1.a. Starting with Maxwell's relation, $\nabla X E = -\frac{\partial B}{\partial t}$, derive the wave equation $\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$. Assume that there are no free charges and currents in the region of the field.

Expected Answer

Starting with Maxwell's relation, $\nabla X E = -\frac{\partial B}{\partial t}$ we take the curl of both sides

$\nabla X \nabla X E = \nabla X(-\frac{\partial B}{\partial t})$

Now $\nabla X \nabla X E = \nabla(\nabla.E) - \nabla^2 E$ (1 mark)

Since there are no free charges the LHS gives $\nabla X \nabla X E = \nabla(0) - \nabla^2 E = -\nabla^2 E$ (1 mark)

Solving for RHS we have $\nabla X(-\frac{\partial B}{\partial t}) = -\frac{\partial}{\partial t}(\nabla X B)$ (1 mark)

Maxwell's 4th equation gives us $\nabla X B = \mu_0(j + \varepsilon_0 \frac{\partial E}{\partial t}) = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ (1 mark)

Substituting for $\nabla X B$ we get

$\nabla X(-\frac{\partial B}{\partial t}) = -\frac{\partial}{\partial t}(\nabla X B) = -\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}) = -(\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2})$ (1 mark)

Equating the LHS and RHS we get $-\nabla^2 E = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$ or

$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$ (1 mark)

1.b. Write an expression for the Compton shift assuming that the scattered photon moves off at an angle ϕ with respect to the direction of travel of the incident photon. If the wavelength of scattered light is 0.15nm estimate the wavelength of the incident light assuming that the angle $\phi = 135°$.

Expected Answer

Expression for Compton shift is $\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos(\phi))$ 2 mark

Given $\lambda' = 0.15nm, \phi = 135°$

Thus $0.15 - \lambda = 2.424\ x\ 10^{-12}(1 + 0.707)$

Thus $\lambda = 0.109nm$ (3 marks)

1.c. Explain the following concepts
(i) Normalization
(ii) Probability density
(iii) Expectation

Expected Answer

(i) Normalization refers to the process of multiplying a wave function by a suitable constant or adjusting its amplitude such that $\int_{-\infty}^{+\infty} \psi^* \psi \, \mathrm{d}x = 1$

(ii) Probabilty density is defined by $\psi^*\psi$ and is the probability per unit length (for 1D case)

(iii) Expectation of an observable A is the average value and is give by $\langle A \rangle = \int_{-\infty}^{+\infty} \psi^* \widehat{A} \, \psi \mathrm{d}x$

where the wavefunction is normalized

(2 marks each)

1.d. The thickness of a piece of paper is measured using interference of light at an air wedge. To observe sustained interference pattern one requires 2 coherent waves. In this experiment light comes from a sodium vapour lamp and this light is hardly coherent. How does one obtain coherent waves whose superposion gives rise to interference pattern?

Expected Answer

Light from sodium vapour lamp is made to fall nearly normally on two glass plates between which we have a wedge. (1 mark)
Light is reflected and refracted from the bottom surface of the top glass plate.
The refracted light is then made to travel through the air column and returns to the glass - air interface where it superposes with the initially reflected light (1 mark)
Since both the reflected and refracted light come from a wave by a process of division of amplitude they are mutualy coherent (1 mark)

2.a. Write the 1D time independent Schrodinger's equation. Show that if ψ_1 is a solution of this equation and ψ_2 is another independent solution for the same potential energy, show that $\psi = a\psi_1 + b\psi_2$ (where a and b are constants) is also a solution to this equation.

