Contents

Beta Gamma Functions

Introduction:

- The Gamma and Beta functions are defined in terms of improper definite integrals.
- These functions belong to the category of the special transcendental functions
- These functions are very useful in many areas like asymptotic series, Riemann-Zeta function, number theory, etc.
- They also have many applications in Engineering and Physics.
- Many integrals which cannot be expressed in terms of elementary functions can be evaluated in terms of beta and gamma functions.

Beta Function:

Beta function is a definite integral whose integrand depends on two variables.

It is also known as Euler's integral of First kind.

Beta function is defined as $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad where \quad m,n>0$

Properties of beta functions:

1. Symmetry: $\beta(m,n)=\beta(n,m)$

Proof: By definition, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ ------------(1)

Put $x=1-t$ in (1)

Then,

$$\beta(m,n)=\int_1^0 (1-t)^{m-1}t^{n-1}\left(-dt\right)=\int_0^1 t^{n-1}(1-t)^{m-1}dt=\beta(n,m)$$

2. Beta function in terms of trigonometric functions:

$$\beta(m,n)=2\int_0^{\frac{\pi}{2}} \sin^{2m-1}x\cos^{2n-1}xdx$$

Proof: By definition, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ ------------(1)

Put $x=\sin^2\theta,\ dx=2\sin\theta\cos\theta d\theta$ in (1)

Then,

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}2\sin\theta\cos\theta d\theta$$

$$=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Note: put $2m-1=p,\ 2n-1=q\Rightarrow m=\frac{p+1}{2},\ n=\frac{q+1}{2}$

$$\therefore\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

3. As an improper integral, $\beta(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx$

Proof: Put $x=\frac{1}{1+y},\ dx=-\frac{1}{(1+y)^2}dy$ in (1)

Then, $\beta(m,n)=\int_{\infty}^{0}\frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$

$$=\int_0^{\infty}\frac{1}{(1+y)^{m-1}}\left(\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}dy$$

$$=\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx\quad(dummy\ variable)$$

Gamma function:

Gamma function is a definite integral whose integrand depends on one variable. It is also known as Euler's integral of second kind. It is the generalization of factorial notation from integer values to real numbers.

Gamma function is defined as $\Gamma(n)=\int_0^{\infty}e^{-x}x^{n-1}dx\ \ where\ \ n>0$ --------------------- (2)

$\Gamma(n)$ is also defined for negative non-integers.

Put $x=t^2,\ dx=2tdt$ in (2)

Then, $\Gamma(n)=\int_0^\infty e^{-t^2}\left(t^2\right)^{n-1}2tdt=2\int_0^\infty e^{-t^2}t^{2n-1}dt$

Therefore, another form of gamma function is $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx$

Evaluation of gamma function:

1. $\Gamma(1)=1$

Proof: Put $n=1$ in (2)

Then, $\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}dx=\int_0^\infty e^{-x}dx=\left[\frac{e^{-x}}{-1}\right]_0^\infty=1$

2. $\Gamma(n+1)=n\Gamma(n)$ where n is a real number. (Reduction formula)

Proof: From the definition, $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$

Therefore, $\Gamma(n+1)=\int_0^\infty e^{-x}x^{(n+1)-1}dx=\int_0^\infty e^{-x}x^n dx$

$$=\left(\frac{x^n e^{-x}}{-1}\right)_0^\infty+n\int_0^\infty e^{-x}x^{n-1}dx=0+n\Gamma(n)$$

$$=n\Gamma(n)$$

3. $\Gamma(n+1)=n!$ where n is a positive integer.

Proof: From Result 2,

$$\Gamma(n+1)=n\Gamma(n)$$

$$=n(n-1)\Gamma(n-1)$$

$$=n(n-1)(n-2)\Gamma(n-2)$$

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$$=n(n-1)(n-2)(n=3)..........2.1.\Gamma(1)=n!$$

Note: $\Gamma(n)$ is not defined when n is zero or a negative integer.

