Special Functions

Elementary Functions

★ <u>Algebraic functions</u>: are the functions obtained by algebraic equations. Types of a. functions:

i) Linear functions
ii) Quadratic functions
iii) Cubic functions
iv) Polynomial functions
v) Rational functions
vi) Radical functions

Gamma Functions

- It is also known as <u>Euler's integral of second kind</u>.
- Let n be any positive integer. Then the definite integral $\int_0^{+\infty} e^{-x} x^{n-1} dx$ is called gamma function of n which is denoted by $\Gamma(n)$ and is defined by,

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad (n>0)} \longrightarrow (1)$$

- $\boxed{\Gamma(1) = 1}$

- <u>Another form of Gamma function</u>:

Put $x = t^2$ in (1), we get,

$dx = 2t\,dt$

When $x=0$, $t=0$, also $x=\infty$, $t=\infty$.

Thus, (1) becomes,

$$\Gamma(n) = \int_0^{\infty} e^{-t^2} t^{2n-2} (2t\,dt)$$

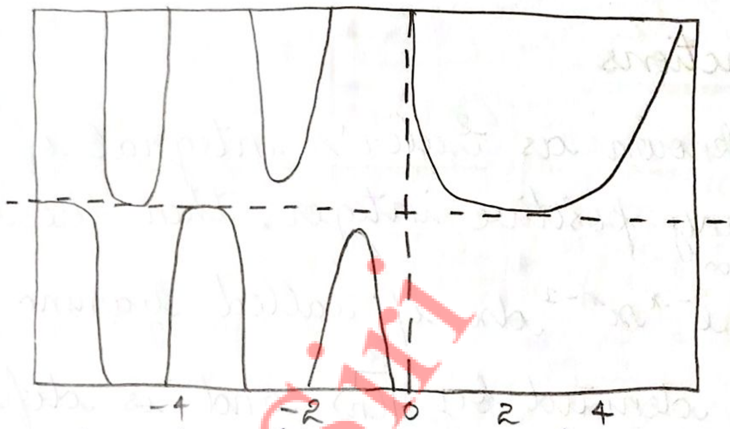
$$\Rightarrow \boxed{\Gamma(n) = 2\int_0^{\infty} e^{-t^2} t^{2n-1} dt}$$

1. $\int_0^\infty e^{-t} t^3 \, dt = \Gamma 4$ $\quad \left.\begin{matrix} n-1=3 \\ n=4 \end{matrix}\right\}$

2. $\int_0^\infty e^{-x} x^7 \, dx = \Gamma 8$ $\quad \left.\begin{matrix} n-1=7 \\ n=8 \end{matrix}\right\}$

3. $\int_0^\infty e^{-x} x^{\frac{2}{3}} \, dx = \Gamma \frac{5}{3}$ $\quad \left.\begin{matrix} n-1=\frac{2}{3} \\ \Rightarrow n=\frac{5}{3} \end{matrix}\right\}$

Graph of Gamma Function



For $n>0$, the function is continuous.

$n=0$ and negative integers, the function is discontinuous.

Beta Function

- $\beta(m,n)$ is the name given by Legendre and Whitaker and Watson for the beta integral. (Euler's integral of the first kind)
- It is defined by,

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx} \quad (m>0, n>0) \longrightarrow 1$$

is convergent for $m>0, n>0$.

- Another form of beta function:

Put $x = \sin^2\theta$ in 1, we get

$dx = 2\sin\theta\cos\theta\, d\theta$

When $x=0,\ \theta=0$, also $x=1,\ \theta=\frac{\pi}{2}$.

$\Rightarrow \beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta\,(2\sin\theta\cos\theta\, d\theta)$

$$\boxed{\Rightarrow \beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}$$

1. $\int_0^1 x^3(1-x)^5 dx = \beta(4,6)$ $\left.\begin{matrix} m=4 \\ n=6. \end{matrix}\right\}$

2. $\int_0^1 \sqrt{x}(1-x)^3 dx = \beta\left(\frac{3}{2},4\right)$ $\left.\begin{matrix} m-1=\frac{1}{2} \Rightarrow m=\frac{3}{2} \\ n-1=3 \Rightarrow n=4. \end{matrix}\right\}$

3. $\int_0^1 x^{-3}(1-x)^5 dx = \beta(m,n)$ does not exist as $m=-2$ which is $m<0$.

