Taylor's and Maclaurin's Series of a function of Two variables / One variable

(FOR SINGLE VARIABLE)

Taylor Series

If a function $f(x)$ has continuous derivatives of order $(n-1)$, then this function can be expressed as a polynomial at a point $x=a$ in the form:

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^n(a)}$$

Maclaurin's Series

If the function is expanded about the origin, we get M. series.

$$\boxed{f(x) = f(0) + (x)f'(0) + \frac{x^2}{2!}f''(0) + \dots \frac{(x)^n}{n!}f^n(0) + \dots}$$

42. Find Taylor series for $f(x) = e^x$ at $x=0$, upto fourth degree terms. (OR) Find Maclaurin's series for $f(x) = e^x$ upto fourth degree terms.

Taylor's series expansion:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f''''(a)$$

Here, $a=0$, $f(x)=e^x$,

$$e^x = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0)$$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$f(x) = e^x$

$f'(x) = f''(x) = f'''(x) = f''''(x) = e^x$

$f'(a) = f''(a) = f'''(a) = f''''(a) = e^a$

At $x=0$, $f(0) = f''(0) = f'''(0) = f''''(0) = 1$

(FOR TWO VARIABLES)

Taylor's series of $f(x, y)$ about (a, b) is given by:

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x-a) f_x(a, b) + (y-b) f_x(a, b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] \ldots$$

Maclaurin's series expansion for function of 2 variable

$$f(x, y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)]$$

$$+ \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \ldots$$

3. Expand $f(x, y) = \sin x \sin y$ in Taylor's series up to second order terms about point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

Taylor's series expansion is given by,

$$f(x,y) = f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \frac{1}{1!}\left[\left(x-\frac{\pi}{4}\right) f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right) f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right] + \frac{1}{2!}\left[\left(x-\frac{\pi}{4}\right)^2 f_{xx}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right) f_{xy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right] \longrightarrow (1).$$

Given, $f(x,y) = \sin x \sin y$

$\Rightarrow f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$\Rightarrow f_x = \cos x \sin y \Rightarrow f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}$

$\Rightarrow f_y = \cos y \sin x \Rightarrow f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}$

$\Rightarrow f_{xx} = -\sin x \sin y \Rightarrow f_{xx}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{2}$

$\Rightarrow f_{yy} = -\sin y \sin x \Rightarrow f_{yy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{2}$

$\Rightarrow f_{xy} = \cos x \cos y \Rightarrow f_{xy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}$

Putting in (1),

$$\sin x \sin y = \frac{1}{2} + \left(x-\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \left(y-\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[\left(x-\frac{\pi}{4}\right)^2\left(\frac{-1}{2}\right) + 2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \left(y-\frac{\pi}{4}\right)^2\left(-\frac{1}{2}\right)\right]$$

$$\Rightarrow \sin x \sin y = \frac{1}{2}\left[1 + \left(x-\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right) - \left[\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 + \left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right)^2\right]\right]$$

44. Expand $f(x,y) = \sin(x+2y)$ in Taylor's series up to third degree about point $(0,0)$.

Taylor's series expansion is given by,

$$\sin(x+2y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0)\right.$$ → ①.

$$\left. + 3xy^2 f_{xyy}(0,0)\right].$$

given, $f(x,y) = \sin(x+2y)$

$\Rightarrow f(0,0) = 0$

$\Rightarrow f_x = \cos(x+2y)(1)$ $\quad\Rightarrow f_x(0,0) = 1$

$\Rightarrow f_y = \cos(x+2y)(2)$ $\quad\Rightarrow f_y(0,0) = 2$

$\Rightarrow f_{xx} = -\sin(x+2y)$ $\quad\Rightarrow f_{xx}(0,0) = 0$

$\Rightarrow f_{yy} = -2\sin(x+2y)(2)$ $\quad\Rightarrow f_{yy}(0,0) = 0$

$\Rightarrow f_{xy} = -\sin(x+2y)(2)$ $\quad\Rightarrow f_{xy}(0,0) = 0$

$\Rightarrow f_{xxx} = -\cos(x+2y)$ $\quad\Rightarrow f_{xxx}(0,0) = -1$

$\Rightarrow f_{yyy} = -4\cos(x+2y)(2)$ $\quad\Rightarrow f_{yyy}(0,0) = -8$.

