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UNIT 4: Partial Differential Equations

Session: 7

Sub Topic: Solutions of PDEs by the method of Separation of

Variables continued

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Solutions of PDEs by the method of Separation of Variables



1. Solve $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$ by the method of separation of variables.

Solution:

Let u = XY, where X = X(x) and Y = Y(y) be the solution of the given PDE.

Substituting into the given PDE, we have

$$x^2 \frac{\partial (XY)}{\partial x} + y^2 \frac{\partial (XY)}{\partial y} = 0$$

$$x^2Y\frac{dX}{dx} + y^2X\frac{dY}{dy} = 0$$

Dividing by XY we have,

$$\frac{x^2}{X}\frac{dX}{dx} = -\frac{y^2}{Y}\frac{dY}{dy}$$

Solutions of PDEs by the method of Separation of Variables



Equating both sides to a common constant k we have,

$$\frac{x^2}{X}\frac{dX}{dx} = k$$

$$\frac{1}{X}dX = \frac{k}{x^2}dx$$

$$; -\frac{y^2}{Y}\frac{dY}{dy} = k$$

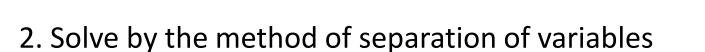
$$; \frac{1}{Y}dY = -\frac{k}{y^2}dy$$

On integrating

$$log_e X = -\frac{k}{x} + c_1$$
 ; $log_e Y = \frac{k}{y} + c_2$
 $X = e^{\left(-\frac{k}{x}\right) + c_1}$; $Y = e^{\left(\frac{k}{y}\right) + c_2}$
 Hence $u = XY = e^{c_1 + c_2} e^{\left(-\frac{k}{x}\right) + \left(\frac{k}{y}\right)}$ and let $c = e^{c_1 + c_2}$

Hence $u = c e^{k\left(\frac{1}{y} - \frac{1}{x}\right)}$ is the required solution.

Solutions of PDEs by the method of Separation of Variables



$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$$
, given that $u(0, y) = 2e^{5y}$.

Solution:

Let u = XY, where X = X(x) and Y = Y(y) be the solution of the given PDE.

Substituting into the given PDE

$$4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$$

Dividing by XY we have,

$$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3 \quad \text{or } \frac{4}{X}\frac{dX}{dx} = 3 - \frac{1}{Y}\frac{dY}{dy}$$



Solutions of PDEs by the method of Separation of Variables



Equating both sides to a common constant k we have,

$$\frac{4}{X}\frac{dX}{dx} = k$$

$$\frac{1}{X}dX = \frac{k}{4}dx$$

$$; 3 - \frac{1}{Y}\frac{dY}{dy} = k$$

$$; \frac{1}{Y}dY = (3 - k)dy$$

On integrating

$$log_e X = \frac{k}{4}x + c_1$$
 ; $log_e Y = (3 - k) y + c_2$
 $X = e^{\frac{k}{4}x + c_1}$; $Y = e^{(3-k)y + c_2}$

Solutions of PDEs by the method of Separation of Variables



Hence
$$u = XY = e^{c_1 + c_2} e^{\frac{k}{4}x + (3-k)y}$$
 and $c = e^{c_1 + c_2}$

Thus $u = u(x, y) = ce^{\frac{k}{4}x + (3-k)y}$ is the general solution.

Further, by data, $u(0, y) = 2e^{5y}$

The general solution becomes

$$2e^{5y} = ce^{(3-k)y}.$$

Comparing we have c = 2 and 3 - k = 5 or k = -2

Thus the required particular solution is given by

$$u = 2e^{-\frac{x}{2} + 5y}$$

Solutions of PDEs by the method of Separation of Variables



3. Solve by the method of separation of variables

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$
, given that $u(x, 0) = 6e^{-3x}$.

Solution:

Let u = XT, where X = X(x) and T = T(t) be the solution of the given PDE.

Substituting into the given PDE, we have

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Dividing by XT we have,

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

Solutions of PDEs by the method of Separation of Variables



Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{dX}{dx} = k$$

$$\frac{1}{X}\frac{dX}{dx} = kdx$$

$$\frac{1}{X}dX = kdx$$

$$\frac{1}{T}dT = \frac{(k-1)}{2}dt$$

On integrating

$$log_e X = kx + c_1$$
 ; $log_e T = \frac{(k-1)}{2} t + c_2$
 $X = e^{kx + c_1}$; $T = e^{(k-1)\frac{t}{2} + c_2}$
 Hence $u = XT = e^{c_1 + c_2} e^{kx + (k-1)\frac{t}{2}}$ and let $c = e^{c_1 + c_2}$

Thus $u = ce^{kx + (k-1)\frac{t}{2}}$ is the general solution.

Solutions of PDEs by the method of Separation of Variables



Further by data,
$$u(x, 0) = 6e^{-3x}$$

That is
$$u = 6e^{-3x}$$
 when $t = 0$

Hence we have
$$6e^{-3x} = ce^{kx}$$
.

Comparing we have
$$c = 6$$
 and $k = -3$

Thus the required particular solution is given by

$$u = 6e^{-3x-2t}$$



THANK YOU

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