Solutions of PDE by the method of Seperation of Variables.

19. Solve $x^2\left(\frac{\partial u}{\partial x}\right)+y^2\left(\frac{\partial u}{\partial y}\right)=0$ by MSV.

Let $u = XY$ where, $X =$ function of $x = X(x)$,
$Y =$ function of $y = Y(y)$.

be the solution of given equation.

Putting $u = XY$ in given eqn,

$$x^2\left[\frac{\partial(XY)}{\partial x}\right]+y^2\left[\frac{\partial(XY)}{\partial y}\right]=0$$

$$\Rightarrow x^2Y\left[\frac{\partial(X)}{\partial x}\right]+y^2X\left[\frac{\partial(Y)}{\partial y}\right]=0$$

$$\Rightarrow x^2Y\left(\frac{dX}{dx}\right)+y^2X\left(\frac{dY}{dy}\right)=0.$$

(because X is a function of single variable x).
(same for Y)

Dividing by XY,

$$\Rightarrow \frac{x^2}{X}\left(\frac{dX}{dx}\right)+\frac{y^2}{Y}\left(\frac{dY}{dy}\right)=0.$$

$$\Rightarrow \frac{x^2}{X}\left(\frac{dX}{dx}\right)=-\frac{y^2}{Y}\left(\frac{dY}{dy}\right)$$

Equating each term to a constant K.

$$\therefore \frac{x^2}{X}\left(\frac{dX}{dx}\right)=K \text{ and } -\frac{y^2}{Y}\left(\frac{dY}{dy}\right)=K$$

$$\Rightarrow \frac{dX}{X}=K\left(\frac{dx}{x^2}\right) \quad \Rightarrow -\frac{dY}{Y}=K\left(\frac{dy}{y^2}\right)$$

$$\Rightarrow \log X=-\frac{K}{x}+C_1 \quad \Rightarrow \log Y=\frac{K}{y}+C_2$$

$\Rightarrow X = e^{-\frac{K}{x} + C_1}$ $\qquad \Rightarrow Y = e^{\frac{K}{y} + C_2}$

Thus, $u = XY$ becomes,

$$u = \left[e^{-\frac{K}{x} + C_1}\right]\left[e^{\frac{K}{y} + C_2}\right]$$

$$\Rightarrow u = e^{-\frac{K}{x} + \frac{K}{y}} \, e^{C_1 + C_2}$$

$$\Rightarrow \underline{\underline{u = C e^{K\left(\frac{1}{y} - \frac{1}{x}\right)}}} \quad \left(\text{where } C = e^{C_1 + C_2}\right)$$

2. Solve $4\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y} = 3u$, given $u(0, y) = 2e^{5y}$

Let $u = XY$ be the solution of the given solution.

$$4\left[\frac{\partial (XY)}{\partial x}\right] + \frac{\partial (XY)}{\partial y} = 3(XY)$$

$$\Rightarrow 4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3(XY)$$

Dividing by XY,

$$\Rightarrow \frac{4}{X}\left(\frac{dX}{dx}\right) + \frac{1}{Y}\left(\frac{dY}{dy}\right) = 3$$

$$\Rightarrow \frac{4}{X}\left(\frac{dX}{dx}\right) = 3 - \frac{1}{Y}\left(\frac{dY}{dy}\right)$$

Equating each term to a constant K,

$\Rightarrow \frac{4}{X}\left(\frac{dX}{dx}\right) = K$ and $\Rightarrow 3 - \frac{1}{Y}\left(\frac{dY}{dy}\right) = K$

$\Rightarrow \frac{dX}{X} = K\left(\frac{dx}{4}\right)$ $\qquad \Rightarrow (3 - K)\,dy = \frac{dY}{Y}$

$\Rightarrow \log X = \frac{Kx}{4} + C_1$ $\qquad \Rightarrow (3 - K)y + C_2 = \log Y$

$\Rightarrow X = e^{\left(\frac{Kx}{4} + C_1\right)}$ $\qquad \Rightarrow Y = e^{\left((3-K)y + C_2\right)}$

Thus, $u = XY$ becomes,

$$u = e^{\frac{Kx}{4}+C_1} e^{(3-K)y+C_2}$$

$$\Rightarrow u = e^{\frac{Kx}{4}+(3-K)y} e^{C_1+C_2}$$

$$\Rightarrow \underline{\underline{u = Ce^{\frac{Kx}{4}+(3-K)y}}} \quad \left(\text{where } C = e^{C_1+C_2}\right)$$

Given $u(0,y) = 2e^{5y}$, i.e., $u = 2e^{5y}$ when $x=0$.

$$\Rightarrow 2e^{5y} = Ce^{(3-K)y}$$

$$\Rightarrow 2e^{5y} = Ce^{3y-Ky}$$

Comparing LHS and RHS, $C=2$, $K=-2$

$$\underline{\underline{u = 2e^{-\frac{x}{2}+5y}}}$$

21. Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given $u(x,0) = 6e^{-3x}$

$$\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial t} = u$$

Let $u = XT$ be the solution of given equation.

