ENGINEERING MATHEMATICS - II

VE23MA141B.

UNIT 4 : FOURIER SERIES & FOURIER TRANSFORMS.

I) Find the Fourier Series expansion of the following functions over the given interval.

1) $f(x) = x - x^2$ from $-\pi$ to π. Hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Solution :

The Fourier expansion of $f(x)$ over $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{--- (1)}$$

where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, 3, \ldots$$

Calculation of a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} - \frac{(-\pi)^2}{2} - \left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right)\right\}.$$

$$a_0 = \frac{1}{\pi}\left\{ \frac{\cancel{\pi^2}}{2} - \frac{\cancel{\pi^2}}{2} - \left(\frac{\pi^3}{3} + \frac{\pi^3}{3} \right) \right\}$$

$$= \frac{1}{\pi}\left(\frac{-2\pi^3}{3} \right) = \frac{-2\pi^2}{3}$$

$$\boxed{a_0 = \frac{-2\pi^2}{3}}$$

Calculation of a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx \cdot dx.$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1-2x) \cdot \frac{1}{n}\left(\frac{-\cos nx}{n} \right) + (-2)\left(\frac{-1}{n^2} \right)\frac{\sin nx}{n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{n^2 \pi}\left[(1-2x)\cos nx \right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{1}{\pi n^2}\left[(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi \right]$$

$$\boxed{a_n = \frac{-4}{n^2}\cos n\pi = \frac{-4}{n^2}(-1)^n.}$$

Calculation of b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[(x - x^2) \cdot \left(-\frac{\cos nx}{n} \right) - (1-2x) \cdot \left(\frac{-1}{n} \right) \cdot \left(\frac{\sin nx}{n} \right) + (-2) \left(\frac{-1}{n^2} \right) \cdot \left(-\frac{\cos nx}{n} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[(x - x^2) \left(-\frac{\cos nx}{n} \right) - \frac{2}{n^3} \cos nx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[(\pi - \pi^2) - \frac{\cos n\pi}{n} - (-\pi - \pi^2) \left(-\frac{\cos n\pi}{n} \right) - \frac{2}{n^3} \left[\cos n\pi - \cos n\pi \right] \right].$$

$$= \frac{1}{\pi} \left\{ -\frac{2\pi \cos n\pi}{n} \right\} = -\frac{2 \cos n\pi}{n} = \frac{-2(-1)^n}{n}.$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}.}$$

The required Fourier series expansion is

$$x - x^2 = \frac{-2\pi^2}{3(2)} + \sum_{n=1}^{\infty} \frac{-4}{n^2} (-1)^n \cos nx - \frac{2(-1)^n}{n} \sin nx.$$

$$x - x^2 = \frac{-\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \frac{2(-1)^n \sin nx}{n} \qquad (2)$$

Deduction part:

To deduce the series, first let us put $x=0$ which is a point of the interval $(-\pi, \pi)$.

Hence (2) becomes

$$0 = \frac{-\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n$$

$$\frac{\pi^2}{3} = -\left[\frac{-4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} \cdots\right]$$

$$\frac{\pi^2}{3} = +4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots}$$

2). $f(x) = -\pi$ for $-\pi < x < 0$

x for $0 < x < \pi$.

Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

Solution

The Fourier expansion of $f(x)$ over $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Calculation of a_0 :

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi \cdot dx + \int_{0}^{\pi} x \cdot dx \right].$$

$$= \frac{1}{\pi} \left[-\pi \left[x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left(-\pi^2 + \frac{\pi^2}{2} \right) = \frac{1}{\pi} \left(\frac{-\pi^2}{2} \right) = \frac{-\pi}{2}.$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cdot\cos nx\,dx + \int_{0}^{\pi} x\cdot\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\cdot\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\cdot\frac{\sin nx}{n}\right]_{0}^{\pi} - \left[1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$a_n = \frac{1}{\pi n^2}\left[\cos nx\right]_{0}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[\cos n\pi - \cos 0\right].$$

$$\boxed{a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right].}$$

Calculation of b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi.\sin nx\,dx + \int_{0}^{\pi} x\cdot\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\cdot\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left[x\cdot\left(\frac{-\cos nx}{n}\right)\right]_{0}^{\pi} - \left[1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi n}\left[\pi\left[\cos nx\right]_{-\pi}^{0} - \left[x\cos nx\right]_{0}^{\pi}\right]$$

$$b_n = \frac{\pi}{\pi n}\left[1 - \cos n\pi - \cos n\pi\right]$$

$$b_n = \frac{1}{n}\left(1 - 2\cos n\pi\right)$$

$$\boxed{b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]}$$

Substituting the Values of a_0, a_n b_n in ①

we get

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\left[1 - (-1)^n\right] \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n}\left[1 - 2(-1)^n\right] \sin nx.$$

