Lagrange's Method of Undetermined Multipliers

Suppose we need to minimise / maximise a function $f(x, y)$ subject to the constraint $\phi(x, y, z) = 0$, then we can introduce an additional unknown constant λ known as Lagrange's Multiplier to ease the process of finding the extremas.

Working Procedure

1. Form an auxillary function / equation,

$$F(x, y) = f(x, y, z) + \lambda \phi(x, y, z) \longrightarrow (1)$$

2. Partially differentiate F in (1) wrt x, y, z respectively.

5. Solve the equation $F_x = 0$, $F_y = 0$, $F_z = 0$ and the constraint for the Lagrange multiplier λ and stationary values x, y, z.

55. Find the extreme value of $x^3 + 8y^3 + 64z^2$, when $xyz = 1$

Let $f(x, y, z) = x^3 + 8y^3 + 64z^2$

Let $\phi(x, y, z) = xyz - 1$

Auxillary eqn. is given by, $F = f + \lambda\phi$

$\Rightarrow F = x^3 + 8y^3 + 64z^3 + \lambda(xyz - 1) \longrightarrow (a)$

$$\left.\begin{aligned} &\Rightarrow F_x = 0 \\ &\Rightarrow 3x^2 + \lambda(yz) = 0. \end{aligned}\right\} \lambda = \frac{-3x^2}{yz} \longrightarrow (1)$$

$$\left.\begin{aligned} &\Rightarrow F_y = 0 \\ &\Rightarrow 24y^2 + \lambda(xz) = 0. \end{aligned}\right\} \lambda = \frac{-24y^2}{xz} \longrightarrow (2)$$

$$\left.\begin{aligned} &\Rightarrow F_z = 0 \\ &\Rightarrow 192z^2 + \lambda(xy) = 0. \end{aligned}\right\} \lambda = \frac{-192z^2}{xy} \longrightarrow (3).$$

$(1) \div (2) \Rightarrow 1 = \frac{-3x^2}{yz} \times \frac{xz}{-24y^2}$

$\Rightarrow x^3 = 8y^3$

$\Rightarrow x = 2y \longrightarrow (4)$

$② \div ③ \Rightarrow 1 = \frac{24y^2}{xz} \times \frac{xy}{192z^2}$

$\Rightarrow y^3 = 8z^3$

$\Rightarrow y = 2z \longrightarrow ⑤$

From ④ and ⑤, $x = 4z$ and $y = 2z$.

Substituting in constraint ϕ,

$xyz = 1$

$\Rightarrow (4z)(2z)(z) = 1$

$\Rightarrow z^3 = \frac{1}{8}$

$\Rightarrow \underline{\underline{z = \frac{1}{2}}}$ $\therefore \underline{\underline{x = 2}}$ and $\underline{\underline{y = 1}}$.

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$\therefore$ Extreme value of $f(x,y,z) = x^3 + 8y^3 + 64z^2$

is $(8 + 8 + \cancel{4}^{16}) = \underline{\underline{32}}\ \cancel{4}$.

56. Find the minimum value of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $xy + yz + xz = 3a^2$

Let $f(x,y,z) = x^2 + y^2 + z^2$

Let $\phi(x,y,z) = xy + yz + xz - 3a^2 \longrightarrow ①$

Auxillary eqn is, $F = f + \lambda\phi$

$F = x^2 + y^2 + z^2 + \lambda(xy + yz + xz - 3a^2)$

$\left.\begin{array}{l}\Rightarrow F_x = 0 \\ \Rightarrow F_x = 2x + \lambda(y+z)\end{array}\right\}$ $\lambda = \frac{-2x}{y+z} \longrightarrow ②$.

$$\left.\begin{aligned} &\Rightarrow F_y = 0 \\ &\Rightarrow F_y = 2y + \lambda(x+z) \end{aligned}\right\} \lambda = \frac{2y}{x+z} \longrightarrow ③$$

$$\left.\begin{aligned} &\Rightarrow F_z = 0 \\ &\Rightarrow F_z = 2z + \lambda(y+x) \end{aligned}\right\} \lambda = \frac{-2z}{y+x} \longrightarrow ④$$

② = ③ $\Rightarrow \dfrac{2x}{y+z} = \dfrac{2y}{x+z}$

$\Rightarrow x^2 + xz = y^2 + yz$

$\Rightarrow x^2 - y^2 + xz - yz = 0$

$\Rightarrow (x+y)(x-y) + z(x-y) = 0$

$\Rightarrow (x-y)(x+y+z) = 0$

$$\left.\begin{aligned} &\therefore x - y = 0 \text{ since } x+y+z \neq 0 \Rightarrow x = y \\ &\text{III}^{ly}, ③ = ④ \Rightarrow y = z \end{aligned}\right\} x = y = z$$

Using ①, $\phi = 3x^2 - 3a^2 = 0$

$\Rightarrow x^2 = a^2 \Rightarrow x = \pm a$

III^{ly}, $y = \pm a$ and $z = \pm a$

$\therefore x = y = z = \pm a$

$\therefore$ Minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ is

$(a^2 + a^2 + a^2) = \underline{\underline{3a^2}}$

57. Find the extreme value of $a^3x^2+b^3y^2+c^3z^2$ wrt to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, where $a>0$, $b>0$, $c>0$.

$$f(x,y,z)=a^3x^2+b^3y^2+c^3z^2$$

$$\phi(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1 \longrightarrow (1)$$

Auxillary eqn is, $F=f+\lambda\phi$.

$$F=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$$\left.\begin{aligned}&\Rightarrow F_x=0\\&\Rightarrow F_x=2a^3x+\lambda\left(\frac{-1}{x^2}\right)\end{aligned}\right\}\ \lambda=2a^3x^3 \longrightarrow (2)$$

$$\left.\begin{aligned}&\Rightarrow F_y=0\\&\Rightarrow F_y=2b^3y+\lambda\left(-\frac{1}{y^2}\right)\end{aligned}\right\}\ \lambda=2b^3y^3 \longrightarrow (3)$$

$$\left.\begin{aligned}&\Rightarrow F_z=0\\&\Rightarrow F_z=2c^3z+\lambda\left(-\frac{1}{z^2}\right)\end{aligned}\right\}\ \lambda=2c^3z^3 \longrightarrow (4)$$

$$\left.\begin{aligned}&(2)=(3) \Rightarrow 2a^3x^3=2b^3y^3\\&\qquad\Rightarrow ax=by\\&(3)=(4) \Rightarrow 2b^3y^3=2c^3z^3\\&\qquad\Rightarrow by=cz\end{aligned}\right\}\ \begin{aligned}&ax=by=cz\\&\Rightarrow y=\frac{ax}{b}\\&\Rightarrow z=\frac{ax}{c}\end{aligned}$$

Putting x, y, z in ①,

$$\Rightarrow \frac{1}{x} + \frac{b}{ax} + \frac{c}{ax} = 1$$

$$\Rightarrow \frac{1}{x}\left(1 + \frac{b}{a} + \frac{c}{a}\right) = 1$$

$$\Rightarrow x = \frac{a+b+c}{a}$$

$$|||^{ly},\ y = \frac{a+b+c}{b}$$

$$|||^{ly},\ z = \frac{a+b+c}{c}$$

$\therefore$ The extreme value of $f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^3$

is $= a^3\left(\frac{a+b+c}{a}\right)^2 + b^3\left(\frac{a+b+c}{b}\right)^2 + \left(\frac{a+b+c}{c}\right)^2 c^3$

$= a(a+b+c)^2 + b(a+b+c)^2 + c(a+b+c)^2$

$= (a+b+c)^2(a+b+c)$

$= \underline{\underline{(a+b+c)^3}}$

58. The temperature T at any point (x, y, z) in space is $400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$.

Let $f(x) = 400xyz^2$

Let $\phi(x) = x^2+y^2+z^2-1 \longrightarrow (1)$.

