ENGINEERING MECHANICS

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ENGINEERING MECHANICS

Distributed Forces-Centroid

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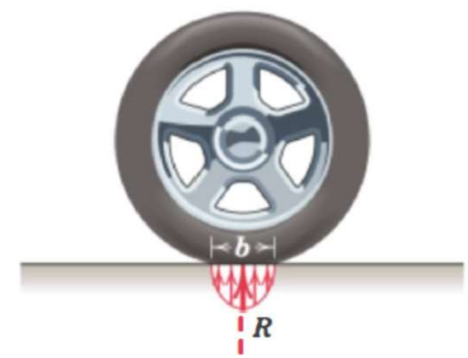
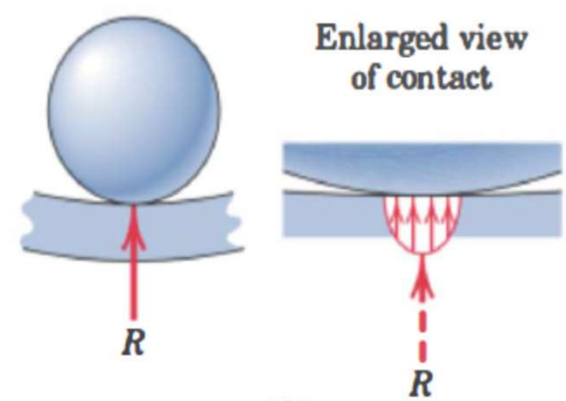
- In the previous discussion we treated all forces as concentrated along their lines of action and at their points of application.
- But "concentrated“ forces do not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area.

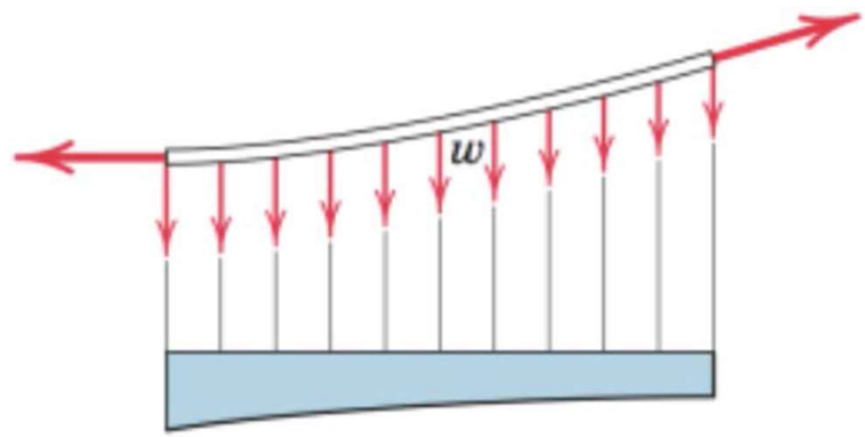
- The force is applied to the tire as shown in the figure, it appears as entire area of contact, which may be appreciable if the tire is soft.
- If the dimensions of b is negligible when compare with other dimensions, the actual force distribution forces is replaced by resultant R.
- Even the force of contact between a hardened steel ball and its race in a loaded ball bearing.



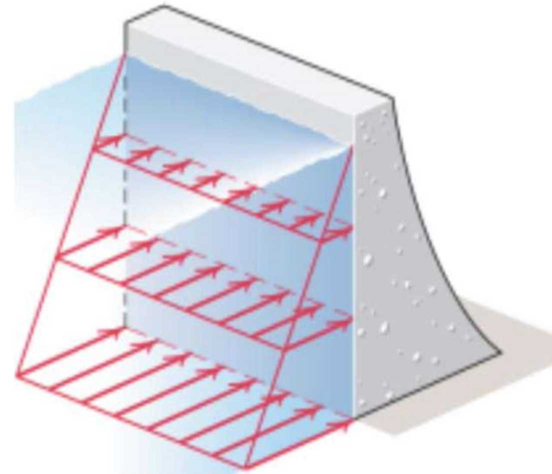


- We want to find the distribution of internal forces in the material near the contact location.
- When the forces are distribution over the region but the dimensions are not negligible, then we consider the actual distribution load.
- We do by summing the effects of the distributed force over the entire region using mathematical integration.
- This requires that we know the intensity of the force at any location. There are three categories of such problems.

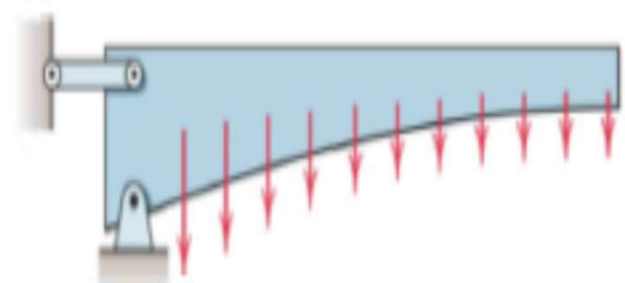
1. Line Distribution. When a force is distributed along a line, as in the continuous vertical load supported by a suspended cable. The intensity w of the loading is expressed as force per unit length of line, newtons per meter (N/m).



2. Area Distribution. When a force is distributed over an area, as with the hydraulic pressure of water against the inner face of a section of dam. The intensity is expressed as force per unit area and called as pressure.

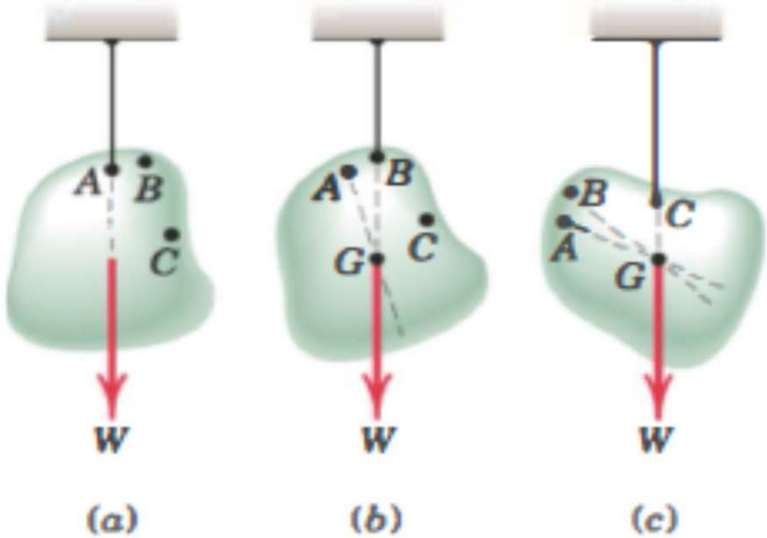


3. Volume Distribution. A force which is distributed over the volume of a body is called a body force. The most common body force is the force of gravitational attraction, which acts on all elements of mass in a body.