Expected Answer

1D time independent Schrodinger's equation is : $\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0$ (1 mark)

If ψ_1 is a solution then $\frac{d^2\psi_1}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi_1 = 0$ ---------- (1)

If ψ_2 is another independent solution then $\frac{d^2\psi_2}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi_2 = 0$ ---------(2) (1 mark)

Multiplying equation 1 by a and equation 2 by b and adding them we have

$a\frac{d^2\psi_1}{dx^2} + b\frac{d^2\psi_2}{dx^2} + a\frac{8\pi^2 m}{h^2}(E - V)\psi_1 + b\frac{8\pi^2 m}{h^2}(E - V)\psi_2 = 0$ (1 mark)

we then have $\frac{d^2 a\psi_1}{dx^2} + \frac{d^2 b\psi_2}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\,a\psi_1 + \frac{8\pi^2 m}{h^2}(E - V)\,b\psi_2 = 0$

Since the differential operator is linear we have

$\frac{d^2(a\psi_1 + b\psi_2)}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)(a\psi_1 + b\psi_2) = 0$ (1 mark)

Since $\psi = a\psi_1 + b\psi_2$ we have $\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)(\psi) = 0$ (1 mark)

2.b. A beam of particles of energy E is incident on a potential step of height V_0 (< E). Express the reflection coefficient in terms of E and V_0. If E = 5V_0 estimate the transmission coefficient.

Expected Answer

The reflection coefficient is given by $R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} \right)^2$ (2 marks)

Substituting for E = 5V we get

$$R = \left(\frac{\sqrt{5V_0} - \sqrt{4V_0}}{\sqrt{5V_0} + \sqrt{4V_0}} \right)^2 = \left(\frac{\sqrt{5} - \sqrt{4}}{\sqrt{5} + \sqrt{4}} \right)^2 = \left(\frac{\sqrt{5} - 2}{\sqrt{5} + 2} \right)^2 = \left(\frac{2.236 - 2}{2.236 + 2} \right)^2$$ (2 marks)

$$= \left(\frac{0.236}{4.236} \right)^2 = 0.003$$

The transmission coefficient is $T = 1 - R = 1 - 0.003 = 0.997$ (1 mark)

2.c. A quantum mechanical oscillator A oscillates with a frequency ν. Mathematically represent the eigenvalues for the ground and second excited quantum states. Assume that oscillator A makes a transition from the second excited quantum state to the ground state and the energy released is absorbed by another oscillator B whose oscillating frequency is 2ν. If oscillator B is in the ground state then what will be the quantum number of the state into which it will be excited?

Expected Answer

For a QM oscillator having frequency ν we have the following
Eigenvalue is $E_n = (n + 0.5)h\nu$ (1mark)
ground state: $E_0 = 0.5h\nu$ (1 mark)
2nd excited state: $E_2 = (2 + 0.5)h\nu$ (1 mark)
For oscillator A: $E_2 - E_1 = (2.5 - 0.5)h\nu = 2h\nu$ (1 mark)
For oscillator B: $E_n = (n + 0.5)h(2\nu) = (n + 0.5)h\nu'$ (1 mark)
and we have $E_n - E_0 = (n - 0.5)h\nu' = 2h\nu$, Thus n = 1 (1 mark)

2.d. 10^{19} electrons of energy 4eV are incident on a potential barrier of height 10eV and width 1nm. What is the quantity of charge that is likely to be detected on the other side of the barrier?

Expected Answer

The transmission coefficient is given by $T = e^{-2\alpha L}$, where $\alpha = \frac{2\pi\sqrt{2m(V-E)}}{h}$ and L is the width of the barrier (1 mark)

The parameter $\alpha = \frac{2\pi\sqrt{2m(V-E)}}{h} = \frac{\sqrt{2 \ x \ 9.11 \ x \ 10^{-31}(6 \ x \ 1.6 \ x \ 10^{-19})}}{1.055 \ x \ 10^{-34}} = 12.54 \ x \ 10^9$ (1 mark)

Thus $T = e^{-2 \ x \ 12.54} = e^{-25.08} = 1.3 \ x \ 10^{-11}$ (1 mark)

Number of electrons that are likely to tunnel = NT = 1.3 x 10^8 and quantity of charge = 1.3 x 10^8 x 1.6 x 10^{-19} = 20.8pC (1 mark)

3.a. What is drift velocity of electrons in a metal and write an expression for it. Show that for a constant electric field applied to a metal the following equation, $v_D\sqrt{T} = constant$, is valid. Here v_D is the drift velocity and T the temperature.