$$\Rightarrow \frac{\partial(XT)}{\partial x} - 2\frac{\partial(XT)}{\partial t} = XT$$

$$\Rightarrow T\frac{dX}{dx} - 2X\frac{dT}{dt} = XT$$

Dividing by XT,

$$\Rightarrow \frac{1}{X}\left(\frac{dX}{dx}\right) - \left(\frac{2}{T}\right)\left(\frac{dT}{dt}\right) = 1$$

$$\Rightarrow \frac{1}{X}\left(\frac{dX}{dx}\right) = 1 + \frac{2}{T}\left(\frac{dT}{dt}\right)$$

Equating each term to constant K,

$$\Rightarrow \frac{1}{X}\left(\frac{dX}{dx}\right) = K \quad \text{and} \quad \Rightarrow 1 + \frac{2}{T}\left(\frac{dT}{dt}\right) = K$$

$$\Rightarrow \frac{dX}{X} = dx\,K \qquad \Rightarrow \frac{dT}{T} = \frac{(K-1)}{2}\,dt$$

$$\Rightarrow \log X = xK + C_1 \qquad \Rightarrow \log T = \frac{t}{2}(K-1) + C_2$$

$$\Rightarrow X = e^{Kx + C_1} \qquad \Rightarrow T = e^{\frac{t}{2}(K-1) + C_2}$$

Putting in $u = XT$,

$$u = e^{Kx + C_1}\, e^{\frac{t}{2}(K-1) + C_2}$$

$$\Rightarrow u = e^{Kx + \frac{t}{2}(K-1)}\, e^{C_1 + C_2}$$

$$\Rightarrow u = C e^{Kx + \frac{t}{2}(K-1)}$$

Given $u(x,0) = 6e^{-3x}$, i.e., $u = 6e^{-3x}$ when $t=0$

$$\Rightarrow 6e^{-3x} = Ce^{Kx}$$

Comparing, $C = 6$, $K = -3$

$\therefore$ The solution is,

$$\underline{\underline{u = 6e^{-3x - 2t}}}$$

Solve

★ Various possible solutions of the one dimensional heat equation $\boxed{u_t = c^2 u_{xx}}$ by the method of seperation of variables:

Consider $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Let $u = XT$, thus,

$$\Rightarrow \frac{\partial (XT)}{\partial t} = \left[\frac{\partial^2 (XT)}{\partial x^2}\right] c^2$$

$$\Rightarrow X\left(\frac{dT}{dt}\right) = c^2 T\left(\frac{d^2X}{dx^2}\right)$$

Divide by XT,

$$\Rightarrow \frac{1}{T}\left(\frac{dT}{dt}\right) = c^2\left(\frac{1}{X}\right)\left(\frac{d^2X}{dx^2}\right)$$

$$\Rightarrow \frac{1}{c^2T}\left(\frac{dT}{dt}\right) = \frac{1}{X}\left(\frac{d^2X}{dx^2}\right)$$

Equating each term to a constant K,

$$\Rightarrow \frac{1}{X}\left(\frac{d^2X}{dx^2}\right) = K \quad \& \quad \frac{1}{c^2T}\left(\frac{dT}{dt}\right) = K$$

$$\Rightarrow \frac{d^2X}{dx^2} - KX = 0 \qquad \Rightarrow \frac{dT}{dt} - Kc^2T = 0$$

The auxillary equation is,

$$(D^2 - K)X = 0 \qquad (D - Kc^2)T = 0$$

$$\cancel{\Rightarrow m^2 - K = 0} \qquad \cancel{\Rightarrow m - Kc^2 = 0}$$

$$\cancel{\Rightarrow m^2 = K} \qquad \cancel{\Rightarrow m = Kc^2}$$

$$\cancel{\Rightarrow m = \pm\sqrt{K}}$$

Case 1 : When $K = 0$

$\Rightarrow D^2 X = 0$ $\qquad\qquad$ $\Rightarrow DT = 0$

$\Rightarrow m^2 = 0$ $\qquad\qquad$ $\Rightarrow m = 0.$

$\Rightarrow m = 0.$

$X = (C_1 + C_2 x) e^{0x}$ $\qquad\qquad$ $T = C_3 e^{0t}$

$\Rightarrow X = C_1 + C_2 x$ $\qquad\qquad$ $\Rightarrow T = C_3$

Thus, the solution of PDE is,

$u = XT \rightarrow \underline{u = (C_1 + C_2 x) C_3}$

$\therefore \underline{u = A + Bx}$ where $A = C_1 C_3$ and $B = C_2 C_3$

Case 2: When K is positive, say $K = p^2$

$(D^2 - p^2) X = 0$ $\qquad\qquad$ $(D - c^2 p^2) T = 0$

$\rightarrow m^2 - p^2 = 0$ $\qquad\qquad$ $\rightarrow m - c^2 p^2 = 0$

$\rightarrow m = \pm p$ $\qquad\qquad$ $\rightarrow m = c^2 p^2$

$X = C_4 e^{px} + C_5 e^{-px}$ $\qquad\qquad$ $\Rightarrow T = C_6 e^{c^2 p^2 t}$

Thus, the solution of PDE is,

$u = XT \Rightarrow \underline{u = (C_4 e^{px} + C_5 e^{-px}) C_6 e^{c^2 p^2 t}}$

