PARTIAL DIFFERENTIAL EQUATIONS

Introduction:

An equation involving partial differential coefficients of a function of two or more variables is known as a **partial differential equation**. If a partial differential equation contains n^{th} and lower order derivatives, it is said to be of n^{th} order PDE. The **degree** of such equation is the greatest exponent of the highest order. Further such equation will be called **linear**, if it is a first degree in the dependent variable and its partial derivatives (i.e. powers or products of the dependent variable and its partial derivatives must be absent). An equation which is not linear is called a **non-linear** differential equation.

In case of two dependent variables we usually assume them to be x and y and z to be dependent on x and y. If there are n-independent variables we take them to be

x_1 , x_2 , x_3 ,……. x_n and z is then regarded as the dependent variable.

Classification of 1st order partial differential equations:

Semi Linear Equations: A first order partial differential equation F(x, y, z, a, b)=0 is known as a semi linear equation if it is linear in a and b and the coefficients of a and b are functions of x and y only.

Quasi linear Equations: A first order partial differential equation F(x, y, z, a, b)=0 is known as quasi linear partial differential equation if it is linear in a and b.

Non Linear Equations: A first order partial differential equation F(x, y, z, a, b)=0 which does not come under above types is known as nonlinear equation.

Examples:

- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is a wave equation , second order, first degree, linear , homogenous.

- $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is a heat equation , second order, linear, homogenous.
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is a Laplace's equation, second order , linear , homogenous.
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ is a second order, linear and non-homogeneous.
- (i) $p^2 + q^2 = 1$ (ii) $pq = z$ is a non-linear PDE.

The following are the standard notations when z = f (x, y) are used:

$$\frac{\partial z}{\partial x} = p \frac{\partial z}{\partial y} = q \frac{\partial^2 z}{\partial x^2} = r \frac{\partial^2 z}{\partial x \partial y} = s \frac{\partial^2 z}{\partial y^2} = t$$

Solution of partial differential equations by direct method or direct integration:

1. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2 \sin y$ when x=0 and z=0 when y is an odd multiple of $\frac{\pi}{2}$.

Solution:

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$$

Integrating with respect to x (keeping y constant)

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When x=0 and $\frac{\partial z}{\partial y} = -2 \sin y$. Then we get f(y)= - siny

Substituting the value of f(y) in equation (1),

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y.$$

Integrating again with respect to y (keeping x constant)

$$z = \cos x \cos y + \cos y + g(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

When y is an odd multiple of $\frac{\pi}{2}$ at z=0. This implies g(x) =0

$$z = \cos x \cos y + \cos y = \cos y (1 + \cos x)$$

2. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = cos(2x + 3y)$

Solution:

$$\frac{\partial^3 z}{\partial x^2 \partial y} = cos(2x + 3y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrate eqn (1) wrt x (keeping y constant)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{sin(2x + 3y)}{2} + f(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Again integrating eqn (2) wrt x (keeping y constant)

$$\frac{\partial z}{\partial y} = -\frac{cos(2x + 3y)}{4} + f(y)x + g(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Again integrating eqn (3) wrt y (keeping x constant)

$$z = -\frac{sin(2x + 3y)}{12} + x\int f(y)dy + \int g(y)dy + \square(x)$$

Linear Partial Differential Equations of 1st Order

The general form of a quasi-linear partial differential equation of 1st order is:

$P(x,y,z)\, z_x + Q(x,y,z) z_y = R(x,y,z)$ ---------------(1)

This equation (1) is known as **Lagrange's linear equation**.

If P and Q are independent of Z and R is linear in Z then (1) is a linear equation. The general solution of Lagrange's linear PDE:

$Pp + Qq = R$ --------------- (1)

ss given by the equation;

$F(u,v)=0$ ---------------(2)

Since the elimination of arbitrary function F from (2) results in (1).

Here $u=u(x,y,z)$, $v=v(x,y,z)$ are known functions.

Method of obtaining general solution:

1) Rewrite the equation in the standard form Pp+Qq=R
2) Form the Lagrange's auxiliary equation (A.E)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{---------------- (3)}$$

3) $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$ are said to be the complete solution of the system of simultaneous equations (provided u_1 and u_2 are linearly independent i.e u_1/u_2 not equal to a constant)

Case 1: One of the variable is either absent or cancells out from the set of auxiliary equations.

Case 2: if $u=c_1$ is known but $v=c_2$ is not possible by case 1, then use $u=c_1$ to get $v=c_2$

Case 3: Introducing Lagrange multipliers P_1, Q_1, R_1 which are functions of x,y z or constants, each fraction in (3) is equal to:

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + Q_1 R} \cdots\cdots\cdots\cdots (4)$$

If P_1 , Q_1 , R_1 are chosen that $P_1P + Q_1Q + R_1R=0$ then $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated.