CENTER OF MASS

Consider a three-dimensional body of any size and shape, having a mass m.
If we suspend the body, as shown in Fig. from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational forces acting on all particles of the body. We repeat the experiment by suspending the body from other points such as B and C, and in each instance we mark the line of action of the resultant force.

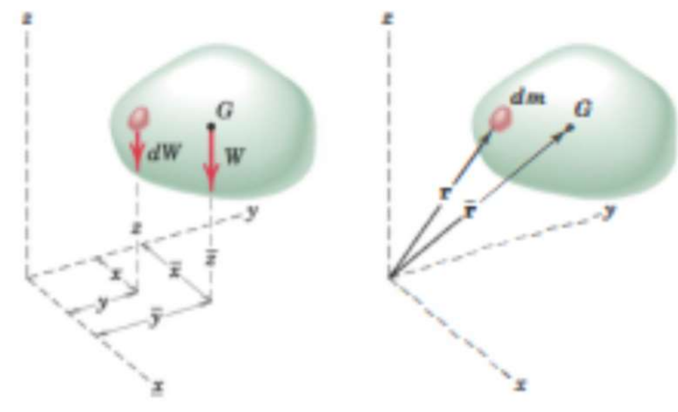


For all practical purposes these lines of action will be concurrent at a single point G, which is called the center of gravity of the body.

Determining the Center of Gravity

To determine mathematically the location of the center of gravity of any body, as shown in Figure, we apply the principle of moments to the parallel system of gravitational forces.

The moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dw acting on all particles treated as infinitesimal elements of the body.



(a) (b)

The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum W = ∫ dw.

Thus, X * W = ∫ x dw.

$$\bar{x} = \frac{\int x\,dW}{W} \qquad \bar{y} = \frac{\int y\,dW}{W} \qquad \bar{z} = \frac{\int z\,dW}{W}$$

To visualize the physical moments of the gravity forces appearing in the “b” equation, we may reorient the body and attached axes so that the z-axis is horizontal. It is essential to recognize that the numerator of each of these expressions represents the sum of the moments, whereas the product of W and the corresponding coordinate of G represents the moment of the sum. This moment principle finds repeated use throughout mechanics.

With the substitution of W = mg and dw= g * dm, the expressions for the coordinates of the center of gravity become

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

Equation (b)

Equations (b) may be expressed in vector form with the aid of Fig. 1(b), in which the elemental mass and the mass center G are located by their respective position vectors $r = X\,i + Y\,j + Z\,k$ and $r = X\,i + y\,j + Z\,k$.
Thus, Equation (b) are the components of the single vector equation

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m}$$ Equation (c)

The density p of a body is its mass per unit volume. Thus, the mass of a differential element of volume dV becomes dm = ρ dV. If ρ is not constant throughout the body but can be expressed as a function of the coordinates of the body, we must account for this variation when calculating the numerators and denominators of Eqs. (b). We may then write these expressions as

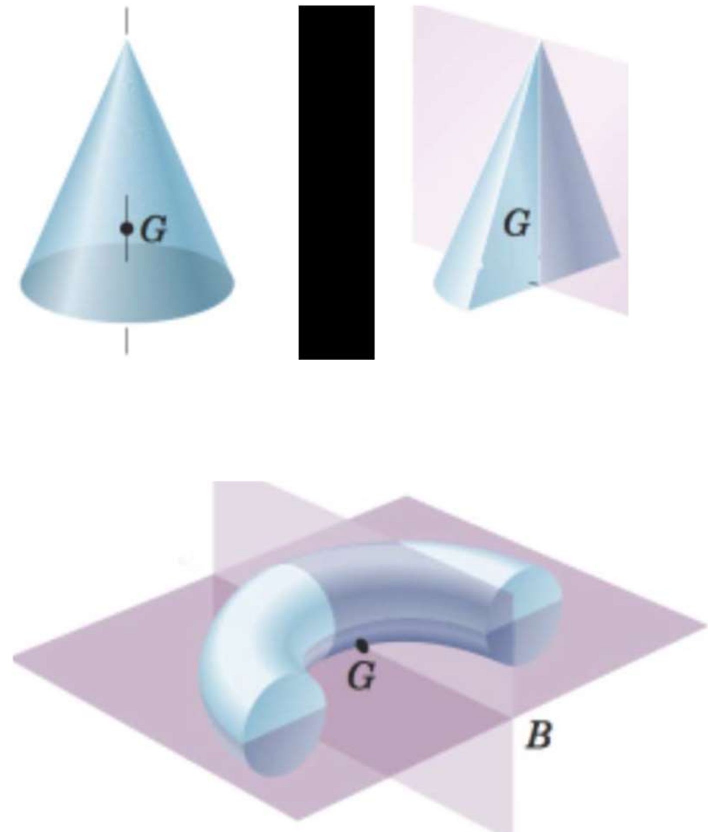
Center of Mass versus Center of Gravity

Equations (b), (c), and (d) are independent of gravitational effects since “g” no longer appears. They therefore define a unique point in the body which is a function solely of the distribution of mass. This point is called the center of mass, and clearly it coincides with the center of gravity as long as the gravity field is treated as uniform and parallel.

It is meaningless to speak of the center of gravity of a body which is removed from the gravitational field of the earth, since no gravitational forces would act on it. The body would, however, still have its unique center of mass. We will usually refer henceforth to the center of mass rather than to the center of gravity. Also, the center of mass has a special significance in calculating the dynamic response of a body to unbalanced forces.

In most problems the calculation of the position of the center of mass may be simplified by an intelligent choice of reference axes. In general, the axes should be placed so as to simplify the equations of the boundaries as much as possible. Thus, polar coordinates will be useful for bodies with circular boundaries.

