Unit 1 class-4

1. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\left(1+x^2+y^2\right)^{3/2}}dxdy$ ans: 2π

$x = r\cos\theta \qquad y = r\sin\theta$

$\Rightarrow$ All 4 quadrants are considered,

So, $\int_0^{2\pi}\int_0^{\infty}\frac{1}{(1+r^2)^{3/2}}\cdot r\,dr\,d\theta$

$t = 1+r^2$

$dt = 2r\,dr \Rightarrow \frac{dt}{2} = r\,dr$

$\Rightarrow \int_0^{2\pi}\int_1^{\infty} t^{-3/2}\frac{dt}{2}\,d\theta$

$= \int_0^{2\pi} -\left[t^{-1/2}\right]_1^{\infty} d\theta$

$= \int_0^{2\pi} d\theta = \left[\theta\right]_0^{2\pi} = 2\pi$

2. $\iint y^2 dxdy$ over the area outside $x^2+y^2-ax=0$ and inside $x^2+y^2-2ax=0$ ans: $\frac{15\pi a^4}{64}$

$x = r\cos\theta \qquad y = r\sin\theta$

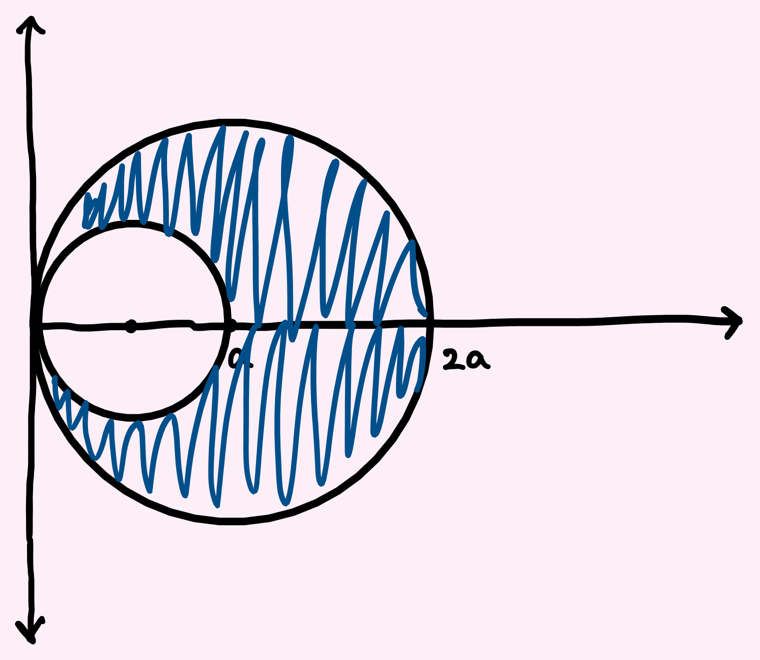
$dxdy = r\,dr\,d\theta$

$x^2+y^2 = ax \Rightarrow r^2 = ar\cos\theta$

$r = a\cos\theta$

$x^2+y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$



θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$\int_{-\pi/2}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta \cdot r \cdot dr \cdot d\theta$

$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta}\cdot\sin^2\theta\,d\theta$

$= \int_{-\pi/2}^{\pi/2}\frac{16-1}{4}\times a^4\cos^4\theta\cdot\sin^2\theta\,d\theta$

$= \frac{15a^4}{4}\int_{-\pi/2}^{\pi/2}\left(\sin\theta\cos\theta\right)^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \frac{15a^4}{4}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 2\theta}{4}\times\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \frac{15a^4}{32}\times 2\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)\left(1+\cos 2\theta\right)d\theta$

$= \frac{15a^4}{32}\int_0^{\pi/2}\left[1-\cos 4\theta+\cos 2\theta-\cos 4\theta\cdot\cos 2\theta\right]d\theta$

$= \frac{15a^4}{32}\int_0^{\pi/2}\left[1-\cos 4\theta+\cos 2\theta-\cos 4\theta\cdot\cos 2\theta\right]$

$= \frac{15a^4}{32}\left[\theta-\frac{\sin 4\theta}{4}+\frac{\sin 2\theta}{2}-\frac{\sin 6\theta+3\sin 2\theta}{12}\right]_0^{\pi/2}$

$= \frac{15a^4}{32}\times\frac{\pi}{2} = \frac{15\pi a^4}{64}$

3. $\iint\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$ the integral being extended over all positive values of x and y subject to $x^2+y^2\le 1$. ans: $\frac{\pi^2}{8}-\frac{\pi}{4}$

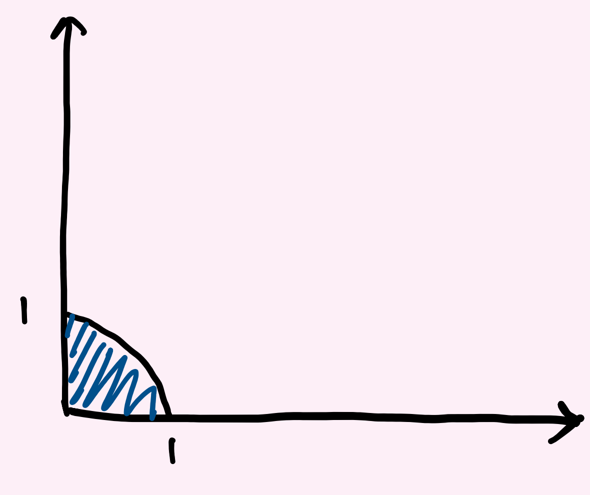
$x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\,dr\,d\theta$

$x^2+y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$

$\hookrightarrow$ No $-$ve cuz only $+$ve values

$x^2+y^2 = 0 \Rightarrow r = 0$



$\int_0^{\pi/2}\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\cdot r\,dr\,d\theta$

$u^2 = 1+r^2$

$\cancel{2}u\,du = \cancel{2}r\,dr$

$\int_0^{\pi/2}\int_1^{\sqrt{2}}\sqrt{\frac{2-u^2}{u^2}}\cdot u\,du\,d\theta$

$= \int_0^{\pi/2}\int_1^{\sqrt{2}}\sqrt{2-u^2}\,du\,d\theta$

$= \int_0^{\pi/2}\left[\frac{u}{2}\sqrt{2-u^2}+\frac{\cancel{2}}{\cancel{2}}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_1^{\sqrt{2}}d\theta$

$= \int_0^{\pi/2}\left[0-\frac{1}{2}+\sin^{-1}(1)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]d\theta$

$= \int_0^{\pi/2}\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{1}{2}\right)d\theta = \int_0^{\pi/2}\left(\frac{\pi}{4}-\frac{1}{2}\right)d\theta$

$= \frac{\pi}{4}\left[\theta\right]_0^{\pi/2}-\left[\frac{\theta}{2}\right]_0^{\pi/2}$

$= \frac{\pi^2}{8}-\frac{\pi}{4}$