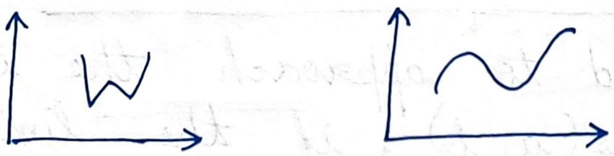
Introduction to Partial Differentiation

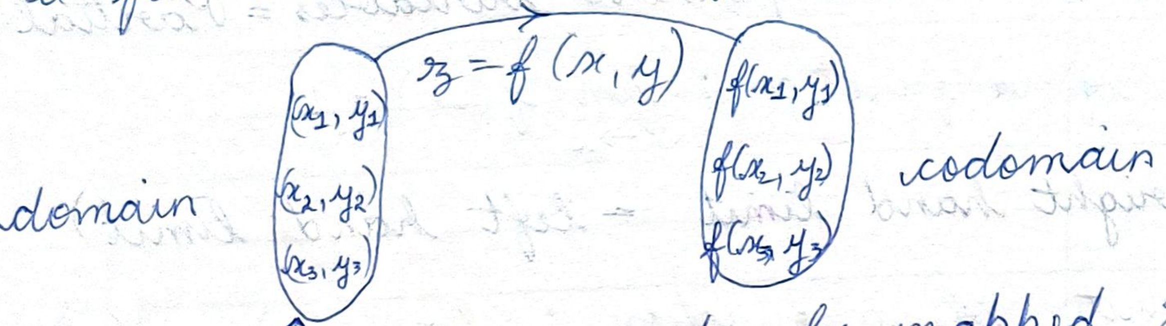
- One independent variable = Differentiation
  Two or more independent variables = Partial diff
- Limit of a function : $\lim_{x \to c} f(x) = L$
  (the right hand limit = left hand limit)
- Continuity of a function : $\lim_{x \to c} f(x) = f(c)$
- $y = f(x)$ is geometrically represented by curves.



not differentiable due to sharp edges.   differentiable

- Quantities which depend on two or more variables are represented using functions with 2 or more variables.
  These quantities are called MULTIVARIABLE FUNCTION
- Ex. of functions with 2 independent variables are area, cylinder, cone, ellipsoid, cube, cuboid.
- Multivariable functions have several independent variables. Eg: Area (l, b)
  Volume (l, b, h)
  Propagation of EMW $(\vec{B}, \vec{E}, \lambda, x)$
  Weather (humidity, pressure, temp.)
- $z = f(x, y)$
  Geometrically represented by surfaces of 3 dimensions

★ Function of two Independent Variables $z = f(x, y)$ is a function that assigns,



a set of ordered pairs to be mapped to a unique real number, denoted by $f(x, y)$

★ Limit of function of two variables: A function $f(x, y)$ is said to approach the limit L as (x, y) approaches (a, b), if the limit remains same along any path of (x, y) approaching (a, b). (a, b) can be approached from (x, y) using infinite paths.

$$\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = L}$$

1. Evaluate limit of function:

$$\lim_{(x,y)\to(2,-1)} \left(x^2 - 2xy + 3y^2 - 4x + 3y - 6\right)$$

The function is defined at $(2, -1)$, thus limit exists.

$$\Rightarrow \lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$$

$$\Rightarrow 4 - 2(2)(-1) + 3(1) - 4(2) + 3(-1) - 6$$

$$\Rightarrow 4 + 4 + 3 - 8 - 3 - 6 = \cancel{3} + \cancel{8} - \cancel{8} - \cancel{3} - 6 = \underline{-6}$$

2. Does the limit exist? $\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{3x^2+y^2}\right)$

The limit does not exist if it is not finite, or if it depends on a particular path.

Consider a path $y = mx$.

As $(x, y) \to 0$, we get $x \to 0$. Thus,

$$\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{3x^2+y^2}\right) = \lim_{x\to 0} \frac{2x(mx)}{3x^2+m^2x^2}$$

$$= \lim_{x\to 0} \left(\frac{2m^2}{3+m^2}\right)$$

which depends on m. For different values of m we obtain different values / limits.

$\therefore$ Limit does not exist.

☆ Continuity of a function with two variables:

A function is continuous at a point if the graph doesn't have holes / breaks at that point.

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$$

Partial differentiation: A partial derivative of a function of several variables is the derivative wrt one of the variables wrt others constant.