Another important clue may be taken from considerations of symmetry. Whenever there exists a line or plane of symmetry in a homogeneous body, a coordinate axis or plane should be chosen to coincide with this line or plane. The center of mass will always lie on such a line or plane, since the moments due to symmetrically located elements will always cancel, and the body may be considered to be composed of pairs of these elements. Thus, the center of mass G of the homogeneous right circular cone of Fig. 2 (a) will lie somewhere on its central axis, which is a line of symmetry. The center of mass of the half right-circular cone lies on its plane of symmetry, Fig. 2 (b). The center of mass of the half ring in Fig. 2 (c) lies in both of its planes of symmetry and therefore is situated on line AB. It is easiest to find the location of G by using symmetry when it exists.



CENTROIDS OF LINES, AREAS, and VOLUMES

When the density p of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Eqs. (d)

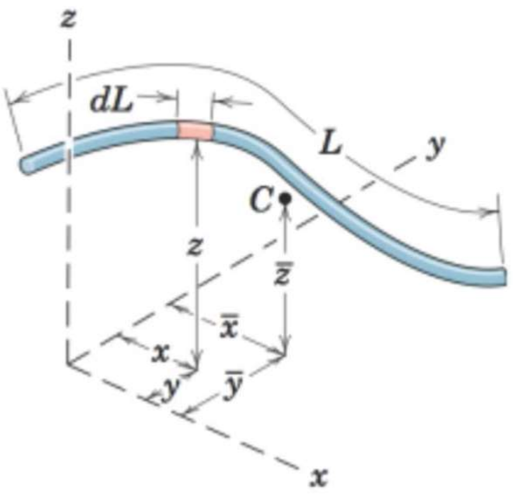
$$\bar{x} = \frac{\int x\rho\, dV}{\int \rho\, dV} \qquad \bar{y} = \frac{\int y\rho\, dV}{\int \rho\, dV} \qquad \bar{z} = \frac{\int z\rho\, dV}{\int \rho\, dV} \qquad \text{Equation (d)}$$

and will therefore cancel. The remaining expressions define a purely geometrical property of the body, since any reference to its mass properties has disappeared. The term centroid is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, we use the term center of mass. If the density is uniform throughout the body, the positions of the centroid and center of mass are identical, whereas if the density varies, these two points will, in general, not coincide.

The calculation of centroids falls within three distinct categories, depending on whether we can model the shape of the body involved as a line, an area, or a volume.

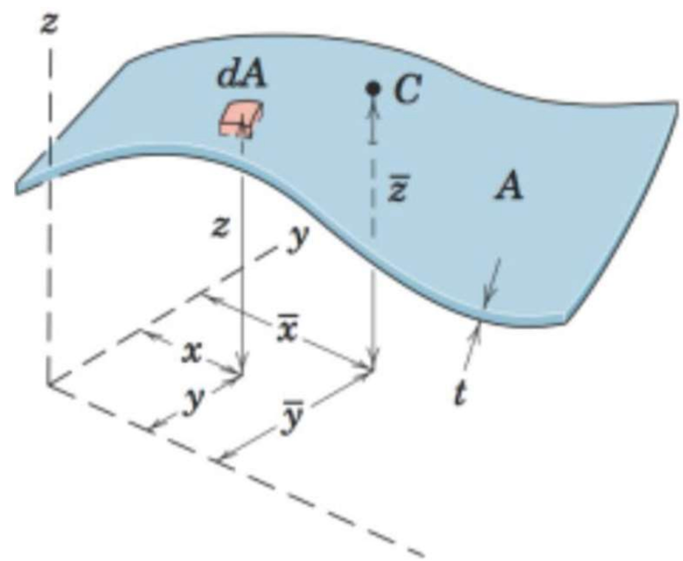
(1) Lines. For a slender rod or wire of length L, cross-sectional area A, and density p, Fig. shown below, the body approximates a line segment, and dm = ρA dL. If ρ and A are constant over the length of the rod, the coordinates of the center of mass also become the coordinates of the centroid C of the line segment, which, from Eqs. (b),

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$



(2) Areas. When a body of density ‘ρ’ has a small but constant thickness ‘t’, we can model it as a surface area A, Fig. shows. The mass of an element becomes dm = ρt dA. Again, if ρ and t are constant over the entire area, the coordinates of the center of mass of the body also become the coordinates of the centroid C of the surface area, and from Eqs. (b) the coordinates may be written

$$\bar{x} = \frac{\int x\,dA}{A} \qquad \bar{y} = \frac{\int y\,dA}{A} \qquad \bar{z} = \frac{\int z\,dA}{A}$$



The numerators in the above Eqs. are called the first moments of area.* If the surface is curved, as illustrated in Fig. with the shell segment, all three coordinates will be involved. The centroid C for the curved surface will in general not lie on the surface. If the area is a flat surface in say, the x-y plane, only the coordinates of C in that plane need to be calculated.

(3) Volumes. For a general body of volume V and density p, the element has a mass dm = p dV . The density p cancels if it is constant over the entire volume, and the coordinates of the center of mass also become the coordinates of the centroid C of the body. From Eqs. (b) they become

$$\bar{x} = \frac{\int x\, dV}{V} \qquad \bar{y} = \frac{\int y\, dV}{V} \qquad \bar{z} = \frac{\int z\, dV}{V}$$

COMPOSITE BODIES and FIGURES

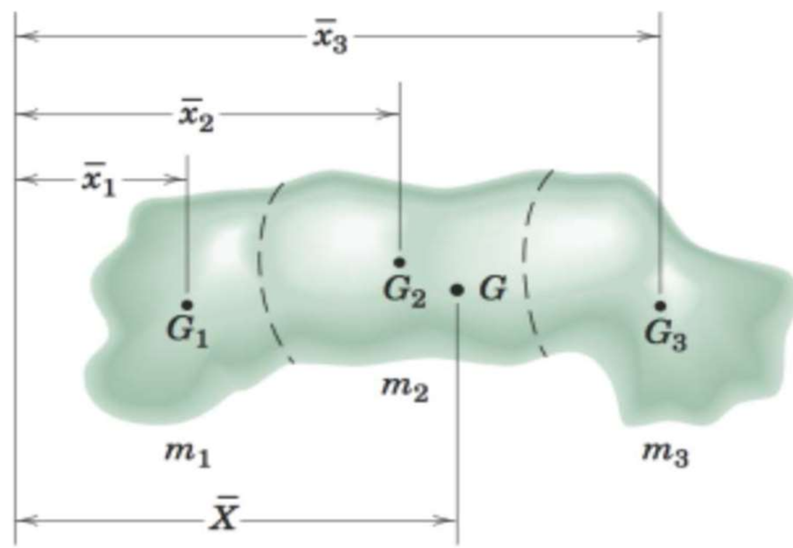
When a body can be conveniently divided into several parts whose mass centers are easily determined, we use the principle of moments and treat each part as a finite element of the whole.