Expected Answer

The average velocity with which the conduction electron moves in metal under the influence of an electric field in a direction opposite to the field. It is expressed as $v_D = \frac{eE\tau}{m}$

(1mark)

Now $\tau = \frac{\lambda}{v_{th}}$ and $v_{th} = \sqrt{\frac{3k_BT}{m}}$ (1 mark)

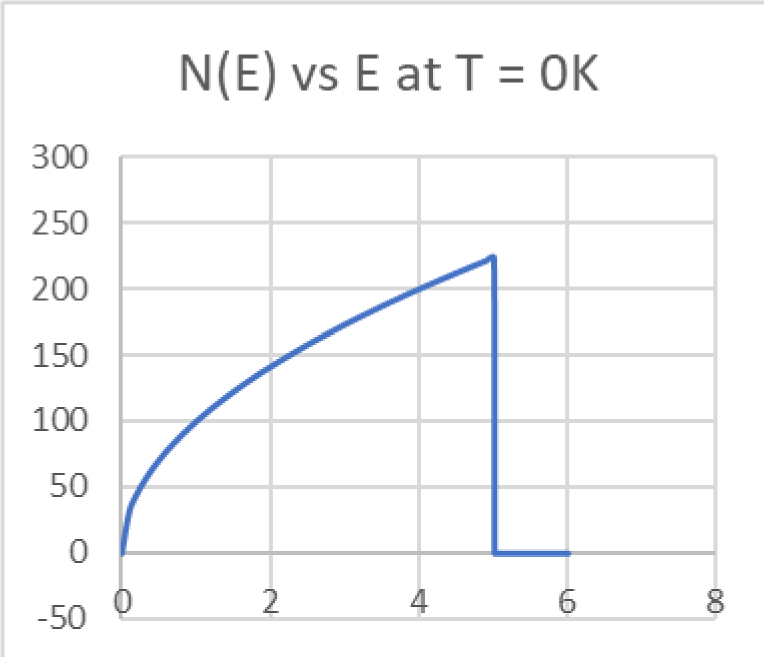
Thus $v_D = \frac{eE\lambda}{mv_{th}} = \frac{eE\lambda}{m\sqrt{\frac{3k_BT}{m}}}$ (1 mark)

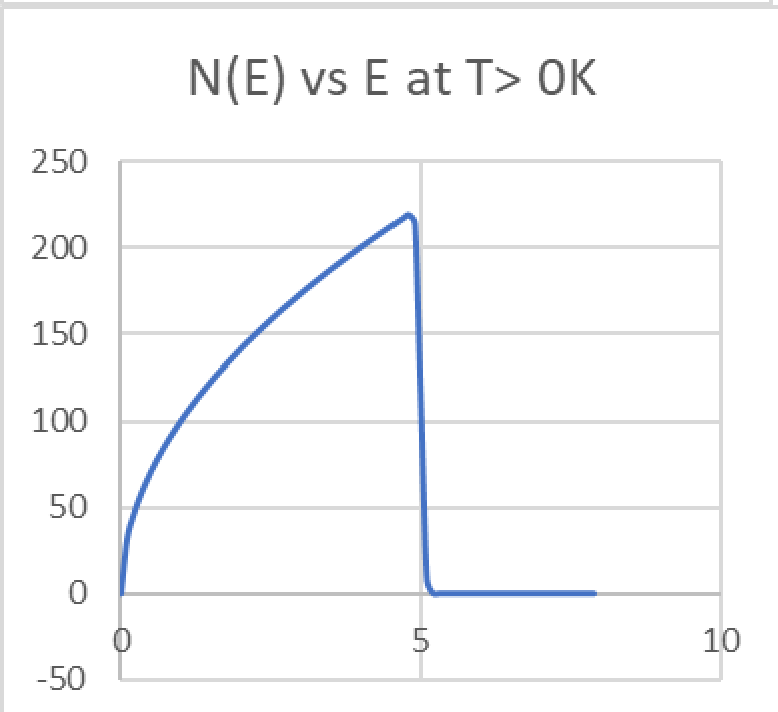
Hence $v_D = \frac{eE\lambda}{\sqrt{3mk_BT}}$ or $v_D\sqrt{T} = \frac{eE\lambda}{\sqrt{3mk_B}} = constant$ (2 marks)

