PARTIAL DIFFERENTIAL EQUATIONS

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ONE DIMENSIONAL WAVE EQUATION

Suppose x and t are the two independent real variables and y is a real-valued function of x and t. Then the second order linear PDE

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{.....(1)}$$

$\left.\begin{array}{l} y = 0 \text{ when } x = 0 \\ y = 0 \text{ when } x = l \end{array}\right\}$ These should be satisfied for every value of t.

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the initial conditions are

(i) $y = f(x)$ when $t = 0$

(ii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$

Solution: Let $y = XT$, where X is a function of x only and T is a function of t only.

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$\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Putting in (1) , we get

$$XT'' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \text{ (Say)}$$

$X'' - kX = 0$ and $T'' - kc^2 T = 0$.....(2)

solving (2), we obtain

(i) when k is positive and is equal to P^2, (say)

$$X = c_1 e^{px} + c_2 e^{-px} \;\&\; T = c_3 e^{cpt} + c_4 e^{-cpt}$$

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(ii) when k is negative and is equal to $-P^2$, (say)

$$X = c_1 \cos px + c_2 \sin px \,\&\, T = c_3 \cos cpt + c_4 \sin cpt$$

(iii) when $k = 0$

$$X = c_1 x + c_2 \,\&\, T = c_3 t + c_4$$

Thus the various possible solution of the wave equation (1) are:

$$y = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{cpt} + c_4 e^{-cpt}\right)$$

$$y = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos cpt + c_4 \sin cpt\right)$$

$$y = \left(c_1 x + c_2\right)\left(c_3 t + c_4\right)$$

THANK YOU

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