ENGINEERING MECHANICS - STATICS

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M:

ENGINEERING MECHANICS

Course Content:

~~Unit 1: Introduction to statics~~

~~Mechanics, Basic Concepts, Scalars and Vectors, Force Systems - Introduction, Force, Rectangular Components, Moment, Numerical.~~

~~**8 Hours**~~

~~Unit 2: Force System~~

~~Force Systems - Couple, Resultants, Numerical.~~

~~**7 Hours**~~

ENGINEERING MECHANICS

Centroid

Department of Civil Engineering

Go through theory part

5/1 – Introduction to distributed forces

5/2 - Center of Mass

5/3 - Centroids of lines, Areas and volumes - Pg 229 to 237

5/4 Composite Bodies and figures - Pg 250 and 251

Go through theory part

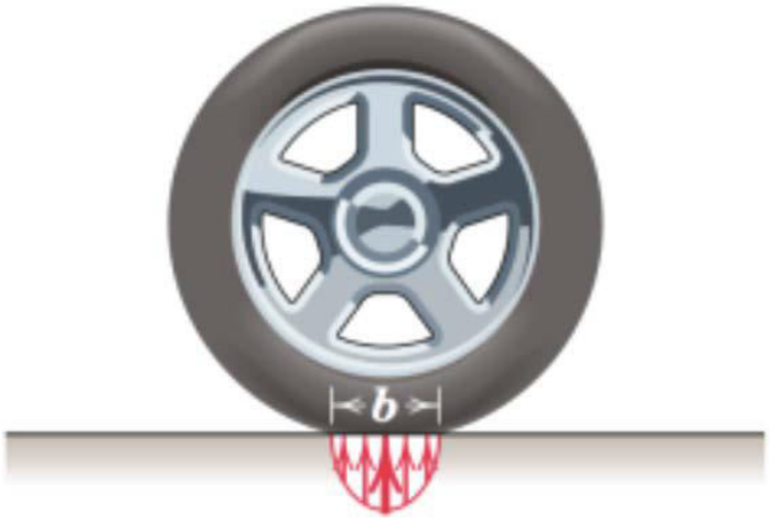
5/1 – Introduction to distributed forces

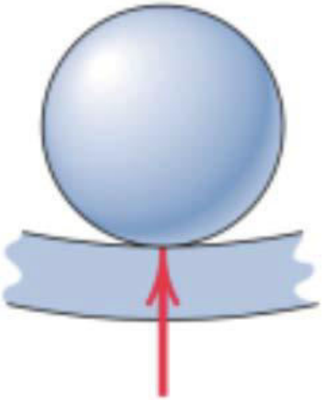
5/2 - Center of Mass

5/3 - Centroids of lines, Areas and volumes - Pg 229 to 237

5/4 Composite Bodies and figures - Pg 250 and 251

- In the previous chapters, we treated all forces as concentrated along their lines of action and at their point of application.
- But in practice, every external force applied mechanically to a body is distributed over a finite contact area.
- The force exerted by the pavement on an automobile tire





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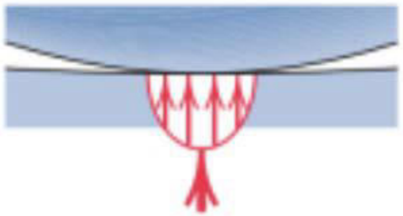
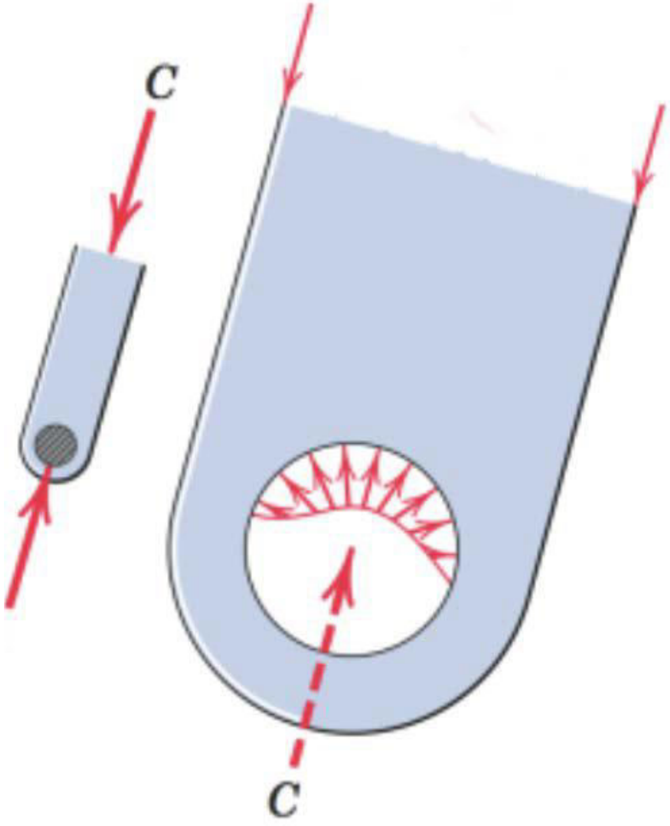
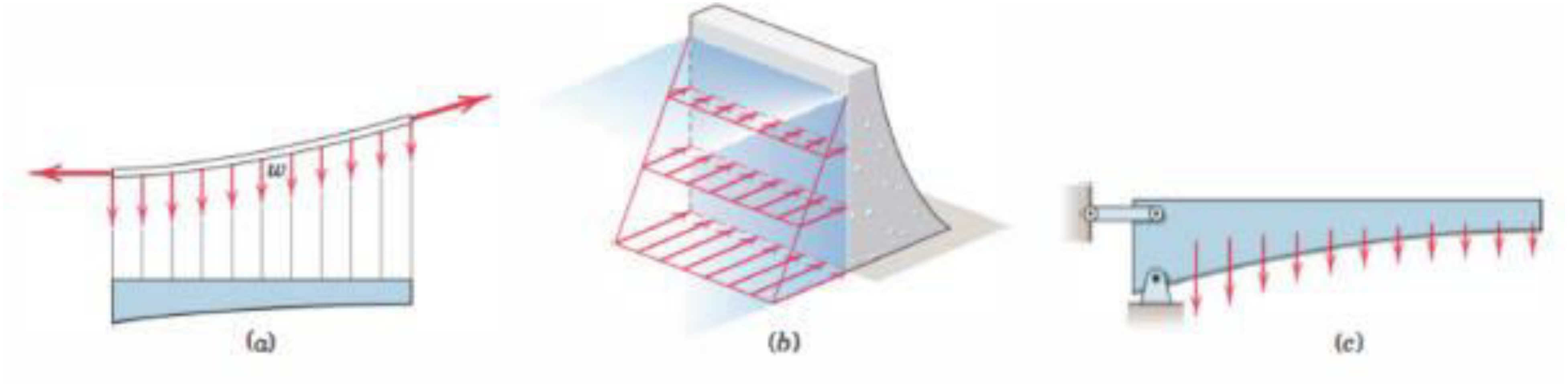




Line distribution Area distribution Volume distribution



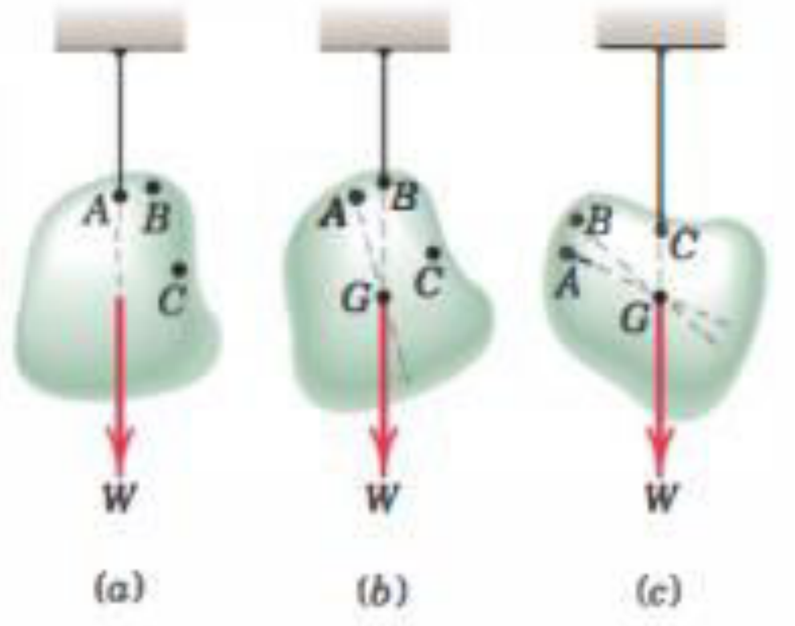
Center of Mass:

Center of Mass is the location or point where the entire mass of the body is supposed to be concentrated.

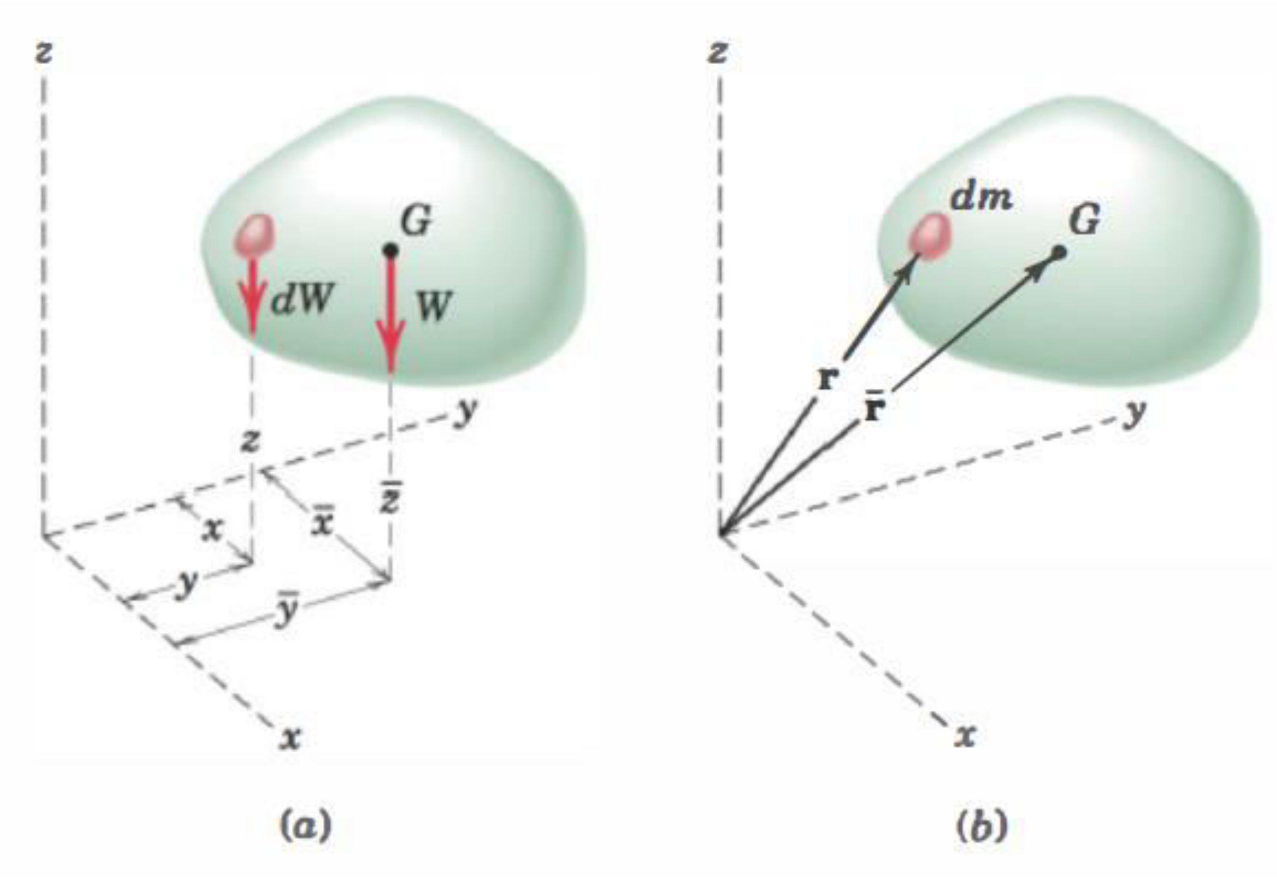
- Consider a three dimensional body of any size and shape having mass m.
- If we suspend the body as shown in figure from any point A, the body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational force acting on all particles of the body.
- This resultant is clearly collinear with the cord.

Center of Mass:

- If we repeat the experiment by suspending the body from other points such as B and C, and in each instance, mark the line of action of the resultant force.
- These lines of action of forces will meet at a single point G.
- This point is called the Center of Gravity of the body.
- The Centre of gravity is a theoretical point in the body where the body's total weight is thought to be concentrated.



Determining the center of gravity:

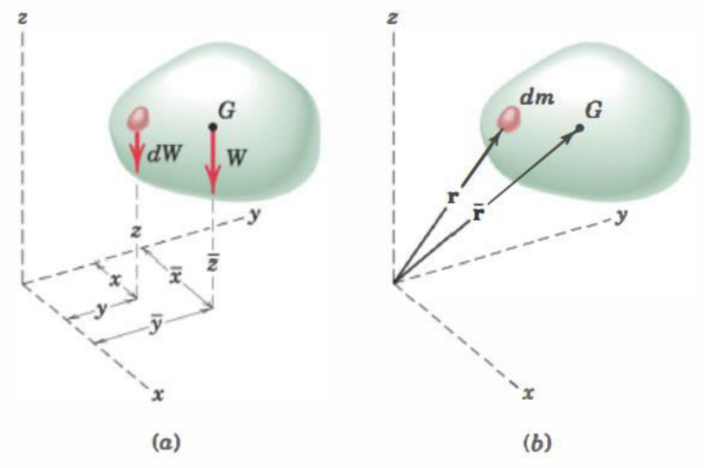


Determining the center of gravity:

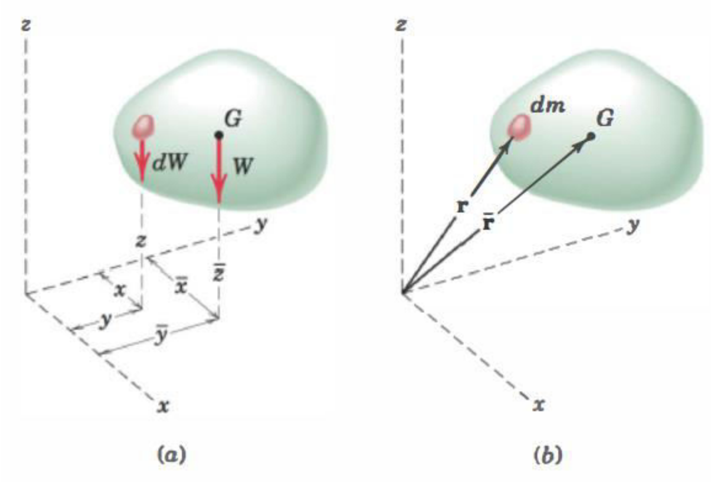
- To determine mathematically the location of the center of gravity of any body, we apply the principle of moments to the parallel system of gravitational forces.

- The moment of resultant gravitational force W about any axis in equal to the sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body.

- The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum W = $\int dW$



Determining the center of gravity



$$\bar{x} = \frac{\int x\, dW}{W} \qquad \bar{y} = \frac{\int y\, dW}{W} \qquad \bar{z} = \frac{\int z\, dW}{W} \qquad (5/1a)$$

$$\bar{x} = \frac{\int x\, dm}{m} \qquad \bar{y} = \frac{\int y\, dm}{m} \qquad \bar{z} = \frac{\int z\, dm}{m} \qquad (5/1b)$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m} \qquad (5/2)$$

Center of Mass and Center of Gravity

Center of Mass is the location or point where the entire mass of the body is supposed to be concentrated.

5/4 Composite Bodies and Figures: Pg 250-251

Sample Problem 5/2

Centroid of a triangular area. Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.

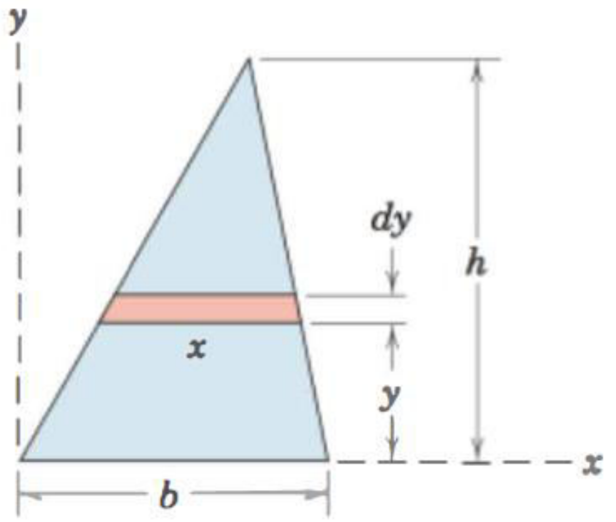
Solution. The x-axis is taken to coincide with the base. A differential strip of
① area $dA = x\,dy$ is chosen. By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. 5/5a gives

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

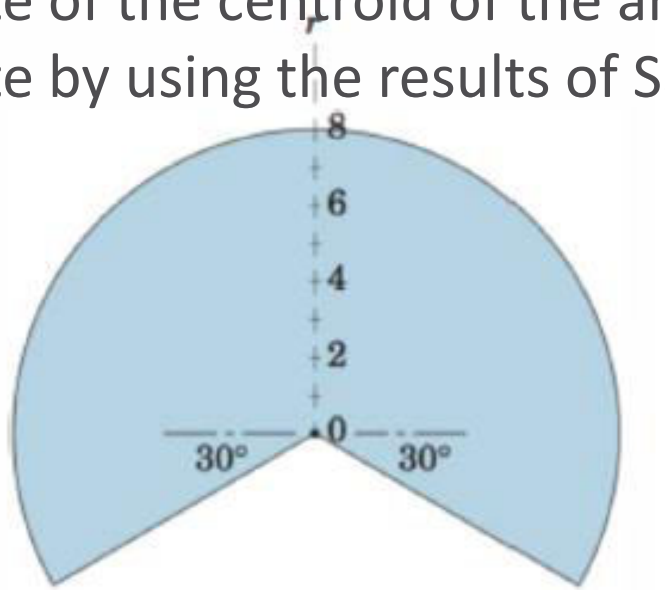
and

$$\bar{y} = \frac{h}{3}$$

Ans.



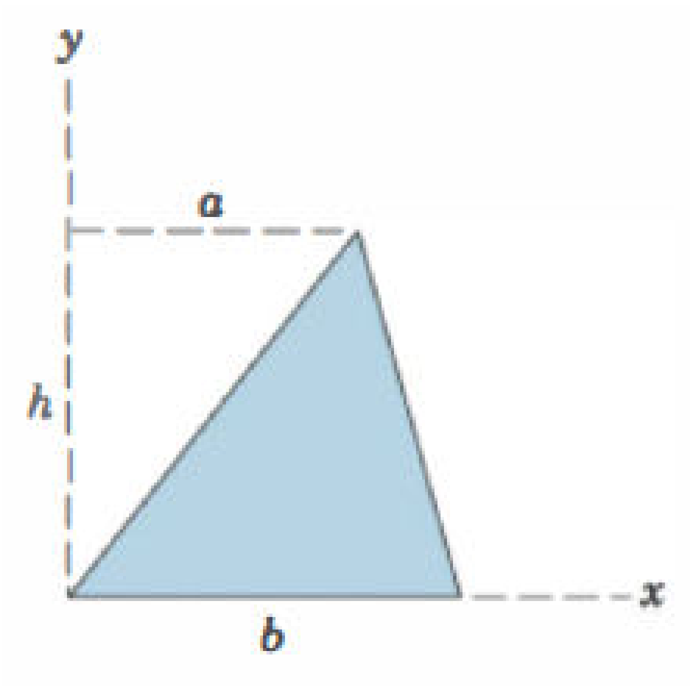
5/2 With your pencil make a dot on the position of your best visual estimate of the centroid of the area of the circular sector. Check your estimate by using the results of Sample Problem 5/3.



5/2 From Sample Problem 5/3 with $r = 8$ and $\alpha = 120° = \frac{2}{3}\pi$:

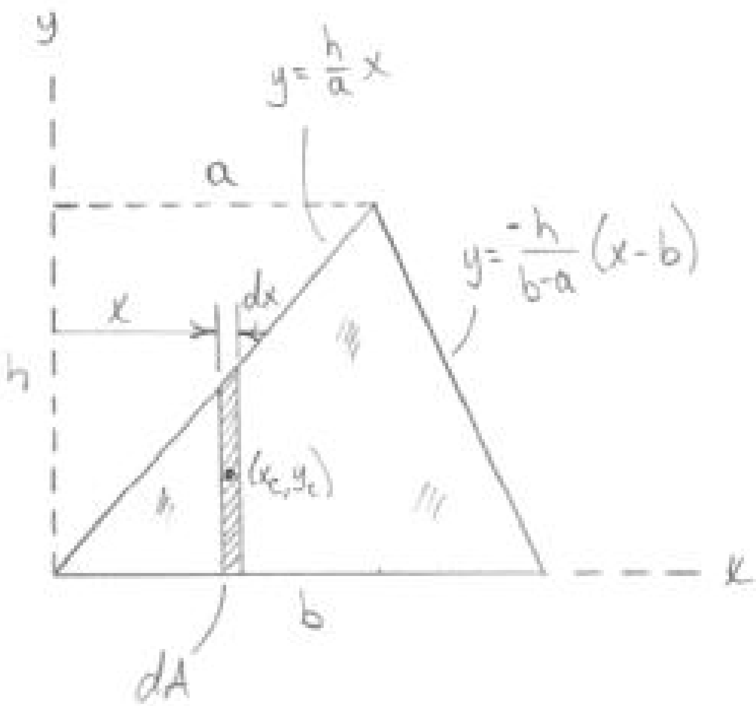
$$\bar{r} = \frac{2}{3}(8)\frac{\sin 120°}{2\pi/3} = \underline{2.21}$$

5/5 Determine the x-coordinate of the centroid of the shaded area.



5/5

UTILIZE A VERTICAL STRIP



$dA = y\,dx$ & $x_c = x$

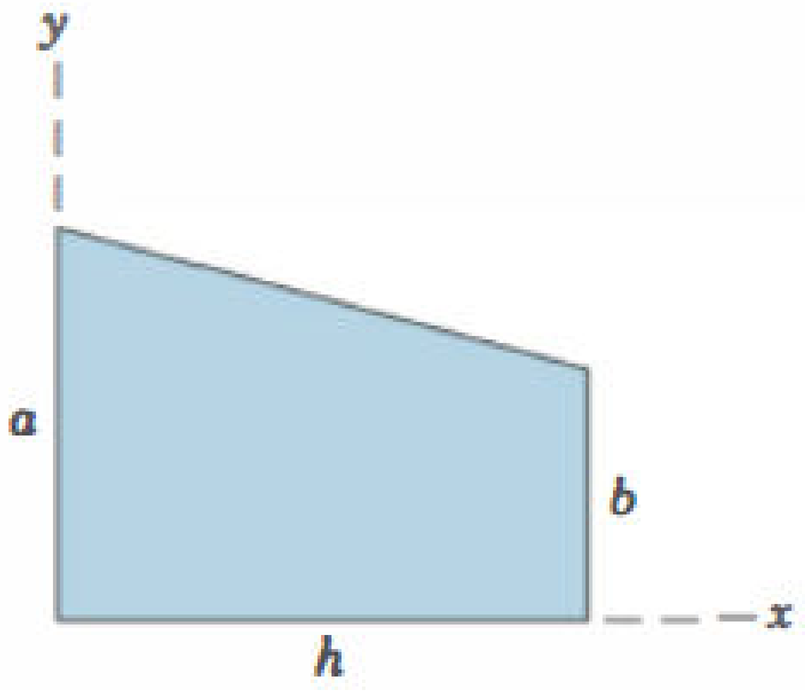
$$A = \int dA = \int_0^a \frac{h}{a} x\,dx + \int_a^b \frac{-h}{b-a}(x-b)\,dx = \frac{h}{2a}x^2\Big|_0^a - \frac{h}{b-a}\left(\frac{x^2}{2} - bx\right)\Big|_a^b$$

$$A = \frac{1}{2}ah + \frac{hb^2}{2(b-a)} + \frac{h}{b-a}\left(\frac{a^2}{2} - ab\right) \longrightarrow A = \frac{1}{2}bh$$

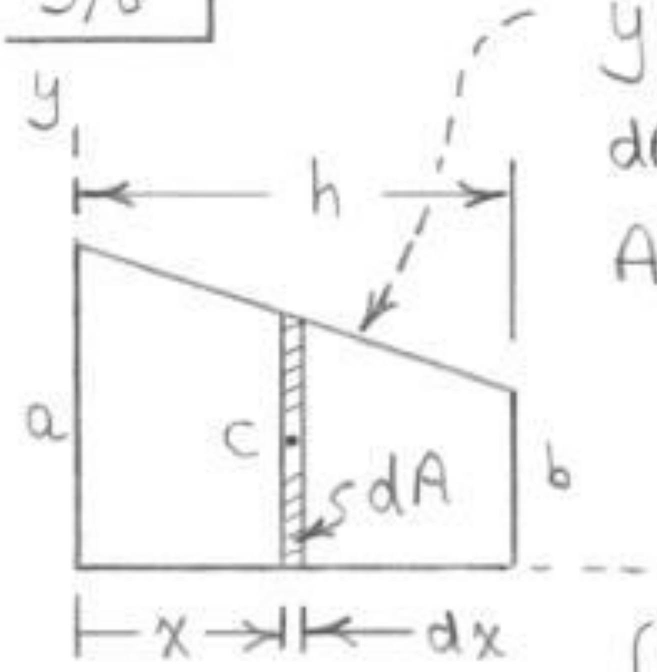
$$\bar{x} = \frac{\int \tilde{x}_c\,dA}{A} = \frac{\int_0^a \frac{h}{a} x^2\,dx + \int_a^b \frac{-h}{b-a}(x^2 - bx)\,dx}{\frac{1}{2}bh}$$

$$= \frac{\frac{1}{3}\frac{hx^3}{a}\Big|_0^a - \frac{h}{b-a}\left(\frac{x^3}{3} - \frac{bx^2}{2}\right)\Big|_a^b}{\frac{1}{2}bh} \longrightarrow \underline{\bar{x} = \frac{1}{3}(a+b)}$$

5/8 Determine the x- and y-coordinates of the centroid of the trapezoidal area.