3.b. Draw separate graphs representing variation of density of occupied states, N(E) as a function of E for (i) T = 0K and (ii) T > 0K and answer the following questions. (assume E_F = 5eV)
(I) Is the area under the graph of N(E) versus E the same for both T = 0K and T > 0K. If yes then why is it so?
(II) In the graph of N(E) versus E for T > 0K, what does the area under the graph beyond the Fermi energy represent? If you compare this area with the total area under the graph what can be commented about the result?

Expected Answer

Two separate graphs for T = 0K and T> 0K (4 marks)





(I) the area under the graph for both T = 0K and T > 0K remains the same as the area represents the total number of "free" electrons in the metal (1 mark)

(II) The area under the graph for E > E_F represents the effective number of conduction electrons and if this area is compared to the total area then this area is a tiny fraction of the total area. This indicates that the effective number of conduction electrons as per the quantum theory is a tiny fraction of the total number of valence electrons (1 mark)

3.c. Explain the following concepts.
(i) effective mass of electrons
(ii) Meissner effect in Superconductivity
(iii) Bloch function

Expected Answer

(i) The electrons in metals behave as if they have variable mass due to the fact that their response is based on both the apploied electric field and the forces due to the environment of the electron. Since the force due to the environment cannot be estimated the mass is made variable to acoount for the behaviour. This is referred to as effective mass and is given by $m^* = \dfrac{h^2}{4\pi^2\left(d^2E/dk^2\right)}$

(ii) When a superconductor is kept in a weak magnetic field and cooled to below the critical temperature then it throws the magnetic field out of its interior

(iii) Bloch function is the eigenfunction of electrons that interact with the periodic potential of metals and is given by $\psi = u(x)e^{ikx}$, where u(x) is a periodic function

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3.d. The resistance of a metal increases with an increase in temperature while the resistance of a semiconductor decreases with an increase in temperature. Explain why the difference in behaviour?

Expected Answer

The resistance of a metal is due to the collisions of electrons with vibrating ions. When the temperature the amplitude of atomic vibrations increases with the result that there are more collisions and hence increase in resistance (1 mark)
In a semiconductor the collisions of charge carriers with atoms increases with increase in temperature. However in semiconductors there is another effect which dominates. This is the increase in the electron-hole pairs with increase in temperature. This effect dominates and hence the overall resistance drops with increase in temperature. (2 marks)

4.a. Derive an expression for the energy density of the Electromagnetic field in terms of Einstein's coefficients under the condition of thermal equilibrium.

Expected Answer

Under the condition of thermal equilibrium the energy emitted by atoms must be equal to the energy absorbed by them when they are in an electromagnetic field. This condition must hold good at all times.

