UE22MA141B - ENG MATH - II

UNIT 5 : FOURIER TRANSFORM

1) Find the complex Fourier transform of the function

$$f(t) = \begin{cases} 1 & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin w}{w} \, dw$.

Solution:

Complex Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-iwt} \, dt$$

$$= \int_{-1}^{1} 1 \cdot e^{-iwt} \, dt$$

Since $f(t) = \begin{cases} 1 & \dots \leq \\ 0 \end{cases}$

$$= \left[\frac{e^{-iwt}}{-iw} \right]_{-1}^{1}$$

$$= \frac{-1}{iw} \left[e^{-iw(1)} - e^{-iw(-1)} \right]$$

$$= \frac{-1}{iw} \left[\cos w - i\sin w - (\cos w + i \sin w) \right]$$

$$= \frac{2i\sin w}{iw} = \frac{2\sin w}{w}.$$

Thus

$$\boxed{F(w) = \frac{2\sin w}{w}.}$$

Let us evaluate $\int_0^\infty \frac{\sin\omega}{\omega} \cdot d\omega$

We have obtained $F(\omega) = \frac{2\sin\omega}{\omega}$

Inverse Fourier transform is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} \cdot d\omega = f(t).$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\omega}{\omega}\, e^{i\omega t} \cdot d\omega = f(t).$$

Put $t = 0$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\,\frac{\sin\omega}{\omega} \cdot d\omega = f(0).$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} \cdot d\omega = 1.$$

$$\frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega}{\omega} \cdot d\omega = 1$$

[Since $\frac{\sin\omega}{\omega}$ is an even fn]

$$\Rightarrow \int_{0}^{\infty} \frac{\sin\omega}{\omega} \cdot d\omega = \frac{\pi}{2}.$$

2) Find the Fourier transform of.

$$f(t) = \begin{cases} 1-t^2 & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1. \end{cases}$$

Hence evaluate $\int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\left(\frac{\omega}{2}\right) \cdot d\omega$.

Solution:

Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} \, dt$$

$$= \int_{-1}^{1} (1-t^2) \, e^{-i\omega t} \, dt$$

$$= \left[(1-t^2) \cdot \frac{e^{-i\omega t}}{-i\omega} - (-2t) \cdot \left(\frac{-1}{i\omega}\right) \cdot \frac{e^{-i\omega t}}{-i\omega} + (-2) \cdot \left(\frac{1}{i^2\omega^2}\right) \cdot \frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1}$$

$$= \left[\frac{2t}{i^2\omega^2} e^{-i\omega t} + \frac{2}{i^3\omega^3} e^{-i\omega t} \right]_{-1}^{1}$$

$$\Rightarrow \left[\frac{2}{i^2\omega^2} \left[e^{-i\omega} - (-1) \cdot e^{-i\omega(-1)} \right] + \frac{2}{i^3\omega^3} \left[e^{-i\omega} - e^{-i\omega(-1)} \right] \right]$$

$$\left[\because \frac{1}{i^2} = -1 \qquad \frac{1}{i^3} = i \right]$$

$$= \left[\frac{-2}{\omega^2} \left[2\cos\omega \right] + \frac{2i}{\omega^3} \left[-2i\sin\omega \right] \right]$$

$$= \frac{-4\cos\omega}{\omega^2} + \frac{4}{\omega^3} \sin\omega \Rightarrow \frac{4}{\omega^3} \left[\sin\omega - \omega\cos\omega \right]$$

To evaluate

$$\int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\left(\frac{\omega}{2}\right)\cdot d\omega.$$

By inverse Fourier transform we have.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega t}\cdot d\omega.$$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cdot(\cos\omega t + i\sin\omega t)\, d\omega$$

$$f(t) = \frac{4}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega t\cdot d\omega.$$

$$+ \frac{i}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\sin\omega t\cdot d\omega$$

The integrand in the second integral on the RHS is an odd function of ω and hence the integral vanishes.

Also, the integrand in the first integral is an even function. Thus

$$f(t) = \frac{4}{2\pi}\times 2\int_0^\infty\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega t\cdot d\omega.$$

Put $t = \frac{1}{2}$

$$f(\tfrac{1}{2}) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\left(\frac{\omega}{2}\right)\cdot d\omega.$$

$$\frac{3}{4} = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\left(\frac{\omega}{2}\right)\cdot d\omega.$$

$$= \frac{1}{-i\omega}[1-0] - \frac{1}{i^2\omega^2}[1-e^{i\omega}]$$

$$+ \frac{1}{-i\omega}[0-1] + \frac{1}{i^2\omega^2}[e^{-i\omega}-1]$$

$$= \cancel{\frac{-1}{i\omega}} + \frac{1}{\omega^2}[1-e^{i\omega}] + \cancel{\frac{1}{i\omega}} - \frac{1}{\omega^2}[e^{-i\omega}-1]$$

$$= \frac{1}{\omega^2}[1] - \frac{1}{\omega^2}(e^{i\omega}+e^{-i\omega}) + \frac{1}{\omega^2}$$

$$= \frac{2}{\omega^2} - \frac{2}{\omega^2}\cos\omega = \frac{2}{\omega^2}(1-\cos\omega)$$

$$= \frac{2}{\omega^2} \times 2\sin^2(\omega/2)$$

$$\boxed{F(\omega) = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)}$$

To evaluate the integral, we use the inverse FT

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega t}\cdot d\omega.$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\omega^2}\cdot\sin^2\left(\frac{\omega}{2}\right) e^{i\omega t}\cdot d\omega.$$

Put $t=0$

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{\omega^2}\cdot\sin^2\left(\frac{\omega}{2}\right)\cdot d\omega.$$

Put $\frac{\omega}{2} = u. \Rightarrow \frac{1}{2}\cdot d\omega = du.$

$$\Rightarrow \int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\left(\frac{\omega}{2}\right)\cdot d\omega = \frac{3\pi}{16}$$

3) Find the Fourier transform of

$$f(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin^2\omega}{\omega^2}\cdot d\omega$

Solution:

Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}\cdot dt$$

$$= \int_{-1}^{0} [1-(-t)]\, e^{-i\omega t}\cdot dt + \int_0^1 [1+(+t)]\, e^{-i\omega t} dt$$

$$= \int_{-1}^{0} (1+t) e^{-i\omega t}\, dt + \int_0^1 (1-t)\cdot e^{-i\omega t}\cdot dt$$

$$= \left[(1+t)\cdot\frac{e^{-i\omega t}}{-i\omega} - (1)\cdot\left(\frac{-1}{i\omega}\right)\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{0}$$

$$+ \left[(1-t)\cdot\frac{e^{-i\omega t}}{-i\omega} - (-1)\cdot\left(\frac{-1}{i\omega}\right)\frac{e^{-i\omega t}}{-i\omega}\right]_0^1$$

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \, 2 \, du$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \, du$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin^2 u}{u^2} \, du$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 u}{u^2} \, du = \frac{\pi}{2}.$$

4) Find the Fourier Sine transform of $f(t) = e^{-|t|}$ and hence evaluate $\int_0^\infty \frac{\omega \sin m\omega}{1+\omega^2} \, d\omega$ where $m > 0$.