Case 4: Multipliers are chosen (more than once) such that the numerator $P_1dx + Q_1dy + R_1dz$ is an exact differential of denominator $P_1P + Q_1Q + R_1R$. Now combine (4) with a fraction of (3) to get an integral.

4) General solution of (1) is $F(u,v) = 0$ or $v = \varphi(u)$.

Examples:

1) Solve: $(mz-ny)\frac{\partial z}{\partial x} + (nx-lz)\frac{\partial z}{\partial y} = ly-mx$

Solution: The given equation is of the form Pp+Qq=R

The auxiliary equations are

$$\frac{dx}{(mz-ny)} = \frac{dy}{(nx-lz)} = \frac{dz}{(ly-mx)} \qquad \text{---------------- (1)}$$

Using multipliers l, m, n each ratio is equal to

$$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}$$
$$\Rightarrow ldx+mdy+ndz=0$$

which on integration gives $lx+my+nz=c_1$

Using multipliers x, y, z each ratio in (1) is equal to $\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}$

$$\Rightarrow xdx+ydy+zdz=0$$

which on integration gives

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2 or x^2+y^2+z^2=c_2$$

Hence the general solution is $\varphi\,(lx+my+nz\,,\,x^2+y^2+z^2)=0$

2) Solve: $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Solution: The given equation is of the form Pp+Qq=R

The auxiliary equations are

$\frac{dx}{(x^2-yz)}=\frac{dy}{(y^2-zx)}=\frac{dz}{(z^2-xy)}$

The subsidiary equation is given by

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)}=\frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$
$$=\frac{dz-dx}{(z^2-xy)-(x^2-yz)}\ldots\ldots\ldots\ldots\ldots(1)$$

Consider first two equations of (1) we get

$$\Rightarrow\frac{dx-dy}{(x^2-yz)-(y^2-zx)}=\frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$
$$\Rightarrow\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$
$$\Rightarrow\frac{dx-dy}{(x-y)}=\frac{dy-dz}{(y-z)}$$

$$\Rightarrow l\,og(\,x-y) = log(y-z) + log\,c_1 \Rightarrow \frac{x-y}{y-z} = c_1$$

Similarly considering last two equations of (1) we get

$$\Rightarrow l\,og(\,y-z) = log(z-x) + log\,c_2$$

$$\Rightarrow \frac{y-z}{z-x} = c_2$$

Hence the required solution is

$$\varphi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

Method of Separation of Variables:

Separation of variables is a powerful technique to solve PDE. For a PDE in the function u of two independent variables x and y, assume that the required solution is separable i.e.

u (x,y)=X(x)Y(y) ---------(1)

where X(x) is a function of x alone and Y(y) is a function of y alone. Then substitution of u from (1) and its derivatives reduces the PDE to the form;

f (X, X', X'',……) = g(Y,Y',Y'',…..) ---------- (2)

which is separable in X and Y. Since the LHS of (2) is a function of x alone and RHS of (2) is a function of y alone, then (2) must be equal to a common constant say "k". Thus (2) reduces to;

f (X,X',X''……)=k ----------(3)

g (X,X',X''……)=k ----------(4)

Thus the determination of solution to PDE reduces to the determination of solutions to two ODE (with appropriate conditions).

Examples:

1. By the method of separation of variables, solve the equation:

$$\frac{\partial z}{\partial x}y^3 + \frac{\partial z}{\partial y}x^2 = 0$$

Solution:

$$\frac{\partial z}{\partial x}y^3 + \frac{\partial z}{\partial y}x^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let the solution of (1) be in the form z=XY where X=X(x) and Y=Y(y)

$$\frac{\partial z}{\partial x} = Y\frac{\partial X}{\partial x}, \frac{\partial z}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$(1) \Rightarrow \left(\frac{\partial X}{\partial x}Y\right)y^3 + \left(X\frac{\partial Y}{\partial y}\right)x^2 = 0$$

$$\left(\frac{dX}{dx}Y\right)y^3 = -\left(X\frac{dY}{dy}\right)x^2$$

$$\frac{1}{X}\left(\frac{dX}{dx}\right)\frac{1}{x^2} = -\frac{1}{Y}\left(\frac{dY}{dy}\right)\frac{1}{y^3}$$

L.H.S is a function of x only and R.H.S is a function of y only.

