ENGINEERING MATHEMATICS - I

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UNIT 4 : Partial Differential Equations

Session : 7

Sub Topic : Solutions of PDEs by the method of Separation of Variables continued

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1. Solve $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}=0$ by the method of separation of variables.

Solution:

Let $u = XY$, where $X = X(x)$ and $Y = Y(y)$ be the solution of the given PDE.

Substituting into the given PDE, we have

$$x^2\frac{\partial(XY)}{\partial x}+y^2\frac{\partial(XY)}{\partial y}=0$$

$$x^2Y\frac{dX}{dx}+y^2X\frac{dY}{dy}=0$$

Dividing by XY we have,

$$\frac{x^2}{X}\frac{dX}{dx}=-\frac{y^2}{Y}\frac{dY}{dy}$$

Equating both sides to a common constant k we have,

$$\frac{x^2}{X}\frac{dX}{dx} = \mathrm{k} \qquad ; -\frac{y^2}{Y}\frac{dY}{dy} = k$$

$$\frac{1}{X}dX = \frac{k}{x^2}dx \qquad ; \frac{1}{Y}dY = -\frac{k}{y^2}\mathrm{dy}$$

On integrating

$$log_e X = -\frac{k}{x} + c_1 \qquad ; log_e Y = \frac{k}{y} + c_2$$

$$X = e^{\left(-\frac{k}{x}\right)+c_1} \qquad ; Y = e^{\left(\frac{k}{y}\right)+c_2}$$

Hence $u = XY = e^{c_1+c_2}\, e^{\left(-\frac{k}{x}\right)+\left(\frac{k}{y}\right)}$ and let $c = e^{c_1+c_2}$

Hence $u = c\, e^{k\left(\frac{1}{y}-\frac{1}{x}\right)}$ is the required solution.

2. Solve by the method of separation of variables

$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given that $u(0, y) = 2e^{5y}$.

Solution:

Let $u = XY$, where $X = X(x)$ and $Y = Y(y)$ be the solution of the given PDE.

Substituting into the given PDE

$$4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$$

Dividing by XY we have,

$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3$ or $\frac{4}{X}\frac{dX}{dx} = 3 - \frac{1}{Y}\frac{dY}{dy}$

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Equating both sides to a common constant k we have,

$\frac{4}{X}\frac{dX}{dx} = \text{k}$; $3 - \frac{1}{Y}\frac{dY}{dy} = k$

$\frac{1}{X}dX = \frac{k}{4}dx$; $\frac{1}{Y}dY = (3 - \text{k})\text{dy}$

On integrating

$log_e X = \frac{k}{4}x + c_1$; $log_e Y = (3 - \text{k})\, y + c_2$

$X = e^{\frac{k}{4}x + c_1}$; $Y = e^{(3 - \text{k})\, y + c_2}$

Hence $u = XY = e^{c_1+c_2}\, e^{\frac{k}{4}x+(3-\mathrm{k})y}$ and $c = e^{c_1+c_2}$

Thus $u = u(x,y) = ce^{\frac{k}{4}x+(3-\mathrm{k})y}$ is the general solution.

Further, by data, $u(0,y) = 2e^{5y}$

The general solution becomes

$2e^{5y} = ce^{(3-k)y}$.

Comparing we have $c = 2$ and $3 - k = 5$ *or* $k = -2$

Thus the required particular solution is given by

$$u = 2e^{-\frac{x}{2}+5y}$$

3. Solve by the method of separation of variables

$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given that $u(x,0) = 6e^{-3x}$.

Solution:

Let $u = XT$, where $X = X(x)$ and $T = T(t)$ be the solution of the given PDE.

Substituting into the given PDE, we have

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Dividing by XT we have,

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

Equating both sides to a common constant k we have,

$\frac{1}{X}\frac{dX}{dx} = \mathrm{k}$; $\frac{2}{T}\frac{dT}{dt} + 1 = k$

$\frac{1}{X}dX = kdx$; $\frac{1}{T}dT = \frac{(k-1)}{2}dt$

On integrating

$log_e X = \mathrm{kx} + c_1$; $log_e T = \frac{(k-1)}{2}\ t + c_2$

$X = e^{kx+c_1}$; $T = e^{(\mathrm{k}-1)\frac{t}{2}+c_2}$

Hence $u = XT = e^{c_1+c_2}\ e^{kx+(\mathrm{k}-1)\frac{t}{2}}$ and let $c = e^{c_1+c_2}$

Thus $u = ce^{kx+(\mathrm{k}-1)\frac{t}{2}}$ is the general solution.

Solutions of PDEs by the method of Separation of Variables

Further by data, $u(x, 0) = 6e^{-3x}$

That is $u = 6e^{-3x}$ when $t = 0$

Hence we have $6e^{-3x} = ce^{kx}$.

Comparing we have $c = 6$ and $k = -3$

Thus the required particular solution is given by

$$u = 6e^{-3x-2t}$$

THANK YOU

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