ENGINEERING MATHEMATICS - I

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UNIT 4 : Partial Differential Equations
Session : 8
Sub Topic : Solution of Homogeneous Linear Partial Differential Equations with Constant Coefficients

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Solution of Homogeneous Linear Partial Differential Equations with constant coefficients

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \dots\dots(1)$$

where $a_0, a_1, \dots, a_n$ are constants, is called **homogeneous linear PDE of the n^{th} order with constants coefficients.**

It is called **homogeneous** because all terms contain derivatives of the same order. Here, all the partial derivatives are of n^{th} order.

Solution of Homogeneous Linear Partial Differential Equations with constant coefficients

On writing, D for $\dfrac{\partial}{\partial x}$ and D' for $\dfrac{\partial}{\partial y}$.

(1) can be written as

$$\left(a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}\left(D'\right)^2 + \ldots. + a_n\left(D'\right)^n\right)z = F(x,y)$$

i.e., $F(D, D')z = F(x,y)$ ... ... (2)

Solution of Homogeneous Linear Partial Differential Equations with constant coefficients

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As in the case of ordinary linear differential equations with constant coefficients the complete solution of equation (2) consists of two parts, namely

(i) the complementary function (CF) which is the complete solution of the equation $F(D, D')z = 0$. It must contain n arbitrary functions, where n is the order of the differential equation.

(ii) the particular integral (PI) which is a particular solution of equation (2).

The complete solution of (2) is $\mathbf{z = CF + PI}$

Rules for finding complementary function

Consider the equation $$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0 \ldots\ldots\ldots (1)$$

which in symbolic form is .

$$(D^2 + a_1 DD' + a_2 (D')^2) z = 0 \ldots\ldots\ldots (2)$$

$$\div (D')^2$$

$$\Rightarrow \left(\left(\frac{D}{D'} \right)^2 + a_1 \frac{D}{D'} + a_2 \right) z = 0$$

Auxiliary equation is $m^2 + a_1 m + a_2 = 0$ where $m = \frac{D}{D'}$

Let its root be m_1, m_2

Rules for finding complementary function

Case(i): If the roots are real and distinct then equation (2) is equivalent to

$$(D - m_1 D')(D - m_2 D')z = 0\ldots..(3)$$

It will be satisfied by the solution of

$$\left(D - m_2 D'\right)z = 0 \Rightarrow p - m_2 q = 0$$

This is a Lagrange's linear differential equation and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

From first and second $\Rightarrow y + m_2 x = a$

From second and third $\Rightarrow\ z = b$

Therefore its solution is $\phi(y + m_2 x, z) = 0$ or $z = \phi(y + m_2 x)$

Rules for finding complementary function

Similarly (2) will also be satisfied by the solution of

$$(D - m_1 D')z = 0 \Rightarrow z = f(y + m_1 x)$$

Hence the complete solution of (1) is $\boldsymbol{z = f(y + m_1 x) + \phi(y + m_2 x)}$

Rules for finding complementary function

Example:

Solve: $(D^2 - D'^2)z = 0$

Solution:

Consider $(D^2 - D'^2)z = 0$

$\div (D')^2$

$$\left(\left(\frac{D}{D'}\right)^2 - 1\right)z = 0$$

Its auxiliary equation is $m^2 - 1 = 0$ where $m = \frac{D}{D'}$

$m = 1, -1$

Here the complete solution is $z = f_1(y + x) + f_2(y - x)$

Rules for finding complementary function

Case(ii): If the roots are equal i.e., $m_1 = m_2$ then (2) is equivalent to

$$(D - m_1 D')^2 z = 0 \ldots\ldots (3)$$

Putting $(D - m_1 D')z = u \ldots\ldots$ (4)

(3) becomes $(D - m_1 D')u = 0$

$$\Rightarrow u = \phi(y + m_1 x)$$

Therefore (4) takes the form $(D - m_1 D')z = \phi(y + m_1 x)$

or $(p - m_1 q) = \phi(y + m_1 x)$

This is again Lagrange's linear and the subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m_1} = \dfrac{dz}{\phi(y + m_1 x)}$

Rules for finding complementary function

Giving $y+m_1x=a$ and $dz=\phi(a)dx \Rightarrow z=\phi(a)x+b$

$$\therefore z-x\phi(y+m_1x)=f(y+m_1x)$$

That is $\boldsymbol{z=f(y+m_1x)+x\phi(y+m_1x)}$

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Example:

Solve: $(4D^2 + 12DD' + 9D'^2)z = 0$

Solution:

