Engineering Mathematics - II
(UE22MA141B)

Unit - 3: Random variables and Probability distributions

Problems on discrete probability distribution

1. In a voice communication system with 50 lines, the random variable is the number of lines in use at a particular time. Answer: {0,1,2,3,...50}=X.

2. A die is tossed thrice. A success is "getting 1 or 6" on a toss. Find the mean and variance of the number of successes. Answer: Mean $\mu = 1$ and Variance $\sigma^2 = \frac{2}{3}$

3. **Home work problem:** A coin is tossed three times. Let X denote the number of heads showing up. Find it's mean and variance. Answer: Mean $\mu = \frac{3}{2}$ and Variance $\sigma^2 = \frac{3}{4}$

4. The probability density function of a random variable X is given as follows:

x:	0	1	2	3	4	5	6
p(x):	k	3k	5k	7k	9k	11k	13k

   Find the value of k. Evaluate,
   (i) $P(X < 4)$; $P(X \geq 5)$; and $P(3 < X \leq 6)$.
   (ii) What will be the maximum value of k so that $P(X \leq 2) > 0.3$?
   Answer: $k = \frac{1}{49}$; $P(X < 4) = \frac{16}{49}$; $P(X \geq 5) = \frac{24}{49}$; and $P(3 < X \leq 6) = \frac{33}{49}$.
   The minimum value of $k = \frac{1}{30}$

5. **Home work problem:** The probability density function of a random variable X is given as follows:

x:	0	1	2	3	4	5	6	7
p(x):	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

   Find the value of k. Evaluate $P(X < 6)$; $P(X \geq 6)$; and $P(0 < X < 5)$.
   Answer: $k = \frac{1}{10}$; $P(X < 6) = \frac{81}{100}$; $P(X \geq 6) = \frac{19}{100}$;$P(0 < X < 5) = \frac{4}{5}$.

6. The Sample space of a random experiment is {a, b, c, d, e, f} and each outcome is equally likely. A random variable is defined as folows:

Outcome	a	b	c	d	e	f
X	0	0	1.5	1.5	2	3

   Determine the probability mass function of X. Use the probability mass function to determine the following probabilities. *a)* $P(X = 1.5)$ *b)* $P(0.5 < X < 2.7)$ *c)* $P(X > 3)$
   d) $P(0 \leq X < 2)$ *e)* $P(X = 0 \text{ or } X = 2)$.
   Answer: (a) $\frac{1}{3}$, (b) $\frac{1}{2}$, (c) 0 , (d) $\frac{2}{3}$, (e) $\frac{1}{2}$

7. The space shuttle flight control system called PASS[primary Avionics software set] uses four independent computers working in parallel. At each critical step, the computers 'vote' to determine the appropriate step. The probability that a computer will ask for roll to the left when a roll to the right is appropriate is 0.0001. let X denote the number of computers that vote for a left roll when a right roll is appropriate. what is the probability mass function of X? what is mean variance of X?
Answer: $P(X = 0) = 0.9996$, $P(X = 1) = 0.0003999$, $P(X = 2) = 5.999 \times 10^{-8}$, $P(X = 3) = 3.9996 \times 10^{-12}$, $P(X = 4) = 10^(- 16)$
$Mean = 0.0004$, $Variance = 4 \times 10^{-4}$

Problems on continuous probability distribution

8. Is the function defined as follows a density function?

$$\begin{aligned} f(x) &= e^{-x} \quad (x \geq 0) \\ &= 0 \quad (x < 0) \end{aligned} \tag{0.1}$$

If so, determine the probability that random variable X having this density will fall in the interval $(1, 2)$.
Answer: $f(x)$ is a density function.
Required probability= $P(1 \leq X \leq 2) = 0.233$. Find F(2)

9. Let X be a continuous random variable with probability density function given by

$$\begin{aligned} f(x) &= kx \quad (0 \leq x \leq 2) \\ &= 2k \quad (2 \leq x < 4) \\ &= -kx + 6k \quad (4 \leq x < 6) \end{aligned}$$

Find the value of k and the mean value of X.
Answer: $k = \frac{1}{8}$ and mean value of X=3.

Problems on Bernoulli distribution

10. A coin has a probability of 0.5 of landing heads when tossed. Let X = 1 if the coin comes up heads, and X = 0 if the coin comes up tails. What is the distribution of X?
Answer: $X \sim Bernoulli(0.5)$.

11. A die has a probability $\frac{1}{6}$ of coming up 6 when rolled. Let X = 1 if the die comes up 6, and X = 0 otherwise. What is the distribution of X?
Answer: The success probability is $p = P(X = 1) = \frac{1}{6}$. Therefore $X \sim Bernoulli\left(\frac{1}{6}\right)$.

12. Ten percent of the components manufactured by a certain process are defective. A component is chosen at random. Let X = 1 if the component is defective, and X = 0 otherwise. What is the distribution of X?
Answer: The success probability is p = P(X = 1) = 0.1. Therefore $X \sim Bernoulli(0.1)$.