Since x and y are independent variables, this expression can hold only if each side is a constant i.e.

$$\frac{1}{X}\left(\frac{dX}{dx}\right)\frac{1}{x^2} = k and -\frac{1}{Y}\left(\frac{dY}{dy}\right)\frac{1}{y^3} = k$$

where k is a constant. These may be written as

$$\frac{1}{X}\left(\frac{dX}{dx}\right) = kx^2 and \frac{1}{Y}\left(\frac{dY}{dy}\right) = -ky^3$$

$$\frac{d}{dx}(\log X) = kx^2 and \frac{d}{dy}(\log Y) = -ky^3$$

Integrating w.r.t x and y we get,

$$log\,X = k\frac{x^3}{3} + log\,C_1\ and\ log\,Y = -k\frac{y^4}{4} + log\,C_2$$

$$\Rightarrow X = C_1 e^{\frac{kx^3}{3}} andY = C_2 e^{\frac{-ky^4}{4}} w\square ereC_1 andC_2 arecons\tan t\,s$$

$$\therefore z = XY = Ae^{k\frac{x^3}{3}} e^{-k\frac{y^4}{4}} w\square ereA = C_1 C_2$$

2. Using method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + uwhereu(x,0) = 6e^{-3x}$$

Solution: Here x and t are independent variables and u is the dependent variable.

Let the solution be in the form u=X(x)Y(t)

$$\therefore \frac{\partial u}{\partial x} = \frac{dX}{dx} Y(t) and \frac{\partial u}{\partial t} = X(x)\frac{dX}{dt}$$

The equation takes the form

$$\frac{dX}{dx} Y(t) = 2X(x)\frac{dY}{dt} + X(x)Y(y)$$

$$\left(\frac{dX}{dx} - X\right) Y = 2X\frac{dY}{dt} \Rightarrow \frac{1}{X}\frac{dX}{dx} - 1 = \frac{2}{Y}\frac{dY}{dt}$$

$$\frac{1}{X}\frac{dX}{dx} - 1 = kand\frac{2}{Y}\frac{dY}{dt} = k \Rightarrow \frac{dX}{X} = (k+1)dxand\frac{dY}{Y} = \frac{k}{2}dt$$

On integration we get,

$$log\,X = (k+1)x + log\,C_1\ and\ log\,Y = \frac{k}{2}t + log\,C_2$$

$$\Rightarrow X = C_1 e^{(k+1)x} andX = C_2 e^{\frac{k}{2}t}$$

Therefore the required solution is$u = XY = Ae^{(k+1)x} e^{\frac{k}{2}t} whereA = C_1 C_2$

But given u(x,0)=$6e^{-3x}$

=> $6e^{-3x} = Ae^{(k+1)x}$

=> A=6 and k+1 = -3

=> A=6 and k=-4

$\therefore u = 6e^{-(3x+2t)}$

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

The general form of a second-order PDE in the function u of the two independent variables x, y is given by

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (1)$$

This equation is linear in second order terms. PDE (1) is said to be "linear or quasi-linear" according as f is linear or non-linear.

PDE (1) is classified as **Elliptic, Parabolic** or **Hyperbolic** according to $B^2 - 4AC < 0, = 0$ or > 0

Example:

Elliptic: ($B^2 - 4AC < 0$)

Laplace's equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Poisson's equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Parabolic: ($B^2 - 4AC = 0$)

One dimensional heat flow equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Hyperbolic: ($B^2 - 4AC > 0$)

One dimensional wave equation

Homogeneous linear partial differential equations with constant coefficients

Consider a partial differential equation of the form

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots.. + A_n \frac{\partial^n z}{\partial y^n} + \cdots$$
$$= F(x, y) \ldots\ldots\ldots\ldots (1)$$

where A_0, A_1 …., A_n are constant coefficients. In this equation the dependent variable z and its derivatives are linear. Since each term (in the LHS) of (1) contains z or its derivatives, equation (1) is known as homogeneous linear partial differential equation of order n with constant coefficients.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by D or D_x and D^1 or D_y respectively. Then (1) can be rewritten as:

$$\left(A_0 {D_x}^n + A_1 {D_x}^{n-1} D_y + \cdots.. + A_n {D_y}^n\right) z = F(x, y)$$

The above equation may rewritten as:

$$F(D_x, D_y)z = F(x,y)\ldots\ldots\ldots(2)$$

Where $F(D_x, D_y) = A_0 {D_x}^n + A_1 {D_x}^{n-1} D_y + \ldots\ldots + A_n {D_y}^n$

Remark:

Equation (1) is called as homogeneous because all terms contain derivatives of **same order.**

Alternative definition

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots\ldots + k_n \frac{\partial^n z}{\partial y^n} + \cdots = F(x,y)\ldots\ldots\ldots\ldots(1)$$

where k's are constant, is called a homogeneous linear partial differential equation of the nth order with constant coefficients. **Equation (1) is called as homogeneous because all terms contain derivatives of same order.**

On writing, $\frac{\partial^r}{\partial x^r} = D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$.

Equation (1) becomes $(D^n + k_1 D^{n-1} D' + \cdots \ldots \ldots \ldots + k_n D'^n)Z = F(x,y)$

Or briefly

$$f(\mathrm{D}, D')z = F(x,y)\ldots\ldots\ldots(2)$$

As in case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely the complementary function and the particular integral.

