Higher Order Differential Equations

Assignment-2

Solve the LDEs:

1. $(D^3 - D^2 - 12D)y = 0.$

The auxillary equation is $m^3 - m^2 - 12m = 0.$

The roots are: $m_1 = 4$, $m_2 = -3$, $m_3 = 0$.

The general solution is given by,

$$\underline{y = C_1 e^{4x} + C_2 e^{-3x} + C_3}$$

2. $(D^4 + 2k^2D^2 + k^4)y = 0.$

The auxillary eqn is $m^4 + 2k^2m^2 + k^4 = 0.$

$\Rightarrow m^4 + k^2m^2 + k^2m^2 + k^4 = 0$

$\Rightarrow m^2(m^2 + k^2) + k^2(m^2 + k^2) = 0.$

$\Rightarrow (m^2 + k^2)^2 = 0$

$\Rightarrow m = \pm ki$

The general solution is given by,

$$\underline{y = \overset{(C_1+C_2x)}{\cancel{C_1}} \cos kx + \overset{(C_3+C_4x)}{\cancel{C_2}} \sin kx}$$

3. $(D^3 + 2D^2 - 5D - 6)y = 0$

The auxillary eqn is $m^3 + 2m^2 - 5m - 6 = 0.$

The roots are $m_1 = 2$, $m_2 = -1$, $m_3 = -3$,

The general solution is,

$$\underline{y = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-3x}}$$

4. $(D^4 + 101D^2 + 25)y = 0$

The auxillary equation is ;

$$m^4 + 101m^2 + 25 = 0$$

The roots are : $m_1 = 10.037i$
$m_2 = 0.498i$
$m_3 = -0.498i$
$m_4 = -10.037i$

The general solution is ,

$$y = C_1 \cos 10.037x + C_2 \sin 10.037x + C_3 \cos 0.498x + C_4 \sin 0.498x$$

5. $(D^3 - 2D^2 - 4D + 8)y = 0$.

The auxillary equation is $m^3 - 2m^2 - 4m + 8 = 0$

The roots are : $m_1 = -2$, $m_2 = 2$, $m_3 = 2$

The general solution is ,

$$y = (C_1 + C_2 x)e^{2x} + C_3 e^{-2x}$$

Assignment - 3

Solve the LDEs :

1. $(D^2 - 5D + 6)y = e^{4x}$

The corresponding homogeneous eqn is $(D^2 - 5D + 6)y = 0$

The auxillary eqn is $m^2 - 5m + 6 = 0$.

$\Rightarrow m_1 = 3, \; m_2 = 2$

$CF = C_1 e^{3x} + C_2 e^{2x}$

To find PI,

$$PI = \frac{1}{f(D)} \times = \left(\frac{1}{D^2-5D+6}\right)e^{4x} = \frac{1}{2}e^{4x}$$
$(D \to 4)$

The general solution is,

$GS = CF + PI$,

$$\underline{\underline{y = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{2}e^{4x}}}$$

2. $(D^2+4D+5)y = -2\cosh x$.
Also find y when $y=0$, $\frac{dy}{dx} = 1$ at $x=0$.

The corresponding homogeneous eqn is
$$(D^2+4D+5)y = 0.$$
The auxiliary eqn is, $m^2+4m+5=0$.
$$\rightarrow m_1 = -2+i, \; m_2 = -2-i$$

$CF = e^{-2x}(C_1\cos x + C_2 \sin x)$.

To find PI,

$$PI = \left(\frac{1}{D^2+4D+5}\right)(-2\cosh x)$$

$$= -\cancel{2}\left[\frac{e^x+e^{-x}}{\cancel{2}}\right]\left(\frac{1}{D^2+4D+5}\right)$$

$$= -e^x\left(\frac{1}{D^2+4D+5}\right)_{(D\to 1)} - e^{-x}\left(\frac{1}{D^2+4D+5}\right)_{(D\to -1)}$$

$$= -e^x\left(\frac{1}{10}\right) - e^{-x}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

The general solution is,

$GS = CF + PI,$

$$y = e^{-2x}(C_1 \cos x + C_2 \sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x} \quad \text{①}$$

At $x=0,\ y=0,$

$$0 = 1[C_1 + 0] - \frac{1}{10} - \frac{1}{2}$$

$$C_1 = \frac{+1+5}{10} = \frac{3}{5}$$

Diff ①,

$$y' = -2e^{-2x}(C_1 \cos x + C_2 \sin x) - \frac{1}{10}e^{x} + \frac{1}{2}e^{-x}$$
$$+ e^{-2x}(-C_1 \sin x + C_2 \cos x)$$

At $x=0,\ y'=1,$

$$1 = -2[C_1+0] - \frac{1}{10} + \frac{1}{2} + 1[0 + C_2]$$

$$1 = -2C_1 - \frac{1}{10} + \frac{1}{2} + C_2$$

$$C_2 = \frac{1}{10} - \frac{1}{2} + 2\left(\frac{3}{5}\right) + 1$$

$$C_2 = \frac{1-5+12}{10} = \frac{4}{5} + 1 = \frac{9}{5}$$

Putting in ①,

$$y = \frac{3}{5}e^{-2x}(\cos x + 3\sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x}$$

3. $(D^3 + 2D^2 + D)y = e^{-x}$

The corresponding homogeneous is,

$$D^3 + 2D^2 + D = 0.$$

The auxillary is $m^3 + 2m^2 + m = 0.$

$$m(m^2 + 2m + 1) = 0$$

$$m(m+1)^2 = 0.$$

$$m_1 = 0,\ m_2 = m_3 = -1$$

$CF = C_1 e^{0x} + (C_2 + C_3 x)e^{-x}$

$\Rightarrow CF = C_1 + (C_2 + C_3 x)e^{-x}$

To find PI,

$$PI = \left(\frac{1}{D^3 + 2D^2 + D}\right)_{(D\to -1)} e^{-x}$$

$$= x\left(\frac{1}{3D^2 + 4D + 1}\right)_{(D\to -1)} e^{-x}$$

$$= x^2\left(\frac{1}{6D + 4}\right)_{(D\to -1)} e^{-x}$$

$$= -\frac{x^2 e^{-x}}{2}$$

The general solution is,

$GS = CF + PI$

$y = C_1 + (C_2 + C_3 x)e^{-x} - \frac{x^2 e^{-x}}{2}$

4. $(D^2+4D+13)\,y = 2e^{-x}$,

given that $y(0)=0$ and $y'(0)=-1$

The corresponding homogeneous ~~so~~ equation,

$$D^2+4D+13=0.$$

The auxillary equation is $m^2+4m+13=0$

$$m_1=-2+3i \quad , \quad m_2=-2-3i$$

$CF = e^{-2x}(C_1\cos 3x + C_2 \sin 3x)$

To find PI,

$$PI = \left(\frac{1}{D^2+4D+13}\right)(2e^{-x})$$

$$= 2\left[\frac{1}{D^2+4D+13}\right]_{(D\to -1)} e^{-x} = \frac{1}{5}e^{-x}$$

The general solution is, $GS = CF + PI$

$$y = e^{-2x}(C_1\cos 3x + C_2\sin 3x) + \frac{1}{5}e^{-x} \rightarrow (1)$$

Given, $y=0$ when $x=0$,

$$0 = 1(C_1+0)+\frac{1}{5} \Rightarrow C_1 = -\frac{1}{5}$$

Differentiating (1) wrt x,

$$y' = -2e^{-2x}(C_1\cos 3x + C_2\sin 3x) + e^{-2x}(-3C_1\sin 3x + 3C_2\cos 3x) - \frac{1}{5}e^{-x}$$

Given $y'=-1$ when $x=0$.

$$-1 = -2(C_1+0) + 1(0+3C_2) - \frac{1}{5}$$

$$\Rightarrow -1 = \frac{2}{5} - \frac{1}{5} + 3C_2 \Rightarrow C_2 = -\frac{2}{5}$$

