Integrals of Bessel's Function

Result 1: $\dfrac{d}{dx}[x^{\nu} J_{\nu}(x)] = +x^{\nu} J_{\nu-1}(x)$

On integrating,

$$\therefore \int x^{\nu} J_{\nu-1}(x)\,dx = x^{+\nu} J_{\nu}(x) + C$$

Put $\nu = 1, 2, 3 \ldots$

$\nu = 1$: $\int x J_0(x)\,dx = x J_1(x) + C$

$\nu = 2$: $\int x^2 J_1(x)\,dx = x^2 J_2(x) + C$

$\nu = 3$: $\int x^3 J_2(x)\,dx = x^3 J_3(x) + C$ and so on.

Result 2: $\dfrac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$

On integrating,

$$\therefore \int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_{\nu}(x) + C$$

Put $\nu = 0, 1, 2, 3 \ldots$

$\nu = 0$: $\int J_1(x) = -J_0(x) + C$

$\nu = 1$: $\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) + C$

$\nu = 2$: $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + C$

$\nu = 3$: $\int x^{-3} J_4(x)\,dx = -x^{-3} J_3(x) + C$ and so on.

Prove that : $\int x J_0(x)\,dx = x J_1(x) + C$

$\int J_1(x)\,dx = -J_0(x) + C$

Note 1:

$\int x^m J_n(x)$ (m and n are integers and $m+n \geq 0$) can be integrated by parts completely when $(m+n)$ is odd and write the answers in terms of J_0 and J_1.

Note 2: When m and n are given and $(m+n)$ is even, the integral depends on the residual integral $\int J_0(x)dx$.

Result 3: $J_\nu'(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)]$

On integrating,

$$2J_\nu(x) = \int J_{\nu-1}(x)\,dx - \int J_{\nu+1}(x)\,dx$$

$$\Rightarrow \int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x)$$

Type 1: Evaluate

1. $\int x^2 J_1(x)\,dx \quad = x^2 J_2(x) = x^2\left[\frac{2}{x}J_1(x) - J_0(x)\right] = \underline{\underline{2xJ_1(x) - x^2J_0(x)}}$
2. $\int x^4 J_1(x)\,dx$ in terms of J_0 and J_1.
3. $\int x^3 J_0(x)\,dx$

Type 2: Evaluate

1. $\int J_3(x)\,dx$
2. $\int J_5(x)\,dx$

Type 3: Evaluate

1. $\int x^3 J_3(x)\,dx$

2. $I = \int x^4 J_1(x)\,dx$

$$= \int x^2\left[x^2 J_1(x)\,dx\right] \quad \left\{ \begin{array}{l} u = x^2 \Rightarrow du = 2x\,dx \\ dv = x^2 J_1(x)\,dx \\ \Rightarrow v = \int x^2 J_1(x) = x^2 J_2(x) + C \end{array} \right.$$

$$= x^2(x^2 J_2) - \int x^2 J_2 (2x\,dx) + C$$

$\int u\,dv = uv - \int v\,du$

$$= x^4 J_2 - 2\int x^3 J_2\,dx + C$$

$$= x^4 J_2 - 2x^3 J_3 + C$$

$J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x) - J_{\nu-1}(x)$ (refer Q28).

$$= x^4\left[\frac{2}{x}J_1 - J_0\right] - 2x^3\left[\left(\frac{8-x^2}{x^2}\right)J_1(x) - \left(\frac{4}{x}\right)J_0(x)\right]$$

$$= 2x^3 J_1 - x^4 J_0 - 16x J_1 + 2x^3 J_1 + 8x^2 J_0$$

$$= \underline{\underline{(4x^3 - 16x)J_1 + (8x^2 - x^4)J_0}}$$

3. $I = \int x^3 J_0(x)\,dx$

$$= \int x^2\left[x J_0(x)\,dx\right] \quad \left\{ \begin{array}{l} u = x^2 \Rightarrow du = 2x\,dx \\ dv = x J_0(x)\,dx \\ \Rightarrow v = \int x J_0(x)\,dx = x J_1(x) \end{array} \right.$$