The complementary function is the complete solution of the equation $f(\mathrm{D}, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

For example:

1. $3\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$ Homogeneous PDE of order 2.
2. $2\frac{\partial^3 z}{\partial x^3}-3\frac{\partial^3 z}{\partial x^2\partial y}+5\frac{\partial^3 z}{\partial x\partial y^2}-8\frac{\partial^3 z}{\partial y^3}=e^{x+y}$ Homogeneous PDE of order 3.
3. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+3\frac{\partial^2 z}{\partial y^2}=x+y$ Homogeneous PDE of order 2.

Complementary functions (C.F.) of homogeneous linear partial differential equation i.e. $F(D_x, D_y)z=0$

Let $z=\emptyset(y+mx)$ be a solution (2) where $\emptyset$ is an arbitrary function and m is is a constant, then

$$D_x z=\phi'(y+mx)\cdot m$$

$$D_x{}^2 z=\phi''(y+mx)\cdot m^2$$

.

.

.

$$D_x{}^n z=\phi^{(n)}(y+mx)\cdot m^n$$

$$D_y z=\phi'(y+mx)$$

$$D_y{}^2 z=\phi''(y+mx)$$

.

.

$$D_y{}^n z=\phi^{(n)}(y+mx)$$

$$D_x D_y z=m\,\phi''(y+mx)$$

$$D^2{}_x D_y z=m^2\,\phi^{(3)}(y+mx)$$

.

.

.

$$D^r{}_x D^s{}_y z = m^r\ \phi^{(r+s)}(y+mx)$$

$$= m^r \phi^{(n)}(y+mx)$$

where r + s = n

Substituting these values in (2) and simplifying, we get:

$$(A_0 m^n\ +\ A_1 m^{n-1} + \ldots\ldots + A_n)\phi^{(n)}(y+mx) = 0 \ldots\ldots(4)$$

Which is true if *m* is a root of the equation:

$$A_0 m^n\ +\ A_1 m^{n-1} + \ldots\ldots + A_n\ =\ 0 \ldots\ldots(5)$$

The equation (5) is known as the (characteristic equation) or the (Auxiliary equation (A.E.)) and is obtained by putting $\boldsymbol{D_x = m}$ **and** $\boldsymbol{D_y = 1}$

In $F(D_x\ , D_y)z = 0$, and it has *n* roots.

Let m_0, m_1 …., m_n be *n* roots of A.E. (5).

Three cases arise:

Case 1: When the roots are distinct:

If m_0, m_1 …., m_n be *n* distinct roots of A.E. (5) then

$\phi_1(y+m_1x), \phi_2(y+m_2x), \ldots\ldots\ldots, \phi_n(y+m_nx)$ are the linear solution corresponding to them and since the sum of any linear solutions is a solution too than the general solution in this case is:

$$z = \phi_1(y+m_1x) +\ \phi_2(y+m_2x) + \ldots\ldots\ldots + \phi_n(y+m_nx)\ \ \ldots\ldots(6)$$

Example:

Find the general solution of:

$$\left(D_x{}^3 + 2D_x{}^2 D_y - 5D_x D_y{}^2 - 6D_y{}^3\right)z = 0$$

Solution. The A.E. is $m^3 + 2m^2 - 5m - 6 = 0$

$$\rightarrow (m+1)(m^2+m-6) = 0$$

$$\rightarrow (m+1)(m+3)(m-2) = 0$$

$$m_1 = -1, m_2 = -3, m_3 = 2$$

Note that m_1, m_2 and m_3 are different roots, then the general solution is

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x)$$

$$\rightarrow\ z = \phi_1(y - x) + \phi_2(y - 3x) + \phi_3(y + 2x)$$

Where $\phi_1, \phi_2\ and\ \phi_3$ are arbitrary functions.

Case 2: When the roots are repeated.

If the root m is repeated k times i.e. $m_1 = m_2 = \ldots = m_k$, then the corresponding solution is :

$$z = \phi_1(y + m_1x) + x\phi_2(y + m_1x) + \ldots. + x^{k-1}\phi_n(y + m_1x) \ldots..(7)$$

where $\phi_1, \ldots, \phi_k$ are arbitrary functions.

Note: If some of the roots m_0, m_1 …., m_n are repeated and the other are not i.e.

$m_1 = m_2 = \ldots = m_k \neq m_{k+1} \neq \ldots. \neq m_n$ then the general solution is :

$$z = \phi_1(y + m_1x) + x\phi_2(y + m_1x) + \ldots + x^{k-1}\phi_k(y + m_1x)$$
$$+ \phi_{k+1}(y + m_{k+1}x) + \ldots + \phi_n(y + m_nx) \ \ldots\ldots(8)$$

Example:

Solve $\left(D_x{}^3 - D_x{}^2D_y - 8D_xD_y{}^2 + 12D_y{}^3\right)z = 0$

Solution: The A.E. is $m^3 - m^2 - 8m + 12 = 0$

$$\rightarrow (m-2)(m-2)(m+3) = 0$$

$m_1 = m_2 = 2$, $m_3 = -3$

Then, the general solution is

$$z = \phi_1(y + 2x) + x\phi_2(y + 2x) + \phi_3(y - 3x)$$

Where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Case 3: When the roots are complex.