Putting in (1),

$$y = -\frac{1}{5}e^{-2x}(\cos 3x + 2\sin 3x) + \frac{1}{5}e^{-x}$$

5. $(D^2-7D+10)y = (1+e^x)^2$

The corresponding homogeneous equation is,

$$D^2-7D+10=0.$$

The auxillary eqn is, $m^2-7m+10=0$

$$m_1 = \frac{7+\sqrt{31}\,i}{4}, \quad m_2 = \frac{7-\sqrt{31}\,i}{4}$$

$$CF = e^{\frac{7}{4}x}\left(C_1 \cos\frac{\sqrt{31}}{4}x + C_2 \sin\frac{\sqrt{31}}{4}x\right)$$

To find PI,

$$PI = \left(\frac{1}{D^2-7D+10}\right)(1+e^x)^2$$

$$= \left(\frac{1}{D^2-7D+10}\right)(1+e^{2x}+2e^x)$$

$$= \underset{(D\to 0)}{\left(\frac{1}{D^2-7D+10}\right)e^{0x}} + \underset{(D\to 2)}{\left(\frac{1}{D^2-7D+10}\right)e^{2x}} + 2\underset{(D\to 1)}{\left(\frac{1}{D^2-7D+10}\right)e^x}$$

$$= \frac{1}{10}e^{0x} + \underset{(D\to 2)}{\left(\frac{x}{2D-7}\right)}e^{2x} + \frac{1}{4}(2e^x)$$

$$= \frac{1}{10} + \frac{1}{2}e^x - \frac{x}{3}e^{2x}$$

The general solution is,

GS = CF + PI

$$y = e^{\frac{7}{4}x}\left(C_1 \cos\frac{\sqrt{31}}{4}x + C_2 \sin\frac{\sqrt{31}}{4}x\right) + \frac{1}{10} + \frac{1}{2}e^x - \frac{x}{3}e^{2x}$$

Assignment - 4

1. Solve $(D^2-5D+6)y = e^{3x} + \sin(2x+1)$

The corresponding homogeneous eqn is,

$$D^2 - 5D + 6 = 0.$$

The auxillary eqn is $m^2 - 5m + 6 = 0$

$$m_1 = 3, \ m_2 = 2$$

$$CF = C_1 e^{3x} + C_2 e^{2x}$$

To find PI,

$$PI = \frac{1}{f(D)} X(x)$$

$$= \left(\frac{1}{D^2-5D+6}\right) e^{3x} + \sin(2x+1)$$

$$= \left[\frac{1}{D^2-5D+6}\right]_{(D \to 3)} e^{3x} + \left[\frac{1}{D^2-5D+6}\right]_{(D^2 \to -4)} \sin(2x+1)$$

$$= x\left[\frac{1}{2D-5}\right]_{(D \to 3)} e^{3x} + \left[\frac{1}{2-5D}\right] \sin(2x+1)$$

$$= \left(\frac{xe^{3x}}{1}\right) + \left[\frac{2+5D}{4-25D^2}\right]_{(D^2 \to -4)} \sin(2x+1)$$

$$= xe^{3x} + \frac{1}{104}\left[2\sin(2x+1) + 10\cos(2x+1)\right]$$

The general solution is given by,

GS = CF + PI

$$y = C_1 e^{3x} + C_2 e^{2x} + xe^{3x} + \frac{1}{104}\left[2\sin(2x+1) + 10\cos(2x\right.$$

2. $(D^2+3D+2)y = 4\cos^2 x$

The auxillary eqn is $m^2+3m+2=0$

$$m_1 = -1, \quad m_2 = -2$$

$CF = C_1 e^{-x} + C_2 e^{-2x}$

To find PI,

$$PI = \left[\frac{1}{D^2+3D+2}\right] 4\cos^2 x$$

$$= \left[\frac{1}{D^2+3D+2}\right]\left[4\left(\frac{\cos 2x+1}{2}\right)\right]$$

$$= \left[\frac{1}{D^2+3D+2}\right] 2\cos 2x + \left[\frac{1}{D^2+3D+2}\right] 2e^{0x}$$

$(D^2 \to -4)$ $\qquad (D \to 0)$

$$= \left[\frac{1}{3D-2}\right] 2\cos 2x + e^{0x}$$

$$= \left[\frac{3D+2}{9D^2-4}\right] 2\cos 2x + 1$$

$(D^2 \to -4)$

$$= \frac{-1}{40}\left[-12\sin 2x + 4\cos 2x\right] + 1$$

$$= -\frac{1}{10}\left[-3\sin 2x + \cos 2x\right] + 1$$

$$= \frac{1}{10}\left[\cos 2x + 3\sin 2x\right] + 1$$

The general solution is,

$GS = CF + PF$

$$y = C_1 e^{-x} + C_2 e^{-2x} + 1 + \frac{1}{10}\left[3\sin 2x - \cos 2x\right]$$

3. $(D^2-4D+3)y = \sin 3x \cos 2x$

The auxillary equation is $m^2-4m+3=0$

$$m_1 = 3, \quad m_2 = 1$$

$CF = C_1 e^{3x} + C_2 e^{x}$

To find PI,

$$PI = \left[\frac{1}{D^2-4D+3}\right] \sin 3x \cos 2x$$

$$= \left[\frac{1}{D^2-4D+3}\right]\left[\frac{1}{2}(\sin 5x + \sin x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\right]\sin 5x + \frac{1}{2}\left[\frac{1}{D^2-4D+3}\right]\sin x$$

$$(D^2 \to -25) \qquad (D^2 \to -1)$$

$$= \frac{1}{2}\left[\frac{1}{-4D-22}\right]\sin 5x + \frac{1}{2}\left[\frac{1}{2-4D}\right]\sin x$$

$$= -\frac{1}{4}\left[\frac{2D-11}{4D^2-121}\right]\sin 5x + \frac{1}{4}\left[\frac{1+2D}{1-4D^2}\right]\sin x$$

$$(D^2 \to -25) \qquad (D^2 \to -1)$$

$$= -\frac{1}{4}\left[\frac{10\cos 5x - 11\sin 5x}{-221}\right] + \frac{1}{4}\left[\frac{\sin x + 2\cos x}{5}\right]$$

$$= \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}(\sin x + 2\cos x)$$

The general solution is,

$GS = CF + PI$

$$y = C_1 e^{3x} + C_2 e^{x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[\sin x + 2\cos x]$$

4. $(D^3+2D^2+D)y = e^{-2x} + \sin 2x$

The auxillary eqn is $m^3+2m^2+m=0$.

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m(m+1)^2=0$$

$$m_1=0,\ m_2=m_3=-1.$$

$$CF = C_1 + (C_2+C_3x)e^{-x}$$

To find PI,

$$PI = \left[\frac{1}{D^3+2D^2+D}\right]_{(D\to -2)} e^{-2x} + \left[\frac{1}{D^3+2D^2+D}\right]_{(D^2\to -4)} \sin 2x$$

$$= -\frac{1}{2}e^{-2x} + \left[\frac{1}{-4D-8+D}\right]\sin 2x$$

$$= -\frac{1}{2}e^{-2x} - \left[\frac{3D-8}{9D^2-64}\right]_{(D^2\to -4)}\sin 2x$$

$$= -\frac{1}{2}e^{-2x} + \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

$$= -\frac{1}{2}e^{-2x} + \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

The general solution is,

$$y = C_1 + (C_2+C_3x)e^{-x} - \frac{1}{2}e^{-2x} + \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

5. $(D^2+2D+1)y = e^{2x} - \cos^2 x$

The auxillary eqn is $m^2+2m+1=0$.

$$m_1 = m_2 = -1$$

$CF = (C_1 + C_2 x)e^{-x}$

To find PI,

$$PI = \left[\frac{1}{D^2+2D+1}\right]\left[e^{2x} - \left(\frac{1+\cos 2x}{2}\right)\right]$$

$$= \left[\frac{1}{D^2+2D+1}\right]e^{2x} - \frac{1}{2}\left[\frac{1}{D^2+2D+1}\right]e^{0x} - \frac{1}{2}\left[\frac{1}{D^2+2D+1}\right]\cos 2x$$