Properties of Beta and Gamma function

i) Recurrence formula for Gamma function:

$\boxed{\Gamma n+1 = n\Gamma n}$, where n is a real number.

Proof: We know, $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

Thus, $\Gamma n+1 = \int_0^\infty e^{-x}x^n dx$

$= \int_0^\infty \underbrace{x^n}_{u}\,\underbrace{e^{-x}}_{v}\, dx$

$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$\underbrace{x^n\left[\frac{e^{-x}}{-1}\right]_0^\infty}_{0} - \int nx^{n-1}\left(\frac{+e^{-x}}{-1}\right)dx$

$= \underbrace{\left[x^n(-e^{-x})\right]_0^\infty}_{\to 0} - \int_0^\infty e^{-x}(nx^{n-1})dx$

$\Rightarrow \Gamma n+1 = n\int_0^\infty e^{-x}x^{n-1}dx = n\Gamma n$

LHS = RHS $\Rightarrow$ Hence, proved

1. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

2. $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right)$

3. $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1+\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right)$

4. $\Gamma\left(\frac{11}{2}\right) = \Gamma\left(1+\frac{9}{2}\right) = \frac{9}{2}\Gamma\left(\frac{9}{2}\right)$

$$= \frac{9}{2}\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{9}{2}\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{9}{2}\left(\frac{7}{2}\right)\left(\frac{15}{8}\Gamma\frac{1}{2}\right) = \frac{945}{32}\Gamma\frac{1}{2}$$

ii) $\boxed{\Gamma(n+1) = n!}$ where n is a positive integer.

We know that, $\Gamma(n+1) = n\Gamma n$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)\ldots(3)(2)(1)\Gamma n$$

$$= n!$$

1. $\Gamma 4 = 3! = 6$
2. $\Gamma 6 = 5! = 120$
3. $\Gamma 10 = 9! = 362880$
4. $\Gamma 15 = 14! = 8.717 \times 10^{10}$

iii) Γn is not defined for $n=0$ and also for the negative integers n.

When n is a negative non-integer, the formula used to find Γn is given by,

$$\Gamma n = \frac{\Gamma n+1}{n}$$

1. $$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\Gamma\frac{1}{2}$$

iv) Symmetry: $\boxed{\beta(m,n) = \beta(n,m)}$

Proof: We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $(1-x) = y \Rightarrow dx = -dy$

When $x=0$, $y=1$, also, when $x=1$, $y=0$.

$$\Rightarrow \beta(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m).$$

LHS = RHS, Hence, proved.

v) <u>Relation between beta and gamma function :</u>

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}}$$

Proof : We know that,

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx$$

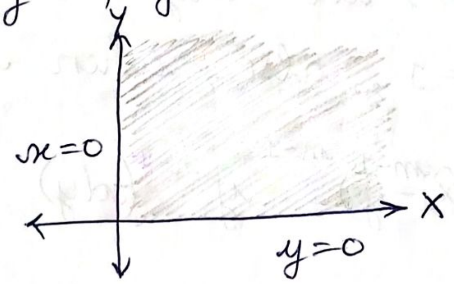
$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$\Rightarrow \Gamma m \, \Gamma n = 4\left[\int_0^\infty e^{-x^2} x^{2m-1} dx\right]\left[\int_0^\infty e^{-y^2} y^{2n-1} dy\right]$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\left[x^{2m-1}\right]\left[y^{2n-1}\right] dx\, dy \longrightarrow (1)$$

The region of integration is bound by the curves

$x=0,\ x=\infty,\ y=0,\ y=\infty$



The region of integration is I coordinate.

