B-Tech- II

Department of Science and Humanities

CLASS -2

UNIT 3

Laplace transform

Laplace Transform of some Elementary Functions

(1) $L(1) = \frac{1}{s}$

Proof : -By Definition

$$L(1) = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, (s > 0)$$

(2) $L(e^{at}) = \frac{1}{s-a}$

Proof : -By Definition

$$L(e^{at}) = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a} \text{ if } s > a$$

$$(3)\ L[e^{-at}] = \frac{1}{s+a},\ s > -a$$

$$(4)\ L[\sinh at] = \frac{a}{s^2 - a^2}$$

Proof : -We have $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$ and $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

By definition

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2 - a^2},\ s > |a|$$

$$(5)\ \text{Similarly, } L[\cosh at] = \frac{s}{s^2 - a^2},\ s > |a|$$

$$(6)\ L[\sin at] = \frac{a}{s^2 + a^2} \text{ and } L[\cos at] = \frac{s}{s^2 + a^2}, s > 0$$

Proof: - We know that $e^{ix} = \cos x + i \sin x$ [Euler's Formula]

$$\therefore e^{iat} = \cos at + i \sin at$$

$$\therefore L[\cos at + i \sin at] = L[e^{iat}] = \frac{1}{s - ia} \qquad \left[\because L(e^{at}) = \frac{1}{s-a}\right]$$

$$= \frac{s + ia}{s^2 + a^2} = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}$$

Equating real and imaginary parts, we get

$$L[\sin at] = \frac{a}{s^2 + a^2} \text{ and } (7) L[\cos at] = \frac{s}{s^2 + a^2}, s > 0$$

$$(8) L(t^n) = \frac{\left|\overline{n+1}\right.}{S^{n+1}} \text{ or } \frac{n!}{S^{n+1}}$$

$$\text{Proof : -L}(t^n) = \int_0^\infty e^{-st}.t^n dt, \textit{ putting } \text{ st} = \text{u}$$

$$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{S^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1} du$$

$$\therefore L(t^n) = \frac{\left|\overline{n+1}\right.}{S^{n+1}}, \text{n} > \text{-1} \qquad \left[\Theta \left|\overline{\text{n}}\right. = \int_0^\infty e^{-x} x^{n-1} dx, n > 0\right]$$

$$\text{or } L(t^n) = \frac{\text{n!}}{S^{n+1}} \qquad [\text{n} = 0,1,2..] \left[\Theta \left|\overline{\text{n}+1}\right. = \text{n!}\right]$$

Definition and results of Gamma function

- Gamma function $$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$\Gamma(n+1) = n!$ when n is a positive integer

- Hence $$L[t^n] = \frac{n!}{s^{n+1}}$$

$\Gamma(n+1) = n\Gamma(n)$ when n is a positive or negative fraction

- $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

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Laplace transform of some elementary functions

$$L\{1\} = \frac{1}{s}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}};\, n \varepsilon N$$

$$L\{e^{at}\} = \frac{1}{s-a};\, s > a$$

$$L\{t^n\} = \Gamma\frac{(n+1)}{s^{n+1}};\, n > -1$$

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

Example:

Find the Laplace transform of

$f(t) = 5e^{-2t} - 3\sin(4t)$ for $t \geq 0$.

Solution:

$$
\begin{aligned}
F(s) &= L\{f(t)\} \\
&= L\left\{5e^{-2t} - 3\sin(4t)\right\} \\
&= 5L\left\{e^{-2t}\right\} - 3L\{\sin(4t)\} \\
&= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0
\end{aligned}
$$

THANK YOU

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