$\therefore \underline{u = e^{c^2 p^2 t} (C e^{px} + D e^{-px})}$ where

$C = C_4 C_6$ and $D = C_5 C_6.$

Case 3: When K is negative, say $K = -p^2.$

$\Rightarrow (D^2 + p^2) X = 0$ $\qquad\qquad$ $\Rightarrow (D + c^2 p^2) T = 0.$

$\Rightarrow m^2 + p^2 = 0$ $\qquad\qquad$ $\Rightarrow m + c^2 p^2 = 0$

$\Rightarrow m^2 = -p^2 \Rightarrow m = \pm ip$ $\qquad\qquad$ $\Rightarrow m = -c^2 p^2$

$m = \pm ip$ $\qquad\qquad m = -c^2p^2$

$X = C_7 \cos px + C_8 \sin px$ $\qquad\qquad T = C_9 e^{-c^2p^2t}$

Thus, the solution of PDE is given by,

$u = XT \rightarrow \underline{u = (C_7 \cos px + C_8 \sin px) C_9 e^{-c^2p^2t}}$

$\therefore \underline{u = (E \cos px + F \sin px) e^{-c^2p^2t}}$, where,

$E = C_7 C_9$ and $F = C_8 C_9$

Out of the three possible solutions, the solution obtained in Case 3 is considered as the suitable solution for solving boundary value problems.

☆ Various possible solutions of the two dimensional Laplace equation $\boxed{u_{xx} + u_{yy} = 0}$ by the method of seperation of variables.

Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Let $u = XY$, thus,

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial^2 y} = 0.$$

$$Y\left(\frac{d^2X}{dx^2}\right) + X\left(\frac{d^2Y}{dy^2}\right) = 0.$$

Divide by XY,

$$\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) + \frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) = 0.$$

$$\Rightarrow \frac{1}{X}\left(\frac{d^2X}{dx^2}\right) = -\frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right)$$

Equate each term to a constant K,

$$\Rightarrow \frac{1}{X}\left(\frac{d^2X}{dx^2}\right) = K \quad \& \quad -\frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) = K$$

$$\Rightarrow \frac{d^2X}{dx^2} - KX = 0 \qquad \Rightarrow -\frac{d^2Y}{dy^2} - KY = 0.$$

$$\Rightarrow (D^2 - K)X = 0 \qquad \Rightarrow (D^2 + K)Y = 0.$$

Case 1: When $K = 0$.

$$\Rightarrow D^2X = 0 \qquad \Rightarrow D^2Y = 0$$

$$\Rightarrow m^2 = 0 \qquad \Rightarrow m^2 = 0.$$

$$\Rightarrow m = 0 \qquad \Rightarrow m = 0$$

$$X = (C_1 + C_2 x)e^{0x} \qquad Y = (C_3 + C_4 y)e^{0t}$$

Thus, the solution of the PDE is,

$$u = XY, \qquad \Rightarrow \underline{\underline{u = (C_1 + C_2 x)(C_3 + C_4 y)}}.$$

Case 2: When $K = +ve$, say $K = p^2$

$$\Rightarrow (D^2 - p^2)X = 0 \qquad \Rightarrow (D^2 + p^2)Y = 0$$

$$\Rightarrow m^2 - p^2 = 0 \qquad \Rightarrow m^2 + p^2 = 0.$$

$$\Rightarrow m = \pm p \qquad \Rightarrow m = \pm ip.$$

$$X = C_5 e^{px} + C_6 e^{-px} \qquad Y = C_7 \cos p xy + C_8 \sin py$$

Thus, the solution of the PDE is, $u = XY$,

$$\underline{\underline{u = (C_5 e^{px} + C_6 e^{-px})(C_7 \cos py + C_8 \sin py)}}.$$

Case 3: When $K = -ve$, say $K = -p^2$

$(D^2 + p^2)X = 0$ $\qquad$ $(D^2 - p^2)Y = 0$

$\Rightarrow m^2 + p^2 = 0$ $\qquad$ $\Rightarrow m^2 - p^2 = 0$

$\Rightarrow m = \pm ip$ $\qquad$ $\Rightarrow m = \pm p$

$X = C_9 \cos px + C_{10} \sin px$ $\qquad$ $Y = C_{11} e^{py} + C_{12} e^{-py}$

Thus, the solution of the PDE is,

$u = XY$

$$\Rightarrow u = (C_9 \cos px + C_{10} \sin px)(C_{11} e^{py} + C_{12} e^{-py})$$

Of these, we take that solution that is consistent with the given boundary conditions.

Solve the non-homogeneous PDE:

1. $(D_x^2 - D_y^2)z = x - y$

The auxillary eqn is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$.

