unit-2 formulas

Laplace Transform

→ Type of integral transform to solve physical problems

→ Let $f(t)$ be defined for $t \geq 0$ & Let s be real/complex number, then Laplace transform of $f(t)$ is $F(s)$ where

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt = L[f(t)]$$

Inverse Laplace Transform

→ If $L[f(t)] = F(s)$ then $f(t)$ is inverse lapace transform of $F(s)$

$$f(t) = L^{-1}[F(s)]$$

Laplace of Standard functions

→ $L(1) = \frac{1}{s}$ $\qquad L^{-1}\left[\frac{1}{s}\right] = 1$

$$L(1) = \int_0^\infty e^{-st} = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-1}{-s} = \frac{1}{s}$$

→ $L(e^{at}) = \frac{1}{s-a}$ $\qquad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$L(\cos at) = \frac{s}{s^2+a^2}$ $\qquad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$L(\sin at) = \frac{a}{s^2+a^2}$ $\qquad L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

$L(\cosh at) = \frac{s}{s^2-a^2}$ $\qquad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$L(\sinh at) = \frac{a}{s^2-a^2}$ $\qquad L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

$L(t^n) = \frac{\boxed{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}$ $\qquad L^{-1}\left[\frac{\boxed{n+1}}{s^{n+1}}\right] = t^n$

Linearity Property

→ $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$

Then $L[c_1 f(t) + c_2 g(t)] = c_1 F(s) + c_2 G(s)$

First Shift / Translation Theorem

→ $L[e^{at} f(t)] = F(s-a)$

→ $L(Ke^{at}) = \frac{K}{s-a}$

$L(e^{bt}\cos at) = \frac{s-b}{(s-b)^2+a^2}$

$L(e^{bt}\sin at) = \frac{a}{(s-b)^2+a^2}$

$L(e^{bt}\cosh at) = \frac{s-b}{(s-b)^2-a^2}$

$L(e^{bt}\sinh at) = \frac{a}{(s-b)^2-a^2}$

$L(e^{bt}t^n) = \frac{\boxed{n+1}}{(s-b)^{n+1}} = \frac{n!}{(s-b)^{n+1}}$

Scaling Property

→ $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Laplace Transform of Derivative

→ $L[f^n(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{n-1}(0)$

ex: $L[f'(t)] = sF(s) - f(0)$

$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

ex: $y'' + 4y = 0$; $y'(0) = 6$; $y(0) = 1$

$L[y''] + 4L[y] = L[0]$

$s^2F(s) - sy(0) - y'(0) + 4F(s) = 0$

$(s^2+4)F(s) = s(1) + 6$

$F(s) = \frac{s+6}{s^2+4}$

$f(t) = L^{-1}\left[\frac{s+6}{s^2+4}\right]$

$= \cos 2t + \frac{6}{2}\sin 2t$

Laplace Transform of Integrals

→ If $L[f(t)] = F(s)$ then,

$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

$L\left[\int_0^t\int_0^t f(t)\,dt\,dt\right] = \frac{F(s)}{s^2}$

$L\left[\int_0^t\int_0^t\int_0^t f(t)\,dt\,dt\,dt\right] = \frac{F(s)}{s^3}$

$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t\int_0^t f(t)\,dt\,dt$

Laplace Transform – multiplication by t^n

→ If $L[f(t)] = F(s)$

then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$ $\Rightarrow$ $L^{-1}[s^n F(s)] = f^n(t)$ ↓ derivatives

if $f(0) = f'(0) = f''(0) = \dots = 0$

ex: $L[tf(t)] = -\frac{d}{ds}F(s)$

$L[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$

$L[t^3 f(t)] = -\frac{d^3}{ds^3}F(s)$

Inverse Laplace Transform of log & ITF functions

→ $L^{-1}[F(s)] = f(t)$ & $L[tf(t)] = -\frac{d\,L[f(t)]}{dt} = -F'(s)$

Then, $L^{-1}[-F'(s)] = tf(t)$

ex: $L^{-1}\left[\cot^{-1}\left(\frac{s}{a}\right)\right]$

$F(s) = \cot^{-1}\left(\frac{s}{a}\right)$

$\frac{dF(s)}{ds} = \frac{-1}{1+\left(\frac{s}{a}\right)^2} \cdot \frac{1}{a} \Rightarrow \frac{-d}{ds}F(s) = \frac{a}{s^2+a^2}$

$L^{-1}\left[\frac{-d}{ds}F(s)\right] = L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$tf(t) = \sin at$

$f(t) = \frac{\sin at}{t}$

Laplace Transform – division by t

→ $L[f(t)] = F(s)$

then, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

$\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

$\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

Laplace Transform of periodic functions

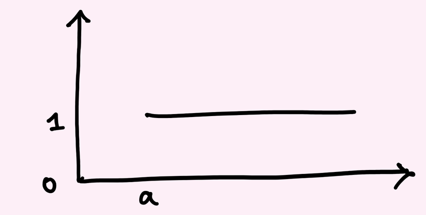
→ $f(t)$ be a periodic function

$$L[f(t)] = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-st} f(t)\,dt \qquad (T = \text{Time period})$$

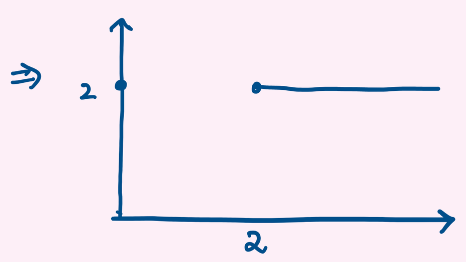
$$\frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \qquad (\text{Geometric Progression})$$

Unit Step Function

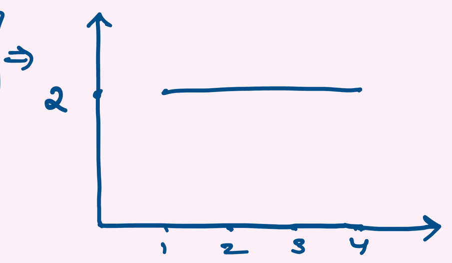
→ $u(t-a)$ or $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$



ex: $2u(t-2) \Rightarrow$



$2[u(t-2) - u(t-4)] \Rightarrow$



Transform of Heaviside unitstep function

→ $L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\,dt$

$= \int_0^a e^{-st}\, 0\,dt + \int_0^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$

$L[u(t-a)] = \frac{e^{-as}}{s}$

Heaviside Shift Theorem

→ $L[f(t)] = F(s)$

then $L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)]$

→ $L[f(t)\,u(t-a)] = e^{-as} L[f(t+a)]$

Inverse Laplace of Heaviside Unitstep function

$L^{-1}[e^{-as}F(s)] = f(t-a)\,u(t-a)$

Convolution Theorem

If $L^{-1}[F(s)] = f(t)$ & $L^{-1}[G(s)] = g(t)$

then $L^{-1}[F(s) \cdot G(s)] = \int_0^t f(u) \cdot g(t-u)\,du$

Dirac Delta / Unit Impulse Function

→ $\delta(t) = \begin{cases} \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$

$\underset{t \to 0}{Lt}\ \delta_e(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$

→ $L[\delta(t-a)] = \int_0^\infty e^{-at}\,\delta_e(t-a) = e^{-as}$

$L\left[\frac{\delta(t-a)}{t}\right] = \frac{e^{-as}}{s}$