By changing into polar co-ordinates,

$x = r\cos\theta,\quad y = r\sin\theta \quad dxdy = r\,dr\,d\theta$

$\Rightarrow r^2 = x^2+y^2$

$\left.\begin{matrix} r \text{ varies from} \\ 0 \text{ to } \infty,\ \theta \\ \text{varies from} \\ 0 \text{ to } \frac{\pi}{2}. \end{matrix}\right\}$

$$\frac{J(x,y)}{(r,\theta)} = \begin{vmatrix} \frac{dx}{dr} & \frac{dy}{dr} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} \end{vmatrix}$$

Thus, (1) becomes,

$$\Rightarrow \Gamma m\,\Gamma n = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m-1}(\cos\theta)^{2m-1} r^{2n-1}(\sin\theta)^{2n-1}(r\,dr\,d\theta)$$

$$= \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right]\left[2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \,\sin^{2n-1}\theta \, d\theta\right]$$

$$\Rightarrow \Gamma m \, \Gamma n = \Gamma(m+n) \, \beta(n, m)$$

$$\Rightarrow \Gamma m \, \Gamma n = \Gamma(m+n) \, \beta(m, n)$$

$$\Rightarrow \beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$\therefore$ LHS = RHS

Hence, proved.

vi) $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Proof : We know that,

$$\beta(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

Here, $m = n = \frac{1}{2}$

$$\Rightarrow \beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^0\theta \, \cos^0\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{2}} = \pi$$

$$\Rightarrow \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

Also, $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \dfrac{\Gamma\frac{1}{2} \, \Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}$

$$\Rightarrow 1 \cdot \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi \quad \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

$\therefore$ LHS = RHS

Hence, proved

vii) $$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$ $p>-1$, $q>-1$

Proof : We know that,

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Put $m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^{p+1-1}\theta \cos^{q+1-1}\theta \, d\theta$$

$$\Rightarrow \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$\therefore$ LHS = RHS

Hence, proved.

viii) Beta function expressed as an improper integral

$$\boxed{\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} \, dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx}$$

Proof: We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx \longrightarrow (1).$$

Put $x = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1 \Rightarrow dx = -\left(\frac{1}{(1+y)^2}\right) dy$

When $x=0$, $y=\infty$, also when $x=1$, $y=0$.

Thus, (1) becomes,

$$\beta(m,n) = \int_{\infty}^{0} \left[\frac{1}{(y+1)^{m-1}}\right]\left[1-\frac{1}{y+1}\right]^{n-1}\left[\frac{-1}{(y+1)^2}\right]dy$$

$$= \int_{0}^{\infty} \left[\frac{1}{(y+1)^{m-1}}\right]\left[\frac{y}{y+1}\right]^{n-1}\left[\frac{1}{(y+1)^2}\right]dy$$

$$= \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}\,dy$$

$$= \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \quad \text{(By symmetry).}$$

(By change of variables)

ix) $\boxed{\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}}$ $(0<n<1)$

Show that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Let $n = \frac{1}{4} \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \dfrac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \pi\sqrt{2}$

LHS = RHS , Hence, proved.

<u>Legendre's duplication formula</u>

For <u>Gamma function</u>,

$$\boxed{\Gamma(2p)\left[\sqrt{\pi}\right] = 2^{2p-1}\,\Gamma p\,\Gamma\left(p+\tfrac{1}{2}\right)}$$

For <u>Beta function</u>,

$$\boxed{\beta\left(p,\tfrac{1}{2}\right) = 2^{2p-1}\,\beta(p,p)}$$

Compute $\Gamma\left(\frac{3}{2}\right), \Gamma\left(\frac{5}{2}\right), \Gamma\left(-\frac{1}{2}\right), \Gamma 6, \Gamma 4, \Gamma 3.5$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\frac{1}{2} = -2\sqrt{\pi}$$

$$\Gamma 6 = 5! = 120$$

$$\Gamma 4 = 3! = 6$$

$$\Gamma(3.5) = \Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

Some important formulae useful for solving problems on Beta and Gamma function.

1. $$\int_0^\infty x^p e^{-ax^q} dx = \frac{\Gamma\left(\frac{p+1}{q}\right)}{q\, a^{\left(\frac{p+1}{q}\right)}}$$

2. $$\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$$

3. $$\int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n \Gamma n+1}{(m+1)^{n+1}}$$ where $n>0$, $m>-1$

4. $$\int_0^1 x^p (1-x^q)^r dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$$

1. Proof: Here, p, q, a are positive constants.

Taking LHS,

$$I = \int_0^\infty x^p e^{-ax^q} dx \longrightarrow (1)$$

Let $y = ax^q \implies x = \left(\frac{y}{a}\right)^{\frac{1}{q}}$

$\Rightarrow dy = [a q x^{q-1}] dx$

When $x=0$, $y=0$, and $x=\infty$, $y=\infty$.