4. When n is a negative non integer, $\Gamma(n)=\frac{\Gamma(n+1)}{n}$

Proof: we have $\Gamma(n+1)=n\Gamma(n)$

$$\therefore \Gamma(n)=\frac{\Gamma(n+1)}{n}=\frac{\Gamma(n+2)}{n(n+1)}=\frac{\Gamma(n+3)}{n(n+1)(n+2)}$$
$$=\ldots\ldots\ldots\ldots=\frac{\Gamma(n+k+1)}{n(n+1)(n+2).....(n+k)}$$

This formula is used to compute $\Gamma(n)$ when n is a negative non-integer.

Some standard results:

1. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Proof: we have, $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx$

Put $n=\frac{1}{2}\Rightarrow \Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-x^2}dx=2\int_0^\infty e^{-y^2}dy$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2=2\int_0^\infty e^{-x^2}dx\times 2\int_0^\infty e^{-y^2}dy=4\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy$$

Put $x=r\cos\theta$ *and* $y=r\sin\theta$, then $x^2+y^2=r^2$ and $dxdy=rdrd\theta$

Since both *x* and *y* vary from 0 to ∞, the region of integration is first quadrant and in first quadrant r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

Therefore,

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2=4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta=4.\frac{\pi}{2}\int_0^\infty e^{-r^2}rdr=2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty=\pi$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

2. **Relationship between beta and gamma functions:**

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof: By definition, $\Gamma(m)=2\int_0^\infty e^{-x^2}x^{2m-1}dx$ and $\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$

Therefore, $\Gamma(m)\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2m-1}dx\times 2\int_0^\infty e^{-y^2}y^{2n-1}dy$

$$= 4\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)} x^{2m-1} y^{2n-1} dxdy$$

Put $x = r\cos\theta \quad y = r\sin\theta$. Then $dxdy = rdrd\theta$

Then, $\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\left(r\cos\theta\right)^{2m-1}\left(r\sin\theta\right)^{2n-1} rdrd\theta$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta \times 2\int_0^\infty e^{-r^2} r^{2m-1} r^{2n-1} rdr$$

Put $r^2 = t,\ rdr = \dfrac{dt}{2}$ in second integral, we get,

$\Gamma(m)\Gamma(n) = \beta(m,n).2.\dfrac{1}{2}\int_0^\infty e^{-t} t^{m+n-1} dt = \beta(m,n)\Gamma\left(m+n\right)$

Thus, $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

3. $\Gamma\left(\dfrac{p+1}{q}\right) = qa^{\frac{p+1}{q}}\int_0^\infty x^p e^{-ax^q} dx,\ p$ and q are positive constants.

Proof: Consider, $\int_0^\infty x^p e^{-ax^q} dx$

Put, $ax^q = y \Rightarrow qax^{q-1}dx = dy$

$$\therefore \int_0^\infty x^p e^{-ax^q} dx = \int_0^\infty x^p e^{-y}\frac{dy}{qax^{q-1}} = \int_0^\infty x^p e^{-y}\frac{dy}{qa\left(\frac{y}{a}\right)^{\frac{q-1}{q}}}$$

$$= \int_0^\infty \left(\frac{y}{a}\right)^{\frac{p}{q}} e^{-y}\frac{dy}{qa\left(\frac{y}{a}\right)^{1-\frac{1}{q}}} = \frac{1}{aq}\int_0^\infty \left(\frac{y}{a}\right)^{\frac{p+1-q}{q}} e^{-y} dy$$

$$= \frac{1}{aq} a^{\frac{q-p-1}{q}} \int_0^\infty y^{\frac{p-q+1}{q}} e^{-y} dy = \frac{1}{q} a^{\frac{-(p+1)}{q}} \int_0^\infty e^{-y} y^{\frac{p+1}{q}-1} dy$$

$$= \frac{1}{a^{\frac{(p+1)}{q}} q}\int_0^\infty e^{-y} y^{\frac{p+1}{q}-1} dy = \frac{1}{a^{\frac{(p+1)}{q}} q}\Gamma\left(\frac{p+1}{q}\right)$$

