Euler's Theorem on Homogeneous Functions

Statement 1: If $u = f(x, y)$ is a homogeneous function of degree n, then:

$$\boxed{x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = nu}$$

Corollary 1: If $u = f(x, y)$ is a homogeneous function of degree n, then:

$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u}$$

Statement 2: If $u = f(x, y, z)$ is a homogeneous function of degree n, then:

$$\boxed{x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) = nu}$$

Deductions from Euler's Theorem

① If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then:

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{f(u)}{f'(u)}$$

② If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then:

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$.

31. If $u = \frac{y^{\frac{1}{3}} - x^{\frac{1}{3}}}{x+y}$, then find the value of

i) $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)$

ii) $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right)$

$u = \frac{y^{\frac{1}{3}} - x^{\frac{1}{3}}}{x+y}$

$\Rightarrow u = \frac{x^{\frac{1}{3}}\left[\left(\frac{y}{x}\right)^{\frac{1}{3}} - 1\right]}{x\left(1 + \left(\frac{y}{x}\right)\right)}$

$\Rightarrow u = \frac{x^{\frac{1}{3}-1}\left[\left(\frac{y}{x}\right)^{\frac{1}{3}} - 1\right]}{\left[1 + \left(\frac{y}{x}\right)\right]}$.

$\Rightarrow u = x^{-\frac{2}{3}}\, g\left(\frac{y}{x}\right)$.

$\therefore$ u is a homogeneous function of degree $= -\frac{2}{3}$.

Applying Euler's theorem to u,

i) $\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \underline{\underline{-\frac{2}{3}u}}$

ii) $\Rightarrow x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = -\frac{2}{3}\left(-\frac{2}{3}-1\right)u$

$= -\frac{2}{3}\left(-\frac{5}{3}\right)u$

$= \underline{\underline{\left(\frac{10}{9}\right)u}}$

If $u = \sqrt{y^2-x^2}\sin^{-1}\left(\frac{x}{y}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$$u = \sqrt{y^2-x^2}\sin^{-1}\left(\frac{x}{y}\right).$$

$$= \sqrt{y^2\left(1-\frac{x^2}{y^2}\right)}\sin^{-1}\left(\frac{x}{y}\right)$$

$$= y\sqrt{1-\left(\frac{x}{y}\right)^2}\sin^{-1}\left(\frac{x}{y}\right)$$

$$= y^1 g\left(\frac{x}{y}\right).$$

$\therefore$ u is a homogeneous function of degree $= 1$.

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \underline{\underline{1u}}$$

32. Verify Euler's theorem for $\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$.

Lets consider $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\therefore z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow z = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\Rightarrow z = x^{\frac{1}{2}} g\left(\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree $= \frac{1}{2}$.

Using Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$$

Verification:

$$\frac{\partial z}{\partial x} = \left[\frac{1+0}{\sqrt{x}+\sqrt{y}}\right] + \left[\frac{x+y}{-1(\sqrt{x}+\sqrt{y})^2}\right]\frac{1}{2}\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{2}\left[\frac{x+y}{(\sqrt{x}+\sqrt{y})^2}\right]\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{2}\left[\frac{x+y}{(\sqrt{x}+\sqrt{y})^2}\right]\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{2(\sqrt{x}+\sqrt{y})\sqrt{x} - x - y}{2(\sqrt{x}+\sqrt{y})^2\sqrt{x}}$$

$$= \frac{2x + 2\sqrt{xy} - x - y}{2(\sqrt{x}+\sqrt{y})^2\sqrt{x}}$$

$$= \frac{x + 2\sqrt{xy} - y}{2(\sqrt{x}+\sqrt{y})^2\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = \frac{0+1}{\sqrt{x}+\sqrt{y}} + \left[\frac{x+y}{-(\sqrt{x}+\sqrt{y})^2}\right]\frac{1}{2}\left(\frac{1}{\sqrt{y}}\right)$$

$$= \frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{2}\left[\frac{x+y}{(\sqrt{x}+\sqrt{y})^2}\right]\left(\frac{1}{\sqrt{y}}\right)$$

$$= \frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{2}\left[\frac{x+y}{(\sqrt{x}+\sqrt{y})^2}\right]\left(\frac{1}{\sqrt{y}}\right)$$