Such a body is illustrated schematically in Figure. Its parts have masses m_1, m_2, m_3 with the respective mass-center coordinates x_1, x_2, x_3 in the x-direction. The moment principle gives.

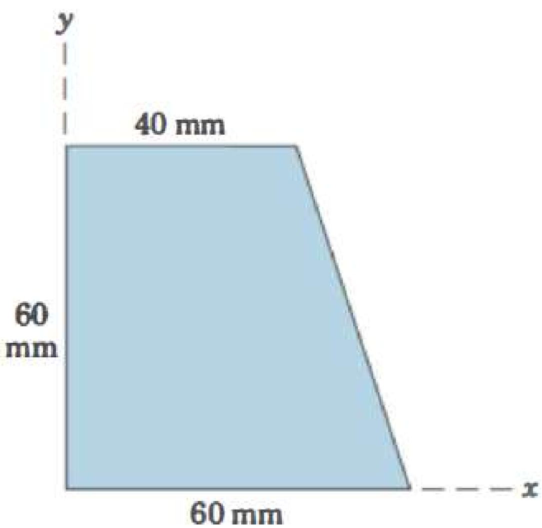
$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

where X is the x-coordinate of the center of mass of the whole. Similar relations hold for the other two coordinate directions.

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$



5/47 Determine the coordinates of the centroid of the trapezoidal area shown.



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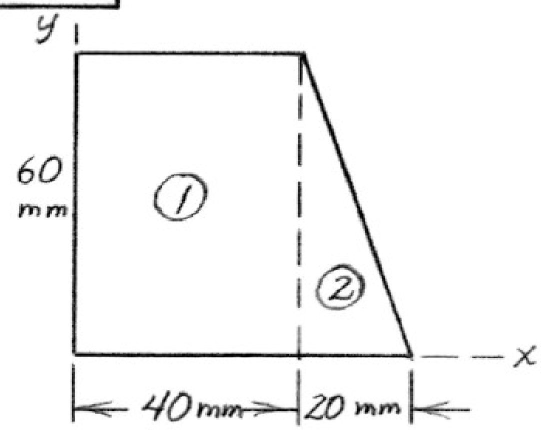


① $A_1 = 40(60) = 2400 \text{ mm}^2$

$\bar{x}_1 = 20 \text{ mm}, \quad \bar{y}_1 = 30 \text{ mm}$

② $A_2 = \frac{1}{2}(20)(60) = 600 \text{ mm}^2$

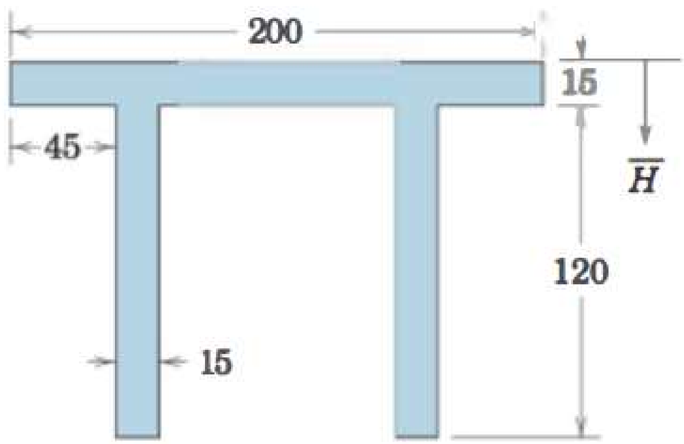
$\bar{x}_2 = 40 + \frac{20}{3} = 46.7 \text{ mm}$

$\bar{y}_2 = \frac{60}{3} = 20 \text{ mm}$

$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{2400(20) + 600(46.7)}{2400 + 600} = \underline{25.3 \text{ mm}}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{2400(30) + 600(20)}{2400 + 600} = \underline{28.0 \text{ mm}}$$

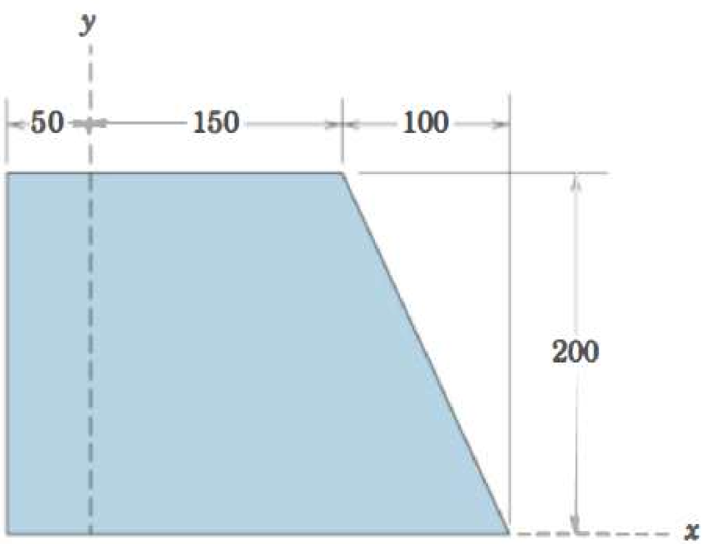
5/48 Determine the distance H from the upper surface of the symmetric double-T beam cross section to the location of the centroid.



$$\bar{H} = \frac{\Sigma A \bar{h}}{\Sigma A} = \frac{200(15)\left(\frac{15}{2}\right) + 2(15)(120)\left(15 + \frac{120}{2}\right)}{200(15) + 2(15)(120)} \longrightarrow \underline{\bar{H} = 44.3 \text{ mm}}$$

ENGINEERING MECHANICS
Centroid

5/49 Determine the x- and y-coordinates of the centroid of the shaded area.