4. $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}, 0<n<1$

Proof: We know that, $\Gamma(n)\Gamma(m)=\beta(n,m)\Gamma(m+n)$

Put $m=1-n \Rightarrow \Gamma(n)\Gamma(1-n)=\beta(n,1-n)\Gamma(1)$

$\Rightarrow \Gamma(n)\Gamma(1-n)=\beta(n,1-n)$

Also, we know that, $\beta(m,n)=\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Therefore, $\beta(n,1-n)=\displaystyle\int_0^\infty \frac{x^{n-1}}{(1+x)}dx=\frac{\pi}{\sin n\pi}$

5. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta=\frac{1}{2}\,\beta\left(\frac{1}{2},\frac{n+1}{2}\right)$

Proof: We know that, $\beta(p,q)=2\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2p-1}\theta\sin^{2q-1}\theta d\theta$

For $p=\dfrac{1}{2}$, $\beta\left(\dfrac{1}{2},q\right)=2\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2q-1}\theta\, d\theta$

Let $2q-1=n \Rightarrow q=\dfrac{n+1}{2}$.

Then, $\beta\left(\dfrac{1}{2},\dfrac{n+1}{2}\right)=2\displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta$

Therefore, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta=\frac{1}{2}\beta\left(\frac{1}{2},\frac{n+1}{2}\right)=\int_0^{\frac{\pi}{2}} \cos^n\theta d\theta$

Also, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta d\theta=\frac{1}{2}\,\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

6. Legendre duplication formula for gamma function: $\sqrt{\pi}\Gamma(2m)=2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$

Proof: We know that, $\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$

Therefore, $\beta(m,m)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2m-1}\theta d\theta$

$$=2\int_0^{\frac{\pi}{2}}\frac{\sin^{2m-1}2\theta}{2^{2m-1}}d\theta=\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}\sin^{2m-1}2\theta d\theta$$

Let $2\theta=t\Rightarrow 2d\theta=dt$.

Then, $\beta(m,m)=\frac{2}{2^{2m-1}}\int_0^{\pi}\sin^{2m-1}t\frac{dt}{2}==\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}\sin^{2m-1}tdt=\frac{1}{2^{2m-1}}\beta\left(\frac{1}{2},m\right)$

i.e. $\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}=\frac{1}{2^{2m-1}}\frac{\Gamma(\frac{1}{2})\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$

$$\Rightarrow\frac{\Gamma(m)}{\Gamma(2m)}=\frac{1}{2^{2m-1}}\frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Rightarrow\sqrt{\pi}\Gamma(2m)=2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$$

7. $\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)$

Proof: RHS= $\int_0^1 x^{m-1}(1-x)^{n+1-1}dx+\int_0^1 x^{m+1-1}(1-x)^{n-1}dx$

$$=\int_0^1[x^{m-1}(1-x)^n dx+\int_0^1 x^m(1-x)^{n-1}]dx=\int_0^1(1-x)^{n-1}x^{m-1}[1-x+x]dx$$

$$=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(m,n)=LHS$$

8. $\beta(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$

Proof: We know that, $\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)$

$$=\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}+\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$=\frac{m!(n-1)!}{(m+n)!}+\frac{(m-1)!n!}{(m+n)!}=\frac{m!(n-1)!+(m-1)!n!}{(m+n)!}$$

$$=\frac{(m-1)!(n-1)\{m+n\}}{(m+n)!}=\frac{(m-1)!(n-1)}{(m+n-1)!}$$

9. $\int_0^1 x^p\left(1-x^q\right)^r dx=\frac{1}{q}\beta\left(\frac{p+1}{q},r+1\right)$

Proof: Take $x^q=y\Rightarrow qx^{q-1}dx=dy$

$$\text{LHS}=\int_0^1 y^{\frac{p}{q}}(1-y)^r\frac{dy}{qy^{\frac{q-1}{q}}}=\frac{1}{q}\int_0^1 y^{\frac{p-q+1}{q}}(1-y)^r dy$$

$$=\frac{1}{q}\beta\left(\frac{p+1}{q},r+1\right)=RHS$$

SOLVED PROBLEMS:

1. Evaluate: $\int_0^{2\pi}\sin^8\theta d\theta$

Ans: $\int_0^{2\pi}\sin^8\theta d\theta=4\int_0^{\frac{\pi}{2}}\sin^8\theta d\theta=4.\frac{1}{2}\beta\left(\frac{9}{2},\frac{1}{2}\right)=2\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)}=2.\frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{4!}=\frac{35\pi}{2}$