5/8



$$y = \left(\frac{b-a}{h}\right)x + a$$

$$dA = y\,dx$$

$$A = \int dA = \int_0^h \left[\left(\frac{b-a}{h}\right)x + a\right] dx$$

$$= \left[\frac{b-a}{h}\frac{x^2}{2} + ax\right]_0^h = \frac{h}{2}(a+b)$$

$$\int x_c\,dA = \int_0^h \left[\left(\frac{b-a}{h}\right)x^2 + ax\right] dx$$

$$= \left[\frac{b-a}{h}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^h = h^2\left(\frac{b}{3} + \frac{a}{6}\right)$$

$$\int y_c\, dA = \int \frac{y}{2}\, y\, dx = \frac{1}{2}\int_0^h y^2\, dx$$

$$= \frac{1}{2}\int_0^h \left[\left(\frac{b-a}{h}\right)^2 x^2 + 2\left(\frac{b-a}{h}\right) a x + a^2\right] dx$$

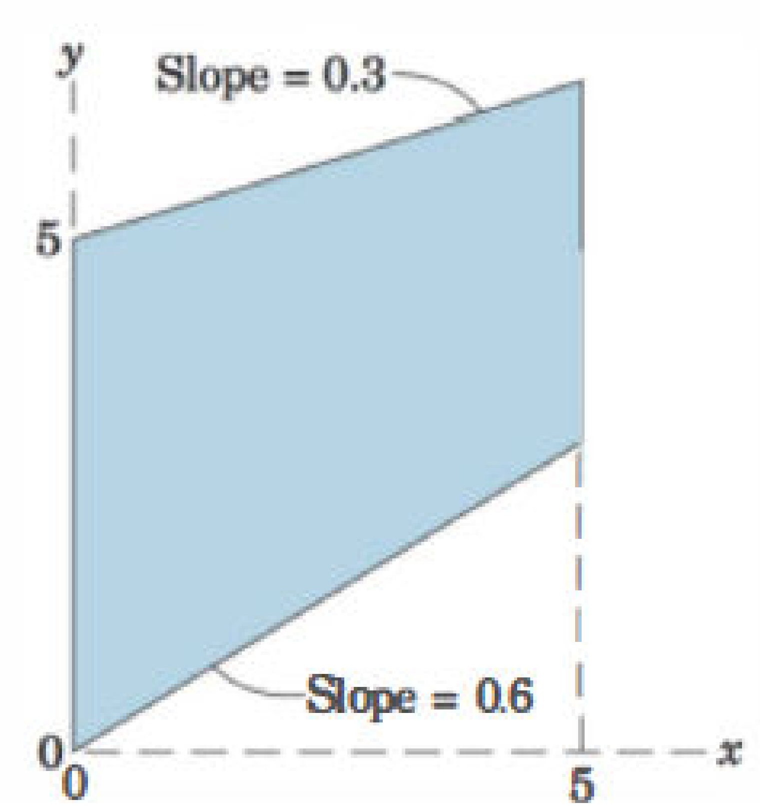
$$= \frac{1}{2}\left[\left(\frac{b-a}{h}\right)^2 \frac{x^3}{3} + 2\left(\frac{b-a}{h}\right) a \frac{x^2}{2} + a^2 x\right]_0^h$$

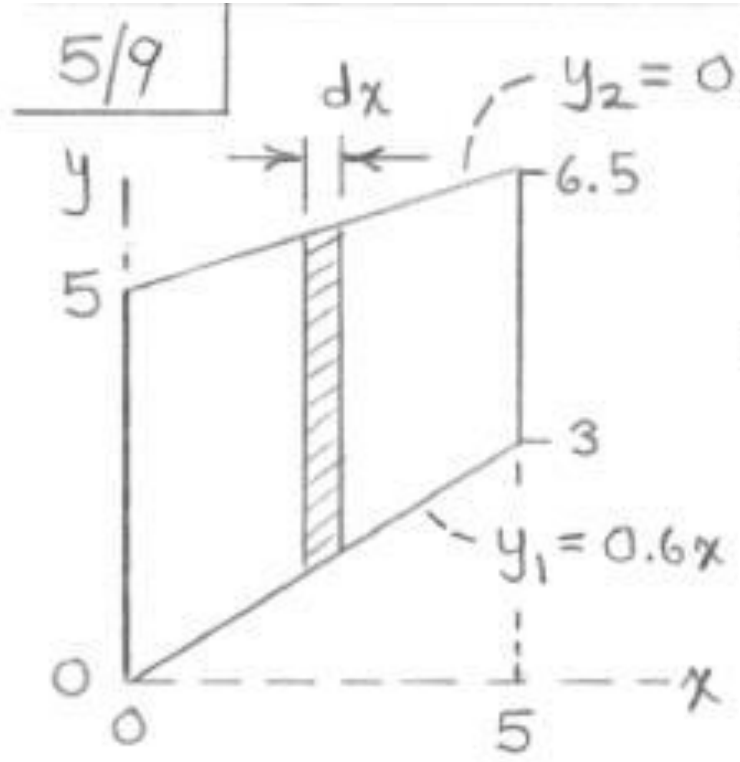
$$= \frac{h}{6}\left[a^2 + ab + b^2\right]$$

$$\bar{x} = \frac{\int x_c\, dA}{A} = \frac{h^2\left(\frac{b}{3} + \frac{a}{6}\right)}{\frac{h}{2}(a+b)} = \frac{h(a+2b)}{3(a+b)}$$

$$\bar{y} = \frac{\int y_c\, dA}{A} = \frac{\frac{h}{6}\left(a^2 + ab + b^2\right)}{\frac{h}{2}(a+b)} = \frac{\left(a^2 + ab + b^2\right)}{3(a+b)}$$

5/9 By direct integration, determine the coordinates of the centroid of the trapezoidal area.





$$A = \int_0^5 (y_2 - y_1)\, dx$$

$$= \int_0^5 (0.3x + 5 - 0.6x)\, dx$$

$$= \int_0^5 (5 - 0.3x)\, dx$$

$$= \left(5x - \frac{0.3x^2}{2}\right)_0^5 = 21.25$$

(Note: Trapezoidal area formula could be used)

$$\int x_c \, dA = \int_0^5 x(5-0.3x)\,dx = \left(\frac{5}{2}x^2 - 0.1x^3\right)_0^5 = 50$$

$$\int y_c \, dA = \int_0^5 \left(\frac{y_1+y_2}{2}\right)(y_2-y_1)\,dx = \frac{1}{2}\int_0^5 (y_2^2 - y_1^2)\,dx$$

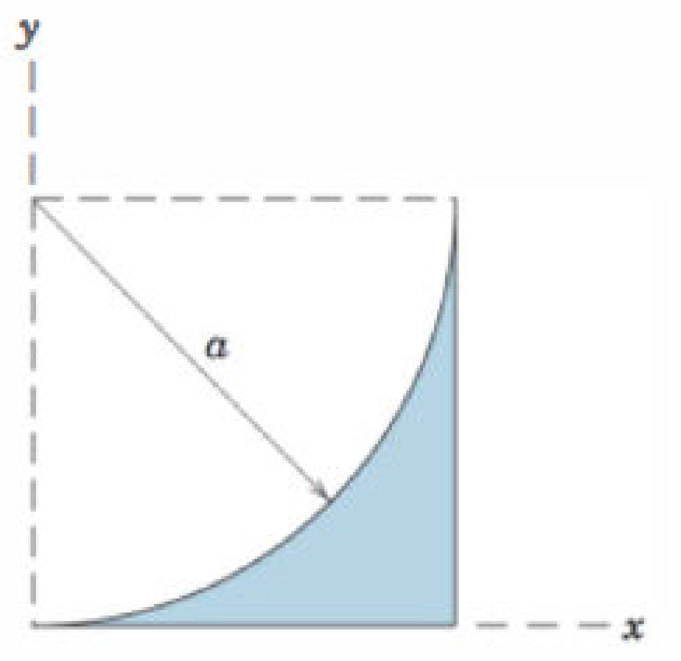
$$= \frac{1}{2}\int_0^5 \left[(0.3x+5)^2 - (0.6x)^2\right]dx = \frac{1}{2}\int_0^5 (25 + 3x - 0.27x^2)\,dx$$

$$= \frac{1}{2}\left[25x + \frac{3x^2}{2} - \frac{0.27x^3}{3}\right]_0^5 = 75.6$$

$$\bar{x} = \int x_c \, dA / A = 50/21.25 = \underline{2.35}$$

$$\bar{y} = \int y_c \, dA / A = 75.6/21.25 = \underline{3.56}$$

5/22 Locate the centroid of the area shown in the figure by direct integration. (Caution: Carefully observe the proper sign of the radical involved.)



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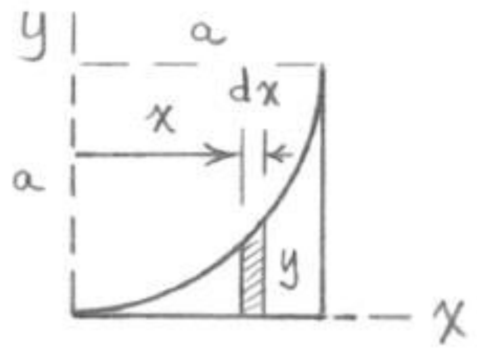


$$x^2 + (y-a)^2 = a^2$$

$$y = a - \sqrt{a^2 - x^2}$$

(use minus sign)

$$A = \int y\,dx = \int_0^a \left[a - \sqrt{a^2 - x^2}\right] dx$$

$$= \left[ax - \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)\right]_0^a = a^2\left(1 - \frac{\pi}{4}\right)$$

$$\int x_c\,dA = \int_0^a xy\,dx = \int_0^a \left[ax - x\sqrt{a^2 - x^2}\right] dx$$

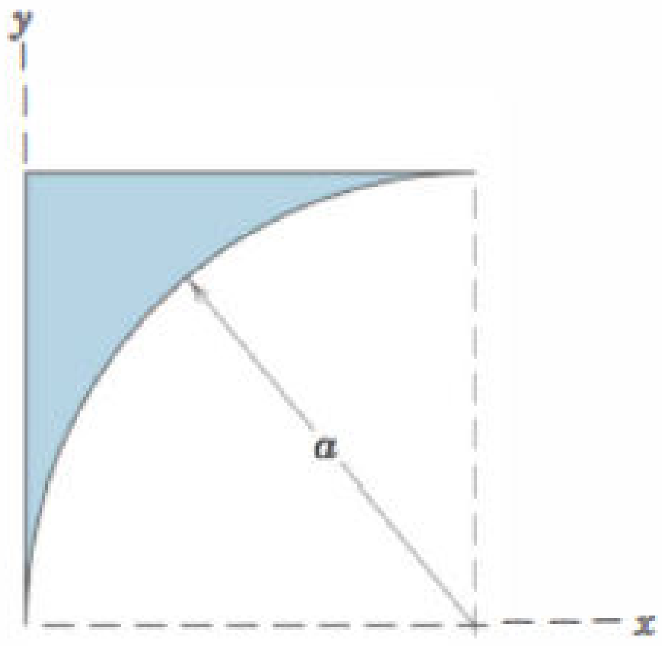
$$= \left[\frac{ax^2}{2} + \frac{1}{3}(a^2 - x^2)^{3/2}\right]_0^a = a^3/6$$

$$\bar{x} = \frac{\int x_c\,dA}{A} = \frac{a^3/6}{a^2\left(1 - \frac{\pi}{4}\right)} = \underline{\frac{2a}{3(4-\pi)}} = \underline{0.777a}$$

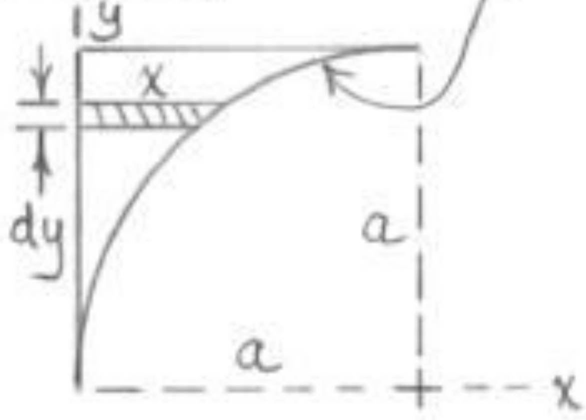
From symmetry, $\bar{y} = a - \bar{x} = a - \frac{2a}{3(4-\pi)} = \underline{\frac{10-3\pi}{3(4-\pi)}a}$

or $\bar{y} = \underline{0.223a}$

5/27 Locate the centroid of the area shown in the figure by direct integration. (Caution: Carefully observe the proper sign of the radical involved.)



5/27



$$(x-a)^2 + y^2 = a^2$$

$$dA = x\,dy = \left(a - \sqrt{a^2-y^2}\right)dy$$

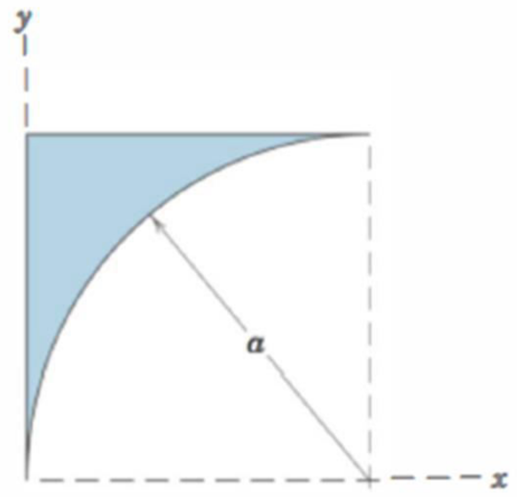
$$\int x_c\,dA = \int \frac{x}{2}\,x\,dy$$

$$= \frac{1}{2}\int_0^a \left(a - \sqrt{a^2-y^2}\right)^2 dy$$

$$\int x_c\,dA = \int_0^a \left(a^2 - a\sqrt{a^2-y^2} - \frac{y^2}{2}\right)dy$$

$$= \left[a^2 y - \frac{a}{2}\left(y\sqrt{a^2-y^2} + a^2 \sin^{-1}\frac{y}{a}\right) - \frac{y^3}{6}\right]_0^a$$

$$= \left(\frac{5}{6} - \frac{\pi}{4}\right)a^3$$



$$\int y_c\,dA = \int y\,x\,dy = \int_0^a \left(ay - y\sqrt{a^2-y^2}\right)dy$$

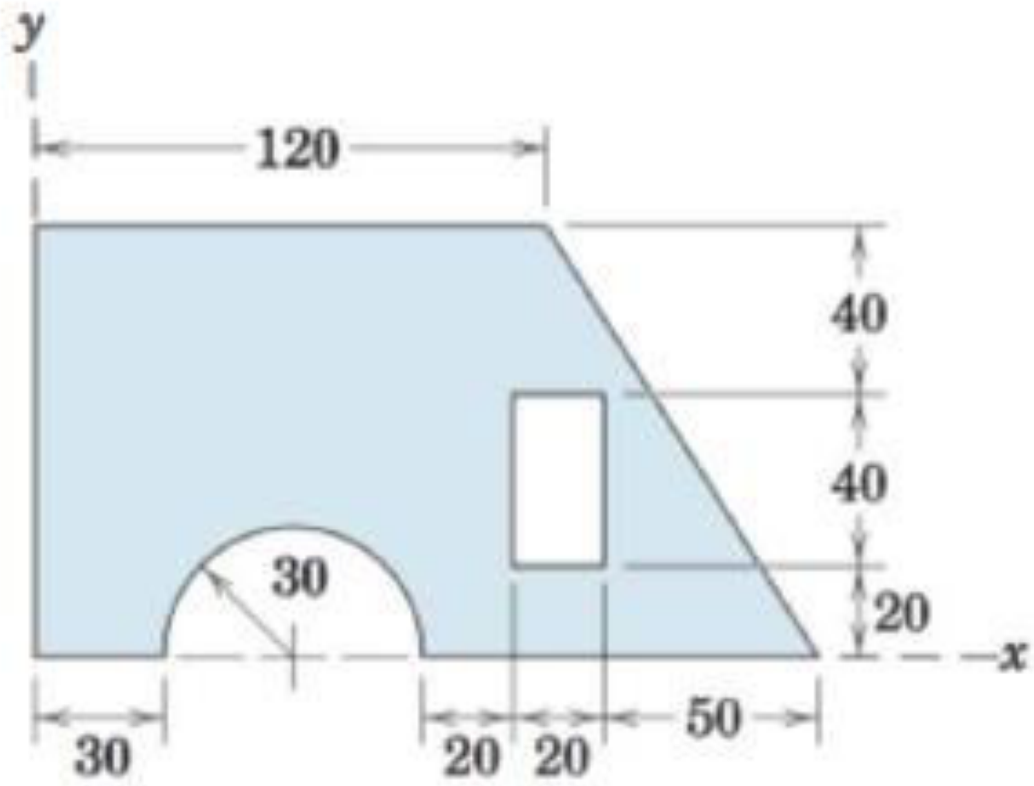
$$= \left[\frac{ay^2}{2} + \frac{1}{3}\sqrt{(a^2-y^2)^3}\right]_0^a = \frac{a^3}{6}$$

$$A = a^2 - \frac{1}{4}\pi a^2 = a^2\left(1-\frac{\pi}{4}\right)$$

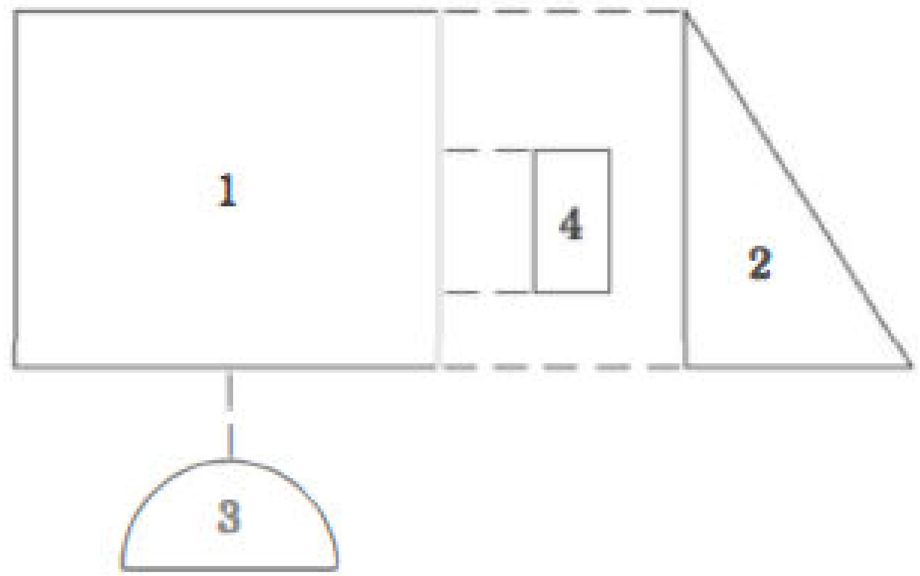
$$\bar{x} = \frac{\int x_c\,dA}{A} = \frac{\left(\frac{5}{6}-\frac{\pi}{4}\right)a^3}{\left(1-\frac{\pi}{4}\right)a^2} = \frac{10-3\pi}{3(4-\pi)}a = \underline{0.223a}$$

$$\bar{y} = \frac{\int y_c\,dA}{A} = \frac{a^3/6}{\left(1-\frac{\pi}{4}\right)a^2} = \frac{2a}{3(4-\pi)} = \underline{0.777a}$$

5/6 Sample Problem: Locate the Centroid of the shaded area:



Dimensions in millimeters



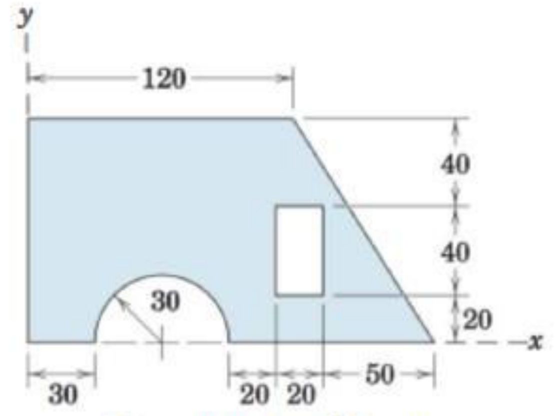
5/6 Sample Problem

PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

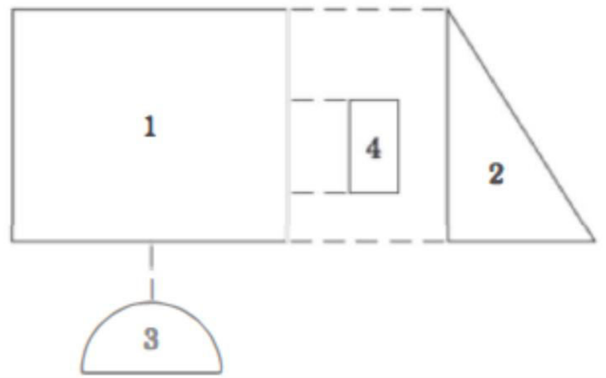
The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\ 000}{12\ 790} = 75.0 \text{ mm} \qquad \textit{Ans.}$$

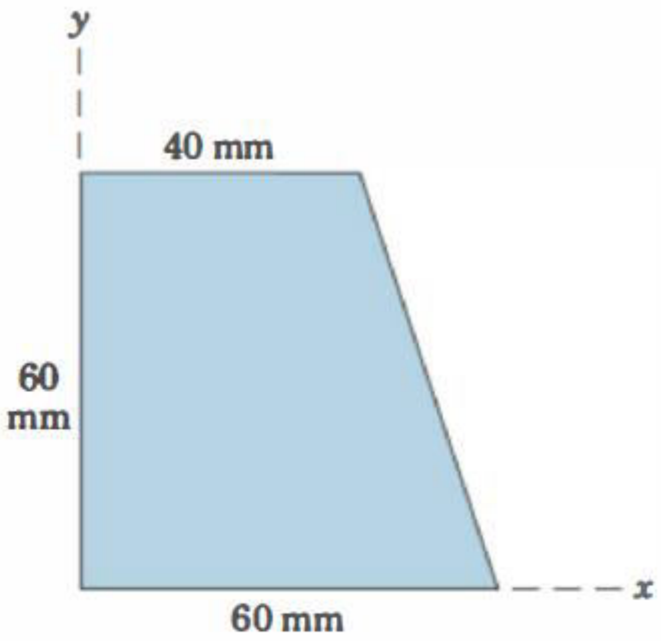
$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\ 000}{12\ 790} = 50.8 \text{ mm} \qquad \textit{Ans.}$$



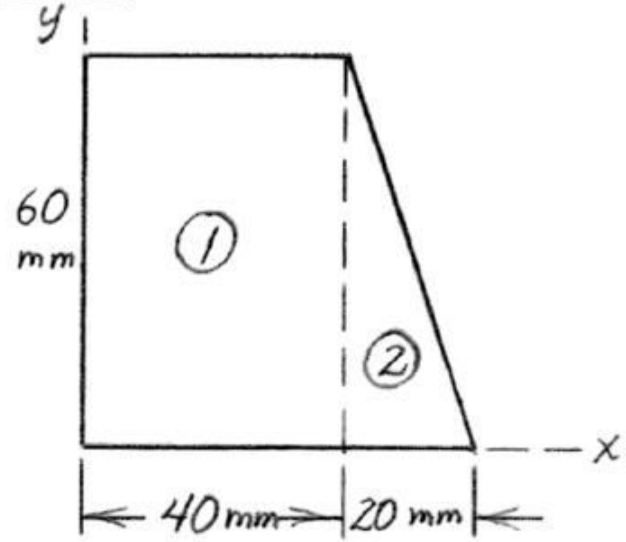
Dimensions in millimeters



5/47 Determine the coordinates of the centroid of the trapezoidal area shown.



5/47



① $A_1 = 40(60) = 2400 \text{ mm}^2$

$\bar{x}_1 = 20 \text{ mm}, \quad \bar{y}_1 = 30 \text{ mm}$

② $A_2 = \frac{1}{2}(20)(60) = 600 \text{ mm}^2$

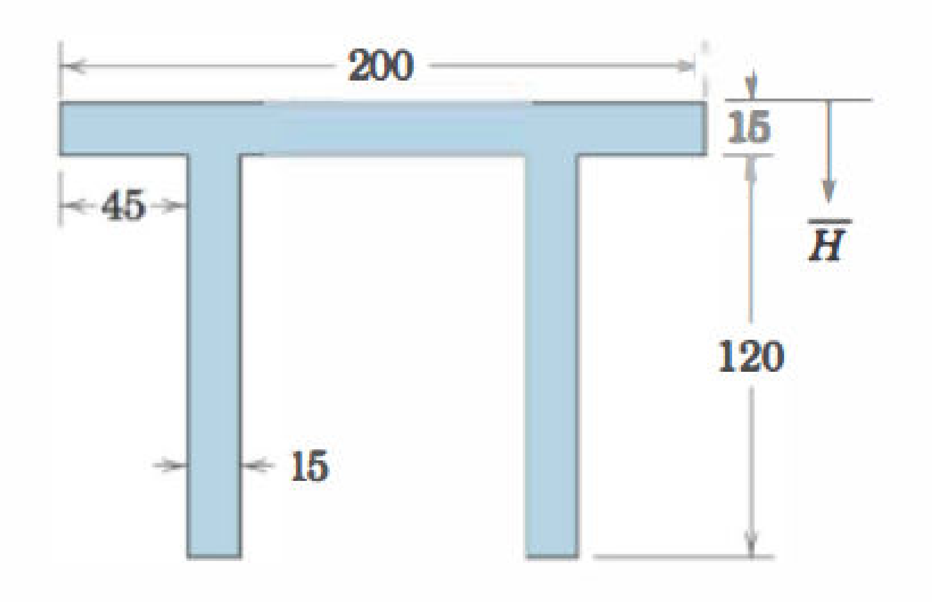
$\bar{x}_2 = 40 + \frac{20}{3} = 46.7 \text{ mm}$

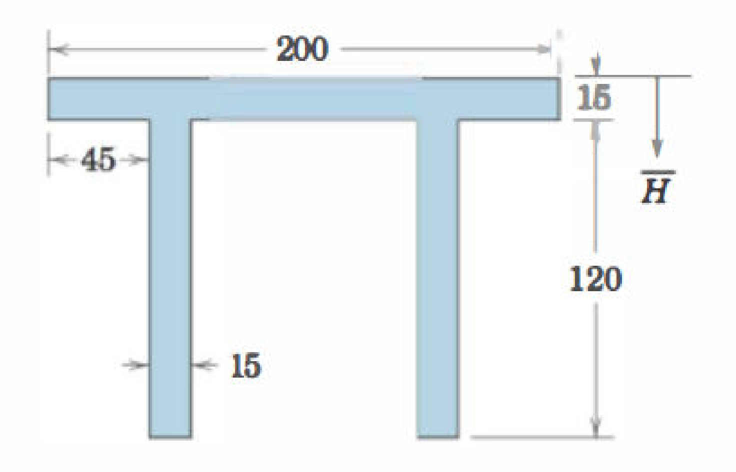
$\bar{y}_2 = \frac{60}{3} = 20 \text{ mm}$

$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{2400(20) + 600(46.7)}{2400 + 600} = \underline{25.3 \text{ mm}}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{2400(30) + 600(20)}{2400 + 600} = \underline{28.0 \text{ mm}}$$

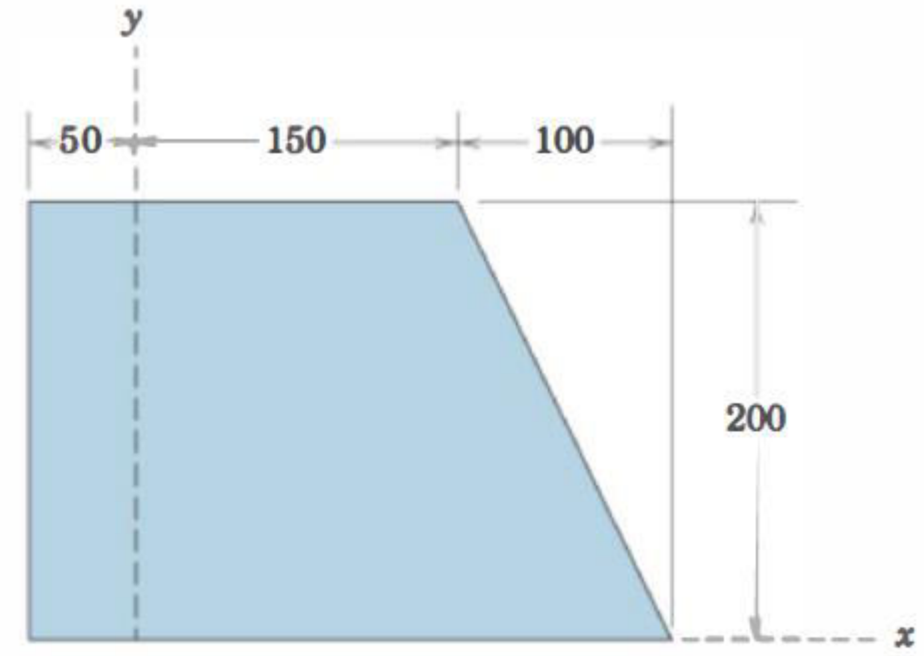
5/48 Determine the distance H from the upper surface of the symmetric double-T beam cross section to the location of the centroid.





$$\bar{H} = \frac{\Sigma A \bar{h}}{\Sigma A} = \frac{200(15)\left(\frac{15}{2}\right) + 2(15)(120)\left(15 + \frac{120}{2}\right)}{200(15) + 2(15)(120)} \longrightarrow \underline{\bar{H} = 44.3 \text{ mm}}$$

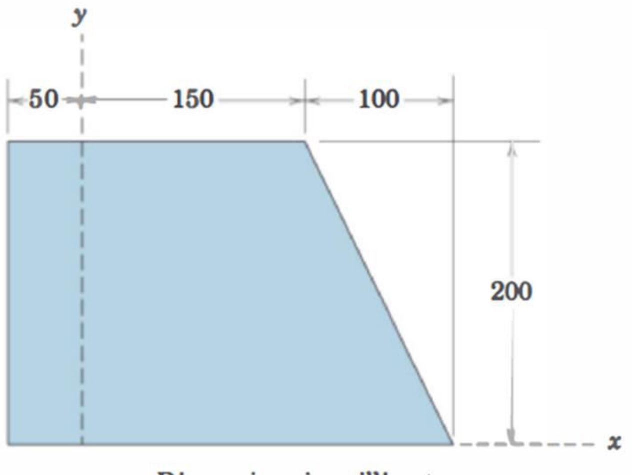
5/49 Determine the x- and y-coordinates of the centroid of the shaded area.



Dimensions in millimeters

$$\bar{X} = \frac{\Sigma A \bar{x}}{\Sigma A} = \frac{50(200)(-25) + 150(200)(75) + \frac{1}{2}(100)(200)\left(150 + \frac{100}{3}\right)}{50(200) + 150(200) + \frac{1}{2}(100)(200)}$$

$$\underline{\bar{X} = 76.7 \text{ mm}}$$

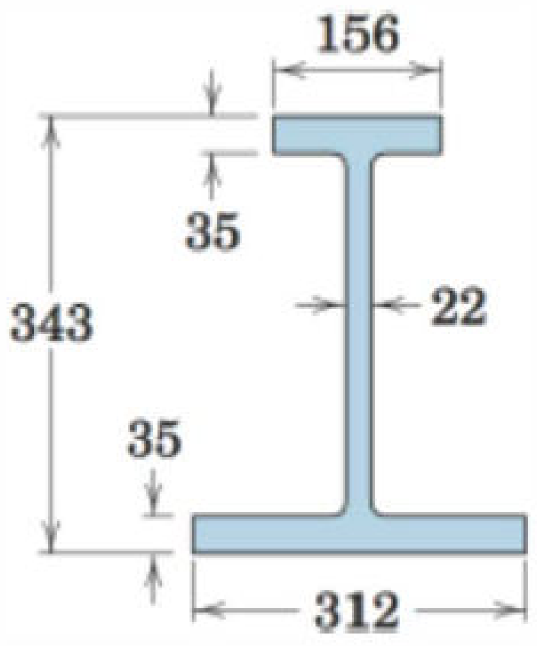


Dimensions in millimeters

$$\bar{Y} = \frac{\Sigma A \bar{y}}{\Sigma A} = \frac{50(200)(100) + 150(200)(100) + \frac{1}{2}(100)(200)\left(\frac{200}{3}\right)}{50(200) + 150(200) + \frac{1}{2}(100)(200)}$$

$$\bar{Y} = 93.3 \text{ mm}$$

5/50 Determine the height above the base of the centroid of the cross-sectional area of the beam. Neglect the fillets.

