1. Evaluate $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy$

<u>Method-1</u>: By substitution,

$$I = \int_0^\infty \sqrt{y}\, e^{-y^2}\, dy$$

Put $y^2 = t \Rightarrow 2y\,dy = dt$

$$\Rightarrow dy = \frac{dt}{2y} = \frac{dt}{2\sqrt{t}}$$

$$\Rightarrow I = \int_0^\infty e^{-t}\, t^{\frac{1}{4}} \left(\frac{dt}{2\sqrt{t}}\right)$$

$$\Rightarrow I = \frac{1}{2}\int_0^\infty e^{-t}\, t^{-\frac{1}{4}}\, dt$$

Comparing with Gamma function,

$$\Rightarrow I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

<u>Method-2</u>: By formula $\int_0^\infty x^p e^{-ax^q}\, dx = \dfrac{\Gamma\left(\frac{p+1}{q}\right)}{q\, a^{\frac{p+1}{q}}}$

Here, $p = \frac{1}{2}, a = 1, q = 2$

$$\Rightarrow \int_0^\infty \sqrt{y}\, e^{-y^2}\, dy = \cancel{\int_0^\infty} \frac{\Gamma\left(\frac{3}{4}\right)}{2} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

2. Evaluate $\int_0^\infty e^{-x^4}\,dx$

Method - 1 : By substitution,

$$I = \int_0^\infty e^{-x^4}\,dx$$

Put $t = x^4 \Rightarrow dt = 4x^3\,dx$

$$\Rightarrow dx = \frac{dt}{4x^3}$$

When $x=0$, $t=0$, and $x=\infty$, $t=\infty$,

$$\Rightarrow I = \int_0^\infty e^{-t}\left(\frac{dt}{4x^3}\right)$$

$$\Rightarrow I = \frac{1}{4}\int_0^\infty e^{-t}\,t^{-\frac{3}{4}}\,dt = \frac{1}{4}\Gamma\frac{1}{4}$$

Method - 2 : By formula $\int_0^\infty x^p e^{-ax^q}\,dx = \dfrac{\Gamma\left(\frac{p+1}{q}\right)}{q\cdot a^{\left(\frac{p+1}{q}\right)}}$

Here, $p=0$, $q=4$, $a=1$

$$\Rightarrow \int_0^\infty e^{-x^4}\,dx = \frac{\Gamma\frac{1}{4}}{4} = \frac{1}{4}\Gamma\frac{1}{4}$$

3. Evaluate $\int_0^\infty 3^{-4x^2} dx$

Method - 1 : By substitution

$$I = \int_0^\infty e^{\log 3^{-4x^2}} dx$$

$$\Rightarrow I = \int_0^\infty e^{-4x^2 \log 3} dx$$

$$\Rightarrow I = \int_0^\infty e^{-(4\log 3)x^2} dx$$

$$\Rightarrow I = \int_0^\infty e^{-kx^2} dx \quad \text{where } k = 4\log 3$$

Put $t = kx^2 \Rightarrow dt = 2xk\,dx$

$$\Rightarrow dx = \frac{dt}{2xk} = \frac{dt}{2k\sqrt{\frac{t}{k}}} = \frac{dt}{2\sqrt{kt}}$$

$$\Rightarrow I = \int_0^\infty e^{-t}\left(\frac{dt}{2\sqrt{k}\sqrt{t}}\right)$$

$$\Rightarrow I = \frac{1}{2\sqrt{k}}\int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \frac{1}{2\sqrt{k}}\Gamma\frac{1}{2} = \frac{1}{4\sqrt{\log 3}}\Gamma\frac{1}{2}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

Method 2: Using formula: $\int_0^\infty x^p e^{-ax^q} dx = \frac{\Gamma\frac{p+1}{q}}{q \cdot a^{\left(\frac{p+1}{q}\right)}}$

Here, $p = 0$, $a = 4\log 3$, $q = 2$

Putting values $\Rightarrow \int_0^\infty x^0 e^{-4\log 3 x^2} dx = \frac{\Gamma\frac{1}{2}}{2(4\log 3)^{\frac{1}{2}}} = \frac{\Gamma\frac{1}{2}}{2\sqrt{4\log 3}} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

4. Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

$$I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$$

Put $t = -\log x \Rightarrow dt = \frac{dx}{-e^{-t}} \quad (\because x = e^{-t})$.