2. Evaluate: $\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^5\theta d\theta$

Ans: $\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^5\theta d\theta=\frac{1}{2}\beta\left(\frac{5}{2},3\right)=\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)}=\frac{\frac{1}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}2}{\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}=\frac{8}{315}$

3. Evaluate: $\int_0^{\frac{\pi}{2}}\sqrt{\cot\theta}\,d\theta$

Ans: $\int_0^{\frac{\pi}{2}}\sqrt{\cot\theta}d\theta=\int_0^{\frac{\pi}{2}}\cot^{\frac{1}{2}}\theta d\theta=\int_0^{\frac{\pi}{2}}\left(\frac{\cos\theta}{\sin\theta}\right)^{\frac{1}{2}}d\theta=\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta d\theta$

$$=\frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}=\frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{\pi}{\sqrt{2}}$$

4. Evaluate: $\int_0^1 \sqrt{\frac{1-x}{x}}dx$

Ans: $\int_0^1 \sqrt{\frac{1-x}{x}}dx=\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}dx=\beta\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}=\frac{\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}=\frac{\frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{1}=\frac{\pi}{2}$

5. Evaluate: $\int_0^\infty \frac{dx}{1+x^4}$

Ans: take $x^2=\tan\theta, 2xdx=\sec^2\theta d\theta \Rightarrow dx=\frac{\sec^2\theta d\theta}{2x}=\frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}$

$$\int_0^\infty \frac{dx}{1+x^4}=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^2\theta}\frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}=\int_0^{\frac{\pi}{2}}\frac{1}{\sec^2\theta}\frac{\sec^2\theta d\theta}{2\sqrt{\tan\theta}}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{2\sqrt{\tan\theta}}=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\tan\theta)^{-\frac{1}{2}}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{\sin\theta}{\cos\theta}\right)^{-\frac{1}{2}}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta d\theta=\frac{1}{2}\frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$=\frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{2\sqrt{2}}$$

BESSEL FUNCTIONS

One of the most important differential equations in applied mathematics is, $x^2y''+xy'+\left(x^2-n^2\right)y=0----$ (1) which is known as Bessel's differential equation of order n.

We employ generalized power series method to find solution in the form of power series of the above differential equation.

Let the power series solution of (1) is of the form

$y=\sum_{r=0}^{\infty}a_r x^{r+k}-----(2)$ where k is a positive real number and a_r's are constants and $a_0\neq 0$.

Differentiating (2) wrt x, we get

$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}-----(3)$ and $\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}-----(4)$

Substituting equation (2), (3) and (4) in equation (1) and simplifying, we get

$$\sum_{r=0}^{\infty}a_r x^{r+k}\left((r+k)^2-n^2\right)+\sum_{r=0}^{\infty}a_r x^{r+k}=0$$

Equating the co-efficients of x^k and x^{k+1} to zero, we get

$a_0\left(k^2-n^2\right)=0$

Since $a_0 \neq 0$, $k^2-n^2=0 \Rightarrow k=\pm n$

Also, $a_1\left((k+1)^2-n^2\right)=0 \Rightarrow a_1=0$

Now, by equating the higher powers of x to zero, we get

$$a_r=-\frac{a_{r-2}}{(r+k)^2-n^2}---------(5)$$

Now Let us discuss two cases.

Case $(i)\,k=n$

Put $k=n$ in equation (5)

Then we get, $a_r=-\dfrac{a_{r-2}}{r^2+2nr}---------(6)$

Put $r=1,2,3,4,........$ equation (6), we get

$$a_2=-\frac{(-1)^1 a_0}{2^2(n+1)1!}---------(7)$$

$$a_4=-\frac{(-1)^2 a_0}{2^4(n+2)(n+1)2!}---------(8)$$

And $a_r=0$ for r = 1,3, 5, …

Writing the values of a_r's in eqn (2) and standardizing the result by taking $a_0=\dfrac{1}{2^n\Gamma(n+1)}$, we get,

$$y_1=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)}\frac{1}{r!}\text{ ---------(9)}$$

(Since this is one solution of equation (1) it is called y_1).