Dimensions in millimeters

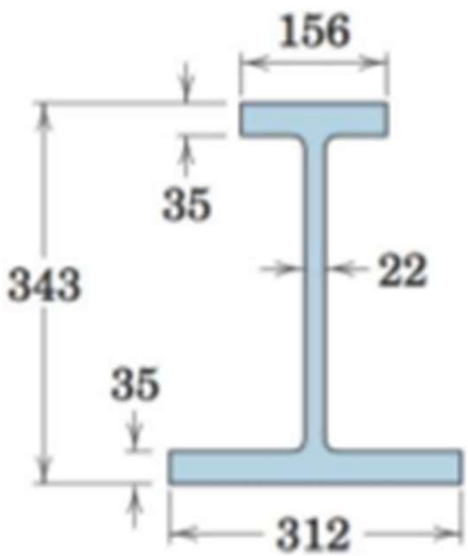
$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{50(200)(-25) + 150(200)(75) + \frac{1}{2}(100)(200)\left(150 + \frac{100}{3}\right)}{50(200) + 150(200) + \frac{1}{2}(100)(200)}$$

$$\bar{X} = 76.7 \text{ mm}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{50(200)(100) + 150(200)(100) + \frac{1}{2}(100)(200)\left(\frac{200}{3}\right)}{50(200) + 150(200) + \frac{1}{2}(100)(200)}$$

$$\bar{Y} = 93.3 \text{ mm}$$

5/50 Determine the height above the base of the centroid of the cross-sectional area of the beam. Neglect the fillets.



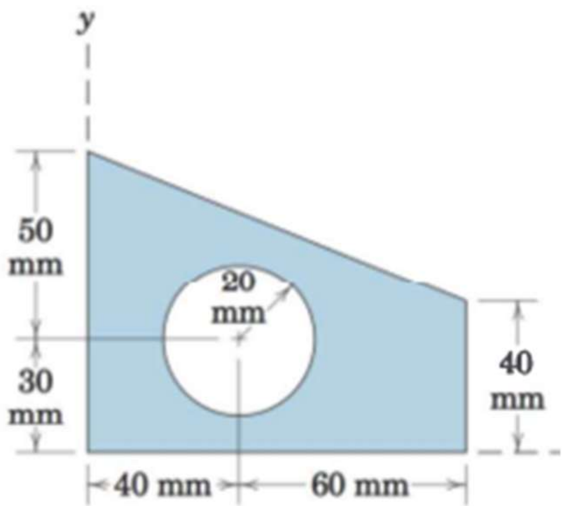
Dimensions in millimeters

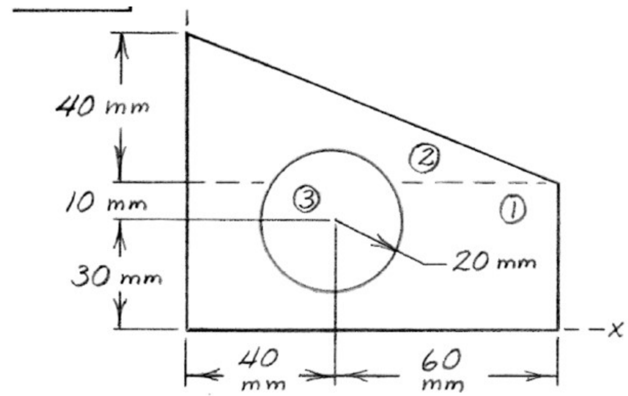
Comp.	$A\ (mm^2)$	$\bar{y}\ (mm)$	$A\bar{y}\ (mm^3)$
1	$312(35)$	$\frac{35}{2}$	191 100
2	$273(22)$	$35+\frac{273}{2}$	1 030 000
3	$156(35)$	$35+273+\frac{35}{2}$	1 777 000

$\Sigma A = 22\,400$ $\qquad$ $\Sigma A\bar{y} = 3\,000\,000$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{3\,000\,000}{22\,400} = \underline{133.9\ mm}$$

5/51 Determine the x- and y-coordinates of the centroid of the shaded area.



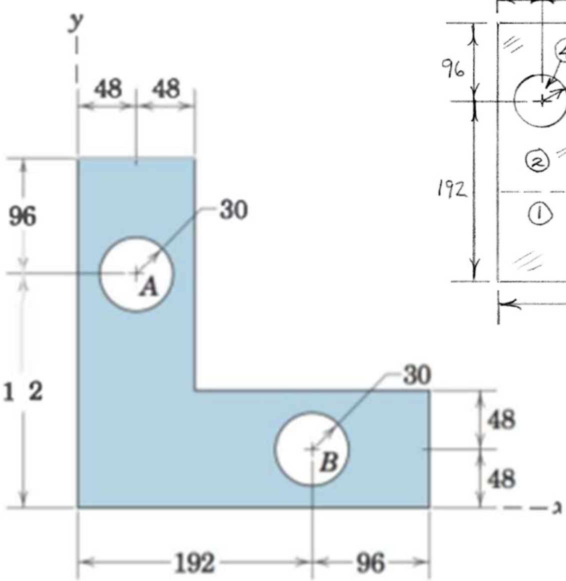


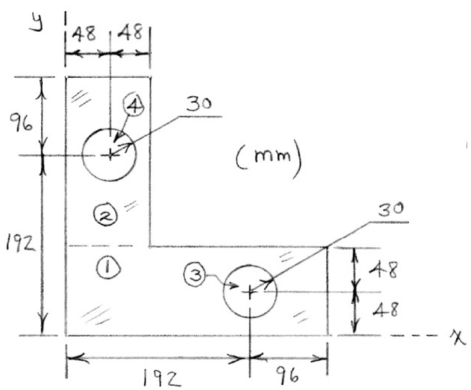
Part	A (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$A\bar{x}$ (mm^3)	$A\bar{y}$ (mm^3)
1	4000	50	20	$200(10^3)$	$80(10^3)$
2	2000	100/3	$40+\frac{40}{3}$	$66.7(10^3)$	$106.7(10^3)$
3	$-\pi(20^2)$	40	30	$-50.3(10^3)$	$-37.7(10^3)$
Totals	4740			$216(10^3)$	$149.0(10^3)$

$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{216(10^3)}{4740} = \underline{45.6 \text{ mm}}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{149.0(10^3)}{4740} = \underline{31.4 \text{ mm}}$$

5/52 Determine the x- and y-coordinates of the centroid of the shaded area.