$$= \frac{2(\sqrt{x}+\sqrt{y})(\sqrt{y}) - (x+y)}{2(\sqrt{x}+\sqrt{y})^2\sqrt{y}} = \frac{2\sqrt{xy} + 2y - x - y}{2(\sqrt{x}+\sqrt{y})^2\sqrt{y}}$$

$$= \frac{y + 2\sqrt{xy} - x}{2(\sqrt{x}+\sqrt{y})^2\sqrt{y}}$$

$$LHS = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)$$

$$= x\left[\frac{x + 2\sqrt{xy} - y}{2(\sqrt{x}+\sqrt{y})^2\sqrt{x}}\right] + y\left[\frac{y + 2\sqrt{xy} - x}{2(\sqrt{x}+\sqrt{y})^2\sqrt{y}}\right]$$

$$= \frac{x\sqrt{x} + 2x\sqrt{y} - y\sqrt{x} + y\sqrt{y} + 2y\sqrt{x} - x\sqrt{y}}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{x\sqrt{x} + x\sqrt{y} + y\sqrt{x} + y\sqrt{y}}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{x(\sqrt{x}+\sqrt{y}) + y(\sqrt{x}+\sqrt{y})}{2(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{x+y}{2(\sqrt{x}+\sqrt{y})} = \frac{1}{2}z = RHS.$$ $\therefore LHS = RHS.$

Hence, proved.

33. If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that

$$xu_x + yu_y + zu_z + 3\tan u = 0.$$

$$\Rightarrow \sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$\Rightarrow \sin u = \frac{x\left[1+2\left(\frac{y}{x}\right)+3\left(\frac{z}{x}\right)\right]}{x^4\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}}$$

$$\Rightarrow \sin u = x^{-3}g\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\therefore$ $\sin u$ is a homogeneous function of degree $= -3$.
(or v)

Using Euler's theorem,

$$x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}\right) = n v$$

(we know, $\sin u = v$).

$$x \frac{\partial(\sin u)}{\partial x} + y \frac{\partial}{\partial y}(\sin u) + z \frac{\partial}{\partial z}(\sin u) = -3 \sin u$$

$$\Rightarrow x \cos u\left(\frac{\partial u}{\partial x}\right) + y \cos u\left(\frac{\partial u}{\partial y}\right) + z \cos u\left(\frac{\partial u}{\partial z}\right) = -3 \sin u$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)x + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) = -3 \tan u$$

$$\Rightarrow x u_x + y u_y + z u_z + 3 \tan u = 0.$$

LHS = RHS

Hence, proved.

Find $\frac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$

$$(\cos x)^y - (\sin y)^x = 0.$$

Let $f(x, y) = (\cos x)^y - (\sin y)^x$

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-f_x}{f_y}$$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1}(\cos y)}\right]$$

$$= \left[\frac{y \sin x (\cos x)^{y-1} + (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x \cos y (\sin y)^{x-1}}\right]$$

$$= \left[\frac{y \tan x (\cos x)^y + (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x \cot y (\sin y)^x}\right]$$

Given, $(\cos x)^y = (\sin y)^x$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y}}$$

55. Find $\frac{\partial y}{\partial x}$ and $\frac{\partial z}{\partial x}$ at $(0, 1, 2)$ at

$z^3 + xy - y^2 z = 6$.

If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x & y,

$$\boxed{\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial z}\right)}} \qquad \boxed{\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = \frac{-\left(\frac{\partial f}{\partial y}\right)}{\left(\frac{\partial f}{\partial z}\right)}}$$

$f_z \neq 0$.

$f = z^3 + xy - y^2 z - 6 = 0$.

$$\frac{\partial y}{\partial x} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-y}{x - 2yz}$$

$$\Rightarrow \frac{\partial y}{\partial x} \text{ at } (0,1,2) = \frac{-1}{0 - 2(1)(2)} = \frac{1}{4}$$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = \frac{-y}{3z^2 - y^2}$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (0,1,2) = \frac{-1}{3(4) - 1} = \frac{-1}{11}$$