Auxillary eqn, $F = f + \lambda\phi$

$\Rightarrow F = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$.

$\left.\begin{array}{l} F_x = 0 \\ \Rightarrow 400yz^2 + \lambda(2x) = 0 \end{array}\right\}$ $\lambda = \dfrac{-400yz^2}{2x} = \dfrac{-200yz^2}{x} \longrightarrow (2)$

$\left.\begin{array}{l} F_y = 0 \\ \Rightarrow 400xz^2 + \lambda(2y) = 0 \end{array}\right\}$ $\lambda = \dfrac{-400xz^2}{2y} = \dfrac{-200xz^2}{y} \longrightarrow (3)$

$\left.\begin{array}{l} F_z = 0 \\ \Rightarrow 800xyz + \lambda(2z) = 0 \end{array}\right\}$ $\lambda = \dfrac{-800xyz}{2z} = -400xy \longrightarrow (4)$

$(2) \div (3) \Rightarrow 1 = \dfrac{-\cancel{200}y\cancel{z^2}}{x} \times \dfrac{y}{-\cancel{200}x\cancel{z^2}} \Rightarrow 1 = \dfrac{y^2}{x^2}$

$\Rightarrow x = y$

$(3) \div (4) \Rightarrow 1 = \dfrac{-\cancel{200}\cancel{x}z^2}{y} \times \dfrac{1}{-\cancel{400}_2\cancel{x}y} \Rightarrow 1 = \dfrac{z^2}{2y^2}$

$\Rightarrow z = \sqrt{2}y$

Putting in (1),

$\Rightarrow y^2 + y^2 + 2y^2 = 1 \Rightarrow y = \dfrac{1}{2} \Rightarrow x = \dfrac{1}{2} \Rightarrow z = \dfrac{1}{\sqrt{2}}$

$\therefore x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{\sqrt{2}}$ is a stationary point

$\therefore$ The maximum/highest temperature on the surface of the unit sphere is $\left(400 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \underline{\underline{50}}$

9. A container with an open top is to have $10\,m^3$ capacity and be made of thin sheet metal. Calculate the dimensions of the box if it is to use the minimum possible amount of metal.

Let x, y, z be the sides of the box.

Volume, $xyz = 10$.

Surface area, $S = xy + 2yz + 2xz$

(because one side is open).

Let $f(x) = xy + 2yz + 2xz$

Let $\phi(x) = xyz - 10. \longrightarrow ①$

Auxillary eqn, $F = f + \lambda\phi$

$\Rightarrow F = xy + 2yz + 2xz + \lambda(xyz - 10)$.

$$\left.\begin{array}{l} F_x = 0 \\ \Rightarrow y + 2z + \lambda(yz) = 0 \end{array}\right\} \lambda = \frac{-(y+2z)}{yz} \longrightarrow ②$$

$$\left.\begin{array}{l} F_y = 0 \\ \Rightarrow x + 2z + \lambda(xz) = 0 \end{array}\right\} \lambda = -\frac{(x+2z)}{xz} \longrightarrow ③$$

$$\left.\begin{array}{l} F_z = 0 \\ \Rightarrow 2y + 2x + \lambda(xy) = 0 \end{array}\right\} \lambda = \frac{-2(y+x)}{xy} \longrightarrow ④$$

$$② - ④ \Rightarrow 0 = -\left(\frac{1}{z} + \frac{2}{y}\right) + \left(\frac{2}{x} + \frac{2}{y}\right)$$

$$\Rightarrow \frac{1}{z} = \frac{2}{x} \quad \Rightarrow x = 2z$$

$$③ - ④ \Rightarrow 0 = -\left(\frac{1}{z} + \frac{2}{x}\right) + \left(\frac{2}{x} + \frac{2}{y}\right)$$

$$\Rightarrow \frac{1}{z} = \frac{2}{y} \quad \Rightarrow y = 2z$$

$\Rightarrow x = y = 2z$

Putting in ①,

$xyz = 10$

$\Rightarrow (2z)(2z)(z) = 10$

$\Rightarrow z^3 = \frac{10}{4} \quad \Rightarrow z = \sqrt[3]{\frac{5}{2}} = 1.357$

$$\cancel{\Rightarrow x(x)\left(\sqrt[3]{\frac{5}{2}}\right) = 10}$$

$$\cancel{\Rightarrow x^2 = \frac{\sqrt{5}\sqrt{2}\sqrt[3]{2}}{\sqrt[3]{5}}}$$

$\Rightarrow x(x)(1.357) = 10$

$\Rightarrow x = \sqrt{\frac{10}{1.357}}$

$\Rightarrow x = y = 2.714$

$\therefore$ The stationary points are $x = 2.714$, $y = 2.714$ and $z = 1.357$.

$\therefore$ The surface area is minimum at the above s. points. Thus, the amount of min. metal used is $= (7.36 + 7.36 + 7.36) = \underline{22.097\ m^2}$

Assignment - 10

1. Determine the critical points and locate any relative minima, maxima and saddle points of function f defined $f(x,y) = x^4 + y^4 - 4xy$.

Given $f(x,y) = x^4 + y^4 - 4xy$.

$\frac{\partial f}{\partial x} = 4x^3 - 4y \qquad \Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4y = 0$

$\Rightarrow y = x^3 \longrightarrow ①$

$\frac{\partial f}{\partial y} = 4y^3 - 4x \qquad \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 - 4x = 0.$

$\Rightarrow x = y^3 \longrightarrow ②$

~~From ①, $4x^3 - 4y = 0$~~

~~$\rightarrow 4x^3$~~

From ①, $y = x^3$

$\Rightarrow y = (y^3)^3$

$\Rightarrow y^9 - y = 0$

$\Rightarrow y(y^8 - 1) = 0$

$\Rightarrow y = 0, 1, -1.$

$\therefore x = 0, 1, -1.$

The critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

$\frac{\partial^2 f}{\partial x^2} = 12x^2 = r \quad \Big| \quad \frac{\partial^2 f}{\partial x \partial y} = -4 = s \quad \Big| \quad \frac{\partial^2 f}{\partial y^2} = 12y^2 = t$

(x,y)	r	s	t	$rt - s^2$	Conclusion
$(0,0)$	0	-4	0	-16	saddle point
$(1,1)$	12	-4	12	128	minimum
$(-1,-1)$	12	-4	12	128	minimum

$\therefore f(x,y)$ is minimum at $(1,1)$ and $(-1,-1)$ and $(0,0)$ is the saddle point.

2\. Divide the number 24 into 3 parts such that the continued product may be maximum.

Let 24 be divided into three parts: x, y and $(24-x-y)$

$$xy(24-x-y) = f(x,y)$$

$$\frac{\partial f}{\partial x} = y(24-x-y) + xy(0-1-0)$$

$$= y(24-x-y) - xy$$

$$= 24y - xy - y^2 - xy$$

$$= 24y - y^2 - 2xy$$

$$\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 24y - y^2 - 2xy = 0.$$

$$\Rightarrow y(24-y-2x) = 0 \quad \text{①}$$

$$\frac{\partial f}{\partial y} = x(24-x-y) + xy(0-0-1)$$

$$= 24x - x^2 - xy - xy$$

$$= 24x - x^2 - 2xy$$

$$\Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow 24x - x^2 - 2xy = 0$$

$$\Rightarrow x(24-x-2y) = 0. \quad \text{②}$$

$\cancel{① - ② \Rightarrow 24y - 24x - y^2 + x^2 = 0.}$

$\cancel{\Rightarrow 24(y-x) + (x+y)(x-y) = 0.}$

$\Rightarrow \cancel{(x-y)(x+y-24)=0}$.

