ENGINEERING MATHEMATICS - II

Random variables and probability distributions

Random variables and probability distributions

- Review of probability
- Random variables
- Functions of random variables

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- The word “probability” is commonly used to denote “the chance of happening of an event”. For example, statements like,

1. I have a fair chance (i.e., a reasonable probability of success) of getting admission.
2. In tossing a coin there is an even chance that a head may come up.

- In each case, we are not sure of the outcome, but we wish to estimate the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions (declaration/positive statement).

Elementary definitions

- Experiment: A process/procedure we perform that generates some result.

  For example, tossing a coin, rolling a die, etc.

- Outcome: An outcome is a possible result of an experiment. For example, in tossing a coin two times, the four possible outcomes are

  (H, T), (T, H), (T, T), and (H, H).

- Sample space: The set of "all possible" distinct outcomes of an experiment is called the sample space and is denoted by S. For example, for tossing a coin, the sample space S={H,T}.

- Note: A sample space S can be either "discrete" or "continuous".

Elementary definitions

- When the sample space S is countable, such as when it includes all integers between 1 and 9, it is known as a discrete sample space.
- When the sample space S is uncountably infinite, such as when it includes all real numbers between 1 and 9, it is known as a continuous sample space.
- Event: An event is a subset of a sample space. For example, for tossing a coin 3 times, the sample space is S={HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

  Let A be the event of getting exactly two heads.

  Then $A = \{\text{HHT, HTH, THH}\}$.

Elementary definitions

- Mutually exclusive events: Two events are said to be “mutually exclusive” if both cannot occur at the same time. For example, in a single coin toss, either a head comes up or a tail comes up, but not both. Mathematically, if two events A and B are mutually exclusive, then $A \cap B = \emptyset$.
- Equally likely event: The events are said to be “equally likely” if the chance of happening is equal of all events. For example, If we toss a coin, there are equal chances of getting a head or a tail. Hence, getting a head or a tail by tossing a coin are equally likely events.

Elementary definitions

- Exhaustive events: Two events A and B are known as exhaustive events if the union of A and B gives the sample space, i.e., $A \cup B = S$.
- For example, if a die is rolled, then the event of getting a prime number and an odd number are exhaustive events or not?

Sample space = {1,2,3,4,5,6}.

Event of getting a prime number = {1,2,3,5}.

Event of getting an odd number = {1,3,5}.

Union of both events = {1,2,3,5}.

Both events together do not equal to the sample space. Hence, these events are not exhaustive.

Definition of probability

- To find the probability of equally likely events, the following formula is used. $P(E) = \frac{n(E)}{n(S)}$

  Here, $n(E)$ is total number of favourable events.

  $n(S)$ is total number of events in sample space.

- Note that,

1. Probability of a certain (sure) event is one.
2. Probability of an impossible event is zero.

Definition of probability

- The probability of non-occurrence of an event E (called its failure) denoted by $P(\bar{E}) = \frac{n(S)-n(E)}{n(S)}$

$$= 1 - \frac{n(E)}{n(S)}$$

$$= 1 - P(E)$$

- Therefore, $P(E) + P(\bar{E}) = 1$

Axioms (statements that are always accepted as true) of probability

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Axiom 1: For any event E, $0 \le P(E) \le 1$

Axiom 2: Let S be a sample space. Then $P(S) = 1$

Axiom 3: If E_1 and E_2 are mutually exclusive events, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

More generally, if $E_1, E_2, \ldots, E_n, \ldots.$ mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$$

For example, consider,

Let A be the event of getting tails when you flip a fair coin.

Axiom 1: $P(A) = \frac{1}{2}$. Since $P(A)$ is between 0 and 1, Axiom 1 holds.

Axiom 2: S ={Head, Tail}. P(S)= $\frac{1}{2} + \frac{1}{2}$ = 1. So, Axiom 2 holds.

Axiom 3: Getting a head and a tail are two mutually exclusive events.