Assignment - 4

1. If $u = e^{x^2+y^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find $\frac{du}{dx}$.

$u \to (x, y)$ and $(y \to x) \Rightarrow u \to x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$= e^{x^2+y^2}(2x)(1) + e^{x^2+y^2}(2y)\left(\frac{dy}{dx}\right)$$

$$= 2e^{x^2+y^2}\left[x + y\left(\frac{dy}{dx}\right)\right] \longrightarrow (1).$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - 1 = \frac{-y^2}{b^2}$$

$$\Rightarrow y^2 = \left(1 - \frac{x^2}{a^2}\right)b^2 \quad \text{(Diff. wrt } x\text{)}$$

$$\Rightarrow \cancel{2}y\left(\frac{dy}{dx}\right) = 0 - \frac{\cancel{2}xb^2}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}\left(\frac{b^2}{a^2}\right).$$

Putting in (1),

$$\frac{du}{dx} = 2e^{x^2+y^2}\left[x + \cancel{y}\left\{-\frac{x}{\cancel{y}}\left(\frac{b^2}{a^2}\right)\right\}\right]$$

$$= 2xe^{x^2+y^2}\left[\frac{a^2-b^2}{a^2}\right]$$

2. If $y \log(\cos x) = x \log(\sin y)$, find $\frac{dy}{dx}$.

Consider $f(x,y) = y\log(\cos x) - x\log(\sin y) = 0$

$$\frac{\partial f}{\partial x} = \frac{y}{\cos x}(-\sin x) - 1\log(\sin y)$$

$$= -y\tan x - \log(\sin y) \longrightarrow ①$$

$$\frac{\partial f}{\partial y} = 1\log(\cos x) - \frac{x}{\sin y}(\cos y)$$

$$= \log(\cos x) - x\cot y \longrightarrow ②$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \underline{\left[\frac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y}\right]}$$

(From ① and ②)

3. If $z = \sqrt{x^2+y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, show that $\frac{dz}{dx} = 0$ when $x = y = a$.

$z \to (x,y)$ and $(y \to x) \Rightarrow (z \to x)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{\cancel{2}}\left(\frac{1}{\sqrt{x^2+y^2}}\right)(\cancel{2}x)(1) + \frac{1}{\cancel{2}}\left(\frac{1}{\sqrt{x^2+y^2}}\right)(\cancel{2}y)\left(\frac{dy}{dx}\right) \longrightarrow ③$$

$$= \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(\frac{dy}{dx}\right)$$

$$x^3 + y^3 + 3axy = 5a^2$$

$$\Rightarrow 3x^2 + 3y^2\left(\frac{dy}{dx}\right) + 3ay + 3ax\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x^2 + y^2\left(\frac{dy}{dx}\right) + ay + ax\left(\frac{dy}{dx}\right) = 0.$$

$$\Rightarrow -\left(\frac{x^2 + ay}{y^2 + ax}\right) = \frac{dy}{dx}$$

Consider $x = y = a$ (given.)

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{a^2 + a^2}{a^2 + a^2}\right) = -1.$$

Putting in ①,

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}$$

Given, $x = y = a$

$$\Rightarrow \frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} = 0.$$

$$\therefore \frac{dz}{dx} = 0$$

Hence, proved.

4. Find the total differential coefficient of $u = x^2y$ wrt x when x, y are connected by $x^2 + xy + y^2 = 1$.

Given, $u = x^2y$

$x^2 + xy + y^2 = 1.$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right) = 2xy(1) + x^2\frac{dy}{dx} \longrightarrow ①.$$

From (a), $\frac{dy}{dx} = \frac{-f_x}{f_y} = -\left(\frac{2x+y}{x+2y}\right) \longrightarrow (2)$.

Putting (2) in (1),

$$\frac{du}{dx} = 2xy - x^2\left(\frac{2x+y}{x+2y}\right).$$

Coefficient of $x^2y = \underline{\underline{-1}}$

5. If $y^3 - 3ax^2 + x^3 = 0$, then prove that

$$\frac{d^2y}{dx^2} = \frac{2a^2x^2}{y^5}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\left[\frac{-6ax+3x^2}{3y^2}\right] = \left[\frac{-x^2+2ax}{+y^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(2a-2x)y^2 - 2y(2ax-x^2)\left(\frac{dy}{dx}\right)}{y^4}$$

$$= \frac{(2a-2x)y^2 - (2ax-x^2)\left(\frac{-x^2+2ax}{y^{\cancel{2}}}\right)(\cancel{2}y)}{y^4}$$

$$= \frac{(2a-2x)y^2 - (2ax-x^2)\frac{(2ax-x^2)}{y}}{y^4}$$

$$= \frac{(2a-2x)y^3 - 2(2ax-x^2)^2}{y^5}$$

$$= \frac{(2a-2x)(3ax^2-x^3) - 2(4a^2x^2+x^4-4x^3a)}{y^5}$$

$$= \frac{6a^2x^2 - \cancel{2ax^3} - \cancel{6ax^3} + \cancel{2x^4} - 8a^2x^2 - \cancel{2x^4} + \cancel{8x^3a}}{y^5}$$

$$\frac{d^2y}{dx^2} = -\frac{2a^2x^2}{y^5}$$

$\therefore$ LHS = RHS

Hence, proved.