$CF = f_1(y - x) + f_2(y + x)$

To find PI,

$$PI = \left[\frac{1}{(D_x^2 - D_y^2)}\right](x - y)$$

$$= \left\{\frac{1}{D_x^2\left[1 - \left(\frac{D_y}{D_x}\right)^2\right]}\right\}(x - y).$$

$$= \left(\frac{1}{D_x}\right)^2 \left[1 - \left(\frac{D_y}{D_x}\right)^2\right]^{-1} (x - y)$$

$$= \frac{1}{(D_x)^2}\left[1+\left(\frac{D_y}{D_x}\right)^2\right](x+y)$$

$$= \frac{1}{(D_x)^2}\left[(x+y)+0\right].$$

$$= \left[\frac{1}{D_x}\right]\int(x+y)+0$$

$$= \int\left(\frac{x^2}{2}+xy\right)dx$$

$$= \frac{x^3}{6}+\frac{x^2y}{2}$$

The general solution is,

$$z = f_1(y-x)+f_2(y+x)+\frac{x^3}{6}+\frac{x^2y}{2}$$

$$x(x^2+y^2-2xy)$$

$$\frac{x^3}{4}+\frac{xy^2}{4}-\frac{\cancel{2}x^2y}{\cancel{4}_2}$$

Solution of Homogeneous Linear Partial Differential Equations

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

where $a_0, a_1, a_2 \dots a_n$ are constants is called homogeneous linear PDE of n^{th} order with constant coefficients.

It is called homogeneous because the terms contain derivatives of the same order. Here, all partial derivatives are of n^{th} order.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by D or D_x or D' or D_y.

On using D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$,

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2}(D')^2 + \dots + a_n (D')^n] z = F(x,y)$$

i.e., $\boxed{F(D, D')z = F(x, y)}$ $\longrightarrow$ (2)

As in the case of ordinary LDE with constant coefficients, the complete solution of eq (2) consists of two parts, namely,

i) The complimentary function (CF) which is the complete solution of the eqn $F(D, D')z = 0$. It must contain n arbitrary functions, where n is the order of the DE.

ii) The particular integral (PI) which is particular solution of eq ②.

The complete solution is $z = CF + PI$.

<u>Rules for finding Complimentary Function</u>

Consider the function $\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} a_2 = 0 \rightarrow ①$

which in symbolic form is $[D^2 + a_1 DD' + a_2 (D')^2] z = 0$

Dividing by $(D')^2$,

$$\Rightarrow \left(\left[\frac{D}{D'}\right]^2 + a_1 \left(\frac{D}{D_1}\right) + a_2 \right) z = 0.$$

Auxillary eqn is $m^2 + a_1 m + a_2 = 0$,

where $m = \frac{D}{D'}$ and roots are m_1, m_2.

(just replace D by m, and D' by 1 to obtain the auxillary equation).

Case 1: If roots are real and distinct, then solution for eqn ①,

$$\boxed{z = f(y + m_1 x) + \phi(y + m_2 x)}$$

22. Solve $(D^2 - D'^2) z = 0$.

Auxillary eqn is $(m^2 - 1) z = 0$.

$\Rightarrow m_1, m_2 = \pm 1$.

The general solution is,

$$z = f_1(y + x) + f_2(y - x)$$

Case 2: If both roots are equal,

$$\boxed{z = f(y + m_1 x) + x\phi(y + m_1 x)}$$

23\. Solve $(4D^2 + 12DD' + 9D'^2)z = 0$

Auxillary eqn is $4m^2 + 12m + 9 = 0$.

$m_1 = -\frac{3}{2}, \quad m_2 = -\frac{3}{2}$

$$z = f_1\left(y - \frac{3}{2}x\right) + x f_2\left(y - \frac{3}{2}x\right)$$

Case 3: If the roots are complex,

$$\boxed{z = f(y + (\alpha + i\beta)x) + \phi(y + (\alpha - i\beta)x)}$$

24\. Solve $(D_x^2 + 5D_y^2 - 2D_x D_y)z = 0$

Auxillary eqn is $m^2 - 2m + 5 = 0$.

$m_1 = 1 + 2i, \quad m_2 = 1 - 2i$.

The general solution is given by,

$$z = f_1[y + (1 + i2)x] + f_2[y + (1 - i2)x]$$

25\. Solve $(D_x^3 - 6D_x^2 D_y + 11 D_x D_y^2 - 6D_y^3)z = 0$.

Auxillary eqn is $m^3 - 6m^2 + 11m - 6 = 0$.

$m_1 = 1, \quad m_2 = 2, \quad m_3 = 3$.

$$z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x).$$

Case 2: If both roots are equal,

$$\boxed{z = f(y + m_1 x) + x\phi(y + m_1 x)}$$

23. Solve $(4D^2 + 12DD' + 9D'^2)z = 0$

Auxillary eqn is $4m^2 + 12m + 9 = 0$.

$m_1 = -\frac{3}{2}$, $m_2 = -\frac{3}{2}$

$$z = f_1\left(y - \tfrac{3}{2}x\right) + x f_2\left(y - \tfrac{3}{2}x\right)$$

Case 3: If the roots are complex,

$$\boxed{z = f\big(y + (\alpha + i\beta)x\big) + \phi\big(y + (\alpha - i\beta)x\big)}$$

24. Solve $(D_x^2 + 5D_y^2 - 2D_xD_y)z = 0$

Auxillary eqn is $m^2 - 2m + 5 = 0$.

$m_1 = 1 + 2i$, $m_2 = 1 - 2i$.

