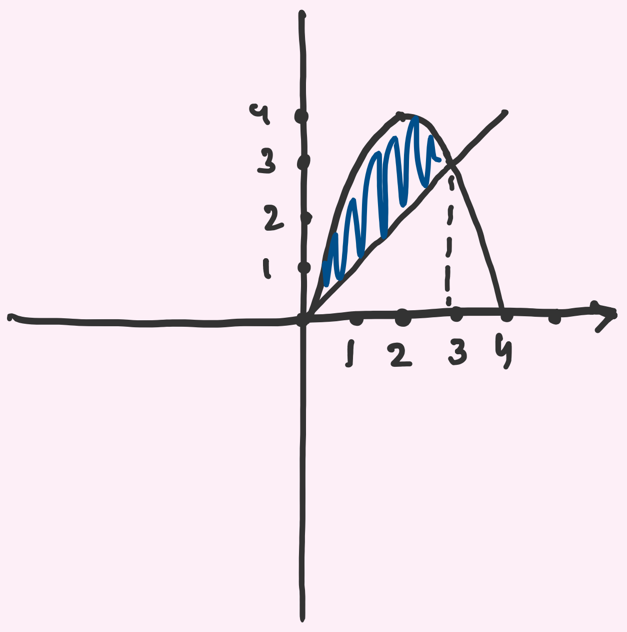
Unit-1 class-2

1. Find by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$
   ans: 4.5

A.



$y = 4x - x^2$ $\qquad$ $y = x$

$x = 0 \quad y = 0$
$x = 1 \quad y = 3$
$x = 2 \quad y = 4$
$x = 3 \quad y = 3$

Parallel to x-axis

$y: x$ to $4x - x^2$

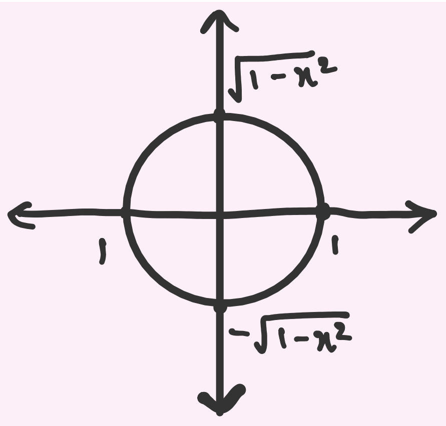
$x: 0$ to 4

$$\int_0^3 \int_x^{4x-x^2} dy\,dx = \int_0^3 (4x - x^2 - x) = \int_0^3 3x - x^2$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2} = 4.5$$

2. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$
   ans: 3π

A.



$x + y + z = 3$
$z = 3 - x - y$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3 - x - y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[3y - xy - \frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left[3\sqrt{1-x^2} - (-3\sqrt{1-x^2}) - \left(x\sqrt{1-x^2} - (-x\sqrt{1-x^2})\right) - \left(\frac{\cancel{1-x^2} - \cancel{(1-x^2)}}{2}\right)\right] dx$$

$$= \int_{-1}^{1} \left(\underbrace{6\sqrt{1-x^2}}_{①} - \underbrace{2x\sqrt{1-x^2}}_{②}\right) dx$$

For ①, $6\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$x = \sin\theta$ $\quad$ $x = 1, \ \theta = \pi/2$
$dx = \cos\theta\,d\theta$ $\quad$ $x = -1, \ \theta = -\pi/2$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta = \frac{\pi}{2}$$

For ②, $\int_{-1}^{1} 2x\sqrt{1-x^2}\,dx$

$u = 1 - x^2$
$du = -2x\,dx$

$$-\int_0^0 \sqrt{u}\,du = 0$$

① − ②

$$6\left(\pi/2\right) - 0 = 3\pi$$

3. Find the average value of the function e^{x+y} over the region $R = [0,2] \times [0,2]$ $\quad$ ans: $\frac{(e^2-1)^2}{4}$

A. $$\int_0^2 \int_0^2 e^{x+y}\,dy\,dx = \int_0^2 \left[e^y\right]_0^2 e^x = \int_0^2 (e^2 - 1) e^x\,dx = (e^2 - 1)^2$$

Now, $$\int_0^2 \int_0^2 dy\,dx = \int_0^2 (2 - 0)\,dx = \int_0^2 2\,dx = 2 \times (2 - 0) = 4$$

Average value $= \dfrac{(e^2 - 1)^2}{4}$