$(D \to 2)$ $(D \to 0)$ $(D^2 \to -4)$

$$= \frac{e^{2x}}{9} - \frac{1}{2} - \frac{1}{2}\left[\frac{1}{2D-3}\right]\cos 2x$$

$$= \frac{e^{2x}}{9} - \frac{1}{2} - \frac{1}{2}\left[\frac{2D+3}{4D^2-9}\right]\cos 2x$$

$(D^2 \to -4)$

$$= \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{50}[-4\sin 2x + 3\cos 2x]$$

The general solution is,

$$y = (C_1 + C_2 x)e^{-x} + \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{50}[3\cos 2x - 4\sin 2x]$$

<u>Assignment - 5</u>

1. Solve $(D^2+3D+2)y = x^2$

The auxillary eqn is $m^2+3m+2=0$.

$$m_1 = -1, \; m_2 = -2$$

$CF = C_1 e^{-x} + C_2 e^{-2x}$

To find PI,

$$PI = \left[\frac{1}{D^2+3D+2}\right] x^2$$

$$\begin{array}{r|l} & \frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4} \\ \hline 2+3D+D^2 & \cancel{x^2} \\ & \overset{(-)}{\cancel{x^2}} \overset{(-)}{+} 3x \overset{(-)}{+} 1 \\ \hline & -\cancel{3x} - 1 \\ & \overset{(+)}{-}\cancel{3x} \overset{(+)}{-} \frac{9}{2} + 0 \\ \hline & \frac{7}{2} \\ & \overset{(-)}{} \frac{7}{2} \\ \hline & 0 \end{array}$$

$$\text{Quotient} = PI = \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

The general solution is,

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

2. $(D^2+4D+4)y = x^3$

The auxillary equation is $m^2+4m+4=0$.

$$m_1 = m_2 = -2$$

$$CF = (C_1 + C_2 x)e^{-2x}$$

To find PI,

$$\begin{array}{r|l} & \frac{x^3}{4} - \frac{3x^2}{4} + \frac{9}{8}x - \frac{3}{4} \\ \hline 4+4D+D^2 & \cancel{x^3} \\ & \overset{(-)}{\cancel{x^3}} \overset{(-)}{+} 3x^2 \overset{(-)}{+} \frac{6x}{4} \\ \hline & -\cancel{3x^2} - \frac{6}{4}x \\ & \overset{(+)}{-}\cancel{3x^2} \overset{(+)}{-} 6x \overset{(+)}{-} \frac{3}{2} \\ \hline & \cancel{\frac{9}{2}x} + \frac{3}{2} \\ & \overset{(-)}{\cancel{\frac{9}{2}x}} \overset{(-)}{+} \frac{9}{2} \\ \hline & -3 \end{array}$$

The general solution is,

$$y = (C_1 + C_2 x)e^{-2x} + \frac{x^3}{4} - \frac{3x^2}{4} + \frac{9}{8}x - \frac{3}{4}$$

3. $(D^2+3D+2)y = 3x+4$

The auxillary eqn is $m^2+3m+2=0$

$$m_1 = -1, \ m_2 = -2$$

$$CF = C_1 e^{-x} + C_2 e^{-2x}$$

To find PI,

$$\begin{array}{r|l} & \frac{3x}{2} - \frac{1}{4} \\ \hline 2+3D+D^2 & 3x + 4 \\ & \underline{\cancel{(-)3x} \ \overset{(-)}{+} \frac{9}{2} + 0} \\ & \quad -\cancel{\frac{1}{2}} \\ & \quad \underline{(+)\cancel{\frac{1}{2}}} \\ & \quad 0 \end{array}$$

The general solution is,

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{3x}{2} - \frac{1}{4}$$

4. $(D^3-6D^2+5D)y = 5+x^2$

The auxillary eqn is $m^3-6m^2+5m=0$

$$\Rightarrow m(m^2-6m+5)=0.$$

$$m_1 = 0, \ m_2 = 5, \ m_3 = 1$$

$$CF = C_1 + C_2 e^{5x} + C_3 e^{x}$$

To find PI,

$$PI = \left(\frac{1}{D^3-6D^2+5D}\right)(5+x^2) = \left[\frac{1}{D}\right]\left[\frac{1}{D^2-6D+5}\right](5+x^2)$$

$$\frac{x^2}{5} + \frac{37}{25}x + \frac{212}{125}$$

$$5 - 6D + D^2 \;\Big)\; x^2 + 5$$

$$\overset{(-)}{\cancel{x^2}} \overset{(+)}{-} \frac{12x}{5} \overset{(-)}{+} \frac{2}{5}$$

$$\frac{37}{5}x - \frac{2}{5}$$

$$\overset{(-)}{\cancel{\frac{37}{5}x}} \overset{(+)}{-} \frac{222}{25}$$

$$\frac{212}{25}$$

$$\overset{(-)}{\cancel{\frac{212}{25}}}$$

$$0$$

$$\text{Quotient} = PI_1 = \frac{x^2}{5} + \frac{37x}{5} + \frac{212}{125}$$

$$PI = \frac{1}{D}\left[\frac{x^2}{5} + \frac{37x}{5} + \frac{212}{125}\right]$$

5. $(D^2+1)y = e^{x} + x^4$

The auxillary eqn is $m^2+1=0$

$$m = \pm i$$

$CF = C_1 \cos x + C_2 \sin x$

To find PI,

$$PI = \left[\frac{1}{D^2+1}\right]e^{x} + \left[\frac{1}{D^2+1}\right]x^4$$

$(D \to 1)$

$$= \frac{1}{2}e^{x} + \left(\frac{1}{D^2+1}\right)x^4$$

$$x^4 - 12x^2 + 24$$

$1+D^2 \,\big)\; x^4$

$\quad \cancel{x^4} + 12x^2$

$\quad -12x^2$

$\quad \cancel{-12x^2} - 24$

$\quad 24$

$\quad 24$

$\quad 0$

The general solution is,

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^{x} + x^4 - 12x^2 + 24$$

Assignment - 6

1. Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^{x}$

The auxillary eqn is $m^3 - 3m^2 + 3m - 1 = 0$

$$m_1 = 1,\ m_2 = 1,\ m_3 = 1$$

$CF = (C_1 + C_2 x + C_3 x^2)e^{x}$

To find PI,

$$PI = \left[\frac{1}{D^3-3D^2+3D-1}\right]x^2e^x$$

$$= e^x\left[\frac{1}{(D+1)^3-3(D+1)^2+3(D+1)-1}\right]x^2$$

$$= e^x\left[\frac{1}{D^3+\cancel{1}+3D^2+3D-3(D^2+1+2D)+3D+3-\cancel{1}}\right]x^2$$

$$= e^x\left[\frac{1}{D^3+\cancel{3D^2}+\cancel{3D}-\cancel{3D^2}-\cancel{3}-\cancel{6D}+\cancel{3D}+\cancel{3}}\right]x^2$$

$$= e^x\left(\frac{1}{D^3}\right)x^2$$

$$= e^x\left(\frac{1}{D^2}\right)\left(\frac{x^3}{3}\right)$$

$$= e^x\left(\frac{1}{D}\right)\left(\frac{x^4}{12}\right) = e^x\left(\frac{x^5}{60}\right)$$

The general solution is given by,

$$y = (C_1+C_2x+C_3x^2)e^x + e^x\left(\frac{x^5}{60}\right)$$

2. $(D^2-2D+1)y = xe^x$

The auxillary eqn is $m^2-2m+1=0$

$m_1 = m_2 = 1$

$CF = (C_1+C_2x)e^x$

To find PI,

$$PI = \left(\frac{1}{D^2-2D+1}\right)xe^x$$

$$= e^x\left[\frac{1}{(D+1)^2-2(D+1)+1}\right]x$$

$$= e^x \left[\frac{1}{D^2+\cancel{1}+\cancel{2D}-\cancel{2D}-\cancel{1}+1-1}\right] x$$

$$= e^x \left[\frac{1}{D^2+\cancel{1-1}}\right] x$$

$$= e^x \left(\frac{x^3}{6}\right)$$

The general solution is given by,

$$y = (C_1 + C_2 x)e^x + \left(\frac{1}{6}\right) e^x x^3$$

3. $(D^2 - 2D + 4)y = e^x \cos x$

The auxillary eqn is $m^2 - 2m + 4 = 0$

$$m = 1 \pm \sqrt{3} i$$

$$CF = e^x (C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x)$$

To find PI,

$$PI = \left[\frac{1}{D^2 - 2D + 4}\right] e^x \cos x$$

$$= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 4}\right] \cos x$$

$$= e^x \left[\frac{1}{D^2 + 1 + \cancel{2D} - \cancel{2D} - 2 + 4}\right] \cos x$$

$$= e^x \left(\frac{1}{D^2+3}\right) \cos x$$

$(D^2 \to -1)$

$$= \frac{1}{2} e^x \cos x$$

The general solution is,

$$y = e^x(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x) + \left(\frac{1}{2}\right) e^x \cos x$$