Equation (9) is a series solution of Bessel differential equation and it is called Bessel function of order n. It is denoted as $J_n(x)$.

Particular solutions of Bessel's equation of order n are called Bessel functions or cylindrical functions of order n denoted by , $J_n(x)=\sum_{r=0}^{\infty}\dfrac{(-1)^r}{r!\Gamma(n+r+1)}\left(\dfrac{x}{2}\right)^{n+2r}$

Case(2): Put $k=-n$ we get $J_{-n}(x)$.

Therefore, $J_{-n}(x)=\sum_{r=0}^{\infty}\dfrac{(-1)^r}{r!\Gamma(-n+r+1)}\left(\dfrac{x}{2}\right)^{-n+2r}$

The complete solution of Bessel differential equation is

$y=C_1J_n(x)+C_2J_{-n}(x)$, provided n is a non- integer.

When n is an integer we get a complete solution of (1) in the form,

$y=C_1Y_n(x)+C_2Y_{-n}(x)$

Where $Y_n(x)$ is called Bessel function of second kind of order n and is given by,

$$Y_n(x) = J_n(x)\int \frac{dx}{x\left(J_n(x)\right)^2}$$

Note: $J_{-n}(x)=(-1)^n J_n(x)$ when 'n' is an integer. i.e., when n is an integer, $J_{-n}(x)$ *and* $J_n(x)$ are not linearly independent.

Proof: We know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

Replacing n by –n, we get,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Because n is an integer and r is an integer, $-n+r+1$ is also an integer.

$$\Gamma(-n+r+1)=(r-n)!$$

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!(r-n)!}\left(\frac{x}{2}\right)^{-n+2r} \qquad (r-n\geq 0)$$

Put $r-n=s$ then, $r=n+s$ and $-n+2r=n+2s$

Therefore, $J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!\ s!}\left(\frac{x}{2}\right)^{n+2s}$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+s+1)\ s!}\left(\frac{x}{2}\right)^{n+2s}$$

Thus, $J_{-n}(x)=(-1)^n J_n(x)$ when n is an integer.

Recurrence relations:

We will now learn some important properties of Bessel functions which are called recurrence relations. Bessel Functions of higher order be expressed by Bessel functions of lower order for all real values of 'n'.

Property (1): $x\frac{d}{dx}J_n(x) = nJ_n(x) - xJ_{n+1}(x)$

Proof: Consider, $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$ -------- (1)

Differentiating wrt x on both sides of equation (1) and then multiplying by x and splitting the summation into two summations, we get

$$x.\frac{d}{dx}J_n(x)=n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}-x.\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+1+2r}$$

i.e. $x.\frac{d}{dx}J_n(x)=nJ_n(x)-xJ_{n+1}(x)$

Property (2): $x\frac{d}{dx}J_n(x)=xJ_{n-1}(x)-nJ_n(x)$

Proof: Consider, $J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$ -------- (1)

Differentiate wrt x and multiply with x , we get,

$$x.\frac{d}{dx}J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}.\frac{x}{2}$$

Now, writing n+2r =2(n+r) – n and splitting the summation, we get

$$x.\frac{d}{dx}J_n(x)=x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}-n.\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

i.e. $x\frac{d}{dx}J_n(x)=xJ_{n-1}(x)-nJ_n(x)$

The remaining four results can be proved using these results.

Property (3): $2nJ_n(x)=x\left(J_{n-1}(x)+J_{n+1}(x)\right)$

Proof: consider property (1) and (2)

$x\frac{d}{dx}J_n(x)=nJ_n(x)-xJ_{n+1}(x)$ and $x\frac{d}{dx}J_n(x)=xJ_{n-1}(x)-nJ_n(x)$

Subtracting one equation from the other, we get,

$$0=-x(J_{n-1}(x)+J_{n+1}(x))+2nJ_n(x)$$

i.e. $2nJ_n(x)=x\left(J_{n-1}(x)+J_{n+1}(x)\right)$

Property (4): $\frac{d}{dx}J_n(x)=\frac{1}{2}\left(J_{n-1}(x)-J_{n+1}(x)\right)$

Proof: consider property (1) and (2)