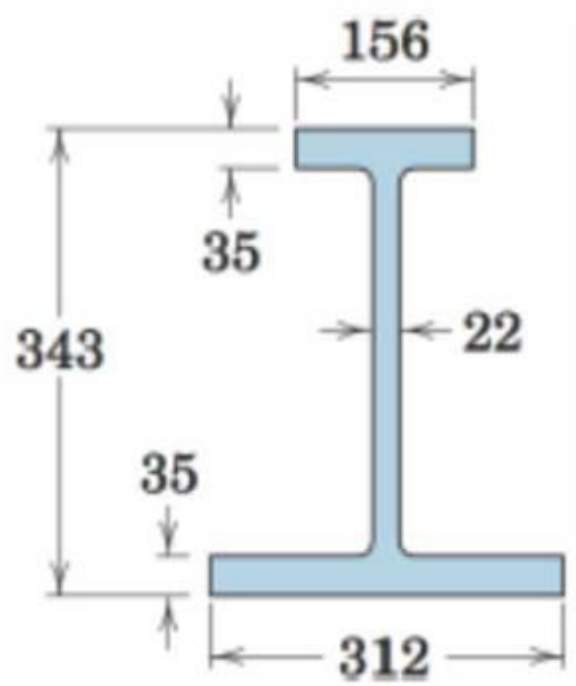


Dimensions in millimeters

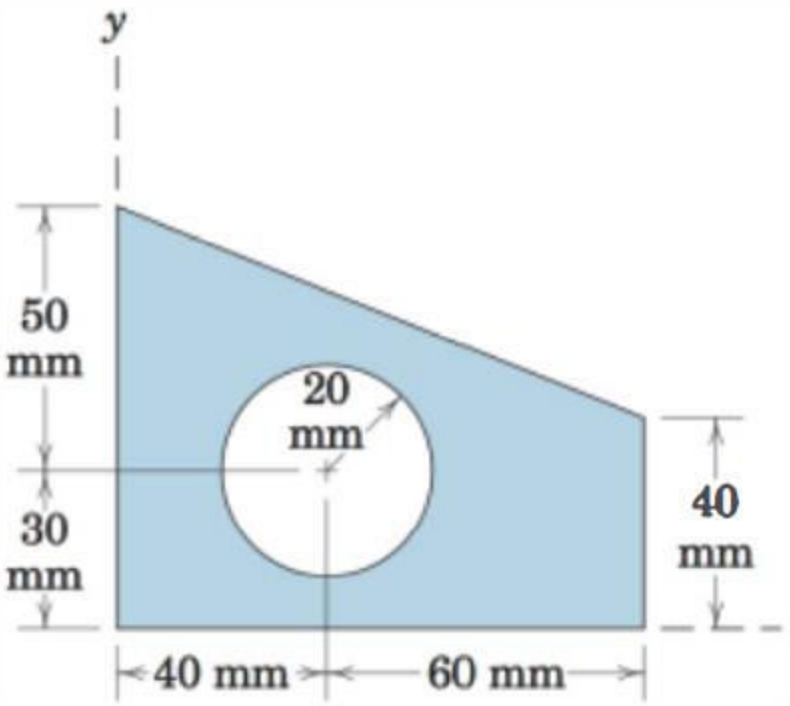
Comp.	$A\ (mm^2)$	$\bar{y}\ (mm)$	$A\bar{y}\ (mm^3)$
①	312(35)	$\frac{35}{2}$	191 100
②	273 (22)	$35 + \frac{273}{2}$	1 030 000
③	156 (35)	$35 + 273 + \frac{35}{2}$	1 777 000

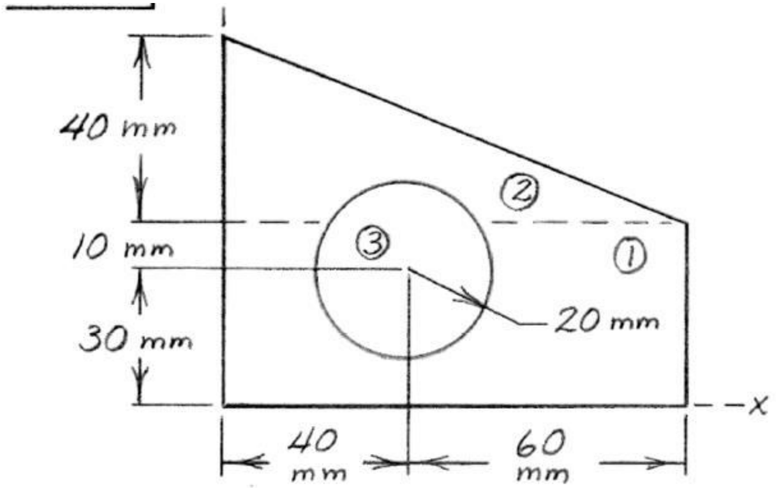
$\Sigma A = 22\,400$ $\Sigma A\bar{y} = 3\,000\,000$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{3\,000\,000}{22400} = \underline{133.9\ mm}$$



5/51 Determine the x- and y-coordinates of the centroid of the shaded area.



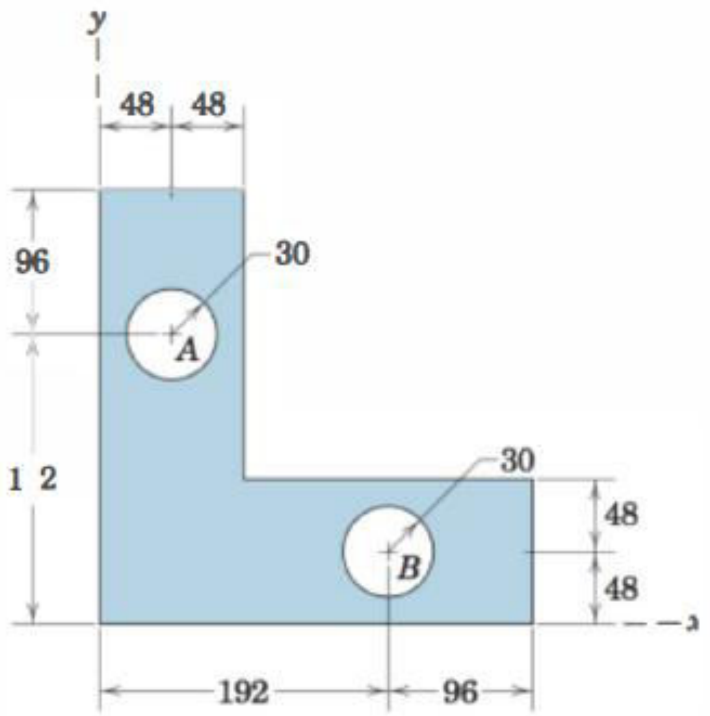


$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{216\,(10^3)}{4740} = \underline{45.6 \text{ mm}}$$

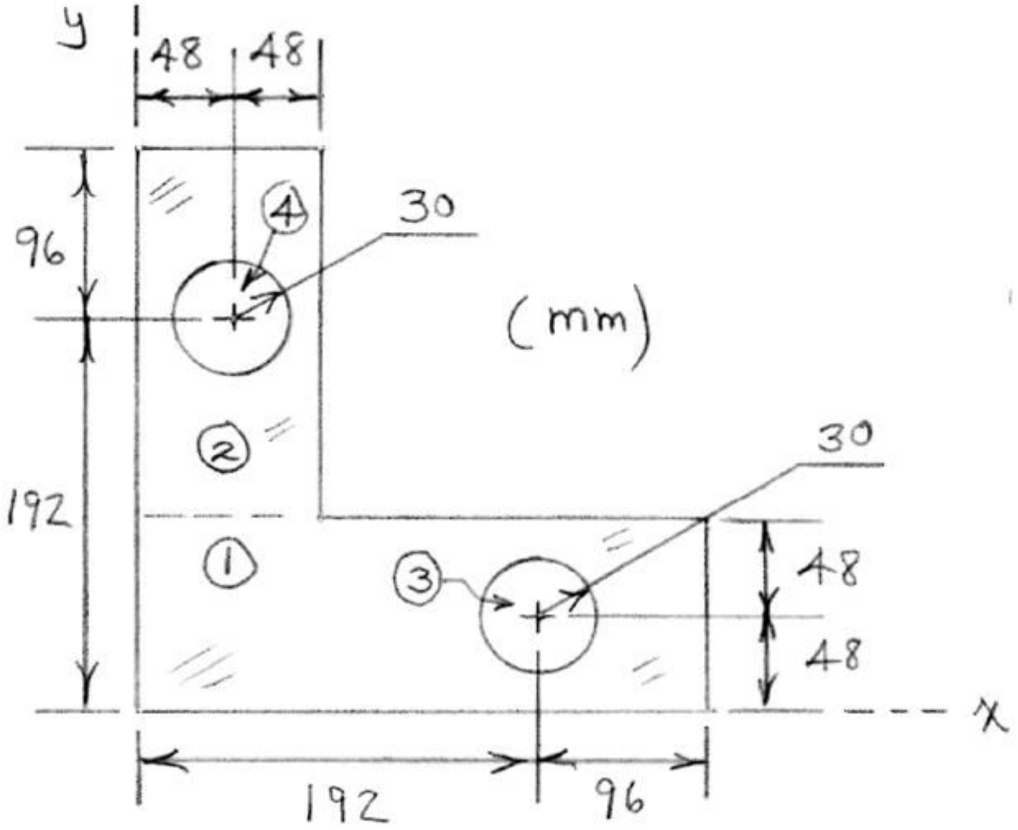
$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{149.0\,(10^3)}{4740} = \underline{31.4 \text{ mm}}$$

Part	A (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$A\bar{x}$ (mm^3)	$A\bar{y}$ (mm^3)
1	4000	50	20	$200(10^3)$	$80(10^3)$
2	2000	100/3	$40+\frac{40}{3}$	$66.7(10^3)$	$106.7(10^3)$
3	$-\pi(20^2)$	40	30	$-50.3(10^3)$	$-37.7(10^3)$
Totals	4740			$216(10^3)$	$149.0(10^3)$

5/52 Determine the x- and y-coordinates of the centroid of the shaded area.



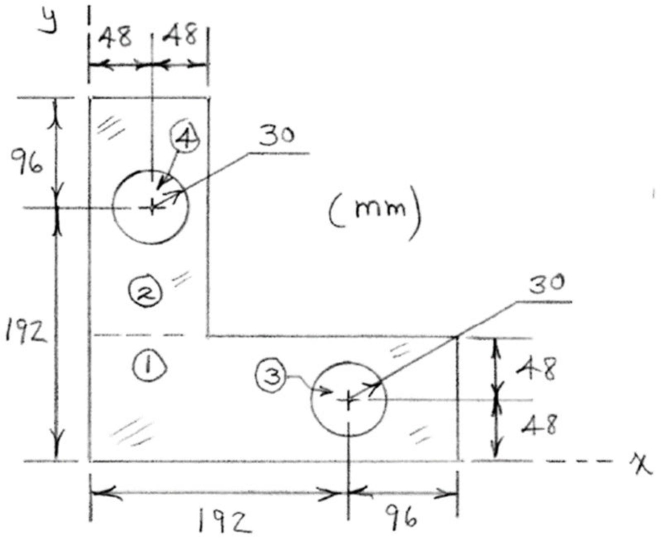
Dimensions in millimeters


y
48
48
96
4
30
(mm)
2
192
1
3
30
48
48
x
192
96

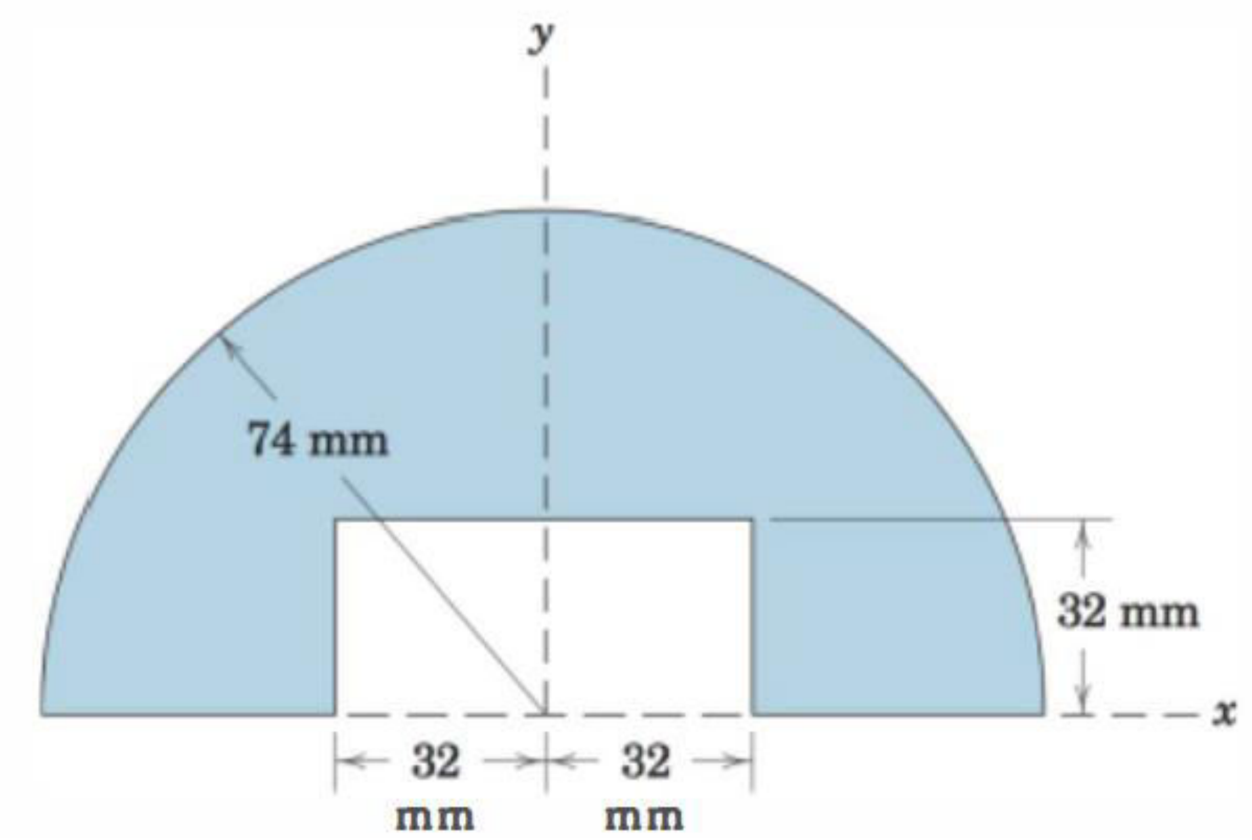
Comp.	A (mm^2)	$\bar{x}$ (mm)	$\bar{x}A$ (mm^3)
1	27 648	144	$3.98(10^6)$
2	18 432	48	$0.885(10^6)$
3	$-$ 2830	192	$-0.543(10^6)$
4	$-$ 2830	48	$-0.1357(10^6)$

$$\Sigma A = 40\,430 \qquad \Sigma \bar{x}A = 4.19(10^6)$$

$$\bar{X} = \bar{Y} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{4.19(10^6)}{40\,430} = \underline{103.6 \text{ mm}}$$

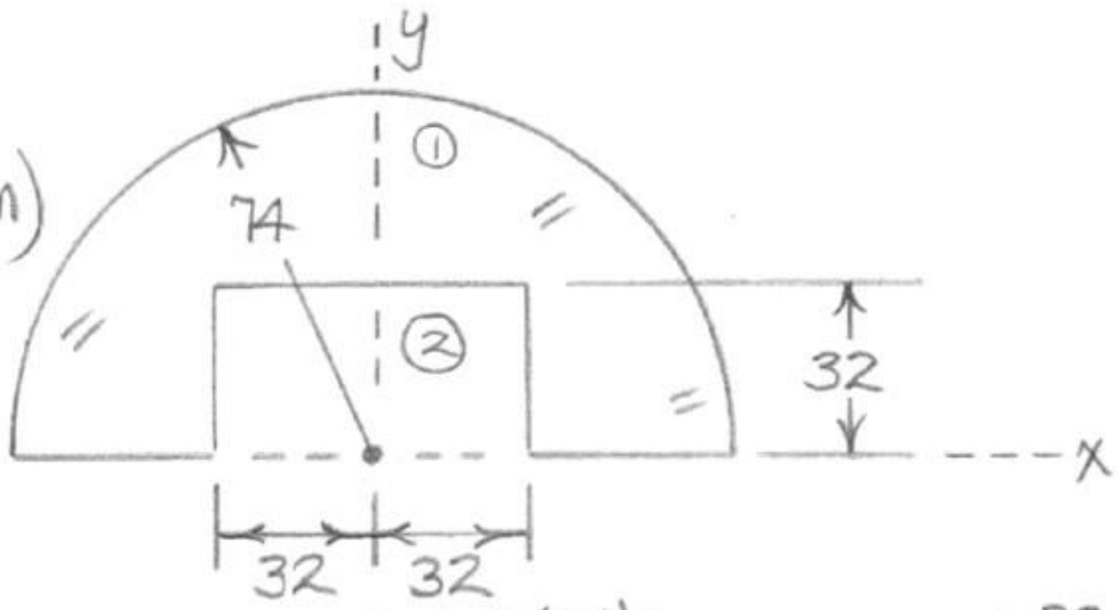


5/53 Calculate the y-coordinate of the centroid of the shaded area.



5/53

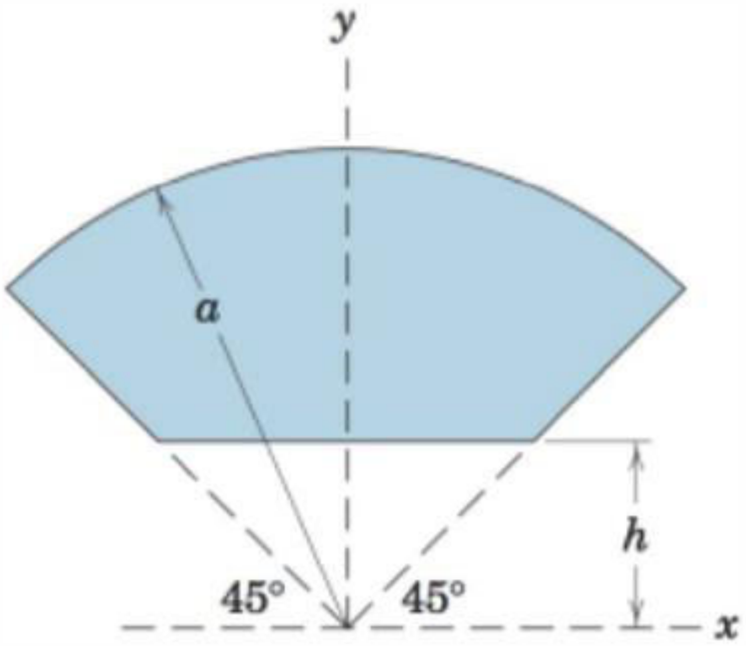
(Dim. in mm)



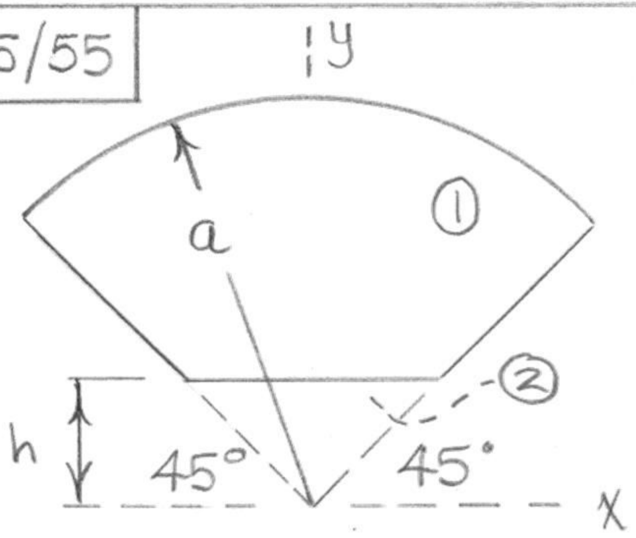
$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{\pi \frac{74^2}{2}\left(\frac{4(74)}{3\pi}\right) - 64(32)\left(\frac{32}{2}\right)}{\pi \frac{74^2}{2} - 64(32)}$$

$$= \underline{36.2 \text{ mm}}$$

5/55 Determine the y-coordinate of the centroid of the shaded area.



5/55



Circular sector (full) 1:

$$A_1 = \frac{\pi}{4}a^2$$

$$\bar{y}_1 = \frac{2}{3}a\frac{\sin 45^\circ}{\pi/4}$$

$$= \frac{4\sqrt{2}}{3\pi}a$$

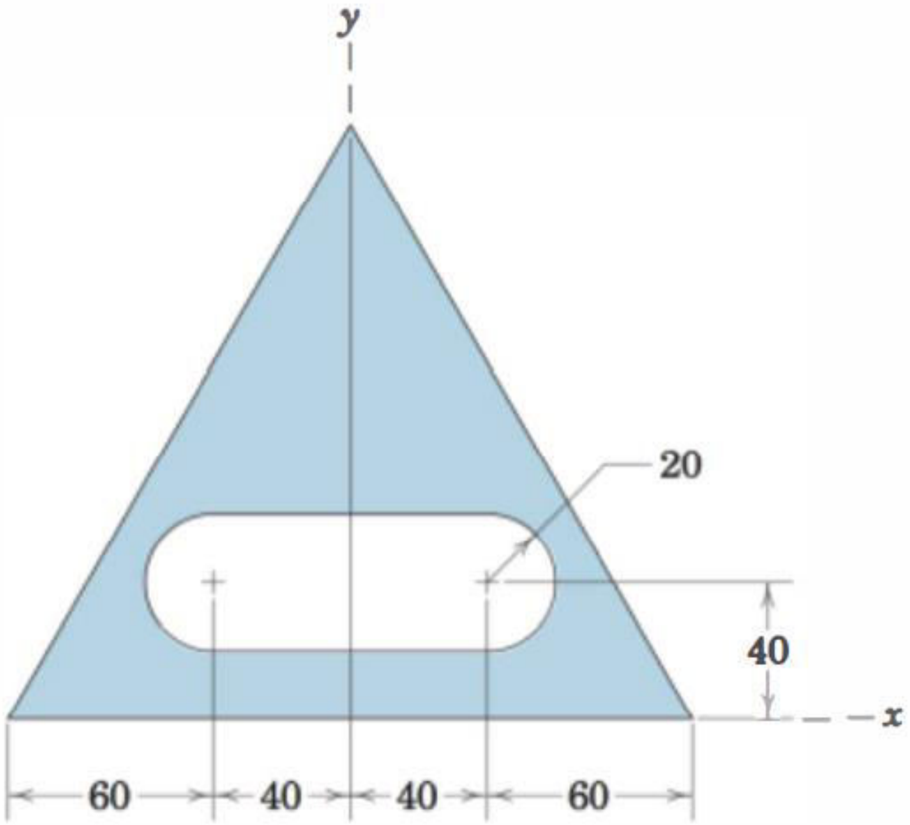
Triangular "hole" 2:

$$A_2 = \frac{1}{2}h(2h) = h^2, \quad \bar{y}_2 = \frac{2}{3}h$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{\frac{\pi}{4}a^2\left(\frac{4\sqrt{2}}{3\pi}a\right) - h^2\left(\frac{2}{3}h\right)}{\frac{\pi}{4}a^2 - h^2}$$

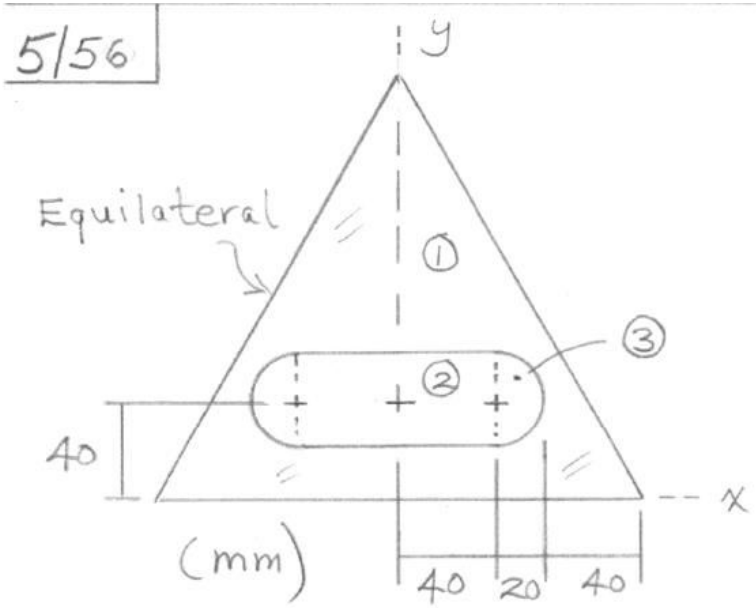
$$= \frac{4(\sqrt{2}a^3 - 2h^3)}{3(\pi a^2 - 4h^2)}$$

5/56 Determine the y-coordinate of the centroid of the shaded area. The triangle is equilateral.



Dimensions in millimeters

5/56

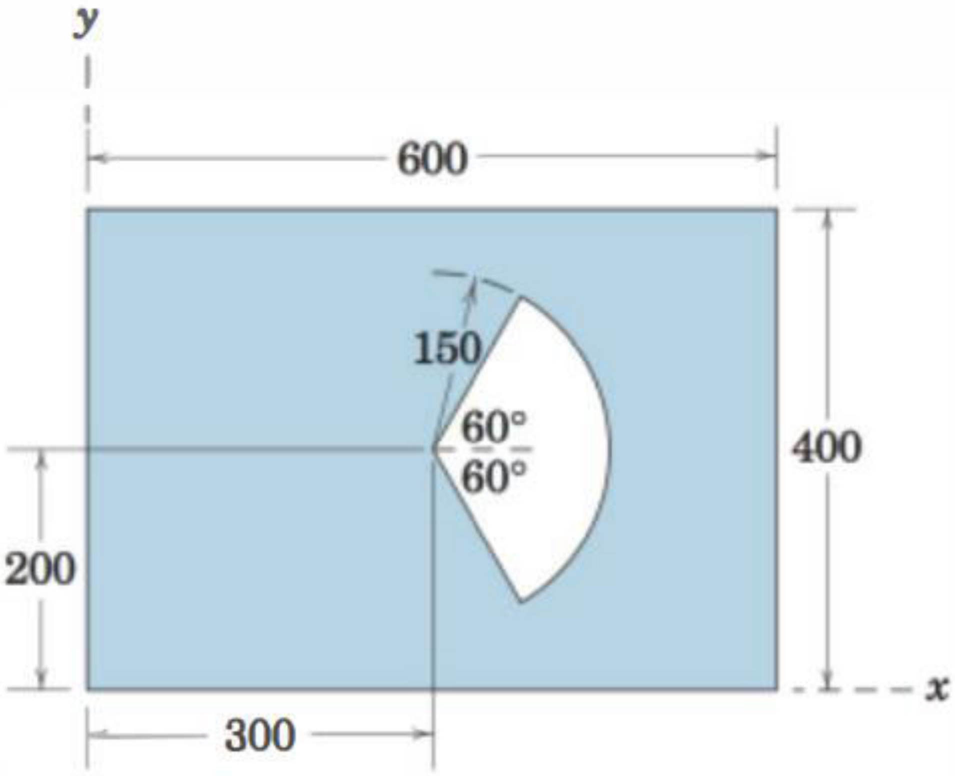


$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{822\,000}{12\,860} = 63.9 \text{ mm}$$

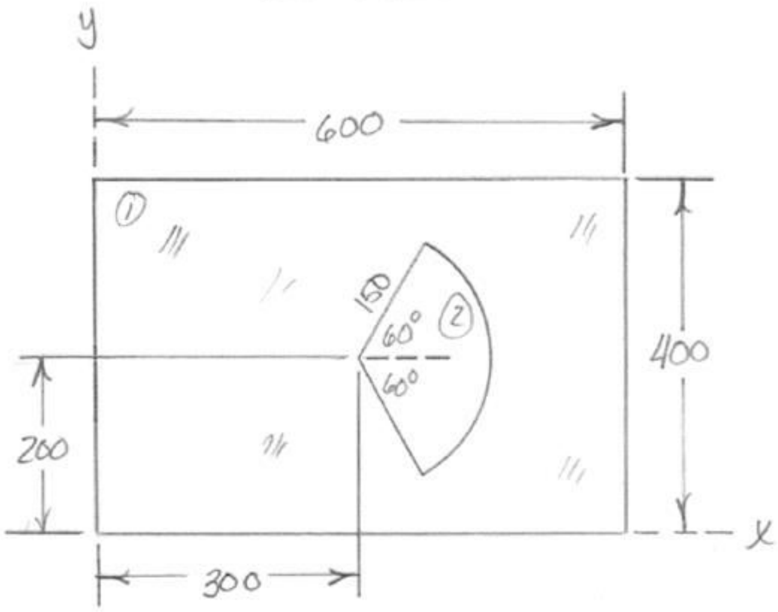
Component	A (mm^2)	$\bar{y}$ (mm)	$\bar{y}A$ (mm^3)
Triangle 1	17 320	57.7	10^6
Rectangle 2	−3200	40	−128 000
2 semicircles 3	−1257	40	−50,300

$\Sigma A = 12\,860$ $\Sigma \bar{y} A = 822\,000$

5/57 Determine the x- and y-coordinates of the centroid of the shaded area.