Solution:

Fourier sine transform is given by

$$F_S(\omega) = \int_0^\infty f(t) \cdot \sin \omega t \cdot dt$$

$$= \int_0^\infty e^{-|t|} \cdot \sin \omega t \cdot dt \qquad |t| = t, \; t > 0.$$

$$= \int_0^\infty e^{-t} \cdot \sin \omega t \cdot dt$$

$$= \left[\frac{e^{-t} \left(-1 \cdot \sin \omega t - \omega \cos \omega t \right)}{(-1)^2 + \omega^2} \right]_0^\infty$$

$$F_s(\omega) = \frac{\omega}{1+\omega^2}.$$

By inverse Fourier Sine transform we have

$$\frac{2}{\pi}\int_0^\infty F_s(\omega)\cdot \sin\omega t \cdot d\omega = f(t)$$

$$\frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2} \sin\omega t \cdot d\omega = f(t).$$

Put $t = m$ where $m > 0$ we have

$$f(t) = e^{-|m|} = e^{-m}$$

$$\frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2} \sin m\omega \cdot d\omega = f(m).$$

$$\frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2} \sin m\omega \cdot d\omega = e^{-m}.$$

$$\Rightarrow \int_0^\infty \frac{\omega}{1+\omega^2} \sin m\omega \cdot d\omega = \frac{\pi}{2} e^{-m}.$$

5) Find the Fourier sine transform of

$\frac{e^{-at}}{t}$, $t \neq 0$, $a > 0$.

$$F_s(\omega) = \int_0^\infty f(t)\ \sin\omega t \cdot dt$$

$$F_s(\omega) = \int_0^\infty \frac{e^{-at}}{t} \cdot \sin\omega t \cdot dt \quad \text{——— (1)}$$

We cannot evaluate this integral directly and hence we employ the rule of differentiation under the integral sign.

$$\frac{d}{d\omega}\left[F_s(\omega)\right] = \int_0^\infty \frac{e^{-at}}{t} \cdot \frac{\partial}{\partial\omega}(\sin\omega t) . dt$$

$$= \int_0^\infty \frac{e^{-at}}{\cancel{t}}\ \cancel{t}\ \cos\omega t \cdot dt$$

$$= \left[\frac{e^{-at}}{a^2+\omega^2}(-a\cos\omega t + \omega\sin\omega t)\right]_0^\infty$$

$$= \frac{1}{a^2+\omega^2}(0+a) = \frac{a}{a^2+\omega^2}$$

$$\frac{d}{d\omega}\left[F_s(\omega)\right] = \frac{a}{a^2+\omega^2}$$

By integrating w.r to ω, we get

$$F_s(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right) + c$$

Put $\omega = 0$,

$$F_s(0) = \tan^{-1}(0) + c$$

$F_s(0) = 0$ from ①

and hence $c = 0$.

Thus $\boxed{F_s(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)}$

6) Find the Fourier Sine transform of e^{-at} and hence show that $\int_0^\infty \frac{\omega \sin k\omega}{a^2+\omega^2} d\omega = \frac{\pi}{2} e^{-ak}$ $(k > 0)$.

Solution:

Fourier sine transform is given by

$$F_s\{f(t)\} = \int_0^\infty f(t) \cdot \sin \omega t \cdot dt$$

$$= \int_0^\infty e^{-at} \cdot \sin \omega t \, dt$$

$$= \left[\frac{e^{-at}}{a^2+\omega^2}(-a \sin \omega t - \omega \cos \omega t)\right]_0^\infty$$

$$= \frac{1}{a^2+\omega^2}(0 - 1(-\omega)) = \frac{\omega}{\omega^2+a^2} = F_s(\omega)$$

$$F_s(\omega) = \frac{\omega}{\omega^2+a^2}.$$

Using inversion formula,

$$f(t) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin\omega t \cdot d\omega$$

$$e^{-at} = \frac{2}{\pi} \int_0^\infty \frac{\omega}{a^2+\omega^2} \sin\omega t \cdot d\omega$$

$$\frac{\pi}{2} e^{-at} = \int_0^\infty \frac{\omega}{a^2+\omega^2} \sin \omega t \cdot d\omega$$

changing the variable from t to k, we have

$$\int_0^\infty \frac{\omega}{a^2+\omega^2} \sin \omega k \cdot d\omega = \frac{\pi}{2} e^{-ak}$$

7) Find the Fourier cosine transform of $\frac{1}{1+t^2}$.

Solution.

$$F_c[f(t)] = \int_0^\infty f(t) \cdot \cos(\omega t) \cdot dt$$

$$F_c(\omega) = \int_0^\infty \frac{\cos(\omega t)}{1+t^2} \cdot dt \quad \text{—— (1)}$$

We cannot evaluate the R.H.S directly and hence we employ Leibnitz rule for differentiation under the integral sign.

$$\frac{dF_c(\omega)}{d\omega} = \int_0^\infty \frac{1}{1+t^2} \frac{\partial}{\partial \omega} (\cos \omega t) \cdot dt$$

$$F_c'(\omega) = \int_0^\infty \frac{-t}{1+t^2} \sin(\omega t) \cdot dt.$$

We cannot evaluate R.H.S even now and hence modify the integrand

$$F_c'(\omega) = -\int_0^\infty \frac{t^2}{t(1+t^2)} \sin(\omega t) \cdot dt$$

$$= -\int_0^\infty \frac{1+t^2-1}{t(1+t^2)} \sin(\omega t)\, dt$$

$$= -\int_0^\infty \frac{\sin \omega t}{t}\, dt + \int_0^\infty \frac{\sin \omega t}{t(1+t^2)}\, dt.$$

We note that $\int_0^\infty \frac{\sin \omega t}{t}\, dt = \frac{\pi}{2}$ and hence

we have

$$F_c'(\omega) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin \omega t}{t(1+t^2)}\, dt \qquad \text{—— (2)}$$