If one of the roots of the given equation is complex let be m_1 , then the conjugate of m_1 is also a root, let be m_2 , so the general solution is:

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \dots\dots\dots + \phi_n(y + m_nx)$$

Where $\phi_1 , \dots , \phi_n$ are arbitrary functions.

Example:

Solve $\left(D_x{}^2 - 2D_xD_y + 5D_y{}^2\right)z = 0$

Solution: The A. E. is $m^2 - 3m + 5 = 0$

$$\rightarrow m = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

$$\therefore\ m_1 = 1 + 2i\ , m_2 = 1 - 2i$$

$$z = \phi_1(y + (1 + 2i)x) + \phi_2(y + (1 - 2i)x)$$

That is the general solution where ϕ_1, ϕ_2 are arbitrary functions.

Particular integral (P.I.) of partial differential equation:

When $f(x, y) \neq 0$ in the equation (2) which it's $f\left(D_x, D_y\right)z = F(x, y)$. Multiplying (3) by the inverse operator $\frac{1}{f\left(D_x, D_y\right)}$

$$\frac{1}{f\left(D_x, D_y\right)} \cdot f\left(D_x, D_y\right)z = \frac{1}{f\left(D_x, D_y\right)}F(x, y)$$

$$z = \frac{1}{f\left(D_x, D_y\right)}F(x, y) \quad \dots\dots\dots (4)$$

which is the particular integral (P.I.)

The operator $F\left(D_x, D_y\right)$ can be written as

$$f\left(D_x, D_y\right) = \left(D_x - m_1D_y\right)\left(D_x - m_2D_y\right)\dots\left(D_x - m_nD_y\right) \quad \dots\dots (5)$$

Substituting (5) in (4) :

$$z = \frac{1}{(D_x - m_1 D_y)(D_x - m_2 D_y)\ldots(D_x - m_n D_y)} F(x, y)$$

Taking $u_1 = \frac{1}{(D_x - m_n D_y)} F(x, y)$

$$\therefore\ (D_x - m_n D_y) u_1 = F(x, y)$$

This equation can be solved by Lagrange's method.

The Lagrange's auxiliary equations are:

$$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{du_1}{F(x, y)} \ \ldots\ldots\ldots (6)$$

Taking the first two fractions of (6)

$$m_n dx + dy = 0 \ \rightarrow m_n x + y = a$$

Taking the first and third fractions of (6)

$$dx = \frac{du_1}{F(x, y)} \ \rightarrow\ F(x, y) dx = du_1$$

On substitution we get

$$F(x, a - m_n x) dx = du_1$$

Integrating the above equation we get

$$u_1 = \int F(x, a - m_n x) dx + b$$

Let b=0, then we have u_1

Similarly, we take

$$u_2 = \frac{1}{D_x - m_{n-1} D_y} u_1$$

And solve it by Lagrange's method, we get u_2,

$$z = u_2 = \frac{1}{D_x - m_1 D_y} u_{n-1}$$

And by solving this equation we get the particular integral (P.I.).

Short methods of finding the P.I. in certain cases:

Case 1: When $F(x,y) = e^{ax+by}$ where a and b are arbitrary constants:

To find the P.I. when $F(a,b) \neq 0$, we derive $F(x,y)$ for x and y n times:

$$D_x e^{ax+by} = a e^{ax+by}$$

$$D_x{}^2 e^{ax+by} = a^2 e^{ax+by}$$

.

.

$$D_x{}^n e^{ax+by} = a^n e^{ax+by}$$

$$D_y e^{ax+by} = b e^{ax+by}$$

$$D^2{}_y e^{ax+by} = b^2 e^{ax+by}$$

.

.

$$D^n{}_y e^{ax+by} = b^n e^{ax+by}$$

$$D_x{}^r D_y{}^s e^{ax+by} = a^r b^s e^{ax+by} \text{ where } r+s = n$$

So

$$f\left(D_x, D_y\right) e^{ax+by} = f(a,b) e^{ax+by}$$

Multiplying both sides by $\frac{1}{f\left(D_x,D_y\right)}$, we get

$$e^{ax+by} = \frac{1}{f\left(D_x, D_y\right)} f(a,b) e^{ax+by}$$

Since $f(a,b) \neq 0$ then we can divide on it:

$$\frac{1}{f(a,b)}e^{ax+by} = \frac{1}{f(D_x,D_y)}e^{ax+by} \qquad \text{.........} (*)$$

Which it is equal to z, then the P. I. is

$z = \frac{1}{f(D_x,D_y)}e^{ax+by} = \frac{1}{f(a.b)}e^{ax+by}$, where $f(a,b) \neq 0$