Assignment - 5

1. Verify Euler's theorem for the following function : $u = x^2yz - 4y^2z^2 + 2xz^3$.

Given, $u = x^2yz - 4y^2z^2 + 2xz^3$.

$$\Rightarrow u = z^4\left(\frac{x^2y}{z^3} - \frac{4y^2}{z^2} + \frac{2x}{z}\right)$$

$$\Rightarrow u = z^4\left[\left(\frac{x^2}{z^2}\right)\left(\frac{y}{z}\right) - 4\left(\frac{y}{z}\right)^2 + 2\left(\frac{x}{z}\right)\right].$$

$$\Rightarrow u = z^4\, g\left(\frac{x}{z}, \frac{y}{z}\right)$$

$\therefore$ u is a homogeneous function of degree $= 4$.

Using Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) = 4u$$

Verification:

$$\frac{\partial u}{\partial x} = 2xyz + 2z^3$$

$$\frac{\partial u}{\partial y} = x^2z - 8yz^2 + 0$$

$$\frac{\partial u}{\partial z} = x^2y - 8y^2z + 6xz^2$$

$LHS = x\left(\frac{\partial}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right)$

$= x(2xyz + 2z^3) + y(x^2z - 8yz^2) + z(x^2y - 8y^2z + 6xz^2)$

$= 2x^2yz + 2xz^3 + x^2yz - 8y^2z^2 + x^2yz - 8y^2z^2 + 6xz^3$

$= 4x^2yz + 8xz^3 - 16y^2z^2$

$= 4[x^2yz - 4y^2z^2 + 2xz^3]$

$= 4u = RHS.$

$\therefore LHS = RHS.$

Hence, proved.

2. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, prove that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 3\tan$

$v = \sin u = \frac{x^2y^2}{x+y} \longrightarrow ①$

$\Rightarrow v = \frac{x^4\left(\frac{y}{x}\right)^2}{x\left(1+\frac{y}{x}\right)}$

$\Rightarrow v = x^3 g\left(\frac{y}{x}\right)$

$\therefore$ v is a homogeneous function of degree $= 1$.

$\therefore$ $\sin u$ is a homogeneous function of degree $= 1$.

Using Euler's theorem,

$x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) = 3v$

$\Rightarrow x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = 3\sin u$

$$\Rightarrow \cos u\left[x\frac{\partial u}{\partial x}\right] + \cos u\left[y\frac{\partial u}{\partial y}\right] = 3\sin u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 3\tan u$$

$\therefore$ LHS = RHS

Hence, proved.

3. If $u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)$, prove that

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0.$$

Given, $u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)$

$$\Rightarrow u = g\left(\frac{x}{y}\right)$$

$\therefore$ u is a homogeneous function.

$$\therefore x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0$$

(do manually proof for more marks)

4. If $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{4}\sin 2u$

$$v = \cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow v = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow v = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$\Rightarrow v = \sqrt{x}\left(\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}\right) = x^{\frac{1}{2}}g\left(\frac{y}{x}\right)$$

$\therefore$ v is a homogeneous function of degree $= \frac{1}{2}$.

Using Euler's theorem,

$$x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) = \frac{1}{2}v$$

$$\Rightarrow x\frac{\partial}{\partial x}(\cot u) + y\frac{\partial}{\partial y}(\cot u) = \frac{1}{2}\cot u$$

$$\Rightarrow -x\,\mathrm{cosec}^2 u\left(\frac{\partial u}{\partial x}\right) - y\,\mathrm{cosec}^2 u\left(\frac{\partial u}{\partial y}\right) = \frac{1}{2}\cot u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{2}\left(\frac{\cot u}{\mathrm{cosec}^2 u}\right)$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{2}(\cos u)(\sin u) \times \frac{2}{2}$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{4}\sin 2u$$

$\therefore$ LHS = RHS

Hence, proved.