Eg: $z = f(x, y)$.

$\boxed{\frac{\partial z}{\partial x}}$ first order partial derivative $\boxed{\frac{\partial z}{\partial y}}$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ $\quad$ $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$

second order partial derivative

Eg: $u = f(x, y, z)$, partial derivatives of first order are $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$. (three

We will get ~~8~~ 9 second order partial derivatives (~~2~~ 3 wrt each x, y, z)

3. Find partial derivatives of $z = x^2 + y^2$.

$z = x^2 + y^2$

$\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$

$\therefore$ Partial derivatives are $\underline{\underline{2x, 2y}}$.

☆ **First Order Partial Derivatives**

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \left[\frac{f(x+h, y) - f(x, y)}{h}\right]$$

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \left[\frac{f(x, y+h) - f(x, y)}{h}\right]$$

☆ **Second Order Partial Derivatives**

Wrt x: $z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

Wrt y: $z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

☆ **Mixed Partial Derivatives**

$z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}$

$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$

★ Clairaut's Theorem : The crossed/mixed partial derivatives are in general, equal.

i.e., $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$.

The order of differentiation is immaterial if second order derivatives are continuous.

I Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the functions:

1. $z = \sin(x^2 y)$

$$\frac{\partial z}{\partial x} = (\cos x^2 y)(2xy) = \underline{\underline{2xy\cos(x^2 y)}}$$

$$\frac{\partial z}{\partial y} = (\cos x^2 y)(x^2) = \underline{\underline{x^2\cos(x^2 y)}}$$

2. $z = 5x^3 y + xy^2$

$$\frac{\partial z}{\partial x} = \underline{\underline{15x^2 y + 1y^2}}$$

$$\frac{\partial z}{\partial y} = \underline{\underline{5x^3 + 2xy}}$$

3. $z = x^y + y^x$

$$\frac{\partial z}{\partial x} = \underline{\underline{yx^{y-1} + y^x \log y}}$$

$$\frac{\partial z}{\partial y} = \underline{\underline{x^y \log x + xy^{x-1}}}$$

4. $z = e^{xy^2}$

$$\frac{\partial z}{\partial x} = e^{xy^2}(1y^2) = \underline{\underline{y^2 e^{xy^2}}}$$

$$\frac{\partial z}{\partial y} = e^{xy^2}(2xy) = \underline{\underline{2xye^{xy^2}}}$$

5. $f(x,y) = x^3y + 5y^2 - x + 7$

$$\frac{\partial f}{\partial x} = \underline{\underline{3x^2y - 1}}$$

$$\frac{\partial f}{\partial y} = \underline{\underline{x^3 + 10y}}$$

6. $f(x,y) = \cos(xy^2) + \sin x$

$$\frac{\partial f}{\partial x} = [-\sin(xy^2)][1y^2] + \cos x$$

$$= \underline{\underline{-y^2\sin(xy^2) + \cos x}}$$

$$\frac{\partial f}{\partial y} = [-\sin(xy^2)][2xy] + 0 = \underline{\underline{-2xy\sin(xy^2)}}$$

7. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

$$\frac{\partial f}{\partial x} = [e^{x^2y^3}(2xy^3)]\sqrt{x^2+1} + [e^{x^2y^3}]\frac{1}{\cancel{2}}(x^2+1)^{\frac{1}{2}-1}(\cancel{2}x)$$

$$= \underline{\underline{2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}}}$$

$$\frac{\partial f}{\partial y} = [e^{x^2y^3}(3y^2x^2)]\sqrt{x^2+1} + e^{x^2y^3}(0)$$

$$= \underline{3x^2y^2e^{x^2y^3}\sqrt{x^2+1}}$$

8. $f(x, y) = x^4 - x^2y^2 + y^4$ at $(-1, 1)$

$$\frac{\partial f}{\partial x} = 4x^3 - 2xy^2$$

At $x = -1$, $\frac{\partial f}{\partial x} = 4(-1) - 2(-1)(1) = -4 + 2 = \underline{\underline{-2}}$

$$\frac{\partial f}{\partial y} = -2x^2y + 4y^3$$

At $y = 1$, $\frac{\partial f}{\partial y} = -2(1)(1) + 4(1) = \underline{\underline{2}}$