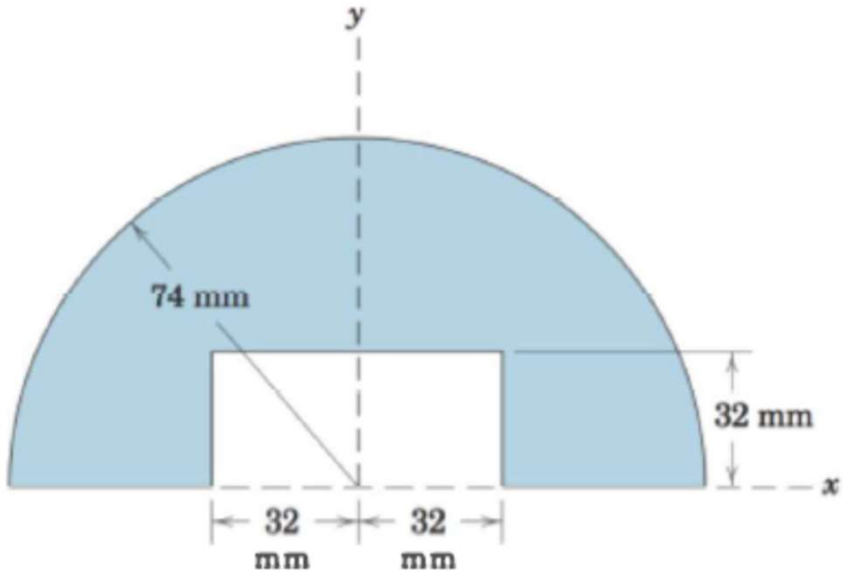


Comp.	A (mm^2)	$\bar{x}$ (mm)	$\bar{x}A$ (mm^3)
1	27648	144	$3.98(10^6)$
2	18432	48	$0.885(10^6)$
3	− 2830	192	$-0.543(10^6)$
4	− 2830	48	$-0.1357(10^6)$

$\Sigma A = 40430$ $\quad$ $\Sigma \bar{x}A = 4.19(10^6)$

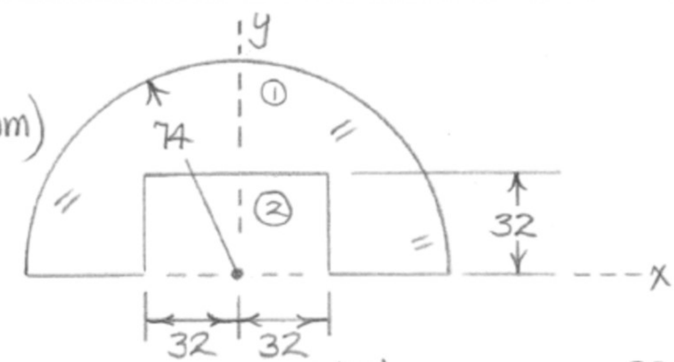
$$\bar{X} = \bar{Y} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{4.19(10^6)}{40430} = \underline{103.6 \text{ mm}}$$

5/53 Calculate the y-coordinate of the centroid of the shaded area.



5/53

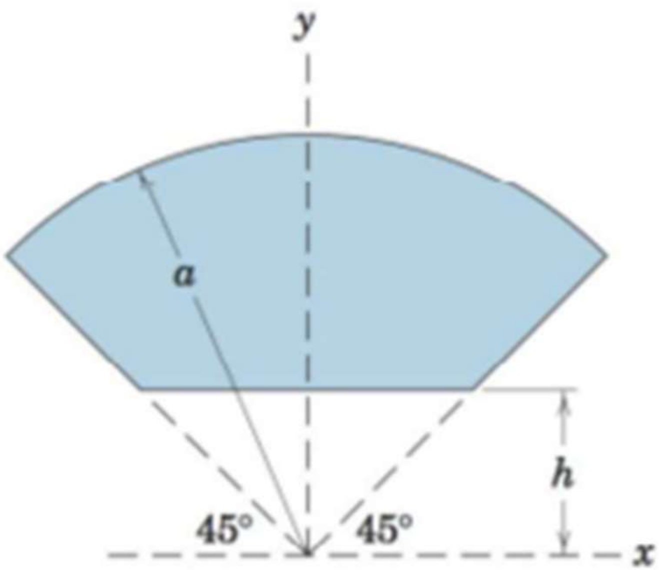
(Dim. in mm)



$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{\pi \frac{74^2}{2}\left(\frac{4(74)}{3\pi}\right) - 64(32)\left(\frac{32}{2}\right)}{\pi \frac{74^2}{2} - 64(32)}$$

$$= \underline{36.2 \text{ mm}}$$

5/55 Determine the y-coordinate of the centroid shaded area.



Circular sector (full) ①:

$$A_1 = \frac{\pi}{4}a^2$$

$$\bar{y}_1 = \frac{2}{3}a\frac{\sin 45°}{\pi/4}$$

$$= \frac{4\sqrt{2}}{3\pi}a$$

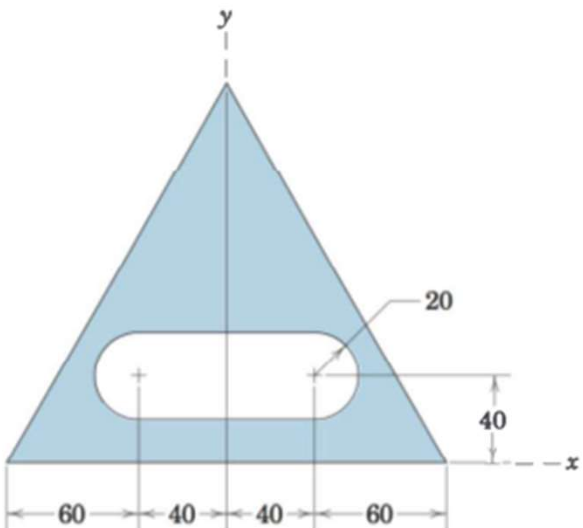
Triangular "hole" ②:

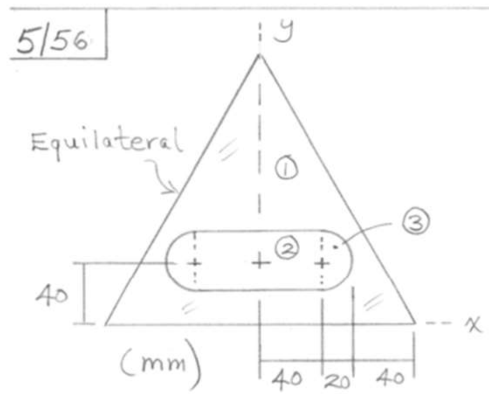
$$A_2 = \frac{1}{2}h(2h) = h^2, \quad \bar{y}_2 = \frac{2}{3}h$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{\frac{\pi}{4}a^2\left(\frac{4\sqrt{2}}{3\pi}a\right) - h^2\left(\frac{2}{3}h\right)}{\frac{\pi}{4}a^2 - h^2}$$

$$= \frac{4(\sqrt{2}a^3 - 2h^3)}{3(\pi a^2 - 4h^2)}$$

5/56 Determine the y-coordinate of the centroid of the shaded area. The triangle is equilateral.