The general solution is given by,

$$z = f_1[y + (1 + i2)x] + f_2[y + (1 - i2)x]$$

25. Solve $(D_x^3 - 6D_x^2D_y + 11D_xD_y^2 - 6D_y^3)z = 0$.

Auxillary eqn is $m^3 - 6m^2 + 11m - 6 = 0$.

$m_1 = 1$, $m_2 = 2$, $m_3 = 3$.

$$z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x).$$

27. Solve $(D_x^2 - D_x D_y - 6D_y^2)z = e^{2x-3y}$

The auxillary eqn is $m^2 - m - 6 = 0$.

$$m_1 = 3,\ m_2 = -2$$

$CF = f_1(y+3x) + f_2(y-2x)$

To find PI,

$$PI = \left[\frac{1}{D_x^2 - D_x D_y - 6D_y^2}\right] e^{2x-3y}$$

$$(D_x \to 2)(D_y \to 3)$$

$$= \left[\frac{1}{4+6-54}\right] e^{2x-3y}$$

$$= -\frac{1}{44} e^{2}$$

The general solution is,

$z = CF + PI$,

$z = f_1(y+3x) + f_2(y-2x) - \frac{1}{44} e^{2x-3y}$

28. Solve $(D_x^3 - 3D_x^2 D_y + 4D_y^3)z = e^{x+2y}$

The auxillary eqn is $m^3 - 3m^2 + 4 = 0$.

$$m_1 = -1, 2, 2.$$

$CF = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

To find PI,

$$PI = \left[\frac{1}{D_x^3 - 3D_x^2 D_y + 4D_y^3}\right] e^{x+2y}$$

$$(D_x \to 1)(D_y \to 2)$$

$$= \frac{1}{27} e^{x+2y}$$

The general solution is,

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

Solve $(D^2 + DD' - 2(D')^2)z = 5e^{x+2y}$

The auxillary eqn is $m^2 + m \mp 2 = 0$

$$m = \cancel{\frac{1 \pm \sqrt{7}i}{2}}$$

$$CF = f_1(y+x) + f_2(y-2x).$$

To find PI,

$$PI = \left[\frac{1}{D^2 + DD' - 2(D')^2}\right] 5e^{x+2y}$$

$(D \to 1)(D' \to 2)$

$$= -\frac{1}{5}(5e^{x+2y}) = -e^{x+2y}$$

The general solution is given by,

$$z = f_1(y+x) + f_2(y-2x) - e^{x+2y}$$

29. Solve $[D^2 + 5DD' + 6(D')^2]z = e^{x-y}$

The auxillary eqn is $m^2 + 5m + 6 = 0$

$$m_1 = -2, \quad m_2 = -3$$

$$CF = f_1(y-2x) + f_2(y-3x).$$

To find PI,

$$PI = \left[\frac{1}{D^2 + 5DD' + 6(D')^2}\right] e^{x-y}$$

$(D \to 1)(D' \to -1).$

$$= \frac{1}{2} e^{x-y}$$

The general solution is,

$$z = f_1(y-2x) + f_2(y-3x) + \frac{1}{2} e^{x-y}$$

Case 2: When $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \left[\frac{1}{F(D^2, DD', (D')^2)}\right] \sin(ax+by).$$

$$= \left[\frac{1}{F(-a^2, -ab, -b^2)}\right] \sin(ax+by) \quad ; F(-a^2, -ab, -b^2) \neq 0$$

30. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos(x+2y).$

The eqn is $z[D^2 - DD'] = \cos(x+2y).$

The auxillary eqn is $m^2 - m = 0$

$$\Rightarrow m_1 = 0, \; m_2 = 1$$

$CF = f_1(y) + f_2(y+x).$

To find PI,

$$PI = \left[\frac{1}{D^2 - DD'}\right] \cos(x+2y)$$

$$= \left[\frac{1}{-1-(-2)}\right] \cos(x+2y) = \cos(x+2y)$$

The general solution is given by,

$$z = f_1(y) + f_2(y+x) + \cos(x+2y)$$

31. Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$

The eqn is $[D^2 + 2DD' + (D')^2]z = \sin(2x+3y)$

The auxillary eqn is $m^2 + 2m + 1 = 0.$

$$m_1 = -1, \; m_2 = -1$$

$CF = f_1(y-x) + x f_2(y-x)$

To find PI,

$$PI = \left[\frac{1}{D^2 + 2DD' + (D')^2}\right] \sin(2x+3y).$$

$D^2 \to -2^2,\ DD' \to -6,\ (D')^2 \to -9$

$$= \left[\frac{1}{-4 + 2(-6) - 9}\right] \sin(2x+3y)$$

$$= -\frac{1}{25} \sin(2x+3y).$$

The general solution is,

$$z = f_1(y-x) + x f_2(y-x) - \frac{1}{25}\sin(2x+3y).$$

☆ General Method to find Particular Integral

Consider the DE,

$F(D,D')z = F(x,y)$ where $F(x,y)$ is a function of x and y.

$$PI = \left[\frac{1}{F(D,D')}\right] F(x,y).$$

$F(D,D')$ can be factorised into n linear factors

$$PI = \left[\frac{1}{(D-m_1D')(D-m_2D')\ldots\ldots(D-m_nD')}\right] F(x,y). \longrightarrow (1)$$

Now in order to find PI given by (1), we find a solution of the equation $(D-mD')z = F(x,y)$ given by

$$\boxed{z = \left[\frac{1}{D-mD'}\right] F(x,y)} \boxed{= F(x, c-mx)\,dx}$$

where after integration, the constant c must be replaced by $(y+mx)$.

