B. Tech – II

CLASS 8

The Unit Step Function (Heaviside Function)

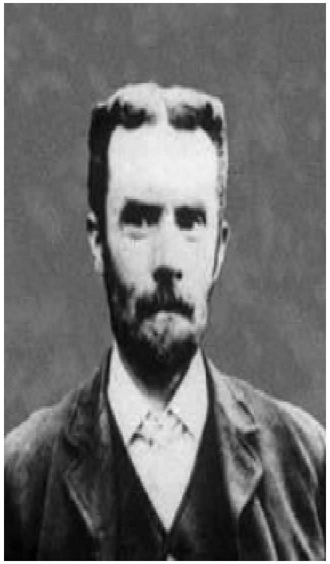
reason for development

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time t. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time t.

The value of $t = 0$ is usually taken as a convenient time to switch on or off the given voltage.

The switching process can be described mathematically by the function called the **Unit Step Function** (otherwise known as the **Heaviside function** after Oliver Heaviside).

WHY NAME HEAVISIDE ?



- Heaviside caught scarlet fever when he was a young child and this affected his hearing.

- At age 16 he left school. He taught himself Morse code and electricity. He was helped by his uncle Charles Wheatstone (after whom the Wheatstone bridge* was named).

- Heaviside introduced **operational calculus** to enable him to solve the ordinary DEs which came out of the theory of electrical circuits. He replaced the differential operator $\frac{d}{dx}$ by a variable p, which transformed differential equations into easier algebraic equations. The solution of the algebraic equation could be transformed back using conversion tables to give the solution of the original differential equation.

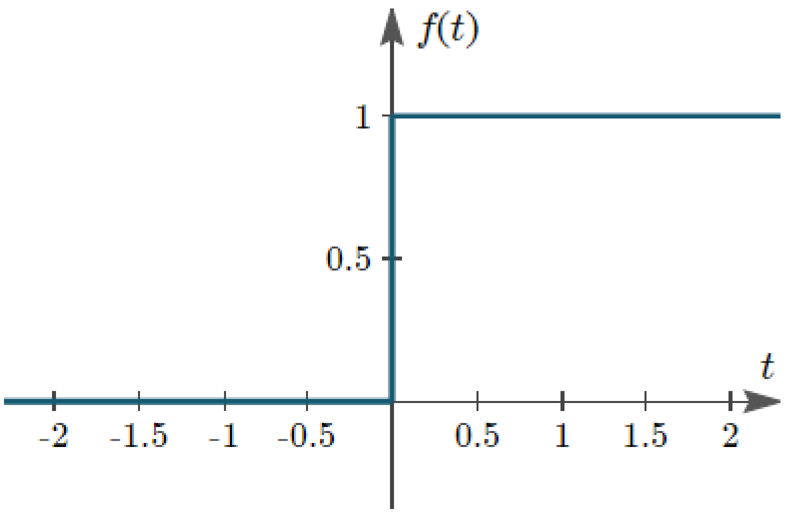
- Had the idea for an **induction coil** to increase induction, but it was patented in 1904 in the United States by AT&T.

UNIT STEP FUNCTION

Definition: The unit step function, $u(t)$, is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

That is, u is a function of time t, and u has value **zero** when time is negative (before we flip the switch); and value **one** when time is positive (from when we flip the switch).

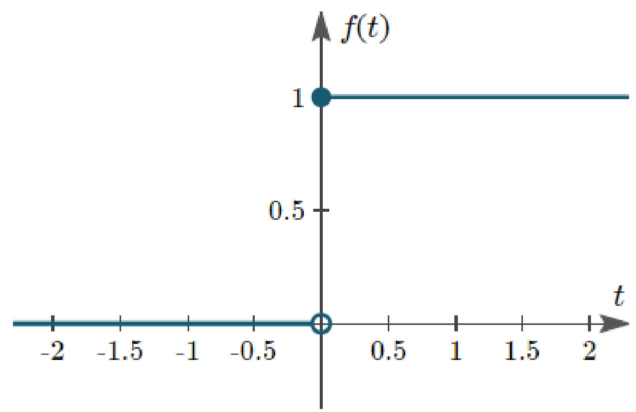


Graph of $f(t) = u(t)$, the unit step function.

In some text books you will see the unit step function defined as having value 1 at $t = 0$, as follows:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We would indicate the discontinuity on our graph like this:



Graph of $f(t) = u(t)$, the unit step function, with $f(0) = 1$.

Also, sometimes you'll see the value given as $f(0) = 0.5$.

In this work, it doesn't make a great deal of difference to our calculations, so we'll continue to use the first interpretation, and draw our graphs accordingly.

SHIFTED UNIT STEP FUNCTION

In many circuits, waveforms are applied at specified intervals other than $t = 0$. Such a function may be described using the **shifted** (aka **delayed**) unit step function.

Definition of Shifted Unit Step Function

A function which has value 0 up to the time $t = a$ and thereafter has value 1, is written:

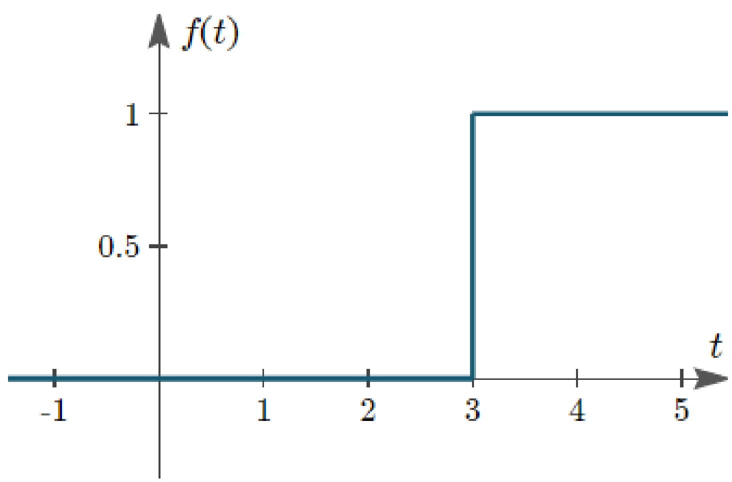
$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t > a \end{cases}$$

Example 1 - Shifted Unit Step Function

$$f(t) = u(t-3)$$

The equation means $f(t)$ has value of 0 when $t < 3$ and 1 when $t > 3$.