Differentiating again by Leibnitz rule

we get

$$F_c''(\omega) = \int_0^\infty \frac{1}{t(1+t^2)} \cdot \frac{\partial}{\partial \omega} (\sin \omega t) \cdot dt$$

$$F_c''(\omega) = \int_0^\infty \frac{1}{\cancel{t}(1+t^2)} \cdot \cancel{t} \cdot \cos\omega t \, dt$$

$$F_c''(\omega) = \int_0^\infty \frac{\cos\omega t}{1+t^2} \cdot dt$$

$$F_c''(\omega) = F_c(\omega) \quad \text{by using (1)}$$

$$F_c''(\omega) - F_c(\omega) = 0.$$

This is a second order D.E of the form

$$(D^2 - 1) F_c(\omega) = 0 \quad \text{where } D = \frac{d}{d\omega}.$$

A.E $\quad m^2 - 1 = 0$

$$m = \pm 1$$

The general solution is given by

$$F_c(\omega) = c_1 e^{\omega} + c_2 e^{-\omega}. \quad \text{——— (3)}$$

We shall find c_1 and c_2

We have $\quad F_c(0) = c_1 + c_2.$

But from (1)

$$F_c(0) = \int_0^\infty \frac{1}{1+t^2} \cdot dt$$

$$= \tan^{-1}(t) \Big]_0^\infty = \frac{\pi}{2}.$$

$$\Rightarrow \quad \pi/2 = c_1 + c_2 \quad \text{——— (4)}$$

Also, from 3

$$F_c'(\omega) = c_1 e^{\omega} - c_2 e^{-\omega}.$$

Put $\omega = 0$

$$F_c'(0) = c_1 - c_2.$$

From 2,
$$F_c'(0) = -\pi/2 + 0 = \frac{-\pi}{2}$$

$\therefore$ we have

$$-\pi/2 = c_1 - c_2 \quad \text{——— (5)}$$

By solving 4 & 5 we have

$$c_1 = 0, \quad c_2 = \pi/2$$

Substituting these values in 3 we have

$$\boxed{F_c(\omega) = \frac{\pi}{2} \cdot e^{-\omega}}.$$

8) Find the Fourier cosine transform of $f(t) = e^{-t^2}$.

Solution:

$$F_c\{f(t)\} = \int_0^\infty f(t) \cdot \cos\omega t \cdot dt.$$

$$F_c(\omega) = \int_0^\infty e^{-t^2} \cdot \cos\omega t \cdot dt. \quad \text{——— (1)}$$

The integral is to be evaluated by using Leibnitz rule for differentiation under integral sign.

Diff. w.r.to ω, we have

$$\frac{dF_c(\omega)}{d\omega} = \int_0^\infty \frac{\partial}{\partial\omega}\left(e^{-t^2}\cos\omega t\right) \cdot dt$$

$$\frac{dF_c(\omega)}{d\omega} = \int_0^\infty e^{-t^2} \cdot -t \cdot \sin(\omega t)\, dt$$

$$2 \cdot \frac{dF_c(\omega)}{d\omega} = \int_0^\infty \left[e^{-t^2} \cdot (-2t) \cdot\right] \sin\omega t \cdot dt.$$

Integrating RHS by parts we have.

$u = \sin\omega t$. $\quad dv = e^{-t^2}(-2t) \cdot dt$

$du = \omega \cdot \cos\omega t \quad v = e^{-t^2}$.

$$2 \cdot \frac{dF_c(\omega)}{d\omega} = \left[\sin\omega t \cdot e^{-t^2}\right]_0^\infty - \int_0^\infty e^{-t^2} \cdot \omega\cos\omega t \cdot dt$$

$$2\frac{dF_c(\omega)}{d\omega} = 0 - \omega \cdot \int_0^\infty e^{-t^2} \cdot \cos\omega t \cdot dt.$$

$$2 \cdot \frac{dF_c(\omega)}{d\omega} = -\omega \cdot F_c(\omega)$$

$$2 \cdot \frac{dF_c(\omega)}{F_c(\omega)} = -\omega \, d\omega.$$

$$\frac{dF_c(\omega)}{F_c(\omega)} = -\frac{\omega}{2} \cdot d\omega.$$

On integration

$$\log F_c(\omega) = -\frac{1}{2} \cdot \frac{\omega^2}{2} + \log K.$$

$$\log\left(\frac{F_c(\omega)}{K}\right) = -\frac{\omega^2}{4}$$

$$\frac{F_c(\omega)}{K} = e^{-\frac{\omega^2}{4}}$$

$$F_c(\omega) = K \cdot e^{-\omega^2/4} \quad \text{——— (2)}$$

To find K, put $\omega = 0$ in (1) & (2).

$$F_c(0) = K \cdot e^0 \qquad (\text{From } (2))$$

$$F_c(0) = K$$

$$F_c(0) = \int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} \quad (\text{From } (1))$$

Hence $K = \dfrac{\sqrt{\pi}}{2}.$

$\therefore$ Sub in (2) we have

$$\boxed{F_c(\omega) = \frac{\sqrt{\pi}}{2} \cdot e^{-\frac{\omega^2}{4}}}$$

NOTE:

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

9) Find the Fourier cosine transform of

$$f(t) = \begin{cases} t & \text{for} \quad 0 < t < 1 \\ 2-t & \text{for} \quad 1 < t < 2 \\ 0 & \text{for} \quad t > 2 \end{cases}$$

Solution :

$$F_c\{f(t)\} = \int_0^{\infty} f(t) \cdot \cos(\omega t)\, dt$$

$$= \int_0^1 t \cdot \cos(\omega t)\, dt \quad \int_1^2 (2-t)\cos(\omega t)\, dt$$

$$= \left[t \cdot \frac{\sin \omega t}{\omega} + \frac{\cos \omega t}{\omega^2} \right]_0^1$$

$$+ \left\{ (2-t) \cdot \frac{\sin \omega t}{\omega} - \frac{\cos \omega t}{\omega^2} \right\}_1^2$$

$$= \left[\cancel{\frac{\sin \omega}{\omega}} + \frac{\cos \omega}{\omega^2} - \frac{1}{\omega^2} \right] + \left[0 - \frac{\cos 2\omega}{\omega^2} - \cancel{\frac{\sin \omega}{\omega}} + \frac{\cos \omega}{\omega^2} \right]$$

$$= \frac{2\cos \omega}{\omega^2} - \frac{1}{\omega^2} - \frac{\cos 2\omega}{\omega^2}$$

$$\boxed{F_c(\omega) = \frac{1}{\omega^2}\left[2\cos \omega - \cos 2\omega - 1 \right]}$$

10) Find the finite Fourier sine transform of

$$f(t) = \begin{cases} -t & \text{for } 0<t<c \\ \pi - t & \text{for } c<t<\pi \end{cases}$$