Considering a bunch of atoms in thermal equilibrium with the surrounding EM field. Let the atoms have two energy levels E_1 and E_2 (> E_1) with populations N_1 and N_2 respectively. Let B_{12}, B_{21} and A_{21} be the Einstein's coefficients for absorption, stimulated emission and spontaneous emission respectively

Thus

The rate of emission = rate of absorption (1 mark)

$B_{12}N_1U_v = A_{21}N_2 + B_{21}N_2U_v$, *where* U_v *is the energy density of EM field* (1 mark)

rearranging the terms we have $B_{12}N_1U_v - B_{21}N_2U_v = A_{21}N_2$

$$(B_{12}N_1 - B_{21}N_2)U_v = A_{21}N_2$$

Simplifying further $U_v = \dfrac{A_{21}N_2}{(B_{12}N_1 - B_{21}N_2)}$ (1 mark)

Dividing the RHS above and below by $B_{21}N_2$ we get $U_v = \dfrac{A_{21}/B_{21}}{(B_{12}/B_{21}\,N_1/N_2 - 1)}$

We know from Boltzmann's relation $\dfrac{N_1}{N_2} = e^{(E_2-E_1)/k_BT} = e^{hv/k_BT}$

Substituting for N_1/N_2 we get $U_v = \dfrac{A_{21}/B_{21}}{(B_{12}/B_{21}e^{hv/k_BT} - 1)}$ (1 mark)

Comparing this expression with the one given by Planck, $U_v = \dfrac{8\pi hv^3/c^3}{(e^{hv/k_BT} - 1)}$ we get

$A_{21}/B_{21} = 8\pi hv^3/c^3$, $B_{12} = B_{21}$ (1 mark)

Thus $U_v = \dfrac{A/B}{(e^{hv/k_BT} - 1)}$ (1 mark)

4.b. What are the requisites of a laser system?

Expected Answer

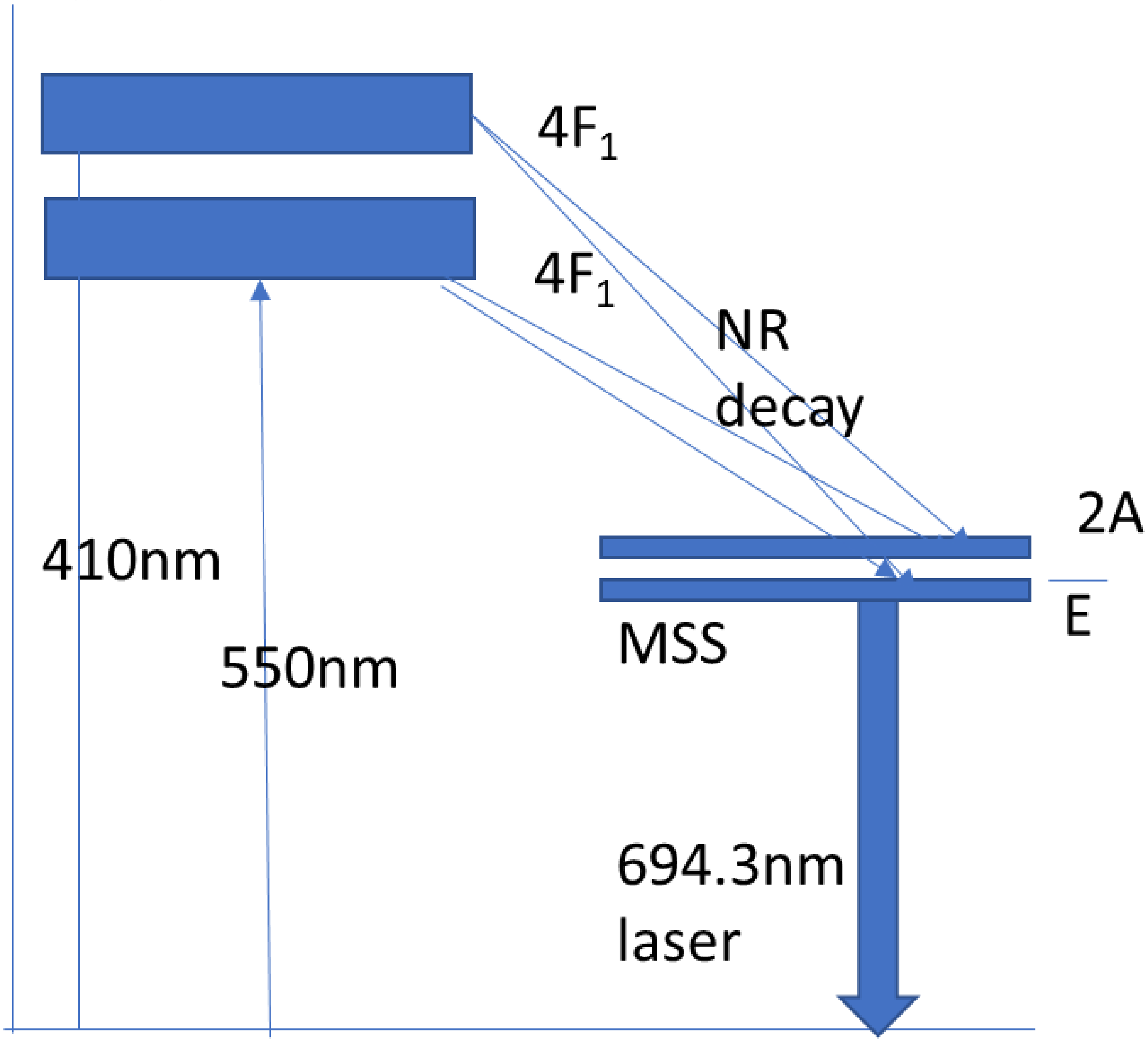
The requirements of a laser system are
1. Suitable material possessing the right energy levels which can support population inversion. The presence of metastable states is favourable
2. Pumping mechanism built into the system which can excite atoms to higher energy states. The mechanism may be optical, electrical, chemical, thermal etc
3. A well designed resonant cavity which will help in selecting the photon states and amplify light.
4. To attain a gain that is above the threshold to keep the laser functioning