When $x = 0, t = \infty$, and $x = 1, t = 0$.

$$\Rightarrow I = \int_\infty^0 \frac{-e^{-t}}{\sqrt{t}}\, dt$$

$$\Rightarrow I = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt = \Gamma\left(\tfrac{1}{2}\right) = \underline{\underline{\sqrt{\pi}}}.$$

5. Evaluate $\int_0^\infty (x^2+4)e^{-2x^2}\,dx$

$$I = \int_0^\infty (x^2+4)e^{-2x^2}\,dx$$

$$\Rightarrow I = \underbrace{\int_0^\infty e^{-2x^2} x^2\,dx}_{p=2,\ a=2,\ q=2} + 4\underbrace{\int_0^\infty e^{-2x^2}\,dx}_{p=0,\ a=2,\ q=2}$$

$$= \left[\frac{\Gamma\left(\frac{3}{2}\right)}{2\times 2^{\frac{3}{2}}}\right] + 4\left[\frac{\Gamma\left(\frac{1}{2}\right)}{2\times 2^{\frac{1}{2}}}\right]$$

$$= \left[\frac{\frac{1}{2}(\sqrt{\pi})}{4\sqrt{2}}\right] + 4\left(\frac{\sqrt{\pi}}{2\sqrt{2}}\right) = \frac{\sqrt{\pi}}{8\sqrt{2}} + \frac{4\sqrt{\pi}}{2\sqrt{2}} = \underline{\underline{\frac{17\sqrt{\pi}}{8\sqrt{2}}}}$$

6. Evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}}\right) dx$

$$I = \int_0^{\frac{\pi}{2}} \sin^{\frac{8}{3}} x \cos^{-\frac{1}{2}} x \, dx$$

Here, $p = \frac{8}{3}$, $q = -\frac{1}{2}$ in formula.

Using formula $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\Rightarrow I = \frac{1}{2} \beta\left(\frac{11}{6}, \frac{1}{4}\right)$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{25}{12}\right)}\right]$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\frac{5}{6}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{13}{12}\Gamma\left(\frac{13}{12}\right)}\right]$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\frac{5}{6}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{13}{12}\left(\frac{1}{12}\right)\Gamma\left(\frac{1}{12}\right)}\right]$$

$$\Rightarrow I = \frac{1}{\cancel{2}} \times \frac{5}{\cancel{6}} \times \frac{\overset{12}{\cancel{\overset{72}{\cancel{144}}}}}{13}\left[\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{12}\right)}\right]$$

$$\Rightarrow I = \frac{60}{13}\left[\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{12}\right)}\right]$$

★ Result : The integrals involving algebraic functions can be converted to trigonometric form by applying substitution.

Algebraic function	Substitution
$a^2 - x^2$	$x^2 = a^2 \sin^2\theta$
$1 - x^4$	$x^4 = \sin^2\theta$
$1 + x^2$	$x^2 = \tan^2\theta$
$1 + x^6$	$x^6 = \tan^2\theta$
$1 - x^n$	$x^n = \sin^2\theta$
$a - x$	$x = a\sin^2\theta$
$1 - x^{\frac{1}{4}}$	$x^{\frac{1}{4}} = \sin^2\theta$

7. Evaluate $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$

Put $x = 2\sin^2\theta$
$\Rightarrow dx = 4\sin\theta\cos\theta\,d\theta$

$x = 0,\ \theta = 0$
$x = 2,\ \theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{4\sin^4\theta}{\sqrt{2-2\sin^2\theta}} (4\sin\theta\cos\theta\,d\theta)$$

$$\Rightarrow I = \frac{16}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \left(\frac{\sin^5\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}}\right) d\theta$$

$$\Rightarrow I = \frac{16}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sin^5\theta\cos^0\theta\,d\theta$$

Using formula $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\Rightarrow I = \frac{16}{\sqrt{2}}\left(\frac{1}{2}\right)\beta\left(3, \frac{1}{2}\right)$

$\Rightarrow I = \frac{16}{2\sqrt{2}}\left[\frac{\Gamma 3 \, \Gamma \frac{1}{2}}{\Gamma \frac{7}{2}}\right]$

$\Rightarrow I = \frac{16}{\cancel{2}\sqrt{2}}\left[\frac{\cancel{2}!\,(\cancel{\sqrt{\pi}})}{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\cancel{\Gamma \frac{1}{2}}}\right]$

$\Rightarrow I = \frac{16}{\sqrt{2}} \times \frac{8}{15} = \frac{16}{15}(4\sqrt{2}) = \frac{64}{15}\sqrt{2}$

8. Evaluate $\int_0^1 x^4(1-x)^3 \, dx$

$\Rightarrow \int_0^1 x^4(1-x)^3 \, dx = \beta(5,4)$

$\Rightarrow \frac{\Gamma 5 \, \Gamma 4}{\Gamma 9} = \frac{4!\,3!}{8!} = \frac{1}{280}$

9. Evaluate $\int_0^1 x^2(1-x^5)^{-\frac{1}{2}} \, dx$

Let $x^5 = \sin^2\theta \Rightarrow x = \sin^{\frac{2}{5}}\theta$

$\Rightarrow dx = \frac{2}{5}\sin^{-\frac{3}{5}}\theta \cos\theta \, d\theta$

When $x = 0, \theta = 0$

$x = 1, \theta = \frac{\pi}{2}$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^{\frac{4}{5}}\theta \left[\cos^2\theta\right]^{-\frac{1}{2}} \left[\frac{2}{5}\sin^{-\frac{3}{5}}\theta \cos\theta \, d\theta\right]$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{2}{5}\left[\sin^{\frac{1}{5}}\theta \cos^{0}\theta \, d\theta\right]$$

$$\Rightarrow I = \frac{2}{5}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{5}}\theta \cos^{0}\theta \, d\theta$$