9. $f(x, y) = ye^{-x}$

$$\frac{\partial f}{\partial x} = ye^{-x}(-1) + e^{-x}(0) = \underline{\underline{-ye^{-x}}}$$

$$\frac{\partial f}{\partial y} = y(0) + 1e^{-x} = \underline{\underline{e^{-x}}}$$

10. $f(x, y) = \sin(3x + 5y)$

$$\frac{\partial f}{\partial x} = [\cos(3x + 5y)](3 + 0) = \underline{\underline{3\cos(3x + 5y)}}$$

$$\frac{\partial f}{\partial y} = [\cos(3x + 5y)][0 + 5] = \underline{\underline{5\cos(3x + 5y)}}$$

Geometrical Interpretation of Partial Differentiation

i) $\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$ represents the slope of the tangent to the curve formed by the intersection of the the plane $y = y_0$ and the surface of any given point (x_0, y_0).

ii) $\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}$ represents the slope of the tangent to the curve formed by the intersection of the plane $x = x_0$ and the surface of any given point (x_0, y_0).

iii) The <u>second order partial derivatives</u> represents the concavity of the surface $z = f(x, y)$.

iv) If $f_{xx} > 0$, then $f(x, y)$ is concave up in the x-direction.

v) If $f_{yy} > 0$, then $f(x, y)$ is concave up in the y-direction.

vi) <u>Mixed partials</u> tell us how a partial in one variable is changing in the direction of the other.

vii) f_{xy} tells how rate of change of $f(x, y)$ in the x-direction is changing as we move in y-direction.

11. Show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all $(x, y) \neq (0,0)$

when $f(x, y) = x^y$.

Given, $f(x, y) = x^y$

$$\Rightarrow \frac{\partial f}{\partial y} = x^y \log x$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x)$$

$$= x^y\left(\frac{1}{x}\right) + y x^{y-1}(\log x) \rightarrow ①$$

$$\Rightarrow \frac{\partial f}{\partial x} = y x^{y-1}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y x^{y-1})$$

$$= y x^{y-1} \log x + 1(x^{y-1})$$

$$= y x^{y-1}(\log x) + x^y\left(\frac{1}{x}\right) \longrightarrow ②$$

① = ②

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$ for all $(x, y) \neq (0,0)$

Hence, proved.

12. Find all the second order partial derivatives of $f(x, y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1, 2)$.

Given, $f(x, y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$.

$$\frac{\partial f}{\partial x} = f_x = \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)}\left(-\frac{1}{x^2} - 0\right)$$

$$= \frac{xy}{(y-x)} \times \left(\frac{-1}{x^2}\right)$$

$$= \frac{-y}{x(y-x)}$$

$$= \frac{y}{x(x-y)}.$$

$$f_y = \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)}\left(0 + \frac{1}{y^2}\right)$$

$$= \frac{xy}{y-x}\left(\frac{1}{y^2}\right)$$

$$= \frac{x}{y(y-x)}$$

At $(1, 2)$ $f_x = \frac{2}{1(1-2)} = -2$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{y}{x(x-y)}\right]$$

$$\Rightarrow f_{xx} = (y)\frac{\partial}{\partial x}\left[\left(\frac{1}{x}\right)\left(\frac{1}{x-y}\right)\right].$$

$$= (y)\left[\left(\frac{-1}{x^2}\right)\left(\frac{1}{x-y}\right) + \left(\frac{1}{x}\right)\left(\frac{-1}{[x-y]^2}\right)(1)\right]$$

$$= y\left[\frac{-1}{x^2(x-y)} - \frac{1}{x(x-y)^2}\right].$$

$$= 2\left[\frac{-1}{1(1-2)} - \frac{1}{1(1-2)^2}\right].$$ (At $(1, 2)$)

$$= 2\left[\left(\frac{-1}{-1}\right) - 1\right] = 0 \qquad \Rightarrow \underline{\underline{f_{xx} = 0}}$$

$$\Rightarrow f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{y(y-x)}\right)$$

$$= \left(\frac{1}{y}\right) \frac{\partial}{\partial x}\left[x \cdot \left(\frac{1}{y-x}\right)\right].$$

$$= \frac{1}{y}\left[1\left(\frac{1}{y-x}\right) + x\left(\frac{-1}{(y-x)^2}\right)(-1)\right].$$