From ① and ②, $x=0, y=0$ is not possible.

$$\left.\begin{aligned} 24-y-2x &= 0 \\ 24-x-2y &= 0. \end{aligned}\right\} \text{(i)} - 2\text{(ii)}$$

$\Rightarrow (24-y-2x)-2(24-x-2y)=0$

$\Rightarrow 24-y-\cancel{2x}-48+\cancel{2x}+4y=0$

$\Rightarrow -24+3y=0$

$\Rightarrow y=8.$

Putting $y=8$ in ① $\Rightarrow 24-8-2x=0$

$\Rightarrow 16=2x$

$\Rightarrow x=8.$

~~∴ The 3 numbers~~

$\dfrac{\partial^2 f}{\partial x^2} = -2y = r \quad \Big| \quad \dfrac{\partial^2 f}{\partial y^2} = -2x = t \quad \Big| \quad \dfrac{\partial^2 f}{\partial x \partial y} = 24-2x-2x$

$= 24-4x = s$

At $(8,8)$

$\Rightarrow rt-s^2 = [(-16)(-16)-(24-\cancel{24}32)]$

$= 256+8 = 264 > 0.$

$\Rightarrow rt-s^2 > 0$ and $r<0$.

∴ The function $f(x)$ is maximum at $(8,8)$.

∴ The required numbers are $x=8, y=8, z=8$.

3. Find local maxima and minima of the function $f(x, y) = x^3 - 12xy + 8y^3$.

Given, $f(x, y) = x^3 - 12xy + 8y^3$

$$\frac{\partial f}{\partial x} = 3x^2 - 12y \Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 12y = 0 \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = 24y^2 - 12x \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow 24y^2 - 12x = 0 \longrightarrow (2).$$

(1) = (2)

$\Rightarrow 3x^2 - 12y = 24y^2 - 12x$

$\Rightarrow x^2 - 4y = 8y^2 - 4x$

$\Rightarrow x^2 + 4x = 8y^2 + 4y$

$\Rightarrow x(x+4) = 4y(\cancel{y} - 2y + 1)$

From (1),

$\Rightarrow 3x^2 = 12y$

$\Rightarrow y = \frac{x^2}{4}$

Putting in (2),

$\Rightarrow 24$

From (2), $x = 2y^2$

Putting in (1), $3(2y^2)^2 - 12y = 0$

$\Rightarrow 12y^4 - 12y = 0$

$\Rightarrow 12y(y^3 - 1) = 0$

$\Rightarrow y = 0$ and $y = 1$.

Putting $y = 0$ in (1), $3x^2 - 0 = 0 \Rightarrow x = 0$

Putting $y = 1$ in (1), $3x^2 - 12 = 0 \Rightarrow x = \pm 2$.

The critical points are $(0,0)$, $(2,1)$, $(-2,1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x = r$$

$$\frac{\partial^2 f}{\partial x \partial y} = -12 = s$$

$$\frac{\partial^2 f}{\partial y^2} = 48y = t$$

(x, y)	r	s	t	$rt - s^2$	Conclusion.
$(0, 0)$	0	-12	0	-144.	saddle point.
$(2, 1)$	12	-12	48.	432.	minimum point
$(-2, 1)$	-12	-12	48	-720.	saddle point.

$\therefore$ The function $f(x, y)$ has local maximum at $(2, 1)$ and, $(0, 0)$ and $(-2, 1)$ are saddle points.

Assignment - 11

1. Let (x, y, z) be any point on the sphere.

Distance b/w (x, y, z) to $(1, 2, 2)$ is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

$$\Rightarrow D^2 = (x-1)^2 + (y-2)^2 + (z-2)^2 = f(x)$$

$$\phi(x) = x^2 + y^2 + z^2 - 36 \longrightarrow (1).$$

Auxillary eqn, $F = f + \lambda\phi$

$$\Rightarrow F = (x-1)^2 + (y-2)^2 + (z-2)^2 + \lambda(x^2 + y^2 + z^2 - 36)$$

Class - 11

Lagrange's Method of Undetermined Multipliers - Problems

1. Find the maximum and minimum distance from the point (1,2,2) to the sphere $x^2 + y^2 + z^2 = 36$. **Ans: Maximum distance =9 and minimum distance=3**

2. If x, y, z are the lengths of the perpendiculars dropped from any point P to the three sides of a triangle of constant area A, find the minimum value of $x^2 + y^2 + z^2$. **Ans:** $\frac{4A^2}{a^2+b^2+c^2}$

3. A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

   Ans: $\frac{b^2}{4(\pi+4)}$

4. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 3x + y$ subject to the condition $x^2 + y^2 = 10$. **Ans: Maximum at (3,1) and the minimum at (-3,-1).**

5. Find the maximum and minimum of $f(x, y, z) = xyz$ on the ellipsoid $x^2 + 2y^2 + 3z^2 = 36$. **Ans: Maximum and Minimum values are** $\pm\frac{2}{\sqrt{3}}$

$F_x = 0$

$\Rightarrow 2(x-1) + \lambda(2x) = 0 \Big\} \quad \lambda = \dfrac{1-x}{x} = \dfrac{1}{x} - 1 \longrightarrow ②$

$F_y = 0$

$\Rightarrow 2(y-2) + \lambda(2y) = 0 \Big\} \quad \lambda = \dfrac{2}{y} - 1 \longrightarrow ③$

$F_z = 0$

$\Rightarrow 2(z-2) + \lambda(2z) = 0 \Big\} \quad \lambda = \dfrac{2}{z} - 1 \longrightarrow ④$

② = ③ $\Rightarrow \quad \dfrac{1}{x} - 1 = \dfrac{2}{y} - 1$

$\Rightarrow \quad y = 2x$

② = ④ $\Rightarrow \quad \dfrac{1}{x} - 1 = \dfrac{2}{z} - 1$

$\Rightarrow z = 2x$

$\Rightarrow y = z = 2x$

Putting in ①, $\Rightarrow x^2 + y^2 + z^2 = 36$

$\Rightarrow x^2 + 4x^2 + 4x^2 = 36$

$\Rightarrow 9x^2 = 36$

$\Rightarrow x = \pm 2$

$\Rightarrow y = z = \pm 4$

$\therefore$ The stationary points are $(2, 4, 4)$, $(-2, -4, -4)$

To find distance,

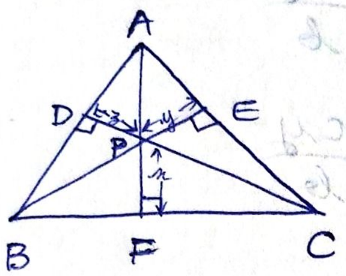
$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$

$= \sqrt{1^2 + 4 + 4} = 3.$

$$D = \sqrt{(-3)^2 + (-6)^2 + (-6)^2} = 9$$

$\therefore$ Maximum distance $= 9$

$\therefore$ Minimum distance $= 3$

2. 

Let $AB = c$, $BC = a$, $CA = b$

$$ar(APB) = \frac{1}{2} cz$$

$$ar(BPC) = \frac{1}{2} ax$$

$$ar(APC) = \frac{1}{2} by$$

$$ar(ABC) = A = \frac{1}{2}(cz + ax + by)$$

(here A = constant)

Let $f(x) = x^2 + y^2 + z^2$

Let $\phi(x) = \frac{1}{2}(cz + ax + by) - A \longrightarrow (1)$.

