Unit-1 class-6

1. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dxdydz$

A. $\int_{-c}^{c}\int_{-b}^{b}\left(\frac{x^3}{3}+y^2x+z^2x\right)_{-a}^{a}dydz$

$= \int_{-c}^{c}\int_{-b}^{b}\left[\left(\frac{a^3}{3}+ay^2+az^2\right)-\left(\frac{-a^3}{3}-ay^2-az^2\right)\right]dydz$

$= 2\int_{-c}^{c}\int_{-b}^{b}\left(\frac{a^3}{3}+ay^2+az^2\right)dydz$

$= 2\int_{-c}^{c}\left[\frac{a^3y}{3}+\frac{ay^3}{3}+ayz^2\right]_{-b}^{b}dz$

$= 2\int_{-c}^{c}\left[\frac{a^3}{3}(b+b)+\frac{a}{3}(b^3+b^3)+az^2(b+b)\right]dz$

$= 4\int_{-c}^{c}\left(\frac{a^3b}{3}+\frac{ab^3}{3}+abz^2\right)dz$

$= 4\left[\frac{a^3bz}{3}+\frac{ab^3z}{3}+\frac{abz^3}{3}\right]_{-c}^{c}$

$= \frac{8abc}{3}\left(a^2+b^2+c^2\right)$

2. $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y}e^{x+y+z}dzdydx$ ans: $\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}+e^{a}-\frac{3}{8}$

A. $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y}e^{x+y+z}dzdydx$

$= \int_{0}^{a}\int_{0}^{x}e^{x+y}.\left(e^{x+y}-1\right)dydx$

$= \int_{0}^{a}\int_{0}^{x}\left(e^{2x+2y}-e^{x+y}\right)dydx$

$= \int_{0}^{a}\left[\frac{e^{2x+2y}}{2}-\left(e^{x+y}\right)\right]_{0}^{x}dx$

$= \int_{0}^{a}\left[\frac{e^{4x}}{2}-\frac{e^{2x}}{2}-e^{2x}+e^{x}\right]dx$

$= \left[\frac{e^{4x}}{8}-\frac{e^{2x}}{4}-\frac{e^{2x}}{2}+e^{x}\right]_{0}^{a}$

$= \left[\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^{x}\right]_{0}^{a}$

$= \frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^{a}-\left(\frac{1}{8}-\frac{3}{4}+1\right)$

$= \frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^{a}-\frac{3}{8}$

3. $\int_{0}^{\frac{\pi}{2}}\int_{0}^{a\sin\theta}\int_{0}^{\frac{a^2-r^2}{a}}rdzdrd\theta$ ans: $\frac{5\pi a^3}{64}$

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$\int_{0}^{\pi/2}\int_{0}^{a\sin\theta}\int_{0}^{\frac{a^2-r^2}{a}}rdzdrd\theta$

$= \int_{0}^{\pi/2}\int_{0}^{a\sin\theta}\left[rz\right]_{0}^{\frac{a^2-r^2}{a}}drd\theta$

$= \int_{0}^{\pi/2}\int_{0}^{a\sin\theta}\left(\frac{ra^2-r^3}{a}\right)drd\theta$

$= \int_{0}^{\pi/2}\left(\frac{r^2a}{2}-\frac{r^4}{4a}\right)_{0}^{a\sin\theta}$

$= \int_{0}^{\pi/2}\left(\frac{a^3\sin^2\theta}{2}-\frac{a^3\sin^4\theta}{4}\right)d\theta$

$= \int_{0}^{\pi/2}\left[\frac{a^3}{2}\left(\frac{1-\cos 2\theta}{2}\right)-\frac{a^3}{4}\left(\left(\sin^2\theta\right)^2\right)\right]d\theta$

$= \int_{0}^{\pi/2}\left[\frac{a^3}{2}\left(\frac{1-\cos 2\theta}{2}\right)-\frac{a^3}{4}\left(\frac{1-\cos 2\theta}{4}\right)^2\right]d\theta$

$= \int_{0}^{\pi/2}\left[\frac{a^3}{4}-\frac{a^3\cos 2\theta}{4}-\frac{a^3}{16}\left(1+\cos^2 2\theta-2\cos 2\theta\right)\right]d\theta$

$= \int_{0}^{\pi/2}\left[\frac{a^3}{4}-\frac{a^3\cos 2\theta}{4}-\frac{a^3}{16}-\frac{a^3}{16}\left(\frac{1+\cos 4\theta}{2}\right)+\frac{a^3}{8}\cos 2\theta\right]d\theta$

$= \int_{0}^{\pi/2}\left(\frac{a^3}{4}-\frac{a^3\cos 2\theta}{4}-\frac{a^3}{16}-\frac{a^3}{32}-\frac{a^3\cos 4\theta}{32}+\frac{a^3\cos 2\theta}{8}\right)d\theta$

$= \left[\frac{a^3\theta}{4}-\frac{a^3\sin 2\theta}{8}-\frac{a^3\theta}{16}-\frac{a^3\theta}{32}-\frac{a^3\sin 4\theta}{128}+\frac{a^3\sin 2\theta}{16}\right]_{0}^{\pi/2}$

$= \frac{a^3\pi}{8}-\frac{a^3}{8}(0)-\frac{a^3\pi}{32}-\frac{a^3\pi}{64}-\frac{a^3}{128}(0)+\frac{a^3}{16}(0)$

$= \frac{8-2-1}{64}\times a^3\pi = \frac{5\pi a^3}{64}$ //