$$= \frac{1}{y}\left[\frac{1}{(y-x)} + \frac{(+x)}{(y-x)^2}\right].$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)} + \frac{(+1)}{(-1+2)^2}\right]. \qquad [At\ (1,2)].$$

$$= \frac{1}{2}\left[+1 + 1\right] = +1 \Rightarrow \underline{\underline{f_{xy} = +1}}$$

$$\Rightarrow f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{x}{y(-x+y)}\right].$$

$$= (x)\frac{\partial}{\partial y}\left[\left(\frac{1}{y}\right)\left(\frac{-1}{x-y}\right)\right].$$

$$= (x)\left[\left(\frac{-1}{y^2}\right)\left(\frac{-1}{x-y}\right) + \left(\frac{1}{y}\right)\left(\frac{+1}{(x-y)^2}\right)(0-1)\right]$$

$$= x\left[\frac{+1}{(x-y)y^2} - \frac{1}{y(x-y)^2}\right].$$

$$= 1\left[\frac{+1}{(1-2)(4)} - \frac{1}{2(1-2)^2}\right]. \qquad [At\ (1,2)].$$

$$= 1\left[\frac{-1}{4} - \frac{1}{2}\right] = \frac{-3}{4} \Rightarrow \underline{\underline{f_{yy} = \frac{-3}{4}}}$$

$$\Rightarrow f_{yx} = \frac{\partial}{\partial y}\left[\frac{y}{x(x-y)}\right]$$

$$= \left(\frac{1}{x}\right)\frac{\partial}{\partial y}\left[\frac{y}{(x-y)}\right]$$

$$= \left(\frac{1}{x}\right)\left[1\cdot\left(\frac{1}{x-y}\right) + y\left(\frac{-1}{(x-y)^2}\right)(0-1)\right]$$

$$= \left(\frac{1}{x}\right)\left[\left(\frac{1}{x-y}\right) + \frac{y}{(x-y)^2}\right]$$

$$= 1\left[\frac{1}{(1-2)} + \frac{2}{(1-2)^2}\right] \qquad [At\ (1,2)]$$

$$= 1\left[-1+2\right] = 1 \qquad \Rightarrow \underline{f_{yx} = 1}$$

<u>Assignment - 1</u>

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following:

i) $z = e^{\frac{x}{y}}$

$$\frac{\partial z}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right) = \underline{\left(\frac{e^{\frac{x}{y}}}{y}\right)} \text{ and } \frac{\partial z}{\partial y} = \underline{e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)}$$

ii) $z = x\cos(xy) + y\sin(xy)$

$$\frac{\partial z}{\partial x} = 1\cos(xy) + x[-\sin(xy)][y+0] + y[\cos(xy)](y+0) +0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \cos(xy) - xy\sin(xy) + y^2\cos(xy)$$

$$\frac{\partial z}{\partial y} = x[-\sin(xy)][x+0]+0+1\sin(xy)+ y[\cos(xy)][x+0]$$

$$\Rightarrow \frac{\partial z}{\partial y} = -x^2\sin(xy)+\sin(xy)+xy\cos(xy).$$

iii) $z = \log(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \left(\frac{1}{x^2+y^2}\right)(2x+0) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \left(\frac{1}{x^2+y^2}\right)(0+2y) = \frac{2y}{x^2+y^2}$$

2. Verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ for the following :

ii) $z = e^{x\cos y + y\cos x}$

$\Rightarrow \dfrac{\partial z}{\partial y} = e^{x\cos y + y\cos x}\left[0(\cos y) + x(-\sin y)\right] + \left[1\cos x - y(0)\right]$

$= e^{x\cos y + y\cos x}\left[-x\sin y + \cos x\right]$

$= (\cos x - x\sin y) \times e^{x\cos y + y\cos x}$

$\Rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$= \dfrac{\partial}{\partial x}\left[(\cos x - x\sin y) \times e^{x\cos y + y\cos x}\right]$