Solution

Finite Fourier sine transform of $f(t)$ is

$$F_s(n) = \int_0^l f(t) \cdot \sin \frac{n\pi t}{l} \cdot dt$$

$$= \int_0^{\pi} f(t) \sin nt \cdot dt$$

$$= \int_0^c -t \cdot \sin nt \, dt + \int_c^{\pi} (\pi - t) \sin nt \cdot dt$$

$$= -\left[t \cdot \left(\frac{-\cos nt}{n} \right) - (1)\left(\frac{-1}{n}\right) \frac{\sin nt}{n} \right]_0^c$$

$$+ \left[(\pi - t)\left(\frac{-\cos nt}{n} \right) - (-1) \cdot \left(\frac{-1}{n} \right) \frac{\sin nt}{n} \right]_c^{\pi}$$

$$= -\left[-\frac{c \cdot \cos nc}{n} + \frac{1}{n^2} \sin nc - 0 \right] + \left[0 + (\pi - c) \frac{\cos nc}{n} + \frac{1}{n^2} \sin nc \right].$$

$$= + \frac{c \cos nc}{n} - \frac{1}{n^2} \sin nc + \frac{\pi \cos nc}{n} - \frac{c \cos nc}{n} + \frac{1}{n^2} \sin nc$$

$$= \frac{\pi \cos nc}{n}.$$

$$\therefore \boxed{F_s(n) = \frac{\pi \cos nc}{n}.}$$

11) If the Fourier sine transform of $f(t) = \frac{1-\cos n\pi}{n^2\pi^2}$, $(0 \le t \le \pi)$, find the inverse finite Fourier sine transform.

Solution

Given $F_S(n) = \frac{1-\cos n\pi}{n^2\pi^2}$

Inverse Finite Fourier sine transform.

$$f(t) = \frac{2}{l} \sum_{n=1}^{\infty} F_S(n) \cdot \sin \frac{n\pi t}{l}$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2\pi^2} \sin nt.$$

$$= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2} \sin nt.$$

12) Find the finite Fourier sine transform of $f(t) = t(\pi - t)$ in $0 < t < \pi$.

Solution.

$$F_S(n) = \int_0^l f(t) \cdot \sin \frac{n\pi t}{l} \cdot dt.$$

$$= \int_0^{\pi} t(\pi - t) \sin nt \cdot dt.$$

$$= \left[t(\pi - t)\cdot\left[\frac{-\cos nt}{n}\right]\right]_0^{\pi} - \left[(\pi - 2t)\left(\frac{-1}{n}\right)\frac{\sin nt}{n}\right]_0^{\pi}$$

$$+ \left[(-2)\left(\frac{-1}{n^2}\right)\cdot\left(\frac{-\cos nt}{n}\right)\right]_0^{\pi}$$

$$= 0 - 0 - \frac{2}{n^3}(\cos n\pi - \cos 0)$$

$$= \frac{-2}{n^3}((-1)^n - 1) = \frac{2}{n^3}(1 - (-1)^n).$$

$$\boxed{F_s(n) = \frac{2}{n^3}(1 - (-1)^n)}$$

13) Find the finite Fourier sine transform of $f(t) = 2t$ in $0 < t < 4$.

Solution.

Finite Fourier Sine transform is defined as.

$$F_s(n) = \int_0^l f(t)\cdot \sin\frac{n\pi t}{l}\,dt$$

$$= \int_0^4 2t\cdot \sin\frac{n\pi t}{4}\cdot dt.$$

$$= \left[2t\left(\frac{-\cos\frac{n\pi t}{4}}{\frac{n\pi}{4}}\right) - 2\cdot\left(\frac{-4}{n\pi}\right)\frac{\sin\frac{n\pi t}{4}}{\frac{n\pi}{4}}\right]_0^4$$

$$= \left[8 \left(\frac{4}{n\pi}\right) \cos n\pi + \left(\frac{8}{n\pi}\right)\left(\frac{4}{n\pi}\right) \sin n\pi \right]$$

$$- \left[9(8)\left(\frac{4}{n\pi}\right) \cos 0 + \left(\frac{8}{n\pi}\right)\left(\frac{4}{n\pi}\right) \sin 0 \right]$$

$$= \frac{-32}{n\pi}(-1)^n + 0 \Rightarrow \frac{32}{n\pi}(-1)^{n+1}$$

$$\boxed{F_s(n) = \frac{32}{n\pi}(-1)^{n+1}}$$

14) Determine the inverse Finite Fourier Sine transform of $\frac{16(-1)^{n-1}}{n^3}$, $n = 1, 2, 3 \ldots$ and $0 < t < 8$

Given

$$F_s(n) = \frac{16(-1)^{n-1}}{n^3}.$$

Inverse Finite Fourier sine transform

$$f(t) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) . \sin \frac{n\pi t}{l}$$

$$= \frac{2}{8} \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{n^3} \sin \frac{n\pi t}{8}$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin \frac{n\pi t}{8}$$

15) Find the finite Fourier sine transform of

$$f(t) = \begin{cases} 0 & 0 \leq t < \pi/2 \\ 1 & \pi/2 \leq t \leq \pi \end{cases}$$

Solution

$$S_n[f(t)] = F_s(n) = \int_0^l f(t) \sin \frac{n\pi t}{l} \, dt$$

$$= \int_0^{\pi} f(t) \sin nt \, dt$$

$$= \int_0^{\pi/2} 0 \cdot \sin nt \, dt + \int_{\pi/2}^{\pi} 1 \cdot \sin nt \, dt$$

$$= \left[\frac{-\cos nt}{n} \right]_{\pi/2}^{\pi} = \frac{-1}{n} \left[\cos n\pi - \cos n\pi/2 \right]$$

$$= \frac{-1}{n} \left[(-1)^n - \cos \frac{n\pi}{2} \right]$$

16) Find the finite Fourier cosine transform of $f(t) = e^{at}$ in $0 < t < l$.

Finite Fourier Cosine transform is given by

$$F_c(n) = \int_0^l f(t) \cdot \cos\left(\frac{n\pi t}{l}\right) dt.$$

$$= \int_0^l e^{at} \cdot \cos \frac{n\pi t}{l} \cdot dt$$

$$= \left[\frac{e^{at}}{a^2 + \left(\frac{n\pi}{l}\right)^2} \left[a \cos \frac{n\pi}{l} t + b \sin \frac{n\pi t}{l} \right] \right]_0^l$$

$$= \frac{e^{al} \cdot l^2}{l^2 a^2 + n^2 \pi^2} (a \cos n\pi) - \frac{l^2}{a^2 + n^2 \pi^2} (a \cos 0)$$

$$\Rightarrow \frac{a l^2}{l^2 a^2 + n^2 \pi^2} \left[e^{al} \cdot (-1)^n - 1 \right]$$

Thus

$$\boxed{F_c(n) = \frac{a l^2}{l^2 a^2 + n^2 \pi^2} \left[e^{al} (-1)^n - 1 \right]}$$

17) Find the finite Fourier cosine transform of $f(t) = t(\pi - t)$ in $0 < t < \pi$.

