Unit I: Integral Calculus

1. Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} dydx$ ans: $\frac{1}{2}$

2. Evaluate $\iint xydxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ ans: $\frac{a^4}{8}$

3. Show that $\iint_R r^2 \sin\theta drd\theta = \frac{2a^2}{3}$ where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

4. Find the area lying between the parabola $y = x^2$ and the line $x + y = 2$ ans: $\frac{7}{6}$

5. Find by double integration area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$
   ans: $a^2\left(1 - \frac{\pi}{4}\right)$

6. Find the area common to the circles $r = a\sin\theta$, $r = a\cos\theta$ by double integration. ans: $\frac{a^4}{4}\left(\frac{\pi}{2} - 1\right)$

7. By double integration find the whole area of the curve $x^2 = y^3(2-y)$ ans: π

8. Find the volume bounded by xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$
   ans: 4π

9. Find the average value of $f(x, y) = e^y\sqrt{x + e^y}$ on the rectangle with vertices (0,0), (4,0), (0,1), (4,1)
   Ans: $\frac{(e^2+8e+16)}{15}\sqrt{e+4} - \frac{5\sqrt{5}}{3} - \frac{e^{5/2}}{15} + \frac{1}{15}$

10. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dxdy$ by changing to polar coordinates. Ans: $\frac{\pi}{8}$

11. Evaluate $\iint_R \frac{x^2 y^2}{x^2 + y^2} dxdy$ where R is the region bounded by the circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 1$, by changing to polar coordinates. Ans: $\frac{3\pi}{16}$

12. Evaluate $\iint_R \frac{1}{\left(1 + x^2 + y^2\right)^2} dxdy$ over one loop of lemniscate $\left(x^2 + y^2\right)^2 = x^2 - y^2$ ans: $\frac{\pi}{4} - 1/2$

13. Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2} dxdy$ by changing to polar coordinates. Ans: $8\left(\frac{\pi}{2} - \frac{5}{3}\right)a^2$

14. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xydydx$ by changing the order of integration. ans: $1/24$

15. Evaluate $\int_0^a \int_0^{\frac{bx}{a}} xdydx$ by changing the order of integration. ans: $a^2b/3$

16. Evaluate $\int_0^c \int_0^b \int_0^a \left(x^2+y^2+z^2\right)dxdydz$ ans: $\frac{abc}{3}\left(a^2+b^2+c^2\right)$

17. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dzdxdy$ ans: $\frac{1}{4}\left(13-8e+e^2\right)$

18. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$ ans: $\frac{4}{35}$

19. Find the volume bounded by xy-plane, the cylinders $x^2+y^2=1$ and the plane $x+y+z=3$ ans: 3π

20. Find the volume bounded by the cylinder $x^2+y^2=4$ and the hyperboloid $x^2+y^2-z^2=1$ ans:$4\sqrt{3}\pi$

21. Find the volume enclosed by the cylinders $x^2+y^2=2ax$ and $x^2+z^2=2ax$ ans: $\frac{16a^3}{3}$

22. Use triple integrals to find the average value of the function $f(x,y,z)=xyz$ over a cube with side length l lying in the first octant with one vertex at (0,0,0) and three sides on coordinate axes. Ans:$\frac{l^3}{8}$

23. Show that $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}=\frac{\pi^2a^2}{8}$, where region of integration is first octant of the sphere $x^2+y^2+z^2=a^2$

24. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using spherical coordinates. Ans:$\frac{4\pi}{3}abc$

25. The density at any point (x,y) of a lamina is $\sigma(x+y)/a$, where σ and a are constants. The lamina is bounded by the lines $x=0, y=0,\ x=a, y=b$. find the position of its center of gravity.
Ans: $\left[\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+b)}{6(a+b)}\right]$

26. If the density at any point of solid octant of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ varies as xyz ,find the coordinates of center of gravity of the solid. Ans: $\left(\frac{16a}{35}, \frac{16b}{35}, \frac{16c}{35}\right)$

27. Find the moment of inertia about z axis of a homogeneous tetrahedron bounded by planes $x=0, y=0, z=x+y\ and\ z=1$ ans:1/30

28. Find the moment of inertia of a hollow sphere about a diameter. its external and internal radii being 51 meters and 49 meters respectively. Ans: 104803770p

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