To deduce the required series let us put $x=0$ in the Fourier series.

$$\frac{f(0^+) + f(0^-)}{2} = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\left[1 - (-1)^n\right]$$

$$\frac{0 + (-\pi)}{2} = \frac{-\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}\left[1 - (-1)^n\right]$$

$$\frac{-\pi}{2} + \frac{\pi}{4} = -\frac{1}{\pi}\left[\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right]$$

$$\frac{-\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

3) $f(x) = x \quad \text{for } 0 \leq x \leq \pi$

$\quad\quad\;\; 2\pi - x. \quad \text{for } \pi \leq x \leq 2\pi$

Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Solution :

The Fourier expansion of $f(x)$ over $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{——— (1)}.$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$$

Calculation of a_0 :

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cdot dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right]$$

$$= \frac{1}{\pi}\left\{ \left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi} \right\}$$

$$a_0 = \frac{1}{\pi}\left\{ \frac{\pi^2}{2} + \left(4\pi^2 - 2\pi^2\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right) \right\}$$

$$= \frac{1}{\pi}\left\{\pi^2\right\} = \pi$$

$$\boxed{a_0 = \pi}$$

Calculation of a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \, dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot \cos nx \cdot dx + \int_{\pi}^{2\pi} (2\pi - x)\cdot \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[x\cdot\left(\frac{\sin nx}{n}\right) - (1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$+ \left[(2\pi - x)\cdot\frac{\sin nx}{n} - (-1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$\Rightarrow \frac{1}{\pi n^2}\left\{ \left(\cos nx\right)_0^{\pi} - \left(\cos nx\right)_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{\pi n^2}\left\{ \cos n\pi - 1 - \left[\cos 2n\pi - \cos n\pi\right] \right.$$

$$= \frac{1}{\pi n^2}\left\{ 2\cos n\pi - 2 \right) \qquad \because \cos 2n\pi = 1$$

$$\boxed{a_n = \frac{+2}{\pi n^2}\{(-1)^n - 1\}}$$

Calculation of b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot\sin nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\cdot\left(\frac{-\cos nx}{n}\right) - (1)\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[(2\pi - x)\cdot\left(\frac{-\cos nx}{n}\right) - (-1)\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{-1}{\pi n}\left\{\left[x\cos nx\right]_0^{\pi} + \left[(2\pi - x)\cos nx\right]_{\pi}^{2\pi}\right\}$$

$$b_n = -\frac{1}{\pi n}\left\{(\pi\cos n\pi - 0) + (0 - \pi\cos n\pi)\right\}$$

$$\boxed{b_n = 0}$$

The Fourier Series representation of $f(x)$ is given by

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{+2}{\pi n^2}\{(-1)^n - 1\}\cos nx. \quad \text{——} (2)$$

Deduction

Put $x = 0$ in (2)

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Fourier Series of even and odd functions

4) $f(x) = x^2$ in $(-\pi, \pi)$ Hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Solution

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$\Rightarrow f(x) \text{ is even.}$$

So, we have $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\,dx.$$

$$b_n = 0.$$

Calculation of a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\,\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

Calculation of a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx \cdot dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cdot \cos nx \, dx.$$

$$= \frac{2}{\pi}\left[x^2 \cdot \left(\frac{\sin nx}{n}\right) - (2x)\cdot\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\left(x\cdot\cos nx\right)_0^{\pi}\right]$$

$$= \frac{4}{\pi n^2}\left[\pi \cdot \cos n\pi\right] = \frac{4}{n^2}(-1)^n.$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$\therefore$ The Fourier series representation is given by

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cdot \cos nx. \qquad \text{--- (2)}$$

Put $x=0$ in (2) we get

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cdot \cos 0$$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$-\frac{\pi^2}{3} = 4 \left(\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

5) $f(x) = x \cos x$ in $(-\pi, \pi)$.

Solution:

The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$f(x) = x \cos x$$

$$f(-x) = -x \cdot \cos(-x) = -x \cos x = -f(x)$$

Hence $f(x)$ is odd function.

