unit-1 notes

Double Integral

$$\iint_R f(x,y)\,dx\,dy = \lim_{\substack{n\to\infty \\ \delta A\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

Jacobian

Let u & v be functions of 2 independant variables x & y

then $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right)$

Change of Variables

Let $x = x(u,v)$ & $y = y(u,v)$ & $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$

then, $\iint_R f(x,y)\,dx\,dy = \iint_{R'} \phi(u,v)\, J\,du\,dv$

where, R is region where x & y vary

R' is region where u & v vary

and, $\phi(u,v) = f(x(u,v), y(u,v))$

Double Integral in Polar form

Let $x = r\cos\theta$, $y = r\sin\theta$

Then, $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_{R'} \phi(r,\theta)\,.\,r\,dr\,d\theta$

Change of Order of Integration

If $\phi(x,y) = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$, then we can also rewrite as

$\phi(x,y) = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(y,x)\,dx\,dy$

Where, a, b, c, d are constants

$x_1(y)$ & $x_2(y)$ are functions dependant on y

$y_1(x)$ & $y_2(x)$ are functions dependant on x

Area, Volume & Average Value of a function

→ $\iint_R z\,dx\,dy$ = volume of solid

→ $\iint_R dx\,dy$ = area of Region R (cartesian form)

→ $\iint_R r\,dr\,d\theta$ = area of Region R (polar form)

→ $\dfrac{\iint_R f(x,y)\,dx\,dy}{\iint_R dx\,dy}$ (or) $\dfrac{\iint_R f(x,y)\,dx\,dy}{A(R)}$ = avg. of $f(x,y)$

Triple Integrals

$$\iiint_V f(x,y,z)\,dV = \sum_{r=1}^{\infty} f(x_r, y_r, z_r)\,dV$$

$$I = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

Change of Variables

Let $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ & $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

then, $\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} \phi(u,v,w)\,J\,du\,dv\,dw$

where, R is region where x, y, z vary

R' is region where u, v, w vary

and, $\phi(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$

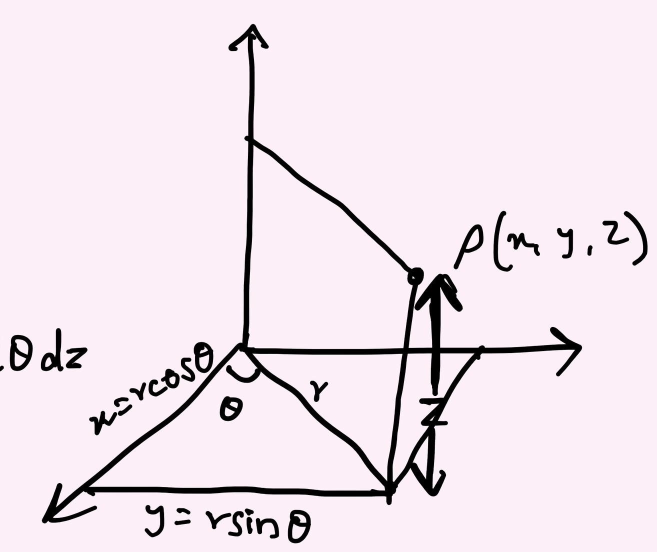
Triple Integrals in cylindrical polar-coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

then (r, θ, z) are cylindrical coordinates

$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

So, $\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} \phi(r,\theta,z)\,.\,r\,.\,dr\,d\theta\,dz$



Triple Integrals in spherical polar-coordinates

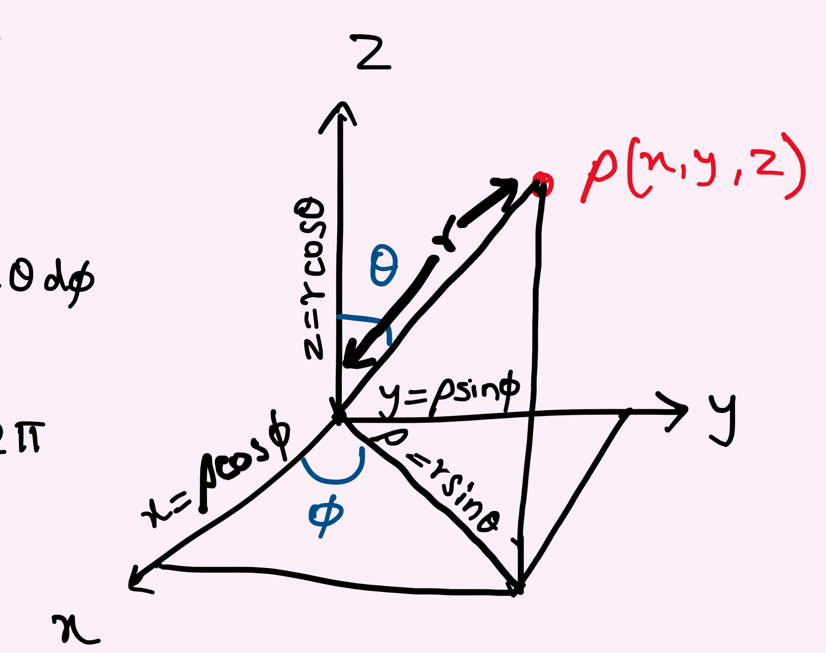
$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

then (r, θ, ϕ) are spherical polar coordinates

$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r^2\sin\theta$

$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} \phi(r,\theta,\phi)\, r^2\sin\theta\,dr\,d\theta\,d\phi$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$



Mass, C.O.G, M.O.I

Let $\rho = f(x,y) > 0$ be surface density of given plane region

Then $m = \iint_D \rho\,dx\,dy = \iint_D f(x,y)\,dx\,dy$

& ρ is continuous density function on planar lamina K,

Moment of mass wrt x & y axis are,

$M_x = \iint_R y\,\rho(x,y)\,dA$ & $M_y = \iint_R x\,\rho(x,y)\,dA$

And if m is mass of lamina,

Then, $COM = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

Similarly for 3D,

$m = \iiint \rho(x,y,z)\,dv$

$M_{xy} = \iiint_Q z\,\rho(x,y,z)\,dv$

$M_{yz} = \iiint_Q x\,\rho(x,y,z)\,dv$

$M_{xz} = \iiint_Q y\,\rho(x,y,z)\,dv$

$\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

$\Rightarrow M.I = \lim_{n\to\infty} \sum_{k=1}^{n} m r^2$

$I_x = \iint \rho\,dA\,.\,y^2$

$I_y = \iint \rho\,dA\,.\,x^2$

$I_x = \iint (y^2+z^2)\,\rho\,dV$

$I_y = \iint (x^2+z^2)\,\rho\,dV$

$I_z = \iint (x^2+y^2)\,\rho\,dV$