5. If $u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$, find the value of :

i) $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)$

ii) $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right)$

$$v = \tan u = \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow v = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} \Rightarrow v = x^{3-\frac{1}{2}}\frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+\sqrt{\frac{y}{x}}\right]} \Rightarrow v = x^{\frac{5}{2}}g\left(\frac{y}{x}\right)$$

$\therefore v$ is a homogeneous function of degree $= \frac{5}{2}$

i) Using Euler's theorem,

$$x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) = \frac{5}{2} v$$

$$\Rightarrow x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = \frac{5}{2}(\tan u)$$

$$\Rightarrow x \sec^2 u \left(\frac{\partial u}{\partial x}\right) + y \sec^2 u \left(\frac{\partial u}{\partial y}\right) = \frac{5}{2} \tan u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \frac{5}{2}\left(\frac{\sin u}{\cancel{\cos u}}\right)\left(\frac{\cos^{\cancel{2}} u}{1}\right)\frac{2}{2}$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \frac{5}{4} \sin 2u \rightarrow (1)$$

) ~~Squaring both sides on (1),~~

$$\cancel{\Rightarrow x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{25}{16}}$$

Differentiating (1) wrt x partially,

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right) + x\left(\frac{\partial^2 u}{\partial x^2}\right) + y\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{5}{4}(2 \cos 2u)\left(\frac{\partial u}{\partial x}\right)$$

Multiply x on both sides,

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{5}{2} \cos 2u \left(x \frac{\partial u}{\partial x}\right) \rightarrow (2)$$

Partially diff. (1) wrt y,

$$\Rightarrow x\left(\frac{\partial^2 u}{\partial y \partial x}\right) + \left(\frac{\partial u}{\partial y}\right) + y\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{5}{4}(2 \cos 2u)\left(\frac{\partial u}{\partial y}\right)$$

Multiply y on both sides,

$$\Rightarrow xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y\left(\frac{\partial u}{\partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{5}{2} \cos 2u \left(y \frac{\partial u}{\partial y}\right) \rightarrow (3)$$

Adding ② + ③ ,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + \left[x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{5}{2}\cos 2u\left[x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right)\right].$$

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$= \frac{5}{2}\cos 2u\left[\frac{5}{4}\sin 2u\right] - \left[\frac{5}{4}\sin 2u\right].$$

$$= \frac{5}{4}\left(\frac{5}{4}\right)(2\sin 2u \cos 2u) - \frac{5}{4}\sin 2u$$

$$= \frac{25}{16}(\sin 4u) - \frac{5}{4}(\sin 2u)$$

Question Answers

What is the degree of homogeneous function

1. $u(x, y) = \dfrac{\sqrt{x^2+y^2}}{x+y}$?

$$u(x, y) = \frac{\sqrt{x^2+y^2}}{x+y} = \frac{x\sqrt{1+\left(\frac{y}{x}\right)^2}}{x\left(1+\frac{y}{x}\right)} = x^0\, g\left(\frac{y}{x}\right)$$

$\therefore$ Degree $= 0$

2. ~~What is~~ $u = \left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$

$$u = \left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{x^{\frac{1}{3}}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right\}}{x^{\frac{1}{2}}\left\{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right\}}\right]^{\frac{1}{2}}$$

$$= x^{-\frac{1}{12}}\, g\left(\frac{y}{x}\right) \qquad \therefore \text{Degree} = \underline{\underline{\frac{-1}{12}}}$$

3. If $u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$, then $x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}\right) = ?$

$v = \sin u = \sqrt{x^2+y^2}$

$\Rightarrow v = \sqrt{x^2+y^2}$

$\Rightarrow v = x\sqrt{1+\left(\frac{y}{x}\right)^2}$

$\therefore$ v is a homogeneous function of degree $= 1$.

Using Euler's theorem,

$$x\left(\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}\right) = 1v$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u) = \sin u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}\right) = \underline{\tan u}$$

4. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that:

i) $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \sin 2u$

ii) $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = (1 - 4\sin^2 u)\sin 2$

$$v = \tan u = \frac{x^3+y^3}{x-y} = f(u)$$

$$\Rightarrow v = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} \Rightarrow v = x^2 g\left(\frac{y}{x}\right)$$

$\therefore$ v (or $\tan u$) is a homogeneous function of degree $= 2$.

