Engineering Mathematics - II (UE23MA141B)

Unit - 2: Laplace Transforms

Problems on Laplace transform of standard functions

1. Find the Laplace transform of $f(t) = \begin{cases} e^t & \text{for } 0 < t < 1 \\ 0 & \text{t} > 0 \end{cases}$

   Answer: $\frac{1}{1-s}\left(1 - e^{-(s-1)}\right)$

2. Find the Laplace transform of $cos^2(at)$.

3. Find the inverse Laplace transform of $\frac{2s+5}{s^2+25}$

4. Find the inverse Laplace transform of $\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right]$

5. Find the inverse Laplace transform of $[\frac{s}{s^4+16}]$ Answer:$\frac{1}{4}sin\sqrt{2}tsinh\sqrt{2}t$

6. **Home work problem**: $f(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$.
   Answer: $\frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{\frac{5}{2}}} + \frac{6}{s^{\frac{3}{2}}} + \frac{12}{s^{\frac{1}{2}}} - \frac{8}{s^{-\frac{1}{2}}}\right)$

7. **Home work problem**: $sin\sqrt{t}$. Answer: $\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$

Problems on Laplace transform of derivatives

8. Find the Laplace transform of $f(t) = sin^2(t)$ using the differentiation formula $L(f^{'}(t))$.

9. Find the Laplace transform of $f(t) = t^3$ using the differentiation formula $L(f^{'''}(t))$.

10. Solve the initial value problem $y^{''} + 4y = 0$, given that $y(0) = 1, y^{'}(0) = 6$.

11. **Home work problem**: Solve the initial value problem $y^{''} + 2y^{'} - 3y = 3$, given that $y(0) = 4, y^{'}(0) = -7$.

12. **Home work problem**: Solve the initial value problem $y^{''} - 5y^{'} + 4y = e^{2t}$, given that $y(0) = \frac{19}{12}$, $y^{'}(0) = \frac{8}{3}$. Answer: $-\frac{1}{2}e^{2t} + \frac{14}{9}e^t + \frac{19}{36}e^{4t}$

Problems on Laplace transform of integrals

13. Prove that $L\left[\int_0^t \int_0^t \int_0^t cosat\,dt\,dt\,dt\right] = \frac{1}{s^2} \cdot \frac{1}{s^2+a^2}$

14. Find the inverse Laplace transform of $\frac{1}{s(s^2+9)}$

15. Find the inverse Laplace transform of $\frac{1}{s^2(s^2+4)}$

Problems on shifting theorem, i.e., $L(e^{at}f(t)) = F(s-a)$

16. Find the Laplace transform of $e^{at}cosbt$ and $e^{\alpha t}sin(\beta t)$.

17. Find the inverse Laplace transform of $\frac{1}{s^2-4s+8}$; $\frac{4}{s^2-s+2}$; and $\frac{s+6}{s^2+6s+13}$.

18. Solve the initial value problem $y^{''}+4y^{'}+4y = 12t^2e^{-2t}$, given that $y(0) = 2, y^{'}(0) = 1$.

19. **Home work problem**: $e^{2t}(3sin4t - 4cos4t)$. Answer: $\frac{20-4s}{s^2-4s+20}$

20. **Home work problem**: $e^{-4t}\int_0^t \frac{sin3t}{t}dt$. Answer: $\frac{1}{s+4}cot^{-1}\left(\frac{s+4}{3}\right)$

Problems on Unit step function (Heaviside function)

Express the following functions in terms of a unit step function and hence find its Laplace transform

21. $f(t) = \begin{cases} \frac{k}{a}t & \text{for } 0 < t < a \\ \frac{k}{a}(t-a) & \text{for } a < t < 2a \\ \frac{k}{a}(t-2a) & \text{for } 2a < t < 3a \end{cases}$

    Answer: $\frac{k}{as^2} - \frac{ke^{-as}}{s}(1-e^{-as})^{-1}$

22. $f(t) = \begin{cases} cost & \text{for } 0 < t < \pi \\ sint & \text{for } t > \pi \end{cases}$

    Answer: $\frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$

23. **Home work problem**: $f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2 \\ 4t & \text{for } t > 2 \end{cases}$

