Solution of Lagrange's linear PDE of first order in the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Working Procedure:

1. Compare the given equation with $Pp + Qq = R$ and write P, Q, R.
2. Form the auxillary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
3. Solve the auxillary eqns by the method of grouping or the method of multipliers or both to get two independent solutions, where a, b are arbitrary constants.
4. Then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

Type 1:

Here, the solution of $Pp + Qq = R$ is obtained by taking two members of the auxillary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ at a time and then integrating to have two independent solutions in variables whose differentials are involved in equation.

Solve the Lagrange's Linear PDE :

1. $2yzp + xzq = 3xy$

This is of the form $Pp + Qq = R$ where

$P = 2yz$, $Q = xz$, $R = 3xy$

The auxillary equation is,

$$\frac{dx}{2yz} = \frac{dy}{xz} = \frac{dz}{3xy}$$

From the first two terms,

$$\Rightarrow \frac{dx}{2y\cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$\Rightarrow x\,dx = 2y\,dy$$

$$\Rightarrow \frac{x^2}{2} = y^2 + C_1$$

$$\Rightarrow x^2 - 2y^2 = 2C_1$$

From the second and last term,

$$\frac{dy}{xz} = \frac{dz}{3\cancel{x}y}$$

$$\Rightarrow 3y\,dy = z\,dz$$

$$\Rightarrow \frac{3y^2}{2} = \frac{z^2}{2} + C_2$$

$$\Rightarrow 3y^2 - z^2 = 2C_2$$

The solution is given by

$$\phi(x^2 - 2y^2,\ 3y^2 - z^2) = 0.$$

2. $p \cot x + q \cot y = \cot z$

The eqn is of the form $Pp + Qq = R$ where

$P = \cot x$, $Q = \cot y$, $R = \cot z$.

The auxillary equation is,

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

From first two terms,

$$\Rightarrow \frac{dx}{\cot x} = \frac{dy}{\cot y}$$

$$\Rightarrow \tan x \, dx = \tan y \, dy$$

$$\Rightarrow \log(\sec x) = \log(\sec y) + \log(C_1)$$

$$\Rightarrow \frac{\sec x}{\sec y} = C_1$$

From second and last term,

$$\Rightarrow \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\Rightarrow \tan y \, dy = \tan z \, dz$$

$$\Rightarrow \log(\sec y) = \log(\sec z) + \log(C_2)$$

$$\Rightarrow \frac{\sec y}{\sec z} = C_2$$

The solution is given by,

$$\phi\left(\frac{\sec x}{\sec y}, \frac{\sec y}{\sec z}\right) = 0$$

3. $y^2 z p = x^2(zq + y)$

$y^2 z p - x^2 z q = x^2 y$

The equation is of the form $Pp + Qq = R$

where $P = y^2 z$, $Q = -x^2 z$, $R = x^2 y$

The auxillary equation is,

$$\frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

From the first two terms,

$$\Rightarrow \frac{dx}{y^2 \cancel{z}} = \frac{dy}{-x^2 \cancel{z}}$$

$$\Rightarrow +x^2 dx = -y^2 dy$$

$$\Rightarrow \frac{+x^3}{3} = \frac{-y^3}{3} + C_1$$

$$\Rightarrow \frac{+x^3}{3} + \frac{y^3}{3} = C_1 \quad \Rightarrow x^3 + y^3 = 3C_1$$

From the second and last terms,

$$\Rightarrow \frac{dy}{-\cancel{x^2} z} = \frac{dz}{\cancel{x^2} y}$$

$$\Rightarrow y\,dy = -z\,dz$$

$$\Rightarrow \frac{y^2}{2} + \frac{z^2}{2} = C_2 \quad \Rightarrow y^2 + z^2 = 2C_2$$

The solution is given by,

$$\phi\left(x^3 + y^3, y^2 + z^2\right) = 0$$

4. $\frac{y^2 z}{x} p + xzq = y^2$

The equation is of the form $Pp + Qq = R$ where

$P = y^2 z$, $Q = x^2 z$, $R = y^2 x$

The auxillary equation is,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

From first two terms,

$$\Rightarrow \frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$\Rightarrow x^2 dx = y^2 dy$$

$$\Rightarrow x^3 - y^3 = 3C_1$$

From first and last terms,

$$\Rightarrow \frac{dx}{\cancel{y^2} z} = \frac{dz}{x \cancel{y^2}}$$

$$\Rightarrow x\,dx = z\,dz$$

$$\Rightarrow x^2 - z^2 = 2C_2$$

The solution is given by,

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

5. $y^2p - xyq = x(z-2y)$

$y^2p - xyq = x(z-2y)$.