Hence, the PI given by the factors in succession starting from the right.

★ Note:

While using general method of finding the PI of $F(D,D')z = F(x,y)$, we shall note the following formulae:

i) $\left[\frac{1}{D-mD'}\right]F(x,y) = \int F(x, c-mx)\,dx$, where $c = y+mx$

ii) $\left[\frac{1}{D+mD'}\right]F(x,y) = \int F(x, c+mx)\,dx$, where $c = y-mx$

32. Solve: $\frac{\partial^2 z}{\partial x^2} - 4\left(\frac{\partial^2 z}{\partial x \partial y}\right) + 4\left(\frac{\partial^2 z}{\partial y^2}\right) = e^{2x+y}$

The eqn is $[D^2 - 4DD' + 4(D')^2]z = e^{2x+y}$

The auxillary eqn is $m^2 - 4m + 4 = 0$.

$m = 2, 2$

$CF = f_1(y+2x) + x f_2(y+2x)$.

To find PI,

$$PI = \left[\frac{1}{D^2 + 4(D')^2 - 4DD'}\right] e^{2x+y}$$

$$= \left[\frac{1}{(D-2D')(D-2D')}\right] e^{2x+y}$$

$$= \frac{1}{D-2D'}\int \left(e^{2x+c-2x}\right) dx$$

$$= \left[\frac{1}{D-2D'}\right] e^{c} x$$

$$= \left[\frac{1}{D-2D'}\right] e^{y+2x} x$$

$$= \int x e^{(c-2x)+2x} dx$$

$$= \frac{x^2}{2} e^{c}$$

$$= \left(\frac{x^2}{2}\right) e^{y+2x}$$

The general solution is,

$$z = f_1(y+2x) + x f_2(y+2x) + \left(\frac{x^2}{2}\right) e^{y+2x}$$

33. Solve $[D^2 + DD' - 6(D')^2] z = \cos(2x+y)$

The auxillary eqn. is $m^2 + m - 6 = 0$.

$$m_1 = 2, \quad m_2 = -3$$

$CF = f_1(y+2x) + f_2(y-3x)$.

To find PI,

$$PI = \left[\frac{1}{D^2 + DD' - 6(D')^2}\right] \cos(2x+y).$$

$$(D^2 \to -2^2, \quad (D')^2 \to -1)$$

$$= \left[\frac{1}{-4-2-6(-1)}\right] \cos(2x+y)$$

$$= \left[\frac{1}{(D+3D')(D-2D')}\right] \cos(2x+y)$$

$$= \frac{1}{(D+3D')}\int \cos(2x + C - 2x)\,dx$$

$$= \left[\frac{1}{D+3D'}\right] x\cos C$$

$$= \left[\frac{1}{D+3D'}\right] x\cos(y+2x).$$

$$= \int x\cos(C + 3x + 2x)\,dx$$

$$= x\frac{\sin(5x+C)}{5} + \frac{\cos(5x+C)}{25}$$

$$= x\left[\frac{\sin(5x+y-3x)}{5}\right] + \left[\frac{\cos(5x+y-3x)}{25}\right]$$

$$= \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y).$$

The general solution is,

$$z = f_1(y+2x) + f_2(y-3x) + \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$

34. Solve $[D_x^3 - 4D_x^2 D_y + 4D_x D_y^2]\,z = 2\sin(3x+2y)$

The auxillary eqn is $m^3 - 4m^2 + 4m = 0.$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$\Rightarrow m = 0, 2, 2$$

$CF = f_1(y) + f_2(y+2x) + x f_3(y+2x).$

To find PI,

$$PI = \left[\frac{1}{D_x^3 - 4D_x^2 D_y + 4D_x D_y^2}\right](2\sin(3x+2y))$$

$(D_x^2 \to -9,\ D_y^2 \to -4)$

$$= 2\left[\cancel{\frac{1}{-9D_x - 4(-9)D_y + 4D_x(-4)}}\right]\cancel{\sin(3x+2y)}$$

$$= 2\left[\cancel{\frac{1}{-25D_x + 36D_y}}\right]\cancel{\sin(3x+2y)}.$$

$$= 2\left[\frac{1}{D_x(-9+24-16)}\right]\sin(3x+2y).$$

$$= -2\int \sin(3x+2y)\,dx$$

$$= \frac{2}{3}\cos(3x+2y).$$

The general solution is,

$$\underline{z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{2}{3}\cos(3x+2y)}$$

<u>Case 3</u>: When $\boxed{F(x,y) = x^m y^n}$, m and n being constants,

$$PI = \left[\frac{1}{F(D,D')}\right]x^m y^n = [F(D,D')]^{-1} x^m y^n$$

If $n < m$, $\left(\frac{1}{F(D,D')}\right)$ is expanded in powers of $\frac{D'}{D}$

If $m < n$, $\left(\frac{1}{F(D,D')}\right)$ is expanded in powers of $\frac{D}{D'}$.

35. Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

The eqn is $[D^2 + 3DD' + 2(D')^2]z = x + y$.

The auxillary eqn is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2.$

$$CF = f_1(y-x) + f_2(y-2x)$$

$$PI = \left[\frac{1}{D^2\left[1 + 3\left(\frac{D'}{D}\right) + 2\left(\frac{D'}{D}\right)^2\right]}\right]x + y$$

$$= \frac{1}{D^2}\left[1+\left\{3\left(\frac{D'}{D}\right)+2\left(\frac{D'}{D}\right)^2\right\}\right]^{-1}(x+y).$$

$$= \frac{1}{D^2}\left[1-3\left(\frac{D'}{D}\right) + 2\left(\frac{D'}{D}\right)^2\right](x+y).$$

$$= \frac{1}{D^2}\left[(x+y)-\frac{3}{D}[D'(x+y)] \mp \frac{2}{D^2}[D'^2(x+y)]\right]$$

$$= \frac{1}{D^2}\left[(x+y)-\frac{3}{D}(1)+\frac{2}{D^2}(0)\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}[y-2x].$$

$$= \frac{1}{D}\int(y-2x)\,dx$$

$$= \frac{1}{D}[xy-x^2]$$

$$= \int xy - x^2$$

$$= \frac{x^2y}{2}-\frac{x^3}{3}$$

The general solution is given by,

$$z = f_1(y-x)+f_2(y-2x)+\frac{x^2y}{2}-\frac{x^3}{3}$$

36. Solve $[D_x^3 - 2D_x^2 D_y]z = 3x^2y$

The auxillary eqn is $m^3-2m^2=0 \implies m=0,0,2.$

$CF = f_1(y)+xf_2(y)+f_3(y+2x).$

To find PI,

$$PI = \left[\frac{1}{D_x^3-2D_x^2D_y}\right]3x^2y$$

$$= 3\left[\frac{1}{D^3\left(1-\frac{2D'}{D}\right)}\right]x^2y$$

$$= \left(\frac{1}{D^3}\right)\left[1-2\left(\frac{D'}{D}\right)\right]^{-1}(3x^2y).$$

$$= \frac{1}{D^3}\left[1+2\left(\frac{D'}{D}\right)+4\left(\frac{D'}{D}\right)^2\right](3x^2y).$$

$$= \frac{1}{D^3}\left[3x^2y+\frac{2}{D}[D'(3x^2y)+\frac{4}{D^2}[D'^2(3x^2y)]\right].$$

$$= \frac{1}{D^3}\left[3x^2y+\frac{2}{D}(3x^2)+\frac{4}{D^2}(0)\right]$$

$$= \frac{1}{D^3}\left[3x^2y+2x^3\right]$$

$$= \frac{1}{D^2}\int(3x^2y+2x^3)\,dx$$

$$= \frac{1}{D^2}\left[\cancel{3}\left(\frac{x^3}{\cancel{3}}\right)y+2\left(\frac{x^4}{4}\right)\right]$$

$$= \frac{1}{D}\int\left(x^3y+\frac{x^4}{2}\right)dx$$

$$= \frac{1}{D}\left[\left(\frac{x^4}{4}\right)y+\left(\frac{x^5}{10}\right)\right]$$

$$= \int\left[\left(\frac{x^4}{4}\right)y+\left(\frac{x^5}{10}\right)\right]dx$$

$$= \frac{x^5y}{20}+\frac{x^6}{60}$$

The general solution is,

$$z = f_1(y) + xf_2(y) + f_3(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60}.$$

Case 4: Exponential Shift

When $\boxed{F(x,y) = e^{ax+by}\,V(x,y)}$ where $V(x,y)$ is any function of x and y.

$$PI = \left[\frac{1}{F(D,D')}\right] e^{ax+by}\, V(x,y)$$

$$\boxed{(D \to D+a)\,(D' \to D'+b).}$$

$$= e^{ax+by}\left[\frac{1}{F(D+a,\, D'+b)}\right] V(x,y)$$

37. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = (y-1)e^{x}$ (or) $z_{xx} - z_{xy} - 2z_{yy} = (y-1)e^{x}$

The eqn is $[D^2 - DD' - 2(D')^2]z = (y-1)e^{x}$

The auxillary eqn is $m^2 - m - 2 = 0$.

$m = 2, -1$.

$CF = f_1(y+2x) + f_2(y-x)$.