Dimensions in millimeters

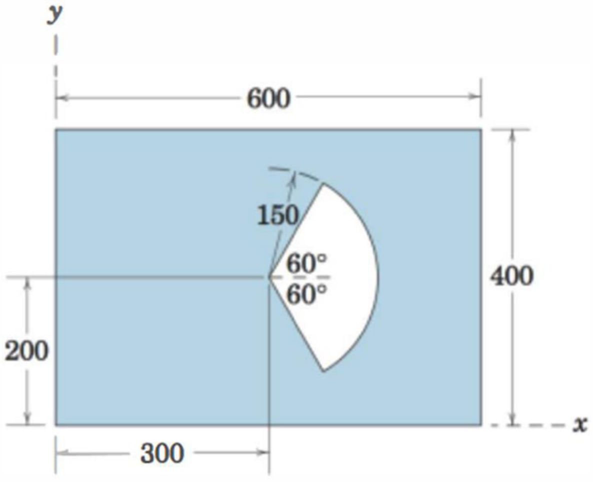


$$\bar{X} = \frac{\Sigma A \bar{x}}{\Sigma A} = \frac{400(600)(300) - \frac{1}{3}\pi(150)^2\left(300 + \frac{2}{3}(150)\frac{\sin 60}{\pi/3}\right)}{400(600) - \frac{1}{3}\pi(150)^2}$$

$\bar{X} = 291$ mm

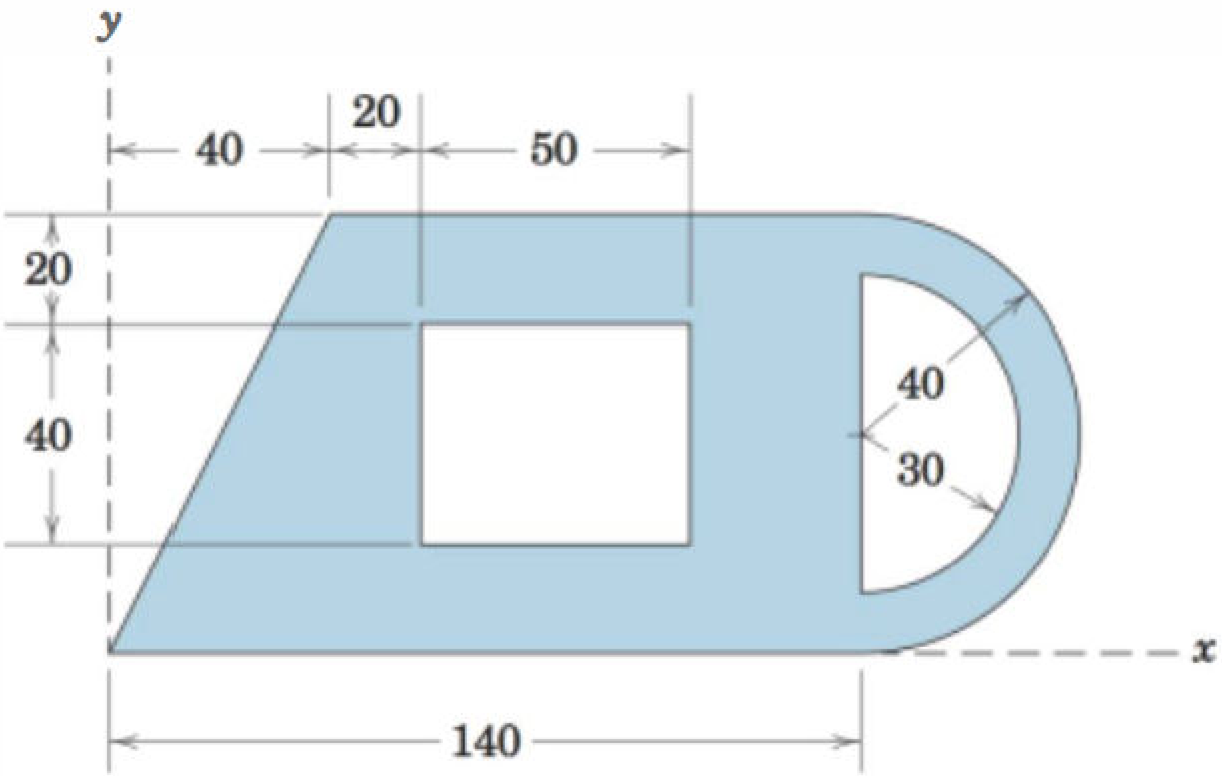
$\bar{Y} = 200$ mm (INSPECTION)

5/57 Determine the x- and y-coordinates of the centroid of the shaded area.



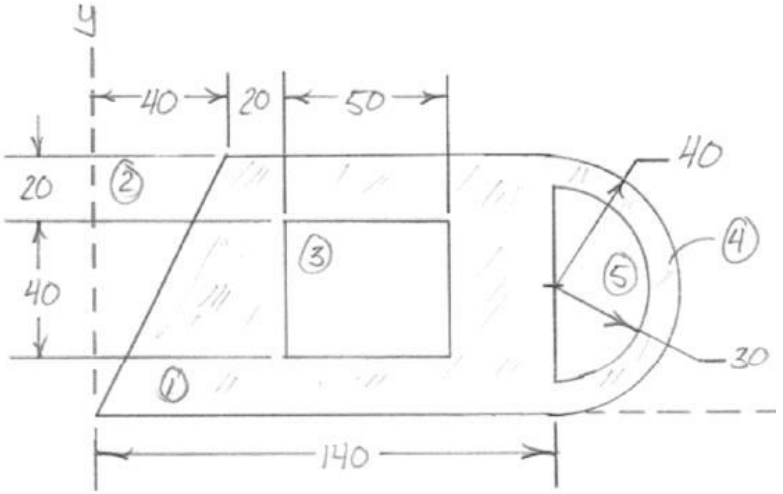
Dimensions in millimeters

5/58 Determine the coordinates of the centroid of the shaded area.



Dimensions in millimeters

Centroid

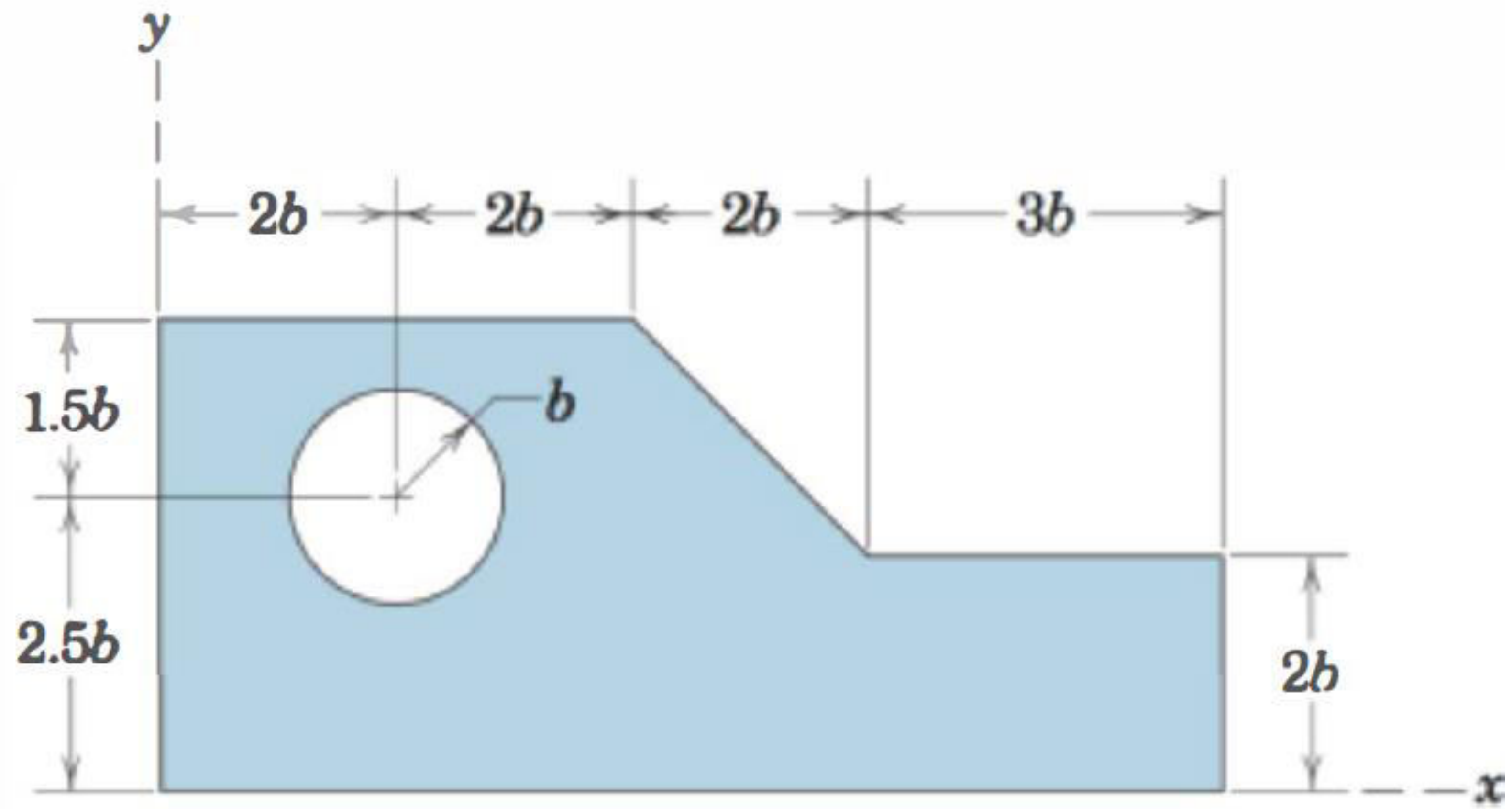


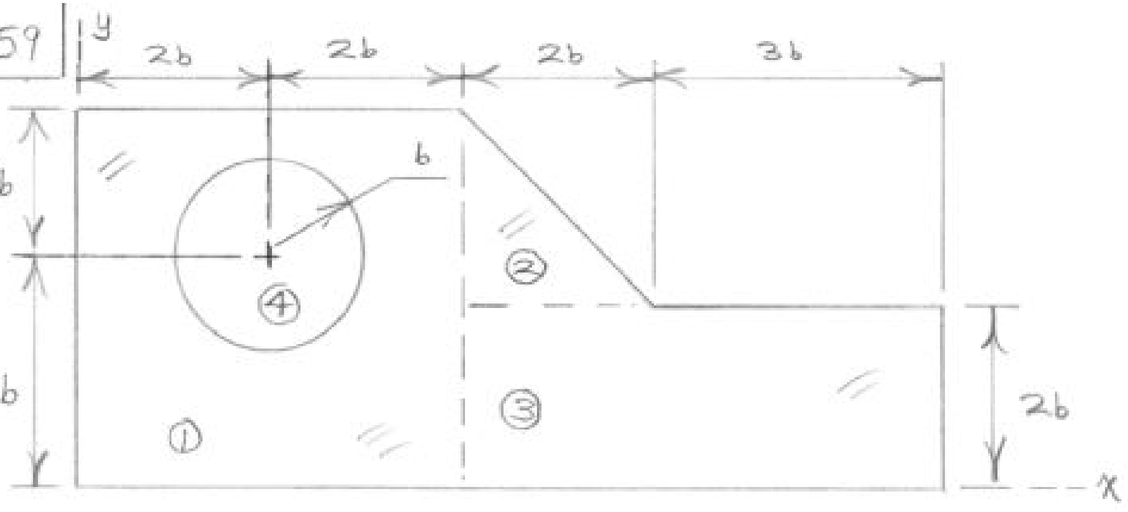
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$A\bar{x}$, mm^3	$A\bar{y}$, mm^3
1	$140(80) = 11200$	70	40	784×10^3	448×10^3
2	$-\frac{1}{2}(40)(80) = -1600$	$\frac{40}{3} = 13.33$	$\frac{2}{3}(80) = 53.3$	-21.3×10^3	-85.3×10^3
3	$-40(50) = -2000$	85	40	-170×10^3	-80×10^3
4	$\frac{\pi(40)^2}{2} = 800\pi$	$140 + \frac{4(40)}{3\pi} = 157.0$	40	395×10^3	32000π
5	$-\frac{\pi(30)^2}{2} = -450\pi$	$140 + \frac{4(30)}{3\pi} = 152.7$	40	-216×10^3	-18000π
Σ	8700			771×10^3	327×10^3

$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{771 \times 10^3}{8700} \rightarrow \underline{\bar{X} = 88.7 \text{ mm}}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{327 \times 10^3}{8700} \rightarrow \underline{\bar{Y} = 37.5 \text{ mm}}$$

5/59 Determine the x- and y-coordinates of the centroid of the shaded area.

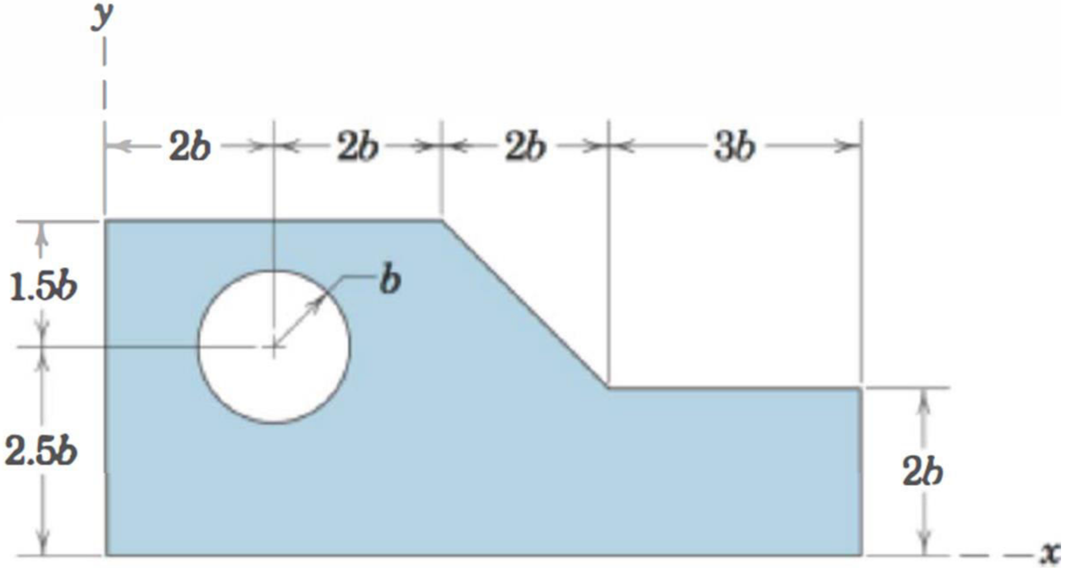




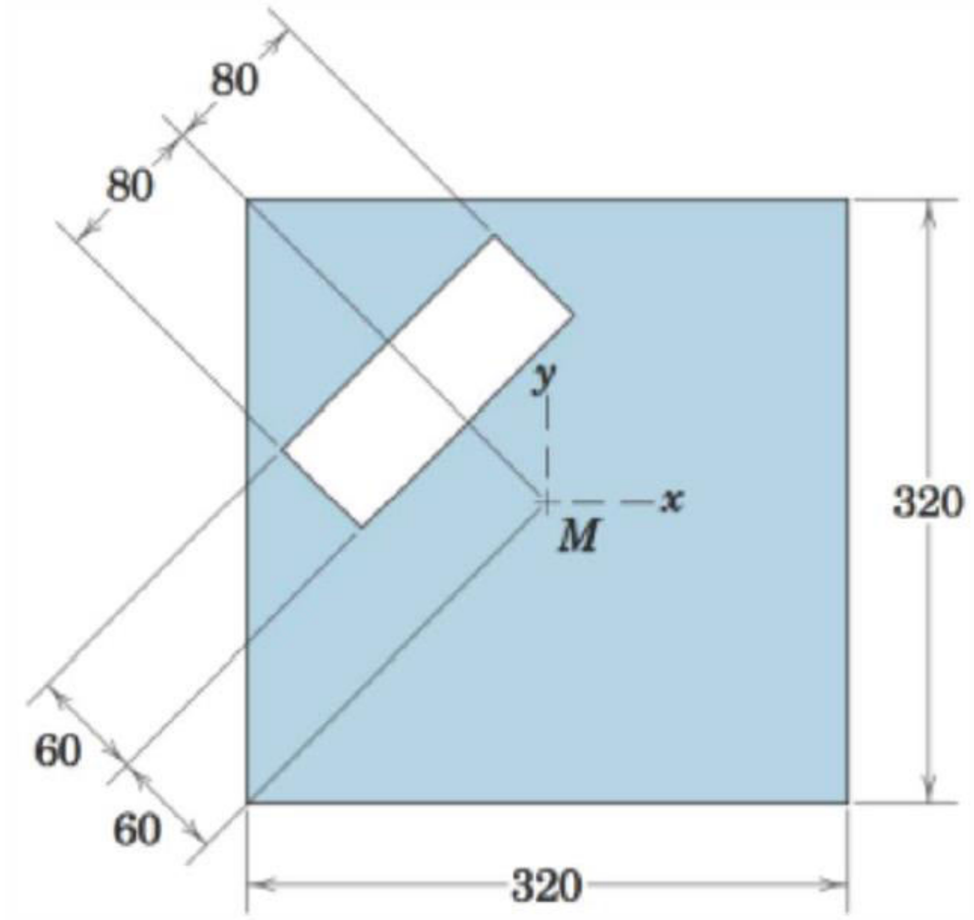
$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{100.1 b^3}{24.9 b^2} = \underline{4.02 b}$$

$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{39.5 b^3}{24.9 b^2} = \underline{1.588 b}$$

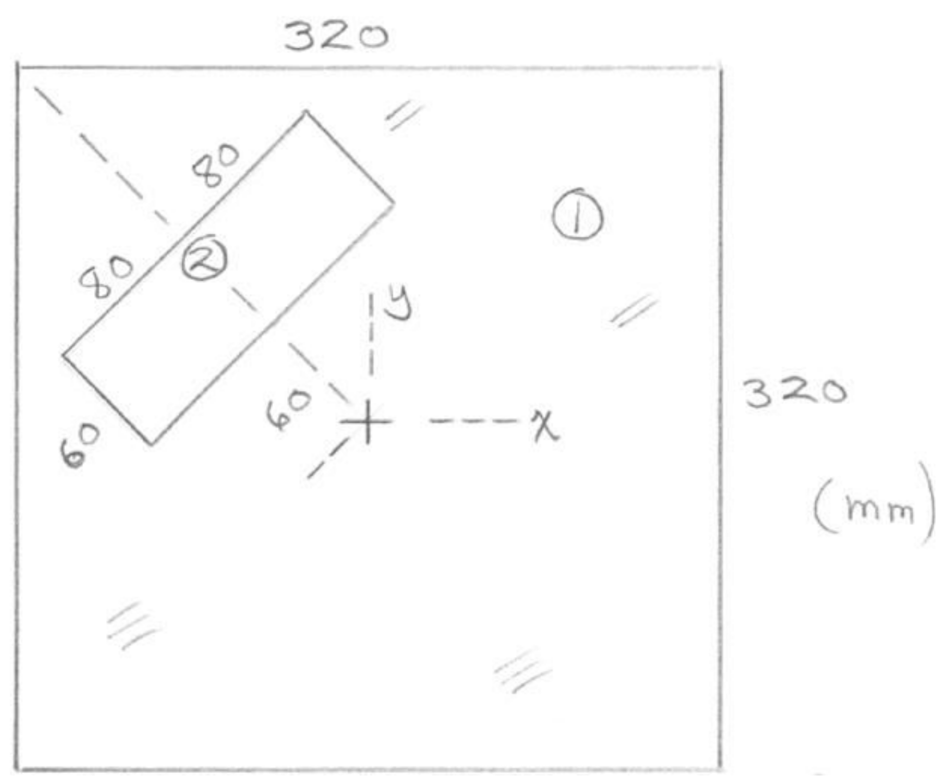
Comp.	A	$\bar{x}$	$\bar{y}$	$\bar{x}A$	$\bar{y}A$
1	$16b^2$	$2b$	$2b$	$32b^3$	$32b^3$
2	$2b^2$	$\left(4b + \frac{2b}{3}\right)$	$\left(2b + \frac{2b}{3}\right)$	$9.33b^3$	$5.33b^3$
3	$10b^2$	$\left(4b + \frac{5b}{2}\right)$	b	$65b^3$	$10b^3$
4	$-\pi b^2$	$2b$	$2.5b$	$-2\pi b^3$	$-2.5\pi b^3$
	$\Sigma A = 24.9b^2$			$\Sigma \bar{x} A = 100.1 b^3$	$\Sigma \bar{y} A = 39.5 b^3$



5/61 By inspection, state the quadrant in which the centroid of the shaded area is located. Then determine the coordinates of the centroid. The plate center is M.



5/61



Comp.	A (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{x}A$ (mm^3)	$\bar{y}A$ (mm^3)
1	$(320)^2$	0	0	0	0
2	$-160(60)$	$-90\frac{\sqrt{2}}{2}$	$90\frac{\sqrt{2}}{2}$	611 000	$-611\,000$

$\Sigma A = 92\,800$

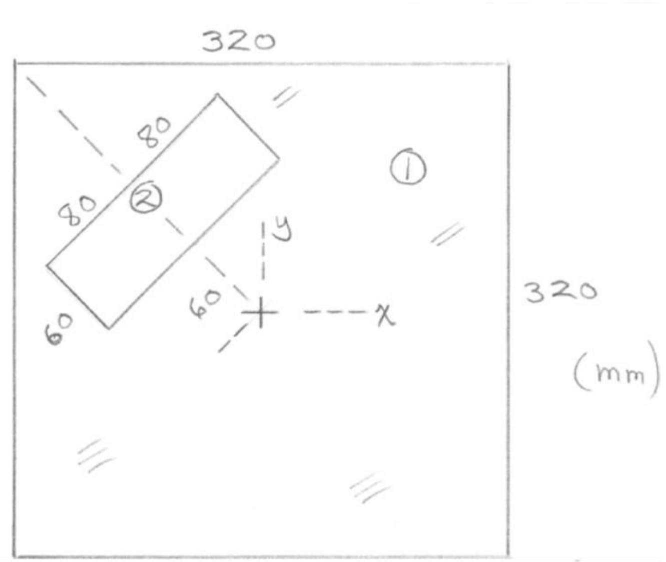
$\Sigma \bar{x}A = 611\,000$

$\Sigma \bar{y}A = -611\,000$

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{611\,000}{92\,800} = 6.58 \text{ mm}$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{-611\,000}{92\,800} = -6.58 \text{ mm}$$

5/61



ENGINEERING MECHANICS

Moment of Inertia

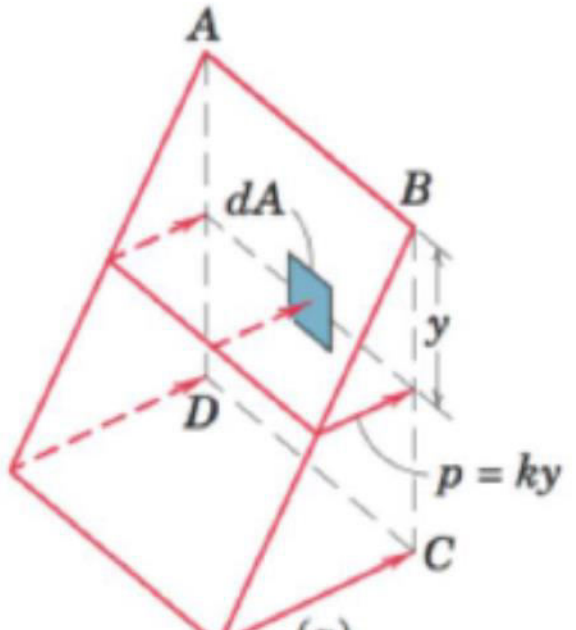
Department of Civil Engineering

Area Moments of Inertia:

- When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area.

- The intensity of the force (pressure or stress) is proportional to the distance of the line of action of the force from the moment axis.

- The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area.



(a)

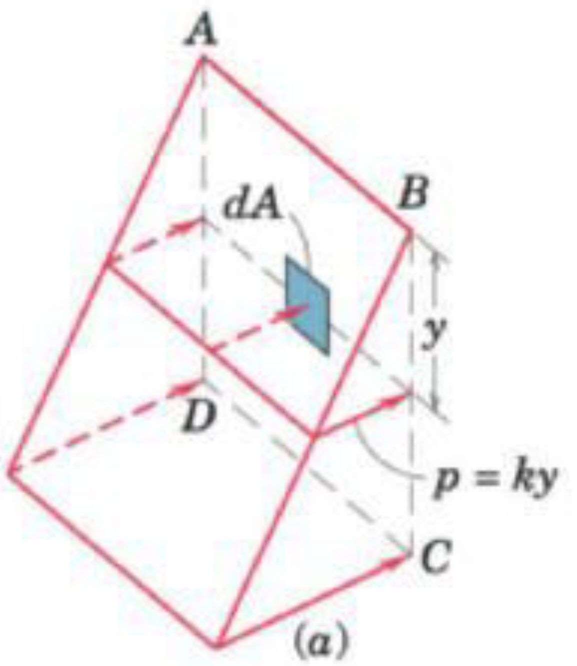
- The total moment involves an integral of form ∫ (distance)² d (area).

This integral is called the moment of inertia or the second moment of the area

- In the figure, the surface area ABCD is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis AB.

- This situation was treated in Art. 519 of Chapter 5, where we described the action of liquid pressure on a plane surface. The moment about AB due to the pressure on the element of area dA is $py\, dA = ky^2\, dA$.

- Thus, the integral in question appears when the total moment $M = k \int y^2\, dA$ is evaluated.