$= \left[-\sin x - 1\sin y + 0\right]e^{x\cos y + y\cos x} +$
$(\cos x - x\sin y)\, e^{x\cos y + y\cos x}(\cos y - y\sin x)\ldots$

$$= \left[(-\sin x - \sin y) + (\cos x - x\sin y) * (\cos y - y\sin x)\right] \times e^{x\cos y + y\cos x}$$

$$= \left[(-\sin x - \sin y) + (\cos x\cos y - y\sin x\cos x - x\sin y\cos y + xy\sin x\sin y)\right] e^{x\cos y + y\cos x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = e^{x\cos y + y\cos x}\left[\cos y - y\sin x\right].$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[e^{x\cos y + y\cos x}\right](\cos y - y\sin x)$$

$$= (-\sin y - 1\sin x)e^{x\cos y + y\cos x} + e^{x\cos y + y\cos x}(-x\sin y + 1\cos x)(\cos y - y\sin x)$$

$$= \left[(-\sin x - \sin y) + (-x\sin y\cos y + xy\sin x\sin y + \cos x\cos y - y\sin x\cos x\right] e^{x\cos y + y\sin x}$$

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$$

Hence, proved.

ii) $z = \dfrac{xy}{y-x}$

$$\frac{\partial z}{\partial y} = \frac{(1x+0)}{(y-x)} + \left[\frac{-1}{(y-x)^2}\right][1-0](xy)$$

$$= \frac{+x}{(y-x)} - \frac{1xy}{(y-x)^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{+x}{(y-x)} - \frac{1xy}{(y-x)^2}\right]$$

$$= \left(\frac{1}{y-x}\right) + \frac{(-x)(-1)}{(y-x)^2} + \frac{(-xy)(-2)}{(y-x)^3}(0-1) - \frac{(y+0)}{(y-x)^2}$$

$$= \frac{+1}{(y-x)} + \frac{2xy}{(y-x)^3} + \frac{x}{(y-x)^2} - \frac{y}{(y-x)^2}$$

$$\frac{\partial z}{\partial x} = \frac{(y+0)}{(y-x)} + \left[\frac{-xy}{(y-x)^2}\right](0-1)$$

$$= \left(\frac{y}{y-x}\right) + \frac{xy}{(y-x)^2}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[\left(\frac{y}{y-x}\right) + \frac{xy}{(y-x)^2}\right]$$

$$= \left(\frac{1}{y-x}\right) + \left[\frac{y(-1)}{(y-x)^2}\right][1-0] + \frac{(x+0)}{(y-x)^2} + \frac{xy(-2)}{(y-x)^3}(1-0)$$

$$= \frac{1}{(y-x)} - \frac{y}{(y-x)^2} + \frac{x}{(y-x)^2} - \frac{2xy}{(y-x)^3}$$

$\therefore \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ Hence, proved.

3. If $\bar{u} = x^3 - 3xy^2 + x + e^x \cos y + 1$,
find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

Given, $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y + 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 1 + e^x \cos y)$$

$$= \underline{\underline{6x + e^x \cos y}}.$$

$$\frac{\partial u}{\partial y} = 0 - 6xy + e^x(-\sin y) = -6xy - e^x \sin y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy - e^x \sin y)$$

$$= \underline{\underline{-6x - e^x \cos y}}$$

Assignment-2

1. If $u = (y-z)(z-x)(x-y)$, prove $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Given $u = (y-z)(z-x)(x-y)$

$$\frac{\partial u}{\partial x} = 0(z-x)(x-y) + (y-z)(0-1)(x-y) + (y-z)(z-x)(1-0)$$

$$= (y-z)(z-x) - (x-y)(y-z)$$

$$\frac{\partial u}{\partial y} = (1-0)(z-x)(x-y) + (y-z)(0)(x-y) + (y-z)(z-x)(0-1)$$

$$= (z-x)(x-y) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (0-1)(z-x)(x-y) + (y-z)(1-0)(x-y) + 0$$

$$= -(z-x)(x-y) + (y-z)(x-y).$$

$$LHS = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= [\cancel{(y-z)(z-x)} - \cancel{(y-z)(x-y)}] + [\cancel{(z-x)(x-y)} - \cancel{(y-z)(z-x)}] + [-\cancel{(z-x)(x-y)} + \cancel{(y-z)(x-y)}].$$

$= 0 =$ RHS

$\Rightarrow$ LHS = RHS

Hence, proved.