To find PI,

$$PI = \left[\frac{1}{D^2 - DD' - 2(D')^2}\right](y-1)e^{x}$$

$$(D \to D+1,\; D' \to D'+0)$$

$$= e^{x}\left[\frac{1}{(D+1)^2 - (D+1)(D') - 2(D')^2}\right](y-1)$$

$$= e^{x}\left[\frac{1}{D^2 + 1 + 2D - DD' - D' - 2(D')^2}\right](y-1)$$

$$= e^{x}\left[1 + (D^2 + 2D - DD' - D' - 2(D')^2)\right]^{-1}(y-1)$$

$$= e^{x}\left[1 - D^2 - 2D + DD' + D' + 2(D')^2\right](y-1)$$

$$= e^{x}\left[(y-1) - 0 - 0 - 0 + 1 + 0\right] \quad = e^{x}y$$

The general solution is,

$$z = f_1(y+2x) + f_2(y-x) + e^{x}y$$

38. Solve $(D-3D'-2)^3 z = 6e^{2x}\sin(3x+y)$.

The auxillary eqn is $(m-3-2)^3 = 0$

$$\Rightarrow (m-5)^3 = 0$$

$$\Rightarrow m = 5, 5, 5.$$

$CF = f_1(y+5x) + x f_2(y+5x) + x^2 f_3(y+5x)$

To find PI,

$$PI = \left[\frac{1}{(D-3D'-2)^3}\right](6e^{2x})[\sin(3x+y)]$$

Replace D by D+2 and D' by D'+0,

$$= \left[\frac{1}{[(D+2)-3D'-2]^3}\right](6e^{2x})[\sin(3x+y)]$$

$$= \left[\frac{1}{(D-3D')^3}\right](6e^{2x})[\sin(3x+y)]$$

$$= \left[\frac{1}{D^3-27(D')^3-3^2DD'(D-3D')}\right]6e^{2x}[\sin(3x+y)]$$

$$= \left[\frac{1}{D^3-27(D')^3-9D^2D'+27D(D')^2}\right]6e^{2x}[\sin(3x+y)].$$

$(D^2\to -9,\ (D')^2\to -1,\ DD'\to -3)$

$$= 6e^{2x}\left[\frac{1}{-9D+27D'+81D'-27D}\right]\sin(3x+y)$$

$$= 6e^{2x}\left[\frac{1}{108D'-36D}\right]\sin(3x+y)$$

$$= \frac{6e^{2x}}{-36}\left[\frac{1}{-3D'+D}\right]\sin(3x+y)$$

$$= \frac{-e^{2x}}{6}\left[\frac{D+3D'}{D^2-9(D')^2}\right]\sin(3x+y)$$

$$= \frac{-e^{2x}}{6}\left[\frac{1}{(D-3D')(D+3D')}\right]\sin(3x+y)$$

$$= -\frac{e^{2x}}{6}\left[\frac{1}{D-3D'}\right]\sin(3x+C-3x)$$

$$= \frac{-e^{2x}}{6}\int \sin C\,dx = \frac{-e^{2x}}{6}\,x\sin(y+3x)$$

$$z = -\frac{e^{2x}}{6}(x)\sin(y+3x) + f_1(y+5x) + x f_2(y+5x) + x^2 f_3(y+5x)$$

39. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x\,\partial y} = \sin x \cos 2y$

The eqn is $z[D^2 - DD'] = \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$

The auxillary eqn is $m^2 - m = 0$

$\Rightarrow m(m-1) = 0$

$\Rightarrow m = 0, 1$

$CF = f_1(y) + f_2(y+x)$

The PI is,

$$PI = \frac{1}{2}\left[\frac{1}{D^2-DD'}\right]\sin(x+2y) + \frac{1}{2}\left[\frac{1}{D^2-DD'}\right]\sin(x-2y)$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\right]\sin(x+2y) + \frac{1}{2}\left[\frac{1}{-1-2}\right]\sin(x-2y)$$

$$= +\frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

The general solution is,

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

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40. Solve $[D^3 - 7D(D')^2 - 6(D')^3]z = \sin(x+2y) + e^{2x+y}$

The auxillary eqn is $m^3 - 7m - 6 = 0$

$$m = 3, -1, -2$$

$CF = f_1(y+3x) + f_2(y-x) + f_3(y-2x)$.

The PI given by, (two parts)

$$PI_1 = \left[\frac{1}{D^3 - 7D(D')^2 - 6(D')^3}\right] \sin(x+2y)$$

$D^2 \to -1$, $(D')^2 \to -4$, $DD' \to -2$

$$= \left[\frac{1}{27D + 24D'}\right] \sin(x+2y)$$

$$= \frac{1}{3}\left[\frac{1}{9D+8D'}\right] \sin(x+2y)$$

$$= \frac{1}{3}\left[\frac{9D-8D'}{81D^2 - 64(D')^2}\right] \sin(x+2y)$$

$$= \frac{1}{525}\left[-9\frac{\cos(x+2y)}{1} + 8\frac{\cos(x+2y)}{2}\right]$$

$$= \frac{1}{525}\left[4\cos(x+2y) - 9\cos(x+2y)\right]$$

$$= \left(\frac{4}{525}\right)\cos(x+2y) - \left(\frac{3}{175}\right)\cos(x+2y).$$

$$PI_2 = \left[\frac{1}{D^3 - 7D(D')^2 - 6(D')^3}\right] e^{2x+y}$$

$D \to 2$, $D' \to 1$

$$= \frac{-1}{12} e^{2x+y}$$

The general solution is given by,

$$z = CF + PI_1 + PI_2$$

$$z = f_1(y+3x) + f_2(y-x) + f_3(y-2x) - \frac{1}{12}e^{2x+y}$$

$$+\left(\frac{4}{525}\right)\cos(x+2y) - \left(\frac{3}{175}\right)\cos(x+2y)$$