Rectangular Area moment of inertia and polar moment of inertia

Rectangular and Polar Moments of Inertia

Consider the area A in the x-y plane, Fig. A/2. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\, dA$ and $dI_y = x^2\, dA$, respectively. The moments of inertia of A about the same axes are therefore

$$\boxed{\begin{aligned} I_x &= \int y^2\, dA \\ I_y &= \int x^2\, dA \end{aligned}} \qquad \text{(A/1)}$$

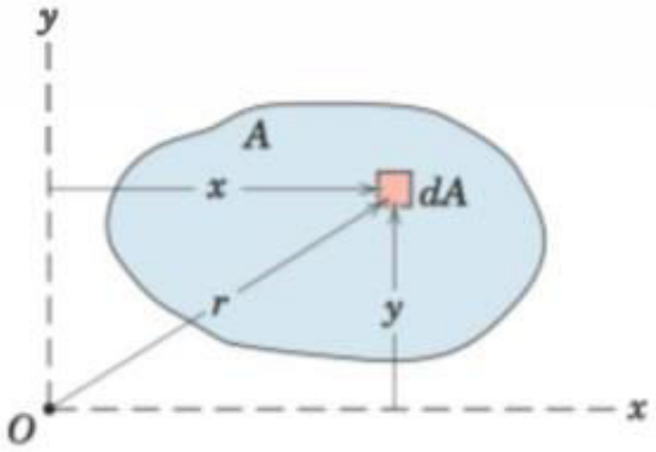


Figure A/2

Rectangular Area moment of inertia and polar moment of inertia

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2\,dA$. The moment of inertia of the entire area about O is

$$\boxed{I_z = \int r^2\,dA} \qquad \text{(A/2)}$$

The expressions defined by Eqs. A/1 are called *rectangular* moments of inertia, whereas the expression of Eq. A/2 is called the *polar* moment of inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$\boxed{I_z = I_x + I_y} \qquad \text{(A/3)}$$

Radius of gyration

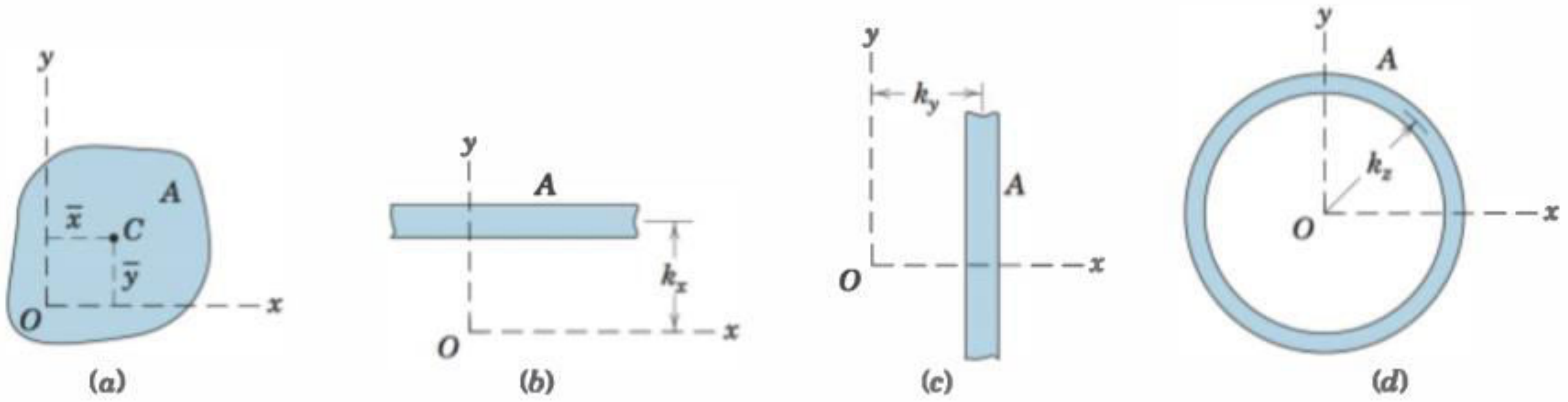


Figure A/3

$$I_x = k_x^2 A$$
$$I_y = k_y^2 A$$
$$I_z = k_z^2 A$$

or

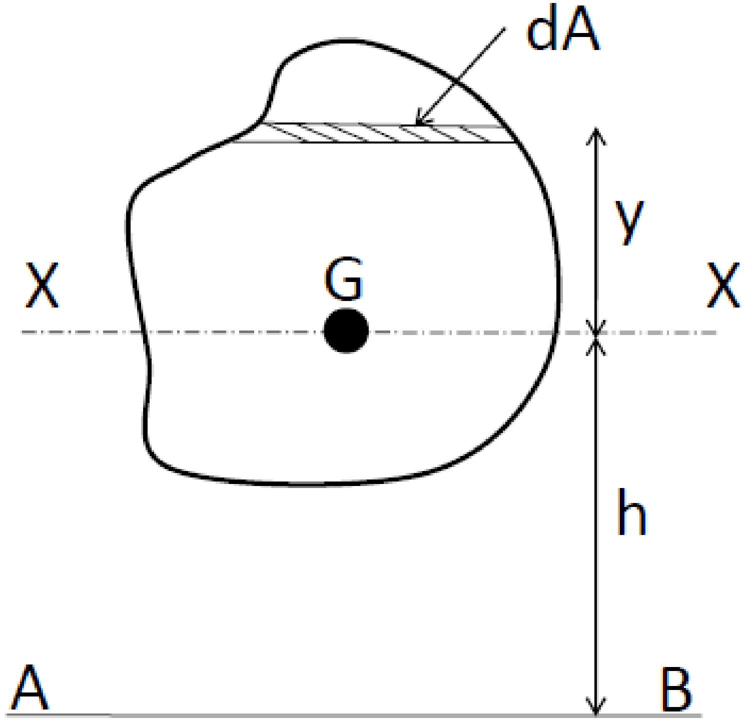
$$k_x = \sqrt{I_x/A}$$
$$k_y = \sqrt{I_y/A}$$
$$k_z = \sqrt{I_z/A}$$

$$k_z^2 = k_x^2 + k_y^2$$

Theorem of the Parallel axis

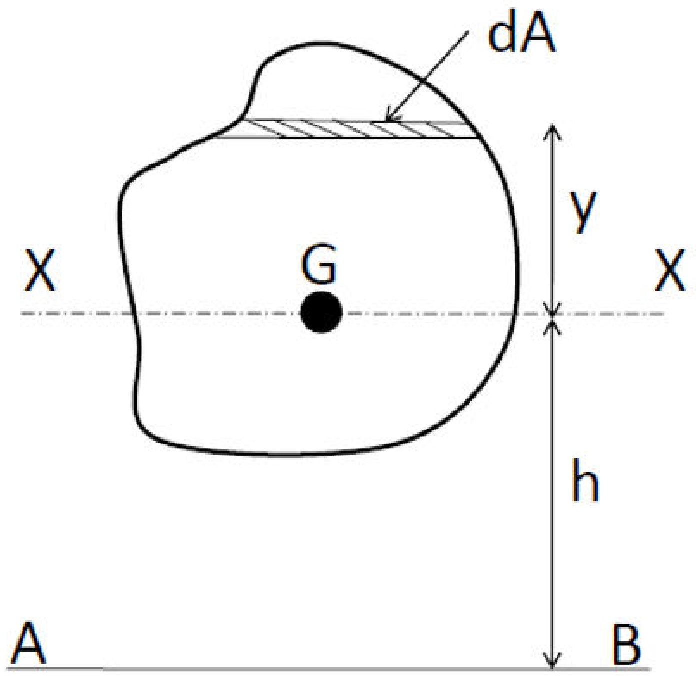
It states that if the MI of a plane area about an axis in the plane of area through the CG of the plane area is I_{GG}, then the MI of the given plane area about a parallel axis AB in the plane of area at a distance h from the CG of the area is given by

$$I_{AB} = I_G + Ah^2$$



- **Conditions for parallel axis theorem**

1. **Two axis should be there and two axes must be parallel to each other**
2. **Between two axes, one axis has to pass through the centroidal axis**



Theorem of the Parallel axis

Consider a strip parallel to XX at a distance y.

$$(I_{XX})_{dA} = dA.y^2$$

$$I_{XX} = I_G = \sum dA.y^2$$

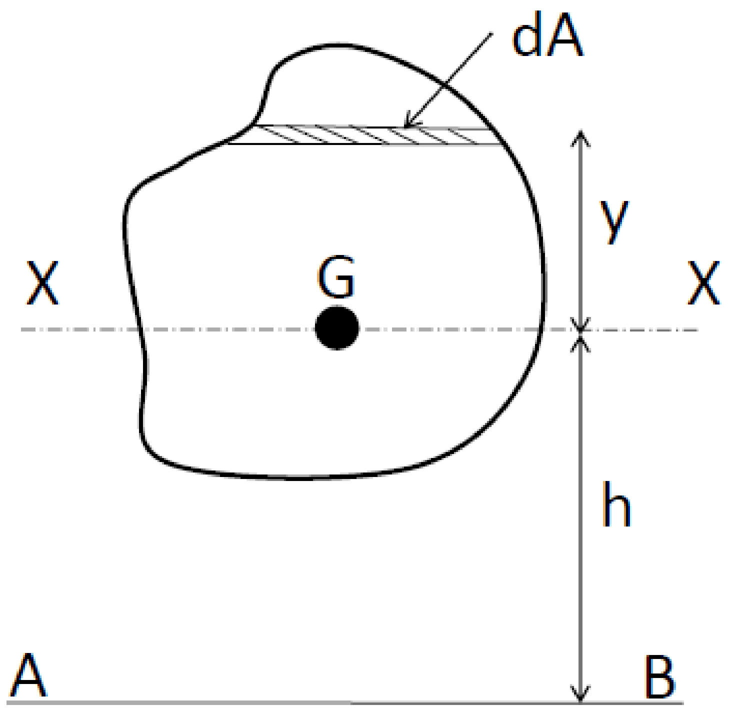
$$(I_{AB})_{dA} = dA.(y+h)^2 = dA(y^2 + h^2 + 2yh)$$

$$I_{AB} = \sum dAy^2 + \sum dAh^2 + \sum 2yhdA$$

$$I_{AB} = h^2 \sum dA + \sum dAy^2 + 2h \sum ydA$$

$$I_{AB} = h^2 A + I_G + 0$$

$$I_{AB} = I_G + Ah^2$$



Transfer axis theorem

$$[I_x = \int y^2\, dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

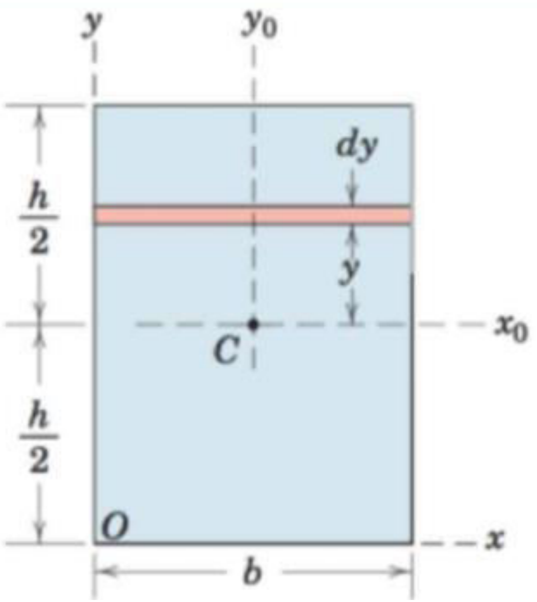
$$\bar{I}_y = \frac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \frac{1}{12}(bh^3 + hb^3) = \frac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem, the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 = \frac{1}{3}Ah^2 \qquad \textit{Ans.}$$



Transfer axis theorem

Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

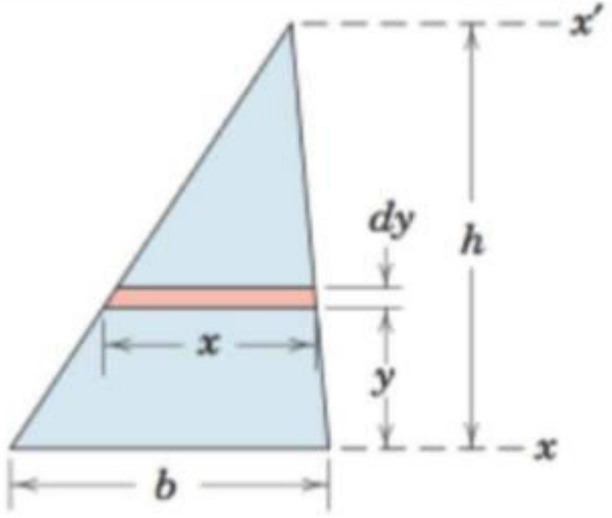
$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem, the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$



Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus,

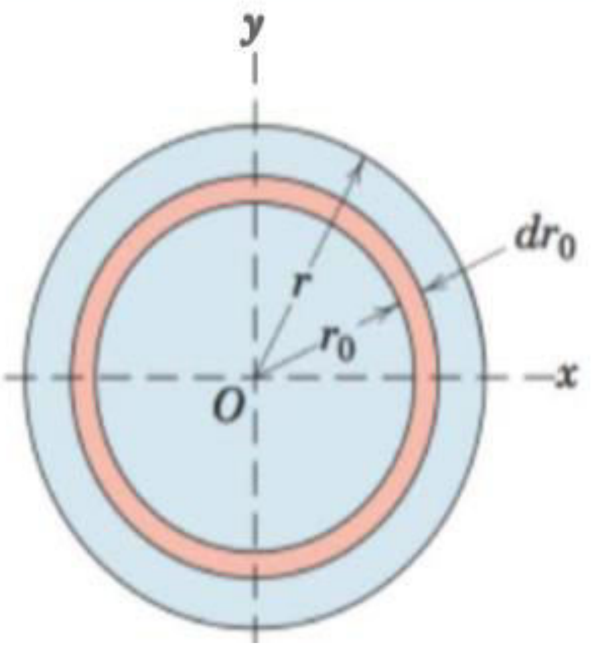
$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^2(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \qquad \textit{Ans.}$$



ENGINEERING MECHANICS

Moment of Inertia

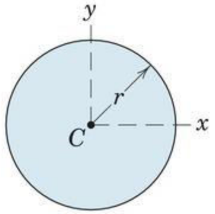
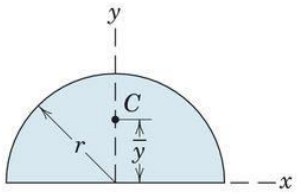
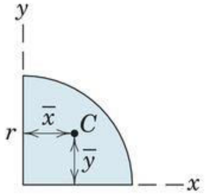
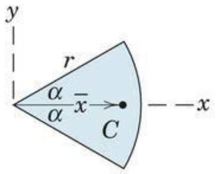
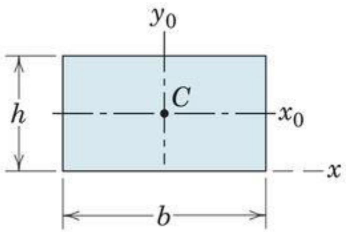
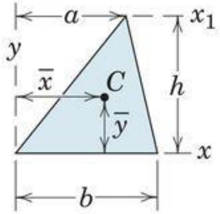
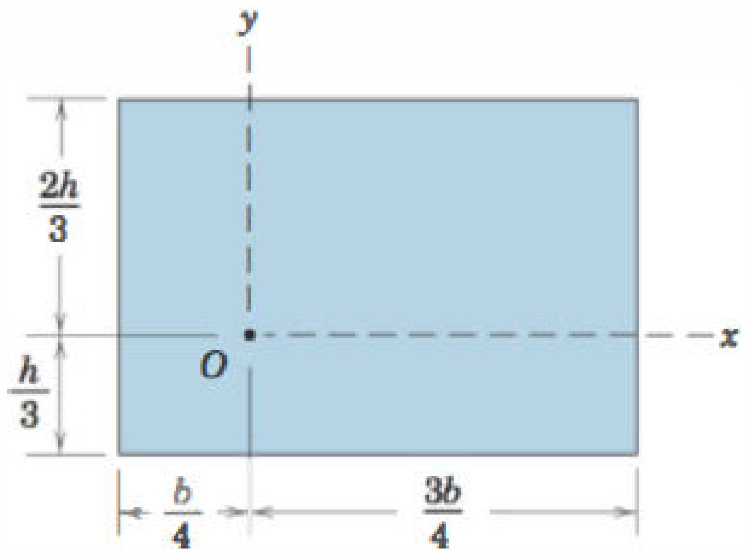
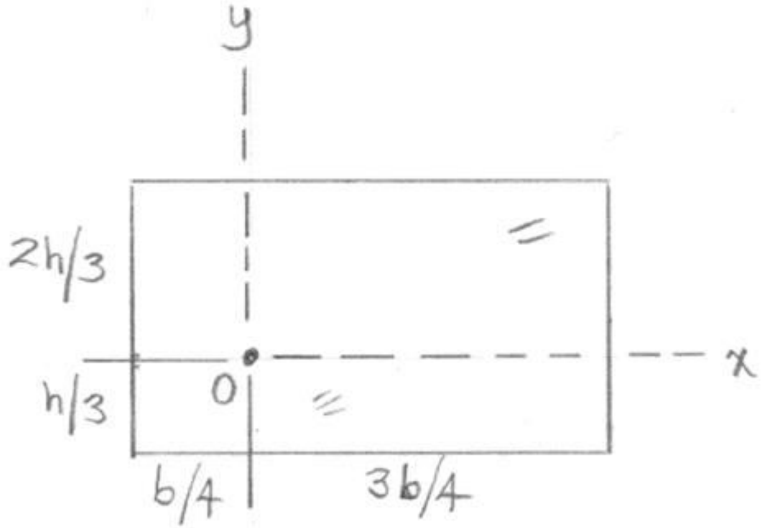
FIGURE		AREA MOMENTS OF INERTIA
Circular Area		$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area		$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area		$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of Circular Sector		$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2}\sin 2\alpha)$ $I_y = \frac{r^4}{4}(\alpha + \frac{1}{2}\sin 2\alpha)$ $I_z = \frac{1}{2} r^4 \alpha$

FIGURE		AREA MOMENTS OF INERTIA
Rectangular Area		$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area		$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

A/1 Determine the moments of inertia of the rectangular area about the x- and y-axes and find the polar moment of inertia about point 0.





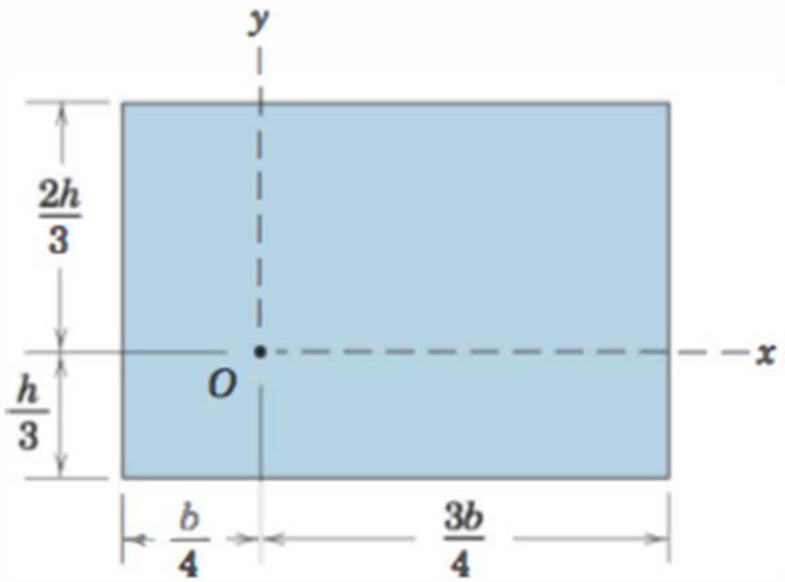
$$I_x = \bar{I}_x + A d_x^2 = \frac{1}{12} b h^3 + b h \left(\frac{h}{6}\right)^2$$

$$= \underline{\frac{1}{9} b h^3}$$

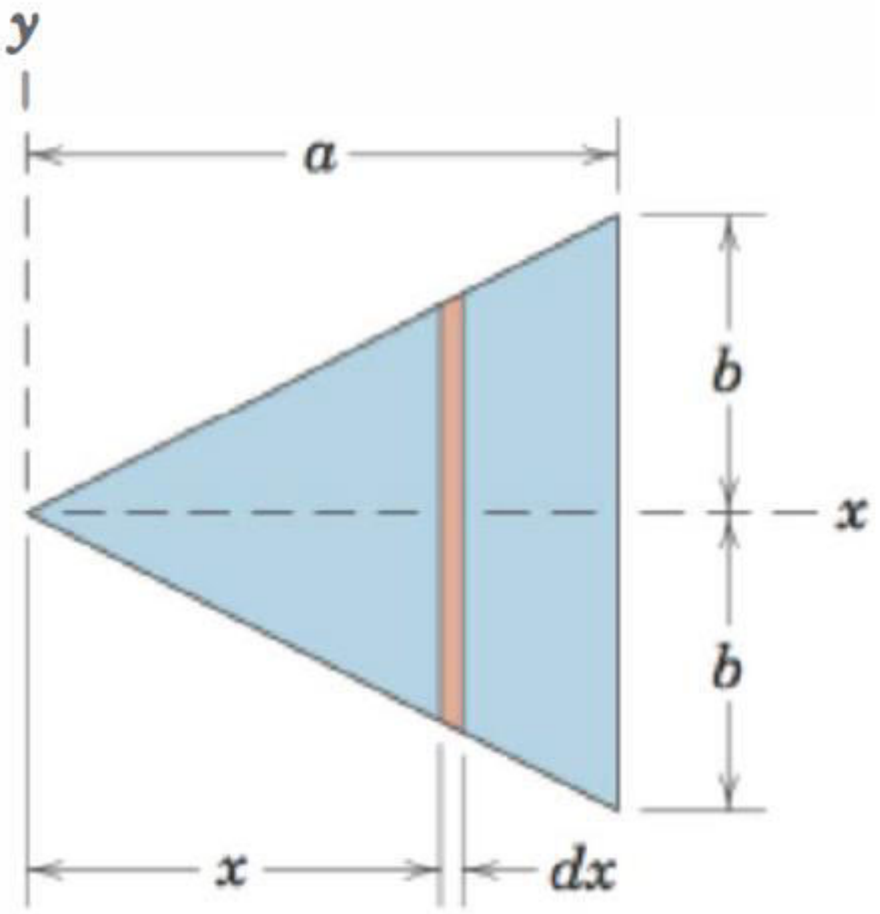
$$I_y = \bar{I}_y + A d_y^2 = \frac{1}{12} h b^3 + b h \left(\frac{b}{4}\right)^2$$

$$= \underline{\frac{7}{48} h b^3}$$

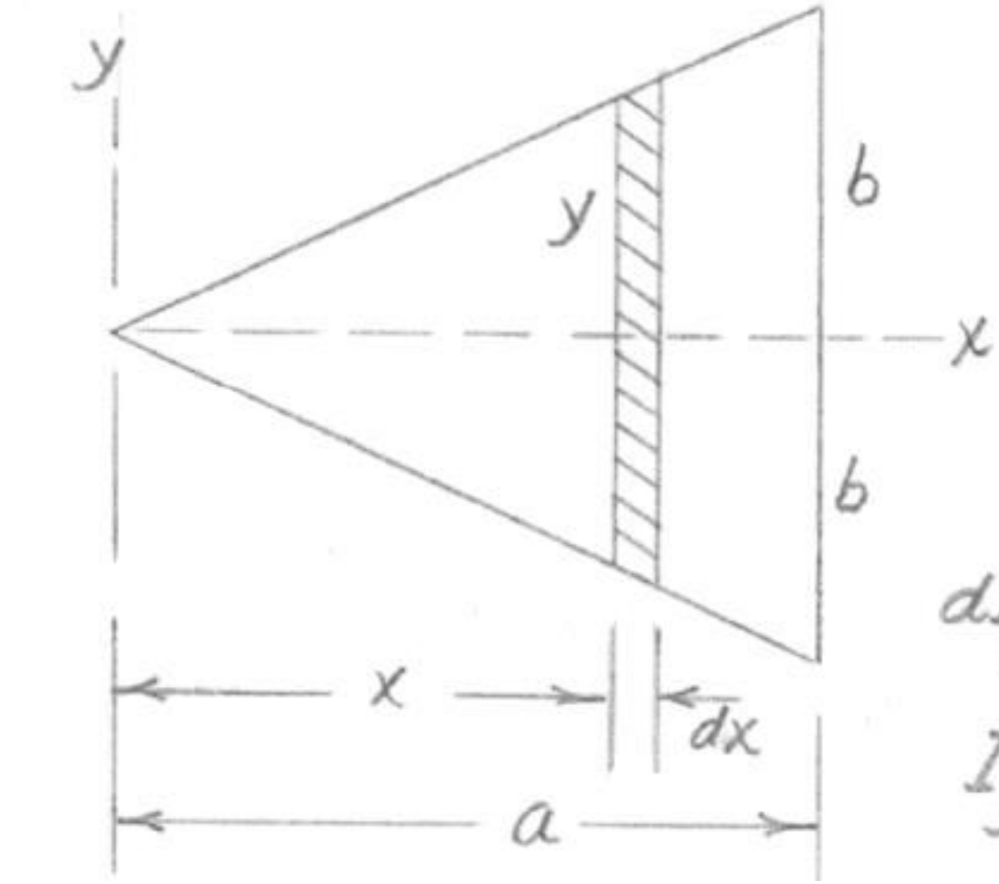
$$I_z = I_x + I_y = \underline{b h \left(\frac{h^2}{9} + \frac{7b^2}{48}\right)}$$



A/2 Use the differential element shown to determine the moment of inertia of the triangular area about the x-axis and about the y-axis.



A/2 Using the results of Sample Prob. A/1



$$dI_x = \frac{1}{12}(2y)^3 dx = \frac{2}{3}y^3 dx$$

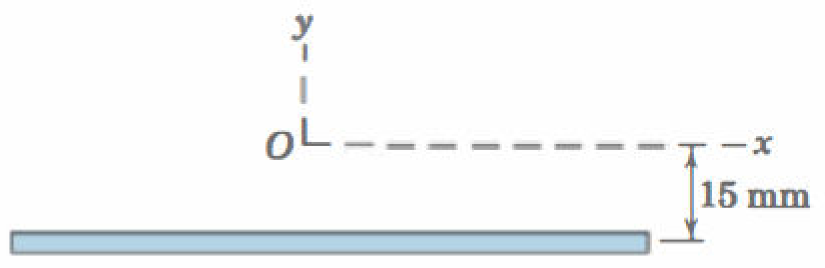
But $y = \frac{b}{a}x$ so

$$I_x = \frac{2b^3}{3a^3}\int_0^a x^3 dx = \underline{\frac{1}{6}ab^3}$$

$$dI_y = x^2(2y\,dx) = 2\frac{b}{a}x^3 dx$$

$$I_y = \frac{2b}{a}\int_0^a x^3 dx = \frac{2b}{a}\frac{a^4}{4} = \frac{1}{2}ba^3$$

A/3 The narrow rectangular strip has an area of 300 mm^2, and its moment of inertia about the y-axis is $35(10^3)$ mm^4. Obtain a close approximation to the polar radius of gyration about point 0.

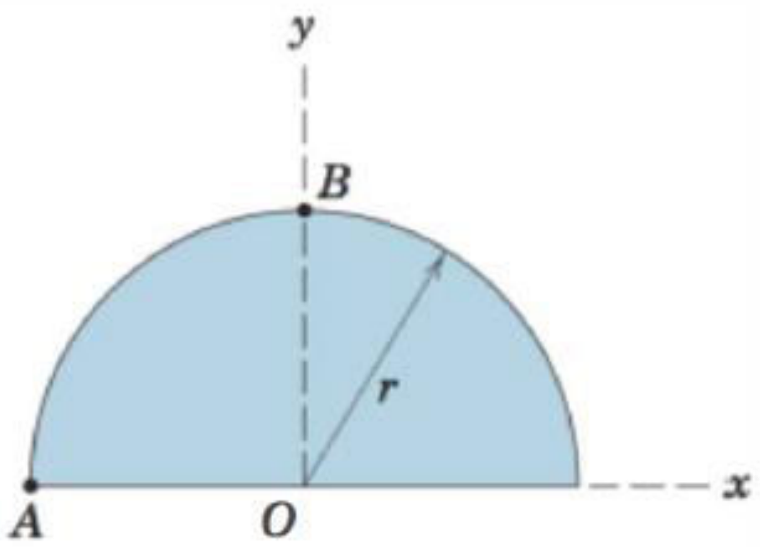


$$I_x \approx Ad^2 = 300(15)^2 = 67.5(10^3)\ mm^4$$

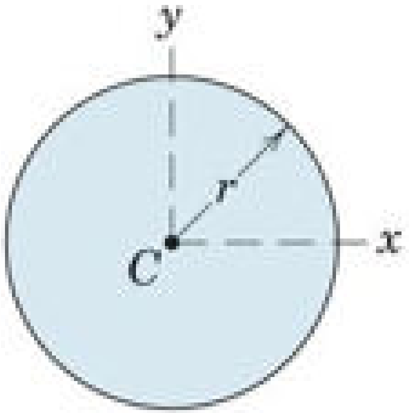
$$J_o = I_x + I_y = 67.5(10^3) + 35(10^3) = 102.5(10^3)\ mm^4$$

$$k_o = \sqrt{J_o/A} = \sqrt{\frac{102.5(10^3)}{300}} = \underline{18.48\ mm}$$

A/6 Determine the polar moments of inertia of the semicircular area about points A and B.

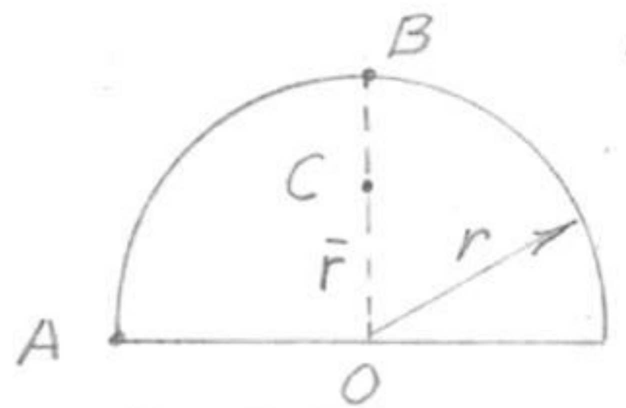


Circular Area



$$I_x = I_y = \frac{\pi r^4}{4}$$

$$I_z = \frac{\pi r^4}{2}$$



$\bar{r} = 4r/3\pi$

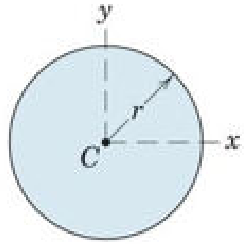
For complete circle

$$I_A = I_0 + Ar^2 = \frac{1}{2}Ar^2 + Ar^2$$
$$= \frac{3}{2}Ar^2$$

For half circle

$$I_A = \frac{1}{2}\left(\frac{3}{2}\pi r^4\right) = \underline{\frac{3}{4}\pi r^4}$$

Circular Area

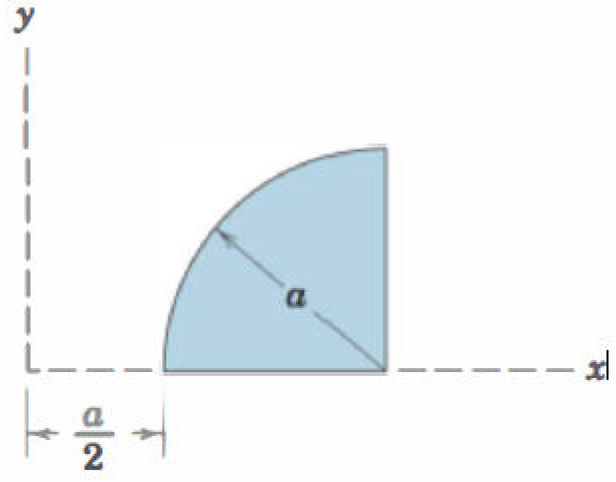


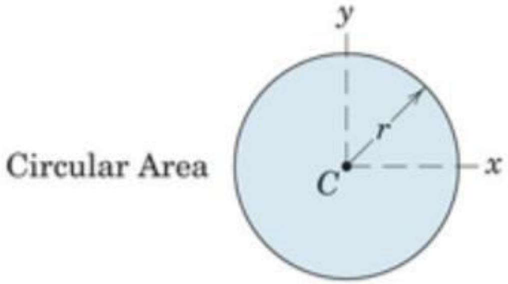
$$I_x = I_y = \frac{\pi r^4}{4}$$
$$I_z = \frac{\pi r^4}{2}$$

For half circle, $I_0 = \frac{1}{4}\pi r^4$

$$I_B = I_C + A(r-\bar{r})^2 = I_0 - A\bar{r}^2 + A(r-\bar{r})^2$$
$$= I_0 + A(r^2 - 2r\bar{r})$$
$$= \frac{1}{4}\pi r^4 + \frac{\pi r^4}{2}\left(1 - \frac{8}{3\pi}\right) = r^4\left(\frac{3\pi}{4} - \frac{4}{3}\right)$$

A/7 Determine the moment of inertia of the quarter circular area about the y-axis.