The eqn. is of the form $Pp + Qq = R$ where

$P = y^2$, $Q = -xy$ and $R = x(z-2y)$.

The auxillary eqn is,

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

From the first two terms,

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\Rightarrow +x\,dx = -y\,dy$$

$$\Rightarrow x^2 + y^2 = 2C_1$$

From second and last terms,

$$\Rightarrow \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow (z-2y)\,dy = -y\,dz$$

$$\Rightarrow y\,dz + z\,dy = 2y\,dy$$

$$\Rightarrow d(yz) = 2y\,dy$$

$$\Rightarrow yz = y^2 + C_2$$

$$\Rightarrow yz - y^2 = C_2$$

The solution is given by,

$$\underline{\phi(x^2+y^2,\ yz-y^2) = 0.}$$

Type-2 :

In this type, the solution of $Pp+Qq=R$ is obtained by taking two members of the auxillary equation and integrate to have an equation (one independent solution) in the variables whose differentials are involved and another independent solution is obtained by making the use of the first solution (integral).

6. $2p+q=\sin(x-2y)$.

The eqn is of the form $Pp+Qq=R$ where $P=2$, $Q=1$, $R=\sin(x-2y)$

The auxillary eqn is,

$$\frac{dx}{2}=\frac{dy}{1}=\frac{dz}{\sin(x-2y)}$$

From first two terms,

$$\Rightarrow \frac{dx}{2}=\frac{dy}{1}$$

$$\Rightarrow dx = 2\,dy$$

$$\Rightarrow x-2y=C_1$$

From second and last term,

$$\Rightarrow \frac{dy}{1}=\frac{dz}{\sin(x-2y)}$$

$$\Rightarrow (\sin C_1)\,dy = dz$$

$$\Rightarrow y\sin C_1 - z = C_2$$

$$\Rightarrow y\sin(x-2y)-z=C_2$$

The solution is given by,

$$\phi(x-2y,\ y\sin(x-2y)-z)=0.$$

7. $p+3q=5z-\tan(3x-y)$.

The equation is of the form $Pp+Qq=R$ where

$P=1,\ Q=3,\ R=5z-\tan(3x-y)$.

The auxillary equation is,

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z-\tan(3x-y)}$$

From first two terms,

$$\Rightarrow \frac{dx}{1}=\frac{dy}{3}$$

$$\Rightarrow 3dx=dy$$

$$\Rightarrow 3x-y=C_1$$

From last two terms,

$$\Rightarrow \frac{dy}{3}=\frac{dz}{5z-\tan C_1}$$

$$\Rightarrow (5z-\tan C_1)dy=3dz$$

$$\Rightarrow 5yz-y\tan C_1=3z+C_2$$

$$\Rightarrow 5yz-y\tan(3x-y)-3z=C_2$$

The solution is given by,

$$\phi[(3x-y),\ 5yz-y\tan(3x-y)-3z]=0.$$

8. $(p-q)(x+y) = z$

$p(x+y) - q(x+y) = z$

The equation is of the form $Pp + Qq = R$ where $P = x+y$ and $Q = -(x+y)$, $R = z$.

The auxillary equation,

$$\frac{dx}{x+y} = \frac{dy}{-(x+y)} = \frac{dz}{z}$$

From first two terms,

$\Rightarrow (\cancel{x+y})\,dx = -dy\,(\cancel{x+y})$

$\Rightarrow dx = -dy$

$\Rightarrow x + y = C_1$

From second and last term,

$$\Rightarrow \frac{dx}{+(x+y)} = \frac{dz}{z}$$

$\Rightarrow z\,dx = +C_1\,dz$

$\Rightarrow x = C_1 \log z + C_2$

$\Rightarrow x - (x+y)\log z = C_2$

The general solution is given by,

$\Rightarrow \phi\left(x+y,\ x-(x+y)\log z\right) = 0.$

9. $xzp + yzq = xy$

The auxillary equation is,

$$\Rightarrow \frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

From first two terms,

$$\Rightarrow \frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}}$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \log C_1$$

$$\Rightarrow \frac{x}{y} = C_1$$

From second and last terms,

$$\Rightarrow \frac{dy}{\cancel{y}z} = \frac{dz}{x\cancel{y}}$$

$$\Rightarrow x\,dy = z\,dz$$

$$\Rightarrow (yC_1)dy = \frac{z^2}{2} + C_2$$

$$\Rightarrow C_1\left(\frac{y^2}{2}\right) = \left(\frac{z^2}{2}\right) + C_2$$

$$\Rightarrow \left(\frac{x}{y}\right)\left(\frac{y^2}{2}\right) = \frac{z^2}{2} + C_2$$

$$\Rightarrow xy - z^2 = 2C_2$$

The solution is given by,

$$\phi\left(\frac{x}{y}, xy - z^2\right) = 0.$$

Type - 3 :