Using formula : $\int_0^{\frac{\pi}{2}} x^{p}(1-x^{q})^{r}\,dx = \frac{1}{q}\,\beta\left(\frac{p+1}{q}, r+1\right)$

$$\Rightarrow I = \frac{2}{5}\left[\frac{1}{2}\,\beta\left(\frac{3}{5}, \frac{1}{2}\right)\right]$$

$$\Rightarrow I = \frac{1}{5}\,\frac{\Gamma\frac{3}{5}\,\Gamma\frac{1}{2}}{\Gamma\frac{11}{10}} \Rightarrow I = \frac{\sqrt{\pi}}{5}\left\{\frac{\Gamma\frac{3}{5}}{\frac{1}{10}\Gamma\frac{1}{10}}\right\}$$

$$\Rightarrow I = 2\sqrt{\pi}\left[\frac{\Gamma\frac{3}{5}}{\Gamma\frac{1}{10}}\right]$$

10. Evaluate $\int_0^1 x^2(1-x^3)^4\,dx$

$$= \frac{1}{3}\,\beta\left(\frac{3}{3}, 4+1\right) = \frac{1}{3}\left(\frac{\Gamma 1\,\Gamma 5}{\Gamma 6}\right) = \frac{1}{3}\left(\frac{4!}{5!}\right) = \frac{1}{15}$$

$$\Rightarrow I = \int_0^1 x^2(1-x^3)^4\,dx$$

Let $x^3 = \sin^2\theta \Rightarrow x = \sin^{\frac{2}{3}}\theta$

$$\Rightarrow dx = \frac{2}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\,d\theta$$

When $x=0$, $\theta=0$,

$x=1$, $\theta=\frac{\pi}{2}$

$$\Rightarrow I = \int_0^1 \sin^{\frac{2}{3}}\theta\,(\cos^2\theta)^4\left[\frac{2}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\,d\theta\right]$$

$$\Rightarrow I = \frac{2}{3}\int_0^1 \sin\theta\cos^9\theta\,d\theta$$

$$\Rightarrow I = \frac{1}{3}\beta(1,5)$$

$$\Rightarrow I = \frac{1}{3}\frac{\Gamma 1\,\Gamma 5}{\Gamma 6}$$

$$\Rightarrow I = \frac{1}{3}\left(\frac{4!}{5!}\right) = \frac{1}{15}$$

11. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\,d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}}\,d\theta = \pi$

$$I = \left[\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^0\theta\, d\theta\right]\left[\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^0\theta\, d\theta\right]$$

$$\Rightarrow I = \left[\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right)\right]\left[\frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)\right]$$

$$\Rightarrow I = \frac{1}{4}\left[\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{4}}\right]\left[\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}\right]$$

$$\Rightarrow I = \frac{1}{4}\left[\frac{\cancel{\Gamma\frac{3}{4}}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{4}}\right]\left[\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}}\right]$$

$$\Rightarrow I = \frac{\cancel{1}}{\cancel{4}}\frac{(\pi\sqrt{2})(\pi)}{\cancel{\frac{1}{4}}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}$$

$$\Rightarrow I = \cancel{\left(\frac{1}{4}\right)}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\cancel{\Gamma\frac{1}{4}}}{\cancel{\left(\frac{1}{4}\right)}\cancel{\Gamma\frac{1}{4}}}$$

$$\Rightarrow I = \frac{\pi^2\sqrt{2}}{\pi\sqrt{2}}$$

$$\Rightarrow I = \sqrt{\pi}\,\sqrt{\pi}$$

$$\Rightarrow I = \pi$$

$$\rightarrow I = \pi$$

LHS = RHS

Hence, proved.

12. Show that $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$

$I_1 = \int_0^\infty x^{\frac{1}{2}} e^{-x^2}\, dx$ Here $p = \frac{1}{2}$, $a = 1$, $q = 2$

Using formula :

$$\Rightarrow I_1 = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2} = \frac{\Gamma\left(\frac{3}{4}\right)}{2} \longrightarrow (1)$$

$$I_2 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \int_0^\infty x^{-\frac{1}{2}} e^{-x^2}\, dx$$

Here $p = -\frac{1}{2}$, $a = 1$, $q = 2$

$$\Rightarrow I_2 = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2} = \frac{\Gamma\left(\frac{1}{4}\right)}{2} \longrightarrow (2)$$

$$LHS = (1) \times (2)$$

$$= \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) \left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\left(\pi\sqrt{2}\right) = \frac{\pi}{2\sqrt{2}} = RHS$$

$LHS = RHS$

Hence, proved.