$x\frac{d}{dx}J_n(x) = nJ_n(x)-xJ_{n+1}(x)$ and $x\frac{d}{dx}J_n(x) = xJ_{n-1}(x)-nJ_n(x)$

Adding the above equations, we get

2x$\frac{d}{dx}J_n(x)$ =x($J_{n-1}(x)$ - $J_{n+1}(x)$

This implies $\frac{d}{dx}J_n(x)=\frac{1}{2}\left(J_{n-1}(x)-J_{n+1}(x)\right)$

Property 5: $\frac{d}{dx}\left(x^nJ_n(x)\right)=x^nJ_{n-1}(x)$

Proof: Consider, $\frac{d}{dx}\left(x^nJ_n(x)\right)$ = $x^n\,\frac{d}{dx}(J_n(x))$ + $J_n(x)\,\mathrm{n}\,x^{n-1}$

Using property (2), $x\frac{d}{dx}J_n(x) = xJ_{n-1}(x)-nJ_n(x)$ in the above equation,

We get $\frac{d}{dx}\left(x^nJ_n(x)\right)=$ $x^n\,(J_{n-1}(x)-\frac{n}{x}J_n(x))$ + $J_n(x)\,nx^{n-1}$

Therefore, $\frac{d}{dx}\left(x^nJ_n(x)\right)=x^nJ_{n-1}(x)$

Propery (6): $\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)$

Proof: Consider, $\frac{d}{dx}\left(x^{-n}J_n(x)\right)$ = $x^{-n}\,\frac{d}{dx}(J_n(x))$ + $J_n(x)$(-n x^{-n-1})

Using property (1), $x\frac{d}{dx}J_n(x) = nJ_n(x)-xJ_{n+1}(x)$

$\frac{d}{dx}\left(x^{-n}J_n(x)\right)$ = x^{-n} ($\frac{n}{x}J_n(x)-J_{n+1}(x)$) + $J_n(x)$(-n x^{-n-1})

Hence, $\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)$

Generating function:

Generating function of a sequence of functions $f_n(x)$ is,

$G(x,t)=\sum_{-\infty}^{\infty} f_n(x)t^n$ which generates $f_n(x)$ i.e., $f_n(x)$ appear as coefficients of various powers of t.

Prove that $\sum_{n=-\infty}^{\infty} J_n(x)t^n = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ **or Prove that Generating function for Bessel function of integral order is** $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$**.**

Proof: $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ = $e^{\frac{xt}{2}}.e^{\frac{-x}{2t}}$

Expanding the exponential functions,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \left(1+\frac{\left(\frac{xt}{2}\right)}{1!}+\frac{\left(\frac{xt}{2}\right)^2}{2!}+\frac{\left(\frac{xt}{2}\right)^3}{3!}+-------+\frac{\left(\frac{xt}{2}\right)^n}{n!}+\frac{\left(\frac{xt}{2}\right)^{n+1}}{(n+1)!}+-----\right)\times$$

$$\left(1-\frac{\left(\frac{x}{2t}\right)}{1!}+\frac{\left(\frac{x}{2t}\right)^2}{2!}-\frac{\left(\frac{x}{2t}\right)^3}{3!}+--------+\frac{\left(\frac{x}{2t}\right)^n}{n!}-\frac{\left(\frac{x}{2t}\right)^{n+1}}{(n+1)!}+------\right)$$

Case (i): positive powers of t

Collecting only co-efficient of t^n in the expansion, we get

Co-efficient of t^n = $\left(\frac{\left(\frac{x}{2}\right)^n}{n!}-\frac{\left(\frac{x}{2}\right)^{n+2}}{1!(n+1)!}+\frac{\left(\frac{x}{2}\right)^{n+4}}{2!(n+2)!}-\frac{\left(\frac{x}{2}\right)^{n+6}}{3!(n+3)!}+-----\right)$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{n+2r}}{r!\Gamma(n+r+1)}$$

i.e. Co-efficient t^n $=J_n(x)$

Case (ii): negative powers of t

Collecting coefficients of t^{-n} in the expansion

We get, coefficients of t^{-n} = $(-1)^n J_n(x)$

Case (iii): collecting terms independent of t (n=0)

Terms independent of t = $\left(1-\frac{\left(\frac{x}{2}\right)^2}{(1!)^2}-\frac{\left(\frac{x}{2}\right)^4}{(2!)^2}+-----+\frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}-----\right)$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{0+2r}}{r!\Gamma(0+r+1)} = J_0(x)$$

Combining the results of three cases,

We get $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

Bessel functions of various orders can be obtained as co-efficient of various powers of t on the power series expansion of $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$.