$f_{xxy} = -2\cos(x+2y) \Rightarrow f_{xxy}(0,0) = -2$

$f_{xyy} = -4\cos(x+2y) \Rightarrow f_{xyy} = -4.$

Putting in ①,

$\sin(x+2y) = 0 + x(1) + y(2) +$

$\frac{1}{2!}[x^2(0) + 2xy(0) + y^2(0)] +$

$\frac{1}{3!}[x^3(-1) + y^3(-8) + 3x^2y(-2) + 3xy^2(-4)]$

$\sin(x+2y) = x + 2y + \frac{1}{2}(0) + \frac{1}{6}[-x^3 - 8y^3 - 6x^2y - 12xy^2]$

$\Rightarrow \sin(x+2y) = x + 2y + \frac{1}{6}[(-x^3 - 8y^3 - 6x^2y - 12xy^2)]$

$\Rightarrow \sin(x+2y) = (x+2y) - \left(\frac{x^3}{6} + \frac{4}{3}y^3 + x^2y + 2xy^2\right)$

45. Compute $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately.

$f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$ for point $(a,b) = (1,1)$.

$f(x,y) = f(1,1) + \frac{1}{1!}[(x-1)f_x(1,1) + (y-1)f_y(1,1)]$

$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy} + (y-1)^2 f_{yy}(1,1)]$ ⟶ ①

$f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$ (given)

$\Rightarrow f(1,1) = \frac{\pi}{4}.$

$$f_x = \left[\frac{1}{1+\left(\frac{x}{y}\right)^2}\right]\left(\frac{1}{y}\right) \Rightarrow f_x(1,1) = \frac{1}{2}$$

$$f_y = \left[\frac{1}{1+\left(\frac{x}{y}\right)^2}\right]\left(-\frac{x}{y^2}\right) \Rightarrow f_y(1,1) = -\frac{1}{2}$$

$\cancel{f_{xy} = \frac{1}{1+\frac{x}{y}}}$ $f_x = \frac{-y}{x^2+y^2}$ | $f_y = \frac{-x}{x^2+y^2}$

$f_{xx} = \frac{-y(2x)}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \frac{-2}{4} = \frac{-1}{2}$

$f_{yy} = \frac{+x(2y)}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{2}{4} = \frac{1}{2}$

$f_{xy} = \frac{-1}{x^2+y^2} + \frac{(+x)(2x)}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1) = \frac{-1}{2} + \frac{2}{4} = 0$

Putting in ①,

$$f(x,y) = \frac{\pi}{4} + 1\left[(x-1)\left(\frac{1}{2}\right) + (y-1)\left(\frac{-1}{2}\right)\right]$$

$$+ \frac{1}{2}\left[(x-1)^2\left(\frac{-1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{x}{y}\right) = \frac{\pi}{4} + \frac{1}{2}\left[(x-1)-(y-1)\right] + \frac{1}{4}\left[(x-1)^2 + (y-1)^2\right]$$

Putting $x = 0.9$ and $y = 1.1$, we get

$$\Rightarrow \tan^{-1}\left(\frac{0.9}{1.1}\right) = \frac{\pi}{4} + \frac{1}{2}\left[-0.1-0.1\right] + \frac{1}{4}\left[(-0.1)^2 + (0.1)^2\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{0.9}{1.1}\right) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{\cancel{4}_2}(0.01)\cancel{(2)}$$

$$\Rightarrow \tan^{-1}\left(\frac{0.9}{1.1}\right) = \frac{\pi}{4} - 0.1 + \frac{0.01}{2}$$

$$= \frac{\pi}{4} - 0.095$$

$$= \underline{\underline{0.690}}$$

6. Find Taylor's series expansion of $f(x,y) = \tan^{-1}(xy)$ and compute approx. value of $f(0.9, -1.2)$

$f(x,y) = \tan^{-1}(xy)$

$\Rightarrow f(x,y) = f(1,-1) + \frac{1}{1!}[(x-1) f_x(1,-1) + (y+1) f_y(1,-1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,-1) + 2(x-1)(y+1) f_{xy}(1,-1) + (y+1)^2 f_{yy}(1,-1)]$ $\longrightarrow$ ①

given, $f(x,y) = \tan^{-1}(xy)$

$f(1,-1) = \tan^{-1}(1) = -\frac{\pi}{4}$

$f_x = \frac{1\,(y)}{1+x^2y^2} \Rightarrow f_x(1,-1) = -\frac{1}{2}$

$f_y = \frac{1\,(x)}{1+x^2y^2} \Rightarrow f_y(1,-1) = \frac{1}{2}$

$f_{xx} = \left(\frac{0y}{1+x^2y^2}\right) + \left(\frac{-y}{(1+x^2y^2)^2}\right)(2y^2x) \Rightarrow f_{xx}(1,-1) = \frac{1}{2}$