13. Show that $\displaystyle\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx \cdot \int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \frac{5\pi^2\sqrt{2}}{128}$

Let $x^4 = \tan^2\theta \Rightarrow x = \tan^{\frac{1}{2}}\theta$

$\Rightarrow dx = \frac{1}{2}\tan^{-\frac{1}{2}}\theta\sec^2\theta\,d\theta$

When $x=0$, $\theta = 0$, and $x=\infty$, $\theta = \frac{\pi}{2}$

$$I_1 = \int_0^{\frac{\pi}{2}} \tan\theta\,(1+\tan^2\theta)^{-3}\left[\frac{1}{2}\tan^{-\frac{1}{2}}\theta\sec^2\theta\,d\theta\right].$$

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{2}\tan^{\frac{1}{2}}\theta\,\sec^{-4}\theta\,d\theta$$

$$\Rightarrow I_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \tan^{\frac{1}{2}}\theta\,\sec^{-4}\theta\,d\theta$$

$$\Rightarrow I_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\cos\theta}\right)^{\frac{1}{2}}\left(\frac{1}{\cos\theta}\right)^{-4} d\theta$$

$$\Rightarrow I_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\,\cos^{\frac{7}{2}}\theta\,d\theta$$

$$\Rightarrow I_1 = \frac{1}{2}\left[\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{\frac{7}{2}+1}{2}\right)\right]$$

$$\Rightarrow I_1 = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{9}{4}\right)$$

$$\Rightarrow I_1 = \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)}$$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{1}{2!}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)\left(\frac{5}{16}\right) = \frac{5}{128}(\pi\sqrt{2})$$

$$I_2 = \int_0^{\infty} x^{-\frac{1}{2}} (1+x)^{-1} dx$$

Let $x = \tan^2\theta \Rightarrow dx = 2\tan\theta \sec^2\theta \, d\theta$

$x=0, \theta=0, x=\infty, \theta=\frac{\pi}{2}$

$$\Rightarrow I_2 = \int_0^{\frac{\pi}{2}} (\tan^2\theta)^{-\frac{1}{2}} (1+\tan^2\theta)^{-1} [2\tan\theta \sec^2\theta \, d\theta].$$

$$\Rightarrow I_2 = \int_0^{\frac{\pi}{2}} 2\tan^{-1}\theta \tan\theta \sec^{0}\theta \, d\theta$$

$$\Rightarrow I_2 = 2\int_0^{\frac{\pi}{2}} \tan^0\theta \sec^0\theta \, d\theta = \pi$$

$$LHS = I_1 I_2$$

$$= \left[\frac{5}{128}\pi\sqrt{2}\right](\pi) = \frac{5}{128}(\pi^2\sqrt{2}) = RHS.$$

LHS = RHS

Hence, proved.

14. Evaluate $\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}} dx \times \int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}}$

Let $I_1 = \int_0^3 x^{\frac{3}{2}} (3-x)^{-\frac{1}{2}} dx$

Let $x = 3\sin^2\theta \Rightarrow dx = 6\sin\theta\cos\theta \, d\theta$ $\begin{cases} x=0, \theta=0 \\ x=3, \theta=\frac{\pi}{2} \end{cases}$

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} (3\sin^2\theta)^{\frac{3}{2}} (3-3\sin^2\theta)^{-\frac{1}{2}} (6\sin\theta\cos\theta \, d\theta)$$

$$\Rightarrow I_1 = (6)(3^{\frac{3}{2}}) \int_0^{\frac{\pi}{2}} \sin^3\theta [3\cos^2\theta]^{-\frac{1}{2}} \sin\theta\cos\theta \, d\theta$$

$$\Rightarrow I_1 = (6)3^{\frac{3}{2}-\frac{1}{2}} \int^{\frac{\pi}{2}} \sin^4\theta \cos^0\theta \, d\theta$$

$$\Rightarrow I_1 = 18\int_0^{\frac{\pi}{2}} \sin^4\theta \cdot \cos^0\theta \cdot d\theta$$

$$\Rightarrow I_1 = 18\left[\frac{1}{2}\beta\left(\frac{5}{2}, \frac{1}{2}\right)\right]$$

$$\Rightarrow I_1 = 18\left(\frac{1}{2}\right)\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$\Rightarrow I_1 = \frac{9}{2}\left[\frac{\frac{3}{2}\left(\frac{1}{2}\right)\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1}\right]$$

$$\Rightarrow I_1 = \frac{9\times 3}{8}(\sqrt{\pi})(\sqrt{\pi}) = \frac{27\pi}{8}.$$

Let $I_2 = \int_0^1 (1-x^{\frac{1}{4}})^{-\frac{1}{2}} dx$

Using formula : $\int_0^1 x^p(1-x^q)^r dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$

Here , $p=0$, $q=\frac{1}{4}$, $r=-\frac{1}{2}$

$$\Rightarrow I_2 = \frac{1}{\left(\frac{1}{4}\right)}\beta\left(\frac{0+1}{\frac{1}{4}}, \frac{1}{2}\right)$$

$$\Rightarrow I_2 = 4\,\beta\left(4, \frac{1}{2}\right)$$

$$\Rightarrow I_2 = 4\left[\frac{\Gamma 4\,\Gamma\frac{1}{2}}{\Gamma\frac{9}{2}}\right]$$

$$= \frac{4(3!)\cancel{\sqrt{\pi}}}{\frac{7}{2}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\cancel{\Gamma\frac{1}{2}}} = \frac{128}{35}$$

Here, $I = I_1 \times I_2$

$$\Rightarrow I = \left(\frac{27\pi}{8}\right)\left(\frac{64}{35}\right) = \underline{\underline{\frac{432\pi}{35}}}$$

