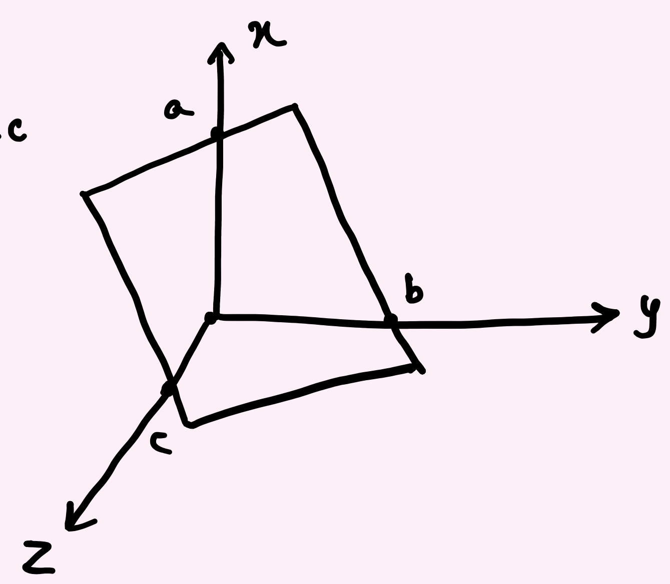
Unit-1 class-7

1. Find the volume of the tetrahedron bounded by co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ans: $\frac{abc}{6}$

A. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b} \Rightarrow z=\left(1-\frac{x}{a}-\frac{y}{b}\right)\cdot c$



On $x-y$ plane, $z=0$

$\frac{x}{a}+\frac{y}{b}=1$

$y=\left(1-\frac{x}{a}\right)\cdot b$

$z \Rightarrow 0$ to $\left(1-\frac{x}{a}-\frac{y}{b}\right)\cdot c$

$y \Rightarrow 0$ to $\left(1-\frac{x}{a}\right)\cdot b$

$x \Rightarrow 0$ to a

$\Rightarrow$ Volume $= \int_0^a \int_0^{(1-\frac{x}{a})b} \int_0^{(1-\frac{x}{a}-\frac{y}{b})c} dz\,dy\,dx$

$= \int_0^a \int_0^{(1-\frac{x}{a})b} \left[z\right]_0^{(1-\frac{x}{a}-\frac{y}{b})c} dy\,dx$

$= \int_0^a \int_0^{(1-\frac{x}{a})b} \left(1-\frac{x}{a}-\frac{y}{b}\right)c\; dy\,dx$

$= \int_0^a \left[cy-\frac{cxy}{a}-\frac{cy^2}{2b}\right]_0^{(1-\frac{x}{a})b}$

$= \int_0^a \left[\left(c-\frac{cx}{a}\right)\left(1-\frac{x}{a}\right)b-\frac{c}{2b}\left(1-\frac{x}{a}\right)^2 b^2\right]$

$= \int_0^a \left[\left(1-\frac{x}{a}\right)^2 bc-\frac{bc}{2}\left(1-\frac{x}{a}\right)^2\right]dx$

$= \int_0^a \frac{bc}{2}\left(1-\frac{x}{a}\right)^2 dx$

$= \frac{bc}{2}\int_0^a \left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)dx = \frac{bc}{2}\left(x+\frac{x^3}{3a^2}-\frac{x^2}{a}\right)_0^a$

$= \frac{bc}{2}\left(\cancel{a}-\cancel{a}+\frac{a^3}{3a^2}\right) = \frac{abc}{6}$

2. Find the volume of the region bounded by $z=x^2+y^2, z=0, x=-a, x=a, y=-a, y=a$ ans: $\frac{8a^4}{3}$

$z \to 0$ to x^2+y^2

$y \to -a$ to a

$x \to -a$ to a

$\int_{-a}^a\int_{-a}^a\int_0^{x^2+y^2} dz\,dy\,dx = \int_{-a}^a\int_{-a}^a (x^2+y^2)\,dy\,dx$

$= \int_{-a}^a \left(x^2y+\frac{y^3}{3}\right)_{-a}^a dx = \int_{-a}^a \left[ax^2+\frac{a^3}{3}-\left(-ax^2-\frac{a^3}{3}\right)\right]dx$

$= 2\int_{-a}^a \left(ax^2+\frac{a^3}{3}\right)dx = 2\left(\frac{ax^3}{3}+\frac{a^3x}{3}\right)_{-a}^a$

$= 2\left(\frac{a^4}{3}+\frac{a^4}{3}-\left(-\frac{a^4}{3}-\frac{a^4}{3}\right)\right)$