When $(a,b) = 0$, then analyze $f(D_x,D_y)$ as follows

$$f(D_x,D_y) = \left(D_x - \frac{a}{b}D_y\right)^r G(D_x,D_y)$$

where $G\ (a,\ b) \neq 0$ we get

$$z = \frac{1}{f(D_x\,,D_y)}e^{ax+by} = \frac{1}{\left(D_x - \frac{a}{b}D_y\right)^r G(D_x,D_y)}e^{ax+by}$$

$= \frac{1}{\left(D_x-\frac{a}{b}D_y\right)^r} \cdot \frac{1}{G(a,b)}e^{ax+by}$ from (*)

Since G (a, b) ≠ 0

$$= \frac{1}{G(D_x,D_y)} \cdot \frac{1}{\left(D_x - \frac{a}{b}D_y\right)^r}e^{ax+by}$$

Then by Lagrange's method r times , we get

$$z = \frac{1}{f(D_x\,,D_y)}e^{ax+by} = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!}e^{ax+by}$$

Which it's the P.I. where

$$f(a,b) = 0, G(a,b) \neq 0$$

<u>Example:</u> Solve $\boldsymbol{(D_x{}^2 - D_xD_y - 6D_y{}^2)z = e^{2x-3y}}$

Solution.

1) To find the general solution

The A.E. of the given equation is

$$m^2 - m - 6 = 0 \rightarrow (m-3)(m+2) = 0$$

$$\therefore m_1 = 3, m_2 = -2$$

$$\therefore z_1 = \phi_1(y+3x) + \phi_2(y-2x)$$

where ϕ_1 and ϕ_2 are arbitrary functions

2) To find the particular Integral (P.I.)

$$a = 2,\ b = -3$$

$$F(a,b) = a^2 - ab - 6b^2$$

$$F(2,-3) = 4 + 6 - 54 = -44 \neq 0$$

$$z_2 = \frac{1}{F(a.b)} e^{ax+by} = \frac{1}{-44} e^{2x-3y}$$

$\therefore$ The general solution is z=C.F+P.I

i.e. $z = z_1 + z_2 = \phi_1(y+3x) + \phi_2(y-2x) - \frac{1}{44} e^{2x-3y}$

Case 2: When $F(x,y) = \sin(ax+by)$ *or* $\cos(ax+by)$ where a and b are arbitrary constants

Let $F(x,y) = \sin(ax+by)$

$$D_x \sin(ax+by) = a\cos(ax+by)$$

$$D_x{}^2 sin(ax+by) = -a^2 sin(ax+by)$$

$$D_y sin(ax+by) = bcos(ax+by)$$

$$D_y{}^2 \sin(ax+by) = -b^2 \sin(ax+by)$$

$$D_x D_y \sin(ax+by) = D_x[b\cos(ax+by)]$$

$$= -ab\sin(ax+by)$$

$$f\left(D_x{}^2, D_x D_y, D_y{}^2\right)\sin(ax+by) = f(-a^2, -ab, -b^2)\sin(ax+by)$$

Multiplying both sides by $\frac{1}{F(D_x{}^2, D_xD_y, D_y{}^2)}$

If $f(-a^2, -ab, -b^2) \neq 0$ then we can divide on it

$$\rightarrow z = \frac{1}{f(D_x{}^2, D_xD_y, D_y{}^2)} \sin(ax + by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by)$$

which it is the particular integral.

And if $f(-a^2, -ab, -b^2) = 0$ then we write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad , \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And follow the solution of the exponential function in case 1.

<u>Example</u>: Solve $(D_x{}^2 - 3D_xD_y + D_y{}^2)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)$

Solution.

1) Finding the general solution z

The A.E. is

$$m^2 - 3m + 0 \;\Rightarrow\; (m-2)(m-1) = 0$$

$$\therefore m_1 = 2, m_2 = 1$$

$$\therefore z_1 = \phi_1(y + 2x) + \phi_2(y + x)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

2) The P.I. of the given equation is

$$\text{P.I.} = z_2 = \frac{1}{F(D_x, D_y)} e^{2x+3y} + \frac{1}{F(D_x, D_y)} e^{x+y} + \frac{1}{F(D_x, D_y)} \sin(x - 2y)$$