Auxillary eqn, $F = f + \lambda\phi$

$$\Rightarrow F = x^2 + y^2 + z^2 + \lambda\left[\frac{1}{2}(ax + by + cz) - A\right].$$

$F_x = 0$

$$\Rightarrow 2x + \frac{1}{2}\lambda[a] = 0 \longrightarrow (2)$$

$F_y = 0$

$$\Rightarrow 2y + \frac{1}{2}\lambda[b] = 0 \longrightarrow (3)$$

$F_z = 0$

$\Rightarrow 2z + \frac{1}{2}\lambda(c) = 0 \longrightarrow$ ④

② = ③ $\Rightarrow \frac{x}{a} = \frac{y}{b} \Rightarrow x = \frac{ay}{b}$

③ = ④ $\Rightarrow \frac{y}{b} = \frac{z}{c} \Rightarrow z = \frac{cy}{b}$

Putting in ①,

$A = \frac{1}{2}\left[c\left(\frac{cy}{b}\right) + by + a\left(\frac{ay}{b}\right)\right].$

$\Rightarrow A = \frac{1}{2}\left[\left(\frac{c^2}{b} + b + \frac{a^2}{b}\right)y\right].$

$\rightarrow A = \frac{y}{2b}(a^2 + b^2 + c^2)$

② x + ③ y + ④ z

$\Rightarrow 2(x^2 + y^2 + z^2) + \frac{1}{2}\lambda[xa + yb + zc] = 0.$

$\Rightarrow 2f + \lambda\left[\frac{1}{2}xa + \frac{1}{2}yb + \frac{1}{2}zc\right] = 0$

$\Rightarrow 2f + \lambda A = 0$

$\Rightarrow \lambda = -\frac{2f}{A}$

From ②, $2x - \left(\frac{2f}{A}\right)\left(\frac{1}{2}a\right) = 0 \Rightarrow x = \frac{af}{2A}$

From ③, $2y - \left(\frac{2f}{A}\right)\left(\frac{1}{2}b\right) = 0 \Rightarrow y = \frac{bf}{2A}$

From ④, $2z - \left(\frac{2f}{A}\right)\left(\frac{1}{2}c\right) = 0 \Rightarrow z = \frac{cf}{2A}$

Putting x, y, z in equation of A,

$$A = \frac{1}{2}a\left(\frac{af}{2A}\right) + \frac{1}{2}b\left(\frac{bf}{2A}\right) + \frac{1}{2}c\left(\frac{cf}{2A}\right).$$

$$\Rightarrow A = \frac{1f}{4A}(a^2+b^2+c^2)$$

$$\Rightarrow f = \frac{4}{a^2+b^2+c^2} \quad \Rightarrow x^2+y^2+z^2 = \frac{4A^2}{a^2+b^2+c^2}$$

3. Let x and y be the two parts into which the given wire is cut. $\Rightarrow b = x+y$

Wire of length x is bent into a square so that each side is $= \frac{x}{4}$ $\Rightarrow$ Area $= \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$.

Wire of length y is bent into circle having a perimeter $= y$ $\Rightarrow y = 2\pi r \Rightarrow r = \frac{y}{2\pi}$

$$\Rightarrow \text{Area} = \pi r^2 = \frac{y^2}{4\pi}$$

$$f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi} \quad \text{and} \quad \phi(x,y) = x+y-b. \longrightarrow (1)$$

Auxillary eqn is $\Rightarrow F = f + \lambda\phi$

$$\Rightarrow F = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x+y-b).$$

$$F_x = 0 \Rightarrow \frac{x}{8} + \lambda = 0. \qquad F_y = 0 \Rightarrow \frac{y}{2\pi} + \lambda = 0$$

$$\Rightarrow x = -8\lambda \qquad \Rightarrow y = -2\pi\lambda.$$

Putting x and y in (1),

$$\Rightarrow b = -8\lambda - 2\pi\lambda \Rightarrow \lambda = \frac{-b}{2\pi+8}.$$

$$\Rightarrow x = +8\left(\frac{b}{2\pi+8}\right) \quad \text{and} \quad y = +2\pi\left(\frac{b}{2\pi+8}\right).$$

$\therefore$ The least value of the sum of the areas of square and circle is :

$$f(x+y) = \frac{\overset{4}{\cancel{64}}\, b^2}{\cancel{16}(8+2\pi)^2} + \frac{\cancel{4\pi^2}\, b^2}{\cancel{4\pi}(8+2\pi)^2}$$

$$= \frac{b^2(4+\pi)}{4(4+\pi)^2} = \frac{b^2}{4(4+\pi)}.$$

4. $f(x,y) = 3x+y$

$\phi(x,y) = x^2+y^2-10. \longrightarrow ①$

Auxillary eqn is, $F = f+\lambda\phi$

$\Rightarrow F = (3x+y)+\lambda(x^2+y^2-10).$

$F_x = 0$

$\Rightarrow 3+\lambda(2x) = 0$ $\Big\}$ $\lambda = \frac{-3}{2x} \longrightarrow ②$

$F_y = 0$

$\Rightarrow 1+\lambda(2y) = 0$ $\Big\}$ $\lambda = \frac{-1}{2y} \longrightarrow ③.$

$② = ③ \Rightarrow \frac{-3}{\cancel{2}x} = \frac{-1}{\cancel{2}y} \Rightarrow x = 3y$

Putting in ①, $(3y)^2+y^2 = 10$

$\Rightarrow 10y^2 = 10 \Rightarrow y = \pm 1.$

For $y=+1$, $x=3$ and for $x=-1$, $x=-3$.

$\therefore$ The stationary points are $(3,1)$ and $(3,-1)$

At $(3,1)$, $f(x,y) = 3(3)+1 = 10. \Rightarrow$ Maximum

At $(-3,-1)$, $f(x,y) = (-3)3+(-1) = -10. \Rightarrow$ Minimum

5. $f(x,y,z) = xyz$

$\phi(x,y,z) = x^2+2y^2+3z^2-36 \longrightarrow (1)$

Auxillary eqn, $F = f + \lambda\phi$

$\Rightarrow F = xyz + \lambda(x^2+2y^2+3z^2-36)$

$\left.\begin{array}{l} F_x = 0 \\ \Rightarrow yz+\lambda(2x)=0 \end{array}\right\} \lambda = \dfrac{-yz}{2x} \longrightarrow (2)$

$\left.\begin{array}{l} F_y = 0 \\ \Rightarrow xz+\lambda(4y)=0 \end{array}\right\} \lambda = \dfrac{-xz}{4y} \longrightarrow (3)$

$\left.\begin{array}{l} F_z = 0 \\ \Rightarrow yx+\lambda(6z)=0 \end{array}\right\} \lambda = \dfrac{-yx}{6z} \longrightarrow (4)$

$(2)=(3) \Rightarrow \dfrac{-y\cancel{z}}{2x} = \dfrac{-x\cancel{z}}{4y} \Rightarrow 2x^2 = 4y^2 \Rightarrow x = \pm y\sqrt{2}$

$(3)=(4) \Rightarrow \dfrac{-\cancel{x}z}{{}_2\cancel{4}y} = -\dfrac{\cancel{x}y}{{}_3\cancel{6}z} \Rightarrow 3z^2 = 2y^2 \Rightarrow z = \pm\sqrt{\dfrac{2}{3}}\,y$

Putting in (1), $2y^2+2y^2+3\left(\frac{2}{3}y^2\right) = 36$

$\Rightarrow 6y^2 = 36 \Rightarrow y = \pm\sqrt{6} \Rightarrow x = \pm\sqrt{6}\sqrt{2}$
$= \pm\sqrt{12}$

$\Rightarrow 3z^2 = (36-12-12) = 12 \Rightarrow z = \dfrac{\sqrt{12}}{\sqrt{3}} = 2$

The two stationary points are $(\sqrt{12}, \sqrt{6}, 2)$ and $(-\sqrt{12}, -\sqrt{6}, -2)$

For $(\sqrt{12}, \sqrt{6}, 2)$, $f(x,y,z) = (\sqrt{12})(\sqrt{6})2$

$= \sqrt{3}(2)(\sqrt{3})(\sqrt{2})(2) = \underline{\underline{12\sqrt{2}}}$

For $(-\sqrt{12}, -\sqrt{6}, -2)$, $f(x,y,z) = (-\sqrt{12})(-\sqrt{6})(-2)$

$= \underline{\underline{-12\sqrt{2}}}$

Find the maximum and minimum distances from the origin to curve $5x^2+6xy+5y^2-8=0$.