15. Evaluate $\int_0^2 (8-x^3)^{-\frac{1}{3}}\,dx$

Let $x^3 = 8\sin^2\theta \Rightarrow dx = \frac{4}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\,d\theta$

$$I = \int_0^2 (8-x^3)^{-\frac{1}{3}}\,dx$$

$$\Rightarrow I = \int_0^2 (8-8\sin^2\theta)^{-\frac{1}{3}}\left[\frac{4}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\,d\theta\right]$$

$$\Rightarrow I = 8^{-1/3}\left(\frac{4}{3}\right)\int_0^2 \cos^{-\frac{2}{3}}\theta\cos\theta\sin^{-\frac{1}{3}}\theta\,d\theta$$

$$\Rightarrow I = \frac{2}{3}\int_0^2 \sin^{-\frac{1}{3}}\theta\cos^{\frac{1}{3}}\theta\,d\theta$$

$$\Rightarrow I = \frac{2}{3}\left[\frac{1}{2}B\left(\frac{1}{3},\frac{2}{3}\right)\right]$$

$$= \frac{\cancel{2}}{3}\left(\frac{1}{\cancel{2}}\right)\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}$$

$$= \underline{\underline{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}}$$

16. Evaluate $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

Put $x^2 = a^2\sin^2\theta \Rightarrow x = a\sin\theta$

$\Rightarrow dx = a\cos\theta$

$\left\{\begin{array}{l} x=0,\ \theta=0 \\ x=a,\ \theta=\frac{\pi}{2} \end{array}\right.$

$$I = \int_0^{\frac{\pi}{2}} a^4\sin^4\theta\,(a^2-a^2\sin^2\theta)^{\frac{1}{2}}(a\cos\theta)$$

$$I = a^6\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta \quad d\theta$$

$$\Rightarrow I = a^6\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta$$

$$\Rightarrow I = a^6\left[\frac{1}{2}\beta\left(\frac{5}{2},\frac{3}{2}\right)\right]$$

$$\Rightarrow I = \frac{a^6}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$$

$$\Rightarrow I = \frac{a^6}{2}\,\frac{\frac{\cancel{3}}{2}\left(\frac{1}{2}\right)\frac{\cancel{1}}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\times\cancel{3}\times 2}$$

$$\Rightarrow I = \frac{\pi a^6}{32}$$

17. Evaluate $\int_0^1 (x\log x)^4\,dx$

Using formula: $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}}$

Here $m = n = 4$

$$\Rightarrow I = \frac{(-1)^4\,\Gamma(5)}{5^5} = \frac{4!}{3125} = \frac{24}{3125}$$

3. Evaluate $\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} dx$

(Whenever the limits are between -1 and 1, always substitute $x = \cos 2\theta$).

$x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$ $\left.\begin{array}{l} x=-1,\ \theta = \frac{\pi}{2} \\ x = 1,\ \theta = 0. \end{array}\right.$

$$\Rightarrow \int_{\frac{\pi}{2}}^{0} \left(\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)^{\frac{1}{2}} (-2\sin 2\theta\, d\theta).$$

$$\Rightarrow 2\int_{0}^{\frac{\pi}{2}} \left(\frac{2\cos^2\theta}{2\sin^2\theta}\right)^{\frac{1}{2}} (\sin 2\theta)\, d\theta$$

$$\Rightarrow 2\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta}{\cancel{\sin\theta}} (2\cancel{\sin\theta}\cos\theta)\, d\theta$$

$$\Rightarrow 4\int_{0}^{\frac{\pi}{2}} \sin^0\theta \cos^2\theta\, d\theta$$

$$\Rightarrow 4\left[\frac{1}{2} \beta\left(\frac{1}{2}, \frac{3}{2}\right)\right]$$

$$\Rightarrow 4\left(\frac{1}{2}\right) \frac{\Gamma\frac{1}{2}\, \Gamma\frac{3}{2}}{\Gamma 2}$$

$$\Rightarrow 2\, \frac{\Gamma\frac{1}{2} \left(\frac{1}{2}\right) \Gamma\frac{1}{2}}{1!}$$

$$\Rightarrow \underline{\underline{\pi}}$$

19. Evaluate $\int_0^{2\pi} \sin^2\theta (1+\cos\theta)^4 d\theta$

$\sin^2\theta = (\sin\theta)^2$

$\Rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

$|||^{ly} \Rightarrow (\sin\theta)^2 = 2^2 \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$

$$\Rightarrow \int_0^{2\pi} 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$\Rightarrow 64\int_0^{2\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\, d\theta$$

Let $\frac{\theta}{2} = x \Rightarrow d\theta = 2dx$ $\left\{\begin{array}{l}\theta=0,\ x=0\\ \theta=2\pi,\ x=\pi\end{array}\right.$

$$\Rightarrow 64\int_0^{\pi} \sin^2 x\cos^{10} x\,(2dx)$$

$$\Rightarrow 128\int_0^{\pi} \sin^2 x\cos^{10} x\, dx$$

Using property $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

if $f(2a-x) = f(x)$

Here $\Rightarrow f(\pi - x) = [\sin(\pi-x)]^2[\cos(\pi-x)]^2$

$= \sin^2 x\cos^2 x = f(x)$.