Thus, (1) becomes,

$$\Rightarrow I = \int_0^\infty \left(\frac{y}{a}\right)^{\frac{p}{q}} e^{-y} \frac{dy}{a\left(\frac{y}{a}\right)^{1-\frac{1}{q}} q}$$

$$\Rightarrow I = \int_0^\infty \frac{y^{\frac{p}{q}}}{a^{\frac{p}{q}}} e^{-y} \frac{dy}{\left(a^{1-1+\frac{1}{q}}\right)\left(y^{1-\frac{1}{q}}\right)q}$$

$$\Rightarrow I = \int_0^\infty \frac{y^{\frac{p}{q}-1+\frac{1}{q}}}{\left(a^{\frac{p}{q}+\frac{1}{q}}\right)q} e^{-y} dy$$

$$\Rightarrow I = \left[q a^{\frac{p+1}{q}}\right]^{-1} \int_0^\infty e^{-y} y^{\frac{p-q+1}{q}} dy$$

$$\Rightarrow I = \left[q a^{\frac{p+1}{q}}\right]^{-1} \Gamma\left(\frac{p-q+1}{q} + 1\right)$$

$$\Rightarrow I = \left[q a^{\frac{p+1}{q}}\right]^{-1} \Gamma\frac{p+1}{q} = \frac{\Gamma\frac{p+1}{q}}{q a^{\frac{p+1}{q}}}$$

(2.) Proof:

Let $I = \int_0^\infty e^{-kx} x^{n-1} dx$

Let $kx = y \Rightarrow x = \frac{y}{k} \Rightarrow dx = \frac{dy}{k}$

When $x=0$, $y=0$

$x=\infty$, $y=\infty$

$$\Rightarrow I = \int_0^\infty e^{-y} \left(\frac{y}{k}\right)^{n-1} \frac{dy}{k}$$

$$\Rightarrow I = \frac{1}{k^{n-1+1}} \int_0^\infty e^{-y} y^{n-1} dy = \frac{1}{k^n} \Gamma n$$

(3.) Proof:

Let $I = \int_0^1 x^m (\ln x)^n dx$

Put $x = e^{-y} \Rightarrow \log x = \log e^{-y} \Rightarrow \log x = -y$

Also $dx = -e^{-y} dy \Rightarrow y = -\log x$

When $x=0$, $y=\infty$

$x=1$, $y=0$.

$$\Rightarrow I = \int_\infty^0 e^{-my} (-y)^n (-e^{-y} dy)$$

$$\Rightarrow I = (-1)^n \int_0^\infty e^{-(m+1)y} y^n dy$$

Let $y(m+1) = t$

$\Rightarrow dy\,(m+1) = dt$

$$\Rightarrow I = (-1)^n \int_0^\infty e^{-t} \frac{t^n}{(m+1)^n} \frac{dt}{m+1}$$

$$\Rightarrow I = \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-t} t^n \, dt$$

$$\Rightarrow I = \frac{(-1)^n}{(m+1)^{n+1}} \overline{|n+1}$$

(4) Proof :

$$I = \int_0^1 x^p (1-x^q)^r \, dx$$

Put $y = x^q \Rightarrow dy = q x^{q-1} dx \Rightarrow dx = \frac{1}{q} y^{\frac{1}{q}-1} dy$

When $x = 0,\ y = 0,$

$x = 1,\ y = 1$

$$\Rightarrow I = \int_0^1 y^{\frac{p}{q}} (1-y)^r \left[\frac{1}{q} y^{\frac{1}{q}-1}\right] dy$$

$$\Rightarrow I = \frac{1}{q} \int_0^1 y^{\frac{p}{q}+\frac{1}{q}-1} (1-y)^r \, dy$$

$$\Rightarrow I = \left(\frac{1}{q}\right) \int_0^1 y^{\frac{p-q+1}{q}} (1-y)^r \, dy$$

$$\Rightarrow I = \frac{1}{q} B\left(\frac{p-q+1}{q}+1,\ r+1\right) = \frac{1}{q} B\left(\frac{p+1}{q},\ r+1\right)$$