Differentiation of Composite Functions of a Single variable

Total Derivative rule (see prev. page for tree shortcut)

If $z = f(x, y)$, where x and y are functions of one independent variable "t", then "z" is said to be a composite function of single variable "t".

The total derivative of z wrt t is given by:

$$\boxed{\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt}}$$

Composite Function of two variables

Chain Rule

It can be used for function of two or more independent variables.

If $z = f(x, y)$, where x and y are functions of two independent variables u, v, then "z" is said to be a composite function of two variables u and v.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u} + \left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial v} + \left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial v} \end{aligned}}$$

16. Find $\frac{du}{dt}$ if $u = x^2 - y^2$ and $x = e^t \cos t$,

$y = e^t \sin t$ at $t = 0$.

Given, $x = e^t \cos t$ At $t=0$, $x = e^0 \cos 0 = 1$

$y = e^t \sin t$ At $t=0$, $y = e^0 \sin 0 = 0$

$u = x^2 - y^2$

Using total derivative rule;

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt}$$

$$= (2x)[e^t \cos t + e^t(-\sin t)] + (-2y)[e^t \sin t + e^t \cos t]$$

$$= 2x(e^t \cos t - e^t \sin t) - 2y(e^t \sin t + e^t \cos t).$$

$$= 2x(e^0 \cos 0 - e^0 \sin 0) - 2y(e^0 \sin 0 + e^0 \cos 0)$$

$$= 2x(1-0) - 2y(0+1).$$

$$= 2(1)(1) - 2(0)(1) = \underline{\underline{2}}.$$

17. Find $\frac{df}{dt}$ at $t=0$ where,

i) $f(x,y) = x \cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$

Given, $x = t^2 + 1$ At $t=0$, $x = 1$

$y = t^3 + t$ At $t=0$, $y = 0$.

$f(x,y) = x \cos y + e^x \sin y$

Using total derivative rule,

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}$$

$= (\cos y + e^x \sin y)(2t) + [(-x \sin y) + e^x \cos y](3t^2+1)$

Substituting $x = 1$, $y = 0$ to above,

$= (\cos 0 + e^1 \sin 0)(2t) + [-1 \sin 0 + e^1 \cos 0](3t^2+1)$

$= [(1+0)(2t)] + [0 + e](3t^2+1)$

$= 2t + 3et^2 + e$

At $t = 0$, $\frac{df}{dt} = e$

i) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$

Given, $x = e^t$ At $t = 0$, $x = 1$

$y = \cos t$ At $t = 0$, $y = 1$

$z = t^3$ At $t = 0$, $z = 0$

Using total derivative rule,

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)\frac{dz}{dt}$$

$= (3x^2 + z^2 + yz)(e^t) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2)$

Substituting $x = 1$, $y = 1$, $z = 0$ to above,

$= (3 + 0 + 0)(e^t) + (3 + 0)(-\sin t) + (0 + 1)(3t^2)$

$= 3e^t - 3\sin t + 3t^2$

At $t = 0$, $\frac{df}{dt} = 3(1) - 3(0) - 3(0) = \underline{3}$

Assignment - 3

1. Given $u = \sin\left(\frac{x}{y}\right)$ where $x = e^t$, $y = t^2$, find the total derivative of u wrt t.

Given, $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$; $\frac{du}{dt} = ?$

Using total derivative rule,

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= \left[\cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)\right]e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)(2t)$$

Substituting x and y,

$$\Rightarrow \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^2}\right) + \cos\left(\frac{e^t}{t^2}\right)\left(\frac{-e^t}{t^4}\right)(2t)$$

$$= \cos\left(\frac{e^t}{t^2}\right)\left[\frac{e^t}{t^2} - \frac{2e^t t}{t^4}\right]$$

$$= \cos\left(\frac{e^t}{t^2}\right)\frac{e^t}{t^2}\left[1 - \frac{2}{t}\right].$$

$$\Rightarrow \frac{du}{dt} = \left[1 - \frac{2}{t}\right]\frac{e^t}{t^2} \times \cos\left(\frac{e^t}{t^2}\right)$$

(answer should be written in terms of t ONLY)

If $z = xy^2 + x^2y$ where $x = at^2$ and $y = 2at$ and $\frac{dz}{dt} = ?$. Verify result by direct substitution.

Given, $z = xy^2 + x^2y$
$x = at^2$
$y = 2at$ $\Big\}$ $\frac{dz}{dt} = ?$

Using total derivative rule,

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$= 2aty^2 + 4atxy + 4axy + 2ax^2$$

~~Verification:~~

$\cancel{z = xy^2 + x^2y}$

~~Substituting x and y,~~

$\cancel{z = (at^2)(2at)^2 + (at^2)^2(2at)}$

$\cancel{\Rightarrow z = 2a^2t^2 + 2a^2t^3}$

$\cancel{\Rightarrow z = 4a^3t^4 + 2a^3t^3}$

Substituting x and y,

$$\Rightarrow \frac{dz}{dt} = 2at(2at)^2 + 4at(at^2)(2at) +$$

$$4a(at^2)(2at) + 2a(at^2)^2$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= \underline{16a^3t^3 + 10a^3t^4} \rightarrow (1)$$

Verification :

$z = xy^2 + x^2y$

Substituting x and y,

$\Rightarrow z = (at^2)(2at)^2 + (at^2)^2(2at)$

$\Rightarrow z = 4a^3t^4 + 2a^3t^5$

$\Rightarrow \frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \longrightarrow ②$

$\therefore$ LHS = RHS and ① = ②

Hence, verified.

3. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm/s and it passes through the value $x = 3$ cm, show that if y is passing through the value $y = 1$ cm, y must be decreasing at the rate of $2\frac{2}{15}$ cm/s in order that z shall remain constant.

Given, $z = 2xy^2 - 3x^2y \longrightarrow ①$

$x = 3$ cm and $\frac{dx}{dt} = 2$ cm/s

For $y = 1$ cm, $\frac{dy}{dt} = ?$

Step-1 : Partial differention

$\frac{\partial z}{\partial x} = 2y^2 - 6xy$ (From ①)

$\Rightarrow \left.\frac{\partial z}{\partial x}\right]_{x=3,y=1} = 2(1) - 6(3)(1) = -16.$

$$\frac{\partial z}{\partial y} = 4xy - 3x^2$$

$$\Rightarrow \left.\frac{\partial z}{\partial y}\right]_{x=3,\, y=1} = 4(3)(1) - 3(9) = 12 - 27 = -15$$

Step 2 : Total derivative rule.

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt}$$

Here, change in $z = 0 \Rightarrow \frac{dz}{dt} = 0$

$$\therefore\ 0 = (-16)(2) + (-15)\frac{dy}{dt}$$

$$\Rightarrow 32 = -15\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dy}{dt} = -\frac{32}{15} \quad \Rightarrow \frac{dy}{dt} = -2\frac{2}{15}\ cm/s$$

(y is decreasing)

Hence, proved.

4. If $z = f(x, y)$, where $x = e^u \cos v$ and $y = e^u \sin v$, show that:

i) $$y\left(\frac{\partial z}{\partial u}\right) + x\left(\frac{\partial z}{\partial v}\right) = e^{2u}\left(\frac{\partial z}{\partial y}\right)$$

ii) $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

Using chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\right)$$

$$\Rightarrow \frac{\partial z}{\partial u} = \left(\frac{\partial z}{\partial x}\right)(e^{u}\cos v) + \left(\frac{\partial z}{\partial y}\right)(e^{u}\sin v) \rightarrow ①$$

Using chain rule,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial v}\right)$$

$$\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{u}\sin v\right) + \left(\frac{\partial z}{\partial y}\right)(e^{u}\cos v) \rightarrow ②$$

i) $$\text{LHS} = y\left(\frac{\partial z}{\partial u}\right) + x\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\left(\frac{\partial z}{\partial x}\right)(e^{u}\cos v) + \left(\frac{\partial z}{\partial y}\right)(e^{u}\sin v)\right] +$$

$$x\left[-\left(\frac{\partial z}{\partial x}\right)(e^{u}\sin v) + \left(\frac{\partial z}{\partial y}\right)(e^{u}\cos v)\right].$$

$$= \cancel{e^{2u}\sin v\cos v}\left(\frac{\partial z}{\partial x}\right) + e^{2u}\sin^2 v\left(\frac{\partial z}{\partial y}\right)$$

$$\cancel{(-e^{2u}\sin v\cos v)}\left(\frac{\partial z}{\partial x}\right) + e^{2u}\cos^2 v\left(\frac{\partial z}{\partial y}\right)$$

$$= e^{2u}\sin^2 v\left(\frac{\partial z}{\partial y}\right) + e^{2u}\cos^2 v\left(\frac{\partial z}{\partial y}\right)$$

$$= e^{2u}\left(\frac{\partial z}{\partial y}\right)\left[\sin^2 v + \cos^2 v\right]$$

$$= e^{2u}\left(\frac{\partial z}{\partial y}\right) = \text{RHS}.$$

$\Rightarrow$ LHS = RHS Hence, proved.

ii) $RHS = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

$$\Rightarrow e^{-2u}\left[\left(\frac{\partial^2 z}{\partial x^2}\right)(e^{2u}\cos^2 v) + \left(\frac{\partial^2 z}{\partial y^2}\right)(e^{2u}\sin^2 v) + \left(\frac{\partial^2 z}{\partial x^2}\right)(e^{2u}\sin^2 v) + \left(\frac{\partial^2 z}{\partial y^2}\right)(e^{2u}\cos^2 v)\right]$$

$$\Rightarrow (e^{-2u})(e^{2u})\left[\left(\frac{\partial^2 z}{\partial x^2}\right)(\cos^2 v + \sin^2 v) + \frac{\partial^2 z}{\partial y^2}(\sin^2 v + \cos^2 v)\right]$$

$$\Rightarrow e^0\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = LHS$$

$\therefore$ LHS = RHS

Hence, proved.

5. If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, show that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) = 0$.

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right) + \left(\frac{\partial u}{\partial s}\right)\left(\frac{\partial s}{\partial x}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial t}{\partial x}\right)$$

(chain rule).

$$\Rightarrow \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial r}\right)\left(\frac{1}{y}\right) + \left(\frac{\partial u}{\partial s}\right)(0) + \left(\frac{\partial u}{\partial t}\right)\left(-\frac{z}{x^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{y}\left(\frac{\partial u}{\partial r}\right) - \frac{z}{x^2}\left(\frac{\partial u}{\partial t}\right) \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial y}\right) + \left(\frac{\partial u}{\partial s}\right)\left(\frac{\partial s}{\partial y}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial t}{\partial y}\right)$$

(chain rule).