$= 2\left(\frac{4a^4}{3}\right) = \frac{8a^4}{3}$

3. Find the volume cut off from the cylinder $x^2+y^2=ax$ by the planes z=0 and z=x ans: $\frac{\pi a^3}{8}$

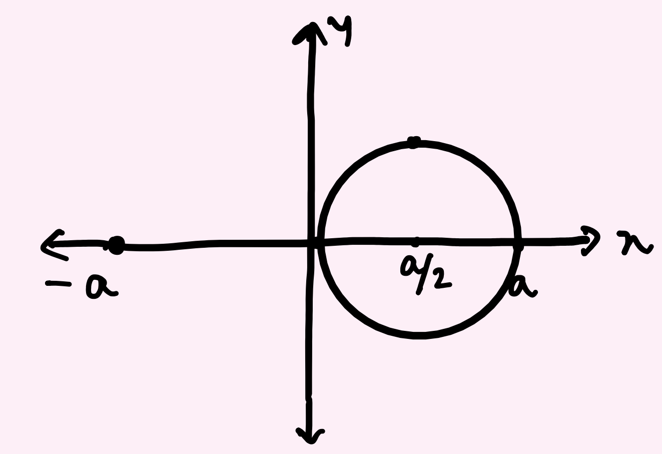
****

$x^2+y^2=ax \Rightarrow y=\sqrt{ax-x^2}$

$z \Rightarrow 0$ to x

$y \Rightarrow -\sqrt{ax-x^2}$ to $\sqrt{ax-x^2}$

$x \Rightarrow 0$ to a



$\int_0^a\int_0^{\sqrt{ax-x^2}}\int_0^x dz\,dy\,dx$

$\Rightarrow \int_0^a\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} x\,dy\,dx = \int_0^a 2x\sqrt{ax-x^2}\,dx = 2\int_0^a \sqrt{ax^3-x^4}\,dx$

$= 2\int_0^a x^{3/2}\sqrt{a-x}\,dx$

$u=\sqrt{x} \Rightarrow du=\frac{1}{2\sqrt{x}}dx \Rightarrow 2du.u=dx$

$\Rightarrow 2\int_0^{\sqrt{a}} u^3\sqrt{a-u^2}\,.\,2u\,du = 4\int_0^{\sqrt{a}} u^4\sqrt{a-u^2}\,du$

$u=\sqrt{a}\sin t \qquad du=\sqrt{a}\cos t\,dt$

$u=\sqrt{a} \Rightarrow t=\pi/2$

$u=0 \Rightarrow t=0$

$\Rightarrow 4\int_0^{\pi/2} a^2\sin^4 t\sqrt{a(1-\sin^2 t)}\,.\,\sqrt{a}\cos t\,dt$

$= 4\int_0^{\pi/2} a^3\sin^4 t\cos^2 t\,dt$

$= 4\int_0^{\pi/2} a^3\sin^4 t\left(1-\sin^2 t\right)dt$

$= 4a^3\int_0^{\pi/2}\left(\sin^4 t-\sin^6 t\right)dt$

$= 4a^3\int_0^{\pi/2}\left(\left(\frac{1+\cos 2t}{2}\right)^2-\left(\frac{1+\cos 2t}{2}\right)^3\right)dt$

$= 4a^3\int_0^{\pi/2}\left(\frac{1+\cos^2 2t+2\cos 2t}{4}-\frac{1+3\cos^2 2t+3\cos 2t+\cos^3 2t}{8}\right)dt$

$= \frac{4a^3}{8}\int_0^{\pi/2}\left[2+\cancel{2}\left(\frac{1+\cos 4t}{\cancel{2}}\right)+4\cos 2t-1-3\left(\frac{1+\cos 4t}{2}\right)-3\cos 2t-\cos^3 2t\right]dt$

$= \frac{a^3}{2}\int_0^{\pi/2}\left[\frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t-\cos 2t-\cos^3 2t\right]dt$

$= \frac{a^3}{2}\left[\frac{t}{2}-\cancelto{0}{\frac{\sin 4t}{8}}+\cancelto{0}{\frac{\sin 2t}{2}}-\cancelto{0}{\frac{\sin 2t}{2}}-\cancelto{0}{\frac{\sin 2t}{2}}+\cancelto{0}{\frac{\sin^3 2t}{6}}\right]_0^{\pi/2}$

$= \frac{a^3}{4}\times\left[\frac{\pi}{2}-0\right] = \frac{\pi a^3}{8}$