$f_{xy} = \left(\frac{1}{1+x^2y^2}\right) + \frac{x\,(-1)}{(1+x^2y^2)^2}(2xy^2) \Rightarrow f_{xy}(1,-1) = 0$

$f_{yy} = \left(\frac{0x}{1+x^2y^2}\right) + \frac{x\,(-1)}{(1+x^2y^2)^2}(2x^2y) \Rightarrow f_{yy}(1,-1) = \frac{1}{2}$

$\therefore \tan^{-1}(xy) = -\frac{\pi}{4} + [-(x-1) + (y+1)]\frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2\right]$

$= -\frac{\pi}{4} + \frac{1}{2}[(x-1) + (y+1)] + \frac{1}{4}[(x-1)^2 + (y-1)^2]$

Putting $x = 0.9$, $y = -1.2$, we get.

$\tan^{-1}(-1.08) = -0.823$

7. Expand $f(x,y) = \log(2x+y+1)$ about $(0,0)$ upto three terms.

Given $f(x,y) = \log(2x+y+1)$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x^{(0,0)} + y f_y^{(0,0)}\right] + \frac{1}{2!}\left[x^2 f_{xx}^{(0,0)} + y^2 f_{yy}^{(0,0)} + 2xy f_{xy}^{(0,0)}\right] + \frac{1}{3!}\left[x^3 f_{xxx}^{(0,0)} + y^3 f_{yyy}^{(0,0)} + 3x^2y f_{xxy}^{(0,0)} + 3xy^2 f_{xyy}^{(0,0)}\right] \rightarrow (1)$$

$f(x,y) = \log(2x+y+1)$ (given)

$\Rightarrow f(0,0) = \log 1 = 0$

$f_x = \left[\frac{1}{2x+y+1}\right](2) \Rightarrow f_x(0,0) = 2$

$f_y = \left[\frac{1}{2x+y+1}\right](1) \Rightarrow f_y(0,0) = 1$

$f_{xx} = \left[\frac{-1(2)}{(2x+y+1)^2}\right](2) \Rightarrow f_{xx}(0,0) = \frac{-4}{1} = -4$

$f_{xy} = \left[\frac{-1}{(2x+y+1)^2}\right](2) \Rightarrow f_{xy}(0,0) = -1(2) = -2$

$f_{yy} = \left[\frac{-1}{(2x+y+1)^2}\right](1) \Rightarrow f_{yy}(0,0) = -1$

$f_{xxx} = (-4)\left[\frac{-2}{(2x+y+1)^3}\right](2) \Rightarrow f_{xxx}(0,0) = \frac{(16)}{1} = 16$

$f_{xxy} = (-2)\left[\frac{-2}{(2x+y+1)^3}\right](2) \Rightarrow f_{xxy}(0,0) = 8$

$f_{xyy} = (-1)\left[\frac{-2}{(2x+y+1)^3}\right](2) \Rightarrow f_{xyy}(0,0) = 2(2) = 4$

$f_{yyy} = (-1)\left[\frac{-2}{(2x+y+1)^3}\right](1) \Rightarrow f_{yyy}(0,0) = 2$

$$\therefore \log(2x+y+1) = (2x+y) - \left(2x^2 + 2xy + \frac{y^2}{2}\right) + \frac{8}{3}x^3 + \frac{y^3}{3} + 4x^2y + 2xy^2$$

48. Expand $f(x,y) = \log(1+x-y)$ about $(0,0)$ about to three terms.

$f(x,y) = \log(1+x-y)$

$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0)$

$+ y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2 y f_{xxy}(0,0) +$

$3xy^2 f_{xyy}(0,0)]$

Given $f(x,y) = \log(1+x-y)$

$\rightarrow f(0,0) = \log(1) = 0$

$f_x = \left[\frac{1}{1+x-y}\right](1) \Rightarrow f_x(0,0) = 1$

$f_y = \left[\frac{1}{1+x-y}\right](-1) \Rightarrow f_y(0,0) = -1$

$f_{xx} = \left[\frac{-1}{(1+x-y)^2}\right](1) \Rightarrow f_{xx}(0,0) = -1$

$f_{xy} = \left[\frac{+1}{(1+x-y)^2}\right](1) \Rightarrow f_{xy}(0,0) = 1$

$f_{yy} = \left[\frac{+1}{(1+x-y)^2}\right](-1) \Rightarrow f_{yy}(0,0) = -1$

$f_{xxx} = \left[\frac{+2}{(1+x-y)^3}\right](1) \Rightarrow f_{xxx}(0,0) = 2$

$f_{yyy} = \left[\frac{+2}{(1+x-y)^3}\right](-1) \Rightarrow f_{yyy}(0,0) = -2$

$f_{xyy} = \left[\frac{+2}{(1+x-y)^3}\right](-1) \Rightarrow f_{xyy}(0,0) = -2$

$f_{xxy} = \left[\frac{-2}{(1+x-y)^3}\right](1) \Rightarrow f_{xxy}(0,0) = -2$

$\therefore \log(1+x-y) = (x - y + xy) - \frac{1}{2}(x^2+y^2) + \frac{1}{3}(x^3-y^3) - (x^2y + xy^2)$