Thus, using this property,

$$\Rightarrow 128(2)\int_0^{\frac{\pi}{2}} \sin^2 x\cos^{10} x\, dx$$

$$\Rightarrow 256\left[\frac{1}{2}B\left(\frac{3}{2},\frac{11}{2}\right)\right] = 128\,\frac{\frac{1}{2}\sqrt{\frac{1}{2}}\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}}}{6!} = \frac{21}{8}$$

(If ν is an integer, then $J_\nu(x)$ and $J_{-\nu}(x)$ will become linearly dependent solutions)

Linear Dependence of Bessel Functions

2. Prove that $J_{-n}(x) = (-1)^n J_n(x)$ where n is +ve integer.

We know that,

$$J_\nu(x) = \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m+\nu}\left[\frac{1}{\Gamma(m+\nu+1)}\right]$$

Put $\nu = n$ here,

$$J_n(x) = \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m+n}\left[\frac{1}{\Gamma(m+n+1)}\right]$$

Put $\nu = -n$ here,

$$J_{-n}(x) = \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m-n}\left[\frac{1}{\Gamma(m-n+1)}\right]$$

For gamma function to exist, $(m-n+1) > 0$

$\Rightarrow m > n-1$

So, $m = 0, 1, 2, 3 \ldots n-1 \Rightarrow$ Gamma function $\to \infty$

(Here m must vary from n to ∞, and not from 0 to ∞, because $m > n-1$ for gamma function to exist)

$$\left(\begin{matrix} n-1 = -2 \Rightarrow 0 > -2 \\ n-1 = -1 \Rightarrow 0 > -1 \end{matrix}\right\} \sum_{n}^{\infty}\Bigg)$$

$$\therefore J_{-n}(x) = \sum_{m=n}^{\infty} \left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m-n}\left[\frac{1}{\Gamma(m-n+1)}\right]$$

Let $s = m-n \Rightarrow m = s+n$ and s varies from 0 to ∞

$$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty}\left[\frac{(-1)^{s+n}}{(s+n)!}\right]\left(\frac{x}{2}\right)^{2(s+n)-n}\left[\frac{1}{\Gamma(s+\cancel{n}-\cancel{n}+1)}\right]$$

$$\Rightarrow J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty}\left[\frac{(-1)^s}{\Gamma(s+1)}\right]\left(\frac{x}{2}\right)^{2(s+n)-n}\left[\frac{1}{(s+n)!}\right]$$

$$\Rightarrow J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^{2s+n}\frac{1}{\Gamma(s+n+1)}$$

By the definition of Bessel function,

$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

When n is a positive integer, $J_{-n}(x)$ and $J_n(x)$ are linearly dependent.

LHS=RHS

Hence, proved.

23. Prove that:

i) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ ii) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

(i) We know that,

$$J_n(x) = \sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m+n}\left[\frac{1}{\Gamma(m+n+1)}\right] \longrightarrow (1)$$

Let $n=\frac{1}{2}$ here,

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}\left[\frac{1}{\Gamma(m+\frac{1}{2}+1)}\right]$$

$$= \sqrt{\frac{x}{2}}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\right]\left(\frac{x}{2}\right)^{2m}\left[\frac{1}{\Gamma(m+\frac{3}{2})}\right]$$

$$= \sqrt{\frac{x}{2}}\left\{\left[\frac{1}{1}\left(\frac{x}{2}\right)^0 \frac{1}{\Gamma\frac{3}{2}}\right] + \left[\frac{(-1)}{1!}\left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\frac{5}{2}}\right] + \left[\frac{(-1)^2}{2!}\left(\frac{x}{2}\right)^4 \frac{1}{\Gamma\frac{7}{2}}\right] + \cdots\cdots\right\}$$

$$= \sqrt{\frac{x}{2}}\left[\left(\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}}\right) - \left\{\left(\frac{x}{2}\right)^2\left(\frac{1}{\frac{3}{2}\left(\frac{1}{2}\right)\left(\Gamma\frac{1}{2}\right)}\right)\right\} + \left\{\frac{1}{2}\left(\frac{x}{2}\right)^4 \frac{1}{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\Gamma\frac{1}{2}\right)}\right\} - \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\left(\frac{4}{3\sqrt{\pi}}\right) + \frac{1}{2}\left(\frac{x^4}{16}\right)\left(\frac{8}{15}\right)\left(\frac{1}{\sqrt{\pi}}\right) - \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{3\sqrt{\pi}} + \frac{1}{4}\left(\frac{x^4}{15}\right)\left(\frac{1}{\sqrt{\pi}}\right) - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots\cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\left(\frac{2}{x}\right)\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\underbrace{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right]}_{\text{expansion of } \sin x}$$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$\therefore$ LHS = RHS

Hence, proved.

ii) We know that,

$$J_{-n}(x) = \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right] \left(\frac{x}{2}\right)^{2m-n} \left[\frac{1}{\Gamma(m-n+1)}\right]$$

