ENGINEERING MATHEMATICS - II

Random Variables and Probability Distributions

Department of Science and Humanities

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UNIT 2 : Random Variables and Probability Distributions

Session: 6

Subtopic: Poisson Distribution

- Poisson distribution is the discrete probability distribution of a discrete random variable x,

  which has no upper bound.

- It is defined for non-negative values of x as follows:

$$f(x,\lambda) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, \dots$$

- Here $\lambda > 0$ is called the parameter of the distribution and

Prerequisite:

- Each trial results in two mutually exclusive outcomes, termed as success and failure.
- The number of trials n is very large i.e., $n \to \infty$.
- The constant probability of success, p, is very small ($p \to 0$).
- $\lambda = np$ is a finite positive real number.

Note:

Binomial distribution can be approximated by Poisson distribution when $n \to \infty$ and $p \to 0$ such that $np = \lambda = constant$.

Examples that give rise to Poisson distribution:

- Number of printing mistakes per page.
- Number of accidents on a highway.
- Number of defectives in a production centre
- Number of telephone calls during a particular time etc…

Poisson Distribution – Derivation of Mean

Mean: $E(X) = \sum_{x=0}^{\infty} xP(x)$

$$= \sum_{x=0}^{\infty} x.\frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Poisson Distribution – Derivation of Variance

Variance: $E(X^2) = \sum_{x=0}^{\infty} x^2 P(x) - \lambda^2$

Consider $\sum_{x=0}^{\infty} x^2 P(x) = \sum_{x=0}^{\infty} x^2 . \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} [x(x-1)+x] . \frac{\lambda^x e^{-\lambda}}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x . x(x-1)}{x!} + e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x . x}{x!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= \lambda^2 e^{-\lambda} \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right) + \lambda\, e^{-\lambda} \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right)$$

$$= \lambda^2 + \lambda$$

Thus variance $E(X^2) = \lambda^2 + \lambda - \lambda^2 = \lambda$

Poisson Distribution – Example

1. Suppose that a book of 600 pages contains 40 printing mistakes. Assume that these errors are randomly distributed throughout the book and the number of errors per page has a Poisson distribution. What is the probability that 10 pages selected at random will be free of errors?

Poisson Distribution – Example

Solution:

$$p = \frac{40}{600} = \frac{1}{15}, \qquad n = 10, \quad \lambda = np = 10 \times \frac{1}{15} = \frac{2}{3}$$

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\left(\frac{2}{3}\right)^x e^{-\frac{2}{3}}}{x!}$$

$$P(x = 0) = \frac{\left(\frac{2}{3}\right)^0 e^{-\frac{2}{3}}}{0!} = e^{-\frac{2}{3}} = 0.51$$

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