$$\Rightarrow \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r}\right)\left(\frac{-x}{y^2}\right) + \left(\frac{\partial u}{\partial s}\right)\left(\frac{1}{z}\right) + \left(\frac{\partial u}{\partial t}\right)(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{x}{y^2}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{z}\left(\frac{\partial u}{\partial s}\right) \longrightarrow 2$$

$$\frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial z}\right) + \left(\frac{\partial u}{\partial s}\right)\left(\frac{\partial s}{\partial z}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial t}{\partial z}\right)$$

(chain rule)

$$\Rightarrow \frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial r}\right)(0) + \left(\frac{\partial u}{\partial s}\right)\left(-\frac{y}{z^2}\right) + \left(\frac{\partial u}{\partial t}\right)\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{y}{z^2}\left(\frac{\partial u}{\partial s}\right) + \frac{1}{x}\left(\frac{\partial u}{\partial t}\right) \longrightarrow 3.$$

$$LHS = x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \cancel{\frac{x}{y}\left(\frac{\partial u}{\partial r}\right)} - \cancel{\frac{z}{x}\left(\frac{\partial u}{\partial t}\right)} - \cancel{\frac{x}{y}\left(\frac{\partial u}{\partial r}\right)} + \cancel{\frac{y}{z}\left(\frac{\partial u}{\partial s}\right)} - \cancel{\frac{y}{z}\left(\frac{\partial u}{\partial s}\right)} + \cancel{\frac{z}{x}\left(\frac{\partial u}{\partial t}\right)} = 0$$

$\therefore$ LHS = RHS

Hence, proved.

Question Bank

(1 MARK QUESTIONS)

1. Find $\frac{\partial^2 u}{\partial x \partial y}$, when $u = e^{xyz}$.

$$\frac{\partial u}{\partial y} = e^{xyz}(xz)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[(xz)e^{xyz}\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = z e^{xyz} + (xz)e^{xyz}(yz)$$

$$= (z + xyz^2)e^{xyz}$$

$$= \underline{\underline{z e^{xyz}(1 + xyz)}}$$

2. Find $\frac{\partial^3 z}{\partial x^2 \partial y}$, when $z = \sin(xy)$

$$\frac{\partial z}{\partial y} = [\cos(xy)]x = x\cos(xy)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\cos(xy) - xy\sin(xy)\right]$$

$$= -y\sin(xy) - xy^2\cos(xy) - y\sin(xy)$$

$$= \underline{\underline{-2y\sin(xy) - xy^2\cos(xy)}}$$

3. If $u = x^y$, then find $\frac{\partial u}{\partial x}$.

$$\frac{\partial u}{\partial x} = y x^{y-1} = y \frac{x^y}{x} = u\left(\frac{y}{x}\right)$$

4. If $u = \sin^{-1}\left(\frac{y}{x}\right)$, then what is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$?

$$u = \sin^{-1}\left(\frac{y}{x}\right).$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{-y}{x^2}\right)$$

$$= \left(\frac{x}{\sqrt{x^2-y^2}}\right)\left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\therefore x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)$$

$$= \cancel{x}\left(\frac{-y}{\cancel{x}\sqrt{x^2-y^2}}\right) + y\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0$$

5. If $u = x^y$, find $\frac{\partial u}{\partial y}$ then.

$$\frac{\partial u}{\partial y} = x^y \log x = u \log_e x$$

9. If $u = x^2y + xy^2$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

$u = x^2y + xy^2$

$\Rightarrow \frac{\partial u}{\partial x} = 2xy + y^2$

$\Rightarrow \frac{\partial u}{\partial y} = x^2 + 2xy$

$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)$

$= x(2xy + y^2) + y(x^2 + 2xy)$

$= 2x^2y + xy^2 + x^2y + 2xy^2$

$= 3x^2y + 3xy^2 = \underline{\underline{3u}}$

10. If $u = xy$, where $x = t^2$ and $y = t$, $\frac{du}{dt} = ?$

$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$

$= y(2t) + x(1)$

$= 2yt + x$

$= 2t^2 + t^2 = \underline{\underline{3t^2}}$

(2 MARK QUESTIONS)

1. If $u = f(x+ay) + g(x-xy)$, what is $\frac{\partial^2 u}{\partial y^2}$?

$u = f(x+ay) + g(x-xy)$

$\frac{\partial u}{\partial y} = a f'(x+ay) - x g'(x-xy)$

$$\frac{\partial^2 u}{\partial y^2} = a^2 f''(x+ay) + x^2 g''(x-xy)$$

2. If $u = x^m y^n$, find value of $\frac{\partial^2 u}{\partial y \partial x}$?

$u = x^m y^n$

$$\frac{\partial u}{\partial x} = m x^{m-1} y^n$$

$$\frac{\partial^2 u}{\partial y \partial x} = mn x^{m-1} y^{n-1}$$

3. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, then what is $\frac{\partial u}{\partial x}$ at $(1,1)$?