49. Find the Taylor's expansion of $\sqrt{1+x+y^2}$ in powers of $(x-1)$ and $(y-0)$ upto second degree terms.

Let $f(x,y) = \sqrt{1+x+y^2}$

Taylor's expansion series of $f(x,y)$ about $(1,0)$:

$$\sqrt{1+x+y^2} = f(1,0) + \frac{1}{1!}\left[(x-1)f_x(1,0) + (y-0)f_y(1,0)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,0) + 2(x-1)(y-0)f_{xy}(1,0) + (y-0)^2 f_{yy}(1,0)\right]. \longrightarrow (1)$$

Given $f(x,y) = \sqrt{1+x+y^2}$

$f(1,0) = \sqrt{2}$

$f_x = \frac{1}{2}\left(\frac{1}{\sqrt{1+x+y^2}}\right)(0+1+0) \Rightarrow f_x(1,0) = \frac{1}{2\sqrt{2}}$

$f_y = \frac{1}{2}\left(\frac{1}{\sqrt{1+x+y^2}}\right)(0+0+2y) \Rightarrow f_y(1,0) = 0$

$f_{xx} = \frac{1}{2}\left[\left(-\frac{1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}\right] \Rightarrow f_{xx}(1,0) = -2^{-\frac{7}{2}}$

$f_{yy} = \frac{1}{2}\left[\left(-\frac{1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}\right](2y) + \frac{1}{\sqrt{1+x+y^2}} \Rightarrow f_{yy}(1,0) = 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$f_{xy} = \frac{1}{2}\left[\left(-\frac{1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}\right](2y) \Rightarrow f_{xy}(1,0) = 0.$

Putting in (1),

$$\sqrt{1+x+y^2} = \sqrt{2} + \frac{1}{1}\left[(x-1)\left(\frac{1}{2\sqrt{2}}\right) + y\cancel{\left(\frac{1}{2\sqrt{2}}\right)}\right]$$
$$+ \frac{1}{2}\left[(x-1)^2\left(-2^{-\frac{7}{2}}\right) + 2(x-1)y(0) + y^2\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$\sqrt{1+x+y^2} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x+y-1) - (x-1)^2\, 2^{-\frac{7}{2}} + \frac{y}{\sqrt{2}}$$

50. Expand $e^{ax} \sin by$ about origin upto third degree term.

$f(x, y) = e^{ax} \sin by.$

$\Rightarrow f(x, y) = f(0, 0) + \frac{1}{1!}\left[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)\right]$

$+ \frac{1}{2!}\left[(x-0)^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + (y-0)^2 f_{yy}(0, 0)\right]$

$+ \frac{1}{3!}\left[x^3 f_{xxx}(0, 0) + y^3 f_{yyy}(0, 0) + 3x^2 y\, f_{xxy}(0, 0)\right.$

$\left. + 3xy^2 f_{xyy}(0, 0)\right]. \longrightarrow (1)$

Given $f(x, y) = e^{ax} \sin by$

$f(0, 0) = \sin(0b)\, e^{0x} = 1(0) = 0$

$f_x = e^{ax}(a) \sin(by) \Rightarrow f_x(0, 0) = 0$

$f_y = e^{ax}(\cos by)(b) \Rightarrow f_y(0, 0) = e^0 (\cos 0) b = b$

$f_{xx} = a^2 e^{ax} \sin(by) \Rightarrow f_{xx}(0, 0) = 0.$

$f_{yy} = -b^2 e^{ax}(+\sin by) \Rightarrow f_{yy}(0, 0) = 0$

$f_{xy} = ab\, e^{ax}(\cos by) \Rightarrow f_{xy}(0, 0) = ab$

$f_{xxx} = a^3 e^{ax} \sin(by) \Rightarrow f_{xxx}(0, 0) = 0$

$f_{yyy} = -b^3 e^{ax}(\cos by) \Rightarrow f_{yyy}(0, 0) = -b^3$

$f_{xxy} = a^2 b\, e^{ax}(\cos by) \Rightarrow f_{xxy}(0, 0) = a^2 b$

$f_{xyy} = -ab^2 e^{ax}(+\sin by) \Rightarrow f_{xyy}(0, 0) = 0$

Putting in ①;