$(D^3 - 2D + 4)y = e^x \sin\left(\frac{x}{2}\right)$

The auxillary eqn is $m^3 - 2m + 4 = 0$

$m_1 = -2, \quad m_2 = 1+i, \quad m_3 = 1-i$

$CF = C_1 e^{-2x} + e^x(C_2 \cos x + C_3 \sin x)$

To find PI,

$$PI = \left[\frac{1}{D^3 - 2D + 4}\right] e^x \sin\left(\frac{x}{2}\right)$$

$$= e^x \left[\frac{1}{(D+1)^3 - 2(D+1) + 4}\right] \sin\left(\frac{x}{2}\right)$$

$$= e^x \left[\frac{1}{D^3 + 1 + 3D^2 + 3D - 2D - 2 + 4}\right] \sin\left(\frac{x}{2}\right)$$

$$= e^x \left[\frac{1}{D^3 + 3D^2 + D + 3}\right] \sin\left(\frac{x}{2}\right)$$

$\left(D^2 \to -\frac{1}{4}\right)$

$$= e^x \left[\frac{1}{-\frac{1}{4}D + 3\left(-\frac{1}{4}\right) + D + 3}\right] \sin\left(\frac{x}{2}\right)$$

$$= e^x \left[\frac{1}{\frac{3}{4}D + \frac{9}{4}}\right] \times \sin\left(\frac{x}{2}\right)$$

$$= \frac{4e^x}{3}\left(\frac{D-3}{D^2 - 9}\right) \sin\left(\frac{x}{2}\right)$$

$\left(D^2 \to -\frac{1}{4}\right)$

$$= \frac{4e^x}{3}\left(\frac{D-3}{\left\{\frac{-37}{4}\right\}}\right) \sin\left(\frac{x}{2}\right)$$

$$= \frac{-16e^x}{111}\left[\frac{1}{2}\cos\left(\frac{x}{2}\right) - 3\sin\left(\frac{x}{2}\right)\right]$$

The general solution is,

$$y = C_1 e^{-2x} + (C_2 \cos x + C_3 \sin x)e^{x} - \frac{16e^{x}}{111}\left[\frac{1}{2}\cos\left(\frac{x}{2}\right) - 3\sin\left(\frac{x}{2}\right)\right]$$

5. $(D^3+8)y = e^{-2x}x^2$

The auxillary eqn is $m^3+8=0$

$$m_1 = -2, \quad m_2 = 1+\sqrt{3}i, \quad m_3 = 1-\sqrt{3}i$$

$$CF = C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x)$$

To find PI,

$$PI = \left(\frac{1}{D^3+8}\right)e^{-2x}x^2$$

$$= e^{-2x}\left[\frac{1}{(D-2)^3+8}\right]x^2$$

$$= e^{-2x}\left[\frac{1}{D^3+12D-6D^2\cancel{-8}\cancel{+8}}\right]x^2$$

$$= \frac{e^{-2x}}{12D}\left[\frac{1}{1-\frac{1}{2}D+\frac{1}{12}D^2}\right]x^2$$

$$= \frac{e^{-2x}}{12D}\left[1-\left(\frac{1}{2}D-\frac{1}{12}D^2\right)\right]^{-1}x^2$$

$$= \frac{e^{-2x}}{12D}\left[1+\left(\frac{1}{2}D-\frac{1}{12}D^2\right)+\left(\frac{1}{2}D-\frac{1}{12}D^2\right)^2\right]x^2$$

$$= \frac{e^{-2x}}{12D}\left[x^2+\frac{1}{2}(2x)-\frac{1}{12}(2)+\frac{1}{4}(2)+0+0\right]$$

$$= \frac{e^{-2x}}{12D}\left[x^2+x+\frac{1}{3}\right]$$

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$$= \frac{e^{-2x}}{12}\left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{3}x\right]$$

$$= \frac{e^{-2x}}{12}\left[\frac{2x^3+3x^2+2x}{6}\right] \times \frac{2}{2}$$

$$= \frac{e^{-2x}}{144}\left[4x^3+6x^2+4x\right]$$

The general solution is,

$$y = C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x) + \frac{e^{-2x}}{144}\left[4x^3+6x^2+4x\right]$$

Cauchy's Linear Differential Equation

Any n^{th} order Cauchy's LDE is of the form

$$x^n\left(\frac{d^n y}{dx^n}\right) + k_1 x^{n-1}\left(\frac{d^{n-1} y}{dx^{n-1}}\right) + \dots + k_n y = X$$

where k's are constants and X is a function of x.

Cauchy's LDE is a special case of Legendre's LDE in which $a = 1$ and $b = 0$.

Such eqns can be reduced to LDE with constant coefficients by substitution.

$$x = e^{z} \Rightarrow z = \log x$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) \Rightarrow x\left(\frac{dy}{dx}\right) = Dy \quad \left(\text{where } D = \frac{d}{dz}\right)$$

IIIly, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

Assignment - 7

1. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 2\cos^2(\log x)$

Put $x = e^z \Rightarrow z = \log x$

As $D = \frac{d}{dz}$,

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$$D(D-1)y + Dy + y = \cos(2\log x) + 1.$$

$$(D^2 - D)y + Dy + y = \cos(2z) + 1$$

$$(D^2+1)y = \cos(2z) + 1.$$

The corresponding homogeneous equation is,

$$D^2+1 = 0.$$

The auxillary equation is $m^2+1=0$.

$$m = \pm i$$

$CF = C_1\cos z + C_2\sin z$

To find PI,

$$PI = \left(\frac{1}{D^2+1}\right)(\cos 2z + 1)$$

$$= \underbrace{\left[\frac{1}{D^2+1}\right]}_{(D^2\to -4)}\cos 2z + \underbrace{\left[\frac{1}{D^2+1}\right]}_{(D\to 0)} e^{0z}$$

$= -\frac{1}{3}\cos 2z + 1$

The general solution is,

$y = C_1 \cos z + C_2 \sin z - \frac{1}{3}\cos 2z + 1$

$\Rightarrow y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{1}{3}\cos(2\log x) + 1$

Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \log x + \cos(\log x)$

Put $x = e^z \Rightarrow z = \log x$

As $D = \frac{d}{dz}$,

$\rightarrow x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\rightarrow x\frac{dy}{dx} = Dy$

$D(D-1)y - Dy + 4y = z + \cos z$

$\rightarrow (D^2 - D)y - Dy + 4y = z + \cos z$

$\rightarrow (D^2 - 2D + 4)y = z + \cos z$

The auxillary eqn is $m^2 - 2m + 4 = 0$

$m_1 = 1 + \sqrt{3}i$

$m_2 = 1 - \sqrt{3}i$

$CF = e^z(C_1 \cos\sqrt{3}z + C_2 \cos\sqrt{3}z)$

To find PI,

$PI = \left[\frac{1}{D^2 - 2D + 4}\right] z + \left[\frac{1}{D^2 - 2D + 4}\right] \cos z$

$(D^2 \to -1)$

division method

$= \frac{1}{4}\left(z + \frac{1}{2}\right)\left[\frac{3+2D}{9-4D^2}\right] \cos z$

