Unit-2 class-3

Q1. Prove that $L\{t^n\} = \frac{n!}{s^n+1}$ where $n = 0, 1, 2, 3, \ldots$

A. $\int_0^\infty e^{-st}.t^n \, dt$

$st = u$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

$= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u}.u^n \, du = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$ if $n = 0, 1, 2, 3 \ldots$

Q2. a) $L[\sin^3 t] = L\left[\frac{3\sin t - \sin 3t}{4}\right]$

$= \frac{3}{4} \cdot \frac{1}{s^2+1} - \frac{1}{4} \cdot \frac{3}{s^2+9} = \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$

$= \frac{3}{4}\left(\frac{\cancel{s^2}+9-\cancel{s^2}-1}{(s^2+1)(s^2+9)}\right) = \frac{6}{(s^2+1)(s^2+9)}$

b) $L[\cos 3t \sin^2 t]$

$\sin^2 t + \cos^2 t$

$= L\left[\cos 3t \cdot \left(\frac{1-\cos 2t}{2}\right)\right]$

$= L\left[\frac{\cos 3t}{2} - \frac{\cos 3t . \cos 2t}{2}\right]$

$\left(\cos A . \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))\right)$

$= \frac{1}{2} \cdot \frac{s}{s^2+9} - \frac{1}{4}\left[\cos 5t + \cos t\right]$

$= \frac{1}{2} \frac{s}{s^2+9} - \frac{1}{4} \cdot \frac{s}{s^2+25} - \frac{1}{4} \cdot \frac{s}{s^2+1}$

$= \frac{1}{4}\left(\frac{2s(s^2+25)(s^2+1) - s(s^2+9)(s^2+1) - s(s^2+9)(s^2+25)}{(s^2+1)(s^2+9)(s^2+25)}\right)$

$= \frac{1}{4}\left(\frac{s(\cancel{2s^4} + 52s^2 + 50 - \cancel{s^4} - 10s^2 - 9 - \cancel{s^4} - 34s^2 - 225)}{(s^2+1)(s^2+9)(s^2+25)}\right)$

$= \frac{1}{4}\left(\frac{(8s^2 - 184) \times s}{(s^2+1)(s^2+9)(s^2+25)}\right) = \frac{2s(s^2-23)}{(s^2+1)(s^2+9)(s^2+25)}$

Q3. $L[\cos^3 2t]$

$= L\left[\frac{3\cos 2t + \cos 6t}{4}\right]$

$= \frac{3}{4} \cdot \frac{s}{s^2+4} + \frac{1}{4} \cdot \frac{s}{s^2+36}$

$= \frac{s}{4}\left(\frac{3}{s^2+4} + \frac{1}{s^2+36}\right)$

$= \frac{s}{4}\left(\frac{3s^2 + 108 + s^2 + 4}{(s^2+4)(s^2+36)}\right)$

$= \frac{s(s^2+28)}{(s^2+4)(s^2+36)}$