Let $u_1 = \frac{1}{F(D_x, D_y)} e^{2x+3y} \quad , a = 2, b = 3$

$$F(D_x, D_y) = a^2 - 3ab + 2b^2$$

$$F(1,1) = 4 - 18 + 18 = 4 \neq 0$$

$$u_1 = \frac{1}{4} e^{2x+3y}$$

$$u_2 = \frac{1}{F(D_x, D_y)} e^{x+y} \quad , a = 1, b = 1$$

$$F(a,b) = a^2 - 3ab + 2b^2$$

$$F(1,1) = 1 - 3 + 2 = 0$$

Analyse $F(D_x, D_y)$,

$$F(D_x, D_y) = (D_x - 2D_y)(D_x - D_y)$$

$$u_2 = \frac{1}{G(a,b)} \frac{x^r}{r!} e^{ax+by}$$

$$= \frac{1}{-1} \frac{x}{1} e^{x+y}$$

$$u_2 = -xe^{x+y}$$

$$u_3 = \frac{1}{F(D_x, D_y)} \sin(x - 2y)$$

$$F(-a^2, -ab, -b^2) = -a^2 + 3ab - 2b^2$$

$$F(-1, 2, -4) = -1 - 6 - 8 = -15 \neq 0$$

$$u_3 = \frac{1}{-15} \sin(x - 2y)$$

Then, the required general solution is

$$z = z_1 + z_2 = \phi_1(y + 2x) + \phi_2(y + x) + \frac{1}{4} e^{2x+3y} - xe^{x+y} - \frac{1}{15} \sin(x - 2y)$$

Where ϕ_1 and ϕ_2 are arbitrary functions.

Case 3: When $F(x,y) = x^a y^b$ where a and b are Non- Negative Integer Number

The particular integral (P.I.) is evaluated by expanding the function $\frac{1}{f(D_x,D_y)}$ in an infinite series of ascending powers of D_x or D_y (i.e) by transfer the function $\frac{1}{f(D_x,D_y)}$ according to the following

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + ...$$

<u>Example:</u> Find P.I. of the equation $(D_x{}^3 - 7D_xD_y{}^2 - 6D_y{}^3)z = x^2y$

Solution: P.I $= \frac{1}{D_x{}^3 - 7D_xD_y{}^2 - 6D_y{}^3} x^2y$

$$= \frac{1}{D_x{}^3\left[1 - \left(\frac{7D_y{}^2}{D_x{}^2} + \frac{6D_y{}^3}{D_{x^3}}\right)\right]} x^2y$$

$$= \frac{1}{D_x{}^3}\left[1 + \left(\frac{7D_y{}^2}{D_x{}^2} + \frac{6D_y{}^3}{D_{x^3}}\right) + \left(\frac{7D_y{}^2}{D_x{}^2} + \frac{6D_y{}^3}{D_{x^3}}\right)^2 + \ldots\right] x^2y$$

$= \frac{1}{D_x{}^3}[x^2y]$

Since $\left(\frac{7D_y{}^2}{D_x{}^2} + \frac{6D_y{}^3}{D_{x^3}}\right)(x^2y) = 0, \left(\frac{7D_y{}^2}{D_x{}^2} + \frac{6D_y{}^3}{D_{x^3}}\right)^2 (x^2y) = 0$

$$= \frac{1}{D_x{}^2}\frac{x^3y}{3} = \frac{1}{D_x}\frac{x^4y}{12} = \frac{x^5y}{60}$$

Case 4 When $F(x,y) = e^{ax+by}V$ where V is a function of x and y

The P.I. in this case is $z = \frac{1}{F(D_x,D_y)} e^{ax+by}V$

$$= e^{ax+by} \frac{1}{F(D_x + a, D_y + b)} V$$

and solving this equation depending on the type of V can get the particular integral (P.I.), as follows:

Example: Find P.I. of the equation $D_x D_y z = e^{2x+3y} x^2 y$

Solution: P.I. $= \frac{1}{D_x D_y} e^{2x+3y} x^2 y \;\; a = 2, b = 3 \; and \; V = x^2 y$

$$= e^{2x+3y} \frac{1}{(D_x+2)(D_y+3)} x^2 y$$

$$= e^{2x+3y} \frac{1}{3(D_x+2)\left(1+\frac{D_y}{3}\right)} x^2 y$$

$$= e^{2x+3y} \frac{1}{3(D_x+2)} \left[1 - \frac{D_y}{3} + \frac{{D_y}^2}{9} - \dots\right] x^2 y$$

$$= e^{2x+3y} \frac{1}{3(D_x+2)} \left[x^2 y - \frac{x^2}{3}\right]$$

$$= e^{2x+3y} \frac{1}{6\left(1+\frac{D_x}{2}\right)} \left[x^2 y - \frac{x^2}{3}\right]$$

$$= \frac{1}{6} e^{2x+3y} \left[1 - \frac{D_x}{2} + \frac{{D_x}^2}{4} - \frac{{D_x}^3}{8} + \dots\right] \left[x^2 y - \frac{x^3}{3}\right], \left(\frac{{D_x}^3}{8} = 0\right)$$

$$= \frac{1}{6} e^{2x+3y} \left[x^2 y - \frac{x^2}{3} - xy + \frac{x}{3} + \frac{y}{2} - \frac{1}{6}\right]$$

$$= e^{2x+3y} \left[\frac{1}{6} x^2 y - \frac{x^2}{18} - \frac{1}{6} xy + \frac{x}{18} + \frac{y}{2} - \frac{1}{36}\right]$$

Non-Homogeneous linear partial differential equations with constant coefficients

A linear partial differential with constant coefficients is known as non-homogeneous l.p.d.e. with constant coefficients if **the order of all the partial derivatives involved in the equation are not all equal.**

For example:

1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + z = x + y$
2) $\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = e^{x+y}$

Definition: A linear differential operator F (D_x, D_y) is known as (reducible), if it can be written as the product of linear factors of the form $aD_x + bD_y + c$ with a, b and c as constants. F (D_x, D_y) is known as (irreducible), if it is not reducible.