$u = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right]\left(\frac{-y}{x^2}\right)$$

$$= \left(\frac{\cancel{x^2}}{x^2+y^2}\right)\left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{x^2+y^2}$$

At $(1,1)$, $\left.\frac{\partial u}{\partial x}\right]_{(1,1)} = \frac{-1}{2}$

4. If $u = e^x \log(1+y)$, then find $\frac{\partial^3 u}{\partial y \partial x^2}$ at $(0,0)$.

$u = e^x \log(1+y)$

$$\frac{\partial u}{\partial x} = e^x \log(1+y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \log(1+y)$$

$$\frac{\partial^3 u}{\partial y \partial x^2} = \frac{e^x}{1+y}$$

At $(0,0)$, $\frac{\partial^3 u}{\partial y \partial x^2} = 1$

If $\sin u = \dfrac{x^2 y^3}{x+y}$, find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

$$\sin u = \frac{x^2y^3}{x+y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2xy^3}{x+y} + \frac{x^2y^3}{-(x+y)^2}(y)$$

$$= \left[\left(\frac{2xy^3}{x+y}\right) - \left(\frac{x^2y^4}{(x+y)^2}\right)\right]\sec u$$

$$\Rightarrow \frac{\partial u}{\partial y} = \left(\frac{3x^2y^2}{x+y}\right) - \left[\frac{x^2y^3}{(x+y)^2}\right]x$$

[Euler's theorem]

$$= \left[\left(\frac{3x^2y^2}{x+y}\right) - \frac{x^3y^3}{(x+y)^2}\right]\sec u$$

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)$$

$$= \left[\frac{2x^2y^3}{(x+y)} - \frac{x^3y^4}{(x+y)^2} + \frac{3x^2y^3}{(x+y)} - \frac{x^3y^4}{(x+y)^4}\right]\sec u$$

$$= \left[\frac{5x^2y^3}{(x+y)} - \frac{2x^3y^4}{(x+y)^2}\right]\sec u$$

$$= \left[\frac{5x^2y^3(x+y) - 2x^3y^4}{(x+y)^2}\right]\sec u$$

$$= \left[\frac{5x^3y^3 + 5x^2y^4 - 2x^3y^4}{(x+y)^2}\right]\sec u$$

$$= \frac{x^2y^3}{(x+y)^2}\left[5x + 5y - 2xy\right]\sec u$$

$$\sin u = \frac{x^2y^3}{x+y}$$

$$= \frac{x^5\left(\frac{y}{x}\right)^3}{x\left(1+\frac{y}{x}\right)}$$

$$= x^4 g\left(\frac{y}{x}\right)$$

$$x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right) = n\frac{g(u)}{g'(u)}$$

$$= 4\left(\frac{\sin u}{\cos u}\right)$$

$$= 4\tan u$$

Question - Answers

1. If $u = x^3y + x^2y^3 + 10xy$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$

Given, $= x^3y + x^2y^3 + 10xy$

$$\frac{\partial u}{\partial x} = 3x^2y + 2xy^3 + 10y$$

$$\frac{\partial^2 u}{\partial x^2} = 6xy + 2y^3$$

$$\frac{\partial u}{\partial y} = x^3 + 3x^2y^2 + 10x$$

$$\frac{\partial^2 u}{\partial y^2} = 6x^2y$$

2. If $u = x^3 + y^3 + 3axy$, prove $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Given, $u = x^3 + y^3 + 3axy$.

LHS:

$$\frac{\partial u}{\partial y} = 3y^2 + 3ax$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = 3a \longrightarrow (1)$$

RHS:

$$\frac{\partial u}{\partial x} = 3x^2 + 3ay$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = 3a \longrightarrow (2)$$

(1) = (2)

$\therefore$ LHS = RHS

Hence, proved.

3. If $u = x^2y + y^2z + z^2x$, prove that

$u_x + u_y + u_z = (x+y+z)^2$

Given, $u = x^2y + y^2z + z^2x$

$$\frac{\partial u}{\partial x} = u_x = 2xy + z^2 \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = u_y = x^2 + 2yz \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = u_z = y^2 + 2zx \rightarrow (3)$$

Adding (1), (2), (3),

$$u_x + u_y + u_z = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$

$$\Rightarrow u_x + u_y + u_z = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$\Rightarrow \underline{\underline{u_x + u_y + u_z = (x+y+z)^2}}$$

$\therefore$ LHS = RHS

Hence, proved.

4. If $u = \log(2x+2y) + \tan(2x-2y)$, prove that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}.$$

Given, $u = \log(2x+2y) + \tan(2x-2y)$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{2x+2y}\right)(2) + [\sec^2(2x-2y)](2)$$

$$= \frac{1}{x+y} + 2\sec^2(2x-2y)$$

$2[2\sec(2x-2y)][\sec(2x-2y)\tan(2x-2y)]2$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \left[\frac{-1}{(x+y)^2}\right](1) + 2[2\sec^2(2x-2y)\tan(2x-2y)(2)]$$

$$= \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\tan(2x-2y) \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \left(\frac{1}{2x+2y}\right)(2) + \sec^2(2x-2y)(-2)$$

$$= \frac{1}{x+y} - 2\sec^2(2x-2y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \left[\frac{-1}{(x+y)^2}\right](1) - 2[2\sec(2x-2y)][\sec(2x-2y)\tan(2x-2y)][-2]$$

$$= \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\tan(2x-2y) \rightarrow (2)$$

(1) = (2)

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$\therefore$ LHS = RHS.

Hence, proved.

5. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Given, $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

It is a symmetrical function.

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}x)$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = (-x)\left[-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{3}{2}-1}(\cancel{2}x)\right] - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= +3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} \rightarrow (1)$$

$|||^{ly}$, $\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} \rightarrow (2)$

$|||^{ly}$, $\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} \rightarrow (3)$

$$LHS = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Substituting ①, ②, ③ in above,

$$= +3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} +$$
$$3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} +$$
$$3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$= 3(x^2+y^2+z^2)^{1-\frac{5}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$= 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}} = 0 = RHS$$

$\therefore$ LHS = RHS

Hence, proved.