$\Rightarrow e^{ax}\sin(by) = \frac{1}{1!}[x(0) + y(b)] + \frac{1}{2!}[x^2(0) + 2xy(ab) + y^2(0)] + \frac{1}{3!}[x^3(0) + y^3(-b^3) + 3x^2y(a^2b) + 3xy^2(0)]$

$\Rightarrow e^{ax}\sin(by) = by + \frac{1}{2}[2xy\,ab] + \frac{1}{6}[-y^3b^3 + 3x^2y\,a^2b].$

$\Rightarrow e^{ax}\sin(by) = by + xy\,ab - \frac{y^3b^3}{6} + \frac{1}{2}x^2y\,a^2b$

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1. Use Taylor's theorem to expand $f(x,y) = x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$ upto third degree terms.

$\Rightarrow f(x,y) = f(1,2) + \frac{1}{1!}[(x-1)f_x(1,2) + (y-2)f_y(1,2)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2)f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,2) + (y-2)^3 f_{yyy}(1,2) + 3(x-1)^2(y-2)f_{xxy}(1,2) + 3(x-1)(y-2)^2 f_{xyy}(1,2)]$

$f(x,y) = x^2+xy+y^2$ (given)

$\Rightarrow f(1,2) = 1+2+4 = 7$

$f_x = 2x+y \Rightarrow f_x(1,2) = 4$

$f_y = x+2y \Rightarrow f_y(1,2) = 5$

$f_{xx} = 1 \Rightarrow f_{xx}(1,2) = 1$

$f_{xx} = 2 \quad \rightarrow f_{xx}(1,2) = 2$

$f_{yy} = 2 \quad \rightarrow f_{yy}(1,2) = 2$

$f_{xxx} = 0$

$f_{yyy} = 0$

$f_{xxy} = 0$

$f_{xyy} = 0$

Putting in ①,

$$x^2 + xy + y^2 = 7 + \frac{1}{1!}[(x-1)4 + (y-2)5]$$

$$+ \frac{1}{2!}[(x-1)^2(2) + (y-2)^2(2) + 2(x-1)(y-2)(1)]$$

$$\rightarrow x^2 + xy + y^2 = 7 + 4(x-1) + 5(y-2) + (x-1)^2 + (y-2)^2 + (x-1)(y-2)$$

Assignment - 7

1. Find Taylor's expansion of e^{xy} in terms of $(x-1)$ and $(y-1)$.

Given, $f(x,y) = e^{xy}$

$$f(x,y) = f(1,1) + \frac{1}{1!}[(x-1)f_x(1,1) + (y-1)f_y(1,1)]$$

$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)]$$

$$+ \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + (y-1)^3 f_{yyy}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1)]$$

$f(x) = e^{xy}$

$\Rightarrow f(1,1) = e$

$f_x = e^{xy}(y) \Rightarrow f_x(1,1) = e$

$f_y = e^{xy}(x) \Rightarrow f_x(1,1) = e$

$f_{xx} = y^2 e^{xy} \Rightarrow f_{xx}(1,1) = e$

$f_{xy} = e^{xy} + xe^{xy}(y) \Rightarrow f_{xy}(1,1) = e + e = 2e$

$f_{yy} = x^2 e^{xy} \Rightarrow f_{yy}(1,1) = e$

$f_{xxx} = y^3 e^{xy} \Rightarrow f_{xxx}(1,1) = e$

$f_{xyy} = 2xe^{xy} + x^2 e^{xy}(y) \Rightarrow f_{xyy}(1,1) = 3e$

$f_{xxy} = ye^{xy} + ye^{xy} + xye^{xy}(y) \Rightarrow f_{xxy}(1,1) = 3e$

$f_{yyy} = x^3 e^{xy} \Rightarrow f_{yyy}(1,1) = e$

$$\therefore e^{xy} = e\left[1 + (x-1) + (y-1) + \frac{(x-1)^2}{2} + \frac{(y-1)^2}{2}\right.$$

$$\left. + (x-1)(y-1) + (x-1)^3 + (y-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)\right]$$

2. Expand $e^x \ln(1+y)$ about origin upto 3^{rd} degree terms.

$f(x,y) = e^x \ln(1+y)$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2}\left[x^2 f_{xx}(0,0)\right.$$

$$\left. + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + \right.$$

$$\left. y^3 f_{yyy}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)\right].$$

①

Given $f(x) = e^{x} \ln(1+y)$.