(1 mark each)

4.c. With the help of a suitable energy diagram explain the working of a Ruby laser.

Expected Answer

Energy diagram - 3 marks



Working:
(i) The Xenon flash tube is activated by applying a voltage. Light from the flash tube bombards the Cr ions in Ruby crystal. The wavelength of light used for exciting the ion is 410 and 550nm. The Cr ions are excited to two levels $4F_1$ and $4F_2$ (2 marks)
(iv) The Cr ions make non radiative transitions to two levels which are designated as $2A$ and $\overline{E}$ (metastable) which can be considered as a single level E. (1 mark)
(v) The Cr ions de-excite to the ground level giving out the laser light of wavelength 694.3nm (1 mark)

Ruby gives out laser light as pulses

4.d. A laser beam is incident normally on a grating of constant $2\mu m$. If the grating is at a distance of 10cm from a screen, estimate the separation between second order spots given that the wavelength of the laser light is 600nm.

Expected Answer

When the laser is incident normally on a grating of constant C the angular location of diffraction spots is given by $C\ Sin(\theta) = n\lambda$ (1 mark)
For the second order diffraction spot, $C\ Sin(\theta) = 2\lambda$
Given $C\ =\ 2\mu m,\ \lambda\ =\ 600nm$
we get the angle for the second order spot as

$$Sin(\theta) = \frac{2\lambda}{C} = \frac{2\ x\ 600}{2000} = 0.6:\ \theta = Sin^{-1}(0.6)\ =\ 36.87^{\circ}$$ (1 mark)

To get the separation between the second order spots we use the relation $\frac{x}{D} = \tan(\theta)$.

$x = D\ \tan(\theta)\ =\ 10\ \tan 36.87\ =\ 10\ x\ 0.75\ =\ 7.5cm$ (1 mark)
$The\ separation\ is\ then\ 2x\ =\ 15cm$

5.a. What is Bohr magneton? Estimate the orbital and spin magnetic moment of an electron given its azimuthal quantum number l = 3 and spin quantum number s =0.5, in terms of the Bohr magneton.