6. If $u = x^y$, then show that $\frac{x}{y}\left(\frac{\partial u}{\partial x}\right) + \frac{1}{\log x}\left(\frac{\partial u}{\partial y}\right) = 2u$.

Given, $u = x^y$. $\rightarrow$ ⓐ

$$\frac{\partial u}{\partial x} = y x^{y-1} \rightarrow ①$$

$$\frac{\partial u}{\partial y} = x^y \log x \rightarrow ②$$

$$LHS = \frac{x}{y}\left(\frac{\partial u}{\partial x}\right) + \frac{1}{\log x}\left(\frac{\partial u}{\partial y}\right)$$

Substituting ① and ②,

$$= \frac{x}{y}\left(y x^{y-1}\right) + \frac{1}{\cancel{\log x}}\left(x^y \cancel{\log x}\right)$$

$$= \frac{\cancel{x}}{\cancel{y}}\left(\frac{\cancel{y} x^y}{\cancel{x}}\right) + x^y$$

$$= 2x^y = 2u = RHS \quad \text{(from ⓐ)}$$

$\therefore$ LHS = RHS Hence, proved.

7. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = +\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

Given, $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \begin{array}{l} x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta) \\ \Rightarrow x^2+y^2 = r^2 \end{array}$

$\longrightarrow$ ①

$\therefore r = \sqrt{x^2+y^2}$

It is a symmetrical function.

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left[\frac{2x}{\sqrt{x^2+y^2}}\right] = \frac{x}{\sqrt{x^2+y^2}}$$

$|||^{ly}$, $$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{\sqrt{x^2+y^2}} + x\left[\left(-\frac{1}{\cancel{2}}\right)\frac{(\cancel{2}x)}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}}$$

$|||^{ly}$, $$\frac{\partial^2 r}{\partial y^2} = \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\text{LHS} = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$$

$$= \frac{2}{\sqrt{x^2+y^2}} - \frac{(x^2+y^2)}{(x^2+y^2)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \underline{\underline{\frac{1}{r}}} \longrightarrow ② \text{ (From ①)}$$

$$RHS = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right].$$

$$= \frac{1}{r}\left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right]$$

$$= \frac{1}{r}\left[\frac{\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}\right] = \frac{1}{r} \rightarrow (3)$$

(2) = (3)

LHS = RHS

Hence, proved.

Which variable is to be treated as constant?

Consider $x = r\cos\theta$ and $y = r\sin\theta$. $\rightarrow (1)$

To find $\left(\frac{\partial r}{\partial x}\right)$, we need to find a relation b/w x & r

It is possible in 2 ways.

$r = x\sec\theta \rightarrow (2)$

$r^2 = x^2 + y^2 \rightarrow (3).$

$\frac{\partial r}{\partial x}$ will be different in (2) and (3).

To avoid confusion,

NOTATION: $\left(\frac{\partial r}{\partial x}\right)_\theta$ means the partial derivative of r wrt x keeping θ constant, in a relation where r is a function of x and θ.

$$\therefore \left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta$$

$|||^{ly}$, $\frac{\partial}{\partial r} = \left(\frac{\partial}{\partial r}\right)_\theta$

$$\frac{\partial}{\partial \theta} = \left(\frac{\partial}{\partial \theta}\right)_r$$

18. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Given, $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ $r^2 = x^2 + y^2$
$\Rightarrow r = \sqrt{x^2+y^2}$

$u = f(r)$.

$$\frac{\partial r}{\partial x} = \left[\frac{1}{2\sqrt{x^2+y^2}}\right](2x) = \frac{x}{\sqrt{x^2+y^2}} \longrightarrow (1)$$

$$\frac{\partial r}{\partial y} = \left[\frac{1}{2\sqrt{x^2+y^2}}\right](2y) = \frac{y}{\sqrt{x^2+y^2}} \longrightarrow (2).$$

From prev ques, $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \longrightarrow (5).$

$$\frac{\partial u}{\partial x} = f'(r)\left(\frac{\partial r}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\left(\frac{\partial^2 r}{\partial x^2}\right) \longrightarrow (3)$$

$|||^{ly}$ $$\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\left(\frac{\partial^2 r}{\partial y^2}\right) \longrightarrow (4)$$

Adding (1) and (2),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Substituting (1), (2), (5) to above,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right] + f'(r)\left[\frac{1}{r}\right].$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

LHS = RHS

Hence, proved.

9. Given $x = r\cos\theta$, $y = r\sin\theta$, find:

$$\left(\frac{\partial x}{\partial r}\right)_\theta, \left(\frac{\partial x}{\partial r}\right)_y, \left(\frac{\partial r}{\partial x}\right)_y, \left(\frac{\partial \theta}{\partial y}\right)_x.$$

Given, $\left.\begin{array}{l} r\cos\theta = x \\ r\sin\theta = y \end{array}\right\} \begin{array}{l} r^2 = x^2 + y^2 \\ \Rightarrow r = \sqrt{x^2+y^2} \end{array} \longrightarrow (1)$

$x = r\cos\theta$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \underline{\underline{\cos\theta}}$$

From (1), $r^2 = x^2 + y^2$

$\Rightarrow x^2 = r^2 - y^2$

$\Rightarrow x = \sqrt{r^2 - y^2}$

$$\left(\frac{\partial x}{\partial r}\right)_y = \frac{1}{2}\left(\frac{1}{\sqrt{r^2-y^2}}\right)(2r) = \frac{r}{\sqrt{r^2-y^2}} = \frac{r}{x} = \underline{\underline{\sec\theta}}$$

↳ we have to express x in terms of r and y.