$\Rightarrow f(0,0) = e^{0}\ln(1) = 0.$

$f_x = e^{x}(\log(1+y)) \quad \Rightarrow f_x(0,0) = 1(0) = 0$

$f_y = \left[\frac{e^{x}}{(1+y)}\right](1) \quad \Rightarrow f_y(0,0) = 1$

$f_{xx} = e^{x}\log(1+y) \quad \Rightarrow f_{xx}(0,0) = 0.$

$f_{yy} = \frac{-e^{x}}{(1+y)^2} \quad \Rightarrow f_{yy}(0,0) = -1$

$f_{xy} = \frac{e^{x}}{1+y} \quad \Rightarrow f_{xy}(0,0) = 1$

$f_{xxx} = e^{x}\log(1+y) \quad \Rightarrow f_{xxx}(0,0) = 0$

$f_{yyy} = \frac{+e^{x}(2)}{(1+y)^3} \quad \Rightarrow f_{yyy}(0,0) = 1(2) = 2$

$f_{xyy} = \frac{-e^{x}}{(1+y)^2} \quad \Rightarrow f_{xyy}(0,0) = -1$

$f_{xxy} = \frac{e^{x}}{1+y} \quad \Rightarrow f_{xxy}(0,0) = 1.$

Putting in ①,

$$e^{x}\ln(1+y) = x(0) + y(1) + \left[x^2(0) + y^2(-1) + 2xy(1)\right]\frac{1}{2}$$
$$+\left[x^3(0) + y^3(2) + 3x^2y(1) + 3xy^2(-1)\right]\frac{1}{6}$$

$$\Rightarrow e^{x}\ln(1+y) = y + \left[\frac{-y^2 + 2xy}{2}\right] + \left[\frac{2y^3 + 3(x^2y - xy^2)}{6}\right]$$

$$\Rightarrow e^{x}\ln(1+y) = y + xy - \frac{y^2}{2} + \frac{1}{2}(x^2y - xy^2) + \frac{y^3}{3} + \ldots$$

3. Find Taylor's expression of $\sqrt{1+x+y^2}$ in powers of $(x-1)$ and $(y-0)$.

$f(x,y) = \sqrt{1+x+y^2}$

$f(x,y) = f(1,0) + \frac{1}{1!}[(x-1)f_x(1,0) + y\, f_y(1,0)]$

$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,0) + y^2 f_{yy}(1,0) + 2y(x-1)f_{xy}(1,0)]$

$\longrightarrow (1)$.

Given $f(x,y) = \sqrt{1+x+y^2}$

$\Rightarrow f(1,0) = \sqrt{1+1+0} = \sqrt{2}$

$f_x = \frac{1}{2}\left(\frac{1}{\sqrt{1+x+y^2}}\right)(1) \Rightarrow f_x(0,0) = \frac{1}{2\sqrt{2}}$

$f_y = \frac{1}{2}\left(\frac{1}{\sqrt{1+x+y^2}}\right)(2y) \rightarrow f_{xy}(0,0) = 0.$

$f_{xx} = \frac{1}{2}\left[\left(-\frac{1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}\right] \rightarrow f_{xx_{(0,0)}} = -2^{-\frac{7}{2}}$

$f_{yy} = \frac{1}{\sqrt{1+x+y^2}} + y\left[\left(-\frac{1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}\right](2y) \Rightarrow f_{yy}(0,0) = \frac{1}{\sqrt{2}}$

$f_{xy} = \frac{y}{(x+1+y^2)^{\frac{3}{2}}}\left(-\frac{1}{2}\right) \rightarrow f_{xy}(0,0) = 0.$

$\therefore \sqrt{1+x+y^2} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) + 0 + \frac{1}{2}\left[(x-1)^2(-2^{-\frac{7}{2}}) + y^2\left(\frac{1}{\sqrt{2}}\right)\right]$

$\Rightarrow \sqrt{1+x+y^2} = \sqrt{2}\left[1 + \left(\frac{x-1}{4}\right) - \frac{(x-1)^2}{32} + \frac{y^2}{4} + \cdots\right]$