Solution of $Pp + Qq = R$ by the method of multipliers. The set of multipliers are given by:

x	y	z
x	$-y$	$-z$
$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{z}$
$\frac{1}{x^2}$	$\frac{1}{y^2}$	$\frac{1}{z^2}$
1	1	1
l	m	n
$-x$	$-y$	$-z$
1	-1	-1
1	-1	0

10. Solve $(y^2 + z^2)p + x(yq - z) = 0$.

$(y^2+z^2)p + xyq = xz$

The equation of the form $Pp + Qq = R$ where $P = y^2 + z^2$, $Q = xy$ and $R = xz$.

The auxillary equation is,

$$\Rightarrow \frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz} \longrightarrow (1)$$

From ~~first~~ two terms, (last)

$$\Rightarrow \frac{dy}{xy} = \frac{dz}{xz} \Rightarrow \frac{y}{z} = C_1$$

Choose the set of multipliers $x, -y, -z$ such that each ~~eqn~~ ratio in ① is equal to

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{xy^2 + xz^2 - xy^2 - xz^2}$$

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$\Rightarrow x\,dx - y\,dy - z\,dz = 0.$$

On integrating,

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = C_2$$

$$\Rightarrow x^2 - y^2 - z^2 = 2C_2$$

The solution is given by,

$$\phi\left(\frac{y}{z}, x^2 - y^2 - z^2\right) = 0.$$

11. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

The auxillary equation is given by,

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \longrightarrow ①$$

Choose the set of multipliers x, y, z such that each ratio in ① is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2}}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

On integrating,

$\Rightarrow x^2 + y^2 + z^2 = 2C_1$

Choose the set of multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ such that each ratio in ① is equal to,

$$\Rightarrow \frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{\cancel{y^2} - \cancel{z^2} + \cancel{z^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}$$

$$\Rightarrow \frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{0}$$

$$\Rightarrow \left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz = 0.$$

On integrating,

$\Rightarrow \log x + \log y + \log z = \log C_2$

$\Rightarrow xyz = C_2$

The solution is given by,

$\underline{\underline{\phi(x^2 + y^2 + z^2,\ xyz) = 0.}}$

12. $(x^2-y^2-z^2)p + 2xy\,q = 2xz$

The auxillary equation is given by,

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \longrightarrow (1).$$

From second and last terms,

$$\Rightarrow \frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log C_1$$

$$\Rightarrow \frac{y}{z} = C_1$$

Choose set of multipliers x, y, z such that each ratio in (1) is equal to,

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

Equating to 2^{nd} term of (1),

$$\Rightarrow \frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{1}{2}\log y = \frac{2x\,dx + 2y\,dy + 2z\,dz}{2(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{1}{2}\log y = \frac{1}{2}\log(x^2+y^2+z^2) + \log C_2$$

$$\Rightarrow \frac{1}{2}\log\left[\frac{y}{x^2+y^2+z^2}\right] = \log C_2$$

$$\Rightarrow \frac{y}{x^2+y^2+z^2} = (C_2)^2$$

The solution is given by,

$$\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0.$$

13. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Auxillary eqn is,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{(x-y)z^2} \longrightarrow (1)$$

Choose set of multipliers, $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ such that each ratio in (1) is,

$$\Rightarrow \frac{\frac{1}{x^2}(dx) + \frac{1}{y^2}(dy) + \frac{1}{z^2}(dz)}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$\Rightarrow \frac{1}{x^2}(dx) + \frac{1}{y^2}(dy) + \frac{1}{z^2}(dz) = 0.$$

On integrating,

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -C_1$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1$$

Choose the set of multipliers, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ such that each ratio in ① is,

$$\Rightarrow \frac{\frac{1}{x}(dx) + \frac{1}{y}(dy) + \frac{1}{z}(dz)}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{xz} - \cancel{zy}}$$

$$\Rightarrow \frac{1}{x}(dx) + \frac{1}{y}(dy) + \frac{1}{z}(dz) = 0$$

On integrating,

$$\Rightarrow \log(x) + \log(y) + \log z = \log C_2$$

$$\Rightarrow xyz = C_2$$

The solution is given by,

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$$

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14. Solve $(x^2 - y^2 - zy)p + (x^2 - y^2 - zx)q = z(x-y)$