$$= x^2(x J_1(x)) - \int x J_1 (2x\,dx) + C$$

$$= x^3 J_1 - 2\int x^2 J_1\,dx + C$$

$$= x^3 J_1 - 2x^2 J_2 + C$$

$$= x^3 J_1 - 2x^2\left[\frac{2}{x}J_1 - J_0\right] + C$$

$$= x^3 J_1 - 4x J_1 + 2x^2 J_0 + C$$

$$= \underline{\underline{(x^3 - 4x)J_1 + 2x^2 J_0 + C}}$$

1. $I = \int J_3(x)\,dx$

$= \int x^2 x^{-2} J_3(x)\,dx$ $\Big\}$ $u = x^2 \Rightarrow du = 2x\,dx$, $dv = x^{-2} J_3(x)\,dx$

(Using Result - 2) $\Rightarrow v = \int x^{-2} J_3(x)\,dx$

$= -x^{-2} J_2(x)$

$= x^2(-x^{-2}J_2) - \int(-x^{-2} J_2)\, 2x\,dx + C$

$= -J_2 + 2\int x^{-1} J_2\,dx + C$

$= -J_2 + 2\left[-x^{-1} J_1\right] + C$

$\Rightarrow I = C - J_2 - \frac{2}{x} J_1$

2. $I = \int J_5(x)\,dx$

$= \int x^4 x^{-4} J_5(x)\,dx$ $\Big\}$ $u = x^4 \Rightarrow du = 4x^3\,dx$, $dv = x^{-4} J_5(x)\,dx$

$\Rightarrow v = -x^{-4} J_4(x)$

$= x^4(-x^{-4}J_4) - \int(-x^{-4}J_4)(4x^3\,dx)$

$= -J_4 + 4\int x^2 x^{-3} J_4\,dx$ $\Big\}$ $u = x^2 \Rightarrow du = 2x\,dx$, $dv = x^{-3} J_4\,dx$

$\Rightarrow v = \int x^{-3} J_4\,dx = x^{-3} J_3$

$= -J_4 + 4\left[x^2 x^{-3} J_3 - \int x^{-3} J_3 (2x\,dx)\right]$

$= -J_4 + 4\left[x^{-1} J_3 - 2x^{-2} J_2\right]$

$= -J_4 + \left(\frac{4}{x}\right) J_3 - \left(\frac{8}{x^2}\right) J_2$

$= -J_4 + \frac{4}{x}\left[\left(\frac{8-x^2}{x^2}\right) J_1 - \left(\frac{4}{x}\right) J_0\right] - \frac{8}{x^2}\left[\frac{2}{x} J_1 - J_0\right]$

$= \left[\frac{-32}{x^3} + \frac{4}{x}\right] J_1 + \left[\frac{16}{x^2} - 1\right] J_0$

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Generating Function for Bessel's Function

If $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ is the generating function of Bessel function, then the coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ are the Bessel's functions of different integral order. That is,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Establish the Jacobi series:

a) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \dots$

b) $\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \dots]$

We know that,

$$e^{+\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$= [J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \dots] + [t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \dots]$$

We know that when n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$

$\therefore J_{-1} = -J_1$ and $J_{-2} = J_2$ and $J_{-3} = -J_3$ and so on,

$$= [J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \dots] + \left[-\frac{1}{t}J_1(x) + \frac{1}{t^2}J_2(x) - \frac{1}{t^3}J_3(x) + \dots\right]$$

$$= J_0(x) + J_1(x)\left[t-\frac{1}{t}\right] + J_2(x)\left[t^2+\frac{1}{t^2}\right] + J_3(x)\left[t^3-\frac{1}{t^3}\right] + \dots \quad (1)$$

Let $t = \cos\theta + i\sin\theta$

$\Rightarrow t^n = \cos n\theta + i\sin n\theta$

$\Rightarrow \frac{1}{t^n} = \cos n\theta - i\sin n\theta$

$$\left.\begin{aligned} t^n + \frac{1}{t^n} &= 2\cos n\theta \\ t^n - \frac{1}{t^n} &= 2i\sin n\theta \end{aligned}\right.$$

From equation ①,

$$e^{\frac{1}{2}x(2\sin\theta)} = J_0(x) + (2i\sin\theta)J_1(x) + (2\cos 2\theta)J_2(x) + (2i\sin 3\theta)J_3 \ldots$$

As we know, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$$

$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 2i\sin 3\theta J_3(x) + \ldots$$

Equating the real and imaginary parts,

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta J_2(x) + \ldots \longrightarrow ②$$

$$\sin(x\sin\theta) = 2\sin\theta J_1(x) + 2\sin 3\theta J_3(x) + \ldots \longrightarrow ③.$$

Equations ② and ③ are called Jacobi series.