From (1), $r = \sqrt{x^2+y^2}$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{2}\left(\frac{1}{\sqrt{x^2+y^2}}\right)(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \underline{\underline{\cos\theta}}$$

↳ we have to express r in terms of x and y.

$$\frac{r\cos\theta}{r\sin\theta} = \frac{x}{y} \Rightarrow \frac{x}{y} = \cot\theta$$

$$\Rightarrow \frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right]\left(\frac{1}{x}\right) = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = \underline{\underline{\frac{x}{x^2+y^2}}}$$

↳ we have to express θ in terms of x and y.

20. If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$, $\frac{\partial y}{\partial \theta} = r\frac{\partial x}{\partial r}$. Hence prove that $\frac{\partial^2 x}{\partial \theta^2} + r\left(\frac{\partial x}{\partial r}\right) + r^2\left(\frac{\partial^2 x}{\partial r^2}\right) = 0$.

$$\left.\begin{aligned} x &= e^{r\cos\theta}\cos(r\sin\theta) \\ y &= e^{r\cos\theta}\sin(r\sin\theta) \end{aligned}\right\} \text{(given).}$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)[\cos(r\sin\theta)] + $$
$$+ e^{r\cos\theta}[-\sin(r\sin\theta)][(r\cos\theta)]$$
$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)].$$
$$= -re^{r\cos\theta}[\sin(\theta + r\sin\theta)]. \longrightarrow (1)$$

$$\frac{\partial x}{\partial r} = e^{r\cos\theta}(1\cos\theta)[\cos(r\sin\theta)]$$
$$+ e^{r\cos\theta}[-\sin(r\sin\theta)](1\sin\theta)$$
$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)].$$
$$= e^{r\cos\theta}[\cos(\theta + r\sin\theta)]. \longrightarrow (2)$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)[\sin(r\sin\theta)] +$$
$$e^{r\cos\theta}[\cos(r\sin\theta)](r\cos\theta)$$
$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)].$$
$$= re^{r\cos\theta}[\cos(\theta + r\sin\theta)] \longrightarrow (3)$$

$$\frac{\partial y}{\partial r} = e^{r\cos\theta}(\cos\theta)[\sin(r\sin\theta)]$$
$$+ e^{r\cos\theta}[\cos(r\sin\theta)](1\sin\theta)$$
$$= e^{r\cos\theta}[\cos\theta\sin(r\sin\theta) + \sin\theta\cos(r\sin\theta)]$$
$$= e^{r\cos\theta}[\sin(\theta + r\sin\theta)]. \longrightarrow (4).$$

From ① and ④, $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$ ⟶ ⑤

From ② and ③, $\dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$ ⟶ ⑥

From ⑤,

$$\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta} \longrightarrow ⑦$$

From ⑥,

$$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 x}{\partial r^2} = \frac{-1}{r^2}\left(\frac{\partial y}{\partial \theta}\right) + \frac{1}{r}\left(\frac{\partial^2 y}{\partial r \partial \theta}\right) \longrightarrow ⑧$$

$$\text{LHS} = \frac{\partial^2 x}{\partial \theta^2} + r\left(\frac{\partial x}{\partial r}\right) + r^2\left(\frac{\partial^2 x}{\partial r^2}\right)$$

$$= -r\left(\frac{\partial^2 y}{\partial r \partial \theta}\right) + r\left[\frac{1}{r}\left(\frac{\partial y}{\partial \theta}\right)\right] + r^2\left[-\frac{1}{r^2}\left(\frac{\partial y}{\partial \theta}\right) + \frac{1}{r}\left(\frac{\partial^2 y}{\partial r \partial \theta}\right)\right]$$

$$= \cancel{-r\left(\frac{\partial^2 y}{\partial r \partial \theta}\right)} + \cancel{\frac{\partial y}{\partial \theta}} - \cancel{\frac{\partial y}{\partial \theta}} + \cancel{r\left(\frac{\partial^2 y}{\partial r \partial \theta}\right)} = 0 = \text{RHS}$$

∴ LHS = RHS

Hence, proved.

22. Find the rate at which ar(rectangle) is increasing at a given instant when the sides of the rectangle are 5 ft and 4 ft and are increasing at the rate of 1.5 ft/s and 0.5 ft/s.

Let x = length and
y = breadth
be the sides of the rectangle
$x = 5 ft$, $y = 4 ft$ (given)

Given, $\frac{dx}{dt} = 1.5 ft/s$

$\frac{dy}{dt} = 0.5 ft/s$

Area $= xy$

$$\frac{dA}{dt} = y\left(\frac{dx}{dt}\right) + x\left(\frac{dy}{dt}\right)$$ (using product rule)

$$= 4(1.5) + 5(0.5)$$

$$= 6 + 2.5 = \underline{\underline{8.5 ft}}$$

23. The altitude of a right circular cone is 15 cm and is increasing at 0.2 cm/s. The radius of the base is 10 cm and is decreasing at 0.3 cm/s. How fast is the volume changing?