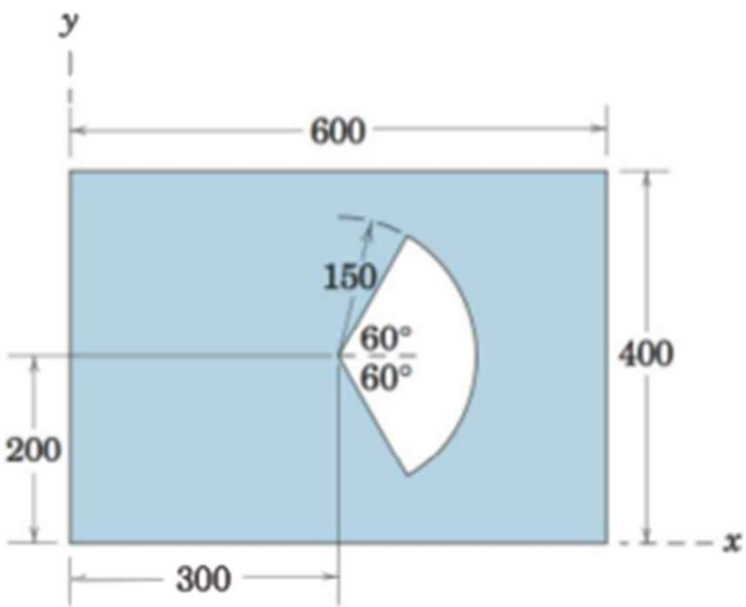




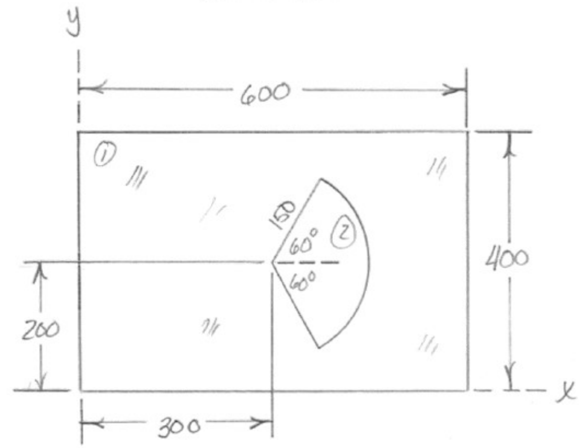
Component	A (mm^2)	$\bar{y}$ (mm)	$\bar{y}A$ (mm^3)
Triangle 1	17 320	57.7	10^6
Rectangle 2	− 3200	40	− 128 000
2 semicircles 3	− 1257	40	− 50,300
	ΣA = 12 860		$\Sigma \bar{y}A$ = 822 000

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} \quad \frac{822\,000}{12\,860} = 63.9 \text{ mm}$$

5/57 Determine the x- and y-coordinates of the centroid of the shaded area.



Dimensions in millimeters

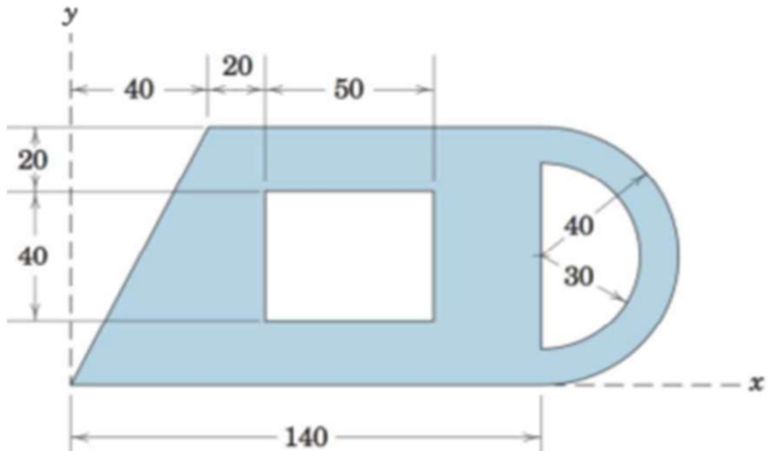


$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{400(600)(300) - \frac{1}{3}\pi(150)^2\left(300 + \frac{2}{3}(150)\frac{\sin 60}{\pi/3}\right)}{400(600) - \frac{1}{3}\pi(150)^2}$$

$\bar{X} = 291$ mm

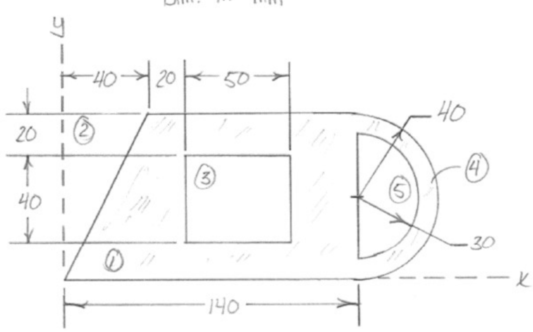
$\bar{Y} = 200$ mm (INSPECTION)

5/58 Determine the coordinates of the centroid of the shaded area.