Using Euler's theorem,

i) $x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) = 2v$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow x\sec^2 u\left(\frac{\partial u}{\partial x}\right) + y\sec^2 u\left(\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 2\left(\frac{\sin u}{\cancel{\cos u}}\right)(\cos^{\cancel{2}} u)$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \sin 2u$$

$\therefore$ LHS = RHS

Hence, proved.

By Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{f(u)}{f'(u)} = 2\frac{\tan u}{\sec^2 u}$$

$$= \underline{\sin 2u}$$

ii) By extension of Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = g(u)[g'(u) - 1]$$

$$= \sin 2u\,[2\cos 2u - 1]$$

$$\cancel{= 2\sin 2u \cos 2u - \sin 2u}$$

$$\cancel{= \sin 4u - \sin 2u}$$

$$= \sin 2u\,[2(1-2\sin^2 u) - 1]$$

$$= \underline{\sin 2u\,[1 - 4\sin^2 u]}$$

LHS = RHS, Hence, proved.

6. If $u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$, show that:

i) $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \tan u$

ii) $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = \tan^3 u$

$$\sin u = \sqrt{x^2+y^2} = f(u)$$

$$\Rightarrow \sin u = x\sqrt{1+\left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \sin u = x^1 h\left(\frac{y}{x}\right)$$

$\therefore$ $\sin u$ is a homogeneous function of degree $= 1$.

i) By extension of Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{f(u)}{f'(u)}$$

$$= 1\,\frac{\sin u}{\cos u}$$

$$= \tan u$$

$g(u) = \tan u$

$\therefore$ LHS = RHS
Hence, proved.

ii) By extension of Euler's theorem,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = g(u)[g'(u)-1]$$

$$= \tan u[\sec^2 u - 1]$$

$$= \tan u(\tan^2 u)$$

$$= \underline{\tan^3 u}$$

$\therefore$ LHS = RHS

Hence, proved.

7. If $u = x^4 \sin^{-1}\left(\frac{y}{x}\right) + y^4 \tan^{-1}\left(\frac{y}{x}\right)$, then evaluate $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right)$ by using extension of Euler's theorem.

$$u = x^4 \sin^{-1}\left(\frac{y}{x}\right) + y^4 \tan^{-1}\left(\frac{y}{x}\right) = p + q.$$

p is a homogeneous function of degree = 4

q ——————— 〃 ——————— of degree = 4.

Using Euler's theorem,

$$x^2 \frac{\partial^2 p}{\partial x^2} + 2xy\left(\frac{\partial^2 p}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 p}{\partial y^2}\right) = 4(4-1)p = 12p \quad \rightarrow (1)$$

$$x^2\left(\frac{\partial^2 q}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 q}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 q}{\partial y^2}\right) = 4(4-1)q = 12q \quad \rightarrow (2)$$

(1) + (2),

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 12(p+q) = \underline{12u}$$

1. Find the exact differential of:

i) $u = \sqrt{x^2+y^2}$

$$du = \left(\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y}\right) dy$$

$$= \left[\frac{1}{2}\left(\frac{1}{\sqrt{x^2+y^2}}\right)(2x)\right] dx + \left[\frac{1}{2}\left(\frac{1}{\sqrt{x^2+y^2}}\right)(2y)\right] dy$$

$$= \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

ii) $u = \frac{y}{x}$

$$du = \left(\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y}\right) dy$$

$$= \left(-\frac{y}{x^2}\right) dx + \left(\frac{1}{x}\right) dy$$

$$= \frac{x\,dy - y\,dx}{x^2}$$

2. If $u = xy(x^2+y^2)$, $x = at^2$, $y = 2at$, $\frac{du}{dt} = ?$

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

$$= \left[y(x^2+y^2) + xy(2x)\right]\left[2at\right] + \left[x(x^2+y^2) + xy(2y)\right][2a]$$

$$= (y^3 + 3x^2y)(2at) + (x^3 + 3xy^2)(2a)$$

$$= \left[8a^3t^3 + 3(a^2t^4)(2at)\right](2at) + \cancel{4a}\left[a^3t^6 + 3(at^2)(4a^2t^2)\right]$$

$$= 16a^4t^4 + 12a^4t^6 + 2a^4t^6 + 24a^4t^4$$

$$= 40a^4t^4 + 14a^4t^6$$

3. If $u = f(x, y)$, $x = s + t$, $y = s - t$,

find $\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \longrightarrow (1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial u}{\partial x}(1) + \frac{\partial u}{\partial y}(-1)$$

$$= \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \longrightarrow (2)$$

From (1) & (2), $\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = \underline{\underline{2\left(\frac{\partial u}{\partial x}\right)}}$