Replace θ by $\left(\frac{\pi}{2} - \theta\right)$,

$$\cos 2\theta = \cos 2\left(\frac{\pi}{2} - \theta\right) = -\cos 2\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\sin 3\theta = \sin 3\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{3\pi}{2} - 3\theta\right) = -\cos 3\theta$$

② becomes,

$$\cos(x\cos\theta) = J_0(x) - 2\cos 2\theta J_2(x) + \ldots$$

③ becomes,

$$\sin(x\cos\theta) = 2\cos\theta J_1(x) - 2\cos 3\theta J_3(x) + \ldots$$

LHS = RHS

Hence, proved.

Prove that $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1$.

We know that, (by Jacobi series).

$$J_0(x) + 2\cos 2\theta J_2(x) + 2\cos 4\theta J_4(x) + \ldots = \cos(x\sin\theta) \rightarrow ①$$

$$2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \ldots = \sin(x\sin\theta) \rightarrow ②$$

Squaring ① on both sides and integrating w.r.t θ from 0 to π, we get,

$$J_0^2(x)\int_0^{\pi} d\theta + 4J_2^2(x)\int_0^{\pi}\cos^2 2\theta\, d\theta + 4J_4^2(x)\int_0^{\pi}\cos^2 4\theta\, d\theta + \dots$$

$$= \int_0^{\pi}\cos^2(x\sin\theta)\,d\theta$$

We know that $\int_0^{\pi}\sin^2 n\theta\, d\theta = \frac{\pi}{2}$ (if n is odd)

$\int_0^{\pi}\cos^2 n\theta\, d\theta = \frac{\pi}{2}$ (if n is even)

$$J_0^2(x)[\theta]_0^{\pi} + 4J_2^2(x)\left[\frac{\pi}{2}\right] + 4J_4^2(x)\left[\frac{\pi}{2}\right] + \dots = \int_0^{\pi}\cos^2(x\sin\theta)\,d\theta$$

$$\Rightarrow J_0^2(x)[\pi] + 2\pi J_2^2(x) + 2\pi J_4^2(x) + \dots = \int_0^{\pi}\cos^2(x\sin\theta)\,d\theta \longrightarrow (3)$$

Squaring (2) on both sides and integrating wrt θ from 0 to π, we get,

$$4J_1^2(x)\int_0^{\pi}\sin^2\theta\, d\theta + 4J_3^2(x)\int_0^{\pi}\sin^2 3\theta\, d\theta + \dots = \int_0^{\pi}\sin^2(x\sin\theta)\,d\theta$$

$$\Rightarrow 4J_1^2(x)\left[\frac{\pi}{2}\right] + 4J_3^2(x)\left[\frac{\pi}{2}\right] + \dots = \int_0^{\pi}\sin^2(x\sin\theta)\,d\theta$$

$$\Rightarrow 2\pi J_1^2(x) + 2\pi J_3^2(x) + \dots = \int_0^{\pi}\sin^2(x\sin\theta)\,d\theta \longrightarrow (4)$$

(3) + (4), we get,

$$\pi\left[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + 2J_3^2(x) + \dots\right] = \int_0^{\pi}\frac{\cos^2(x\sin\theta) + \sin^2(x\sin\theta)}{2}\,d\theta$$

$$\Rightarrow \pi\left[J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots\right] = \pi$$

$$\Rightarrow J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots = 1$$

LHS = RHS

Hence, proved.

Prove that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$ where n is a positive integer.