Consequently $a_0 = 0$, $a_n = 0$.

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} x \cdot \cos x \cdot \sin nx \, dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \sin nx \cdot \cos x \, dx$$

We know

$$\sin A \cdot \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right].$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{1}{2}\left[\sin(nx+x) + \sin(nx-x)\right] dx.$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} x \cdot \sin(n+1)x \, dx + \int_0^{\pi} x \cdot \sin(n-1)x \, dx\right\}.$$

Applying Bernoulli's rule to each of the integrals

$$b_n = \frac{1}{\pi}\left[x \cdot \left(\frac{-\cos(n+1)x}{n+1}\right) - (1) \cdot \left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\left[x \cdot \left(-\frac{\cos(n-1)x}{n-1}\right) - (1) \cdot \left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left\{\frac{-1}{n+1}\left[\pi \cos(n+1)\pi - 0\right] - \frac{1}{n-1}\left[\pi \cos(n-1)\pi - 0\right]\right\}$$

Here $\sin(n+1)\pi = 0 = \sin(n-1)\pi$, since $n = 1, 2, 3,$

$$b_n = -\left\{\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}$$

$$b_n = -\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}.$$

$$b_n = (-1)^n \left\{ \frac{1}{n+1} + \frac{1}{n-1} \right\}$$

$$\boxed{b_n = (-1)^n \frac{2n}{n^2-1}} \quad (n \neq 1)$$

We shall now find b_n when $n=1$. That is to find b_1

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \cdot \cos x \, dx$$

Put $n=1$,

$$b_1 = \frac{2}{\pi} \int_0^{\pi} x \cdot \sin x \cos x \, dx$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} x \cdot \frac{\sin 2x}{2} \, dx$$

$$b_1 = \frac{1}{\pi} \int_0^{\pi} x \cdot \sin 2x \, dx$$

$$b_1 = \frac{1}{\pi} \left[x \cdot \left(\frac{-\cos 2x}{2} \right) - (1) \cdot \left(\frac{-\sin 2x}{4} \right) \right]_0^{\pi}$$

$$= \frac{-1}{2\pi} \left[x \cdot \cos 2x \right]_0^{\pi}$$

$$b_1 = \frac{-1}{2\pi} \left[\pi \cdot \cos 2\pi - 0 \right]$$

$$\boxed{b_1 = \frac{-1}{2}}$$

Fourier Series representation is given by

$$x \cos x = \frac{-1}{2} \sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx.$$

6) $f(x) = |\cos x|$ in $(-\pi, \pi)$

Solution:

$$f(x) = |\cos x| \quad , \quad x \in (-\pi, \pi)$$

$$f(-x) = |\cos(-x)| = |\cos x| = f(x).$$

$\Rightarrow f(x)$ is even function.

$\therefore b_n = 0$.

$\therefore$ The Fourier series representation is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx.$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot dx$$

We know

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & x < 0. \end{cases}$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cdot dx.$$

$$\begin{cases} \cos x \geq 0 & 0 \leq x \leq \frac{\pi}{2} \\ \cos x < 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx\right]$$

$$= \frac{2}{\pi}\left\{ [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi}\left\{ \sin \pi/2 - 0 - (\sin \pi - \sin \pi/2)\right\}.$$

$$= \frac{2}{\pi}(1+1) = \frac{4}{\pi}$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} -\cos x \cdot \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/2} 2 \cos nx \cdot \cos x \, dx - \int_{\pi/2}^{\pi} 2 \cos nx \cdot \cos x \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x] \, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0 \right] - \left[0 - \left(\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right) \right] \right\}, \quad n \neq 1$$

$$a_n = \frac{2}{\pi} \left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}\right]$$

$$= \frac{2}{\pi}\left\{\frac{\sin(\pi/_2 + n\pi/_2)}{n+1} - \frac{\sin(\pi/_2 - \frac{n\pi}{2})}{n-1}\right\}.$$

$$= \frac{2}{\pi}\left\{\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right\}.$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\cdot\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}.$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\cdot\left\{\frac{-2}{n^2-1}\right\}$$

$$a_n = -\frac{4}{\pi(n^2-1)}\cdot\cos\frac{n\pi}{2} = \underline{\underline{\frac{-4\cos n\pi/_2}{\pi(n^2-1)}}}$$

$n \neq 1.$

<u>Calculation</u> <u>of a_1</u>:

$$a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\cdot\cos x\cdot dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cdot\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\cdot\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x\,dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi + \frac{\sin 2\pi}{2} - \left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right)\right)\right]$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right] = 0.$$

∴ The Fourier series is represented by

$$\boxed{f(x) = \frac{2}{\pi} + \sum \frac{-4\cos n\pi/2}{\pi(n^2-1)} \cdot \cos nx}$$

Fourier series expansion of $f(x)$ over an arbitrary interval $(-l, l)$ and $(0, 2l)$.