Assigment - 8

1. Expand $f(x,y) = \cos(2x+y) + 3\sin(x+y)$ about the origin upto second degree terms.

$f(x,y) = \cos(2x+y) + 3\sin(x+y)$

$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)]$

$+ \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$

$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2 y f_{xxy}(0,0)$

$+ 3xy^2 f_{xyy}(0,0)]$

$\longrightarrow (1)$

Given, $f(x,y) = \cos(2x+y) + 3\sin(x+y)$

$\Rightarrow f(0,0) = \cos 0 + 3\sin 0 = 1$

$f_x = -\sin(2x+y)(2) + 3\cos(x+y)(1)$

$= -2\sin(2x+y) + 3\cos(x+y)$ $\Rightarrow f_x(0,0) = 3$

$f_y = -\sin(2x+y) + 3\cos(x+y)$ $\Rightarrow f_y(0,0) = 3$

$f_{xx} = -2\cos(2x+y)(2) - 3\sin(x+y)$

$= -4\cos(2x+y) - 3\sin(x+y)$ $\Rightarrow f_{xx}(0,0) = -4$

$f_{yy} = -\cos(2x+y) - 3\sin(x+y)$ $\Rightarrow f_{yy}(0,0) = -1$

$f_{xy} = -2\cos(2x+y) - 3\sin(x+y)$ $\Rightarrow f_{xy}(0,0) = -2$

Putting in (1),

$\cos(2x+y) + 3\sin(x+y) = 3x + 3y - 2x^2 - 2xy - \frac{1}{2}y^2$

2. Find the Taylor's polynomial of degree 2 at the point $\left(1, \frac{\pi}{2}\right)$ for the function $f(x,y) = xy^2 + \cos xy$

Given $f(x,y) = xy^2 + \cos xy$

$$f(x,y) = f\left(1,\frac{\pi}{2}\right) + \frac{1}{1!}\left[(x-1) f_x\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right) f_y\left(1,\frac{\pi}{2}\right)\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(1,\frac{\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right) f_{xy}\left(1,\frac{\pi}{2}\right)\right] \rightarrow (1).$$

Given $f(x,y) = xy^2 + \cos xy.$

$$f\left(1,\frac{\pi}{2}\right) = 1\left(\frac{\pi^2}{4}\right) + \cos\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$f_x = y^2 - y\sin xy \quad \Rightarrow f_x\left(1,\frac{\pi}{2}\right) = \left(\frac{\pi^2}{4} - \frac{\pi}{2}\right)$

$f_y = 2xy - x\sin xy \quad \Rightarrow f_y\left(1,\frac{\pi}{2}\right) = (\pi - 1)$

$f_{xx} = 0 - y^2\cos xy \quad \Rightarrow f_{xx}\left(1,\frac{\pi}{2}\right) = 0.$

$f_{xy} = 2y - \sin xy - xy\cos xy \Rightarrow f_{xy}\left(1,\frac{\pi}{2}\right) = (\pi - 1)$

$f_{yy} = 2x - xy\cos xy \quad \Rightarrow f_{yy}\left(1,\frac{\pi}{2}\right) = 2$

$$\Rightarrow xy^2 + \cos xy = \frac{\pi^2}{4} + \left[(x-1)\left(\frac{\pi^2}{4} - \frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)(\pi-1)\right]$$

$$+ \frac{1}{2}\left[(x-1)^2(0) + \left(y-\frac{\pi}{2}\right)^2(2) + 2(x-1)\left(y-\frac{\pi}{2}\right)(\pi-1)\right]$$

$$\Rightarrow xy^2 + \cos xy = \frac{\pi^2}{4} + (x-1)\left(\frac{\pi^2}{4} - \frac{\pi}{2}\right) + (\pi-1)\left(y-\frac{\pi}{2}\right)$$

$$+ \left(y-\frac{\pi}{2}\right)^2 + (x-1)\left(y-\frac{\pi}{2}\right)(\pi-1)$$

3. Expand the function $f(x, y) = x^2 + xy - y^2$ by Taylor's theorem in powers of $(x-1)$ and $(y+2)$.

$f(x, y) = x^2 + xy - y^2$

$f(x,y) = f(1,-2) + \frac{1}{1!}\left[(x-1)f_x(1,-2) + (y+2)f_y(1,-2)\right]$

$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + (y+2)^2 f_{yy}(1,-2) + 2(x-1)(y+2)f_{xy}(1,-2)\right]$ $\longrightarrow$ ①

Given $f(x, y) = x^2 + xy - y^2$

$\Rightarrow f(1,-2) = 1 - 2 - 4 = -5$.

$f_x = 2x + y$ $\qquad\Rightarrow f_x(1,-2) = 0$.

$f_y = x - 2y$ $\qquad\Rightarrow f_y(1,-2) = 5$

$f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 1$

Putting in ①,

$x^2 + xy - y^2 = -5 + (x-1)(0) + (y+2)(5) +$

$\frac{1}{2}\left[(x-1)^2(2) + (y+2)^2(-2) + 2(x-1)(y+2)(1)\right]$.