Expected Answer

The orbital magnetic moment of the electron in the first Bohr orbit of the H-atom is referred to as Bohr magneton and is given by $\mu_B = \frac{eh}{4\pi m}$ (2 marks)

For the electron with l= 3 the angular momentum is $L = \frac{\sqrt{l(l+1)}h}{2\pi} = \frac{2\sqrt{3}h}{2\pi}$. (1 mark)

The orbital magnetic moment is then $\mu_L = \frac{e}{2m}L = \frac{e}{2m}\frac{2\sqrt{3}h}{2\pi} = 2\sqrt{3}\mu_B$ (1 mark)

For the spin we have $\mu_S = \frac{2e}{2m}S = \frac{2e}{2m}\frac{\sqrt{3}h}{8\pi} = \frac{e}{2m}\frac{\sqrt{3}h}{4\pi} = \sqrt{3}\mu_B$ (1 mark)

5.b. Explain how Weiss was able to give a model for paramagnetic materials which could explain the relation $\chi = \frac{C}{T-\theta}$?

Expected Answer

(i) According to Weiss, in some paramagnetic substances the dipole - dipole interaction needs to be taken into account to determine the magnetic field at the site of a dipole. (1 mark)
(ii) According to him a dipole is surrounded by other dipoles and the magnetic field as detected by any dipole is the sum of the applied field and the field due to all other dipoles (1 mark)
(iii) Weiss suggested that the field due to other dipoles - molecular field, H_m - is proportional to the existing magnetization.
(iv) Since H_m is proportional to M we write it as
$H_m = \gamma M$, *where γ is called the Weiss' constant*. (1 mark)

we thus write $\frac{M}{H + H_m} = \frac{C}{T}$ where H_m is the molecular field (1 mark)

(v) Thus $\frac{M}{H + \gamma M} = \frac{C}{T}$ which is rearranged to give $\frac{M}{H} = \frac{C}{T - \gamma C}$ which is written as $\chi = \frac{C}{T - \theta}$, *where* $\theta = \gamma C$ (1 mark)

5.c. Explain the following polarisation mechanisms
(i) ionic and
(ii) dipolar or orientational
Which one of these is temperature dependent and why?

Expected Answer

(i) ionic polarization - This is due to the shifting of ions in an ionic material on application of electric field. Before application of field the net polarisation is zero and after the field is applied due to the shift in the position of positive and negative ions a net dipole moment is created which leads to this polarisation (2 marks)
(ii) dipolar or orientational polarization occurs in materials where the molecule is polar. In the absence of the applied field the dipoles point in random directions leading to net zero dipole moment. On application of the field the dipoles tend to align themselves as much as possible along the direction of the field leading to net dipole moment and hence polarisation (2 marks)

Of the two, the dipolar polarisation is temperature dependent as the molecular agitation increases with temperature and this leads to disruption in the alignment process leading to a decrease in polarisation with increase in temperature (1 mark)

5.d. Distinguish between direct piezoelectric effect and inverse piezoelectric effect
A pieoelectric material is subjected to an alternating electric field and it is found that the resonant frequency is 5KHz and the antiresonant frequency is 5.4KHz. If the input energy is U_E and the output energy is U_M, what fraction (as a percentage) of U_E is U_M?

Expected Answer

Direct piezoelectric effect - This refers to the phenomenon that when a mechanical deformation is applied to a piezoelectric material, an electric field develops within it
Inverse piezoelectric effect - This refers to the phenomenon that when an electric field is applied to a piezoelectric material it undergoes a mechanical deformation

The ratio of the output energy to input energy is given by $\frac{U_M}{U_E} = k^2 = 1 - \left(\frac{f_r}{f_a}\right)^2$. Here k is the piezoelectric coupling coefficient and f_r and f_a are the resonant and antiresonant frequencies. Thus $\frac{U_M}{U_E} = 1 - \left(\frac{f_r}{f_a}\right)^2 = 1 - \left(\frac{5}{5.4}\right)^2 = 0.143$
Thus U_M is 14.3% of U_E