7) Expand $f(x) = e^{-x}$ as a Fourier Series in the interval $(-l, l)$.

Solution:

The Fourier series of $f(x)$ over $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx.$$

Calculation of a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}.\,dx = \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{-l}-e^{l}}{-1}\right]$$

$$a_0 = \frac{e^{l}-e^{-l}}{l} = \frac{2\sinh l}{l}.$$

Calculation of a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}.\cos\frac{n\pi x}{l}.dx.$$

$$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\therefore a_n = \frac{1}{l}\left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{l^2}}\left[-\cos\frac{n\pi}{l}x + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{l^2.e^{-l}}{l^2+n^2\pi^2}\left[-(-1)^n\right] - \frac{l^2 e^{l}}{l^2+n^2\pi^2}\left[-(-1)^n\right]\right]$$

$$= \frac{l\,(-1)^n}{l^2+n^2\pi^2}\left[-e^{-l}+e^{l}\right]$$

$$\boxed{a_n = \frac{2l\,(-1)^n \cdot \sinh l}{l^2+n^2\pi^2}}$$

Calculation of b_n:

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot \sin\frac{n\pi x}{l}\cdot dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\ \sin\frac{n\pi x}{l}\ dx$$

$$\int e^{ax}\cdot \sin bx\ dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$b_n = \frac{1}{l}\left[\frac{e^{-x}}{1^2+\frac{n^2\pi^2}{l^2}}\left[-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right]\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{l^2\cdot e^{-l}}{l^2+n^2\pi^2}\left[-\frac{n\pi}{l}\cdot(-1)^n\right] - \frac{l^2 e^{+l}}{l^2+n^2\pi^2}\left[-\frac{n\pi}{l}(-1)^n\right]\right]$$

$$= \frac{l\,(-1)^n}{l^2+n^2\pi^2}\cdot\left[\frac{n\pi}{l}\right]\cdot\left[-e^{-l}+e^{l}\right]$$

$$\boxed{b_n = \frac{2n\pi\,(-1)^n\cdot \sinh l}{l^2+n^2\pi^2}}$$

∴ The Fourier Series expansion given by

$$f(x) = \frac{\sinh l}{l} + \sum_{n=1}^{\infty} \frac{2l(-1)^n \sinh l}{l^2+n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n \sinh l}{l^2+n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right)$$

8) $$f(x) = \begin{cases} \pi x & \text{for } 0 < x < 1 \\ \pi(2-x) & \text{for } 1 < x < 2 \end{cases}$$

Solution.

$f(x)$ is defined in $(0, 2)$. Hence period $= 2$

$$2l = 2$$

$$\boxed{l = 1}$$

∴ The Fourier Series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x.$$

Calculation of a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \quad \text{becomes}$$

$$= \frac{1}{1}\int_0^{2} f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx.$$

$$a_0 = \pi\left\{\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2\right\}$$

$$= \pi\left[\frac{1}{2} + (4-2) - (2-\tfrac{1}{2})\right\} = \pi$$

$$\boxed{a_0 = \pi}$$

Calculation of a_n

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 \pi(2-x)\cos n\pi x\, dx$$

$$= \pi\left[\int_0^1 x\cos n\pi x\, dx + \int_1^2 (2-x)\cos n\pi x\, dx\right]$$

$$a_n = \pi\left\{\left[x\cdot\frac{\sin n\pi x}{n\pi} - (1)\cdot\frac{1}{n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)\right]_0^1\right.$$

$$\left.+\left[(2-x)\frac{\sin n\pi x}{n\pi} - (-1)\cdot\frac{1}{n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)\right]_1^2\right\}$$

$$= \pi\left\{+\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{-\cos 2n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}$$

$$= \frac{\pi}{n^2\pi^2}\left\{2(-1)^n - 2\right\} \quad \left[\because \cos 2n\pi = (-1)^{2n} = 1\right]$$

$$\therefore \boxed{a_n = \frac{1}{n^2\pi}\left(2(-1)^n - 2\right)}$$

$\therefore$ The Fourier series expansion is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x .$$