Let $P(x, y)$ be any point on the curve.

The distance b/w the point (x, y) and $(0, 0)$,

$D = \sqrt{(x-0)^2+(y-0)^2}$

$\Rightarrow D = \sqrt{x^2+y^2}$

$\Rightarrow D^2 = x^2+y^2 = f(x)$.

$\therefore f(x) = x^2+y^2$

$\phi(x) = 5x^2+6xy+5y^2-8 \longrightarrow ①$

Auxillary eqn, $F = f + \lambda\phi$

$\Rightarrow F = x^2+y^2 + \lambda(5x^2+6xy+5y^2-8)$

$F_x = 0$

$\Rightarrow 2x + \lambda(10x+6y) = 0$ $\Big\}$ $\lambda = \dfrac{-2x}{10x+6y} = \dfrac{-x}{5x+3y} \longrightarrow ②$

$F_y = 0$

$\Rightarrow 2y + \lambda(6x+10y) = 0$ $\Big\}$ $\lambda = \dfrac{-2y}{6x+10y} = \dfrac{-y}{3x+5y} \longrightarrow ③$.

$② = ③ \Rightarrow \dfrac{-x}{5x+3y} = \dfrac{-y}{3x+5y}$

$\Rightarrow 3x^2 + \cancel{5xy} = \cancel{5xy} + 3y^2$

$\Rightarrow x = \pm y$

Putting in ①, $\Rightarrow 5x^2+6xy+5y^2-8=0$.

$(x=+y)$ $\Rightarrow 5y^2+6y^2+5y^2-8=0$

$\Rightarrow 16y^2 = 8 \Rightarrow y = \pm\frac{1}{\sqrt{2}} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Putting in ①, $\Rightarrow 5x^2+6xy+5y^2-8=0$
$(x=-y)$

$$\Rightarrow 5y^2-6y^2+5y^2-8=0$$
$$\Rightarrow 4y^2-8=0$$
$$\Rightarrow y=\pm\sqrt{2} \Rightarrow x=\mp\sqrt{2}$$

The stationary points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

(x, y)	Distance
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$D=\sqrt{x^2+y^2}=1$
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$D=\sqrt{x^2+y^2}=1$.
$(\sqrt{2}, -\sqrt{2})$	$D=\sqrt{x^2+y^2}=2$
$(-\sqrt{2}, \sqrt{2})$	$D=\sqrt{x^2+y^2}=2$

$\therefore$ The points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ are at a minimum distance from the origin and the min. distance $=1$.

$\therefore$ The points $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ are at a maximum distance from the origin and the max. distance $=2$.

Find the maximum and minimum distance of the point (3, 4, 12) from the unit sphere with centre at the origin.

Let (x, y, z) be any point on the unit sphere.

Distance b/w (x, y, z) and $(3, 4, 12)$.

$\Rightarrow D = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$

$\Rightarrow D^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x)$

$\phi(x) = x^2 + y^2 + z^2 - 1$. $\longrightarrow$ ①

Auxillary eqn, $F = f + \lambda\phi$

$\Rightarrow F = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$

$F_x = 0$

$\Rightarrow 2(x-3) + \lambda(2x) = 0$ $\Big\}$ $\lambda = -\left(\frac{x-3}{2}\right)2 = \frac{3-x}{x}$ $\longrightarrow$ ②

$F_y = 0$

$\Rightarrow 2(y-4) + \lambda(2y) = 0$ $\Big\}$ $\lambda = \left(\frac{4-y}{2y}\right)2 = \frac{4-y}{y}$ $\longrightarrow$ ③

$F_z = 0$

$\Rightarrow 2(z-12) + \lambda(2z) = 0$ $\Big\}$ $\lambda = \frac{12-z}{z}$ $\longrightarrow$ ④.

② = ③ $\Rightarrow 3-x = 4-y \Rightarrow x = 3-4+y$

$\Rightarrow x = y-1$ $\longrightarrow$ ⑤

② = ④ $\Rightarrow 3-x = 12-z \Rightarrow x = 3-12+z$

$\Rightarrow x = z-9$ $\longrightarrow$ ⑥.

Putting in (1)

$\Rightarrow \cancel{x^2 + y^2 + z^2 = 1}$

$\Rightarrow \cancel{x^2 + (x+1)^2 + (x+9)^2 = 1}$

$\Rightarrow \cancel{x^2 + x^2 + 2x + 1 + x^2 + 81 + 18x = 1}$

$(2) = (3) \Rightarrow \dfrac{3-x}{x} = \dfrac{4-y}{y}$

$\Rightarrow 3y - xy = 4x - xy$

$\Rightarrow x = \dfrac{3y}{4}$

$\bigcirc = (4) \Rightarrow \dfrac{4-y}{y} = \dfrac{12-z}{z}$

$\Rightarrow 4z - yz = 12y - yz$

$\Rightarrow z = 3y.$

Putting in (1) $\Rightarrow x^2 + y^2 + z^2 = 1$

$\Rightarrow \left(\dfrac{3y}{4}\right)^2 + y^2 + (3y)^2 = 1$

$\Rightarrow \dfrac{9y^2}{16} + y^2 + 9y^2 = 1$

$\Rightarrow 9y^2 + 16y^2 + 144y^2 = 16$

$\Rightarrow 169y^2 = 16 \Rightarrow y = \pm\dfrac{4}{13}$

$\therefore x = \dfrac{3}{4}\left(\pm\dfrac{4}{13}\right) = \pm\dfrac{3}{13}$

$\therefore z = 3\left(\pm\dfrac{4}{13}\right) = \pm\dfrac{12}{13}.$

The stationary points are $\left(\dfrac{3}{13}, \dfrac{4}{13}, \dfrac{12}{13}\right)$ & $\left(\dfrac{-3}{13}, \dfrac{-4}{13}, \dfrac{-12}{13}\right)$

For $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$, putting in ①,

$$\Rightarrow D = \sqrt{\left(\frac{3}{13}-3\right)^2+\left(\frac{4}{13}-4\right)^2+\left(\frac{12}{13}-12\right)^2}$$

$$\Rightarrow D = \sqrt{9\left(\frac{1}{13}-1\right)^2+16\left(\frac{1}{13}-1\right)^2+144\left(\frac{1}{13}-1\right)^2}$$

$$\Rightarrow D = \sqrt{169\left(\frac{1}{13}-1\right)^2}$$

$$\Rightarrow D = 13\left(\frac{1}{13}-1\right) = 1-13 = -12.$$

For $\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$ putting in ①,

$$\Rightarrow D = \sqrt{\left(\frac{-3}{13}-3\right)^2+\left(\frac{-4}{13}-4\right)^2+\left(-\frac{12}{13}-12\right)^2}$$

$$\Rightarrow D = \sqrt{169\left(-\frac{1}{13}-1\right)^2}$$

$$\Rightarrow D = 13\left(-\frac{1}{13}-1\right) = -1-13 = -14$$

$\therefore$ The sphere is at maximum distance of 14 at $\left(\frac{-3}{13}, -\frac{4}{13}, \frac{-12}{13}\right)$ and is at minimum distance of 12 at $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$.