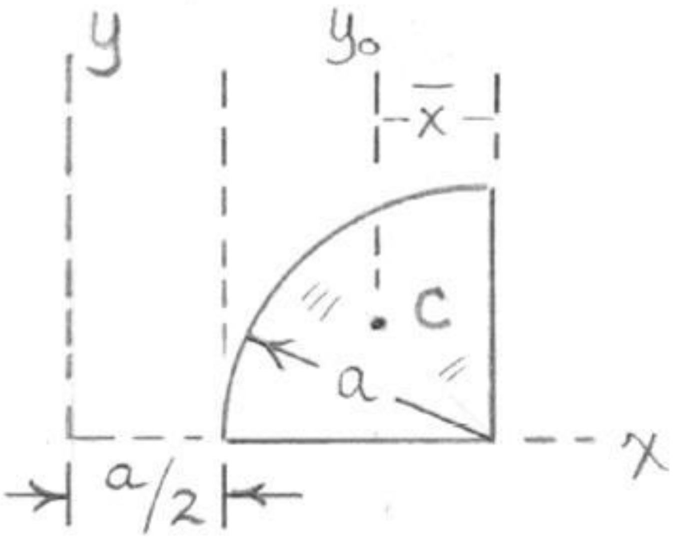




$$I_x = I_y = \frac{\pi r^4}{4}$$

$$I_z = \frac{\pi r^4}{2}$$

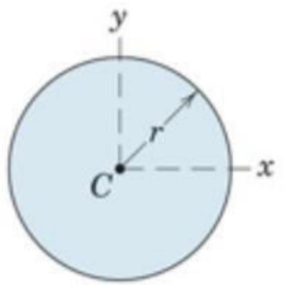
A/7



From Table D/3:

$$\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^4$$

$$\bar{x} = \frac{4a}{3\pi}$$



$$I_x = I_y = \frac{\pi r^4}{4}$$

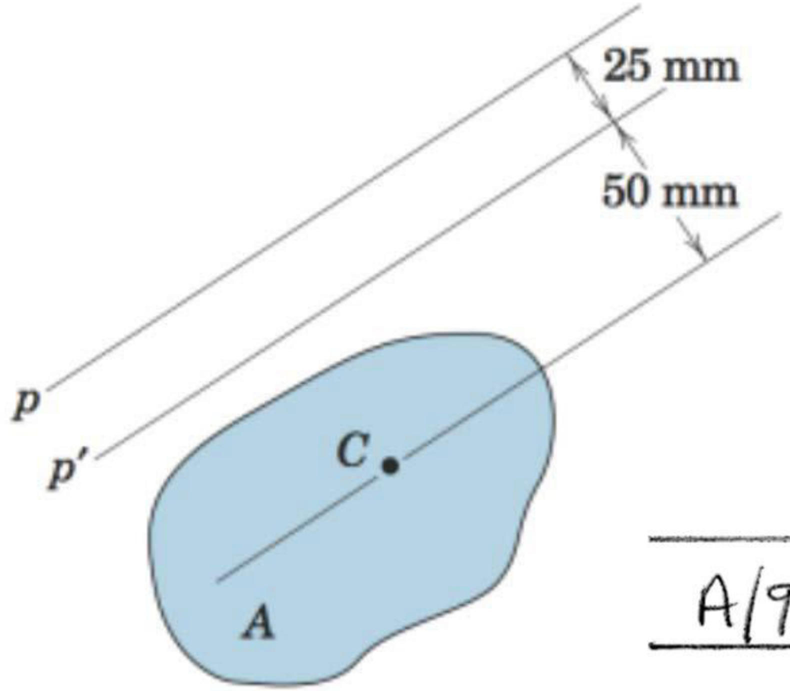
$$I_z = \frac{\pi r^4}{2}$$

$$I_y = \bar{I}_y + Ad_y^2$$

$$= \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^4 + \frac{\pi a^2}{4}\left[\frac{a}{2} + \left(a - \frac{4a}{3\pi}\right)\right]^2$$

$$= \left[\frac{5\pi}{8} - 1\right]a^4$$

A/9 The moments of inertia of the area A about the parallel p- and p'-axes differ by $15(10^6)$ mm^4. Compute the area A, which has its centroid at C.



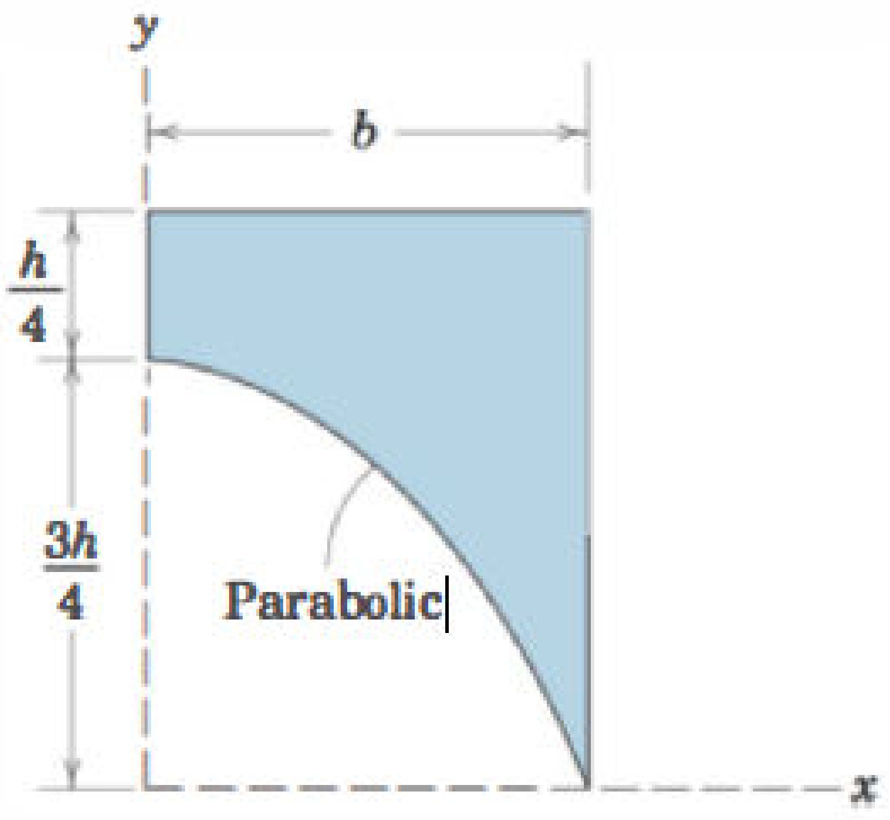
A/9

$$I_p = I_c + A(75)^2, \quad I_{p'} = I_c + A(50)^2$$

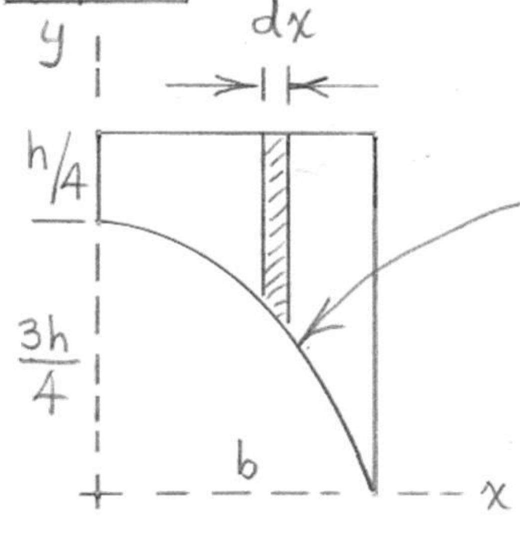
$$I_p - I_{p'} = 15(10^6) = A[(75)^2 - (50)^2]$$

$$\underline{A = 4800 \text{ mm}^2}$$

A/12 Determine the moment of inertia of the shaded area of the previous problem about the x-axis.



A/12



$$y = \frac{3h}{4} - \frac{3h}{4b^2}x^2$$

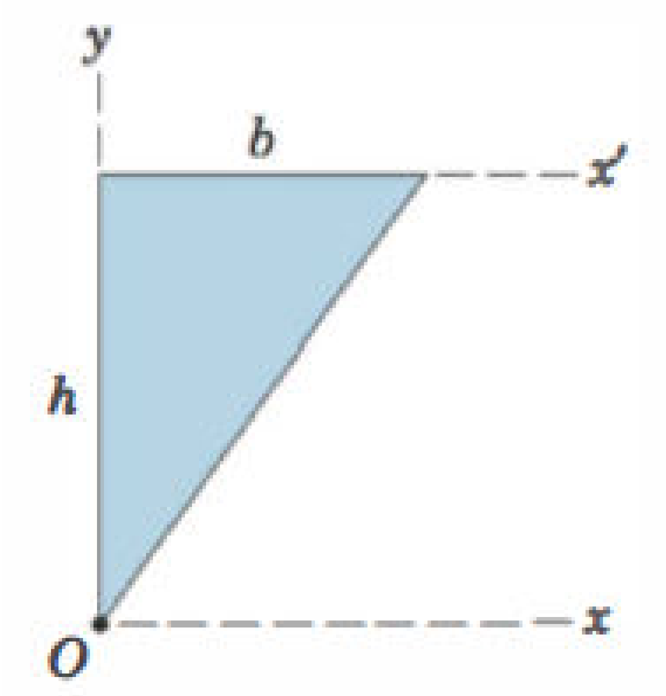
$$dI_x = \frac{1}{3}h^3 dx - \frac{1}{3}y^3 dx$$

$$= \frac{1}{3}dx\left[h^3 - \left(\frac{3h}{4} - \frac{3h}{4b^2}x^2\right)^3\right]$$

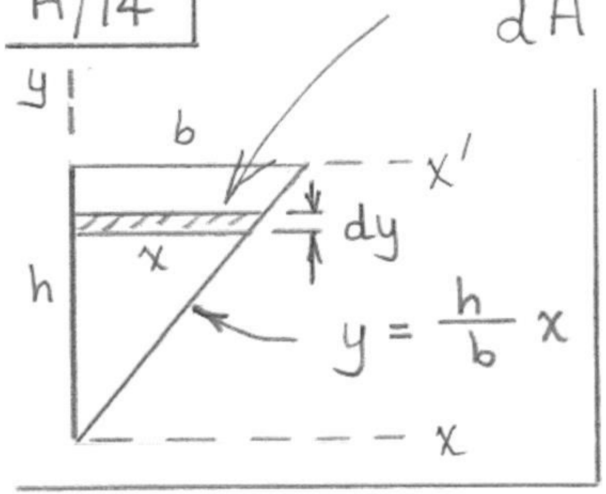
$$= \frac{1}{3}dx\left[\frac{37}{64}h^3 + \frac{81}{64}\frac{h^3}{b^2}x^2 - \frac{81}{64}\frac{h^3}{b^4}x^4 + \frac{27}{64}\frac{h^3}{b^6}x^6\right]$$

$$I_x = \int_0^b dI_x \quad \text{to obtain} \quad \underline{0.269bh^3}$$

A/14 By direct integration, determine the moments of inertia of the triangular area about the x- and x' -axes.



A/14



$$dA = x\,dy = \left(\frac{b}{h}y\right)dy$$

$$I_x = \int y^2 dA = \int_0^h y^2\left(\frac{b}{h}y\right)dy$$

$$= \frac{b}{h}\frac{y^4}{4}\Bigg|_0^h = \underline{\frac{1}{4}bh^3}$$

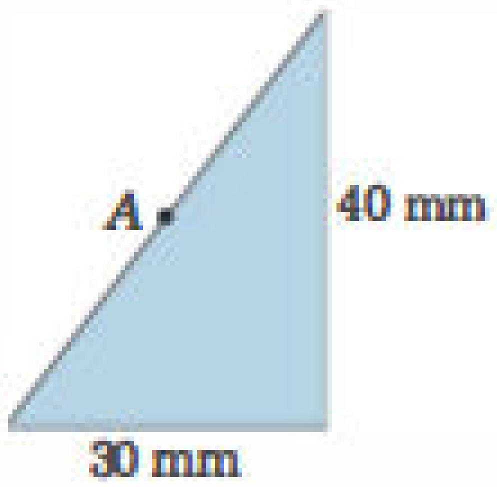
$$I_{x'} = \int (h-y)^2 dA = \int (h^2 - 2hy + y^2)\left(\frac{b}{h}y\,dy\right)$$

$$= \frac{b}{h}\int_0^h (h^2y - 2hy^2 + y^3)\,dy$$

$$= \frac{b}{h}\left[\frac{h^2y^2}{2} - \frac{2}{3}hy^3 + \frac{y^4}{4}\right]_0^h$$

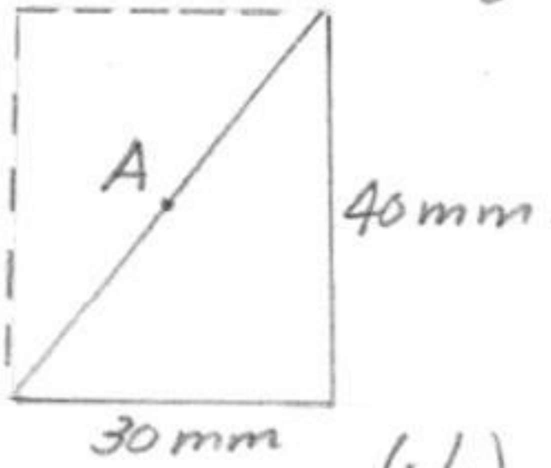
$$= \underline{\frac{1}{12}bh^3}$$

A/16 Determine the radius of gyration about a polar axis through the midpoint A of the hypotenuse of the right-triangular area. (Hint: Simplify your calculation by observing the results for a 30 x 40-mm rectangular area.)



A/16

$$(J_A)_{triangle} = \frac{1}{2}(J_A)_{rectangle}$$



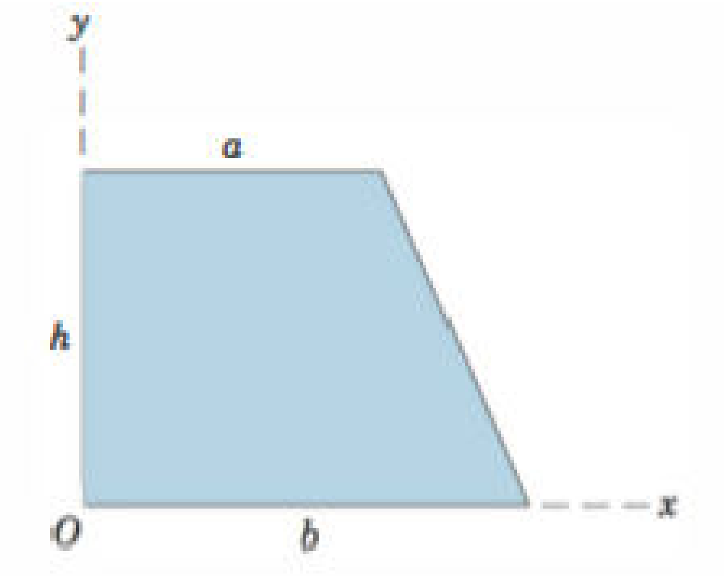
$$= \frac{1}{2}\left[\frac{1}{12}A(b^2+h^2)\right]$$ from Sample Prob. A/1

$$= \frac{1}{24}(30)(40)(\overline{30}^2+\overline{40}^2) = 12.5(10^4)\ \text{mm}^4$$

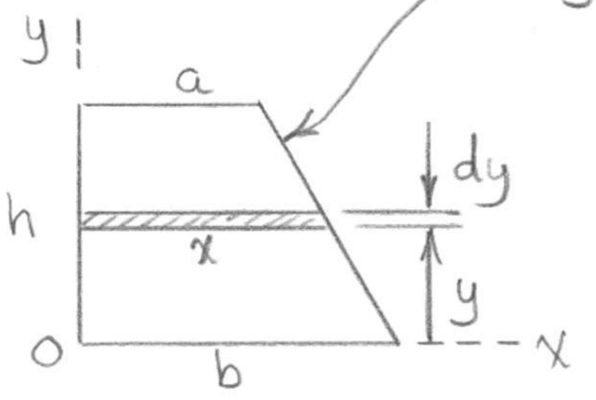
$$(J_A)_{triangle} = k_A^2 A$$

So $k_A = \sqrt{\dfrac{12.5(10^4)}{30(40)/2}} = \sqrt{208.4} = \underline{14.43\ \text{mm}}$

A/17 Determine by direct integration the moments of inertia of the trapezoidal area about the x- and y-axes. Find the polar moment of inertia about point 0.



A/17



$$y = -\frac{h}{b-a}x + \frac{bh}{b-a}$$

$$\Rightarrow x = -\frac{b-a}{h}y + b$$

$$dI_x = y^2 dA = y^2(x\,dy) = y^2\left(-\frac{b-a}{h}y + b\right)dy$$

$$= \left(-\frac{b-a}{h}y^3 + by^2\right)dy$$

$$I_x = \int dI_x = \int_0^h \left(-\frac{b-a}{h}y^3 + by^2\right)dy$$

$$= h^3\left(\frac{a}{4} + \frac{b}{12}\right)$$

$$dI_y = \frac{1}{3} dA\, x^2 = \frac{1}{3}(x\,dy)x^2 = \frac{1}{3}x^3 dy$$

$$= \frac{1}{3}\left[-\frac{b-a}{h}y + b\right]^3 dy$$

$$I_y = \int dI_y = \frac{1}{3}\int\left(-\frac{b-a}{h}y + b\right)^3 dy$$

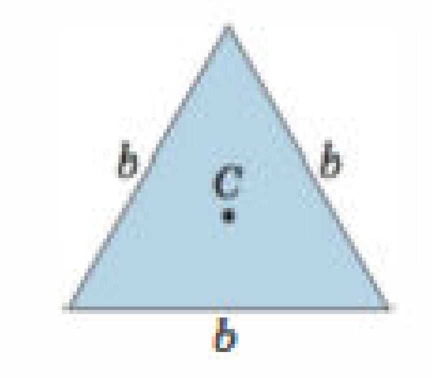
$$\vdots$$

$$= \underline{\frac{h}{12}\left(a^3 + a^2 b + a b^2 + b^3\right)}$$

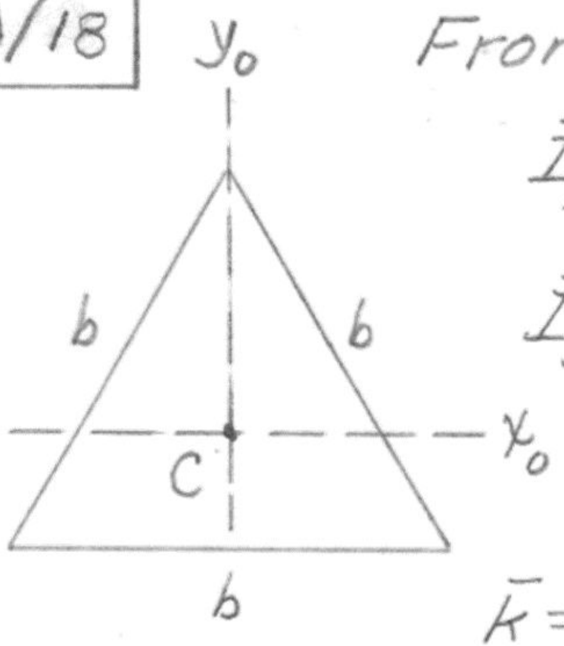
$$I_o = I_x + I_y$$

$$= \underline{\frac{h}{12}\left[h^2(3a+b) + a^3 + a^2 b + a b^2 + b^3\right]}$$

A/18 Determine the polar radius of gyration of the area of the equilateral triangle of side b about its centroid C.



A/18



From Sample Problem A/2

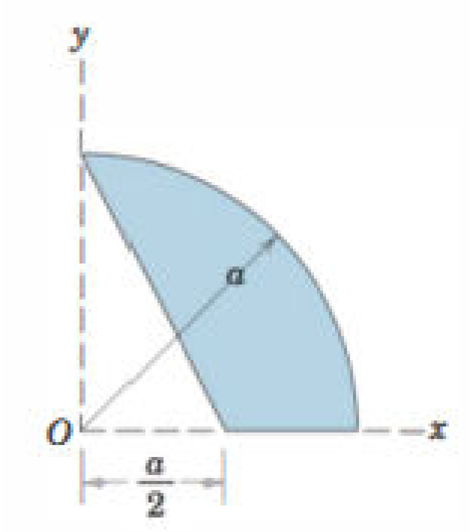
$$\bar{I}_x = \frac{1}{36} b\left(b\frac{\sqrt{3}}{2}\right)^3 = \frac{b^4}{96}\sqrt{3}$$

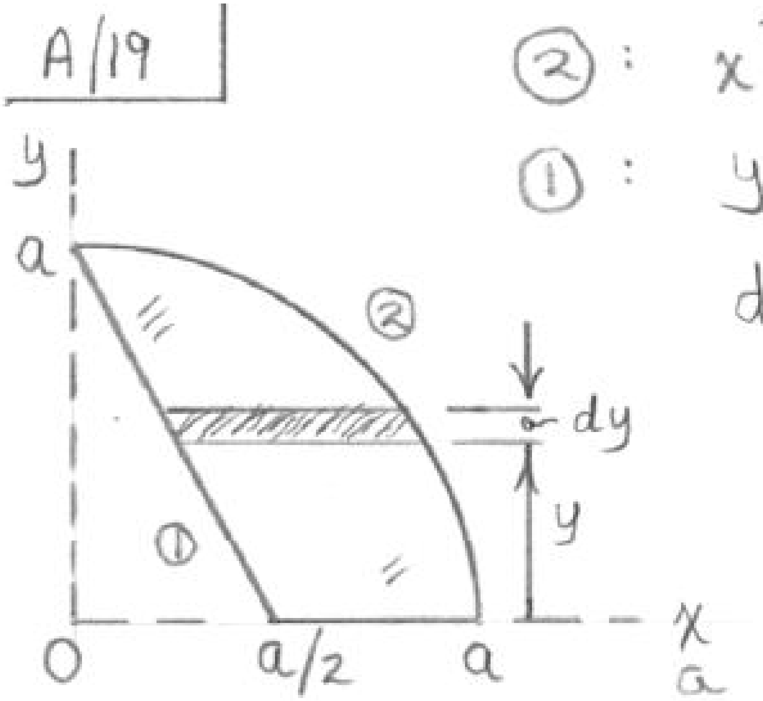
$$\bar{I}_y = 2\left(\frac{1}{12} b\frac{\sqrt{3}}{2}\left[\frac{b}{2}\right]^3\right) = \frac{b^4}{96}\sqrt{3}$$

$$\bar{J} = \bar{I}_x + \bar{I}_y = \frac{b^4}{48}\sqrt{3}$$

$$\bar{k} = \sqrt{\bar{J}/A} = \sqrt{\frac{b^4\sqrt{3}}{48} \Big/ \left(\frac{b^2}{2}\frac{\sqrt{3}}{2}\right)} = \underline{\frac{b}{2\sqrt{3}}}$$

A/19 Determine the moment of inertia of the shaded area about the x-axis.





② : $x^2 + y^2 = a^2$

① : $y = -2x + a$

$$dA = (x_2 - x_1)\,dy$$

$$= \left[\sqrt{a^2 - y^2} - \frac{a-y}{2}\right] dy$$

$$I_x = \int y^2 dA = \int_0^a y^2\left[\sqrt{a^2-y^2} - \frac{a-y}{2}\right] dy$$

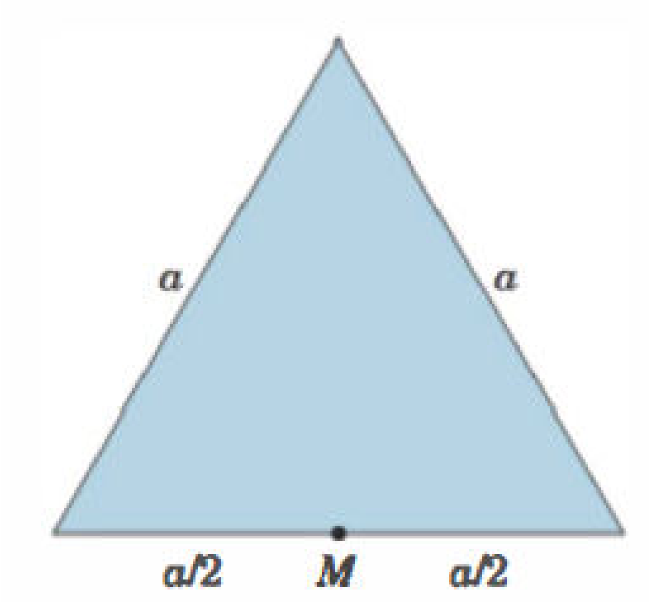
A/19

$$= \int_0^a \left[y^2\sqrt{a^2-y^2} - \frac{ay^2-y^3}{2} \right] dy$$

$$= \left[-\frac{y}{4}\sqrt{(a^2-y^2)^3} + \frac{a^2}{8}\left(y\sqrt{a^2-y^2} + a^2\sin^{-1}\frac{y}{a}\right) - \frac{ay^3}{6} + \frac{y^4}{8} \right]_0^a = a^4\left[\frac{\pi}{16} - \frac{1}{24}\right]$$

$$= \frac{a^4}{8}\left[\frac{\pi}{2} - \frac{1}{3}\right]$$

A/25 Determine the polar radius of gyration of the area of the equilateral triangle about the midpoint M of its base.



A/25

$$I_z = I_x + I_y, \quad I_z = Ak_z^2$$

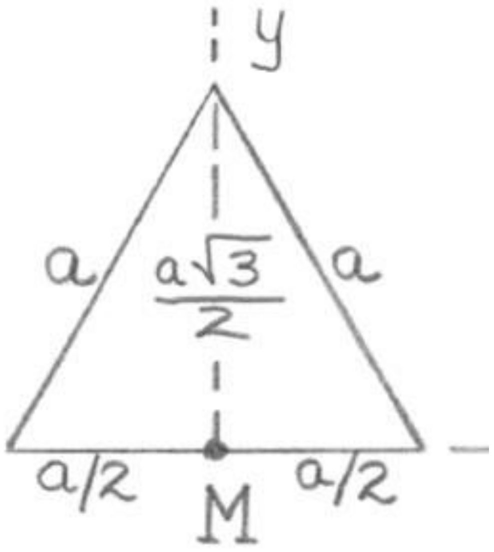
$$\therefore \quad k_M = \sqrt{(I_x + I_y)/A}$$

$$I_x = \frac{1}{12} bh^3 = \frac{1}{12} a \left(\frac{a\sqrt{3}}{2}\right)^3$$

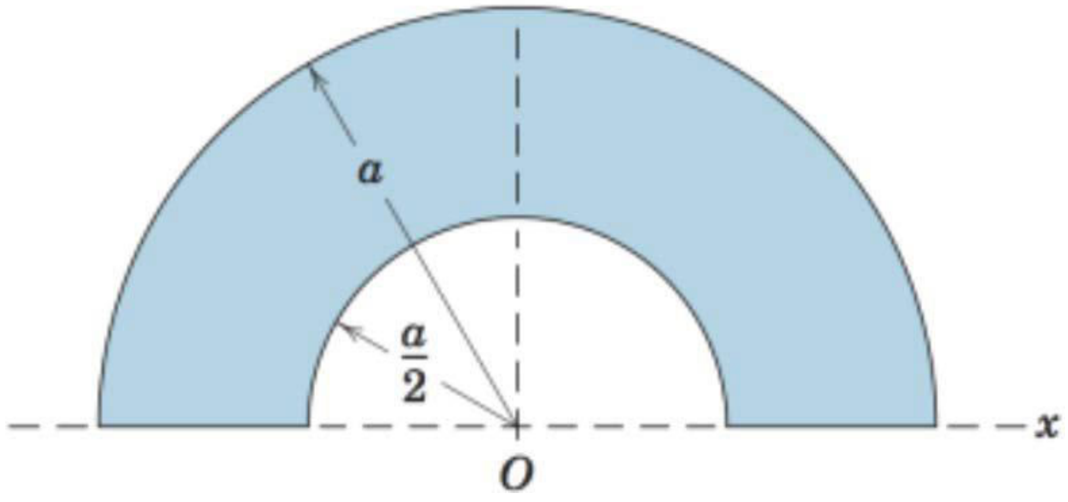
$$= \frac{\sqrt{3}}{32} a^4$$

$$I_y = 2\left(\frac{1}{12} \frac{a\sqrt{3}}{2} \left(\frac{a}{2}\right)^3\right) = \frac{\sqrt{3}}{96} a^4$$

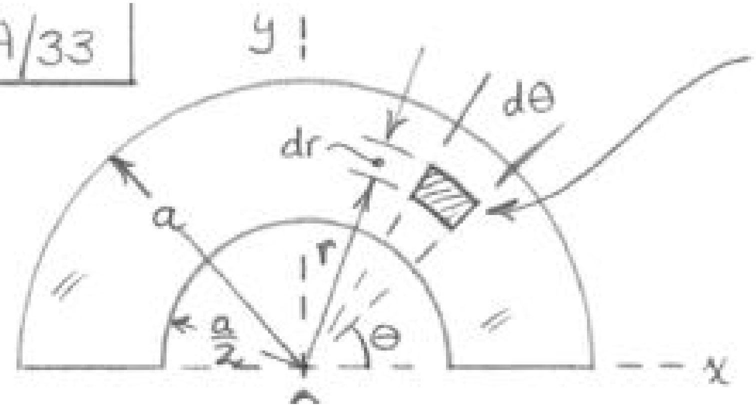
$$k_M = \left[\frac{\frac{\sqrt{3}}{32} a^4 + \frac{\sqrt{3}}{96} a^4}{\frac{a}{2} a \frac{\sqrt{3}}{2}}\right]^{1/2} = \frac{a}{\sqrt{6}}$$



A/33 By the methods of this article, determine the rectangular and polar radii of gyration of the shaded area about the axes shown.