Dimensions in millimeters

Dim. in mm



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$A\bar{x}$, mm^2	$A\bar{y}$, mm^2
1	$140(80) = 11200$	70	40	784×10^3	448×10^3
2	$-\frac{1}{2}(40)(80) = -1600$	$\frac{40}{3} = 13.33$	$\frac{2}{3}(80) = 53.3$	-21.3×10^3	-85.3×10^3
3	$-40(50) = -2000$	85	40	-170×10^3	-80×10^3
4	$\frac{\pi(40)^2}{2} = 800\pi$	$140+\frac{4(40)}{3\pi} = 157.0$	40	395×10^3	32000π
5	$-\frac{\pi(30)^2}{2} = -450\pi$	$140+\frac{4(30)}{3\pi} = 152.7$	40	-216×10^3	-18000π
Σ	8700			771×10^3	327×10^3

$$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{771\times10^3}{8700} \rightarrow \bar{X} = 88.7 \text{ mm} \qquad \bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{327\times10^3}{8700} \rightarrow \bar{Y} = 37.5 \text{ mm}$$

ENGINEERING MECHANICS
Centroid

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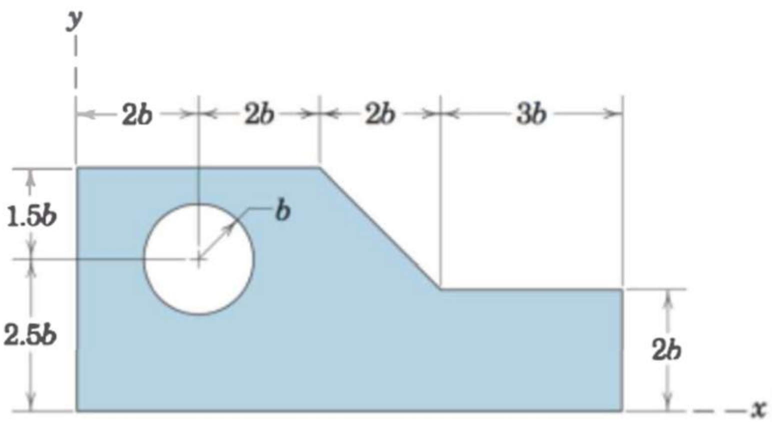
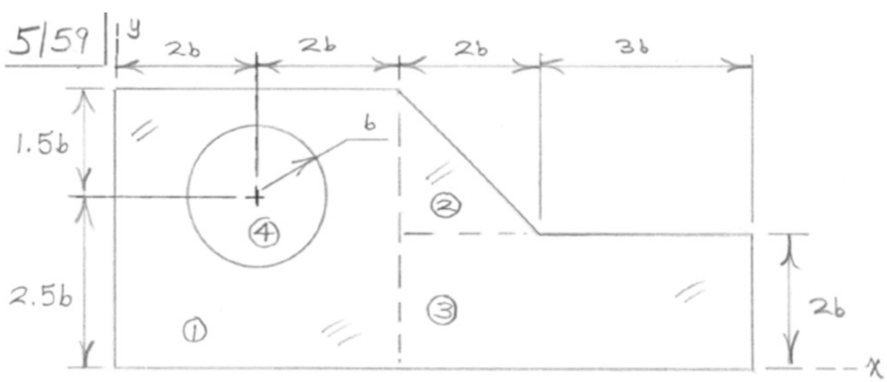
5/59 Determine the x and y coordinates of the centroid of the shaded area.



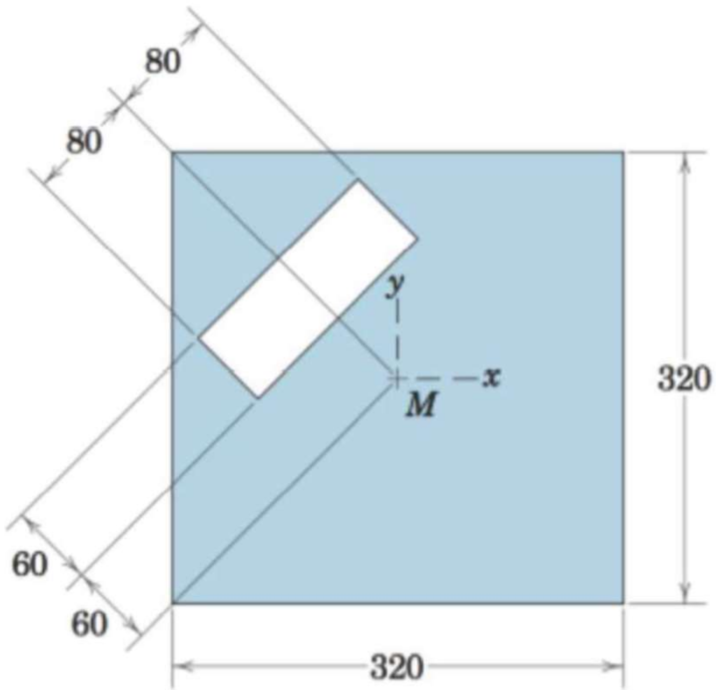


Comp.	A	$\bar{x}$	$\bar{y}$	$\bar{x}A$	$\bar{y}A$
1	$16b^2$	$2b$	$2b$	$32b^3$	$32b^3$
2	$2b^2$	$\left(4b+\frac{2b}{3}\right)$	$\left(2b+\frac{2b}{3}\right)$	$9.33b^3$	$5.33b^3$
3	$10b^2$	$\left(4b+\frac{5b}{2}\right)$	b	$65b^3$	$10b^3$
4	$-\pi b^2$	$2b$	$2.5b$	$-2\pi b^3$	$-2.5\pi b^3$
	$\Sigma A = 24.9b^2$			$\Sigma \bar{x}A = 100.1b^3$	$\Sigma \bar{y}A = 39.5b^3$

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{100.1b^3}{24.9b^2} = 4.02b$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{39.5b^3}{24.9b^2} = 1.588b$$

5/61 By inspection, state the quadrant in which the centroid of the shaded area is located. Then determine the coordinates of the centroid. The plate center is M.



Comp.	A (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{x}A$ (mm^3)	$\bar{y}A$ (mm^3)
1	$(320)^2$	0	0	0	0
2	$-160(60)$	$-90\frac{\sqrt{2}}{2}$	$90\frac{\sqrt{2}}{2}$	611 000	$-611\ 000$

$\Sigma A = 92\ 800$

$\Sigma \bar{x}A = 611\ 000$

$\Sigma \bar{y}A = -611\ 000$

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{611\ 000}{92\ 800} = \underline{6.58 \text{ mm}}$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{-611\ 000}{92\ 800} = \underline{-6.58 \text{ mm}}$$

THANK YOU

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