Calculation of b_n.

$$b_n = \frac{1}{1} \int_0^2 f(x) \sin n\pi x \cdot dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx .$$

$$= \pi \left\{ \left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\cdot\left(\frac{-1}{n\pi}\right)\cdot \frac{\sin n\pi x}{n\pi} \right]_0^1 + \left[(2-x)\cdot\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\cdot\left(\frac{-1}{n\pi}\right) \frac{\sin n\pi x}{n\pi} \right]_1^2 \right\} .$$

$$= \pi \cdot \left\{ \frac{-(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} \right\} = 0 .$$

$\therefore$ The required fourier series is given by

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(n\pi x) .$$

9) $f(x) = x^2$ in $(-l, l)$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is an even function

Hence $b_n = 0$

$\therefore$ The required Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Calculation of a_0

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad \text{becomes}$$

$$= \frac{2}{l}\int_{0}^{l} x^2 \cdot dx$$

$$= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2 \cdot l^3}{3l} = \frac{2l^2}{3}$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

Calculation of a_n

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{becomes}$$

$$= \frac{2}{l}\int_{0}^{l} x^2 \cdot \cos\frac{n\pi x}{l} \cdot dx$$

$$a_n = \frac{2}{l}\left[x^2 \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (2x).\frac{l}{n\pi}.\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} + 2\left(\frac{-l^2}{n^2\pi^2}\right).\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l$$

$$a_n = \frac{2}{l}\left[2l^2.\frac{l}{n^2\pi^2}\cos n\pi\right]$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}(-1)^n.}$$

The Fourier expansion of $f(x)$ is given by

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{l}.$$

Problems on Half-Range Fourier Series

10) Find the half-range Fourier sine and cosine series of $f(x) = \begin{cases} x & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$.

Solution

The sine half-range Fourier Series of the function in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

Calculation of b_n.

$$b_n = \frac{2}{\pi} \left[\int_0^{\pi/2} x \cdot \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right]$$

$$b_n = \frac{2}{\pi} \left[\left[x \cdot \left(-\frac{\cos nx}{n} \right) - (1) \cdot \left(\frac{-1}{n} \right) \cdot \frac{\sin nx}{n} \right]_0^{\pi/2} + \left[(\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \cdot \left(\frac{-1}{n} \right) \frac{\sin nx}{n} \right]_{\pi/2}^{\pi} \right].$$

$$= \frac{2}{\pi} \left[\left(\frac{-1}{n} \cdot \frac{\pi}{2} \cdot \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right) + \left(0 - \left(-\frac{\pi}{2n} \cos n\pi/2 - \frac{1}{n^2} \sin n\pi/2 \right) \right) \right]$$

$$b_n = \frac{2}{\pi} \left[\frac{2}{n^2} \cdot \sin \frac{n\pi}{2} \right] \Rightarrow \boxed{b_n = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}.}$$

The required half range sine series is given by

$$\boxed{f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin nx.}$$

(ii) The cosine half-range Fourier series of $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx\, dx.$$

Calculation of a_0:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx.$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx \right]$$

$$= \frac{2}{\pi} \left\{ \left[\frac{x^2}{2} \right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2} \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8} \right) \right\}$$

$$= \frac{2}{\pi} \left\{ 2\frac{\pi^2}{8} \right) = \frac{\pi}{2} \qquad \boxed{a_0 = \frac{\pi}{2}}$$

Calculation of a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cdot \cos nx \, dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\cdot \frac{\sin nx}{n} - (1)\cdot\frac{1}{n}\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi/2} + \left[(\pi - x)\cdot \frac{\sin nx}{n} - (-1)\cdot\frac{1}{n}\left(-\frac{\cos nx}{n}\right)\right]_{\pi/2}^{\pi}\right]$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{1}{n}\cdot\frac{\pi}{2}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cdot\cos\frac{n\pi}{2} - \left(\frac{1}{n^2}\right)\right] + \left[\frac{-1}{n^2}\cdot\cos n\pi - \left[\frac{1}{n}\cdot\frac{\pi}{2}\cdot\sin\frac{n\pi}{2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2}{n^2}\cdot\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{1}{n^2}\cos n\pi\right\}.$$

$$\Rightarrow \frac{2}{\pi n^2}\cdot\left\{2\cos\frac{n\pi}{2} - (1 + (-1)^n)\right\}.$$

The required cosine half range Fourier Series is given by

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}\cos nx.}$$

11) Find the half range Fourier Sine Series of

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{for } 0 \le x \le 1/2 \\ x - \frac{3}{4} & \text{for } 1/2 \le x \le 1 \end{cases}$$

Solution:

$f(x)$ is defined in $(0,1)$. Comparing with half

The range $(0, l)$ we have $l = 1$.

