PES UNIVERSITY ONLINE

B. Tech – II

Department of Science and Humanities

Introduction to Unit Impulse function

- An airplane making a “hard” landing
- A mechanical system being hit by a hammer blow,
- A ship being hit by a single high wave
- A tennis ball being hit by a racket
- These are the phenomena of an impulsive nature where actions of forces are applied over short intervals of time.
- We can model such phenomena by “Dirac’s delta function,” and solve them very effectively by the Laplace transform

Description of Unit Impulse function

To model such situations, we consider the function

$$\delta_{\varepsilon}(t-a) = \begin{cases} \frac{1}{\varepsilon}, & if\ a \le t \le a+\varepsilon \\ 0, & otherwise \end{cases}$$

This function represents, a force of magnitude $1/\varepsilon$ acting from $t = a$ to $t = a + \varepsilon$ where ε is positive and small.

In mechanics, the integral of a force acting over a time interval $a \le t \le a + \varepsilon$ is called the **impulse of the force.** Similarly for electromotive forces *E(t) acting on circuits.*

Definition of Unit Impulse function

The impulse of $\delta_\varepsilon(t-a)$ in (1) is

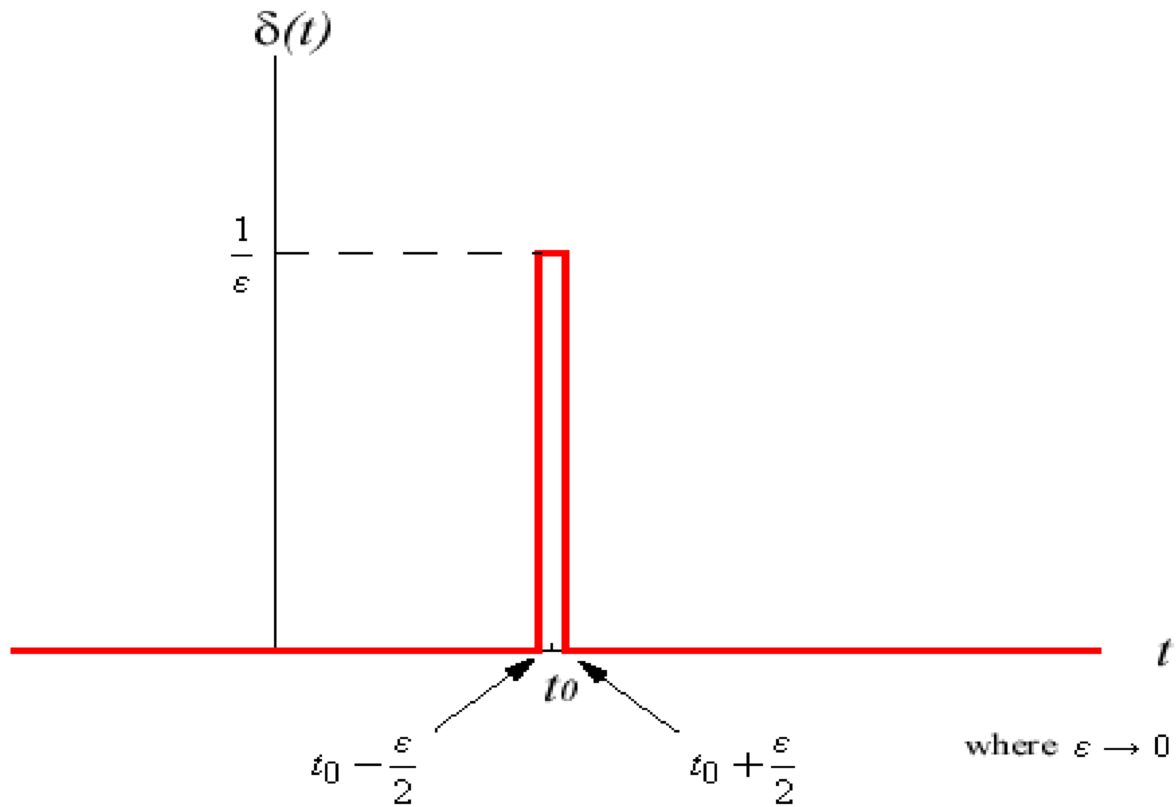
$$I_\varepsilon = \int_0^\infty \delta_\varepsilon(t-a)dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon} dt = 1 \qquad \text{------(2)}$$

- To find out what will happen if ε becomes smaller and smaller,
- Take the limit of δ_ε as $\varepsilon \to 0 (\varepsilon > 0)$
- This limit is denoted by $\delta(t-a)$ that is,