$(D^2 \to -1)$

$$= \frac{1}{4} z + \frac{1}{13}[3\cos z - 2\sin z]$$

The general solution is,

$$y = x[C_1 \cos(\sqrt{3}\log x) + C_2 \sin(\sqrt{3}\log x)] + \frac{1}{4}\left(\log x + \frac{1}{2}\right)$$

$$+ \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)]$$

3. $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$

The LDE becomes,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$$

Let $x = e^z \Rightarrow z = \log x$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$$\Rightarrow D(D-1)y + Dy + y = z(\sin z)$$

$$\Rightarrow (D^2 + 1)y = z(\sin z)$$

The auxillary eqn is $m^2 + 1 = 0$

$m = \pm i$

$$CF = C_1 \cos z + C_2 \sin z$$

To find PI,

$$PI = \left[\frac{1}{D^2+1}\right] z \sin z$$

$$= \left[z - \left(\frac{2D}{D^2+1}\right)\right]\left[\frac{\sin z}{D^2+1}\right]$$

$(D^2 \to -1)$

$$= \cancel{z - \left(\frac{2D}{}\right.}$$

$$= z\left(\frac{\sin z}{D^2+1}\right) - \left(\frac{2D}{D^2+1}\right)\left(\frac{\sin z}{D^2+1}\right)$$

$$= z^2\left(\frac{\sin z}{2D}\right) - \frac{\cancel{2D}(\sin z)\, z}{\cancel{2}(D^2+1)(2\cancel{D})}$$

?

$$= z^2\left(\frac{\sin z}{2D}\right) - \frac{1}{2}\left[\frac{z(\sin z)}{D^2+1}\right]$$

$$= z^2\left(\frac{\sin z}{2D}\right) - \frac{1}{2}\left[\frac{z^2 \sin z}{2D}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 \sin z}{2D}\right]$$

$$= \frac{1}{4} z^2(-\cos z)$$

$$= -\frac{1}{4}(\log x)^2 \cos(\log x).$$

4. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2\sin(\log x)$.

Let $x = e^z \Rightarrow z = \log x$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$$D(D-1)y - 3Dy + 5y = e^{2z}\sin z$$

$$\Rightarrow D^2y - Dy - 3Dy + 5y = e^{2z}\sin z$$

$$\Rightarrow (D^2 - 4D + 5)y = e^{2z}\sin z$$

The auxillary equation is $m^2 - 4m + 5 = 0$.

$$m = 2 \pm i$$

$$CF = e^{2z}[C_1\cos z + C_2\sin z]$$

To find PI,

$$PI = \left[\frac{1}{D^2-4D+5}\right]e^{2z}\sin z$$

$$= \left[\frac{1}{(D+2)^2-4(D+2)+5}\right]e^{2z}\sin z$$

$$= e^{2z}\left[\frac{1}{D^2+4+\cancel{4D}-\cancel{4D}-8+5}\right]\sin z$$

$$= e^{2z}\left[\frac{1}{D^2+1}\right]\sin z$$

$(D^2 \to -1)$

$$= e^{2z}z\left(\frac{1}{2D}\right)\sin z$$

$$= e^{2z}z\left(\frac{1}{2}\right)(-\cos z)$$

$$= -\frac{1}{2}e^{2z}z\cos z$$

The general solution is given by,

$$y = e^{2z}(C_1 \cos \cancel{z} + C_2 \sin \cancel{z}) - \frac{1}{2} z e^{2z} \cos z$$

$$y = x^2[C_1 \cos(\log x) + C_2 \sin(\log x)]$$
$$- \frac{1}{2} x^2 (\log x).\cos(\log x)$$

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 + \frac{1}{x}$$

Let $x = e^z$, $z = \log x$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$$\Rightarrow D(D-1)y - 3Dy + 4y = 2e^{2z} + e^{-z}$$

$$\Rightarrow D^2y - Dy - 3Dy + 4y = 2e^{2z} + e^{-z}$$

$$\Rightarrow (D^2 - 4D + 4)y = 2e^{2z} + e^{-z}$$

The auxillary eqn is $m^2 - 4m + 4 = 0$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m_1 = m_2 = 2$$

$$CF = (C_1 + C_2 \cancel{z})e^{2z}$$

To find PI,

$$PI = \left[\frac{1}{D^2 - 4D + 4}\right][2e^{2z} + e^{-z}]$$

$$= 2\left[\frac{1}{D^2 - 4D + 4}\right]e^{2z} + \left[\frac{1}{D^2 - 4D + 4}\right]e^{-z}$$

$(D \to 2)$ $\qquad$ $(D \to -1)$

$$= 2z\left[\frac{1}{2D-4}\right]_{(D\to 2)} e^{2z} + \frac{1}{9}e^{-z}$$

$$= 2z^2\left(\frac{1}{2D}\right)_{(D\to 2)} e^{2z} + \frac{1}{9}e^{-z}$$

$$= \frac{1}{2}z^2 e^{2z} + \frac{1}{9}e^{-z}$$

The general solution is,

$$y = (C_1 + C_2 z)e^{2z} + \frac{1}{2}z^2 e^{2z} + \frac{1}{9}e^{-z}$$

$$= [C_1 + C_2(\log x)]x^2 + \frac{1}{2}x^2(\log x)^2 + \frac{1}{9x}$$

<u>Assignment-8</u>

1. Solve $(x+1)^2 y'' - 2(x+1)y' - 10y = 3x$

Let $(x+1) = e^t \Rightarrow \log(x+1) = t$

$(x+1)^2 y'' = 1D(D-1)y$

$(x+1)y' = 1Dy$

$\Rightarrow D(D-1)y - 2Dy - 10y = 3x$

$\Rightarrow D^2y - Dy - 2Dy - 10y = 3(e^t - 1)$

$\Rightarrow (D^2 - 3D - 10)y = 3(e^t - 1)$.

The auxillary equation is $m^2 - 3m - 10 = 0$

$m_1 = 5,\ m_2 = -2$

$CF = C_1 e^{5t} + C_2 e^{-2t}$

To find PI,

$$PI = \left[\frac{1}{D^2 - 3D - 10}\right][3e^t - 3]$$

$$= 3\left[\frac{1}{D^2-3D-10}\right]e^{t} - 3\left[\frac{1}{D^2-3D-10}\right]e^{0t}$$

$$(D\to 1) \qquad\qquad (D\to 0).$$

$$= \frac{-3}{12}e^{t} + \frac{3}{10}$$

The general solution is given by,

$$y = C_1 e^{5t} + C_2 e^{-2t} - \frac{3}{12}e^{t} + \frac{3}{10}.$$

$$= C_1(x+1)^5 + \frac{C_2}{(x+1)^2} - 3\left[\left(\frac{x+1}{12}\right) - \frac{1}{10}\right].$$

$$(5+2x)^2 y'' - 6(5+2x)y' + 8y = \log(5+2x)$$

Let $(2x+5) = e^{t} \Rightarrow t = \log(2x+5).$

$$\Rightarrow (2x+5)^2 y'' = 2^2 D(D-1)y$$

$$\Rightarrow (2x+5)y' = 2Dy$$

Thus, the eqn becomes,

$$4D(D-1)y - 6(2Dy) + 8y = \cancel{\log}\, t$$

$$\Rightarrow 4D^2y - 4Dy - 12Dy + 8y = t$$

$$\Rightarrow (4D^2 - 16D + 8)y = t$$

The auxillary eqn is $4m^2 - 16m + 8 = 0.$

$$m_1 = 2+\sqrt{2},\ m_2 = 2-\sqrt{2}$$

$$CF = C_1 e^{(2+\sqrt{2})t} + C_2 e^{(2-\sqrt{2})t}$$

$$= e^{2t}\left[C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}\right]$$

$$PI = \left[\frac{1}{4D^2-16D+8}\right] t$$

$$\begin{array}{r|l} & \frac{t}{8}+\frac{1}{4} \\ \hline 8-16D+4D^2 & t \\ & \overset{(-)}{t} \overset{(+)}{-} 2 + 0 \\ \hline & 2 \\ & \overset{(-)}{2} + 0 + 0 \\ \hline & 0 \end{array}$$

The general solution is,

$$y = e^{2t}\left[C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}\right] + \frac{t}{8} + \frac{1}{4}$$

where $t = \log(5+2x)$.