Show that the rectangular solid of max. volume that can be inscribed in a given sphere is a cube.

Let $2x, 2y, 2z$ be the sides of the rectangular solid.

$\Rightarrow$ Volume $= (2x)(2y)(2z) = 8xyz. = f(x, y, z)$

$x^2+y^2+z^2 = r^2$ is the sphere.

$\Rightarrow \phi(x, y, z) = x^2+y^2+z^2-r^2 \longrightarrow (1)$

Auxillary eqn is $F = f + \lambda\phi$

$\Rightarrow F = 8xyz + \lambda(x^2+y^2+z^2-r^2)$

$$\left.\begin{aligned} &F_x = 0 \\ &\Rightarrow 8yz + \lambda(2x) = 0 \end{aligned}\right\} \lambda = -\frac{8yz}{2x} = -\frac{4yz}{x} \rightarrow (2)$$

$$\left.\begin{aligned} &F_y = 0 \\ &\Rightarrow 8xz + \lambda(2y) = 0 \end{aligned}\right\} \lambda = -\frac{8xz}{2y} = -\frac{4xz}{y} \rightarrow (3)$$

$$\left.\begin{aligned} &F_z = 0 \\ &\Rightarrow 8xy + \lambda(2z) = 0 \end{aligned}\right\} \lambda = -\frac{8xy}{2z} = \cancel{-8x} -\frac{4xy}{z} \rightarrow (4)$$

$(2) = (3) \Rightarrow -\frac{4yz}{x} = -\frac{4xz}{y}$

$\Rightarrow y^2 z = x^2 z$

$\Rightarrow y = \pm x \quad (\because z \neq 0)$

$(3) = (4) \Rightarrow -\frac{4xz}{y} = -\frac{4xy}{z}$

$\Rightarrow xz^2 = y^2 x$

$\Rightarrow y = \pm z \quad (\because x \neq 0)$

(Dimensions cannot be negative)

Putting in ①, $x^2 + y^2 + z^2 = r^2$

$$\Rightarrow y^2 + y^2 + y^2 = r^2 \Rightarrow y = \frac{r}{\sqrt{3}}.$$

$$x = z = + \frac{r}{\sqrt{3}}.$$

The stationary points are $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$

$\therefore$ All dimensions are equal.

Volume $= 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} r^3$

$\therefore$ The rectangular solid of max. volume that can be inscribed in a sphere is a cube.

Hence, proved.

<u>Question Bank</u> (4 MARK QUESTIONS)

11. If $u = x^3 \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{y^3} \sin^{-1}\left(\frac{x}{y}\right)$, find the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + x u_x + y u_y$.

$$u = x^3 \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{y^3} \sin^{-1}\left(\frac{x}{y}\right).$$

$$u = x^3\left[\tan^{-1}\left(\frac{y}{x}\right) + \left(\frac{x}{y}\right)^3 \operatorname{cosec}^{-1}\left(\frac{y}{x}\right)\right]. \Rightarrow u = x^3 f\left(\frac{y}{x}\right)$$

Degree $= 3$.

Using Euler's theorem,

$$x u_x + y u_y = 3u$$

$$x^2 u_{xx} + 2xy\, u_{yx} + y^2 u_{yy} = 3(3-1)u = 6u$$

$\therefore$ Answer $= 9u$

If $u = x^3 \tan^{-1}\left(\frac{y}{x}\right) + y^3 \cos^{-1}\left(\frac{x}{y}\right)$, then find

$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

$$u = x^3 \tan^{-1}\left(\frac{y}{x}\right) + y^3 \cos^{-1}\left(\frac{x}{y}\right).$$

$$u = x^3 \tan^{-1}\left(\frac{y}{x}\right) + y^3 \sec^{-1}\left(\frac{y}{x}\right).$$

$$u = x^3\left[\tan^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^3 \sec^{-1}\left(\frac{y}{x}\right)\right].$$

$$\Rightarrow u = x^3 f\left(\frac{y}{x}\right). \longrightarrow \text{Degree} = 3.$$

$$\Rightarrow x u_x + y u_y = 3u$$

$$\Rightarrow x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 3(3-1)u = (2u)3 = \underline{6u}$$

Find the exact differential of:

i) $u = \log_e (x^2 + y^2)$

$$du = \frac{\partial u}{\partial x}(dx) + \frac{\partial u}{\partial y}(dy).$$

$$= \frac{1(2x)}{x^2+y^2}\, dx + \frac{(2y)\, dy}{x^2+y^2}$$

$$= \underline{\frac{2x\, dx + 2y\, dy}{x^2+y^2}}$$

ii) $u = \pi x^2 y$

$$du = \frac{\partial u}{\partial x}(dx) + \frac{\partial u}{\partial y}(dy)$$

$$= 2\pi xy\, dx + \pi x^2\, dy \Rightarrow \underline{\pi x\left[2y\, dx + x\, dy\right]}$$

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4. If $u = xy(x+y)$ where $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right).$$

$$= (2xy + y^2)(2at) + (x^2 + 2xy)(2a).$$

$$= [2(at^2)(2at) + 4a^2t^2](2at) + [(at^2)^2 + 2(at^2)(2at)](2a)$$

$$= [4a^2t^3 + 4a^2t^2](2at) + [a^2t^4 + 4a^2t^3](2a).$$

$$= 8a^3t^4 + 8a^3t^3 + 2a^3t^4 + 8a^3t^3$$

$$= \underline{\underline{10a^3t^4 + 16a^3t^3}}$$

5. If $u = f(y-z, z-x, x-y)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $p = y-z$, $q = z-x$ and $r = x-y$. $\to u = f(p,q,r)$.

$\frac{\partial p}{\partial y} = 1$ and $\frac{\partial p}{\partial z} = -1$

$|||^{ly}$, $\frac{\partial q}{\partial z} = 1$ and $\frac{\partial q}{\partial x} = -1$

$|||^{ly}$, $\frac{\partial r}{\partial x} = 1$ and $\frac{\partial r}{\partial y} = -1$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial x}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \to (1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \to (2)$$

$$\to \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial z}\right) = -\frac{\partial u}{\partial p} + \frac{\partial q}{\partial z} \to (3)$$

Adding ① + ② + ③,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

$\Rightarrow$ LHS = RHS   Hence, proved.