Solution :

$$F_c(n) = \int_0^l f(t) \cdot \cos \frac{n\pi t}{l} \, dt$$

NOTE: (For previous problem)

$$\int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$F_c(n) = \int_0^{\pi} t(\pi - t) \cdot \cos nt \, dt$$

$$= \left[t(\pi - t) \frac{\sin nt}{n} - (\pi - 2t) \cdot \left(\frac{1}{n}\right) \left(\frac{-\cos nt}{n}\right) + (-2)\left(\frac{-1}{n^2}\right) \cdot \frac{\sin nt}{n} \right]_0^{\pi}$$

$$= + \frac{1}{n^2} \left[(-\pi) \cdot \cos n\pi - \pi \cos 0 \right]$$

$$= \frac{\pi}{n^2} \left[-(-1)^n - 1 \right] = \frac{-\pi}{n^2} \left[1 + (-1)^n \right]$$

$$\therefore \boxed{F_c(n) = \frac{-\pi}{n^2} \left[1 + (-1)^n \right]}$$

18) Find $f(t)$ if the finite Fourier cosine transform of $f(t)$ is

a) $F_c(n) = \begin{cases} \frac{1}{2n} \sin \frac{n\pi}{2} & \text{for } n = 1, 2, 3, \ldots \\ \frac{\pi}{4} & \text{for } n = 0 \end{cases}$

in $0 < t < 2\pi$

b) $F_c(n) = \begin{cases} \cos \frac{2n\pi}{3} & \text{for } n = 1, 2, 3 \ldots \\ 1 & \text{for } n = 0 \end{cases}$ in $0 < t < 1$

Inverse Finite Fourier cosine transform of $F_c(n)$

a) $$f(t) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cdot \cos \frac{n\pi t}{l}$$

Here $F_c(0) = \frac{\pi}{4}$, $l = 2\pi$

$$f(t) = \frac{1}{2\pi} \cdot \frac{\pi}{4} + \frac{2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{2n} \sin \frac{n\pi}{2} \cdot \cos \frac{n\pi t}{2\pi}$$

$$f(t) = \frac{1}{8} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos \frac{n\cancel{\pi} t}{2\cancel{\pi}}$$

b) The inverse Finite Fourier cosine transform of $F_c(n)$ is

$$f(t) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cdot \cos \frac{n\pi t}{l}$$

Here $F_c(0) = 1$ and $l = 1$.

$$= 1 + 2 \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi}{3}\right) \cos n\pi t.$$

19) Find the finite Fourier cosine transform of $f(t) = 2t$ in $0 < t < 4$.

$$F_c(n) = \int_0^l f(t) \cos \frac{n\pi t}{l} \, dt.$$

$$= \int_0^4 2t \cdot \cos \frac{n\pi t}{4}\, dt$$

$$= \left[2t\left(\frac{\sin\frac{n\pi t}{4}}{\frac{n\pi}{4}}\right) - 2\cdot\left(\frac{4}{n\pi}\right)\cdot \frac{-\cos\frac{n\pi t}{4}}{\frac{n\pi}{4}} \right]_0^4$$

$$= +\frac{32}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{32}{n^2\pi^2}\left[(-1)^n - 1\right]$$

20) Find the finite Fourier sine and cosine transforms of $f(t) = \cos at$ in $0 < t < \pi$

a) Fourier sine transform

$$F_s(n) = \int_0^{\pi} \cos at \ \sin nt\, dt$$

$$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{1}{2}\int_0^{\pi} \sin(n+a)t + \sin(n-a)t \cdot dt$$

$$= -\frac{1}{2}\left[\frac{\cos(n+a)t}{n+a} + \frac{\cos(n-a)t}{n-a}\right]_0^{\pi}$$

$$= -\frac{1}{2}\left[\frac{\cos(n+a)\pi - 1}{n+a} + \frac{\cos(n-a)\pi - 1}{n-a}\right]$$

Note

$$\cos(n+a)\pi = \cos(n\pi + a\pi)$$

$$= \cos n\pi \cos a\pi + \sin n\pi \sin a\pi$$

$$= (-1)^n \cos a\pi$$

$$\cos(n-a)\pi = \cos(n\pi - a\pi)$$

$$= \cos n\pi \cos a\pi - \sin n\pi \sin a\pi$$

$$= (-1)^n \cos a\pi$$

Therefore

$$F_s(n) = \frac{-1}{2}\left[\frac{(-1)^n \cos a\pi - 1}{n+a} + \frac{(-1)^n \cos a\pi - 1}{n-a}\right]$$

$$= -\left[\frac{(-1)^n \cos a\pi - 1}{2}\right]\left[\frac{1}{n+a} + \frac{1}{n-a}\right]$$

$$= -\left[\frac{(-1)^n \cos a\pi - 1}{2}\right]\left[\frac{n-\cancel{a}+n+\cancel{a}}{n^2-a^2}\right]$$

$$= \frac{-2n}{2(n^2-a^2)}\left[(-1)^n \cos a\pi - 1\right]$$

$$= \frac{-n}{n^2-a^2}\left[(-1)^n \cos a\pi - 1\right]$$

$$F_s(n) = \frac{+n}{n^2-a^2}\left[1 - (-1)^n \cos a\pi\right]$$

(OR)

$$F_S(n) = \frac{-1}{2}\left[\frac{(-1)^{n+a}-1}{n+a} + \frac{(-1)^{n-a}-1}{n-a}\right]$$ if a is an integer

If $n-a$ is even then $n+a$ is also even

$$\therefore F_S(n) = \frac{-1}{2}\left[0\right] = 0$$

If $n-a$ is odd then $n+a$ is also odd

$$= \frac{-1}{2}\left[\frac{-2}{n+a} \quad \frac{-2}{n-a}\right]$$

$$F_S(n) = \frac{2n}{n^2-a^2}.$$

$$\therefore F_S(n) = \begin{cases} 0 & \text{if } n \pm a \text{ is even} \\ \dfrac{2n}{n^2-a^2} & \text{if } n \pm a \text{ is odd} \end{cases}$$

