Partial Differential Equations

Definition of Partial Differential Equations

A partial DE is an equation involving two (or more) independent variables x, y and a dependent variable z and its partial derivatives such as $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$ etc.

Ex: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial y}\right)^3$$

Order and Degree of PDE

The order of a PDE is the order of the highest derivative appearing in the equation.

The degree of a PDE is the degree of the highest order PD.

Notations

$p = \frac{\partial z}{\partial x}$ $\qquad r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$

$q = \frac{\partial z}{\partial y}$ $\qquad s = \frac{\partial^2 z}{\partial x \partial y}$

General forms of first and second order PDE

An equation containing x, y, z, p, q defines a first order PDE, i.e., $f(x, y, z, p, q) = 0$.

An equation containing x, y, z, p, q, r, s, t, defines a second order PDE, i.e.,

$$f(x, y, z, p, q, r, s, t) = 0.$$

Formation of PDE

1) Elimination of arbitrary constants
2) Elimination of arbitrary functions.

Elimination of Arbitrary Constants

Consider a relation of the form:

$$f(x, y, z, a, b) = 0, \longrightarrow ①$$

where z is a function of x and y and a, b are constants.

Diff ① partially wrt x and y, we obtain a first order PDE.

Note:

i) If the number of arbitrary constants equals to the no. of independent variables in ①, then the PDE obtained by elimination is of first order.

It may not always be possible to eliminate the arbitrary constants, then we will find the second order partial derivatives and eliminate the arbitrary constants. Also, the higher order PDE may not be unique.

ii) If the number of constants exceeds the number of independent variables, then PDE of second order and higher orders arise.

Form a PDE by eliminating a constant from the relations :

1. $z = (x-a)^2 + (y-b)^2$

$$z = (x-a)^2 + (y-b)^2 \longrightarrow ①$$

Diff. p. wrt x,

$$\frac{\partial z}{\partial x} = 2(x-a) \Rightarrow p = 2(x-a)$$

$$\Rightarrow x-a = \frac{p}{2}$$

Diff. p wrt y,

$$\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b)$$

$$\Rightarrow y-b = \frac{q}{2}$$

Thus, ① becomes,

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$\Rightarrow \underline{4z = p^2 + q^2} \text{ (or)}$$

$$\Rightarrow 4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

2. $z = (x+a)(y+b)$

Given, $z = (x+a)(y+b) \longrightarrow ①$

Diff ① p wrt x,

$$\frac{\partial z}{\partial x} = p = (y+b)$$

Diff ① p. wrt y,

$$\frac{\partial z}{\partial y} = q = (x+a)$$

Thus, becomes ①,

$$\underline{z = pq} \text{ (or)}$$

$$\Rightarrow z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

3. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \longrightarrow ①$

Diff wrt x partially,

$\frac{\partial z}{\partial x} = 2\left(\frac{x}{a^2}\right) \Rightarrow p = 2\left(\frac{x}{a^2}\right) \Rightarrow \frac{1}{a^2} = \frac{p}{2x}$

Diff wrt y partially,

$\frac{\partial z}{\partial y} = 2\left(\frac{y}{b^2}\right) \Rightarrow q = 2\left(\frac{y}{b^2}\right) \Rightarrow \frac{1}{b^2} = \frac{q}{2y}$

Thus, ① becomes,

$z = x^2\left(\frac{p}{2x}\right) + y^2\left(\frac{q}{2y}\right)$

$z = \frac{px}{2} + \frac{qy}{2} \longrightarrow 2z = px + qy$

$2z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$

4. $z = ce^{-\omega t} \sin \omega x \longrightarrow ①$

Here, ω and c are constants

Diff p wrt x,

$\frac{\partial z}{\partial x} = (ce^{-\omega t})\, \omega \cos \omega x \longrightarrow ②$

Diff p wrt t,

$\frac{\partial z}{\partial t} = -\omega c e^{-\omega t} (\sin \omega x) \longrightarrow ③$

Diff ② p. wrt x,

$\frac{\partial^2 z}{\partial x^2} = (ce^{-\omega t})(-\omega^2 \sin \omega x) \longrightarrow ④$

Diff ③ p. wrt t,

$\frac{\partial^2 z}{\partial t^2} = (\omega^2 c e^{-\omega t})(\sin \omega x) \longrightarrow ⑤$

④ + ⑤

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0 \quad \longrightarrow r + t = 0$$

5. Form the PDE of the family of all the spheres whose centres lie on the xy-plane and have constant radius r.

The eqn of the sphere whose centres lie on xy-plane ($z=0$) and radius $= r$ is given by,

$$(x-a)^2 + (y-b)^2 + z^2 = r^2 \quad \longrightarrow ①$$

Diff p wrt x,

$$\Rightarrow 2(x-a) + 2z\left(\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow z\left(\frac{\partial z}{\partial x}\right) = -(x-a)$$

$$\Rightarrow (x-a) = -z\left(\frac{\partial z}{\partial x}\right) \quad \longrightarrow ②$$