The auxillary equation is,

$$\frac{dx}{x^2 - y^2 - zy} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x-y)} \longrightarrow ①$$

Choose set of multipliers, 1, -1, -1 such that each ratio in ① is,

$$\frac{dx - dy - dz}{\cancel{x^2} - \cancel{y^2} - \cancel{zy} - \cancel{x^2} + \cancel{y^2} + \cancel{zx} - \cancel{zx} + \cancel{zy}}$$

$$\Rightarrow dx - dy - dz = 0.$$

$$\Rightarrow x - y - z = C_1$$

Choose set of multipliers 1, −1, 0 such that each ratio in ① is,

$$\Rightarrow \frac{dx - dy + 0}{\cancel{x^2} - \cancel{y^2} - yz - \cancel{x^2} + \cancel{y^2} + zx + 0}$$

$$\Rightarrow \frac{dx - dy}{z(x-y)}.$$

This cannot be simplified,

Choose set of multipliers $x, -y, 0$ such that each ratio of ① is,

$$\Rightarrow \frac{x\,dx - y\,dy + 0}{x^3 - xy^2 - \cancel{xyz} - yx^2 + y^3 + \cancel{xyz}}$$

$$\Rightarrow \frac{x\,dx - y\,dy}{x^2(x-y) - y^2(x-y)}$$

$$\Rightarrow \frac{x\,dx - y\,dy}{(x+y)(x-y)(x-y)}$$

Equating this to ①,

$$\frac{x\,dx - y\,dy}{(x+y)(x-y)\cancel{(x-y)}} = \frac{dz}{z\cancel{(x-y)}}$$

On integrating,

$$\frac{1}{2}\log(x^2-y^2) = \log z + \log C_2$$

$$\log(x^2-y^2) = 2\log z + \log C_2^2$$

$$C_2^2 = \frac{x^2-y^2}{z^2}$$

The general solution is,

$$\phi\left(x-y-z, \frac{x^2-y^2}{z^2}\right) = 0.$$

15. $(y+z)p + (z+x)q = x+y$ (★★★★)

Auxillary equation is,

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \longrightarrow (1).$$

Choose the set of multipliers $(1,-1,0)$ and $(1,0,-1)$ such ~~so~~ that each ratio in (1) is,

$$\Rightarrow \frac{dx-dy+0}{y+\cancel{z}-\cancel{z}-x+0}$$

$$\Rightarrow \frac{dx-dy}{y-x} \longrightarrow (2)$$

$$\Rightarrow \frac{dx+0-dz}{\cancel{y}+z+0-x-\cancel{y}}$$

$$\Rightarrow \frac{dx-dz}{z-x} \longrightarrow (3).$$

From (2) and (3),

$$\frac{dx-dy}{-(x-y)} = \frac{dx-dz}{-(x-z)}.$$

On integrating,

$$-\log(x-y) = -\log(x-z) - \log C_1$$

$$\log(x-y) = \log(x-z) + \log C_1$$

$$\left(\frac{x-y}{x-z}\right) = C_1 \longrightarrow (4).$$

Choose the set of multipliers $(1,-1,0)$ and $(1,1,1)$ such that each ratio in (1) is,

$$\frac{dx-dy}{y+\cancel{z}-\cancel{z}-x}$$

$$\Rightarrow \frac{dx-dy}{-(x-y)}$$

$$\Rightarrow \frac{dx+dy+dz}{2(x+y+z)}$$

On integrating,

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) - \log C_2$$

$$\Rightarrow -\log(x-y)^2 - \log(x+y+z) = +(2)\log C_2$$

$$\Rightarrow (x-y)^2(x+y+z) = C_2$$

The solution is,

$$\phi\left(\frac{x-y}{x-z},\ (x-y)^2(x+y+z)\right) = 0.$$

16. Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Auxillary eqn is,

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \longrightarrow ①$$

Choose the set of multipliers (y, z, x) such that each ratio in ① is,

$$\frac{y\,dx + z\,dy + x\,dz}{\cancel{x^2y} - \cancel{y^2z} + \cancel{y^2z} - \cancel{z^2x} + \cancel{z^2x} - \cancel{x^2y}}$$

