Unit-1 class-10

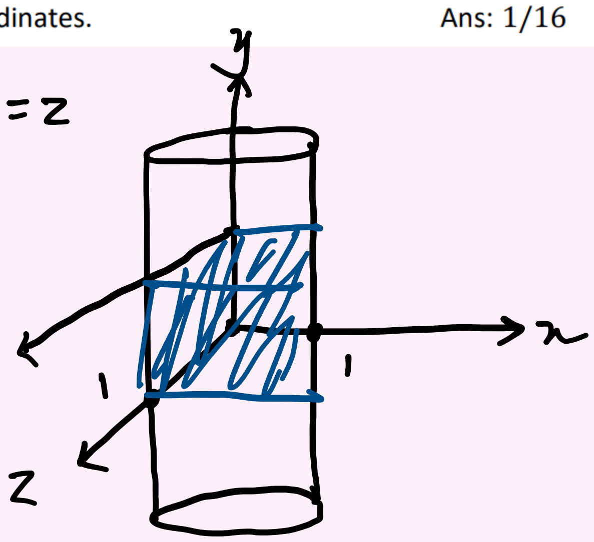
1. If R is the region bounded by $x = 0, y = 0, z = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$ evaluate $\iiint xyz\,dx\,dy\,dz$ by changing to cylindrical coordinates. Ans: 1/16

$x = r\cos\theta$; $y = r\sin\theta$; $z = z$

$z \to 0$ to 1

$r \to 0$ to 1

$\theta \to 0$ to $\pi/2$



$$\int_0^{\pi/2}\int_0^1\int_0^1 r^2\cos\theta\sin\theta \times z\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3\left[\frac{z^2}{2}\right]_0^1 \cos\theta\sin\theta\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\,d\theta$$

$t = \sin\theta \Rightarrow dt = \cos\theta\,d\theta$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/2 \Rightarrow t = 1$

$$= \int_0^1 \frac{1}{8}\times t\,dt = \left[\frac{t^2}{2}\right]_0^1 \times \frac{1}{8} = \frac{1}{16}$$

2. Evaluate $\iiint (x^2 + y^2)\,dx\,dy\,dz$ over the region bounded by the paraboloid $x^2 + y^2 = 3z$ and the plane $z = 3$ ans: $\frac{81\pi}{2}$

$x = r\cos\theta$; $y = r\sin\theta$

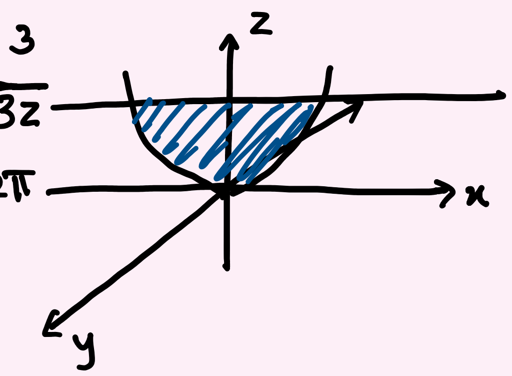
$x^2 + y^2 = r^2$

$dx\,dy\,dz = r\,dz\,dr\,d\phi$

$z \Rightarrow 0$ to 3

$r \Rightarrow 0$ to $\sqrt{3z}$

$\theta \Rightarrow 0$ to 2π



$$\int_0^{2\pi}\int_0^3\int_0^{\sqrt{3z}} r^2 \cdot r\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left[\frac{r^4}{4}\right]_0^{\sqrt{3z}} dz\,d\theta = \int_0^{2\pi}\int_0^3 \frac{9z^2}{4}\,dz\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{9z^3}{12}\right]_0^3 d\theta = \int_0^{2\pi}\frac{9\times\cancel{27}\,9}{\cancel{12}\,4}\times d\theta = \frac{81}{4}\times 2\pi = \frac{81\pi}{2}$$

3. Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{x^2 + y^2 + z^2}}$ over the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ $a > b > 0$ ans: $2\pi(a^2 - b^2)$

$x = r\sin\theta\cos\phi$

$y = r\cos\theta\sin\phi$

$z = r\cos\theta$

$x^2 + y^2 + z^2 = 1$

$r \to b$ to a

$\theta \to 0$ to π

$\phi \to 0$ to 2π

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\int_0^{2\pi}\int_0^{\pi}\int_b^a \frac{r^{\cancel{2}}\sin\theta\,dr\,d\theta\,d\phi}{\cancel{r}} = \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^2}{2}\right]_b^a \sin\theta\,d\theta\,d\phi$$

$$= \frac{a^2 - b^2}{2}\int_0^{2\pi}(-\cos\theta)_0^{\pi}\,d\phi = (a^2 - b^2)\pi\,(1+1) = 2\pi(a^2 - b^2)$$