$\Rightarrow x^2 + xy - y^2 = -5 + 5(y+2) + (x-1)^2 - (y+2)^2 + (x-1)(y+2)$.

Question - Answers

1. If $u = x^2 y$ and $x^2 + xy + y^2$, find $\frac{du}{dx}$.

$f(x, y) = x^2 + xy + y^2$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{du}{dx} = (2xy)(1) + x^2\left(\frac{dy}{dx}\right) \longrightarrow ①.$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(2x+y)}{(x+2y)}. \longrightarrow ②.$$

From ① and ②,

$$\Rightarrow \frac{du}{dx} = 2xy - x^2\left(\frac{2x+y}{x+2y}\right).$$

2. If $u = e^{xy}$ and $x + xy + y = 1$, find $\frac{du}{dx}$.

$f(x, y) = x + xy + y = 1$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right).$$

$$\Rightarrow \frac{du}{dx} = ye^{xy}(1) + xe^{xy}\left(\frac{dy}{dx}\right) \longrightarrow ①.$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\left(\frac{1+y}{x+1}\right) \longrightarrow ②$$

From ① and ②,

$$\frac{du}{dx} = ye^{xy} - xe^{xy}\left(\frac{1+y}{1+x}\right)$$

3. If x and y are connected by $x^2 - y^2 = 2$ and $u = \tan(x^2 + y^2)$, find $\frac{du}{dx}$.

$u = \tan(x^2 + y^2)$.

Let $f(x, y) = x^2 - y^2 = 2$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{du}{dx} = 2x \sec^2(x^2+y^2)(1) + \sec^2(x^2+y^2)(2y)\frac{dy}{dx} \longrightarrow ①.$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\left(\frac{2x}{-2y}\right) = \frac{2x}{2y} = \frac{x}{y} \longrightarrow ②$$

From ① and ②,

$$\Rightarrow \frac{du}{dx} = 2x \sec^2(x^2+y^2) + \cancel{2y} \sec^2(x^2+y^2)\left(\frac{x}{\cancel{y}}\right)$$

$$\Rightarrow \frac{du}{dx} = \underline{\underline{4x \sec^2(x^2+y^2)}}$$

1. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem/series.

Given, $f(x) = x^2y + 3y - 2$

$$f(x,y) = f(1,-2) + \frac{1}{1!}\left[(x-1) f_x(1,-2) + (y+2) f_y(1,-2)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + (y+2)^2 f_{yy}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2)\right] \longrightarrow ①$$

Given, $f(x) = x^2y + 3y - 2$

$\Rightarrow f(1,-2) = 1(-2) + 3(-2) - 2 = -6 - 4 = -10$

$f_x = 2xy \qquad \Rightarrow f_x(1,-2) = -2(2) = -4$

$f_y = x^2 + 3 \quad \Rightarrow f_y(1,-2) = 4.$

$f_{xx} = 2y \quad \Rightarrow f_y(1,-2) = -4.$

$f_{xy} = 2x \quad \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy} = 0.$

Putting in ①,

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(4)]$$
$$+ \frac{1}{2}[(x-1)^2(-4) + (y+2)^2(0) + 2(x-1)(y+2)(2)].$$

$$\Rightarrow \underline{x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2}$$
$$\underline{+ 2(x-1)(y+2)}$$

2. Explain $f(x,y) = e^x \sin y$ in powers of x and y upto the terms containing 3rd degree.

Using Maclaurin's series,

$$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] +$$
$$\frac{1}{3!}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)]$$

Given, $f(x) = e^x \sin y$

$\Rightarrow f(0,0) = 0.$

$f_x = e^x \sin y \quad \Rightarrow f_x(0,0) = 0$

$f_y = +e^x \cos y \quad \Rightarrow f_y(0,0) = 1.$

$f_{xx} = e^x \sin y \quad \Rightarrow f_{xx}(0,0) = 0$

$f_{yy} = -e^x \sin y \Rightarrow f_{yy}(0,0) = 0$

$f_{xy} = e^x \cos y \Rightarrow f_{xy}(0,0) = 1$

$f_{xxx} = e^x \sin y \Rightarrow f_{xxx}(0,0) = 0$

$f_{yyy} = -e^x \cos y \Rightarrow f_{yyy}(0,0) = -1$

$f_{xxy} = e^x \cos y \Rightarrow f_{xxy}(0,0) = 1$

$f_{xyy} = -e^x \sin y \Rightarrow f_{xyy}(0,0) = 0.$

Putting in ①,

$$e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \cdots$$