3. $(1+x)^2 y'' + (1+x)y' + y = 4\cos(\log(1+x))$

Let $e^t = 1+x \Rightarrow \log(1+x) = t$

$(1+x)^2 y'' = 1D(D-1)y$

$(1+x)y' = 1Dy$

Thus, eqn becomes,

$D(D-1)y + Dy + y = 4\cos t$

$\Rightarrow (D^2+1)y = 4\cos t$

The auxillary eqn is $m^2+1=0 \Rightarrow m = \pm i$

$CF = C_1\cos t + C_2 \sin t$

To find PI,

$$PI = \left[\frac{1}{D^2+1}\right] 4\cos t$$

$$D^2 \to -1$$

$$= 4t\left(\frac{1}{2D}\right)\cos t = 2t\sin t$$

The general solution is given by,

$$y = C_1 \cos t + C_2 \sin t + 2t \sin t$$

where $t = \log(1+x)$

4. $(1+2x)^2 y'' - 2(1+2x) y - 12y = 6x$

Let $(1+2x) = e^t \implies t = \log(1+2x)$

Thus, the eqn becomes,

$$4D(D-1)y - 2(2Dy) - 12y = 6x$$

$$\Rightarrow (4D^2 - 4D)y - 4Dy - 12y = 6\left(\frac{e^t - 1}{2}\right)$$

$$\Rightarrow (4D^2 - 8D - 12)y = 3(e^t - 1)$$

The auxillary eqn is $4m^2 - 8m - 12 = 0$

$$m_1 = 3, \; m_2 = -1$$

$$CF = C_1 e^{3t} + C_2 e^{-t}$$

To find PI,

$$PI = \left[\frac{1}{4D^2 - 8D - 12}\right] 3(e^t - 1).$$

$$= 3\left[\frac{1}{4D^2 - 8D - 12}\right]_{(D \to 1)} e^t - 3\left[\frac{1}{4D^2 - 8D - 12}\right]_{(D \to 0)} e^{0t}$$

$$= -\frac{3}{16} e^t + \frac{3}{12}$$

$$= -\frac{3}{16} e^t + \frac{1}{4}$$

The general solution is,

$$y = C_1(1+2x)^3 + \frac{C_2}{(1+2x)} - \frac{3}{8}x + \frac{1}{16}.$$

5. $(x+1)^2 y'' + (x+1) y = (2x+3)(2x+4)$

Let $e^t = x+1$, thus $\log(x+1) = t$

$D(D-1)y + Dy = (2x+3)(2x+4)$

$\Rightarrow D^2 y = [2(e^t-1)+3][2(e^t-1)+4]$

$\Rightarrow D^2 y = [2e^t+1][2e^t+2]$.

$\Rightarrow D^2 y = 4e^{2t} + 4e^t + 2e^t + 2$

$\Rightarrow D^2 y = 4e^{2t} + 6e^t + 2$

The auxillary eqn is $m^2 = 0 \Rightarrow m = 0$

$CF = (C_1 + C_2 t)e^{0t}$

To find PI,

$$PI = \left[\frac{1}{D^2}\right]4e^{2t} + \left[\frac{1}{D^2}\right]6e^t + \left[\frac{2}{D^2}\right]e^{0t}$$

$(D \to 2)$ $(D \to 1)$ $(D \to 0)$

$= e^{2t} + 6e^t + t^2$

The general solution is given by,

$y = (C_1 + C_2 t)e^{0t} + e^{2t} + 6e^t + t^2$ where $t = \log(x+1)$

<u>Assignment - 9</u>

1. Solve $y'' + y = \sec x \tan x$ by MVP.

$\Rightarrow (D^2+1) y = \sec x \tan x$

The auxillary equation is $m^2 + 1 = 0$. $\Rightarrow m = \pm i$

$CF = C_1 \cos x + C_2 \sin x$ $\Big\}$ $y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

To find PI,

$$PI = -\cos x \int \frac{\sin x}{1} (\sec x \tan x)\, dx + \sin x \int \frac{\cos x}{1} (\sec x \tan x)\, dx$$

$$= -\cos x \int (\tan^2 x)\, dx + \sin x \int \tan x\, dx$$

$$= -\cos x \int (\sec^2 x - 1)\, dx + \sin x [\log(\sec x) + (c_2)]$$

$$= -\cos x [\tan x - x \cancel{+ c_1}] + \sin x [\log(\sec x) \cancel{+ c_2}]$$

$$= \cos x [x - \tan x \cancel{+ c_1}] + \sin x [\log(\sec x) \cancel{+ c_2}].$$

The general solution is,

$$y = \cos x [x - \tan x + C_1] + \sin x [\log(\sec x) + C_2].$$

2. $x^2 y'' + x y' - y = x^2 \log x$ by MVP

Let $x = e^z$, thus $z = \log x$

$x^2 y'' = D(D-1)y$ and $x y' = Dy$

Thus, the eqn becomes,

$$D(D-1)y + Dy - y = e^{2z} z$$

$$\Rightarrow (D^2 - 1)y = e^{2z} z$$

The auxillary eqn is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$CF = C_1 e^{z} + C_2 e^{-z} \quad \left.\begin{matrix} y_1 = e^{z} \\ y_2 = e^{-z} \end{matrix}\right.$$

$$W = \begin{vmatrix} e^{z} & e^{-z} \\ e^{z} & -e^{-z} \end{vmatrix} = -e^{0} + (-e^{0}) = -2$$

To find PI,

$$PI = -e^{z}\int \frac{e^{-z}(e^{2z})z}{(-2)} dz + e^{-z}\int \frac{e^{z}(e^{2z}z)}{(-2)} dz$$

$$= -e^{z}\int \left(\frac{e^{z}z}{-2}\right) dz + e^{-z}\int \left(\frac{e^{3z}z}{-2}\right) dz$$

$$= +\frac{e^{z}}{2}\left[ze^{z} - e^{z} + 0\right] - \frac{e^{-z}}{2}\left[z\left(\frac{e^{3z}}{3}\right) - 1\left(\frac{e^{3z}}{9}\right)\right]$$

$$= +\frac{ze^{2z}}{2} - \frac{e^{2z}}{2} - \frac{ze^{2z}}{6} + \frac{e^{2z}}{18}.$$

$$= ze^{2z}\left(\frac{-1}{6} + \frac{1}{2}\right) + e^{2z}\left(\frac{-1}{2} + \frac{1}{18}\right).$$

$$= ze^{2z}\left(\frac{-1+3}{6}\right) + e^{2z}\left(\frac{-9+1}{18}\right)$$

$$= -\frac{4}{9}e^{2z} + \frac{1}{3}ze^{2z}$$

The general solution is, $y = CF + PI$

$$y = C_1 x + \frac{C_2}{x} + \frac{1}{3}x^2 \log x - \frac{4}{9}x^2$$

$y'' + 2y' + 2y = e^{-x}\sec^3 x$ by MVP.

The auxillary eqn is $(D^2 + 2D + 2)y = e^{-x}\sec^3 x$

The eqn is $\Rightarrow m^2 + 2m + 2 = 0$.

$\Rightarrow m_1 = -1 + i$

$\Rightarrow m_2 = -1 - i$.

$CF = e^{-x}(C_1 \cos x + C_2 \sin x)$ $\Big\}$ $y_1 = \cos x (e^{-x})$

$y_2 = \sin x (e^{-x})$.

$$W = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ (-e^{-x}\cos x - e^{-x}\sin x) & (-e^{-x}\sin x + e^{-x}\cos x) \end{vmatrix}$$

$= e^{-x}(\cos x)[-e^{-x}\sin x + e^{-x}\cos x] + (e^{-x}\cos x + e^{-x}\sin x)(e^{-x}\sin x$

$= -e^{-2x}\cancel{\cos x \sin x} + e^{-2x}\cos^2 x + e^{-2x}\cancel{\cos x \sin x} + e^{-2x}\sin^2 x$

$= e^{-2x}(1) = e^{-2x}$

To find PI,

$$PI = -e^{-x}\cos x \int \frac{\cancel{e^{-x}}\sin x(\cancel{e^{-x}}\sec^3 x)}{\cancel{e^{-2x}}}dx$$

$$+ e^{-x}\sin x \int \frac{\cancel{e^{-x}}\cos x(\cancel{e^{-x}}\sec^3 x)}{\cancel{e^{-2x}}}dx$$

$= -e^{x}\cos x \int \tan x \sec^2 x \, dx + e^{-x}\sin x \int \sec^2 x \, dx$

$u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$

$= -e^{x}\cos x \int u \, du + e^{-x}\sin x(\tan x)$

$= -e^{x}\cos x \left(\frac{\tan^2 x}{2}\right) + e^{-x}\sin(\tan x)$.