A/33



$$dA = r\,dr\,d\theta$$

$$A = \frac{1}{2}\left[\pi a^2 - \pi\left(\frac{a}{2}\right)^2\right] = \frac{3}{8}\pi a^2$$

$$I_x = \int y^2 dA = \int_0^{\pi}\int_{a/2}^{a} (r\sin\theta)^2\, r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{15}{64} a^4 \sin^2\theta\, d\theta = \frac{15}{128}\pi a^4$$

$$I_y = \int x^2 dA = 2\int_0^{\pi/2}\int_{a/2}^{a} (r\cos\theta)^2\, r\,dr\,d\theta$$

$$= 2\int_{a/2}^{a} \frac{15}{64} a^4 \cos^2\theta\, d\theta = \frac{15}{128}\pi a^4$$

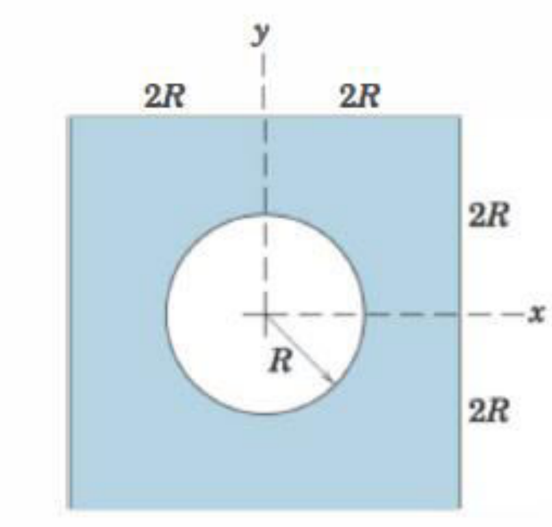
$$= 2\int_{a/2}^{a} \frac{15}{64} a^4 \cos^2\theta\, d\theta = \frac{15}{128}\pi a^4$$

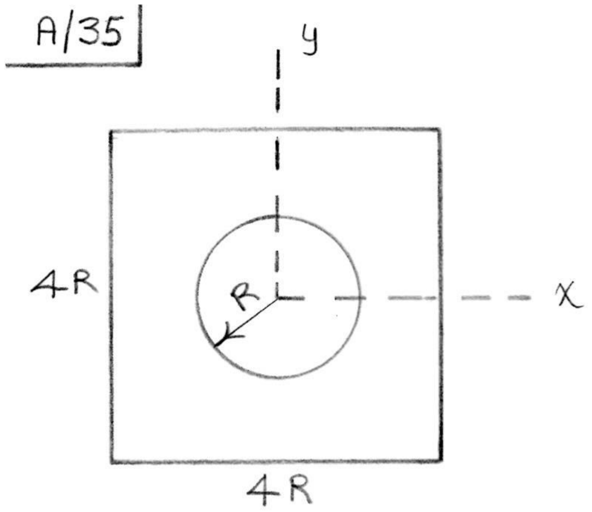
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{15}{128}\pi a^4}{\frac{3}{8}\pi a^2}} = \frac{\sqrt{5}}{4} a = k_y$$

$$k_z^2 = k_x^2 + k_y^2 = 2\left(\frac{5}{16} a^2\right)$$

$$k_z = \frac{\sqrt{10}}{4} a$$

A/35 Determine the moment of inertia about the x-axis of the square area without and with the central circular hole.





Without hole, $I_x = \frac{1}{12}(4R)(4R)^3 = \frac{64}{3}$

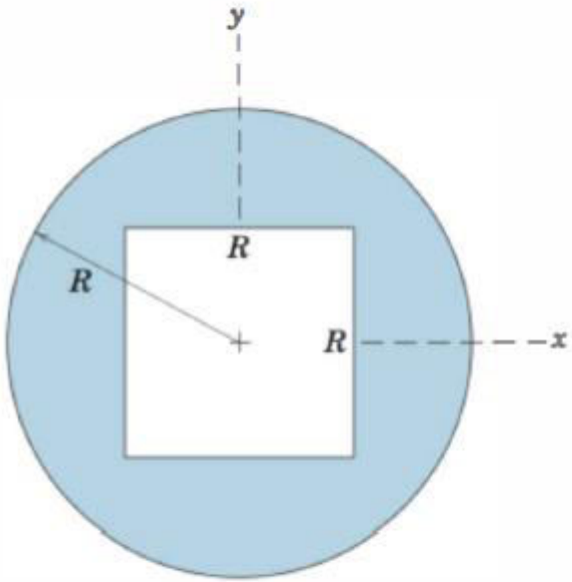
$\underline{(21.3\,R^4)}$

With hole, $I_x = \frac{64}{3}R^4 - \frac{1}{4}(\pi R^2)R^2$

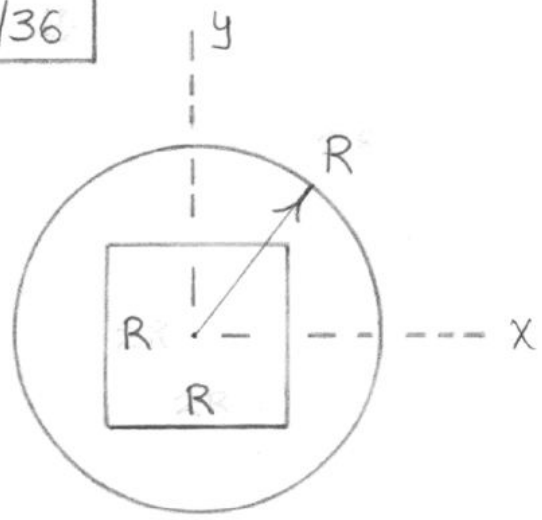
$= \underline{20.5\,R^4}$

(a 3.68% reduction)

A/36 Determine the polar moment of inertia of the circular area without and with the central square hole.



A/36



Without square hole:

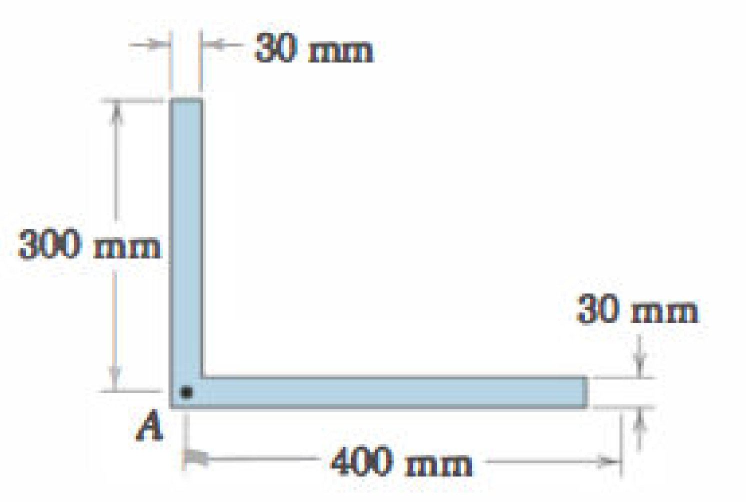
$$I_z = 2I_x = 2\left(\frac{1}{4}\pi R^2 \cdot R^2\right) = \underline{1.571R^4}$$

With hole:

$$I_z = 1.571R^4 - 2\left(\frac{1}{12}R \cdot R^3\right) = \underline{1.404R^4}$$

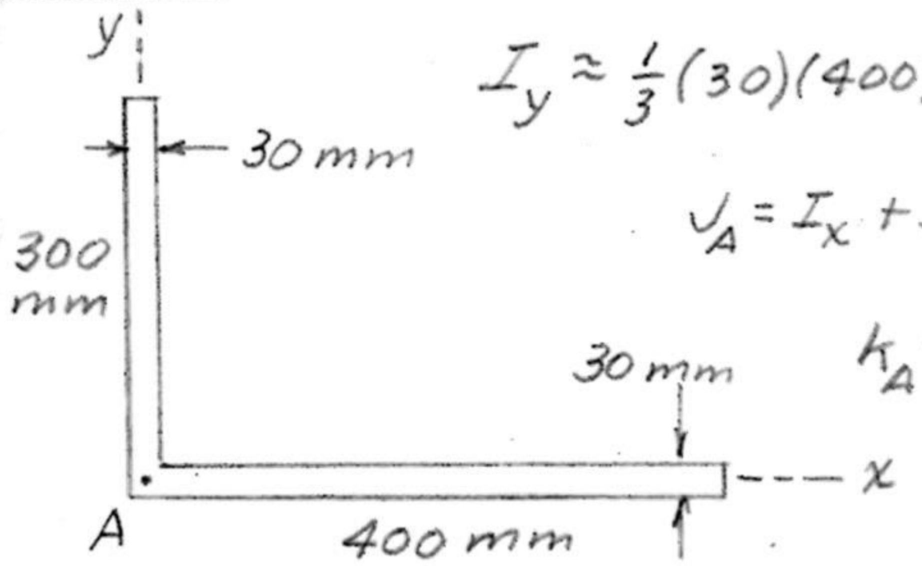
(a reduction of 10.61%)

A/37 Calculate the polar radius of gyration of the area of the angle section about point A Note that the width of the legs is small compared with the length of each leg.



A/37

$$I_x \approx \frac{1}{3}(30)(300)^3 + 0 = 270(10)^6 \text{ mm}^4$$



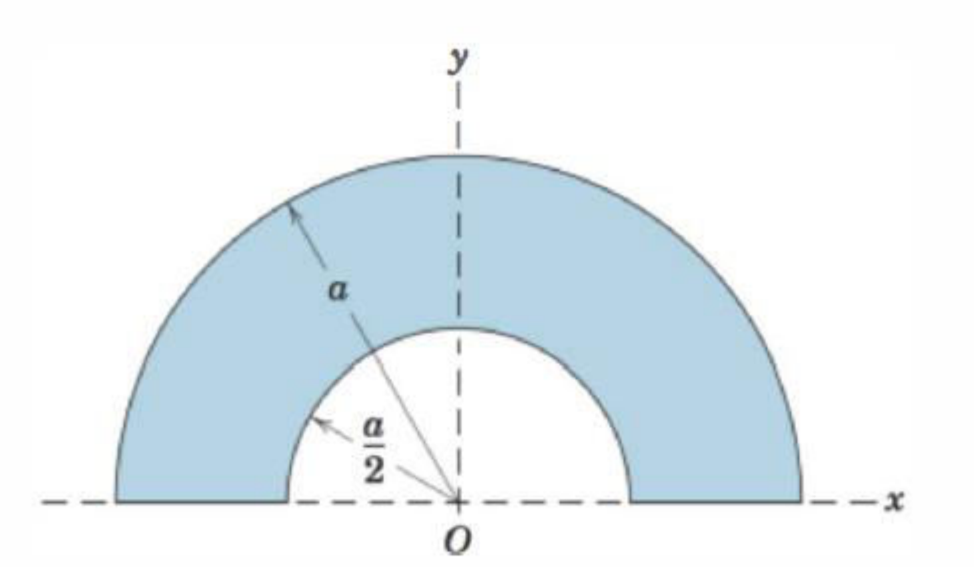
$$I_y \approx \frac{1}{3}(30)(400)^3 + 0 = 640(10)^6 \text{ mm}^4$$

$$J_A = I_x + I_y = 910(10)^6 \text{ mm}^4$$

$$k_A = \sqrt{J_A/A} = \sqrt{\frac{910(10)^6}{30(300+400)}}$$

$$k_A = \underline{208 \text{ mm}}$$

A/38 Calculate the polar radius of gyration of the area of the angle section about point A Note that the width of the legs is small compared with the length of each leg.



A/38

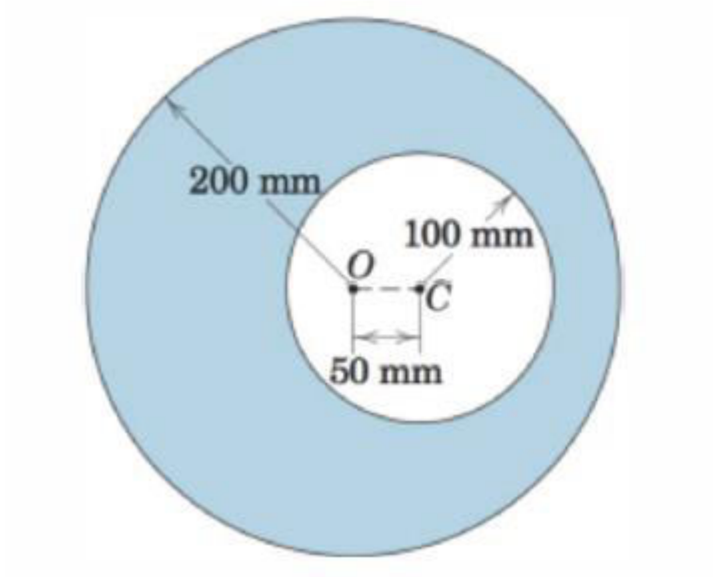
$$I_z = \frac{1}{2}\left[\frac{\pi a^4}{2} - \frac{\pi\left(\frac{a}{2}\right)^4}{2}\right] = \frac{15}{64}\pi a^4$$

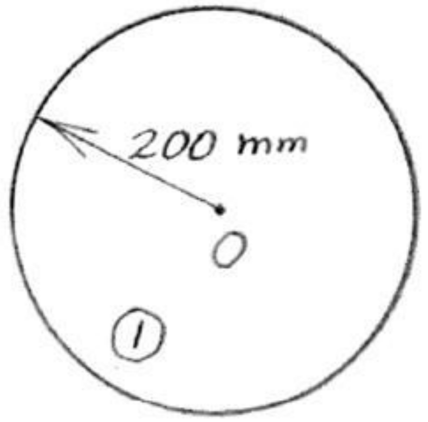
$$k_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{\frac{15}{64}\pi a^4}{\frac{3}{8}\pi a^2}} = \underline{\frac{\sqrt{10}}{4}a}$$

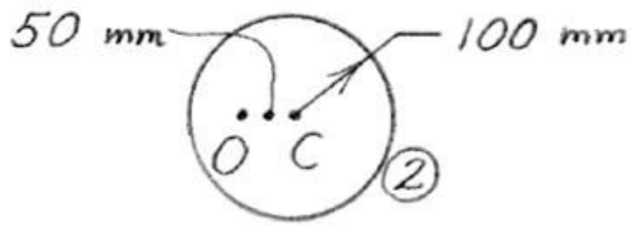
From $k_x^2 + k_y^2 = k_z^2$ and the fact that $k_x = k_y$ for the present case,

$$2k_x^2 = \left(\frac{\sqrt{10}}{4}a\right)^2, \quad k_x = k_y = \underline{\frac{\sqrt{5}}{4}a}$$

A/39 Calculate the polar radius of gyration of the shaded area about the center 0 of the larger circle.







Area $A = A_1 - A_2 = \pi(200^2 - 100^2) = 3(10^4)\pi \ \text{mm}^2$

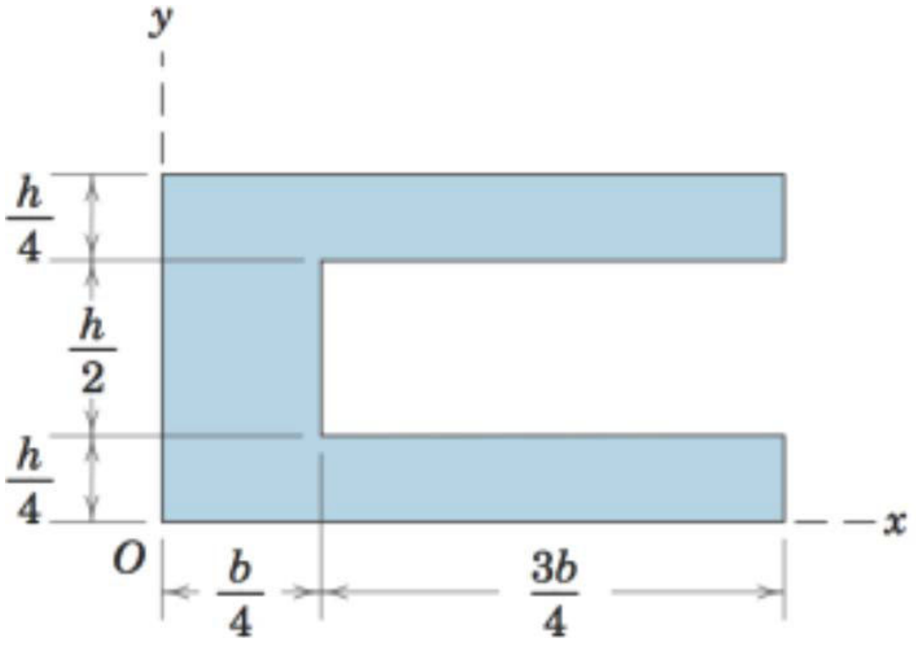
① $I_{O_1} = \frac{1}{2}(\pi \cdot 200^2)(200^2) = 8(10^8)\pi \ \text{mm}^4$

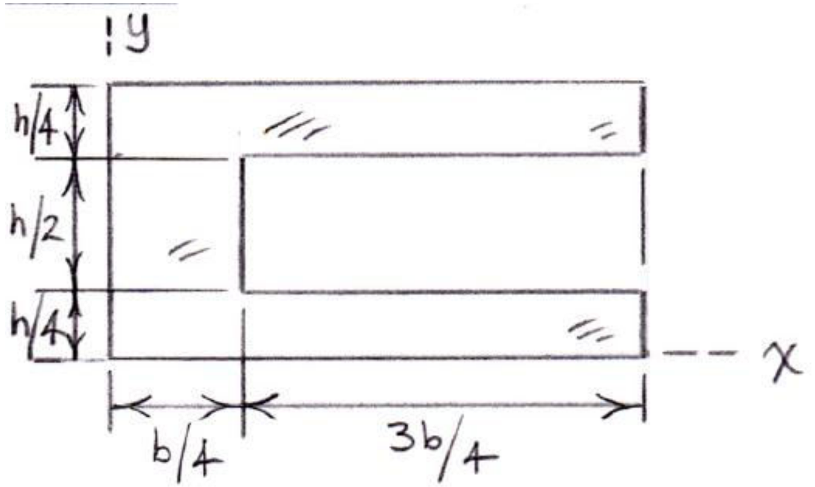
② $I_{O_2} = \frac{1}{2}(\pi \cdot 100^2)(100^2) + \pi(100^2)(50^2) = 0.75(10^8)\pi \ \text{mm}^4$

So $I_O = I_{O_1} - I_{O_2} = 7.25(10^8)\pi \ \text{mm}^4$

$k_O = \sqrt{I_O / A} = \sqrt{\dfrac{7.25(10^8)\pi}{3(10^4)\pi}} = \underline{155.5 \ \text{mm}}$

A/40 Determine the percent reductions in both area and area moment of inertia about the y-axis caused by removal of the rectangular cutout from the rectangular plate of baseband height h.





Percent reductions :

$$n_A = \frac{bh - \frac{5}{8}bh}{bh}(100\%) = \underline{37.5\%}$$

$$n_{I_y} = \frac{\frac{1}{3}hb^3 - \frac{65}{384}hb^3}{\frac{1}{3}hb^3} = \underline{49.2\%}$$

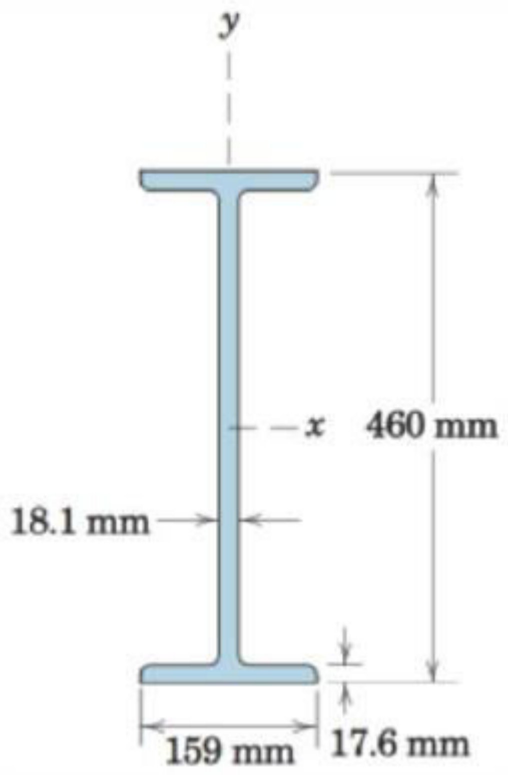
Full rectangle : $A = bh, \quad I_y = \frac{1}{3}hb^3$

With cutout : $A = bh - \frac{3b}{4}\left(\frac{h}{2}\right) = \frac{5}{8}bh$

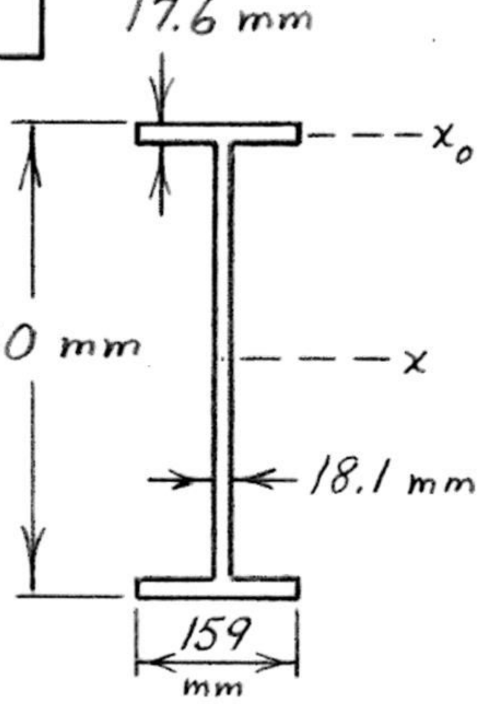
$$I_y = \frac{1}{3}hb^3 - \left[\frac{1}{12}\frac{h}{2}\left(\frac{3b}{4}\right)^3 + \frac{3}{8}bh\left(\frac{b}{4} + \frac{3b}{8}\right)^2\right]$$

$$= \frac{65}{384}hb^3$$

A/41 The cross-sectional area of an I-beam has the dimensions shown. Obtain a close approximation to the handbook value of Ix = 385(10^6) mm^4 by treating the section as being composed of three rectangles.



A/41



Flanges: $\bar{I}_x = I_{x_0} + Ad^2$

$$= 2\left\{\frac{1}{12}(159)(17.6^3) + 159(17.6)\left(230 - \frac{17.6}{2}\right)^2\right\}$$

$$= 2\{7.22(10^4) + 1.369(10^8)\}\ mm^4$$

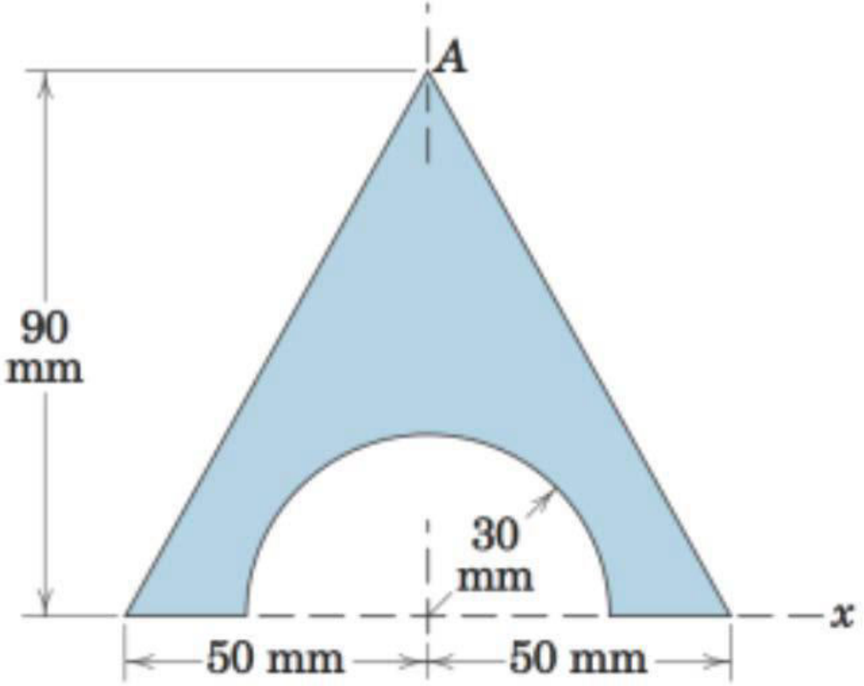
$$= 2.74(10^8)\ mm^4$$

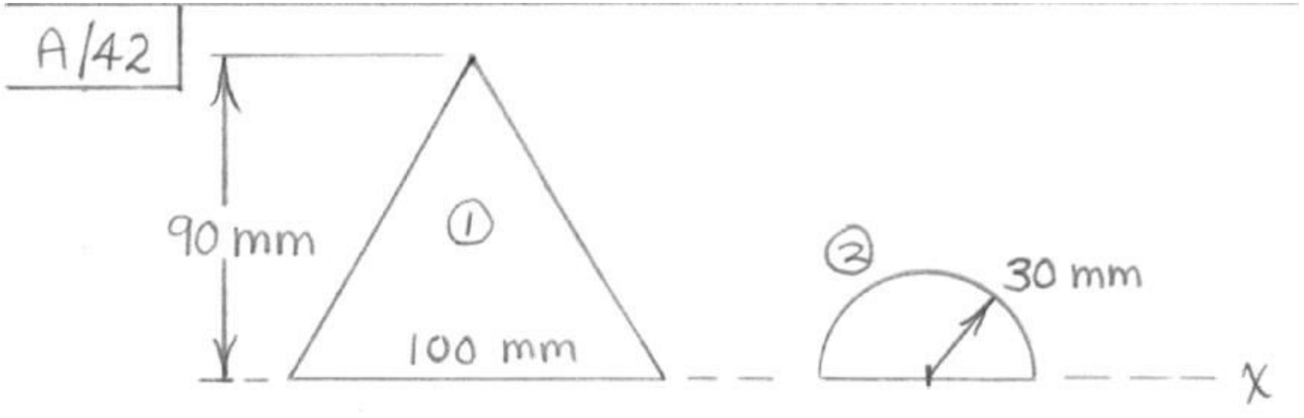
Web: $\bar{I}_x = \frac{1}{12}(18.1)(460 - 2[17.6])^3$

$$= 1.156(10^8)\ mm^4$$

Total $\underline{\bar{I}_x = 3.90(10^8)\ mm^4}$

A/42 Calculate the moment of inertia of the shaded area about the x-axis.



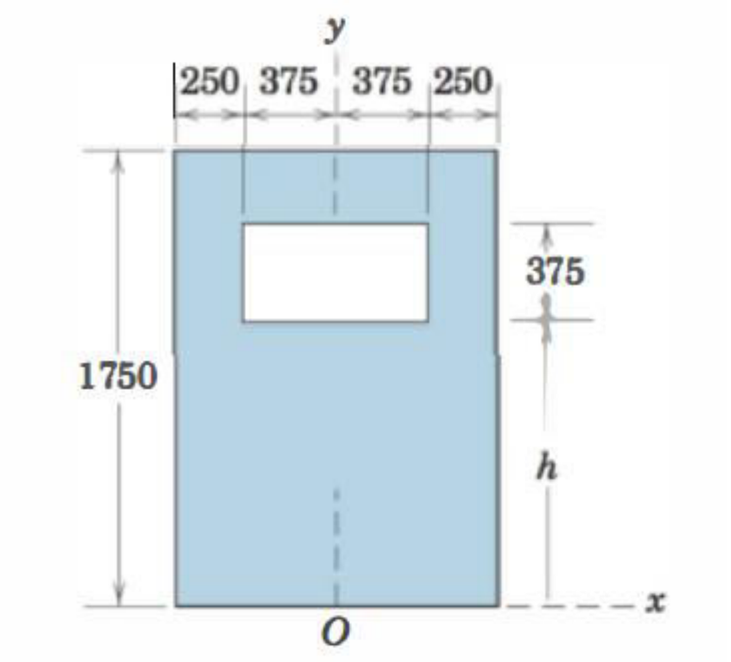


$$I_{x_1} = \frac{1}{12}(100)(90^3) = 6.08(10^6)\ \text{mm}^4$$

$$I_{x_2} = -\frac{\pi(30^4)}{8} = -0.318(10^6)\ \text{mm}^4$$

So $I_x = (6.08 - 0.318)10^6 = \underline{5.76(10^6)\ \text{mm}^4}$

A/43 The variable h designates the arbitrary vertical location of the bottom of the rectangular cutout within the rectangular area. Determine the area moment of inertia about the x-axis for (a) h = 1000 mm and (b) h = 1500 mm.



Dimensions in millimeters

(a) $h = 1000$ mm (hole complete)

$$I_x = \frac{1}{3}(1250)(1750^3) - \left[\frac{1}{12}(750)(375)^3 + 750(375)\left(1000 + \frac{375}{2}\right)^2\right]$$

$$= \underline{1.833(10^{12})\ \text{mm}^4} \text{ or } \underline{1.833\ \text{m}^4}$$

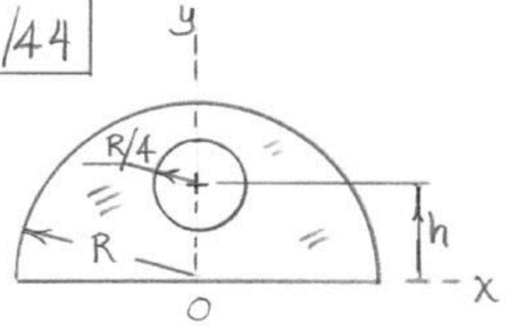
(b) $h = 1500$ mm (250 mm of hole in play)

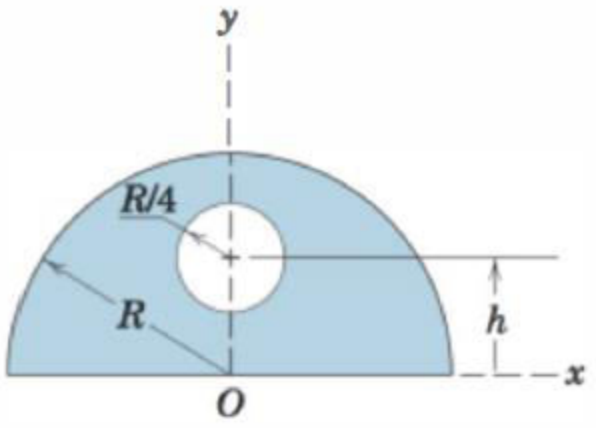
$$I_x = \frac{1}{3}(1250)(1750^3) - \left[\frac{1}{12}(750)(250)^3 + 750(250)\left(1500 + \frac{250}{2}\right)^2\right]$$

$$= \underline{1.737(10^{12})\ \text{mm}^4 \text{ or } 1.737\ \text{m}^4}$$

A/44 The variable h designates the arbitrary vertical location of the center of the circular cutout within the semicircular area. Determine the area moment of inertia about the x-axis for (a) h = 0 and (b) h = R/2.