The corresponding sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where

$$b_n = \frac{2}{1} \int_0^1 f(x) \cdot \sin n\pi x \, dx.$$

Calculation of b_n:

$$b_n = 2 \cdot \left[\int_0^{1/2} \left(\frac{1}{4} - x\right) \sin n\pi x \, dx + \int_{1/2}^{1} \left(x - \frac{3}{4}\right) \cdot \sin n\pi x \, dx \right]$$

$$= 2 \cdot \left[\left[\left(\frac{1}{4} - x\right) \cdot \left(\frac{-\cos n\pi x}{n\pi}\right) - (-1) \cdot \left(\frac{-1}{n\pi}\right) \frac{\sin n\pi x}{n\pi} \right]_0^{1/2} + \left[\left(x - \frac{3}{4}\right) \cdot \left(\frac{-\cos n\pi x}{n\pi}\right) - (1) \cdot \left(\frac{-1}{n\pi}\right) \cdot \frac{\sin n\pi x}{n\pi} \right]_{1/2}^{1} \right]$$

$$b_n = 2 \left\{ \left[\frac{+1}{4n\pi} \cdot \cos \frac{n\pi}{2} - \frac{1}{n^2\pi^2} \sin \frac{n\pi}{2} - \left(\frac{-1}{4n\pi} - 0\right) \right] + \left[\frac{-1}{4n\pi} \cos n\pi - \left[\left(\frac{+1}{4n\pi}\right) \cos \frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin \frac{n\pi}{2} \right] \right] \right\}$$

$$b_n = 2\left\{\left[\frac{1}{4n\pi} - \frac{1}{4n\pi}\cos n\pi\right] - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}.$$

$$\boxed{b_n = \frac{1}{2n\pi}\left\{1-(-1)^n\right\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Thus the Sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x$$

12) Find the half-range Fourier cosine series of

$$f(x) = \begin{cases} Kx & \text{for } 0 \le x \le l/2 \\ K(l-x) & \text{for } l/2 \le x \le l \end{cases}$$

Solution:

$f(x)$ is defined in $(0, l)$ and the cosine half-range series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\frac{n\pi x}{l}\,.dx.$$

Calculation of a_0:

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx.$$

$$a_0 = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2}{l}\left\{k\cdot\left[\frac{x^2}{2}\right]_0^{l/2} + kl\cdot[x]_{l/2}^{l} - k\left[\frac{x^2}{2}\right]_{l/2}^{l}\right\}.$$

$$a_0 = \frac{2}{l}\left\{k\cdot\frac{l^2}{8} + kl\left[\frac{l}{2}\right] - \frac{k}{2}\left[l^2 - \frac{l^2}{4}\right]\right\}.$$

$$= \frac{2}{l}\left\{\frac{kl^2}{8} + \frac{kl^2}{2} - \frac{kl^2}{2} + \frac{kl^2}{8}\right\}.$$

$$a_0 = \frac{2}{l}\left\{\frac{2\cdot kl^2}{8}\right\}$$

$$\boxed{a_0 = \frac{kl}{2}}$$

Calculation of a_n.

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cdot\cos\frac{n\pi x}{l}\cdot dx.$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cdot\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{l}\left[\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{l}{n\pi}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\cdot\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\cdot\left(\frac{l}{n\pi}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_{l/2}^{l}\right]$$

$$a_n = \frac{2k}{l}\left\{\left[\left(\frac{l}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right) - \left(\frac{l^2}{n^2\pi^2}\right)\right.\right.$$

$$\left.\left. + \left[-\frac{l^2}{n^2\pi^2}\cos n\pi - \left(\frac{l}{2}\frac{l}{n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right)\right]\right]\right\}$$

$$a_n = \frac{2k}{l}\left[\frac{2l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2} - \frac{l^2}{n^2\pi^2}\cos n\pi\right]$$

$$a_n = \frac{2k}{l}\left[\frac{l^2}{n^2\pi^2}\right]\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}.$$

$$\boxed{a_n = \frac{2kl}{\pi^2 n^2}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right)}$$

The half-range cosine series is given by.

$$f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty} \frac{2kl}{\pi^2 n^2}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right).\cos\frac{n\pi x}{l}.$$

Problems on Parseval's Identity.