The sketch of the waveform is as follows:



Graph of $f(t) = u(t-3)$, a shifted unit step function.

Rectangular Pulse

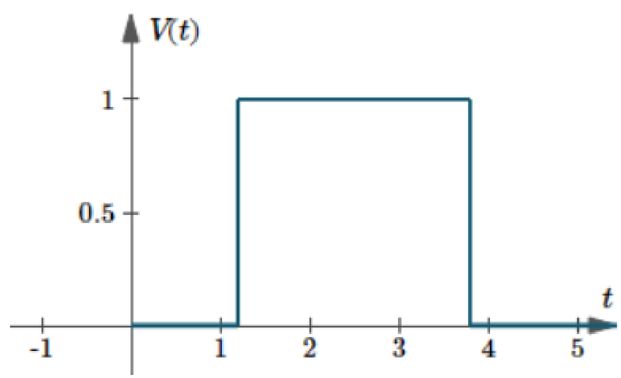
A common situation in a circuit is for a voltage to be applied at a particular time (say $t = a$) and removed later, at $t = b$ (say). We write such a situation using unit step functions as:

$$\boxed{V(t) = u(t-a) - u(t-b)}$$

This voltage has strength 1, duration $(b-a)$.

Example 2 - Rectangular Pulse

The graph of $V(t) = u(t-1.2) - u(t-3.8)$ is as follows. Here, the duration is $3.8 - 1.2 = 2.6$.



Write the following functions in terms of **unit step** function(s). Sketch each waveform.

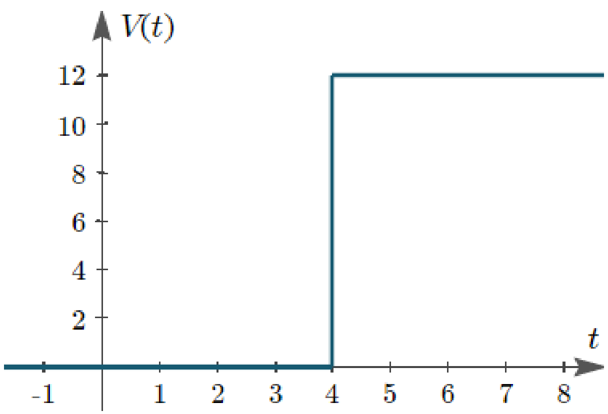
(a) A 12-V source is switched on at $t = 4$ s.

Since the voltage is turned on at $t = 4$, we need to use $u(t-4)$. We multiply by 12 since that is the voltage.

We write the function as follows:

$V(t) = 12 \cdot u(t-4).$

Here's the graph:



Graph of $V(t) = 12 \cdot u(t-4)$, a shifted step function.

LAPLACE TRANSFORM UNIT STEP AND SHIFTED UNIT STEP FUNCTION

Note that u(t) is also denoted as h(t) and u(t-a) as $u_a(t)$ or h(t-a)

Taking the Laplace transform of $h(t)$ we find

$$\mathcal{L}[h(t)] = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

A Heaviside function at $\alpha \geq 0$ is the shifted function $h(t-\alpha)$ (α units to the right). For this function, the Laplace transform is

$$\mathcal{L}[h(t-\alpha)] = \int_0^\infty h(t-\alpha)e^{-st}dt = \int_\alpha^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_\alpha^\infty = \frac{e^{-s\alpha}}{s},\ s > 0.$$

Standard results

- 1. $\mathcal{L}\{u(t)\} = \dfrac{1}{s}$

  2. $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

Various discontinuous function in terms of Unit step function

If $f(t) = \begin{cases} f_1(t) & , 0 < t < a \\ f_2(t) & , t > a \end{cases}$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u[t-a]$$

proof: $(0 < t < a)$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, \cancel{u[t-a]}^{\,0}$$

$$f(t) = f_1(t) \qquad 0 < t < a$$

$\underline{t > a}$ $\qquad u(t-a) = 1$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, \cancel{u(t-a)}^{\,1}$$

$$f(t) = f_2(t) \qquad t > a$$

In General

$$f(t) = f_1(t) \quad \text{for } 0 < t < a_1$$
$$= f_2(t) \quad \text{for } a_1 < t < a_2$$
$$\vdots$$
$$= f_{n-1}(t) \quad \text{for } a_{n-2} < t < a_{n-1}$$
$$= f_n(t) \quad \text{for } t > a_{n-1}$$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u[t-a_1]$$
$$+ [f_3(t) - f_2(t)]\,u[t-a_2] + \cdots$$
$$\cdots\cdots [f_n(t) - f_{n-1}(t)]\,u[t-a_{n-1}]$$

Express f (t) in terms of the Heavisides unit step function and find its Laplace transform:

$f(t) = \{(t^2, 0 < t < 2)\ (4t, 2 < t < 4)\ (8, t > 4)$

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$$

We get

$f(t) = t^2 + (4t - t^2)\, u(t-2) + (8 - 4t)\, u(t-4)$

$f(t) = t^2 + [4 - (t-2)^2]\, u(t-2) + [-4(t-4) - 8]\, u(t-4)$

Thanks all