4. If $f(x, y) = x^2 y$, $x = t$, $y = t^2$, find $\frac{du}{dt}$.

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

$$= (2xy)(1) + x^2(2t)$$

$$= [2(t)(t^2)] + (t^2)(2t) = \underline{\underline{4t^3}}$$

- If $u(x, y) = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$, prove that $x u_x + y u_y = 6u$.

$$u(x, y) = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$$

$$= x^6 \left(\frac{y}{x}\right)^2 \sin^{-1}\left(\frac{y}{x}\right)$$

$$= x^6 \, g\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree $= 6$.

Using Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 6u$$

$$\Rightarrow x u_x + y u_y = 6u$$

Hence, proved.

- If $u(x, y) = \frac{x^3 + y^3}{\sqrt{x - y}}$, prove that $x u_x + y u_y = \frac{5}{2} u$

$$u(x, y) = \frac{x^3 + y^3}{\sqrt{x - y}}$$

$$= \frac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{\sqrt{x}\left[\sqrt{1 - \left(\frac{y}{x}\right)}\right]}$$

$$= x^{\frac{5}{2}} \, g\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree $= \frac{5}{2}$.

Using Euler's theorem, $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \frac{5}{2} u$

$$\Rightarrow x u_x + y u_y = \frac{5}{2} u$$

• If $u = \sqrt{x^4+y^4}\ \tan^{-1}\left(\frac{y}{x}\right)$, prove that

i) $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 2u$

ii) $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 2u$

$$u = x^2\sqrt{1+\left(\frac{y}{x}\right)^4}\ \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^2 g\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree $= 2$.

Using Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 2u$$

Using corollary of Euler's theorem,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = n(n-1)u$$

$$= 2(2-1)u$$

$$= \underline{\underline{2u}}$$

36. If $\sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right) = u$, show that

$x u_x + y u_y + z u_z + 3\tan u = 0$.

Let $\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$

$$\Rightarrow \sin u = \frac{x\left[1+2\left(\frac{y}{x}\right)+3\left(\frac{z}{x}\right)\right]}{x^4\left[\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}\right]}$$

$$\Rightarrow \sin u = x^{-3}\, g\left(\frac{y}{x}, \frac{z}{x}\right).$$

$\therefore$ $\sin u$ is a homogeneous function of degree $=-3$.

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -3\sin u$$

(Using Euler's theorem).

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) = -3\tan u$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + z\left(\frac{\partial u}{\partial z}\right) + 3\tan u = 0$$

LHS = RHS

Hence, proved.

(OR) By using deductions of Euler's theorem,

$$x u_x + y u_y + z u_z = n\frac{f(x)}{f'(u)} \qquad \text{where } f(u) = \sin u$$

$$= n\frac{\sin u}{\cos u}$$

$$= -3\tan u$$

37. If $u = e^{\frac{x^3+y^3}{3x+4y}}$, show that $x u_x + y u_y = 2u \log u$.

$u = e^{\frac{x^3+y^3}{3x+4y}}$

Taking log on both sides,

$$\log u = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[3+4\left(\frac{y}{x}\right)\right]}.$$

$\Rightarrow \log u = x^2 f\left(\frac{y}{x}\right).$

$\therefore \log u$ is a homogeneous function of degree $= 2$

Using Euler's theorem,

$$x u_x + y u_y = n \frac{f(u)}{f'(u)}$$

$$\Rightarrow x u_x + y u_y = 2 \frac{\log u}{\left(\frac{1}{u}\right)}$$

$$\Rightarrow x u_x + y u_y = \underline{\underline{2u \log u}}$$

$\therefore$ LHS = RHS

Hence, proved.

38. If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^3 \tan^{-1}\left(\frac{x}{y}\right)$, find the value of $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right)$

$$\Rightarrow u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^3 \cot^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow u = x^3\left[\sin^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^3 \cot^{-1}\left(\frac{y}{x}\right)\right].$$

$$\Rightarrow u = x^3 f\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree $= 3$.