A/44





(a) h = 0 (One-half of hole considered)

$$I_x = \frac{\pi R^4}{8} - \frac{\pi (R/4)^4}{8} = \frac{255}{2048}\pi R^4$$

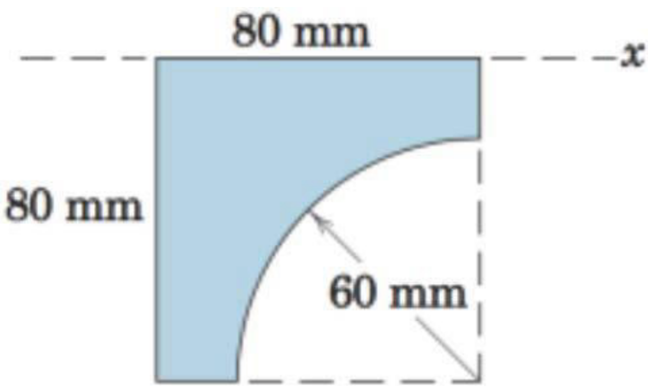
$(0.391 R^4)$

(b) $h = \frac{R}{2}$ (Entire hole now in play)

$$I_x = \frac{\pi R^4}{8} - \left[\frac{\pi (R/4)^4}{4} + \pi\left(\frac{R}{4}\right)^2\left(\frac{R}{2}\right)^2\right]$$

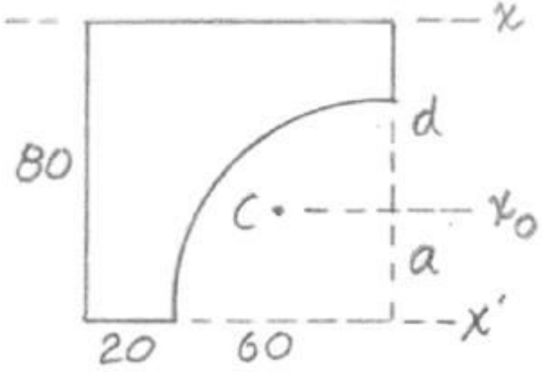
$$= \frac{111}{1024}\pi R^4 \quad (0.341 R^4)$$

A/45 Calculate the moment of inertia of the shaded area about the x-axis.



A/45

Dimen. in mm



Square: $I_x = \frac{1}{3}b^4 = \frac{1}{3}(80)^4 = 13.65(10^6)\ \text{mm}^4$

Quarter-circle: $a = \frac{4r}{3\pi} = \frac{4(60)}{3\pi}$

$= 25.46\ \text{mm}$

$d = 80 - 25.46 = 54.54\ \text{mm}$

$$I_x = I_{x_0} + Ad^2 = I_{x'} - Aa^2 + Ad^2$$

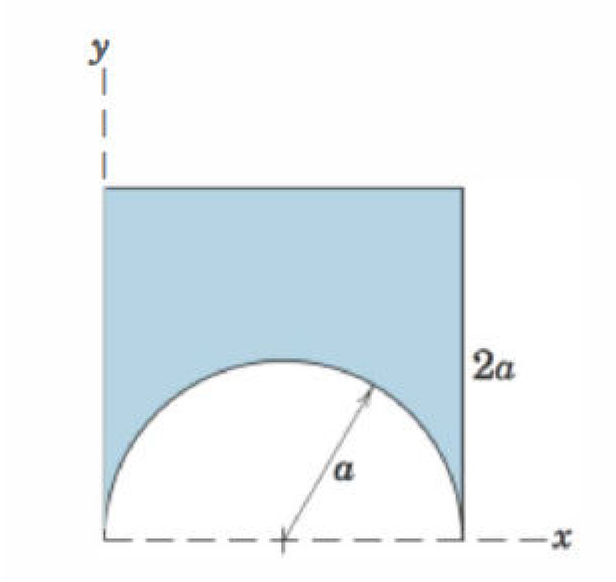
$$= -\frac{1}{4}\frac{\pi r^4}{4} - \frac{\pi r^2}{4}(d^2 - a^2) = -\frac{\pi r^2}{4}\left(\frac{r^2}{4} + d^2 - a^2\right)$$

$$= -\frac{\pi (60)^2}{4}\left[\frac{60^2}{4} + (54.54)^2 - (25.46)^2\right]$$

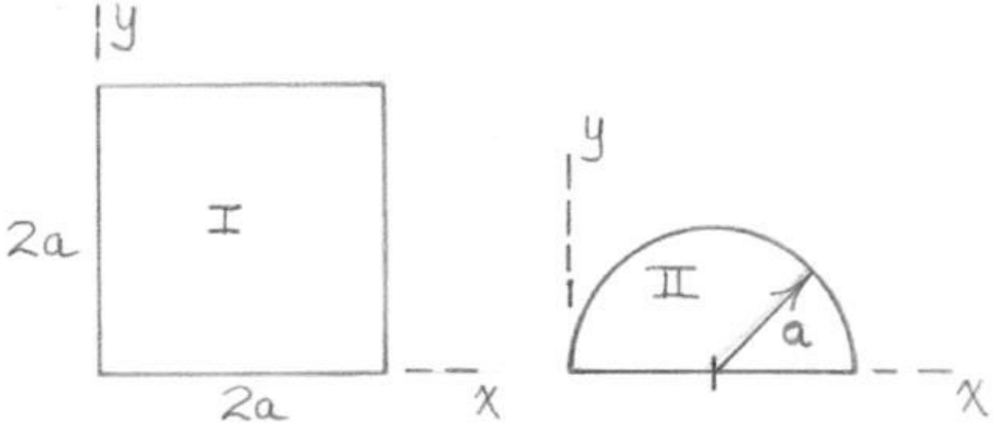
$$= -9.120(10^6)\ \text{mm}^4$$

Total $I_x = (13.65 - 9.120)(10^6) = \underline{4.53(10^6)\ \text{mm}^4}$

A/46 Determine the moments of inertia of the shaded area about the x- and y-axes.



A/46



I. Square $I_x = I_y = \frac{1}{3}(4a^2)(2a)^2 = \frac{16}{3}a^4$

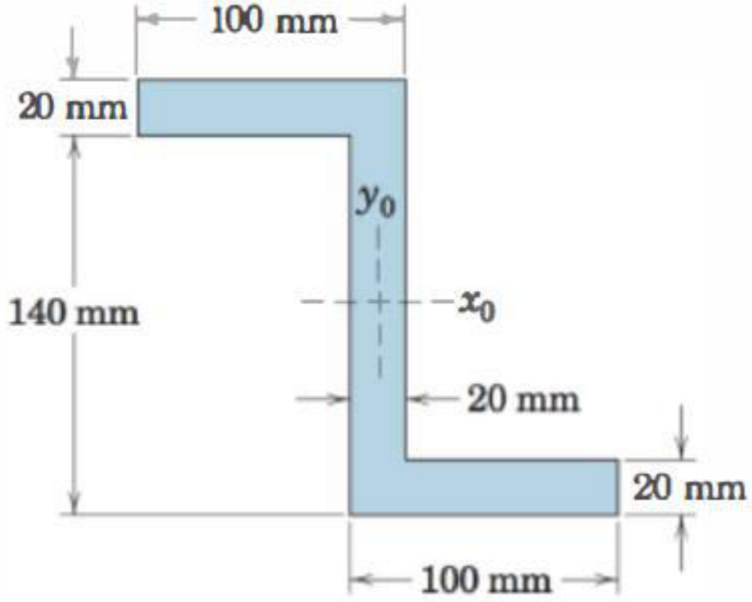
II. Semicircle $I_x = \frac{1}{8}\pi a^4$

$$I_y = \frac{1}{8}\pi a^4 + \frac{1}{2}\pi a^2(a^2) = \frac{5}{8}\pi a^4$$

Combined: $I_x = \frac{16}{3}a^4 - \frac{\pi}{8}a^4 = \underline{4.94a^4}$

$$I_y = \frac{16}{3}a^4 - \frac{5}{8}\pi a^4 = \underline{3.37a^4}$$

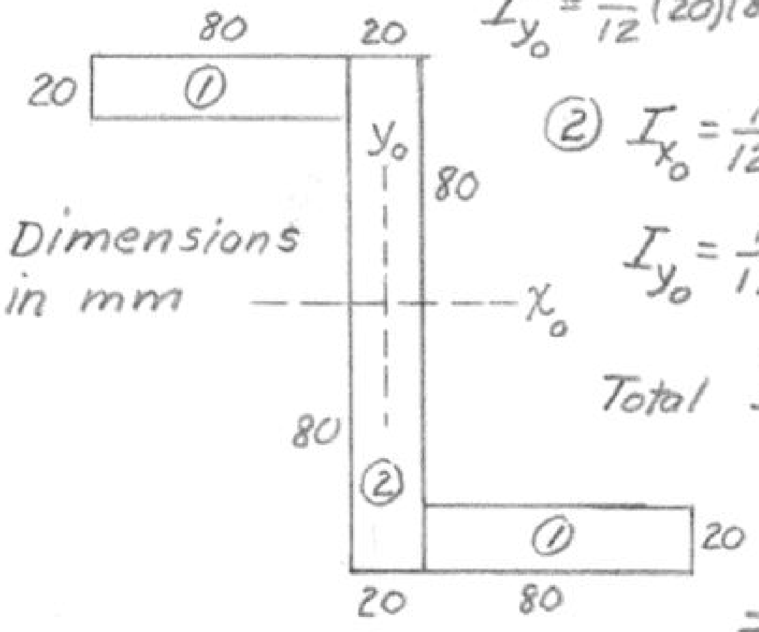
A/48 Determine the moments of inertia of the Z-section about its centroidal xo- and yo-axes.



A/48

① $I_{x_0} = \frac{1}{12}(80)(20)^3 + (80)(20)(70)^2 = 7.89(10^6)\ mm^4$

$I_{y_0} = \frac{1}{12}(20)(80)^3 + (20)(80)(50)^2 = 4.85(10^6)\ mm^4$



Dimensions in mm

② $I_{x_0} = \frac{1}{12}(20)(160)^3 = 6.83(10^6)\ mm^4$

$I_{y_0} = \frac{1}{12}(160)(20)^3 = 0.1067(10^6)\ mm^4$

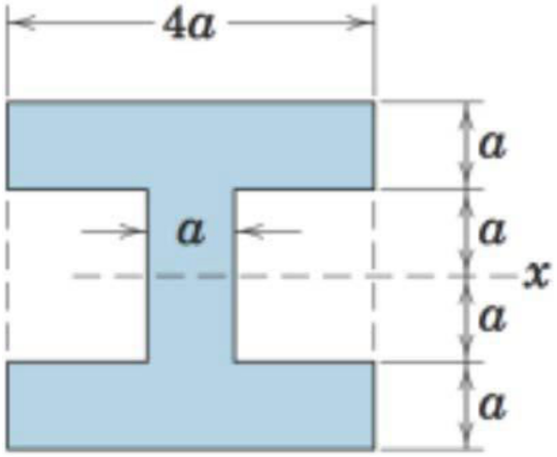
Total $\bar{I}_x = [2(7.89) + 6.83](10^6)$

$= \underline{22.6(10^6)\ mm^4}$

$\bar{I}_y = [2(4.85) + 0.1067](10^6)$

$= 9.81(10^6)\ mm^4$

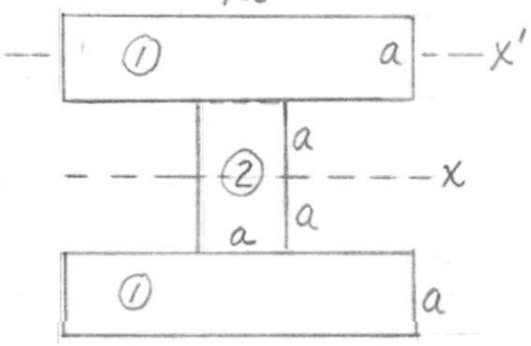
A/49 Determine the moment of inertia of the shaded area about the x-axis in two different ways.



A/49

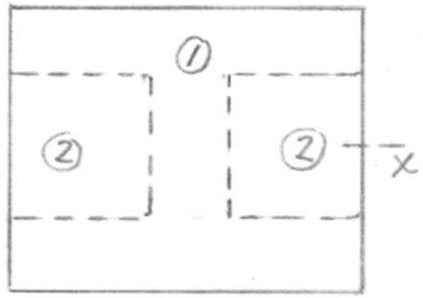
Sol. I

$$I_{1} = 2\left[\frac{1}{12}4a(a^3) + 4a^2\left(\frac{3a}{2}\right)^2\right] = \frac{56}{3}a^4$$



$$I_{2} = \frac{1}{12}a(2a)^3 = \frac{2}{3}a^4$$

Total $I_x = \frac{58}{3}a^4$



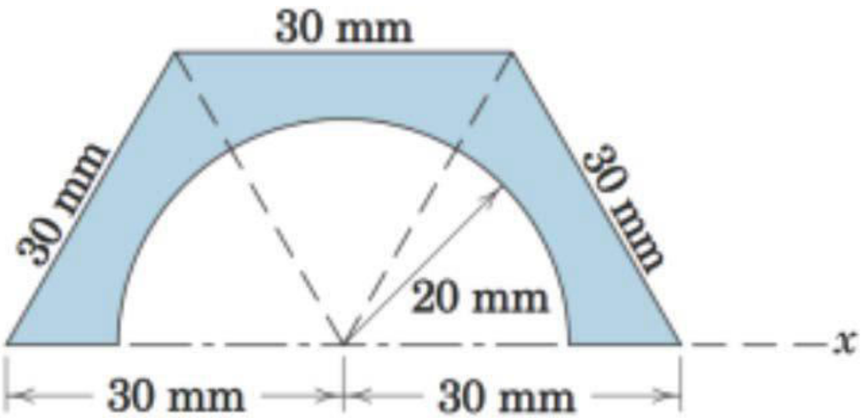
Sol. II

$$I_{1} = \frac{1}{12}(4a)(4a)^3 = \frac{64}{3}a^4$$

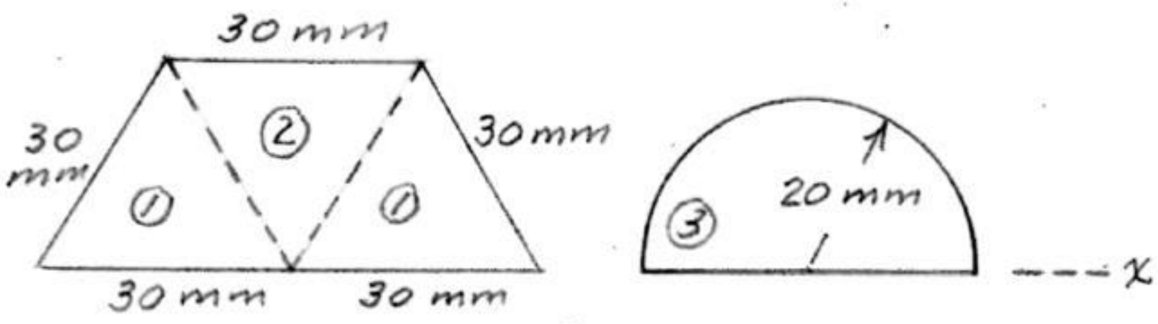
$$I_{2} = -\frac{1}{12}(3a)(2a)^3 = -2a^4$$

$$\text{Total} = \left(\frac{64}{3} - \frac{6}{3}\right)a^4 = \frac{58}{3}a^4$$

A/51 Calculate the moment of inertia of the shaded area about the x-axis.



A/51



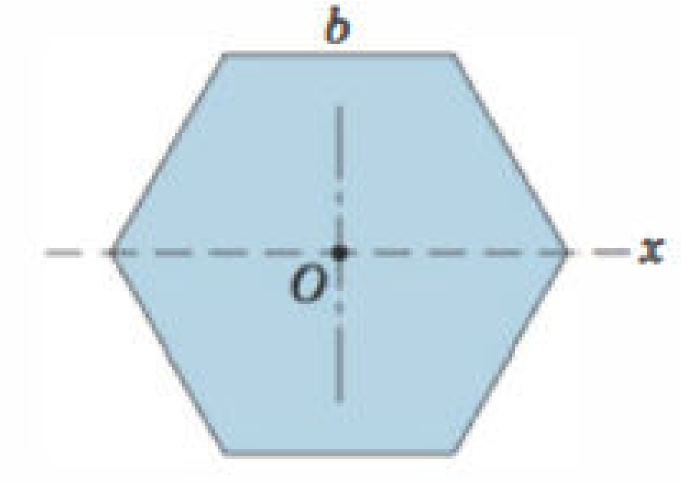
① $I_x = 2\left(\frac{1}{12}\right)(30)\left(30\frac{\sqrt{3}}{2}\right)^3 = \frac{81}{16}\sqrt{3}\,(10^4)\ mm^4$

② $I_x = \frac{1}{4}(30)\left(30\frac{\sqrt{3}}{2}\right)^3 = \frac{243}{32}\sqrt{3}\,(10^4)\ mm^4$

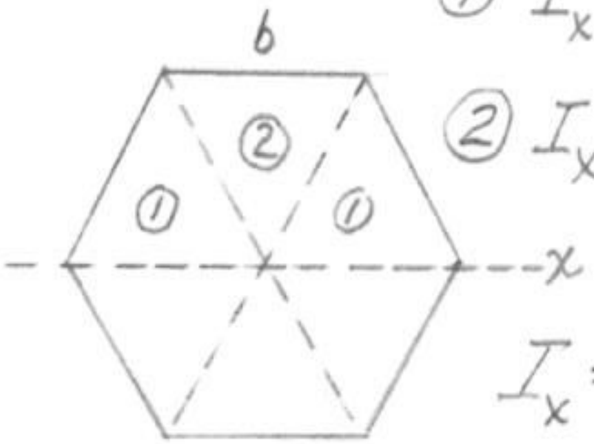
③ $I_x = -\frac{1}{2}\left(\frac{1}{4}\pi[20]^4\right) = -2\pi(10^4)\ mm^4$

Total $I_x = \underline{15.64(10^4)\ mm^4}$

A/53 Develop a formula for the moment of inertia of the regular hexagonal area of side b about its central x-axis.



A/53 From Sample Problem A2



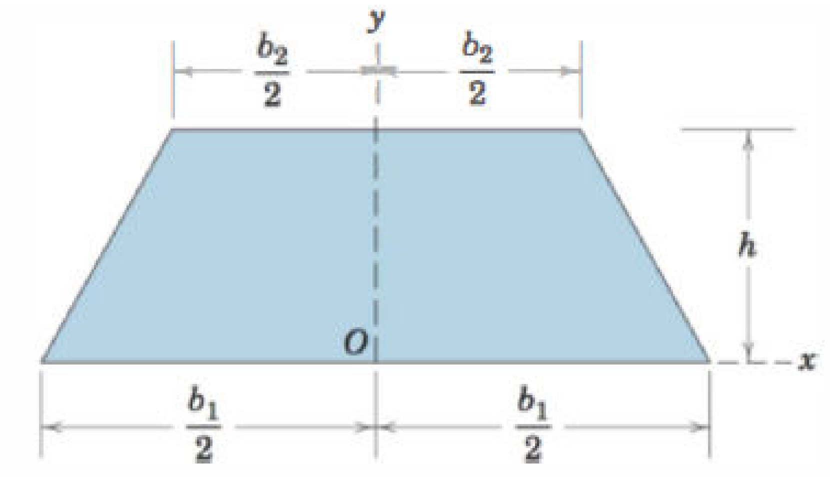
① $I_x = \frac{1}{12} b (b\sqrt{3}/2)^3 = \frac{\sqrt{3}}{32} b^4$

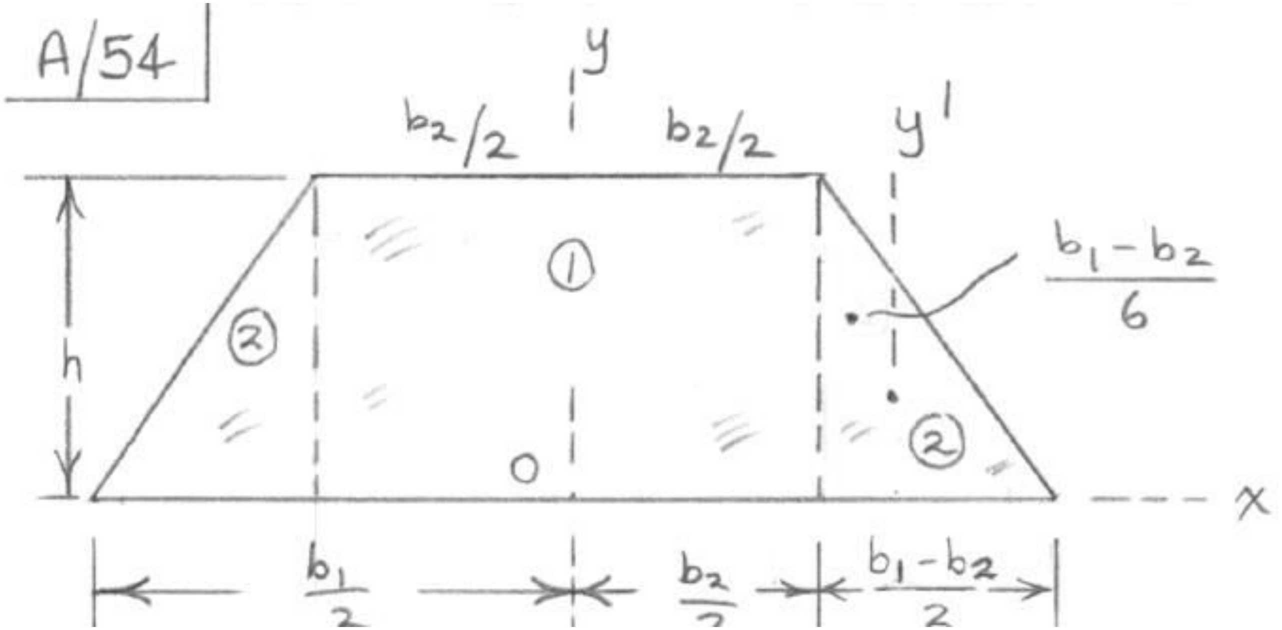
② $I_x = \frac{1}{4} b h^3 = \frac{1}{4} b (b\sqrt{3}/2)^3 = \frac{3\sqrt{3}}{32} b^4$

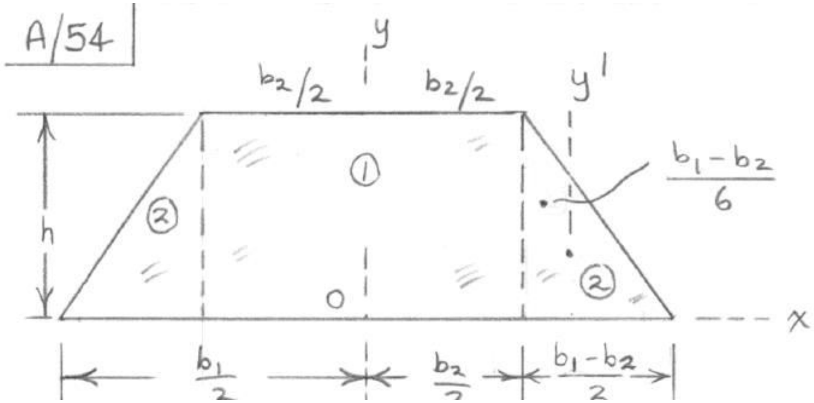
$$I_x = 4 I_{①} + 2 I_{②} = \frac{\sqrt{3}}{8} b^4 + \frac{3\sqrt{3}}{16} b^4$$

$$= \underline{\frac{5\sqrt{3}}{16} b^4}$$

A/54 By the method of this article, determine the moments of inertia about the x· and y-axes of the trapezoidal area.







$$I_x = I_{x_1} + 2I_{x_2} = \frac{1}{3}b_2h^3 + 2\left[\frac{1}{12}\frac{b_1-b_2}{2}h^3\right]$$

$$= h^3\left(\frac{b_1}{12} + \frac{b_2}{4}\right)$$

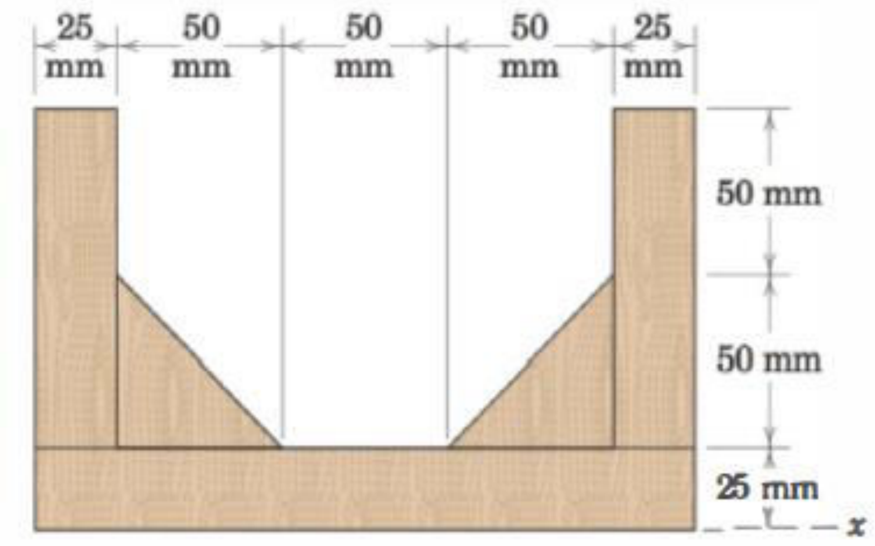
$$I_y = I_{y_1} + 2I_{y_2}$$

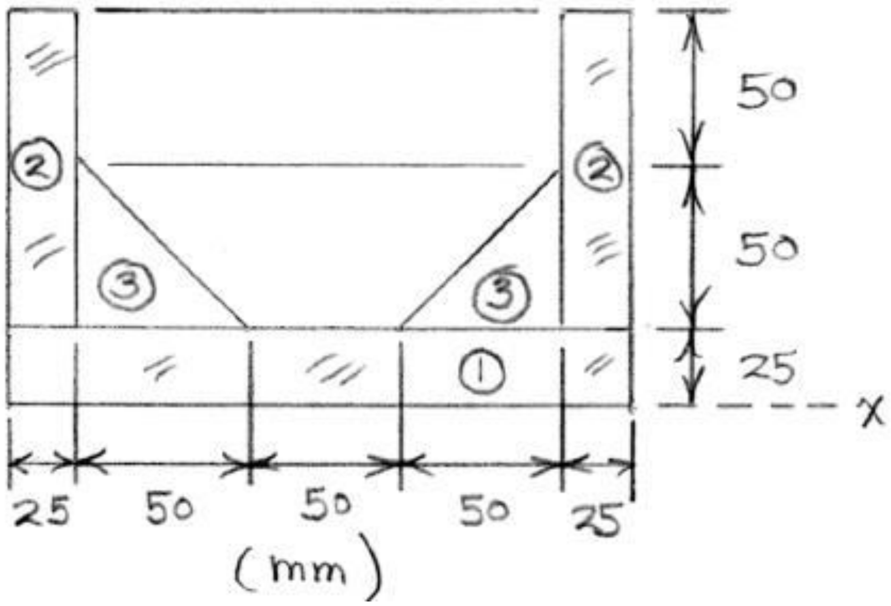
$$= \frac{1}{12}hb_2^3 + 2\left[\frac{1}{36}h\left(\frac{b_1-b_2}{2}\right)^3 + \frac{1}{2}h\left(\frac{b_1-b_2}{2}\right)\left(\frac{b_2}{2} + \frac{b_1-b_2}{6}\right)^2\right]$$

$$\vdots$$

$$= \frac{h}{48}\left(b_1^3 + b_1^2b_2 + b_1b_2^2 + b_2^3\right)$$

A/55 Determine the moment of inertia of the cross sectional area of the reinforced channel about the x-axis.





Comp.	A	d_x	$\bar{I}_x$	Ad_x^2
1	$200(25)$	$25/2$	$\frac{1}{12}(200)(25)^3$	781 250
2	$2[100(25)]$	75	$2[\frac{1}{12}(25)(100^3)]$	28 125 000
3	$2[\frac{1}{2}(50)(50)]$	$(25+\frac{50}{3})$	$2[\frac{1}{36}(50)(50^3)]$	+ 340 278

$$\Sigma \bar{I}_x = 4\ 774\ 306$$

$$\Sigma A d_x^2 = 33\ 246\ 528 \text{ mm}^4$$

$$I_x = \Sigma \bar{I}_x + \Sigma A d_x^2 = 38{,}020{,}833 \text{ mm}^4$$

$$\text{or } 38.0(10^6) \text{ mm}^4$$

THANK YOU

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