2\cos^2\frac{\theta}{2}\right]^4\times 4\sin^2\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}\,d\theta$$

$$= \frac{\mu a^4}{\cancel{2}}\times 16\times\cancel{4}^{2}\int_0^{\pi}\sin^2\frac{\theta}{2}\cdot\cos^{10}\frac{\theta}{2}\,d\theta$$

$$= 32\mu a^4\int_0^{\pi}\sin^2\frac{\theta}{2}\cdot\cos^{10}\frac{\theta}{2}\,d\theta$$

$t = \frac{\theta}{2} \Rightarrow 2dt = d\theta$

$$= 32\mu a^4\int_0^{\pi/2}\sin^2 t\cdot\cos^{10}t\;2dt$$

$$= 64\mu a^4\times\frac{1}{2}\beta\left(\frac{3}{2},\frac{11}{2}\right)$$

$$= 32\mu a^4\times\frac{\Gamma\frac{3}{2}\,\Gamma\frac{11}{2}}{\Gamma 7} = \cancel{32}\mu a^4\times\frac{\frac{\sqrt{\pi}}{2}\times\left(\frac{\cancel{9}^{3}\times 7\times\cancel{5}\times\cancel{3}\times\sqrt{\pi}}{\cancel{32}}\right)}{\cancel{6}^{2}\times\cancel{5}\times 4\times\cancel{3}\times 2}$$

$$= \frac{21\pi\mu a^4}{32}$$

3. Find the mass of a solid in the form of the positive octant of the sphere $x^2+y^2+z^2=9$ if the density at any point is 2xyz. ans:30.375

$\rho = 2xyz$

$z \Rightarrow 0$ to $\sqrt{9-y^2-x^2}$

$y \Rightarrow 0$ to $\sqrt{9-x^2}$

$x \Rightarrow 0$ to 3

$$m = \int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} 2xyz\,.\,dz\,dy\,dx$$

$$= \int_0^3\int_0^{\sqrt{9-x^2}}\left[xyz^2\right]_0^{\sqrt{9-x^2-y^2}} dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}}\left(xy(9-x^2-y^2)\right)dy\,dx$$

$$= \int_0^3\int_0^{\sqrt{9-x^2}}\left(9xy - x^3y - xy^3\right)dy = \int_0^3\left[\frac{9xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_0^{\sqrt{9-x^2}}dx$$

$$= \int_0^3\left[\frac{9x(9-x^2)}{2}-\frac{x^3(9-x^2)}{2}-\frac{x\underbrace{(9-x^2)^2}_{81+x^4-18x^2}}{4}\right]dx$$

$$= \int_0^3\frac{162x-18x^3-18x^3+2x^5-81x-x^5+18x^3}{4}dx$$

$$= \int_0^3\left(\frac{x^5-18x^3+81x}{4}\right)dx = \left[\frac{x^6}{24}-\frac{9x^4}{8}+\frac{81x^2}{8}\right]_0^3 = \frac{243}{8}$$

4. Find the centroid of the area enclosed by the parabola $y^2=4ax$, the axis of x and its latus rectum. Ans: $\left(\frac{3a}{20},\frac{3a}{16}\right)$

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No answer found, please contact me if you do find one

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