Unit-1 class-8

1. Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ ans: $\frac{16a^3}{3}$

y varies from 0 to $\sqrt{a^2-x^2}$
z varies from 0 to $\sqrt{a^2-x^2}$
x varies from 0 to a

$$8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} [z]_0^{a^2-x^2}\, dy\,dx$$

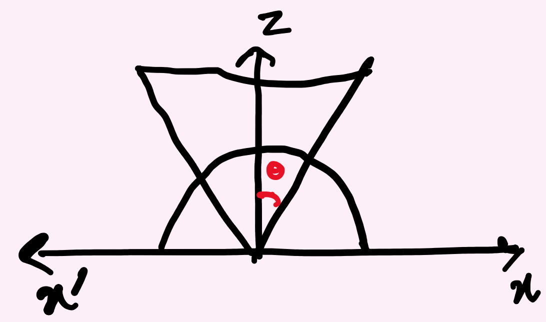
$$= 8\int_0^a (a^2-x^2)[y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= 8\int_0^a (a^2-x^2)\, dx \quad = 8\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \left(a^3-\frac{a^3}{3}\right)\times 8 = \frac{16a^3}{3}$$

2. Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$ above XY plane. ans: $\frac{\pi a^3}{3}(2-\sqrt{2})$

$x = r\sin\theta\cos\phi$
$y = r\cos\theta\sin\phi$
$z = r\cos\theta$



$r \to 0$ to a
$\phi \to 0$ to 2π
$\theta \to 0$ to $\pi/4$

$x^2+y^2=z^2$
$x^2+y^2+z^2 = 2z^2$
$r^2 = 2r^2\cos^2\theta$
$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^a r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^a \sin\theta\, d\theta\, d\phi$$

$$= \frac{a^3}{3}(-\cos\theta)_0^{\pi/4} \times 2\pi$$

$$= \frac{\pi a^3}{3}\left(\frac{-1}{\sqrt{2}}+1\right)\times 2$$

$$= \frac{\pi a^3}{3}\left(2-\sqrt{2}\right)$$

3. Find the average value of $f(x,y,z)=x+y+z$, using triple integrals over the region D= $\{(x,y,z)\ |0\le x\le 1, 0\le y\le 3, 0\le z\le 5\}$. ans: $\frac{9}{2}$

$$\text{Avg} = \frac{\int_0^1\int_0^3\int_0^5 (x+y+z)\,dz\,dy\,dx}{\int_0^1\int_0^3\int_0^5 dz\,dy\,dx}$$

Num $\Rightarrow \int_0^1\int_0^3\left(xz+yz+\frac{z^2}{2}\right)_0^5 dy\,dx$

$$= \int_0^1\int_0^3\left(5x+5y+\frac{25}{2}\right)dy\,dx$$

$$= \int_0^1\left(5xy+\frac{5y^2}{2}+\frac{25y}{2}\right)_0^3 dx$$

$$= \int_0^1\left(15x+\frac{45}{2}+\frac{75}{2}\right)dx$$

$$= \left[\frac{15x^2}{2}+60x\right]_1^1 = \frac{135}{2}$$

Den: $\int_0^1\int_0^3\int_0^5 dz\,dy\,dx$

$= 5\times 3\times 1$

$= 15$

$$\text{Avg} = \frac{\cancel{135}\ \cancel{27}\ 9}{2\times\cancel{15}\ \cancel{3}} = \frac{9}{2}$$