Finite Fourier Sine transform

$$F_S(n) = \frac{n}{n^2-a^2}\left(1-(-1)^n \cos a\pi\right)$$

$$= \begin{cases} 0 & \text{if } n \pm a \text{ is even} \\ \dfrac{2n}{n^2-a^2} & \text{if } n \pm a \text{ is odd.} \end{cases}$$ and if a is an integer

b) $$F_c(n) = \int_0^{\pi} \cos at \cos nt \, dt$$

$$= \frac{1}{2}\int_0^{\pi} \left(\cos(n+a)t + \cos(n-a)t\right) dt$$

$$= \frac{1}{2}\left[\frac{\sin(n+a)t}{n+a} + \frac{\sin(n-a)t}{n-a}\right]_0^{\pi}$$

Note

$$\sin(n \pm a)\pi = \sin n\pi \cos a\pi \pm \cos n\pi \sin a\pi$$

$$= \pm(-1)^n \sin a\pi$$

$$F_c(n) = \frac{1}{2}\left[\frac{(-1)^n \sin a\pi}{n+a} - \frac{(-1)^n \sin a\pi}{n-a}\right]$$

$$= \frac{(-1)^n \sin a\pi}{2}\left[\frac{1}{n+a} - \frac{1}{n-a}\right]$$

$$\frac{\cancel{n}-a-\cancel{n}-a}{(n+a)(n-a)}$$

$$= \frac{(-1)^n \sin a\pi}{2}\left[\frac{-2a}{n^2-a^2}\right]$$

$$\boxed{F_c(n) = \frac{a}{n^2-a^2}(-1)^{n+1}\sin a\pi}$$

$F_c(n) = 0$ if a is an integer.

Fourier transform of elementary functions

1) Find the fourier transform of dirac delta function $\delta(t)$.

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t}\, dt.$$

Dirac delta function $\delta(t)$ is defined as.

$$\delta(t) = \left\{ \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[u(t) - u(t-\epsilon)\right] \right.$$

Where $u(t)$ is the unit step function.

We have

$$u(t) - u(t-\epsilon) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t < \epsilon \\ 0 & t \geq \epsilon \end{cases}.$$

Now,

$$F\{f(t)\} = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} (u(t) - u(t-\epsilon)) \cdot e^{-i\omega t}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{0}^{\epsilon} 1 \cdot e^{-i\omega t}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{\epsilon}.$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{e^{-i\omega\epsilon} - e^{0}}{-i\omega}\right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{1 - e^{-i\omega\epsilon}}{i\omega} \right].$$

$$= \lim_{\epsilon \to 0} \frac{+\cancel{i\omega} \cdot e^{-i\omega\epsilon}}{\cancel{i\omega}}.$$ L Hospital Rule.

$$= \underline{\underline{1}}$$

$$\boxed{\begin{aligned} \therefore\ F\{\delta(t)\} &= 1 \\ F^{-1}\{1\} &= \delta(t). \end{aligned}}$$

NOTE:

We Know That $\int_0^\infty f(t) \cdot \delta(t-a)\, dt = f(a).$

Now, this gets modified as

$$\int_{-\infty}^{\infty} f(t) \cdot \delta(t-a)\, dt = f(a).$$

2) Find the Fourier transform of $e^{-a|t|}$; $-\infty < t < \infty$ $a > 0$.

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} \cdot e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} \cdot e^{-i\omega t}\, dt.$$

$$= \int_{-\infty}^{0} e^{(a-i\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+i\omega)t}\, dt$$

$$= \left[\frac{e^{(a-i\omega)t}}{a-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega}.$$

$$= \frac{a+i\omega+a-i\omega}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$$

$$\boxed{F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}.}$$

Properties of Fourier transform.

Linearity property:

$$F\{af(t) + b\,g(t)\} = a\cdot F\{f(t)\} + b\cdot F\{g(t)\}.$$

provided the Fourier transform of $f(t)$ and $g(t)$ exists.

3) Find the Fourier transform of Constant function $f(t) = 1$

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} \, dt$$

$$F\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-i\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} (\cos\omega t - i\sin\omega t) \, dt$$

$$= \int_{-\infty}^{\infty} \cos\omega t \cdot dt - i \int_{-\infty}^{\infty} \sin\omega t \cdot dt.$$

$$F\{1\} = 2\int_{0}^{\infty} \cos\omega t \cdot dt \qquad \text{—— (1)}$$

$\sin\omega t \rightarrow$ odd fn

$\Rightarrow \int_{-\infty}^{\infty}$ odd fn $= 0$

$\cos\omega t \rightarrow$ even fn

$\Rightarrow \int_{-\infty}^{\infty}$ even fn $= 2\int_{0}^{\infty}$ even fn

By definition of inverse fourier transform we have.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \cdot d\omega.$$

We know. $F\{\delta(t)\} = 1$

$f(t) = 1 \Rightarrow F(\omega) = 1.$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \; e^{i\omega t} \, d\omega$$

$$2\pi \, \delta(t) = \int_{-\infty}^{\infty} (\cos\omega t + i \sin\omega t) \, d\omega$$

$$= \int_{-\infty}^{\infty} \cos\omega t \, d\omega + i \underbrace{\int_{-\infty}^{\infty} \sin\omega t \, d\omega}_{0}$$

$$2\pi \delta(t) = 2 \int_{0}^{\infty} \cos\omega t \cdot d\omega \quad \text{——— } (2)$$

Substituting in (1) we have.

$$\boxed{F\{1\} = 2\pi \delta(\omega)}$$

NOTE:

$$2\pi \, \delta(t) = 2 \int_{0}^{\infty} \cos\omega t \cdot d\omega$$

Put $\omega = \lambda$

$$2\pi \delta(t) = 2 \int_{0}^{\infty} \cos\lambda t \cdot d\lambda$$

Put $t = \omega$

$$2\pi \delta(\omega) = 2 \int_{0}^{\infty} \cos\lambda\omega \cdot d\lambda.$$

Shifting property

(Shifting on t-axis).

If $F\{f(t)\} = F(\omega)$ and t_0 is any real number then

$$F\{f(t-t_0)\} = F(\omega)\cdot e^{-i\omega t_0}.$$

Problem:

Find the Fourier transform of

$f(t) = e^{-a|t-2|}$.

We know that

$$F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}$$

Also,

$$F\{f(t-t_0)\} = F(\omega)\, e^{-i\omega t_0}$$

$$\therefore F\{e^{-a|t-2|}\} = \frac{2a}{a^2+\omega^2}\cdot e^{-i\omega(2)}.$$

$$= e^{-2i\omega}\cdot\left(\frac{2a}{a^2+\omega^2}\right).$$

Frequency Shifting:

If $F\{f(t)\} = F(\omega)$ and ω_0 is any real number, then

$$F\{e^{i\omega_0 t} f(t)\} = F(\omega-\omega_0).$$

Problem

* Find the Fourier transform of e^{iat}.