We have by Jacobi series,

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta\, J_2(x) + \dots \longrightarrow ①$$

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \dots \longrightarrow ②$$

Multiplying by $\cos n\theta$ on both sides of ① and integrate wrt θ b/w limits 0 and π,

$$\Rightarrow \int_0^{\pi} \cos n\theta \cos(x\sin\theta)\,d\theta = \underbrace{J_0(x)\int_0^{\pi}\cos n\theta\, d\theta}_{0} + \overbrace{2J_2(x)\int_0^{\pi}\cos 2\theta \cos n\theta\,d\theta}^{n=2 \Rightarrow \frac{\pi}{2}} + 2J_4(x)\int_0^{\pi}\underbrace{\cos 4\theta \cos n\theta\, d\theta}_{n=4 \Rightarrow \frac{\pi}{2}} + \dots$$

$$\boxed{\int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = 0 \quad \text{if } m \neq n \text{ and, } m \text{ and } n \text{ are integers}}$$

$$\int_0^{\pi} \sin^2 n\theta\, d\theta = \frac{\pi}{2} \quad \text{if } n \text{ is odd}$$

$$\int_0^{\pi} \cos^2 n\theta\, d\theta = \frac{\pi}{2} \quad \text{if } n \text{ is even}$$

$$\Rightarrow \int_0^{\pi} \cos n\theta \cos(x\sin\theta)\,d\theta = 0 + 2J_2(x)\left[\frac{\pi}{2}\right] + 2J_4(x)\left[\frac{\pi}{2}\right] + \dots$$

$$\Rightarrow \int_0^{\pi} \cos n\theta \cos(x\sin\theta)\,d\theta = \pi J_2(x) + \pi J_4(x) + \pi J_6(x) + \dots$$

$$\Rightarrow \int_0^{\pi} \cos n\theta \cos(x\sin\theta)\,d\theta = \begin{cases} \pi J_n(x) & \text{when } n \text{ is even} \longrightarrow ③ \\ 0 & \text{when } n \text{ is odd} \longrightarrow ④ \end{cases}$$

Multiplying by $\sin n\theta$ on both sides of ② and integrating wrt θ b/w limits 0 to π,

$$\Rightarrow \int_0^{\pi} \sin n\theta \sin(x\sin\theta)\,d\theta = 2J_1(x)\underbrace{\int_0^{\pi} \sin\theta \sin n\theta\, d\theta}_{n=1 \Rightarrow \frac{\pi}{2}} + 2J_3(x)\underbrace{\int_0^{\pi} \sin 3\theta \sin n\theta\, d\theta}_{n=3 \Rightarrow \frac{\pi}{2}} + \dots$$

$$\Rightarrow \int_0^{\pi} \sin n\theta \sin(x\sin\theta)\,d\theta = 2J_1(x)\left[\frac{\pi}{2}\right] + 2J_3(x)\left[\frac{\pi}{2}\right] + 2J_5(x)\left[\frac{\pi}{2}\right] + \dots$$

$$= \begin{cases} \pi J_n(x) & \text{when } n \text{ is odd} \longrightarrow ⑤ \\ 0 & \text{when } n \text{ is even} \longrightarrow ⑥ \end{cases}$$

Adding ③ and ⑥, we get,

$$\int_0^{\pi} [\cos n\theta \cos(x\sin\theta) + \sin n\theta \sin(x \sin\theta)]\, d\theta = \pi J_n(x)$$

$$\Rightarrow \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta = \pi J_n(x)$$

$$\Rightarrow J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

LHS = RHS

Hence, proved.

Orthogonality property of Bessel Functions

Prove that $\int_0^{a} x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{n+1}^2(a\alpha) & \text{if } \alpha = \beta \end{cases}$

where α and β are roots of $J_n(ax) = 0$.

Consider the differential equation (of Bessel function):

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \longrightarrow ①$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \longrightarrow ②$$

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ be the solutions of eqns ① and ② respectively.