Diff p wrt y,

$$\Rightarrow 2(y-b) + 2z\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow (y-b) = -z\left(\frac{\partial z}{\partial y}\right) \quad \longrightarrow ③$$

Putting ② in ③ in ①,

$$\Rightarrow \left[-z\left(\frac{\partial z}{\partial x}\right)\right]^2 + \left[-z\left(\frac{\partial z}{\partial y}\right)\right]^2 + z^2 = r^2$$

$$\Rightarrow z^2(p^2 + q^2 + 1) = r^2$$

6i) $z = xy + y\sqrt{x^2-a^2} + b \longrightarrow (1)$

Diff. p. wrt x,

$$\frac{\partial z}{\partial x} = y + y\left[\frac{1}{\cancel{2}}(x^2-a^2)^{-\frac{1}{2}}(\cancel{2}x)\right] + 0.$$

$$\frac{\partial z}{\partial x} = y + xy(x^2-a^2)^{-\frac{1}{2}} \longrightarrow (2)$$

Diff. p wrt y,

$$\frac{\partial z}{\partial y} = x + \sqrt{x^2-a^2} + 0 \longrightarrow (3).$$

$$\Rightarrow \frac{\partial z}{\partial y} - x = \sqrt{x^2-a^2}$$

Putting in (2),

$$\frac{\partial z}{\partial x} = y + xy\left(\frac{\partial z}{\partial y} - x\right)^{-1}$$

$$\Rightarrow p = y + \left(\frac{xy}{q-x}\right) \text{ or } \underline{pq = px + qy}$$

6ii) $z = ax + by + ab$

Partially diff wrt x, $p = a$

Partially diff. wrt y, $q = b$

Substituting in above eq,

$$\underline{z = px + qy + pq}$$

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow (1).$

Diff p wrt x,

$$2\left(\frac{x}{a^2}\right) + 2\left(\frac{z}{c^2}\right)\left(\frac{\partial z}{\partial x}\right) = 0.$$

$$\Rightarrow \frac{x}{a^2} = \frac{-z}{c^2}\left(\frac{\partial z}{\partial x}\right) \longrightarrow (2)$$

Diff p wrt y,

$$2\left(\frac{y}{b^2}\right) + 2\left(\frac{z}{c^2}\right)\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{y}{b^2} = \frac{-z}{c^2}\left(\frac{\partial z}{\partial y}\right) \longrightarrow (3).$$

Diff (2) wrt x p,

$$\Rightarrow \frac{1}{a^2} = -\frac{1}{c^2}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z}{c^2}\left(\frac{\partial^2 z}{\partial x^2}\right).$$

$$\Rightarrow \frac{1}{a^2} = \frac{-1}{c^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + z\left(\frac{\partial^2 z}{\partial x^2}\right)\right] \longrightarrow (4)$$

From (1), $\frac{1}{a^2} = \left(\frac{-z}{xc^2}\right)\left(\frac{\partial z}{\partial x}\right)$

Putting in (4),

$$\left(\frac{-z}{x\cancel{c^2}}\right)\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{\cancel{c^2}}\left[\left(\frac{\partial z}{\partial x}\right)^2 + z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]$$

$$z\left(\frac{\partial z}{\partial x}\right) = x\left[\left(\frac{\partial z}{\partial x}\right)^2 + z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]$$

$$\underline{\underline{zp = x[p^2 + zr]}}$$

$z = a \log(x^2+y^2) + b$

Diff p wrt x,

$$\frac{\partial z}{\partial x} = a\left(\frac{1}{x^2+y^2}\right)(2x) + 0$$

$$\Rightarrow p = \frac{2ax}{x^2+y^2} \longrightarrow (1)$$

Diff p wrt y,

$$\Rightarrow \frac{\partial z}{\partial y} = a\left(\frac{1}{x^2+y^2}\right)(2y)$$

$$\Rightarrow q = \frac{2ay}{x^2+y^2} \longrightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{p}{q} = \left(\frac{\cancel{2a}x}{\cancel{x^2+y^2}}\right)\left(\frac{\cancel{x^2+y^2}}{\cancel{2a}y}\right)$$

$$\Rightarrow \frac{p}{q} = \frac{x}{y}$$

$$\Rightarrow \underline{\underline{py - qx = 0}}$$

Solutions of PDE by direct integration method

Solve the eqn by DIM.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + a$$

Integrate on both sides wrt x,

$$\frac{\partial z}{\partial y} = \int\left(\frac{x}{y}\right)dx + \int a\,dx + f(y).$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y).$$

Integrating wrt y,

$$z = \int \left(\frac{x^2}{2y}\right) dy + \int (ax)\, dy + \int f(y)\, dy + g(x)$$

$$\Rightarrow z = \left(\frac{x^2}{2}\right) \log y + axy + F(y) + g(x)$$

where $F(y) = \int f(y)\, dy$

10. $$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \cos(2x + 3y)$$

Integrating wrt x,

$$\frac{\partial^2 z}{\partial x \partial y} = \int \cos(2x + 3y)\, dx + f(y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\sin(2x + 3y)}{2} + f(y)$$

Integrating wrt x,

$$\frac{\partial z}{\partial y} = \int \left[\frac{\sin(2x + 3y)}{2}\right] dx + \int f(y)\, d\cancel{y}^{x} + g(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\cos(2x + 3y)}{4} + x f(y) + g(y)$$

Integrating wrt y,

$$z = -\frac{\sin(2x + 3y)}{12} + x F(y) + G(y) + h(x)$$

where $F(y) = \int f(y)\, dy$ and $G(y) = \int g(y)\, dy$

11. $\frac{\partial^2 z}{\partial x \partial y} = (\sin x)(\sin y)$ given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$, $z = 0$ if y is an odd multiple of $\frac{\pi}{2}$.