    Answer: $L(f(t)) = \frac{2}{s^3} + e^{-2s}\left[\frac{4}{s} - \frac{2}{s^3}\right]$

24. $L^{-1}[\frac{se^{-4s}}{s^2-5s+6}]$ Answer: $u(t-4)e^{\frac{5}{2}(t-4)}[cos\frac{h}{2}(t-4) + 5sinh\frac{1}{2}(t-4)]$

Problems on Laplace transform of Dirac-Delta function

25. Prove that $L(\delta(t-a)) = e^{-as}$.

26. Find $L(te^{-2t}\delta(t-2))$. Answer: $2e^{-2(s+2)}$

27. Find $L\left(\frac{\delta(t-a)}{t}\right)$. Answer: $\frac{e^{-as}}{s}$

28. Find the solution of the initial value problem $y^{''} + 2y^{'} + 5y = \delta(t-2)$, given that $y(0) = 0, y^{'}(0) = 0$.

Problems on differentiation of Laplace transform: Multiplication by t^n

29. $t^2 sinat$. Answer: $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

30. $t(3sin2t - 2cos2t)$. Answer: $\frac{-2s^2+12s+8}{(s^2+4)^2}$

31. **Home work problem**: $t^3 cost$. Answer: $\frac{-2s^3+54s}{(s^2+9)^3}$

32. Find the inverse Laplace transform of $\frac{2(s+1)}{(s^2+2s+2)^2}$ and $\frac{1}{(s+5)^4}$

33. Evaluate $\int_0^\infty te^t sintdt$ Answer: $\frac{-1}{2}$

Problems on the integration of Laplace transform: Division by t

34. $\frac{1-e^{-t}}{t}$. Answer: $log\left(\frac{s-1}{s}\right)$

35. $\frac{sin3t \cdot cost}{t}$. Answer: $\frac{1}{2}\left(\pi - tan^{-1}(\frac{s}{4}) - tan^{-1}(\frac{s}{2})\right)$

36. **Home work problem**: $\frac{e^{3t}}{t}$. Answer: Does not exist.

37. Find the inverse Laplace transform of $\frac{s}{(s^2-9)^2}$

38. $L^{-1}[log(1+\frac{1}{s^2})]$. Answer: $\frac{2(1-cost)}{t}$

39. **Home work problem**: $L^{-1}[slog(\frac{s-a}{s+a})]$. Answer: $\frac{-2a}{t}coshat+\frac{2}{t^2}sinhat$

40. Evaluate $\int_0^\infty e^{-2t}[\frac{2sint-3sinht}{t}]dt$. Answer: $2cot^{-1}(2)-\frac{3}{2}log3$

Problems on convolution theorem

41. $F(s)=\frac{1}{(s+1)(s^2+1)}$ Answer: $\frac{1}{2}\left(sint-cost+e^{-t}\right)$

42. $F(s)=\frac{s}{(s^2+1)^2}$ Answer: $\frac{tsint}{2}$

43. $F(s)=\frac{1}{(s+2)^2(s-2)}$ Answer: $\frac{-te^{-2t}}{4}-\frac{1}{16}(e^{-2t}-e^{2t})$

44. **Home work problem**: $F(s)=\frac{1}{(s^2+4)(s+1)^2}$ Answer: $\frac{2}{25}e^{-t}+\frac{te^{-t}}{5}-\frac{1}{50}(3sin2t+4cos2t)$

Problems on periodic functions

45. $f(t)=\begin{cases} cost & \text{for } 0<t\leq\pi \\ -1 & \text{for } \pi\leq t\leq 2\pi \end{cases}$

    Answer: $L(f(t))=\frac{s}{(1+s^2)(1-e^{-\pi s})}-\frac{e^{-\pi s}}{s(1-e^{-\pi s})}$

46. $f(t)=\begin{cases} 1+t & \text{for } 0\leq t<1 \\ 3-t & \text{for } 1\leq t<2 \end{cases}$

    Answer: $L(f(t))=\frac{1}{s}+\frac{1-e^{-s}}{s^2(1+e^{-s})}$

47. **Home work problem**: $f(t)=\begin{cases} sint & \text{for } 0<t<\pi \\ 0 & \text{for } \pi<t<2\pi \end{cases}$

    Answer: $L(f(t))=\frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}$