ENGINEERING MATHEMATICS - II

Random Variables and Probability Distributions

Department of Science and Humanities

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UNIT 2 : Random Variables and Probability Distributions

Session: 5

Subtopic: Binomial Distribution

Binomial Distribution

- Binomial distribution is a discrete probability distribution.
- The binomial random variable X is the number of successes in Bernoulli trials with the number of times that the event has occurred out of n trials. The possible values of X are $0, 1, 2, \dots, n$.
- Since X take only integer values X is discrete.
- Thus binomial distribution is a discrete probability distribution.
- For a random experiment E, if event A happens, then we call it a success otherwise it is a failure.
- We associate a probability of success $P(A) = p$, and the probability of failure defined as $P(\bar{A}) = q = 1 - p$.

Prerequisite

- Each trial results in two disjoint outcomes (a success or a failure).
- The number of trials made (n) is finite.
- The trials are independent.
- $P(success) = p$ is a constant for trial.

Examples that give rise to binomial distribution:

- Throwing a dice.
- Tossing of coins.
- Drawing a card/cards etc…

A random variable X is said to follow the binomial distribution if the probability distribution function is given by

$P(X = r) = P(r) = nC_r p^r q^{n-r}, r = 0, 1, 2, \ldots, n$ where $q = 1 - p$.

Note:

1. $\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n} P(r) = \sum_{r=0}^{n} nC_r p^r q^{n-r} = (p + q)^n = 1$.

2. The frequency function of the binomial distribution is defined by

   $f(r) = N * P(r)$ where N is the number of times the experiment is repeated.

Binomial Distribution – Derivation of Mean

$$\text{Mean} = \text{expectation} = \mu = E(x)$$

$$= \sum_{x=0}^{n} xP(x)$$

$$= \sum_{x=0}^{n} x * nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x\, \frac{n!}{x!(n-x)!}\, p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} (n-1)C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np(p+q)^{n-1} = np(1) = np.$$

Binomial Distribution – Derivation of Variance

$$\text{Variance} = \sigma^2 = \sum_{x=0}^{n}(x-\mu)^2 P(x)$$

$$= \sum_{x=0}^{n}(x^2 - 2\mu x + \mu^2)P(x)$$

$$= \sum_{x=0}^{n} x^2 \; P(x) - 2\mu \sum_{x=0}^{n} x \; P(x) + \mu^2 \sum_{x=0}^{n} P(x)$$

$$= \sum_{x=0}^{n} x^2 \; P(x) - 2\mu . np + \mu^2 . 1$$

$$= \sum_{x=0}^{n} x^2 \; P(x) - (np)^2 \quad \underline{\hspace{4cm}} (1)$$

Binomial Distribution – Derivation of Variance Cont...

Consider $\sum_{x=0}^{n} x^2 \; P(x)$

$$= \sum_{x=0}^{n} x^2 . nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \; \frac{n!}{x!(n-x)!} \; p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \; [x(x-1)+x] \frac{n!}{x!(n-x)!} \; p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \;\; x(x-1). \frac{n!}{x!(n-x)!} \; p^x q^{n-x} + \sum_{x=1}^{n} \;\; x \frac{n!}{x!(n-x)!} \; p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \; \frac{n!}{(x-2)!(n-x)!} \; p^x q^{n-x} + np$$

Binomial Distribution – Derivation of Variance Cont...

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,((n-2)-(x-2))!}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \,.\, (n-2)C_{(x-2)}\; p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$= n(n-1)p^2 + np, \text{ since } p+q=1$$

Thus variance $= \sigma^2 = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq.$

Binomial Distribution – Example

1. Assume that 50% of all engineering students are good in mathematics.

   Determine the probabilities that among 18 engineering students

   (i) exactly 10,

   (ii) at least 10,

   (iii) at most 8,

   (iv) at least 2 and at most 9, are good in Maths.

Binomial Distribution – Example

Solution:

Let X = number of engineering students who are good in Maths.

p = probability of a student good in Maths $= 50\% = \frac{50}{100} = \frac{1}{2}$

$n = 18$ and $q = 1 - p = \frac{1}{2}$

$$P(x) = nC_x p^x q^{n-x} = 18C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{18-x}$$

(i) Exactly 10 students are good in Maths out of 18

$$P(x = 10) = 18C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{18-10} = 0.1670$$

Binomial Distribution – Example

(ii) At least 10 students are good in Maths out of 18

$$P(x \geq 10) = P(11) + P(12) + \cdots + P(18)$$

$$= 18C_{11}\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right)^{7} + \cdots + 18C_{18}\left(\frac{1}{2}\right)^{18}\left(\frac{1}{2}\right)^{0} = 0.4073$$

(iii) At most 8 students are good in Maths out of 18

$$P(x \leq 8) = P(1) + P(1) + \cdots + P(1)$$

$$= 18C_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{17} + \cdots + 18C_{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{10} = 0.4073$$

(iv) At least 2 and at most 9 are good in Maths out of 18

$$P(2 \leq x \leq 9) = P(2) + P(3) + \cdots + P(9)$$

$$= 18C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{16} + \cdots + 18C_{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{9} = 0.5920$$

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