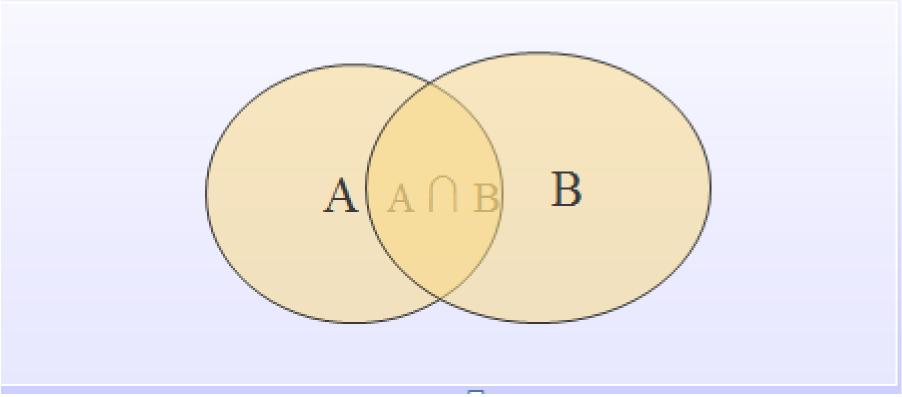
Then P({Head} U {Tail}) = P({Head})+P({Tail})= $\frac{1}{2} + \frac{1}{2}$= 1 .

So, Axiom 3 holds.

Additive theorem for probability

Statement: If A and B are any two arbitrary events (not necessarily mutually exclusive) of a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Additive theorem for mutually exclusive events

Statement: If A and B are two mutually exclusive events, then the probability of either A or B is given by $P(A \cup B) = P(A) + P(B)$.

For example, consider,

A card is drawn from a pack of 52, what is the probability that it is a king or a queen?

Solution: Let Event A = Draw of a card of king. Event B = Draw of a card of queen.

P (card draw is king or queen) = P (card is king) + P (card is queen)

$$P(A \cup B) = P(A) + P(B) \quad = \quad \frac{4}{52} \quad + \quad \frac{4}{52}$$

$$= \frac{2}{13}$$

Random variables

Let us take an example of tossing a coin.

The sample space is S =$\{\text{heads}, \text{tails}\}$

Let us give them the values: heads=0 and tails=1, and here we have a random variable X. That is,

$$X = \begin{cases} 0, & if\, \omega \in S \text{ is heads} \\ 1, & if\, \omega \in S \text{ is tails} \end{cases}$$

In short, $X = \{0,1\}$

Random variables

So,

- We have an **experiment** (such as tossing a coin).
- We give **values** to each event (subset of sample space).
- The **set of values** is a **Random Variable.**

Random variables

- A Random variable is a set of **possible values** from a random experiment.

- A random variable X on a sample space S is a function $X: S \longrightarrow R$ from S to the set of real numbers R, which assigns a real number $X(S)$ to each sample point ω of sample space S .

- A random variable X is a function from a sample space S into the real numbers R.

Example 1: Rolling a die once.

Solution: Now, Sample space Ω = {1, 2, 3, 4, 5, 6}.

Random Variable X = The number shown on the top face.

Here X = {1, 2, 3, 4, 5, 6}.

Let's find the probability value:

$P(X = 1) = \frac{1}{6}$; $P(X = 2) = \frac{1}{6}$; $P(X = 3) = \frac{1}{6}$; $P(X = 4) = \frac{1}{6}$; $P(X = 5) = \frac{1}{6}$

$P(X = 6) = \frac{1}{6}$

Note that the sum of the probabilities = **1**.

Types of random variables

- Random variables are classified into the following two categories:
- Discrete random variable
- Continuous random variable

Discrete random variable

- Discrete random variable: A random variable X is said to be discrete if it assumes a finite or countable number of distinct possible values.

Example:

- The number of people going to a cricket match.
- Number of people born in January.

Note that in all the examples mentioned above, we cannot have values like 1.1 or 1.2 or 2.1, and so on.

Continuous random variable

- Continuous random variable: A random variable X is said to be continuous if it assumes infinite number of distinct possible values.

Example:

- Average age of people born in January.
- Average weight of people born in January.

Functions of random variables

- A random variable is a rule that assigns a numeric value to every possible outcome in a sample space S.
- A function of a random variable is another random variable that is derived from the original variable by applying a mathematical function to it.
- These functions transform the original random variable into a new one with different properties, such as its mean, variance, and probability distribution.
- The main functions of random variables include: Probability Density Function (PDF); Cumulative Distribution Function (CDF); Expected Value; Variance; Standard Deviation; and Moment Generating Function.

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