2. If $u = \log(\tan x + \tan y + \tan z)$, then show that

$$\sin 2x\left(\frac{\partial u}{\partial x}\right) + \sin 2y\left(\frac{\partial u}{\partial y}\right) + \sin 2z\left(\frac{\partial u}{\partial z}\right) = 2.$$

Given, $u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{\tan x + \tan y + \tan z}\right)(\sec^2 x + 0 + 0)$$

$$= \frac{\sec^2 x}{\tan x + \tan y + \tan z} \rightarrow ①.$$

$$\frac{\partial u}{\partial y} = \left(\frac{1}{\tan x + \tan y + \tan z}\right)(0 + \sec^2 y + 0) = \frac{\sec^2 y}{\tan x + \tan y + \tan z} \rightarrow ②$$

$$\frac{\partial u}{\partial z} = \left(\frac{1}{\tan x + \tan y + \tan z}\right)(0 + 0 + \sec^2 z)$$

$$= \frac{\sec^2 z}{\tan x + \tan y + \tan z} \longrightarrow (3).$$

$$LHS = \sin 2x\left(\frac{\partial u}{\partial x}\right) + \sin 2y\left(\frac{\partial u}{\partial y}\right) + \sin 2z\left(\frac{\partial u}{\partial z}\right)$$

Substituting (1), (2) and (3), we get,

$$= \frac{(2\sin x \cos x)\sec^2 x}{\tan x \tan y + \tan z} + \frac{2\sin y \cos y(\sec^2 y)}{\tan x + \tan y + \tan z} + \frac{2\sin z \cos z(\sec^2 z)}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2 = RHS$$

$\therefore$ LHS = RHS

Hence, proved.

3. If $u = \log \rho$, where $\rho^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho^2}$.

$$\rho^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \longrightarrow (1).$$

Partially differentiating wrt x,

$$2\rho\left(\frac{\partial \rho}{\partial x}\right) = \frac{\partial}{\partial x}[(x-a)^2 + (y-b)^2 + (z-c)^2].$$

$$\Rightarrow 2\rho\left(\frac{\partial \rho}{\partial x}\right) = 2(x-a)(1-0) + 0 + 0.$$

$$\Rightarrow \frac{\partial \rho}{\partial x} = \left(\frac{x-a}{\rho}\right) \longrightarrow (2)$$

Partially differentiating ① wrt y,

$$2\rho\left(\frac{\partial \rho}{\partial y}\right) = 0 + 2(y-b)(1-0) + 0$$

$$\Rightarrow \frac{\partial \rho}{\partial y} = \left(\frac{y-b}{\rho}\right) \longrightarrow ③.$$

Partially differentiating wrt z,

$$2\rho\left(\frac{\partial \rho}{\partial z}\right) = 0 + 0 + 2(z-c)(1-0)$$

$$\Rightarrow \frac{\partial \rho}{\partial z} = \left(\frac{z-c}{\rho}\right) \longrightarrow ④.$$

Given, $u = \log \rho$. $\longrightarrow$ ⑤.

Partially diff. wrt x,

$$\frac{\partial u}{\partial x} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right).$$

(From ②)

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\rho}\left(\frac{x-a}{\rho}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{(x-a)}{\rho^2}\right] = \frac{1}{\rho^2}$$

Partially diff. ⑤ wrt y,

$$\frac{\partial u}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial y}\right).$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{\rho}\left(\frac{y-b}{\rho}\right)$$ (From ③)

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{(y-b)}{\rho^2}\right] = \frac{1}{\rho^2}$$

Partially diff. ⑤ wrt z,

$$\frac{\partial u}{\partial z} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial z}\right)$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{1}{\rho}\left(\frac{z-c}{\rho}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{(z-c)}{\rho^2}\right] = \frac{1}{\rho^2}$$

$$LHS = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{1}{\rho^2} + \frac{1}{\rho^2} + \frac{1}{\rho^2} = \frac{3}{\rho^2}$$

$\therefore$ LHS = RHS

Hence, proved.

7. If $z = f(y-3x) + \phi(y+2x) + \sin x - y\cos x$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

$$z = f(y-3x) + \phi(y+2x) + \sin x - y\cos x \quad \rightarrow ①$$

Partially diff wrt x,

$$\frac{\partial z}{\partial x} = -3f'(y-3x) + 2\phi'(y+2x) + \cos x + y\sin x$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = +9f''(y-3x) + 4\phi''(y+2x) - \sin x + y\cos x \quad \rightarrow ②$$

Partially diff ① wrt y,

$$\frac{\partial z}{\partial y} = f'(y-3x) + \phi'(y+2x) + 0 - \cos x$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = f''(y-3x) + \phi''(y+2x) \quad \rightarrow ③$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -3f''(y-3x) + 2\phi''(y+2x) + \sin x \quad \rightarrow ④$$