Given, height $= h = 15 cm$
radius $= r = 10 cm$

Given, $\frac{dh}{dr} = 0.2 cm/s$ and $\frac{dr}{dt} = -0.3 cm/s$

Volume $= \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2r\left(\frac{dr}{dt}\right)h + r^2\left(\frac{dh}{dt}\right)\right]$$ (using product rule)

$$= \frac{1}{3}\pi\left[20 \times 15(-0.3) + 100(0.2)\right]$$

$$= \frac{1}{3}\pi\left[300(-0.3) + 20\right]$$

$$= \frac{1}{3}\pi\left[-90+20\right] = -\frac{70\pi}{3} \text{ cm}^3/\text{s}$$

4. If $z = \log(u^2+v)$, where $u = e^{x+y^2}$, $v = x+y^2$, then find $2y\left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial z}{\partial y}\right)$.

Given, $z = \log(u^2+v)$

$u = e^{x+y^2}$ and $v = x+y^2$

Using chain rule,

$$\frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \left(\frac{1}{u^2+v^2}\right)(2u)\left[e^{x+y^2}(1)\right] + \left(\frac{1}{u^2+v^2}\right)(2v)[1]$$

$$= \frac{2ue^{x+y^2} + 2v}{u^2+v^2} \longrightarrow (1).$$

$$\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{2u}{u^2+v^2}\right)\left[e^{x+y^2}(2y)\right] + \left(\frac{2v}{u^2+v^2}\right)(2y)$$

$$= \frac{2u(2y)e^{x+y^2} + 4vy}{u^2+v^2}$$

$$= \frac{4uye^{x+y^2} + 4vy}{u^2+v^2}$$

$\therefore 2y\left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial z}{\partial y}\right)$

$$\Rightarrow \left[\frac{4uye^{x+y^2} + 4vy}{u^2+v^2}\right] - \left[\frac{4uye^{x+y^2} + 4vy}{u^2+v^2}\right]$$

$= \underline{\underline{0}}$

25. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that

$$x^2\left(\frac{\partial u}{\partial x}\right) + y^2\left(\frac{\partial u}{\partial y}\right) + z^2\left(\frac{\partial u}{\partial z}\right) = 0.$$

Let $p = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$q = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

$u \to (p, q) \to (x, y\, z) \Rightarrow u \to (x, y, z)$

Using chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \left(\frac{-1}{x^2}\right)\left(\frac{\partial u}{\partial p}\right) + \left(-\frac{1}{x^2}\right)\left(\frac{\partial u}{\partial q}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{x^2}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right)$$

$$\Rightarrow +x^2\left(\frac{\partial u}{\partial x}\right) = -\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right) \longrightarrow (1).$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial y}\right)$$

$$= \frac{\partial u}{\partial p}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial q}(0)$$

$$\Rightarrow y^2\left(\frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial p} \longrightarrow (2).$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial z}\right)$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)$$

$$\Rightarrow z^2\left(\frac{\partial u}{\partial z}\right) = \frac{\partial u}{\partial q} \longrightarrow (3).$$

$$\text{LHS} = x^2\left(\frac{\partial u}{\partial x}\right) + y^2\left(\frac{\partial u}{\partial y}\right) + z^2\left(\frac{\partial u}{\partial z}\right)$$

Substituting (1), (2), (3),

$$= -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0 = \text{RHS}.$$

$\therefore$ LHS = RHS Hence, proved.

5. Show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$,

where u is a function of x, y, where $x = r\cos\theta$ and $y = r\sin\theta$.

Given, $u = f(x, y)$
$x = r\cos\theta$
$y = r\sin\theta$.

$u \rightarrow (x, y) \rightarrow (r, \theta) \Rightarrow u \rightarrow (r, \theta)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}\right).$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \longrightarrow (1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial \theta}\right)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = r\left[-\sin\theta\left(\frac{\partial u}{\partial x}\right) + \cos\theta\left(\frac{\partial u}{\partial y}\right)\right]$$

$$\Rightarrow \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) = -\sin\theta\left(\frac{\partial u}{\partial x}\right) + \cos\theta\left(\frac{\partial u}{\partial y}\right) \longrightarrow (2).$$

Squaring and adding (1) and (2),

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$+ \cancel{2\cos\theta\sin\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)} + \sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$\cancel{-2\sin\theta\cos\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 (\sin^2\theta + \cos^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$\therefore$ LHS = RHS

Hence, proved.

Implicit functions for functions of two variables.

If x is an independent variable and y is a dependent variable and if y is not explicitly expressed in terms of x, then the functional relationship between x and y is called implicit function.

Any implicit function $f(x, y) = c$ is an implicit functions where y is a function of x and c is constant.

If $\boxed{f(x, y) = c}$, then $\boxed{\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-f_x}{f_y}}$

27. Find $\frac{dy}{dx}$ at the point $(1, 1)$ for $e^y - e^x + xy = 1$.

Given, $f(x, y) = e^y - e^x + xy$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{-e^x + y}{e^y + x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x + y}{e^y + x}$$

At $(1, 1)$,

$$\frac{dy}{dx} = \left(\frac{e+1}{e+1}\right) = \underline{\underline{1}}$$

28. Find $\frac{du}{dx}$ if $u = \tan(x^2 + y^2)$ and x, y are connected by the relation $x^2 - y^2 = 2$.

$u \to (x, y)$ and $(y \to x)$ $\Rightarrow u \to x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$= \sec^2(x^2 + y^2)(2x)(1) + \sec^2(x^2 + y^2)(2y)\left(\frac{dy}{dx}\right) \longrightarrow (1).$$

$x^2 - y^2 = 2$ (given)

$\Rightarrow y^2 = x^2 - 2$

$\Rightarrow 2y\left(\frac{dy}{dx}\right) = 2x$ (differentiate w.r.t. x)

