ENGINEERING MATHEMATICS - I

Random variables and Probability Distributions

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UNIT 2 : Random Variables and Probability Distributions

Session : 3

Sub Topic : Continuous Probability Distribution

Continuous Distribution

- Discrete random variables have their support as finite or countable.
- There also exist random variables whose support is uncountable.
- For example :

  a) The time that a bus arrives at a bus stop.

  b) The lifetime of a radioactive material. Etc.

- Variables typically modelled with continuous distributions:

  Area, Length, Volume, Distance, Time, Mass and Weight, Energy, Force

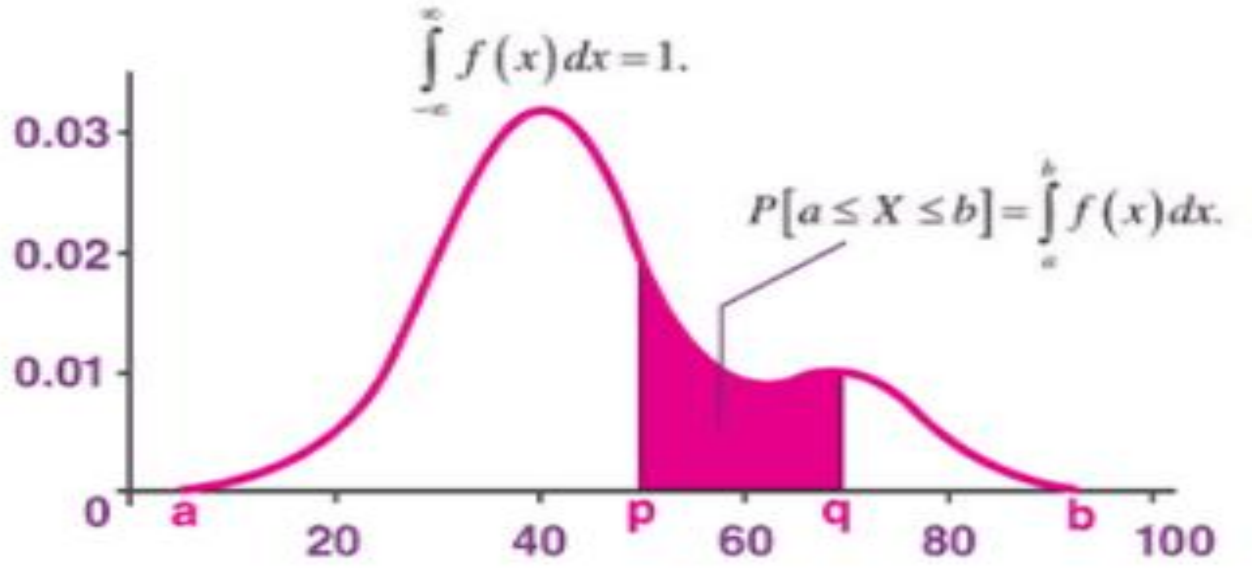
Let X be such a random variable.

We say that X is a continuous random variable if there exists a non negative function f such that for all real numbers $x \in (-\infty, +\infty)$, having the property that , for any set B of real numbers,

$$P(x \in B) = \int_{x \in B} f(x)\, dx \tag{1}$$

The function f is called the density function of the random variable X.

Density Function of a Continuous Random Variable



Density Function of a Continuous Random variable

This means that the probability that a continuous random variable X assumes values in an interval $[p,q]$ is equal to the area of the region bounded by the density function $y = f(x)$, the $X-axis$ and the lines $x = p$ and $x = q$.

$$P(p \leq X \leq q) = \int_p^q f(x)dx \qquad (2)$$

Remarks on the density of a Continuous Random variable

Remark 1. For all numbers x, the density function $f(x) \geq 0$, so that the graph of $y = f(x)$ never drops below the $x - axis$.

Remark 2. The area under the density function $y = f(x)$ and above the $x - axis$ is equal to 1.

Remark 3. Because area of a line segment is zero, the probability that a continuous random variable takes on exactly one value is zero.

Remark 4. Thus whether or not the end points of an interval are included makes no difference concerning the probability of the interval. i.e

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) \tag{3}$$

Example

The amount of time, in hours, before a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

What is the probability that

a) a computer will function between 50 and 150 hours before breaking down;

b) it will function less than 100 hours

Example

Solution:

a) Since $\int_{-\infty}^{\infty} f(x)dx = 1$, we have, $\lambda \int_0^{\infty} e^{-x/100} = 1 \implies \lambda = \dfrac{1}{100}$.

Hence the probability that a computer will function between 50 and 150 hours before breaking down is:

$$P(50 < X < 150) = \frac{1}{100}\int_{50}^{150} e^{-x/100}dx = -\left. e^{-x/100}\right|_{50}^{150} = e^{-1/2} - e^{-3/2} \approx 0.384$$

b) The probability that it will function less than 100 hours = $P(X < 100) = \int_0^{100} f(x)dx = \dfrac{1}{100}\int_0^{100} e^{-x/100}dx =$

$$-\left. e^{-x/100}\right|_{0}^{100} = 1 - e^{-1} \approx 0.633$$

Expectation and Variance of Continuous Random variables

Expectation of a continuous random variable

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx \tag{4}$$

Expectation of a function g of a continuous random variable

If X is a continuous random variable having density function $f(x)$ then the expectation of any real-valued function $g(x)$ of X is defined as:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{5}$$

This is also known as the *Law of unconscious statistician*

Example

Find $E[X]$ if the density function of X is defined as:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$E[X] = 2\int_0^1 x dx = 2\{\frac{x^3}{3}\}\Big|_0^1 = \frac{2}{3}$$

Example

If X is a continuous random variable with distribution function $F_X(x)$ and density function $f_X(x)$, find the density function of the random variable Y defined by $Y = 2X$.

Solution: Given that $F_X(x) = P(X \leq x)$.

Therefore,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(2X \leq y) \\ &= P(X \leq \frac{y}{2}) \\ &= F_X(\frac{y}{2}) \end{aligned}$$

Chebyshev's Inequality

Let X be a random variable with mean $E[X] = \mu_X$ and standard deviation σ_X. Then,

$$P(|X - \mu_X| \geq k\sigma_X) \leq \frac{1}{k^2} \tag{6}$$

This means that, the probability that a random variable differs from its mean by k standard deviations or more is never greater than $\frac{1}{k^2}$.

Chebyshev's Inequality - Example

The length of a rivet manafactured by a certain process has mean $\mu_X = 50mm$ and standard deviation $\sigma_X = 0.45mm$. What is the largest possible value for the probability that the length of the rivet is outside the interval $(49.1, 50.9)mm$?

Solution: From Chebyshev's Inequality, we have

$$P(X \leq 49.1 \;\; or \;\; X \geq 50.9) = ?$$

$$P(X \leq 49.1 - 50 \;\; or \;\; X \geq 50.9 - 50) = \frac{1}{2^2}$$

$$P(|X - 50| \geq 0.9) =$$

$$P(|X - \mu_X| \geq 2(\sigma_X)) \leq \frac{1}{2^2}$$

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Thank You