Taking LHS,

$$\text{LHS} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

Substituting ②, ③ and ④ to LHS,

$$= [9f''(y-3x) + 4\phi''(y+2x) - \sin x + y\cos x] + [-3f''(y-3x) + 2\phi''(y+2x) + \sin x] - 6[f''(y-3x) + \phi''(y+2x)]$$

$$= f''(y-3x)[9-3-6] + \phi''(y-2x)[4+2-6] - \cancel{\sin x} + y\cos x + \cancel{\sin x} = \underline{y\cos x} = \text{RHS}$$

$\therefore$ LHS = RHS

Hence, proved.

5. At what rate is the area of a rectangle changing if its length is 15 m and increasing at 3 m/s while its width is 6 m and increasing at 2 m/s ?

Given, $l = 15\,m$
$b = 6\,m$

Also given, $\frac{dl}{dt} = 3\,m/s$ and $\frac{db}{dt} = 2\,m/s$.

We know, $A = lb$

Partially diff wrt l,

$$\frac{\partial A}{\partial l} = b = 6m \rightarrow (1)$$

Partially diff wrt b,

$$\frac{\partial A}{\partial b} = l = 15m \rightarrow (2).$$

$A = lb$

Diff wrt t,

$$\frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt}$$

From (1) and (2),

$$\Rightarrow \frac{dA}{dt} = \frac{\partial A}{\partial b}\left(\frac{db}{dt}\right) + \frac{\partial A}{\partial l}\left(\frac{dl}{dt}\right)$$

$$= 15(2) + 6(3) = 30 + 18 = \underline{48\,m^2/s}$$

(OR) $$\frac{dA}{dt} = l\left(\frac{db}{dt}\right) + b\left(\frac{dl}{dt}\right)$$

$$= 15(2) + 6(3) = 30 + 18 = \underline{48\,m^2/s}$$

13. If $f(x, y, z) = e^{x^2+y^2+z^2}$, then find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}$

Given $f(x, y, z) = e^{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x} = e^{x^2+y^2+z^2}(2x + 0 + 0) = 2x\, e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2x\, e^{x^2+y^2+z^2}\right)$$

$$= 2e^{x^2+y^2+z^2} + 2x\, e^{x^2+y^2+z^2}(2x)$$

$$= \underline{\underline{(4x^2 + 2)\, e^{x^2+y^2+z^2}}}$$

$$\frac{\partial f}{\partial y} = e^{x^2+y^2+z^2}(0 + 2y + 0) = 2y\, e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(2y\, e^{x^2+y^2+z^2}\right)$$

$$= 2e^{x^2+y^2+z^2} + 2y\, e^{x^2+y^2+z^2}(2y)$$

$$= \underline{\underline{(4y^2 + 2)\, e^{x^2+y^2+z^2}}}$$

$$\frac{\partial f}{\partial z} = e^{x^2+y^2+z^2}(0 + 0 + 2z) = 2z\, e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(2z\, e^{x^2+y^2+z^2}\right)$$

$$= \underline{\underline{(4z^2 + 2)\, e^{x^2+y^2+z^2}}}$$

14. $f(x, y, \cancel{z}) = e^x \log y + \cos y \log x$, then find f_{xxx}, f_{xyy}, f_{xxy} and f_{yyy} at $\left(1, \frac{\pi}{2}\right)$.

Given, $f(x, y) = e^x \log y + \cos y \log x$.

$$\frac{\partial f}{\partial x} = e^x \log y + 0 + 0 + \frac{1}{x}(\cos y).$$

$$\Rightarrow f_x = e^x \log y + \frac{1}{x}(\cos y).$$

$$\Rightarrow f_{xx} = e^x \log y - \frac{1}{x^2}(\cos y).$$

$$\Rightarrow f_{xxx} = e^x \log y + \frac{2\cos y}{x^3}$$

At $\left(1, \frac{\pi}{2}\right)$, $f_{xxx} = e \log\left(\frac{\pi}{2}\right) + \frac{2\cos\left(\frac{\pi}{2}\right)}{1}$.