$$\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$$

- $\delta(t-a)$ is called the **Dirac delta function or unit impulse function.**

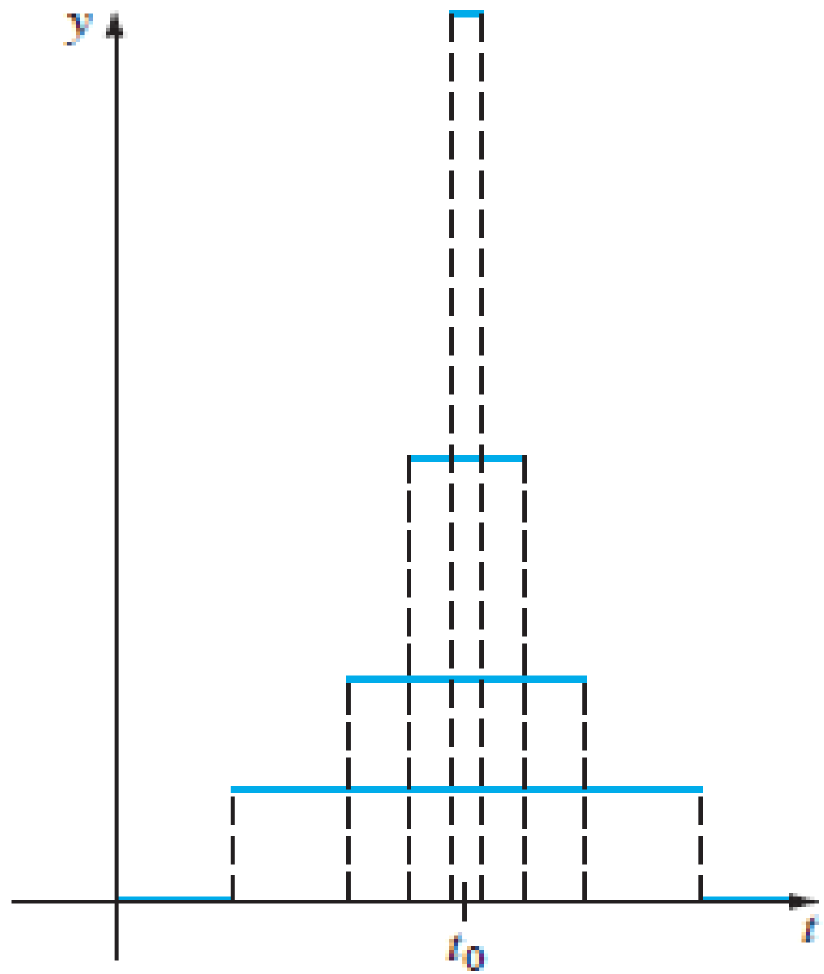
Representation of Unit Impulse function



The function

$$\delta_\varepsilon(t-a)$$

Behavior of δ_ε as $\varepsilon \to 0$



Characterization of Unit Impulse function

- $\delta(t-a)$ is not a function in the ordinary sense as used in calculus.
- It is a so-called generalized function(also called as distributions).
- By the definition

$$\delta(t-a) = \begin{cases} \infty & if\ t = a \\ 0 & otherwise \end{cases}$$

and

$$\int_0^\infty \delta(t-a)\,dt = 1$$

Laplace Transform of Unit Impulse function

we have $\delta_\varepsilon(t-a) = \frac{1}{\varepsilon}[u(t-a) - u(t-(a+\varepsilon))]$

$$L[\delta_\varepsilon(\mathrm{t}-a)] = \frac{1}{\varepsilon}L[u(t-a) - u(t-(a+\varepsilon))]$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right] = \frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}$$

$$L[\delta(t-a)] = L\left[\lim_{\varepsilon\to 0}\delta_\varepsilon(\mathrm{t}-\mathrm{a})\right] = \lim_{\varepsilon\to 0}L[\delta_\varepsilon(\mathrm{t}-\mathrm{a})]$$

$$L[\delta(t-a)] = e^{-as}\lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{\varepsilon s} = e^{-as}\lim_{\varepsilon\to 0}\frac{se^{-\varepsilon s}}{s} = e^{-as}$$

Property of Unit Impulse function

If the function f is integrable and continuous at a then

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

Proof:

$$\int_0^\infty f(t)\delta_\varepsilon(t-a)dt$$

$$= \int_0^\infty f(t)\frac{1}{\varepsilon}[u(t-a)-u(t-a-\varepsilon)]dt$$

$$= \frac{1}{\varepsilon}\int_0^\infty f(t)\frac{1}{\varepsilon}[u(t-a)-u(t-a-\varepsilon)]dt$$

$$= \frac{1}{\varepsilon}\int_0^\infty f(t)u(t-a)dt - \frac{1}{\varepsilon}\int_0^\infty f(t)u(t-a-\varepsilon)\,dt$$

Property of Unit Impulse function...

$$= \frac{1}{\varepsilon}\left[\int_a^\infty f(t)dt - \int_{a+\varepsilon}^\infty f(a)dt\right] = \frac{1}{\varepsilon}\left[F(t)_0^\infty - F(t)_{a+\varepsilon}^\infty\right]$$

$$= \frac{1}{\varepsilon}\left[F(\infty) - F(a) - \left(F(\infty) - F(a+\varepsilon)\right)\right]$$

$$= \frac{1}{\varepsilon}\left[F(a+\varepsilon) - \mathrm{F(a)}\right]$$

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{\varepsilon\to 0}\int_0^\infty f(t)\delta_\varepsilon(t-a)dt$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[F(a+\varepsilon) - \mathrm{F(a)}\right] \quad \text{[by L'Hospital's rule]}$$

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{\varepsilon\to 0} F'(a+\varepsilon) = \lim_{\varepsilon\to 0} f(a+\varepsilon) = f(a)$$

Problems on Unit Impulse function

NOTE

$$L\left[f(t)\delta(t-a)\right]=\int_0^{\infty} e^{-st} f(t)\delta(t-a)dt =\int_0^{\infty} g(t)\delta(t-a)dt = g(a) = e^{-sa} f(a)$$

Find the Laplace transform of

1. $L[\sin 2t\delta(t-2)] = L\left[f(t)\delta(t-2)\right] = e^{-2s} f(2) = e^{-2s} \sin 4$

2. $L[t^n \delta(t-a)] = L\left[f(t)\delta(t-a)\right] = e^{-sa} f(a) = e^{-sa} a^n$

3. $L\left[\dfrac{2\delta(t-3)+3\delta(t-2)}{t}\right] = L\left[\dfrac{2\delta(t-3)}{t}\right] + L\left[\dfrac{3\delta(t-2)}{t}\right] = \dfrac{2e^{-3s}}{3} + \dfrac{3e^{-2s}}{2}$

thanks