The general solution is,

$$y = +e^{-x}\cos x\left[C_1 - \frac{\tan^2 x}{2}\right] + e^{-x}\sin x\,[C_2 + \tan x].$$

5. $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$ by MVP.

The auxillary eq is $m^2 - 3m + 2 = 0$.

$m_1 = 2, \quad m_2 = 1$

$CF = C_1 e^x + C_2 e^{2x}$ $\left\}\begin{array}{l} y_1 = e^x \\ y_2 = e^{2x} \end{array}\right.$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$= 2e^{3x} - e^{3x} = e^{3x}$$

To find PI,

$$PI = -e^x \int \cancel{\frac{e^{2x}}{e^{3x}}} \left(\frac{e^x}{1+e^x}\right) dx + e^{2x} \int \frac{e^x}{e^{3x}} \left(\frac{e^x}{1+e^x}\right) dx$$

$$= -e^x \int \left(\frac{1}{1+e^x}\right) dx + e^{2x} \int \left(\frac{1}{e^x(1+e^x)}\right) dx$$

$$= -e^x \int \left[\frac{-e^x}{1+e^x} + 1\right] dx + e^{2x} \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$= -e^x\left[-\log(1+e^x) + x\right] + e^{2x}(-e^{-x}) - e^{2x}\left[-\log(1+e^x) + x\right]$$

$$= +e^x \log(1+e^x) - xe^x - e^{-x}e^{2x} + e^{2x}\log(1+e^x) - xe^{2x}$$

$$= e^x\left[\log(1+e^x) - x - 1\right] + e^{2x}\left[\log(1+e^x) - x\right].$$

The general solution is,

$y = CF + PI$

$$\Rightarrow y = C_1 e^x + C_2 e^{2x} + e^x\left[\log(1+e^x) - x - 1\right] + e^{2x}\left[\log(1+e^x) - x\right]$$

Assignment-10

1. Solve $y'' + y = \tan x$ by MVP.

The auxillary eqn is $(D^2+1)\,y = \tan x$

$$\longrightarrow m^2 + 1 = 0$$

$$m = \pm i$$

$$CF = C_1 \cos x + C_2 \sin x \quad \left.\begin{matrix} y_1 = \cos x \\ y_2 = \sin x \end{matrix}\right.$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

To find PI,

$$PI = -\cos x \int \frac{\sin x(\tan x)}{1}\,dx + \sin x \int \frac{\cos x \tan x}{1}\,dx$$

$$= -\cos x \int \sin x \tan x\,dx + \sin x \int \sin x\,dx$$

$$= -\cos x \int \left(\frac{1-\cos^2 x}{\cos x}\right) dx + \sin x(-\cos x)$$

$$= -\cos x \int (\sec x - \cos x)\,dx + \sin x(-\cos x)$$

$$= -\cos x\left[\log(\tan x + \sec x)\right] - \cancel{\cos x \sin x} + \cancel{\cos x \sin x}$$

$$= -\cos x\left[\log(\tan x + \sec x)\right].$$

The general solution,

$$y = C_1 \cos x + C_2 \sin x - (\cos x)\log(\tan x + \sec x).$$

2. $y'' - 2y' + 2y = e^{x}\tan x$ by MVP

The auxillary eqn is $m^2 - 2m + 2 = 0$

$$m_1 = 1+i$$
$$m_2 = 1-i$$

$$CF = e^{x}(C_1\cos x + C_2 \sin x) \Big\} \begin{array}{l} y_1 = e^{x}\cos x \\ y_2 = e^{x}\sin x \end{array}$$

$$W = \begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ (e^{x}\cos x - e^{x}\sin x) & (e^{x}\sin x + e^{x}\cos x) \end{vmatrix}$$

$$= \cancel{e^{2x}\cos x\sin x} + e^{2x}\cos^2 x - \cancel{e^{2x}\cos x\sin x} + e^{2x}\sin^2 x$$

$$= e^{2x}(1) = e^{2x}$$

To find PI,

?

$$PI = -e^{x}\cos x\int \frac{\cancel{e^{x}}\sin x(\cancel{e^{x}}\tan x)}{\cancel{e^{2x}}}dx$$

$$+ e^{x}\sin x\int \frac{\cancel{e^{x}}\cos x(\cancel{e^{x}}\tan x)}{\cancel{e^{2x}}}dx$$

$$= -e^{x}\cos x\int \sin x\tan x\,dx + e^{x}\sin x\int \sin x\,dx$$

$$= -e^{x}\cos x\int(\sec x - \cos x)dx + e^{x}\sin x(-\cos x)$$

$$= -e^{x}\cos x\log(\sec x\tan x) + e^{x}\cancel{\cos x\sin x} - e^{x}\cancel{\sin x\cos x}$$

The general solution is,

$$y = CF + PI$$

$$y = e^{x}(C_1\cos x + C_2\sin x) - e^{x}\cos x\log(\sec x\tan x)$$

3. $y'' - y = e^{-2x}\sin e^{-x}$ by MVP

The auxillary eqn is $m^2 - 1 = 0$. $\Rightarrow m = \pm 1$.

$$CF = C_1 e^{x} + C_2 e^{-x} \Big\} \begin{array}{l} y_1 = e^{x} \\ y_2 = e^{-x} \end{array}$$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{0} - e^{0} = -2$$

To find PI,

$$PI = -e^{x}\int \frac{e^{-x}(e^{-2x}\sin e^{-x})}{(-2)}dx + e^{-x}\int \frac{e^{x}(e^{-2x}\sin e^{-x})}{(-2)}dx$$

$$= \frac{+e^{x}}{2}\int (e^{-3x}\sin e^{-x})dx + \left(\frac{-e^{-x}}{2}\right)\int (e^{-x}\sin e^{-x})dx$$

$u = +e^{-x} \Rightarrow u^2 = +e^{-2x}$

$du = -e^{-x}dx \rightarrow$ Thus, $\int -u^2 \sin u \, du$

$-[u^2(-\cos u) - 2u(-\sin u) + 2(\cos u)]$

$\int -\sin u \, du$

$+\cos u$

$\cos e^{-x}$

$$= \frac{e^{x}}{2}\left[e^{-2x}\cos e^{-x} - 2e^{-x}\sin e^{-x} - 2\cos e^{-x}\right] - \frac{e^{-x}}{2}(\cos e^{-x})$$

$$= \cancel{\frac{e^{-x}}{2}\cos e^{-x}} - \sin e^{-x} - e^{x}\cos e^{-x} - \cancel{\frac{e^{-x}}{2}\cos e^{-x}}$$

$$= -\sin e^{-x} - e^{x}\cos e^{-x}$$

The general solution is

$$y = C_1 e^{x} + C_2 e^{-x} - (\sin e^{-x} + e^{x}\cos e^{-x})$$

Assignment - 12

1. $L\left(\frac{d^2q}{dt^2}\right) + R\left(\frac{dq}{dt}\right) + \frac{q}{C} = E$

$$\Rightarrow 0.05\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + \left(\frac{q}{100\times 10^{-6}}\right) = 0.$$

$$\Rightarrow \frac{d^2q}{dt^2} + 400\frac{dq}{dt} + (2\times 10^5).q = 0.$$

The auxillary eqn is $m^2 + 400m + (2\times 10^5) = 0.$

$$m_1 = -200 + 400i$$

$$m_2 = -200 - 400i$$

$$CF = q = e^{-200t}(C_1\cos 400t + C_2 \sin 400t) \longrightarrow (1).$$

Given, at $t=0$, $q = 0.01$

$\Rightarrow 0.01 = 1[C_1] \rightarrow C_1 = 0.01$

Differentiating wrt ①,

$$\frac{dq}{dt} = -200e^{-200t}(C_1 \cos 400t + C_2 \sin 400t)$$
$$+ e^{-200t}(-400C_1 \sin 400t + 400C_2 \cos 400t)$$

Given $i=0$ at $t=0$,

$\Rightarrow 0 = -200(C_1 + 0) + 1(0 + 400C_2)$

$\Rightarrow C_2 = 0.0005$

$q = e^{-200t}(0.01 \cos 400t - 0.0005 \sin 400t)$

$I = 5e^{-220t} \sin 400t$?

