Engineering Mathematics - II (UE22MA141B)

Unit - 4: Fourier Series and Fourier Transforms

Find the Fourier series expansion of the following functions over the given interval

1. $f(x) = x - x^2$ from $-\pi$ to π. Hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$
   Answer: $\frac{a_0}{2} = \frac{-\pi^2}{3}; a_n = \frac{-4(-1)^n}{n^2}; b_n = \frac{-2(-1)^n}{n}$

2. $f(x) = \begin{cases} -\pi & \text{for } -\pi < x < 0 \\ x & \text{for } 0 < x < \pi \end{cases}$
   Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$
   Answer: $\frac{a_0}{2} = \frac{-\pi}{4}; a_n = \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right); b_n = \frac{1}{n}\left(1 - 2(-1)^n\right)$

3. $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \pi \\ 2\pi - x & \text{for } \pi \leq x \leq 2\pi \end{cases}$
   Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$
   Answer: $\frac{a_0}{2} = \frac{\pi}{2}; a_n = \frac{2}{\pi n^2}\left((-1)^n - 1\right); b_n = 0$

Fourier series of even and odd functions

4. $f(x) = x^2$ in $(-\pi, \pi)$. Hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$
   Answer: $\frac{a_0}{2} = \frac{\pi^2}{3}; a_n = \frac{4(-1)^n}{n^2}; b_n = 0$

5. $f(x) = xcosx$ in $(-\pi, \pi)$.
   Answer: $a_0 = 0; a_n = 0; b_n = \frac{2n(-1)^n}{n^2-1}$ for $n \neq 1$; $b_1 = -\frac{1}{2}$

6. **Home work** $f(x) = |cosx|$ in $(-\pi, \pi)$.
   Answer: $\frac{a_0}{2} = \frac{2}{\pi}; a_n = \frac{-4cos\frac{n\pi}{2}}{\pi(n^2-1)}$ for $n \neq 1$; $a_1 = 0$; $b_n = 0$

Fourier series expansion of $f(x)$ over an arbitrary interval $(-l, l)$ and $(0, 2l)$

7. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.
   Answer: $\frac{a_0}{2} = \frac{sinhl}{l}; a_n = \frac{2l(-1)^n sinhl}{l^2+n^2\pi^2}; b_n = \frac{2n\pi(-1)^n sinhl}{l^2+n^2\pi^2}$

8. $f(x) = \begin{cases} \pi x & \text{for } 0 < x < 1 \\ \pi(2-x) & \text{for } 1 < x < 2 \end{cases}$
   Answer: $\frac{a_0}{2} = \frac{\pi}{2}; a_n = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right); b_n = 0$

9. **Home work problem**: $f(x) = x^2$ in $(-l, l)$. Answer: $\frac{a_0}{2} = \frac{l^2}{3}; a_n = \frac{4l^2(-1)^n}{n^2\pi^2}; b_n = 0$

Problems on Half-range Fourier series

10. Find the half-range Fourier sine and cosine series of $f(x) = \begin{cases} x & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$

    Answer: $\frac{a_0}{2} = \frac{\pi}{4}; a_n = \frac{2}{\pi n^2}\left(2cos\frac{n\pi}{2} - 1 - (-1)^n\right); b_n = \frac{4}{\pi n^2} sin\frac{n\pi}{2}$

11. Find the half-range Fourier sine series of $f(x) = \begin{cases} \frac{1}{4} - x & \text{for } 0 \le x \le \frac{1}{2} \\ x - \frac{3}{4} & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$

    Answer: $b_n = \frac{1}{2n\pi}(1 - (-1)^n) - \frac{4}{n^2\pi^2} sin\frac{n\pi}{2}$

12. **Home work** Find the half-range Fourier cosine series of $f(x) = \begin{cases} kx & \text{for } 0 \le x \le \frac{l}{2} \\ k(l - x) & \text{for } \frac{l}{2} \le x \le l \end{cases}$

    Answer: $\frac{a_0}{2} = \frac{kl}{4}; a_n = \frac{2kl}{\pi^2 n^2}\left(2cos\frac{n\pi}{2} - 1 - (-1)^n\right)$

Problems on Parseval's Identity

13. Obtain the Fourier series for $y = x^2$ in $-\pi < x < \pi$ and hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{90}$. Answer: $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

14. Expand $f(x) = x - \frac{x^2}{2}$ in $(0, 2)$ as Fourier sine series and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$ Answer: $f(x) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} sin\frac{(2n-1)\pi x}{2}; \frac{\pi^4}{960}$

15. **Home work** Using the Fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$ show that:

    $(i) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ $\quad (ii) \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Problems on complex Fourier series

16. $f(x) = e^{-x}$ in $-1 \le x \le 1$. Answer: $e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n sinh1}{(1 + in\pi)} e^{in\pi x}$

17. $f(x) = cosax$ in $-\pi \le x \le \pi$. Answer: $cosax = \sum_{n=-\infty}^{\infty} \frac{(-1)^n asina\pi}{\pi(a^2 - n^2)} e^{inx}$

18. **Home work problem**: $f(x) = e^{ax}$ in $-\pi \le x \le \pi$. Answer: $e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n sinha\pi}{\pi(a - in)} e^{inx}$

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1. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$. Hence evaluate $\int_0^{\infty} \frac{sinx}{x} dx$.