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Substitititing in ①,

$$\frac{du}{dx} = \sec^2(x^2+y^2)(2x) + \sec^2(x^2+y^2)(2y)\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{du}{dx} = 4x\sec^2(x^2+y^2) \quad \therefore ②$$

$$\Rightarrow \frac{du}{dx} = 4x\sec^2(x^2+x^2-2)$$

$$\Rightarrow \frac{du}{dx} = \underline{4x\sec^2(2x^2-2)}$$

29. Find $\frac{du}{dx}$ given $u = x\log(xy)$ and $x^3+y^3+3xy-1=0$

$u \to (x,y)$ and $(y \to x) \Rightarrow (u \to x)$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$= \left[\log(xy) + \left(\frac{x}{\cancel{xy}}\right)(\cancel{y})\right](1) + \left[\left(\frac{x}{xy}\right)x\right]\left(\frac{dy}{dx}\right)$$

$$= [\log(xy)+1] + \frac{x}{y}\left(\frac{dy}{dx}\right) \longrightarrow ①$$

Given, $x^3+y^3+3xy-1=0$.

Differentiate wrt x,

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) + 3y + 3x\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x^2 + y^2\left(\frac{dy}{dx}\right) + y + x\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x^2 + y = -(y^2+x)\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x^2+y}{y^2+x}\right) \longrightarrow (2).$$

(Use $\frac{dy}{dx} = -\frac{f_x}{f_y}$ always)

Substituting (2) in (1),

$$\frac{du}{dx} = \log(xy) + 1 - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

30. For the curve $xe^y + ye^x = 0$, find the equation of the tangent line at the origin.

$$xe^y + ye^x = 0.$$

$$\text{Slope} = \frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{e^y + ye^x}{xe^y + e^x}\right)$$

$$\left.\frac{dy}{dx}\right]_{\substack{(x=0)\\(\text{and } y=0)}} = -1 = m$$

Equation of tangent is given by,

$$y = mx \quad \Rightarrow \quad \underline{\underline{y = -x}}$$

Homogeneous Functions

i) A function $u = f(x,y)$ is said to be a homogenous function of degree n, if it can be expressed in the form $x^n g\left(\frac{y}{x}\right)$ or $y^n g\left(\frac{x}{y}\right)$, g being the arbitrary function.

Eg 1: Consider $u = 3x + 4y$

$\Rightarrow u = 3x + 4y$

$\Rightarrow u = x\left(3 + \frac{4y}{x}\right)$.

$\Rightarrow u = x^1 \cdot g\left(\frac{y}{x}\right)$

function g has terms only in the form of $\left(\frac{y}{x}\right)$.

$\therefore$ u is a homogeneous function of degree $= 1$

(OR) $u = 3x + 4y$

$\Rightarrow u = y\left(3\frac{x}{y} + 4\right)$.

$\Rightarrow u = y^1 \cdot g\left(\frac{x}{y}\right)$.

$\therefore$ u is a homogeneous function of degree $= 1$.

Eg 2: $u = x^2 y \tan^{-1}\left(\frac{x}{y}\right) + xy^2 \sec^{-1}\left(\frac{x}{y}\right)$

$u = x^3\left[\left(\frac{y}{x}\right)\tan^{-1}\left(\frac{x}{y}\right) + \left(\frac{y}{x}\right)^2 \sec^{-1}\left(\frac{x}{y}\right)\right]$.

$\Rightarrow u = x^3 g\left(\frac{y}{x}\right)$

$\therefore$ u is a homogeneous function of degree $= 3$.

(OR) $u = y^3\left[\left(\frac{x}{y}\right)^2 \tan^{-1}\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right) \sec^{-1}\left(\frac{x}{y}\right)\right]$

$\Rightarrow u = y^3 g\left(\frac{x}{y}\right)$

$\therefore$ u is a homogeneous function of degree $= 3$.

3: $u = x^3 y \tan^{-1}\left(\frac{x}{y}\right) + x y^3 \sec^{-1}\left(\frac{x}{y}\right)$.

$u = x^4\left[\left(\frac{y}{x}\right)\tan^{-1}\left(\frac{x}{y}\right) + \left(\frac{y}{x}\right)^3 \sec^{-1}\left(\frac{x}{y}\right)\right]$.

$\Rightarrow u = x^4 g\left(\frac{y}{x}\right)$

$\therefore$ u is a homogeneous function of degree $= 4$.

$u = y^4\left[\left(\frac{x}{y}\right)^3 \tan^{-1}\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)\sec^{-1}\left(\frac{x}{y}\right)\right]$.

$\Rightarrow u = y^4 g\left(\frac{x}{y}\right)$

$\therefore$ u is a homogeneous function of degree $= 4$.

ii) A function $u = f(x, y, z)$ is said to be a homogeneous function of degree n if it can be expressed in the form $x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ or $y^n g\left(\frac{x}{y}, \frac{z}{y}\right)$ or $z^n g\left(\frac{x}{z}, \frac{y}{z}\right)$, g being the arbitrary function.

Eg 1: Consider $u = x^3 + y^3 + z^3 + 3xyz$

$\Rightarrow u = x^3\left[1 + \left(\frac{y}{x}\right)^3 + \left(\frac{z}{x}\right)^3 + 3\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)\right]$

$\Rightarrow u = x^3 g\left(\frac{y}{x}, \frac{z}{x}\right)$

$\therefore$ u is a homogeneous function of degree $= 3$.