$$\Rightarrow f_{xxx} = e \log\left(\frac{\pi}{2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{y} + (-\sin y) \log x$$

$$\Rightarrow f_y = \frac{e^x}{y} - \sin y \log x.$$

$$\Rightarrow f_{yy} = -\frac{e^x}{y^2} - \cos y \log x.$$

$$\Rightarrow f_{yyy} = \frac{2e^x}{y^3} + \sin y \log x$$

At $\left(1, \frac{\pi}{2}\right)$, $f_{yyy} = \frac{2e}{\left(\frac{\pi}{2}\right)^3} + \sin\left(\frac{\pi}{2}\right) \log 1.$

$$= \frac{16e}{\pi^3} + \log 1 (1) = \frac{16e}{\pi^3}$$

$$f_{yy} = -\frac{e^x}{y^2} - \cos y \log x$$

$$f_{xyy} = -\frac{e^x}{y^2} - \frac{\cos y}{x}$$

At $\left(1, \frac{\pi}{2}\right) \Rightarrow f_{xyy} = -\frac{e}{\left(\frac{\pi}{2}\right)^2} - \frac{\cos\left(\frac{\pi}{2}\right)}{1}$

$$\Rightarrow f_{xyy} = \underline{\underline{\frac{-4e}{\pi^2}}}$$

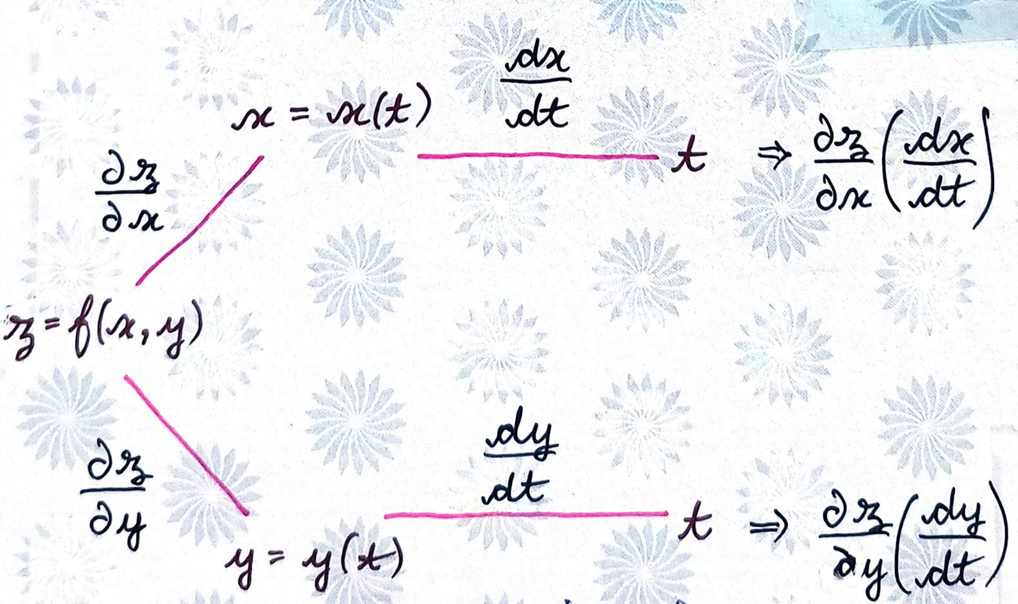
$$f_y = \frac{e^x}{y} - \sin y \log x$$

$$f_{xy} = \frac{e^x}{y} - \frac{\sin y}{x}$$

$$f_{xxy} = \frac{e^x}{y} + \frac{1 \sin y}{x^2}$$

At $\left(1, \frac{\pi}{2}\right) \Rightarrow f_{xxy} = \underline{\underline{\frac{2e}{\pi} - 1}}$

tree diagram



$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right)$$

Total Differential: If $z = f(x, y)$ is a function of two independent variables, then the total differential dz is defined by:

$$\boxed{dz = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy}$$

15. Find the total differential of:

i) $u = xy$

$u = xy.$

$$\Rightarrow du = \left(\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y}\right) dy$$

$$\Rightarrow du = y\,dx + x\,dy.$$

ii) $u = \dfrac{x}{y}$

$u = \dfrac{x}{y}$

$$\Rightarrow du = \left(\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y}\right) dy$$

$$\Rightarrow du = \frac{dx}{y} - \left(\frac{x}{y^2}\right) dy$$

Simplifying further,

$$\Rightarrow du = \frac{y\,dx - x\,dy}{y^2}$$