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = (\sin x)(\sin y)$.

Integrating wrt x,

$\frac{\partial z}{\partial y} = \int (\sin x)(\sin y)\,dx + f(y)$.

$\Rightarrow \frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$. $\longrightarrow$ ①

Given, $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$,

$-2\sin y = -\cos x \sin y + f(y)$.

$-2\sin y = -\sin y + f(y)$

$f(y) = -\sin y$.

Thus, ① becomes,

$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$.

Integrate wrt y,

$z = \cos x \cos y + \cos y + g(x)$. $\longrightarrow$ ②

Given, y = odd multiple of $\frac{\pi}{2}$ like $\frac{\pi}{2}, \frac{3\pi}{2}$ etc,

when $z = 0$.

$0 = 0 + 0 + g(x) \Rightarrow g(x) = 0$.

Thus, ② becomes,

$z = (\cos x)(\cos y) + (\cos y)$

$\Rightarrow \underline{z = \cos y(\cos x + 1)}$

12. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, given that $u=0$ when $t=0$ and $\frac{\partial u}{\partial t} = 0$ at $x=0$.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t} \cos x$$

Integrating wrt x,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \longrightarrow (1)$$

Given, $\frac{\partial u}{\partial t} = 0$ when $x=0$,

$$0 + 0 = f(t) \Rightarrow f(t) = 0$$

Thus (1) becomes,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x$$

Integrating wrt t,

$$u = -e^{-t} \sin x + f(x). \longrightarrow (2)$$

Given, $u=0$ when $t=0$,

$$0 = -1 \sin x + f(x) \Rightarrow f(x) = \sin x$$

Thus, (2) becomes,

$$u = -e^{-t} \sin x + \sin x$$

$$\Rightarrow \underline{\underline{u = (e^{-t} + 1) \sin x}}$$

13. Solve $\frac{\partial^2 z}{\partial x^2} = xy$, given that $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$ and $z=0$, when $x=0$.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$$

Integrating wrt x,

$$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \longrightarrow ①$$

Given, $\frac{\partial z}{\partial x} = \log(1+y)$ when $x = 1$,

$$\Rightarrow \log(1+y) = \frac{y}{2} + f(y).$$

$$\Rightarrow f(y) = \log(1+y) - \frac{y}{2}$$

Thus, ① becomes,

$$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + \log(1+y) - \frac{y}{2}$$

Integrating wrt x,

$$z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2} + g(y) \longrightarrow ②$$

Given, $z = 0$ when $x = 0$.

$$\Rightarrow 0 = 0 + 0 + 0 + g(y) \Rightarrow g(y) = 0.$$

Thus, ② becomes,

$$z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2}$$

14. Solve $\frac{\partial^2 u}{\partial x^2} = x + y$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x + y$$

Integrating wrt x,

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + xy + f(y).$$

Integrating wrt x,

$$u = \frac{x^3}{6} + \frac{1}{2}x^2 y + x f(y) + g(y).$$

- $$\frac{\partial^2 z}{\partial y^2} = \sin xy$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \sin xy$$

Integrating wrt y,

$$\frac{\partial z}{\partial y} = \frac{-\cos xy}{x} + f(x)$$

Integrating wrt y,

$$z = -\frac{\sin xy}{x^2} + y f(x) + g(x).$$

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15. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$, given $\frac{\partial z}{\partial x} = x$ when $y=1$ and $z=0$ when $x=0$.

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y} + a$$

Integrating wrt ~~x~~ y,

$$\frac{\partial z}{\partial x} = x \log y + ya + f(x) \longrightarrow (1)$$

Given, $\frac{\partial z}{\partial x} = x$ and $y = 1$,

$$x = x \log y + ya + f(x). \Rightarrow f(x) = x - a$$

Thus, (1) becomes, $\frac{\partial z}{\partial x} = x \log y + ya + x - a$

Integrating wrt x,

$$z = \frac{x^2}{2} \log y + xya + \frac{x^2}{2} - xa + g(y). \rightarrow (2)$$

Given, $z=0$ when $x=0$.

$$0 = 0 + 0 + 0 + 0 + g(y) \Rightarrow g(y) = 0.$$

Thus (2) becomes,

$$z = \frac{x^2}{2} \log y + xya + \frac{x^2}{2} - xa$$