Let $f(t) = 1$. .. Then $F\{1\} = 2\pi\,\delta(\omega)$

By shifting property we have

$$F\{f(t)\cdot e^{iat}\} = F(\omega-\omega_0).$$

$$\boxed{F\{1\cdot e^{iat}\} = 2\pi\,\delta(\omega-a)}$$

* Find the Fourier transform of $\cos at$

$$f(t) = \cos at$$

$$F\{\cos at\} = F\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$$

$$= \frac{1}{2}\left[F\{e^{iat}\} + F\{e^{-iat}\}\right]$$ by linearity property.

$$= \frac{1}{2} \left[2\pi\, \delta(\omega - a) + 2\pi\, \delta(\omega + a) \right]$$

$$\boxed{F\{\cos at\} = \pi \left[\delta(\omega - a) + \delta(\omega + a) \right]}$$

* Find the Fourier transform of $\sin at$.

$$F\{\sin at\} = F\left\{ \frac{e^{iat} - e^{-iat}}{2i} \right\}$$

$$= \frac{1}{2i} \left[F\{e^{iat}\} - F\{e^{-iat}\} \right]$$

$$= \frac{1}{2i} \left[2\pi\, \delta(\omega - a) - 2\pi\, \delta(\omega + a) \right]$$

$$= -i\pi \left[\delta(\omega - a) - \delta(\omega + a) \right]$$

$$\boxed{F\{\sin at\} = i\pi \left[\delta(\omega + a) - \delta(\omega - a) \right]}$$

Modulation Theorem.

If $F\{f(t)\} = F(\omega)$ and ω_0 is any real number then

$$F\{f(t)\cdot \cos(\omega_0 t)\} = \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]$$

and

$$F\{f(t)\ \sin(\omega_0 t)\} = \frac{i}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right]$$

Problem

* Find Fourier transform of

$f(t) = 2e^{-3|t|} \cos 4t$.

$$F\{2e^{-3|t|}\} = 2\cdot F\{e^{-3|t|}\}$$

$$= 2\cdot \frac{2(3)}{3^2+\omega^2} \qquad \left[F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}\right]$$

$$= \frac{12}{9+\omega^2}.$$

$$F\{2e^{-3|t|}\cos 4t\} = \frac{1}{2}\left[F(\omega+4) + F(\omega-4)\right]$$

$$= \frac{1}{2}\left\{\frac{12}{9+(\omega+4)^2} + \frac{12}{9+(\omega-4)^2}\right\}$$

by Modulation Theorem

$$= \frac{12}{2}\left\{\frac{\omega^2-8\omega+25+\omega^2+8\omega+25}{(\omega^2+8\omega+25)(\omega^2-8\omega+25)}\right\}$$

$$= \frac{2(\omega^2+25)}{6\ (\omega^4}$$

$$= 6\left\{ \frac{2\omega^2 + 50}{(\omega^2+25)^2 - 64\,\omega^2} \right\}$$

$$\boxed{F\{e^{-3|t|}\cos 4t\} = \frac{12(\omega^2+25)}{\omega^4 - 14\omega^2 + 625}}$$

* **Fourier transforms of derivatives**

If $F\{f(t)\} = F(\omega)$ then

$$F\{f^{(n)}(t)\} = (i\omega)^n \cdot F(\omega).$$

NOTE:

$$F\{f'(t)\} = i\omega \cdot F(\omega)$$

$$F\{f''(t)\} = -\omega^2 \cdot F(\omega).$$

$$F\{f'''(t)\} = -i\omega\, F(\omega) \text{ and so on.}$$

Problem.

Find the Solution of the differential equation

$$y' - 2y = H(t)\, e^{-2t} \qquad -\infty < t < \infty.$$

using Fourier transforms, where $H(t) = u_0(t)$ is the unit step function.

Applying the Fourier transform to the differential equation, we get

$$F\{y'\} - 2F\{y\} = F\{H(t) \cdot e^{-2t}\}.$$

$$i\omega\, Y(\omega) - 2Y(\omega) = \frac{1}{2+i\omega}$$

$$(i\omega - 2)Y(\omega) = \frac{1}{2+i\omega}$$

where $F\{y(t)\} = Y(\omega)$

$$Y(\omega) = \frac{1}{(2+i\omega)(i\omega-2)}.$$

$$Y(\omega) = \frac{-1}{(2+i\omega)(2-i\omega)}$$

$$F^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{-1}{4+\omega^2}\right\}$$

$$y(t) = \frac{-1}{4} e^{-2|t|}.$$

This solution can also be written as.

$$y(t) = \begin{cases} \frac{-1}{4} e^{2t} & t < 0 \\ \frac{-1}{4} e^{-2t} & t > 0 \end{cases}.$$

NOTE :

$$H(t) \cdot e^{-2t} = \begin{cases} 0 & t < 0 \\ e^{-2t} & t \geq 0. \end{cases}$$

$$F\{H(t) \cdot e^{-2t}\} = \int_0^\infty e^{-2t} \cdot e^{-i\omega t}\, dt = \left[\frac{e^{-(2+i\omega)t}}{-(2+i\omega)}\right]_0^\infty$$

$$= \frac{1}{2+i\omega}.$$

Symmetry Property of Fourier transforms

Let $F\{f(t)\} = F(\omega)$ Then

$$F\{F(t)\} = 2\pi f(-\omega)$$

Example

$$F\left\{\frac{6}{t^2+9}\right\} = 2\pi e^{-3\omega}$$

$F(\omega)$ $f(t)$

$$\frac{6}{\omega^2+9} = e^{-3|t|}$$

That is, If $e^{-3|t|} \xleftrightarrow{F.T} \frac{6}{\omega^2+9}$

$$\frac{6}{t^2+9} \xleftrightarrow{F.T} 2\pi e^{-3|-\omega|}$$

$$2\pi e^{-3\omega}$$

Problem.

Find the Fourier transform of $f(t) = \frac{1}{5+it}$

We shall use the symmetry property to find the Fourier transform.

We know That

$$F\{H(t)e^{-5t}\} = \frac{1}{5+i\omega}$$

$$F\left\{\frac{1}{5+it}\right\} = 2\pi \cdot H(-\omega)e^{-5(-\omega)}$$

$$= 2\pi H(-\omega)e^{5\omega}$$

Therefore

$$F\left\{\frac{1}{5+it}\right\} = F(\omega) = \begin{cases} 2\pi e^{5\omega} & \omega \le 0 \\ 0 & \omega > 0 \end{cases}$$

* Find the Fourier transform of $e^{-a^2t^2}$, Hence deduce That $e^{-t^2/2}$ is self reciprocal in respect of Fourier transform.