13) Obtain the Fourier series for $y = x^2$ in $-\pi < x < \pi$ and hence show That $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$.

x^2 is an even function

$\Rightarrow b_n = 0$.

Calculation of a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

Calculation of a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\cdot\frac{\sin nx}{n} - (2x)\frac{1}{n}\cdot\left(-\frac{\cos nx}{n}\right) + 2\left(\frac{-1}{n^2}\right)\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n}\cdot\cos n\pi\right] = \frac{4}{n^2}(-1)^n$$

∴ The Fourier Cosine Series is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cdot\cos nx.$$

The Fourier coeff a_n contain terms $\Sigma\frac{1}{n^2}$ But we require the sum of their squares $\Sigma\frac{1}{n^4}$.

So, we use Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{4}\cdot\frac{4\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{4\pi^4}{45} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

14 Expand $f(x) = x - \frac{x^2}{2}$ in $(0, 2)$ as Fourier Sine series and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$.

Solution

The Fourier Sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx$$

Here. $l = 2$.

$$\therefore \quad b_n = \frac{2}{2} \int_0^2 \left(x - \frac{x^2}{2}\right) \sin \frac{n\pi x}{2} \cdot dx$$

$$= \left[\left(x - \frac{x^2}{2}\right)\left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \left(1 - \frac{\cancel{2}x}{\cancel{2}}\right)\left(\frac{-2}{n\pi}\right) \cdot \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} + (-1) \cdot \left(\frac{-4}{n^2\pi^2}\right) \cdot \left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) \right]_0^2.$$

$$= \left[\left(0 - 0 - \frac{8}{n^3\pi^3} \cos n\pi\right) - \left(0 - 0 - \frac{8}{n^3\pi^3}\right) \right]$$

$$= \frac{8}{n^3\pi^3}\left[-(-1)^n + 1\right] = \frac{8}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$\boxed{b_n = \frac{8}{n^3\pi^3}\left[1-(-1)^n\right]}$$

The Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^3\pi^3}\left[1-(-1)^n\right] \sin \frac{n\pi x}{2}$$

But

$$1-(-1)^n = \begin{cases} 1-(-1) = 2 & \text{when } n \text{ is odd} \\ 1-(+1) = 0 & \text{when } n \text{ is even} \end{cases}$$

$$\therefore\ f(x) = x - \frac{x^2}{2} = \sum_{n=1,3,5} \frac{8}{n^3\pi^3}(2) \sin \frac{n\pi x}{2}.$$

But 1, 3, 5, ... are odd numbers represented in general as $(2n-1)$ where $n = 1, 2, 3, \ldots$

Thus we have

$$f(x) = x - \frac{x^2}{2} = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{1 . \sin \frac{(2n-1)\pi x}{2}}{(2n-1)^3}.$$

Thus

$$x - \frac{x^2}{2} = \frac{16}{\pi^3} \sum_{n=}^{\infty} \frac{1}{(2n-1)^3} \cdot \sin \frac{(2n-1)\pi x}{2}.$$

For Half-range sine series, Parseval's identity is

$$2\int_0^l \left[f(x)\right]^2 dx = l\left[\sum_{n=1}^{\infty} b_n^2\right]$$

$$\Rightarrow \cancel{2}\int_0^2 \left(x-\frac{x^2}{2}\right)^2 dx = \cancel{2}\sum_{n=1}^{\infty}\frac{(16)^2}{\pi^6\cdot(2n-1)^6}$$

$$\int_0^2 x^2 - x^3 + \frac{x^4}{4}\,dx = \frac{(16)^2}{\pi^6}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

$$\left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{4}\cdot\frac{x^5}{5}\right]_0^2 = \frac{(16)^2}{\pi^6}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

$$\frac{\cancel{16}}{60}\times\frac{\pi^6}{(16)^{\cancel{2}}} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.$$

$$\boxed{\frac{\pi^6}{960} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.}$$

15) Using the Fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$. Show that

1) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ (ii) $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$

Solution.