Problems on Binomial distribution

13. The phone lines to an airline reservation system are occupied 40% of the time. Assume that the events that the lines are occupied on succesive calls are independent. Assume that 10 calls are placed to the airline.
(i) what is the probability that for exactly three calls the lines are occupied?
(ii) what is the probability that for atleast one call the lines are not occupied?
(iii) what is the expected number of calls in which the lines are all occupied.
Answer: (i)$P(X = 3) = 0.2149$, (ii)$P(z \geq 1) = 0.9999$, (iii)$E(X) = 4$

14. Heart failure is due to either natural occuerence(87%) or outside factors(13%). outside factors are releated to induce substances or foreign objects. Natural occurence are caused by arterial blockage, disease and infection. Suppose that 20 patients will visit an emergency room with heart failure. Assume that cause of heart failure between individuals are independent.
(a) what is the probability that three individuals have conmditions caused by outtside factors?
(b) what is the probability that three or more individuals have conditions caused by outside factors?
(c) what is the mean and standard deviation of the number of individuals with conditions caused by outside factors?
Answer: (a) 0.235, (b) 0.492, (c) 2.6,
$V(X) = 2.262$, Standard deviation=1.504

15. In eight throws of a fair dies, 5 or 6 is considered a success. Find the mean of the number of success and the standard deviation. Answer: Mean=$\frac{8}{3}$, standard deviation=$\frac{4}{3}$

16. The probability that a man hits a target is $\frac{1}{3}$ how many times must he fire so that the probability of hitting the target atleast once is more than 90% ? Answer: Man should fire 6 times.

Problems on Poisson distribution

17. Suppose that X has poisson distribution with a mean of 4. Determine the following probabilities:
(a) $P(X = 0)$ (b) $P(X \leq 2)$, (c) $P(X = 4)$, (d) $P(X = 8)$
Answer: (a) 0.0183, (b) 0.2375, (c) 0.1954, (d) 0.0298

18. The number of telephone calls that arrive at a phone exchange is often modelled as a Poisson random variable. Assume that on the average there are 10 calls per hour.
(a) What is the probability that there are exactly 5 calls in one hour?
(b) What is the probability that there are 3 or fewer calls in one hour?
(c) What is the probability that there are exactly 15 calls in two hour?
(d) What is the probability that there are exactly 5 calls in 30 minutes?
Answer: (a) 0.0378, (b) 0.0103, (c) 0.0516, (d) 0.1755

19. Fit a Poisson distribution for the following data and calculate the theoretical frequencies

X	0	1	2	3	4
f	122	60	15	2	1

Answer: The theoritical frequencies are 121, 61, 15, 3, 0

20. In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing
(i) no defective (ii) one defective (iii) two defective blades, in a consignment of 10000 packets.
Answer: (i) $f(0)$=9802, (ii) $f(1)$=196, (iii) $f(2)$=2

Problems on Normal distribution

21. If X is normally distributed with mean 6 and standard deviation 5, Find.
(i) $P(0 \leq X \leq 8)$
(ii) $P(|X-6| < 10)$
Answer: (i) 0.5403, (ii) 0.9544.

22. The life of army shoes is normally distributed with mean 8 months and Standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months?
Answer: 4886.

23. If the height of 300 students is normally distributed with 64.5 inches and Standard deviation 3.3 inches. Find the height below which 99% of the students lie.
Answer: X=72.25

24. In a Normal distribution 30.85% of the items are over 64 and 8% are under 45. Find the mean and Standard deviation
Answer: Mean=59 and standard deviation=10

Problems on exponential distribution

25. If X is an exponential variate with mean 3 find
(i) $P(X > 1)$, (ii) $P(X < 3)$
Answer: (i) 0.7165, (ii) 0.6321

26. The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth
(i) ends less than 5 minutes
(ii) ends between 5 and 10 minutes.
Answer: (i) 0.6321, (ii) 0.2325.

27. In a certain town the duration of a shower is exponential distribution with mean 5 minutes. What is the probability that a shower will last for
(i) 10 minutes or more
(ii) less than 10 minutes
(iii) between 10 and 12 minutes.
Answer:(i) 0.1353, (ii) 0.8647, (iii) 0.0446.

Problems on Normal Approximation to binomial distribution

28. The manufacturing of semiconductor chips produces 2 % defective chips. Assume the chips are independent and that a lot contains 1000 chips.
(a) Approximate the probability that more than 25 chips are defective.
(b) Approximate the probability that between 20 and 30 chips are defective.
Answer: (a) $P(X > 25) = 0.107485$.
(b)$P(20 < X < 30) = 0.4401$.

29. There were 49.7 million people with some type of long-lasting condition or disability living in the United States in 2000. This represented 19.3 percent of the majority of civilians aged five and over. A sample of 1000 persons is selected at random.
(a) Approximate the probability that more than 200 persons in the sample have a disability.
(b) Approximate the probability that between 180 and 300 people in the sample have a disability.
Answer: (a) $P(X > 200) = 0.2743$. (b)$P(180 < X < 300) = 0.84134$.

Problems on Normal Approximation to Poisson distribution

30. Suppose that X is a Poisson random variable with $\lambda = 6$.
(a) Compute the exact probability that X is less than 4.
(b) Approximate the probability that X is less than 4 and compare to the result in part(a).
(c) Approximate the probability that $8 < X < 12$.

Answer: (a) $P(X < 4) = 0.1512$.
(b) $P(X < 4) = 0.1538$.
(c) $P(8 < X < 12) = 0.1413$.

31. Hits to a high-volume web site are assumed to follow a Poisson distribution with a mean of 10000 per day. Approximate each of the following:
(a) The Probability of more than 20000 hits in a day.
(b) The Probability of less than 9900 hits in a day.
(c) The value such that the probability that the number of hits in a day exceed the value is 0.01.
Answer: (a) $P(X > 20000) = 0$.
(b) $P(X < 9900) = 0.1562$.
(c) $X = 10233.5$.