Also, find the Fourier transform of

(i) $e^{-2(t-3)^2}$ (ii) $e^{-t^2/2}\cos 3t$ (iii) $e^{-t^2}\cos 3t$

Solution.

$$F\{e^{-a^2t^2}\} = \int_{-\infty}^{\infty} e^{-a^2t^2} \cdot e^{-i\omega t}\, dt$$

$A = at$

$2AB = i\omega t$

$2at\, B = i\omega t$

$B = \frac{i\omega \cancel{t}}{2a\cancel{t}}$

$$= \int_{-\infty}^{\infty} e^{-(a^2t^2 + i\omega t)}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\left(a^2t^2 + i\omega t + \left(\frac{i\omega}{2a}\right)^2 - \left(\frac{i\omega}{2a}\right)^2\right)}\, dt$$

$$= e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(at + \frac{i\omega}{2a}\right)^2}\, dt$$

Put

$$at + \frac{i\omega}{2a} = u.$$

$$a \cdot dt = du.$$

$$\therefore F\{e^{-a^2t^2}\} = e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \cdot \frac{du}{a}.$$

We know that

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$\therefore F\{e^{-a^2t^2}\} = \frac{e^{\frac{-\omega^2}{4a^2}} \cdot \sqrt{\pi}}{a}.$$

Taking $a^2 = 1/2$

$$F \cdot \{e^{-t^2/2}\} = \frac{e^{\frac{-\omega^2}{2}} \cdot \sqrt{\pi}}{\sqrt{1/2}}$$

$$F \cdot \{e^{-t^2/2}\} = \sqrt{2\pi} \cdot e^{\frac{-\omega^2}{2}}.$$

Fourier transform of $e^{-t^2/2}$ is a constant times $e^{-\omega^2/2}$. Also the functions $e^{-t^2/2}$ and $e^{-\omega^2/2}$ are the same.

Hence it follows that $e^{-t^2/2}$ is self-reciprocal under the Fourier transform.

b) i) $F\{e^{-2(t-3)^2}\}$

$f(t) = e^{-t^2/2}$

$f(2t) = e^{-\frac{(2t)^2}{2}}$

$$F\{e^{-2t^2}\} = F\{e^{-\frac{(2t)^2}{2}}\}$$

∴ By change of scale property

$$F\{f(2t)\} = \frac{1}{2}F\left(\frac{\omega}{2}\right)$$

$$F\cdot\{e^{-\frac{(2t)^2}{2}}\} = \frac{1}{2}\cdot\left(\sqrt{2\pi}\, e^{-\frac{(\omega/2)^2}{2}}\right)$$

$$F\cdot\{e^{-2t^2}\} = \sqrt{\frac{\pi}{2}}\, e^{-\omega^2/8}$$

By shifting property we have

$$F\cdot\{f(t-t_0)\} = F(\omega)\cdot e^{-i\omega t_0}$$

$$F\cdot\{e^{-2(t-3)^2}\} = \sqrt{\frac{\pi}{2}}\, e^{-\frac{\omega^2}{8}}\cdot e^{-i\omega(3)}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\left(\frac{\omega^2}{8}+i3\omega\right)}$$

ii) Also by modulation theorem

$$F\{f(t)\cdot\cos\omega_0 t\} = \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]$$

$$F\cdot\{e^{-t^2/2}\cos 3t\} = \frac{1}{2}\left[\sqrt{2\pi}\cdot e^{-\frac{(\omega+3)^2}{2}} + \sqrt{2\pi}\, e^{-\frac{(\omega-3)^2}{2}}\right]$$

$$= \frac{\sqrt{2\pi}}{2}\left[e^{-\frac{(\omega+3)^2}{2}} + e^{-\frac{(\omega-3)^2}{2}}\right]$$

$$= \sqrt{\frac{\pi}{2}}\left[e^{-\frac{(\omega+3)^2}{2}} + e^{-\frac{(\omega-3)^2}{2}}\right]$$

(OR)

b) (i) $F\{e^{-2(t-3)^2}\}$

We know

$$F\{e^{-a^2t^2}\} = \frac{\sqrt{\pi}}{a} e^{\frac{-\omega^2}{4a^2}} \quad \text{———} (*)$$

Put $a^2 = 2$

$$\therefore F\{e^{-2t^2}\} = \frac{\sqrt{\pi}}{\sqrt{2}} e^{-\frac{\omega^2}{8}}$$

$$F\{e^{-2(t-3)^2}\} = \sqrt{\frac{\pi}{2}}\, e^{-\left(\frac{\omega^2}{8} + 3i\omega\right)}$$

Since $F\{f(t-t_0)\} = F(\omega)e^{-i\omega t_0}$.

(ii) $F\{e^{-t^2}\cos 3t\}$.

$$F\{e^{-t^2}\} = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} \quad \left(\text{Put } a^2 = 1 \text{ in } (*)\right)$$

By Modulation.

$$\therefore F\{e^{-t^2}\cos 3t\} = \frac{1}{2}\left\{\sqrt{\pi}\cdot e^{-\frac{(\omega+3)^2}{4}} + \sqrt{\pi}\, e^{-\frac{(\omega-3)^2}{4}}\right\}$$

$$= \frac{\sqrt{\pi}}{2}\left\{e^{-\frac{(\omega+3)^2}{4}} + e^{-\frac{(\omega-3)^2}{4}}\right\}$$

* Find the Fourier sine transform of

$$f(t) = \begin{cases} \sin t & 0 < t < a \\ 0 & t > a \end{cases} \quad \text{for } \omega = 1$$

Solution

$$F_s\{f(t)\} = \int_0^{\infty} f(t) \sin \omega t \cdot dt$$

$$= \int_0^{a} \sin t \cdot \sin \omega t \, dt$$

$$\sin A \cdot \sin B = \frac{1}{2} \cos(A-B) - \cos(A+B)$$

$$= \frac{1}{2} \int_0^{a} \cos(\omega t - t) - \cos(\omega t + t) \, dt$$

$$= \frac{1}{2} \left[\frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1} \right]_0^a$$

$$F_s\{f(t)\} = \frac{1}{2} \; \frac{\sin(\omega-1)a}{\omega-1} - \frac{\sin(\omega+1)a}{\omega+1}.$$

For $\omega = 1$.

$$F_s\{f(t)\} = \frac{1}{2} \cdot \left[\lim_{\omega \to 1} \frac{\sin(\omega-1)a}{\omega-1} - \frac{\sin 2a}{2} \right]$$

$$F_s\{f(t)\} = \frac{1}{2}\left[\lim_{\omega \to 1} \frac{a\cos(\omega-1)a}{1} - \frac{\sin 2a}{2}\right]$$

By L Hospital Rule

$$= \frac{1}{2}\left[a - \frac{\sin 2a}{2}\right]$$