Using corollary of Euler's theorem,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) = n(n-1)u$$

$$= 3(3-1)u$$

$$= \underline{6u}$$

39. If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^{-3}\tan^{-1}\left(\frac{x}{y}\right)$, find the value of $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right)$

$$u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{y^3}\tan^{-1}\left(\frac{x}{y}\right)$$

$$\Rightarrow u = x^3 \operatorname{cosec}^{-1}\left(\frac{x}{y}\right) + \frac{1}{y^3}\cot^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow u = x^3\left[\operatorname{cosec}^{-1}\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)^3 \cot^{-1}\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow u = x^3 f\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree $= 3$.

Using Euler's theorem,

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = nu = 3u \longrightarrow (1)$$

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) = n(n-1)u$$

$$= 6u \longrightarrow (2)$$

Adding (1) + (2),

Answer $= 9u$

40. If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \tan^{-1}\left(\frac{y}{x}\right)$, find $x\left(\frac{\partial u}{\partial x}\right)$

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right)$$

at $x = 1$, $y = 1$.

Let $u = v + w$, where

$v = x^3 \sin^{-1}\left(\frac{y}{x}\right)$ At $x=1, y=1$, $v = \frac{\pi}{2}$

$w = x^4 \tan^{-1}\left(\frac{y}{x}\right)$ At $x=1, y=1$, $w = \frac{\pi}{4}$.

$\therefore$ v is a homogeneous function of degree $= 3$.

$\therefore$ w is a homogeneous function of degree $= 4$

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v \longrightarrow (1)$$

$$x^2\left(\frac{\partial^2 v}{\partial x^2}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x \partial y}\right) = 6v \longrightarrow (2)$$

$$x\left(\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial w}{\partial y}\right) = 4w \longrightarrow (3).$$

$$x^2\left(\frac{\partial^2 w}{\partial x^2}\right) + y^2\left(\frac{\partial^2 w}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 w}{\partial x \partial y}\right) = 12w \longrightarrow (4).$$

(1) + (2) + (3) + (4)

$$\Rightarrow x\left[\frac{\partial(v+w)}{\partial x}\right] + y\left[\frac{\partial}{\partial y}(v+w)\right] + x^2\left[\frac{\partial^2}{\partial x^2}(v+w)\right] +$$

$$y^2\left[\frac{\partial^2}{\partial y^2}(v+w)\right] + 2xy\left[\frac{\partial^2}{\partial x \partial y}(v+w)\right] = 9v + 16w$$

$$\Rightarrow x u_x + y u_y + x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy}$$

$$= 9\left(\frac{\pi}{2}\right) + 16\left(\frac{\pi}{4}\right)$$

$$= \frac{18\pi + 16\pi}{4}$$

$$= \frac{34\pi}{4} = \frac{17\pi}{2}$$

41. If $(\sqrt{x}+\sqrt{y})\sin^2 u - x^{\frac{1}{3}} - y^{\frac{1}{3}} = 0$, prove that

$$12x\left(\frac{\partial u}{\partial x}\right) + 12y\left(\frac{\partial u}{\partial y}\right) + \tan u = 0$$

$$(\sqrt{x}+\sqrt{y})\sin^2 u = x^{\frac{1}{3}} + y^{\frac{1}{3}}$$

$$\Rightarrow \sin^2 u = \frac{x^{\frac{1}{3}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}$$

$$\Rightarrow \sin^2 u = x^{-\frac{1}{6}}\, g\left(\frac{y}{x}\right)$$

$\therefore \sin^2 u$ is a homogeneous function of degree $= \frac{-1}{6}$

Using extension of Euler's theorem,

$$x\left(\frac{du}{dy}\right) + y\left(\frac{du}{dy}\right) = n\frac{f(u)}{f'(u)} \quad \text{where } f(u) = \sin^2 u$$

$$= \left(-\frac{1}{6}\right)\frac{\sin^2 u}{2\sin u \cos u}$$

$$\Rightarrow 12x\left(\frac{\partial u}{\partial x}\right) + 12y\left(\frac{\partial u}{\partial y}\right) + \tan u = 0$$

$\therefore$ LHS = RHS Hence, proved.