3. $L\left(\frac{d^2q}{dt^2}\right) + R\left(\frac{dq}{dt}\right) + \frac{q}{C} = E$

$\Rightarrow 0.2\left(\frac{dq^2}{dt^2}\right) + 40\left(\frac{dq}{dt}\right) + \frac{q}{(10^{-4})} = 0.$

$\Rightarrow \frac{dq^2}{dt^2} + 200\left(\frac{dq}{dt}\right) + 5\times10^4(q) = 0.$

The homogeneous eqn is $D^2 + 200D + (5\times10^4) = 0.$

The auxillary eqn is $m^2 + 200m + (5\times10^4) = 0$

$$m = -100 \pm 200i$$

$CF = q = e^{-100t}(C_1 \cos 200t + C_2 \sin 200t) \longrightarrow$ ①

Given, at $t=0$, $q = 1C$.

$\Rightarrow 1 = C_1$

Diff. wrt t to ①,

$$i = -100e^{-100t}(C_1 \cos 200t + C_2 \sin 200t) + e^{-100t}[-200C_1 \sin 200t$$
$$+ 200C_2 \cos 200t]$$

Given, $i = 2$ when $t=0$ $\Big\}$ $-100C_1 + 200C_2 = 2$

$\Rightarrow C_2 = \frac{51}{100}$

Thus, the solution is given by,

$$q = e^{-100t}\left(\cos 200t + \frac{51}{100}\sin 200t\right).$$

The eqn for current is given by,

$$i = -100e^{-100t}\left(\cos 200t + \frac{51}{100}\sin 200t\right)$$
$$+ e^{-100t}\left[-200\sin 200t + 200\left(\frac{51}{100}\right)\cos 200t\right].$$

$$\Rightarrow i = +e^{-100t}\left[(-100+102)\cos 200t + \left(\frac{51}{100}\right)-(200)\sin 200t\right].$$

$$\Rightarrow i = e^{-100t}\left[2\cos 200t - 251\sin 200t\right]$$

2. $L\left(\frac{dq^2}{dt}\right) + R\left(\frac{dq}{dt}\right) + \frac{q}{C} = E$

$$\Rightarrow 2\left(\frac{dq^2}{dt^2}\right) + 4\left(\frac{dq}{dt}\right) + \left(\frac{q}{0.05}\right) = 100.$$

$$\Rightarrow \frac{dq^2}{dt^2} + 2\left(\frac{dq}{dt}\right) + 10q = 50.$$

The auxillary equation is $m^2 + 2m + 10 = 0$

$$m = -1 \pm 3i$$

$$CF = e^{-t}(C_1\cos 3t + C_2\sin 3t).$$

To find PI,

$$PI = \left(\frac{50}{D^2+2D+10}\right)e^{0t} = 5.$$
$$D \to 0$$

The general solution is,

$$q = e^{-t}(C_1\cos 3t + C_2\sin 3t) + 5. \longrightarrow (1)$$

Diff (1) wrt t,

$$i = -e^{-t}(C_1\cos 3t + C_2\sin 3t) + e^{-t}(-3C_1\sin 3t + 3C_2\cos 3t) \longrightarrow (2)$$

Given, $q = 0$, $t = 0$ $\Rightarrow$ $0 = C_1 + 5 \Rightarrow C_1 = -5.$

Given, $i = 0$, $t = 0 \Rightarrow 0 = -C_1 + (-3)(0) + 3C_2 \Rightarrow C_2 = -\frac{5}{3}$

The eqn for charge q is,

$$q = 5 - \frac{5}{3} e^{-t} (3\cos 3t + \sin 3t)$$

The eqn for current i is,

$$i = -e^{-t}\left(-5\cos 3t - \frac{5}{3}\sin 3t\right) + e^{-t}(15\sin 3t - 5\cos 3t)$$

$$\Rightarrow i = +e^{-t}\left(\cancel{5\cos t} + \frac{5}{3}\sin 3t + 15\sin 3t - \cancel{5\cos 3t}\right)$$

$$\Rightarrow i = \frac{50}{3} e^{-t} \sin 3t$$

4. $L\left(\frac{di}{dt}\right) + Ri = 0$

$$\Rightarrow L\left(\frac{d^2q}{dt^2}\right) + R\left(\frac{dq}{dt}\right) = 0.$$

$$\Rightarrow 3\left(\frac{d^2q}{dt^2}\right) + 2\left(\frac{dq}{dt}\right) = 0.$$

The auxillary eqn is $3m^2 + 2m = 0$.

$$\Rightarrow m(3m+2) = 0$$

$$\Rightarrow m = 0, \frac{-2}{3}$$

$$CF = C_1 e^{0t} + C_2 e^{-\frac{2}{3}t} = q \longrightarrow ①$$

Current at $t = 0$ is $i = 1A$.

Diff. ① wrt t,

$$i = -\frac{2}{3} C_2 e^{-\frac{2}{3}t}$$

$$\Rightarrow 1 = -\frac{2}{3} C_2 \quad \Rightarrow C_2 = \frac{-3}{2} \quad \Rightarrow \underline{i = e^{-\frac{2}{3}t}} \text{ (general)}$$

Current at $t = 2$ is, $\underline{i(2) = e^{-\frac{4}{3}}}$

Question Bank

1. If $y = e^{2t}$ is a solution to $\frac{d^2y}{dt^2} - 5\left(\frac{dy}{dt}\right) + ky = 0$ then find k.

$y = e^{2t}$ is a solution, means the roots are $m_1 = 0$ and $m_2 = 2$

Thus, the auxillary eqn is $m^2 - 5m + k = 0$.

Putting $m = 0 \Rightarrow \underline{k = 0}$.

Putting $m = 2 \Rightarrow 0 = 4 - 10 + k \Rightarrow \underline{k = 6}$

2. If $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$ where a, b and g are positive numbers and $x = a'$, $\frac{dx}{dt} = 0$, when $t = 0$, show that $x = a + (a' - a)\cos\left(\sqrt{\frac{g}{b}}\, t\right)$.

$$\frac{d^2x}{dt^2} + \left(\frac{g}{b}\right)x = \left(\frac{g}{b}\right)a$$

The homogeneous eqn is $\left(D^2 + \frac{g}{b}\right)x = \left(\frac{g}{b}\right)a$

The auxillary eqn is $m^2 + \frac{g}{b} = 0 \rightarrow m = \pm i\sqrt{\left(\frac{g}{b}\right)}$

$$CF = e^{ax}\left(C_1 \cos\sqrt{\frac{g}{b}}\, t + C_2 \sin\sqrt{\frac{g}{b}}\, t\right).$$

To find PI,

$$PI = \left[\frac{\left(\frac{g}{b}\right)a}{\left(D^2 + \frac{g}{b}\right)}\right]_{D \to 0} = a$$

The general solution is,

$$x = C_1 \cos\sqrt{\frac{g}{b}}\, t + C_2 \sin\sqrt{\frac{g}{b}}\, t + a$$

Given, $x = a'$, $\frac{dx}{dt} = 0$ when $t = 0$.

Thus,

$a' = C_1 + a \Rightarrow C_1 = (a' - a)$.

Also,

$$x' = -C_1\sqrt{\frac{g}{l}}\left[\sin\left(\sqrt{\frac{g}{l}}\right)t\right] + C_2\left[\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t\right].$$

$$\Rightarrow 0 = C_2\sqrt{\frac{g}{l}} \rightarrow C_2 = 0.$$

Thus, the complete solution is,

$$x = a + (a' - a)\cos\left(\sqrt{\frac{g}{l}}\right)t$$

LHS = RHS

Hence, proved.