$$\Rightarrow y\,dx + z\,dy + x\,dz = 0$$

On integrating,

$$\Rightarrow xy + yz + xz = C_1 \longrightarrow ②$$

From ①, $\dfrac{dx-dy}{x^2-y^2-yz+zx} = \dfrac{dy-dz}{y^2-z^2-zx+xy}$

$$\Rightarrow \frac{d(x-y)}{(x+y)(x-y)+z(x-y)} = \frac{d(y-z)}{(y+z)(y-z)+x(y-z)}$$

$$\Rightarrow \frac{d(x-y)}{\quad} = \frac{d(y-z)}{\quad}$$

$|||^{ly}$ from ①, $\dfrac{dy-dz}{y^2-z^2-zx+xy} = \dfrac{dz-dx}{z^2-x^2-xy+yz}$

$\Rightarrow \dfrac{d(y-z)}{(y-z)\cancel{(x+y+z)}} = \dfrac{d(z-x)}{(z-x)\cancel{(z+x+y)}}$

$\rightarrow \log(y-z) = \log(z-x) + \log C_3$

$\rightarrow \left(\dfrac{y-z}{z-x}\right) = C_3 \longrightarrow$ ④.

From ②, ③, ④,

$\phi\left(\dfrac{x-y}{y-z}, \dfrac{y-z}{z-x}\right) = 0$ OR $\phi\left(xy+yz+zx, \dfrac{x-y}{y-z}\right) = 0$

OR $\phi\left(xy+yz+zx, \dfrac{y-z}{z-x}\right) = 0$

17. Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

The equation is of the form $Pp + Qq = R$ where

$P = x(y^2+z)$, $Q = -y(x^2+z)$, $R = (x^2-y^2)z$

The auxillary eqn is,

$\dfrac{dx}{x(y^2-z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{(x^2-y^2)z} \longrightarrow$ ①

Choose the set of multipliers $\left(\frac{1}{x}, \frac{1}{y}, +\frac{1}{z}\right)$ such that the ratio in ① is,

$\dfrac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{\cancel{y^2} - z - \cancel{x^2} - z + \cancel{x^2} - \cancel{y^2}}$

$= \dfrac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{-2z}$

Equating with ①,

$$\frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{-2\cancel{z}} = \frac{dz}{(x^2-y^2)\cancel{z}}$$

$$\Rightarrow \left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz = \left[\frac{-2}{x^2-y^2}\right]dz$$

$$\Rightarrow \log x + \log y + \log z = \frac{-2z}{x^2-y^2} + \cancel{lo}\, C_1$$

$$\Rightarrow xyz + \frac{2z}{x^2-y^2} = C_1$$

Choosing set of multipliers $(x, y, -1)$,

$$\frac{x\,dx + y\,dy - dz}{\cancel{x^2y^2} - x^2z - \cancel{y^2x^2} - \cancel{y^2z} - \cancel{x^2z} + \cancel{y^2z}}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{-2x^2z}$$

Equating with (1),

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{-2x^2\cancel{z}} = \frac{dz}{(x^2-y^2)\cancel{z}}$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = \left(\frac{-2x^2}{x^2-y^2}\right)z + C_2$$

$$\Rightarrow x^2 + y^2 - 2z + \frac{4x^2z}{x^2-y^2} = +2C_2$$

Then, we can conclude,

$$\phi\left(xyz + \frac{2z}{x^2-y^2},\ x^2+y^2-2z+\frac{4x^2z}{x^2-y^2}\right) = 0.$$

18. Solve $(x+2z)p + (4zx-y)q = 2x^2+y$

Auxillary eqn is,

$$\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y} \longrightarrow (1)$$

Choose set of multipliers $(y, x, -2z)$ and $(2x, -1, -1)$ such that ratio of (1) is,

$$\frac{y\,dx + x\,dy - 2z\,dz}{\cancel{xy} + \cancel{2zy} + \cancel{4zx^2} - \cancel{xy} + 2z\cancel{(2x^2)} - \cancel{2zy}}$$

$\Rightarrow y\,dx + x\,dy - 2z\,dz = 0.$

On integrating,

$xy - z^2 = C_1$

Using other set of multipliers,

$$\frac{2x\,dx - dy - dz}{2\cancel{x^2} + \cancel{4xz} - \cancel{4zx} + \cancel{y} - \cancel{2x^2} - \cancel{y}}$$

$\Rightarrow 2x\,dx - dy - dz = 0$

On integrating,

$x^2 - y - z = C_2$

The solution is given by,

$\phi(xy - z^2, x^2 - y - z) = 0.$