Consider $(4D^2 + 12DD' + 9D'^2)z = 0$

$\div (D')^2$

$$\left(4\left(\frac{D}{D'}\right)^2 + 12\frac{D}{D'} + 9\right)z = 0$$

Its auxiliary equation is $4m^2 + 12m + 9 = 0$ where $m = \frac{D}{D'}$

$$m = -\frac{3}{2}, -\frac{3}{2}$$

Here the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$

Rules for finding the Particular Integral

Consider the equation

$$(D^2 + a_1 DD' + a_2 (D')^2)z = F(x,y)$$

i.e., $F(D,D')z = F(x,y)$

$$\therefore \text{P.I.} = \frac{1}{F(D,D')} F(x,y)$$

Case (i) : When $F(x,y) = e^{ax+by}$

$$\text{P.I.} = \frac{1}{F(D,D')} e^{ax+by}$$

$$= \frac{1}{F(a,b)} e^{ax+by}; \quad F(a,b) \neq 0$$

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Case (ii) : When $F(x, y) = sin(ax + by)$ or $cos(ax + by)$

$$\mathrm{P.I.} = \frac{1}{F(D^2, DD', (D')^2)} \sin(ax + by)$$

$$\mathrm{P.I.} = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by)\,; F(-a^2, -ab, -b^2) \neq 0$$

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Case (iii) : When $F(x, y) = x^m y^n, m \ and \ n \ being \ constants$

$$\mathrm{P.I.} = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$$

(a) If $n < m, \frac{1}{F(D,D')}$ is expanded in powers of $\frac{D'}{D}$

(b) If $\mathrm{m} < n, \frac{1}{F(D,D')}$ is expanded in powers of $\frac{D}{D'}$

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Case (iv) : Exponential shift

When $F(x,y) = e^{ax+by}V(x,y), where\ V(x,y)\ is\ any\ function\ of\ x\ and\ y$

$$\text{P. I.} = \frac{1}{F(D,D')}e^{ax+by}V(x,y) = e^{ax+by}\frac{1}{F(D+a,D'+b)}V(x,y)$$

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Case (v): When $F(x, y)$ is any function of x and y

$$\text{P. I.} = \frac{1}{F(D, D')} V(x, y)$$

Resolve $\dfrac{1}{F(D, D')}$ into partial fractions considering $F(D, D')$ as a function of D alone and operate each partial fractions on $V(x, y)$ remembering that

$$\frac{1}{D - mD'} V(x, y) = \int V(x, c - mx)\,dx$$

where c is replaced by $y + mx$ after integration.

Case (i) : When $F(x, y) = e^{ax+by}$

$$\text{P. I.} = \frac{1}{F(D, D')} e^{ax+by}$$

$$= \frac{1}{F(a,b)} e^{ax+by}; \quad F(a, b) \neq 0$$

1. Find the general solution of the partial differential equation $(D^2 + DD' - 2(D')^2)z = 5e^{x+2y}$.

Solution:

Consider $(D^2 + DD' - 2(D')^2)z = 0$

$\div (D')^2$

$$\left(\left(\frac{D}{D'}\right)^2 + \frac{D}{D'} - 2\right)z = 0$$

Auxiliary equation is $m^2 + m - 2 = 0$

$m = 1, -2$

$\therefore\ CF = f_1(y + x) + f_2(y - 2x)$

$$PI = \frac{1}{D^2+DD'-2(D')^2} 5e^{x+2y}$$

$F(a,b) = F(1,2) = 1^2 + 1 * 2 - 2(2)^2 = -5$ (Replacing D by 1 and D' by 2)

$$\therefore PI = \frac{1}{-5} 5e^{x+2y}$$

$$PI = -e^{x+2y}$$

Therefore, the general solution of the given differential equation is given by

$$z = f_1(y+x) + f_2(y-2x) - e^{x+2y}$$

2. Solve: $(D^2 + 5DD' + 6(D')^2)z = e^{x-y}$.

Solution:

Consider $(D^2 + 5DD' + 6(D')^2)z = 0$

$\div (D')^2$

$$\left(\left(\frac{D}{D'}\right)^2 + 5\frac{D}{D'} + 6\right)z = 0$$

Auxiliary equation is $m^2 + 5m + 6 = 0$

$m = -2, -3$

$\therefore \; CF = f_1(y - 2x) + f_2(y - 3x)$

$$PI = \frac{1}{D^2+5DD'+6(D')^2}\, e^{x-y}$$

$F(a,b) = F(1,-1) = 1^2 + 5(1)(-1) + 6(-1)^2 = 2$ (Replacing D by 1 and D' by -1)

$$\therefore PI = \frac{1}{2}\, e^{x-y}$$

Therefore, the general solution of the given differential equation is given by

$$z = f_1(y-2x) + f_2(y-3x) + \frac{1}{2}\, e^{x-y}$$

THANK YOU

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