$f(x) = |x|$ is an even function

$\Rightarrow b_n = 0.$

Calculation of a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot dx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \cdot dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

$$\boxed{a_0 = \pi}$$

Calculation of a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \cos nx \, dx$$

$$a_n = \frac{2}{\pi}\left[x \cdot \frac{\sin nx}{n} - (1)\left(\frac{1}{n}\right)\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[+\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].$$

$\therefore$ The Fourier series is given by

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos nx.$$

Parseval Equation.

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx.$$

If n is even, Then $a_n = 0$. $\therefore$

so, n is odd, and in general odd numbers are represented as $(2n-1)$

$$\frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left[\frac{-4}{\pi(2n-1)^2}\right]^2 = \frac{1}{\pi}\,\frac{2}{3}\pi^3$$

$$\frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{2\pi^2}{3}$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{2\pi^2}{3} - \frac{\pi^2}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^2}{6}\times\frac{\pi^2}{16} = \frac{\pi^4}{96}$$

ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$

Consider

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots$$

$$= \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}+\frac{1}{2^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) \quad \left[\begin{matrix}\text{From}\\ \text{Part (i)}\end{matrix}\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}+\frac{1}{2^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\left(1-\frac{1}{2^4}\right)\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}\times\frac{16}{15}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$$

Problems on Complex Fourier Series

16) $f(x) = e^{-x}$ in $-1 \leq x \leq 1$

Solution:

The complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{l}}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) \cdot e^{-\frac{in\pi x}{l}} \cdot dx.$$

Calculation of Fourier Coefficients:

$$c_n = \frac{1}{2} \int_{-1}^{1} f(x) \cdot e^{-in\pi x} dx.$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1+in\pi)x} dx.$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)} - e^{1+in\pi}}{-(1+in\pi)} \right]$$

$$\Rightarrow \frac{1}{2} \left[\frac{e^{1+in\pi} - e^{-(1+in\pi)}}{1+in\pi} \right]$$

$$= \frac{e^{1} . e^{in\pi} - e^{-1}.e^{-in\pi}}{2(1+in\pi)}$$

$e^{in\pi}$
$= \cos n\pi + i \sin n\pi$
$= (-1)^n.$

$$c_n = \frac{(-1)^n}{1+in\pi} \left\{ \frac{e^{1} - e^{-1}}{2} \right\}$$

$e^{-in\pi}$
$= \cos n\pi - i\sin n\pi$
$= (-1)^n.$

$$\boxed{c_n = \frac{(-1)^n . \sinh 1}{1+in\pi}}$$

$\therefore$ The Complex Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n . \sinh 1}{1+in\pi} . e^{inx}.$$

17) $f(x) = \cos ax$ in $-\pi \le x \le \pi$. where 'a' is not an integer.

Solution:

The Complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n . e^{inx}.$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Calculation of Fourier Coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) . e^{-inx} dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cdot e^{-inx}\, dx.$$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right].$$

Here $a = -in$, $b = a$.

$$\therefore C_n = \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+a^2}(-in\cos ax + a\sin ax)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\cos a\pi + a\sin a\pi) - \frac{e^{in\pi}}{a^2-n^2}(-in\cos a(-\pi) + a\sin a(-\pi))\right]$$

$$C_n = \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\,(-e^{-in\pi} + e^{in\pi}) + a\sin a\pi\,(e^{-in\pi} + e^{in\pi})\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\,(2i\sin n\pi) + a\sin a\pi\,(2\cos n\pi)\right]$$

$$C_n = \frac{1}{2\pi(a^2-n^2)}(2a\sin a\pi\cos n\pi)$$

$$\boxed{C_n = \frac{(-1)^n \cdot a\sin a\pi}{\pi(a^2-n^2)}}$$

$\therefore$ The complex FS is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot a\sin a\pi}{\pi(a^2-n^2)}\, e^{inx}.$$

18) $f(x) = e^{ax}$ in $-\pi \leq x \leq \pi$

Solution

The complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

where
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\, dx$$

Calculation of Fourier Coefficients :

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\, dx.$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(in-a)} \cdot dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-(in-a)x}}{-(in-a)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)} \left[e^{-(in-a)\pi} - e^{(in-a)\pi} \right]$$

$$= \frac{(-1)^n}{2\pi(a-in)} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$\boxed{c_n = \frac{(-1)^n}{\pi(a-in)} \sinh a\pi}$$

∴ The Fourier Series is given by

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot \sinh a\pi}{\pi(a-in)} e^{inx}}$$