16. If $u = f(x, y)$, $x = s + t$, $y = s - t$, find the value of $\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$. 8

$$x = s + t \Rightarrow \frac{\partial x}{\partial s} = 1 \text{ and } \frac{\partial x}{\partial t} = 1.$$

$$y = s - t \Rightarrow \frac{\partial y}{\partial s} = 1 \text{ and } \frac{\partial y}{\partial t} = -1.$$

$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\right) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \rightarrow ①$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \rightarrow ②$$

Adding ① + ② $\Rightarrow$ $\underline{\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 2\frac{\partial u}{\partial x}}$

17. $w = e^{x+y}\cos 2z$, $x = \log t$, $y = \log(t^2+1)$, $z = t$, find $\frac{dw}{dt}$.

$$w = e^{x+y}\cos 2z \cancel{\Rightarrow w = e^{x+y}\cos 2t}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial w}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial w}{\partial z}\left(\frac{dz}{dt}\right)$$

$$= \left(e^{x+y}\cos 2z\right)\left(\frac{1}{t}\right) + \left(e^{x+y}\cos 2z\right)\left(\frac{2t}{t^2+1}\right) - 2\sin 2t\, e^{x}$$ 19.

$$= e^{x+y}\left[\frac{\cos 2z}{t} + \left(\frac{2t}{t^2+1}\right)\cos 2z - 2\sin 2t\right] \rightarrow$$

$$e^{x+y} = e^{\log_e t + \log_e(t^2+1)} = e^{\log_e[t(t^2+1)]} = t(t^2+1) \rightarrow$$

Substituting ② and $z = t$ in ①,

$$\frac{dw}{dt} = t(t^2+1)\left[\frac{\cos 2t}{t} + \left(\frac{2t}{t^2+1}\right)\cos 2t - 2\sin 2t\right]$$

Obtain Taylor's series of expansion $f(x,y) = xy^2 + y\cos(x-y)$ about the point $(1,1)$.

$$f(x,y) = f(1,1) + \frac{1}{1!}\left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] \longrightarrow ①$$

Given, $f(x,y) = xy^2 + y\cos(x-y)$

$f(1,1) = 1 + 1\cos 0 = 2$

$f_x = y^2 - y\sin(x-y) \Rightarrow f_x(1,1) = 1 - 1(0) = 1$

$f_y = 2xy + \cos(x-y) + y\sin(x-y) \Rightarrow f_y(1,1) = 3$.

$f_{xx} = -y\cos(x-y) \Rightarrow f_{xx}(1,1) = -1(1) = -1$

$f_{xy} = 2y - \sin(x-y) + y\cos(x-y) \Rightarrow f_{xy}(1,1) = 3$.

$f_{yy} = 2x + \sin(x-y) + \sin(x-y) - y\cos(x-y) \Rightarrow f_{yy} = 1$

Putting in ①,

$$f(x,y) = 2 + [(x-1) + 3(y-1)] + \frac{1}{2}\left[-(x-1)^2 + 6(x-1)(y-1) + (y-1)^2\right] + \ldots$$

Expand $f(x,y) = \sin(xy)$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ upto second degree terms.

$$f(x,y) = f\left(1,\frac{\pi}{2}\right) + \frac{1}{1!}\left[(x-1)f_x\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right) + f_{yy}\left(1,\frac{\pi}{2}\right)\left(y-\frac{\pi}{2}\right)^2\right] \rightarrow ①$$

Given, $f(x,y) = \sin(xy)$.

$f(1, \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$.

$f_x = [\cos(xy)]y \quad \rightarrow f_x(1, \frac{\pi}{2}) = 0$.

$f_y = [\cos(xy)]x \quad \rightarrow f_y(1, \frac{\pi}{2}) = 0$

$f_{xx} = -y^2 \sin(xy) \quad \rightarrow f_{xx}(1, \frac{\pi}{2}) = -\frac{\pi^2}{4}$.

$f_{xy} = \cos(xy) - [x \sin(xy)]y \rightarrow f_{xy}(1, \frac{\pi}{2}) = -\frac{\pi}{2}$

$f_{yy} = -x^2 \sin(xy) \quad \Rightarrow f_{yy}(1, \frac{\pi}{2}) = -1$

Putting in ①,

$f(x,y) = \sin(xy)$

$\Rightarrow \sin(xy) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right.$

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20. Find minimum and maximum values of :

i) $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

Given, $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

$\frac{\partial f}{\partial x} = 4x - 4x^3 \quad \Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 4x(1 - x^2) = 0$

$\Rightarrow x = \pm 1, 0$.

$\frac{\partial f}{\partial y} = -4y + 4y^3 \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow 4y(-1 + y^2) = 0$.

$\Rightarrow y = \pm 1, 0$.

The stationary points are $(\pm 1, \pm 1)$, $(\pm 1, 0)$, $(0, \pm 1)$, $(0,$

$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

$\Rightarrow r = 4 - 12x^2$

$\Rightarrow s = 0$ and $\Rightarrow t = -4 + 12y^2$

(x, y)	$r = 4 - 12x^2$	$s = 0$	$t = 12y^2 - 4$	$rt - s^2$	Conclusion.
$(\pm 1, \pm 1)$	-8	0	8	-64	saddle point.
$(\pm 1, 0)$	-8	0	-4	32	maximum
$(0, \pm 1)$	4	0	8	32	minimum.
$(0, 0)$.	4	0	-4	-16	saddle point.

$\therefore$ The maximum value of $f(x,y)$ is at the point $(\pm 1, 0)$ and its max. value is 1.

The minimum value of $f(x, y)$ is at the point $(0, \pm 1)$ and its min. value is -1.

4. Find the extreme value of xyz, when $x + y + z = a$ where $a > 0$.

$f(x, y, z) = xyz$.

$\phi(x, y, z) = x + y + z - a \longrightarrow (1)$

Auxillary eqn is, $F = f + \lambda \phi$

$\Rightarrow F = xyz + \lambda(x + y + z - a)$.

$\left.\begin{array}{l} F_x = 0 \\ \Rightarrow yz + \lambda = 0 \end{array}\right\} \lambda = -yz \longrightarrow (2)$

$\left.\begin{array}{l} F_y = 0 \\ \Rightarrow xz + \lambda = 0 \end{array}\right\} \lambda = -xz \longrightarrow (3)$

$\left.\begin{array}{l} F_z = 0 \\ \Rightarrow xy + \lambda = 0 \end{array}\right\} \lambda = -xy. \longrightarrow (4)$

$(2) = (3) = (4) \Rightarrow yz = xz = xy \Rightarrow x = y = z$

Putting in (1), $x + y + z = a$

$\Rightarrow x + x + x = a \Rightarrow x = \frac{a}{3}$.

$\Rightarrow x = y = z = \frac{a}{3}$.

The extreme value of function $f(x,y,z) = xyz$ at point $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ is $\frac{a^3}{27}$.

25. Find the extreme value of $x^p + y^p + z^p$ on the surface $x^q + y^q + z^q = 1$, where $0 < p < q$, $x > 0$, $y > 0$, $z > 0$.

$f(x,y,z) = x^p + y^p + z^p$

$\phi(x,y,z) = x^q + y^q + z^q - 1. \rightarrow (1)$

Auxillary eqn is, $F = f + \lambda\phi$.

$\Rightarrow F = x^p + y^p + z^p + \lambda(x^q + y^q + z^q - 1)$.