For example, the operator $D_x^2 - D_y^2$ which can be written in the form (D_x - D_y) (D_x + D_y) is reducible, whereas the operator $D_x^2 - D_y^3$ which cannot be decomposed into linear factors is irreducible.

Note: A l.p.d.e. with constant coefficient F (D_x, D_y) z = f (x, y) is known as reducible, if F (D_x, D_y) reducible, and is known as irreducible, if F (D_x, D_y) is irreducible.

OR

If in the equation

$$f(D_x, D_y)z = F(x,y)\ldots\ldots\ldots(1)$$

The polynomial expression $f(D_x, D_y)$ is not homogeneous, then equation (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F+P.I.

Note:

The methods to find P.I are the same as those for homogeneous linear equations.

Determination of Complementary function (C.F.) (the general solution) of a reducible non-homogeneous linear partial differential equation with constant coefficients

Let F (D_x, D_y) = ($aD_x + bD_y + c)^k$, where a, b, c are constants and k is a natural number, then the equation F (D_x, D_y) z = 0 will be $(aD_x + bD_y + c)^k z = 0$ and the solution is

$$z = e^{-\frac{c}{a}x}\phi(ay - bx) \qquad ; a \neq 0, k = 1$$

Or

$$z = e^{-\frac{c}{b}y}\phi(ay - bx) \qquad ; b \neq 0, k = 1$$

For any k > 1, the solution is

$$z = e^{-\frac{c}{b}y}[\phi_1(ay - bx) + x\phi_2(ay - bx) + \ldots + x^{k-1}\phi_k(ay - bx)]; \ b \neq 0$$

Or

$$z = e^{-\frac{c}{a}x}[\phi_1(ay - bx) + x\phi_2(ay - bx) + \ldots + x^{k-1}\phi_k(ay - bx)]; \ a \neq 0$$

Where ϕ_1,…, ϕ_n are arbitrary functions.

Example:

Solve (D_x – 2 D_y +1)^4 = 0

Solution: We have a = 1, b = -2, c = 1, k = 4

Then

$$z = e^{-\frac{1}{2}y}[\phi_1(y + 2x) + x\phi_2(y + 2x) + x^2\phi_2(y + 2x) + x^2\phi_3(y + 2x) + x^3\phi_4(y + 2x)]$$

Where ϕ_1,…, ϕ_4 are arbitrary functions.

Example:

solve $\underbrace{(\mathbf{2Dx - 3\,Dy + 1})}_{\textbf{linear}}\ \underbrace{(\mathbf{Dx + 2\,Dy - 2})}_{\textbf{linear}}$ **z = 0**

Solution: The given equation is reducible, then we have

$a_1 = 1,\ b_1 = -3,\ c_1 = 1,\ k_1 = 1$

$z_1 = e^{-\frac{1}{2}x}[\phi_1(2y + 3x)]$

$a_2 = 1,\ b_2 = 2,\ c_2 = -2,\ k_2 = 1$

$z_2 = e^{2x}[\phi_2(y - 2x)]$

The general solution is

$$z = z_1 + z_2 \ \rightarrow\ z = e^{-\frac{1}{2}}\phi_1(2y + 3x) + e^{2x}\phi_2(y - 2x)$$

Where ϕ_1, ϕ_2 are two arbitrary functions.

Example:

Solve $(D_x^2 - D_y)\, z = \sin(x-2y)$

Solution:

(1) The general solution z_1 of $(D_x^2 - D_y)\, z = 0$ is

$$F(a, b) = a^2 - b = 0 \rightarrow a_i^2 = b_i$$

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x + a_i^2 y}$$

(2) To find the P.I. of the given equation

$$\text{P.I.} = z = \frac{1}{D_x^2 - D_y}\sin(x - 2y)$$

$$a = 1,\ b = -2 \ \rightarrow\ D_x^2 = -a^2 = -1$$

$$= \frac{1}{-1 - D_y}\sin(x - 2y)$$

Multiplying by $\frac{1}{-1+D_y}$

$$=\frac{-1+D_y}{1-D_y^2}\sin(x-2y)$$

D_y^2 = -b² = -4

$$=\frac{-1+D_y}{1+4}\sin(x-2y)$$

$$=\frac{1}{5}[-\sin(x-2y)-2\cos(x-2y)]$$