Here, $n = \frac{1}{2}$, thus,

$$J_{-\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right] \left(\frac{x}{2}\right)^{2m-\frac{1}{2}} \left[\frac{1}{\Gamma(m-\frac{1}{2}+1)}\right]$$

$$= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!}\right] \left(\frac{x}{2}\right)^{2m} \left[\frac{1}{\Gamma(m+\frac{1}{2})}\right]$$

$$= \sqrt{\frac{2}{x}} \left[\left\{\frac{1}{1}\left(\frac{x}{2}\right)^0 \frac{1}{\Gamma\frac{1}{2}}\right\} + \left\{\frac{(-1)}{1!}\left(\frac{x}{2}\right)^2 \frac{1}{\Gamma\frac{3}{2}}\right\} + \left\{\frac{1}{2!}\left(\frac{x}{2}\right)^4 \frac{1}{\Gamma\frac{5}{2}}\right\} + \dots \right]$$

$$= \sqrt{\frac{2}{x}} \left[\frac{1}{\sqrt{\pi}} + \frac{x^2}{4}\left(\frac{-1}{\frac{1}{2}\sqrt{\pi}}\right) + \frac{1}{2}\left(\frac{x^4}{16}\right)\left(\frac{1}{\frac{3}{2}\left(\frac{1}{2}\right)\sqrt{\pi}}\right) + \dots \right]$$

$$= \sqrt{\frac{2}{x\pi}} \left[1 - \frac{x^2}{2} + \left(\frac{1}{24}\right)x^4 - \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \underbrace{\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right]}_{\text{expansion of } \cos x}$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

LHS = RHS

Hence, proved.

4. Prove that $\int_0^{\frac{\pi}{2}} \sqrt{x}\, J_{\frac{1}{2}}(2x)\,dx$, as $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$\int_0^{\frac{\pi}{2}} \sqrt{x}\, J_{\frac{1}{2}}(2x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cancel{\sqrt{x}} \sqrt{\frac{2}{\pi(2\cancel{x})}} \sin 2x\, dx$$

$$= \frac{1}{\sqrt{\pi}}(2) \int_0^{\frac{\pi}{2}} \sin x \cos x\, dx$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2} \beta(1,1)\right]$$

$$= \frac{\cancel{2}}{\sqrt{\pi}}\left(\frac{1}{\cancel{2}}\right) \frac{\boxed{1}\,\boxed{1}}{\boxed{2}} = \frac{1}{\sqrt{\pi}}$$

$$\frac{1}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \sin 2x\, dx$$

$$= \frac{1}{\sqrt{\pi}}\left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2\sqrt{\pi}}[-(-1)+1] = \frac{1}{\sqrt{\pi}}$$

Recurrence Relations

1. $\frac{d}{dx}[x^{\nu} J_{\nu}(x)] = x^{\nu} J_{\nu-1}(x)$ OR $[x^{\nu} J_{\nu}(x)]' = x^{\nu} J_{\nu-1}(x)$

2. $\frac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$ OR $[x^{-\nu} J_{\nu}(x)]' = -x^{-\nu} J_{\nu+1}(x)$

3. $\frac{d}{dx}[J_{\nu}(x)] = J_{\nu-1}(x) - \left(\frac{\nu}{x}\right) J_{\nu}(x)$ OR

$$x J_{\nu}'(x) = x J_{\nu-1}(x) - \nu J_{\nu}(x)$$

4. $J_{\nu}'(x) = \left(\frac{\nu}{x}\right) J_{\nu}(x) - J_{\nu+1}(x)$ OR

$$x J_{\nu}'(x) = \nu J_{\nu}(x) - x J_{\nu+1}(x)$$

(Multiplying x on both sides in (3) and (4)

5. $J'_\nu(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)]$ OR

$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$

6. $J_{\nu-1}(x) + J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x)$ OR

$(2\nu) J_\nu(x) = x[J_{\nu-1}(x) + J_{\nu+1}(x)]$

25. Prove that $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

By Recurrence relations, (1st formula/relation)

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$$

Put $\nu = \frac{1}{2}$,

$$\frac{d}{dx}[x^{\frac{1}{2}} J_{\frac{1}{2}}(x)] = x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}\left[x^{\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \sin x\right] = x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$$

$$\Rightarrow \frac{d}{dx}\left[\sqrt{\frac{2}{\pi}} \sin x\right] = \sqrt{x}\, J_{-\frac{1}{2}}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} (\cos x) = x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

LHS = RHS

Hence, proved

26. Express $J_{\frac{5}{2}}(x)$ and $J_{-\frac{5}{2}}(x)$ in terms of sine and cosine functions.

By 6^{th} recurrence relation,

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x)$$

$$\Rightarrow J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x) - J_{\nu-1}(x) \longrightarrow ①$$

Put $\nu = \frac{3}{2}$ in ①,

$$\Rightarrow J_{\frac{5}{2}}(x) = \frac{\cancel{2}}{x}\left(\frac{3}{\cancel{2}}\right) J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{5}{2}}(x) = \left(\frac{3}{x}\right) J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x).$$

$$\Rightarrow J_{\frac{5}{2}}(x) = \left(\frac{3}{x}\right) J_{\frac{3}{2}}(x) - \sqrt{\frac{2}{\pi x}} \sin x \longrightarrow ②.$$