$F_x = 0$
$\Rightarrow p x^{p-1} + \lambda(q x^{q-1}) = 0$ $\Big\}$ $\lambda = \frac{-p x^{p-1}}{q x^{q-1}} = -\frac{p}{q} x^{p-q} \rightarrow (2)$

$F_y = 0$
$\Rightarrow p y^{p-1} + \lambda(q y^{q-1}) = 0$ $\Big\}$ $\lambda = -\frac{p}{q} y^{p-q} \longrightarrow (3)$

$F_z = 0$
$\Rightarrow p z^{p-1} + \lambda(q z^{q-1}) = 0$ $\Big\}$ $\lambda = -\frac{p}{q} z^{p-q} \longrightarrow (4)$

$(2) = (3) = (4) \Rightarrow -\frac{p}{q} x^{p-q} = \frac{p}{q} y^{p-q} = \frac{p}{q} z^{p-q}$

$\Rightarrow x^{p-q} = y^{p-q} = z^{p-q}$

$\Rightarrow x = y = z$

Putting in (1), $x^q + y^q + z^q = 1$

$\Rightarrow 3x^q = 1 \Rightarrow x^q = \frac{1}{3}$

$\Rightarrow x = 3^{-\frac{1}{q}}$

$\Rightarrow x = y = z = 3^{-\frac{1}{q}}$

The extreme value of given function $f(x, y, z)$ is $\left(3^{\left(-\frac{1}{q}\right)p} + 3^{p\left(\frac{1}{q}\right)} + 3^{p\left(-\frac{1}{q}\right)}\right)$.

$$= 3^{1}\, 3^{-\frac{p}{q}}$$

$$= 3^{1-\frac{p}{q}} = \underline{\underline{3^{\left(\frac{q-p}{q}\right)}}}$$

Question – Answers.

1. Find central points or stationary points of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Given, $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$.

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$\Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0 \rightarrow x(x^2-1) + y = 0. \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x$$

$$\Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow y(y^2-1) + x = 0. \longrightarrow (2)$$

$$(1) + (2) \Rightarrow x(x^2-1) + y(y^2-1) + (x+y) = 0.$$

$$\Rightarrow x^3 - \cancel{x} + y^3 - \cancel{y} + \cancel{x} + \cancel{y} = 0$$

$$\Rightarrow x^3 + y^3 = 0.$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0.$$

$\Rightarrow \cancel{x=}\ y = -x$ (the other eqn. cannot be factorised)

Putting in (1),

$$x^3 - x - x = 0$$

$$x^3 = 2x \Rightarrow x = 0, \pm\sqrt{2}$$

The stationary points are $\underline{(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})}$

2. Find the extreme value of the function $f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$.

Given, $f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

$$\frac{\partial f}{\partial x} = 2xy^2 - 10x - 8y$$

$$\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow xy^2 - 5x - 4y = 0 \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = 2yx^2 - 8x - 10y$$

$$\Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow x^2y - 5y - 4x = 0. \longrightarrow (2)$$

$$x(1) - (2)y \Rightarrow \cancel{-x^2y^2} + \cancel{x^2y^2} - 5x^2 + 5y^2 \cancel{-4yx + 4xy} = 0$$

$$\Rightarrow 5x^2 = 5y^2 \longrightarrow x = \pm y$$

When $x = +y$ in (1),

$$y^3 - 5y - 4y = 0 \Rightarrow y(y^2 - 9) = 0$$

$$\Rightarrow y = 0, \pm 3. \Rightarrow x = 0, \pm 3$$

When $x = -y$ in (1),

$$-y^3 + 5y - 4y = 0 \Rightarrow y(-y^2 + 1) = 0$$

$$\Rightarrow y = 0, \pm 1 \Rightarrow x = 0, \mp 1$$

The stationary points are $(0,0), (3,3), (-3,3), (1,-1)$ and $(-1,1)$.

$$r = \frac{\partial^2 f}{\partial x^2} = 2y^2 - 10.$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4xy - 8.$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^2 - 10$$

(x, y)	$r = 2y^2 - 10$	$s = 4xy - 8$	$t = 2x^2 - 10$	$rt - s^2$	Conclusion
$(0,0)$	-10	-8	-10	36	maximum.
$(3,3)$	8	28	8	-720	saddle point
$(-3,-3)$	8	28	8	-720	saddle point.
$(1,-1)$	-8	-12	-8	-80	saddle point.
$(-1,1)$	-8	-12	-8	-80	saddle point.

The function $f(x, y)$ is maximum at $(0,0)$ and its max. value is 0.

The function has no extreme values.

3. Find the dimensions of a rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm.

Let x, y, z be the dimensions of a rectangular box, thus, volume $= xyz$.

$\Rightarrow f(x, y, z) = xyz$

$\Rightarrow \phi(x, y, z) = xy + 2yz + 2xz - 432. \longrightarrow (1)$

Auxillary eqn is, $F = f + \lambda\phi$

$\Rightarrow F = xyz + \lambda(xy + 2yz + 2xz - 432)$

$$\left.\begin{aligned} &F_x = 0 \\ &\Rightarrow yz + \lambda(y + 2z) = 0 \end{aligned}\right\} \Rightarrow \lambda = \frac{-yz}{y + 2z} \longrightarrow (2)$$

$$\left.\begin{aligned} &F_y = 0 \\ &\Rightarrow xz + \lambda(x + 2z) = 0 \end{aligned}\right\} \Rightarrow \lambda = \frac{-xz}{x + 2z} \longrightarrow (3)$$

$$\left.\begin{aligned} &F_z = 0 \\ &\Rightarrow xy + \lambda(2y + 2x) = 0 \end{aligned}\right\} \Rightarrow \lambda = \frac{-xy}{2y + 2x} \longrightarrow (4)$$

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$$\Rightarrow \frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-xy}{2x+2y}.$$

$$\Rightarrow \frac{y\cancel{z}}{y+2z} = \frac{x\cancel{z}}{x+2z} \Rightarrow xy + 2yz = xy + 2xz$$

$$\Rightarrow x = y$$

$$\Rightarrow \frac{xz}{x+2z} = \frac{xy}{2x+2y} \Rightarrow 2x^2z + 2xyz = x^2y + 2xyz$$

$$\Rightarrow y = 2z$$

$$\Rightarrow x = y = 2z$$

Putting in ①,

$$xy + 2yz + 2xz = 432$$

$$\Rightarrow y^2 + 2y\left(\frac{y}{2}\right) + 2y\left(\frac{y}{2}\right) = 432$$

$$\Rightarrow y^2 = \frac{432}{3} = 144 \Rightarrow y = \pm 12.$$

$$\Rightarrow x = y = \pm 12 \quad \Rightarrow z = \pm 6.$$

Since, dimensions cannot be -ve,

Hence, $x = 12\,cm$, $y = 12\,cm$, $z = 6\,cm$ gives the maximum volume.

Assignment - 5.

1. State and prove Euler's theorem for a function of two variables.

Any function $f(x)$ which can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ is called a homogeneous function of degree n in terms of x and y. Euler's theorem for the above is given by: $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = nu.$

Let u be a homogeneous function of degree n in x and y.

$\therefore \quad u = x^n f\left(\frac{y}{x}\right).$

$$\Rightarrow \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + \left[x^n f'\left(\frac{y}{x}\right)\right]\left(-\frac{y}{x^2}\right).$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} y f'\left(\frac{y}{x}\right).$$

$$\Rightarrow \frac{\partial u}{\partial y} = \left[x^n f'\left(\frac{y}{x}\right)\right]\left(\frac{1}{x}\right) = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$LHS = x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$$= \left[nx^n f\left(\frac{y}{x}\right) - \cancel{x^{n-1} y f'\left(\frac{y}{x}\right)}\right] + \cancel{x^{n-1} y f'\left(\frac{y}{x}\right)}$$

$$= n\left[x^n f\left(\frac{y}{x}\right)\right] = nu = RHS.$$

Hence, $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = nu$

$LHS = RHS \Rightarrow$ Hence, proved.

$$u = \frac{1}{\sqrt{x^2+y^2}} \Rightarrow u = \frac{1}{x\sqrt{1+\left(\frac{y}{x}\right)^2}} \Rightarrow u = x^{-1} f\left(\frac{y}{x}\right).$$

$\therefore$ u is a homogeneous function of degree $= -1$.

Using Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -1u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{x^2+y^2}}$$

Hence, proved.