   Answer: $F(s) = \sqrt{\frac{2}{\pi}} \frac{sins}{s}; \int_0^{\infty} \frac{sinx}{x} dx = \frac{\pi}{2}$

2. Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$.

   Hence evaluate $\displaystyle\int_0^\infty \left(\frac{xcosx - sinx}{x^3}\right) \cdot cos\frac{x}{2}dx.$

   Answer: $F(s) = -2\sqrt{\frac{2}{\pi}}\left(\frac{scoss - sins}{s^3}\right)$; $\displaystyle\int_0^\infty \left(\frac{xcosx - sinx}{x^3}\right) \cdot cos\frac{x}{2}dx = -\frac{3\pi}{16}$

3. Find the Fourier transform of $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$. Hence evaluate $\displaystyle\int_0^\infty \frac{sin^2x}{x^2}dx.$

   Answer: $F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1 - coss}{s^2}\right)$; $\displaystyle\int_0^\infty \frac{sin^2x}{x^2}dx = \frac{\pi}{2}$

Problems on Fourier sine transform and its inversion formula

4. Find the Fourier sine transform of $e^{-|x|}$ and hence evaluate $\displaystyle\int_0^\infty \frac{xsinmx}{1+x^2}dx.$

   Answer: $F_s(f(x)) = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}$; $\displaystyle\int_0^\infty \frac{xsinmx}{1+x^2}dx = \frac{\pi e^{-x}}{2}$

5. Find the Fourier sine transform of $\frac{e^{-ax}}{x}; x \ne 0; a > 0$. (Differentiation under integral sign needs to be used). Answer: $F_s(f(x)) = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{s}{a}\right)$

6. Find the Fourier sine transform of $\frac{e^{-ax}}{x}; x \ne 0; a > 0$; hence evaluate $\displaystyle\int_0^\infty \frac{xsinkx}{a^2+x^2}dx.$

   Answer: $F_s(f(x)) = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$; $\displaystyle\int_0^\infty \frac{xsinkx}{a^2+x^2}dx = \frac{\pi}{2}e^{-ak}$

Problems on Fourier cosine transform and its inversion formula

7. Find the Fourier cosine transform of $\frac{1}{1+x^2}$. Answer: $F_c(f(x)) = \sqrt{\frac{\pi}{2}}e^{-s}$

8. Find the Fourier cosine transform of e^{-x^2}. (Differentiation under integral sign needs to be used). Answer: $F_c(f(x)) = \frac{1}{\sqrt{2}}e^{\frac{-s^2}{4}}$

9. Find the Fourier cosine transform of $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$.

   Answer: $F_c(f(x)) = \frac{2}{\pi}\left(\frac{2coss - cos2s - 1}{s^2}\right)$

Problems on properties of Fourier transform

10. Find the Fourier transform of $e^{-a^2x^2}$, $a < 0$. Hence deduce that $e^{-\frac{x^2}{2}}$ is self-reciprocal in respect of the Fourier transform.
    Also, find the Fourier transform of (i) $e^{-2(x-3)^2}$ and (ii) $e^{-x^2}cos3x$. (Please refer to Grewal's book).

Problems on finite Fourier sine and cosine transform

11. Find the finite Fourier sine transform of $f(x) = \begin{cases} -x & \text{for } 0 < x < c \\ \pi - x & \text{for } c < x < \pi \end{cases}$
    Answer: $F_s(f(x)) = \frac{\pi cosnc}{n}$
12. Find the finite Fourier sine transform of $f(x) = \frac{1-cosn\pi}{n^2\pi^2}$.
    Answer: $F_s(f(x)) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\dfrac{sin(2n-1)x}{(2n-1)^2}$
13. **Home work problem**: Find the finite Fourier sine transform of $f(x) = x(\pi - x)$ in $0 < x < \pi$.
    Answer: $F_s(f(x)) = \frac{2}{n^3}\left(1 - (-1)^n\right)$
14. Find the finite Fourier cosine transform of $f(x) = 2x$ in $0 < x < 4$.
    Answer: $F_c(f(x)) = \frac{32}{(n\pi)^2}\left((-1)^n - 1\right)$
15. Find the finite Fourier cosine transform of $f(x) = x(\pi - x)$ in $0 < x < \pi$.
    Answer: $F_c(f(x)) = -\frac{\pi}{n^2}\left(1 + (-1)^n\right)$
16. **Home work problem**: Find the finite Fourier cosine transform of $f(x) = e^{ax}$ in $0 < x < l$. Answer: $F_c(f(x)) = -\frac{al^2}{a^2l^2+(n\pi)^2}\left(e^{al}(-1)^n - 1\right)$