To find $J_{\frac{3}{2}}(x)$, put $\nu = \frac{1}{2}$ in ①,

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{\cancel{2}}{x}\left(\frac{1}{\cancel{2}}\right) J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x\right] \longrightarrow ③$$

Putting ③ in ②,

$$\Rightarrow J_{\frac{5}{2}}(x) = \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\right] - \left[\sqrt{\frac{2}{\pi x}} \sin x\right]$$

$$\Rightarrow J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right]$$

$$\Rightarrow J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin x\left(\frac{3-x^2}{x^2}\right) - \left(\frac{3}{x}\right)\cos x\right]$$

Now, to find $J_{-\frac{5}{2}}(x)$, from ①,

$$J_{\nu-1}(x) = \left(\frac{2\nu}{x}\right)J_\nu(x) - J_{\nu+1}(x) \longrightarrow ④$$

Put $\nu = -\frac{3}{2}$ in ①,

$$\Rightarrow J_{-\frac{5}{2}}(x) = \frac{2}{x}\left(-\frac{3}{2}\right)J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{5}{2}}(x) = \left(-\frac{3}{x}\right)J_{-\frac{3}{2}}(x) - \sqrt{\frac{2}{\pi x}}\cos x \longrightarrow ⑤.$$

Put $\nu = -\frac{1}{2}$ in ④,

$$\Rightarrow J_{-\frac{3}{2}}(x) = \frac{2}{x}\left(-\frac{1}{2}\right)J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = \left(-\frac{1}{x}\right)\left[\sqrt{\frac{2}{\pi x}}\cos x\right] - \left(\sqrt{\frac{2}{\pi x}}\sin x\right)$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[-\frac{\cos x}{x} - \sin x\right] \longrightarrow ⑥$$

Putting ⑥ in ⑤,

$$\Rightarrow J_{-\frac{5}{2}}(x) = \left(-\frac{3}{x}\right)\left[\sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$\Rightarrow J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right]$$

$$\Rightarrow J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos x\left(\frac{3-x^2}{x^2}\right) + \left(\frac{3}{x}\right)\sin x\right]$$

Prove that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ and

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

Using Recurrence relation-6,

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x).$$

$$\Rightarrow J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right) J_\nu(x) - J_{\nu-1}(x) \longrightarrow (1)$$

Let $\nu = \frac{1}{2}$, putting in (1),

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{2}{x}\left(\frac{1}{2}\right) J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \left(\frac{1}{x}\right)\left[\sqrt{\frac{2}{\pi x}} \sin x\right] - \left[\sqrt{\frac{2}{\pi x}} \cos x\right]$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

LHS = RHS $\Rightarrow$ Hence, proved.

Let $\nu = -\frac{1}{2}$, putting in (1),

$$\Rightarrow J_{-\frac{3}{2}}(x) = \left(\frac{2}{x}\right)\left(\frac{-1}{2}\right) J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = \left(\frac{-1}{x}\right)\left[\sqrt{\frac{2}{\pi x}} \cos x\right] - \left[\sqrt{\frac{2}{\pi x}} \sin x\right].$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

LHS = RHS

Hence, proved.

28. Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

By 6^{th} recurrence relation,

$$\left(\frac{2\nu}{x}\right)J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x).$$

$$\Rightarrow J_{\nu+1}(x) = \left(\frac{2\nu}{x}\right)J_\nu(x) - J_{\nu-1}(x)$$

Put $\nu = 1, 2, 3$ in ①,

$$\therefore J_2(x) = \left(\frac{2}{x}\right)(1)\,J_1(x) - J_0(x) \longrightarrow ①$$

$$\therefore J_3(x) = \left(\frac{2}{x}\right)(2)\,J_2(x) - J_1(x) \longrightarrow ②$$

$$\therefore J_4(x) = \left(\frac{2}{x}\right)(3)\,J_3(x) - J_2(x) \longrightarrow ③.$$

Putting ② in ①,

$$J_3(x) = \left(\frac{2}{x}\right)(2)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)$$

$$\Rightarrow J_3(x) = \frac{4}{x}\left(\frac{2}{x}\right)J_1(x) - \left(\frac{4}{x}\right)J_0(x) - J_1(x)$$

$$\Rightarrow J_3(x) = \left(\frac{8 - x^2}{x^2}\right)J_1(x) - \left(\frac{4}{x}\right)J_0(x) \longrightarrow ④$$

Putting ① and ④ in ③,

$$J_4(x) = \left(\frac{2}{x}\right)(3)\left[\left(\frac{8-x^2}{x^2}\right)J_1(x) - \left(\frac{4}{x}\right)J_0(x)\right] - \left[\frac{2}{x}J_1(x)\right] + J_0(x)$$

$$\Rightarrow J_4(x) = \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right)J_1(x) - \frac{24}{x^2}J_0(x) + J_0(x)$$

$$\Rightarrow J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$