Multiplying ① by $\left(\frac{v}{x}\right)$ and ② by $\left(\frac{u}{x}\right)$, we get,

$$\Rightarrow x u'' v + u' v + v \alpha^2 x u - n^2 u \left(\frac{v}{x}\right) = 0 \longrightarrow ③$$

$$\Rightarrow x v'' u + v' u + u \beta^2 x v - n^2 v \left(\frac{u}{x}\right) = 0 \longrightarrow ④$$

Subtracting ③ and ④, we get,

$$\underbrace{x(u''v - v''u) + (u'v - v'u)} + \underbrace{uvx(\alpha^2 - \beta^2)} - \overbrace{\frac{n^2}{x}(uv - vu)}^{\to 0} = 0$$

$$d[x(u'v - v'u)] = uvx(\beta^2 - \alpha^2)$$

$$~~x[(u'v' + vu'') - (v'u' + v''u)] + (u'v - v'u)~~$$

$$~~\Rightarrow x[vu'' - v''u] + (u'v - v'u)~~$$

Integrate on both sides wrt x between limits 0 and a,

$$\int_0^a d(x(u'v - v'u))\,dx = \int_0^a xvu(\beta^2 - \alpha^2)\,dx$$

$$\Rightarrow [x(u'v - v'u)]_0^a = (\beta^2 - \alpha^2)\int_0^a [xvu]\,dx$$

$$\Rightarrow \int_0^a (xvu)\,dx = \left[\frac{a}{\beta^2 - \alpha^2}\right][u'v - v'u]_0^a$$

Now, $u = J_n(\alpha x)$ and $v = J_n(\beta x)$.

$$\Rightarrow u' = \alpha J_n'(\alpha x) \qquad \Rightarrow v' = \beta J_n'(\beta x)$$

Substituting in above eqn,

$$\Rightarrow \int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \left[\frac{a}{\beta^2 - \alpha^2}\right][\alpha J_n'(\alpha x) J_n(\beta x) - \beta J_n'(\beta x) J_n(\alpha x)]$$

$$\Rightarrow \int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \left[\frac{a}{\beta^2 - \alpha^2}\right][\alpha J_n'(a\alpha) J_n(a\beta) - \beta J_n'(a\beta) J_n(a\alpha) - 0] \longrightarrow 5$$

Given, α and β are roots of $J_n(\alpha x) = 0$

$$\Rightarrow J_n(a\alpha) = 0 \text{ and } J_n(a\beta) = 0.$$

Substituting in eqn 5,

$$\Rightarrow \int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

only when $\beta^2 - \alpha^2 \neq 0 \Rightarrow \beta^2 \neq \alpha^2 \Rightarrow \beta \neq \alpha$

$$\therefore \int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = 0, \text{ when } \alpha \neq \beta.$$

When $\alpha = \beta$, the RH side of eqn 5, it becomes $\left(\frac{0}{0}\right)$. (indeterminate form). Thus, to evaluate RHS of 3, we assume that α is a root of $J_n(\alpha x) = 0$.

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We also assume β as a variable approaching α, i.e., $\beta \to \alpha$

Thus, eqn ⑤ becomes,

$$\lim_{\beta\to\alpha}\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = \lim_{\beta\to\alpha}\left[\frac{a}{\beta^2-\alpha^2}\right]\left[\alpha J_n'(a\alpha) J_n(\beta x) - 0\right].$$

$$\Rightarrow \int_0^a x J_n(\alpha x) J_n(\alpha x)\, dx = \lim_{\beta\to\alpha}\left[\left(\frac{a}{2\beta}\right)\left(\alpha J_n'(a\alpha) J_n'(a\beta)\, a\right]\right.$$

$$= \left[\frac{a}{2}\right]\left[J_n'(a\alpha) J_n'(a\alpha)\, a\right].$$

$$\Rightarrow \int_0^a x J_n^2(\alpha x)\, dx = \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2 \longrightarrow ⑥$$

Using Recurrence relation - ④,

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x).$$

Replace x by $a\alpha$,

$$J_n'(a\alpha) = \underbrace{\left[\frac{n}{a\alpha}\right] J_n(a\alpha)}_{\to 0} - J_{n+1}(a\alpha)$$

$$\Rightarrow J_n'(a\alpha) = -J_{n+1}(a\alpha).$$

Thus, ⑥ becomes,

$$\int_0^a x J_n^2(\alpha x)\, dx = \frac{a^2}{2}\left[-J_{n+1}(a\alpha)\right]^2.$$

$$\Rightarrow \int_0^x x J_n^2(\alpha x)\, dx = \frac{a^2}{2} J_{n+1}^2(a\alpha). \text{, when } \alpha = \beta.$$

Hence, proved.