Important outcomes of Generating Function:

1. $\cos(x\sin\theta)=J_0+2\left(J_2\cos 2\theta+J_4\cos 4\theta+..........\right)$
2. $\sin(x\sin\theta)=2\left(J_1\sin\theta+J_3\sin 3\theta+.J_5\sin 5\theta.........\right)$
3. $J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos\left(n\theta-x\sin\theta\right)d\theta$ **(integral form of Bessel function)**

Let us first derive results (1) and (2) which are called Jacobi Series.

Jacobi Series:

$$\cos(x\sin\theta)=J_0+2\left(J_2\cos 2\theta+J_4\cos 4\theta+..........\right)$$

$$\sin(x\sin\theta)=2\left(J_1\sin\theta+J_3\sin 3\theta+.J_5\sin 5\theta.........\right)$$

Proof: we know that $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

Expanding the summation,

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = J_0(x)+tJ_1(x)+t^2J_2(x)+\ t^3J_3(x)+--------+t^{-1}J_{-1}(x)+t^{-2}J_{-2}(x)+\ t^{-3}J_{-3}(x)+----$$

$$=J_0(x)+(t-t^{-1})J_1(x)+(t^2+t^{-2})J_2(x)+\ (t^3-t^{-3})J_3(x)+------ \qquad (1)$$

$$(\because \text{when } n \text{ is an integer}, J_{-n}(x)=(-1)^n J_n(x))$$

$$Let \quad t=\cos\theta+i\sin\theta, then \frac{1}{t}=\cos\theta-i\sin\theta$$

$$t+\frac{1}{t}=2\cos\theta; \quad t-\frac{1}{t}=2i\sin\theta$$

$$t^n+\frac{1}{t^n}=2\cos n\theta; \quad t^n-\frac{1}{t^n}=2i\sin n\theta$$

Using in eqn (1), we get

$$e^{\frac{1}{2}x(2i\sin\theta)} = J_0(x)+2(J_2\cos 2\theta+J_4\cos 4\theta+-----)+2i(J_1\sin\theta+J_3\sin 3\theta+-----)$$

i.e. $\cos(x\sin\theta)+i\sin(x\sin\theta)=J_0(x)+2(J_2\cos 2\theta+J_4\cos 4\theta+--)+2i(J_1\sin\theta+J_3\sin 3\theta+---)$

Equating the real and imaginary parts, we get

$$\cos(x\sin\theta)=J_0+2(J_2\cos 2\theta+J_4\cos 4\theta+..........)$$
$$\sin(x\sin\theta)=2(J_1\sin\theta+J_3\sin 3\theta+.J_5\sin 5\theta.........)$$

Integral form of Bessel Function:

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-x\sin\theta)d\theta$$

Proof: Consider Jacobi Series

$$\cos(x\sin\theta)=J_0+2\left(J_2\cos 2\theta+J_4\cos 4\theta+..........\right)----(1)$$
$$\sin(x\sin\theta)=2\left(J_1\sin\theta+J_3\sin 3\theta+.J_5\sin 5\theta.........\right)----(2)$$

Let n be an even number.

Multiply equation (1) with $\cos n\theta$ and integrate wrt θ between the limits 0 to π

Multiply equation (2) with $\sin n\theta$ and integrate wrt θ between the limits 0 to π

Adding the equations, we get $LHS = \int_0^{\pi} \cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta = \int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$

$$RHS = \int_0^{\pi} J_0 \cos n\theta\, d\theta + 2\int_0^{\pi} \left(J_2\cos 2\theta + J_4\cos 4\theta + ...\right)\cos n\theta d\theta$$

$$+2\int_0^{\pi} \left(J_1\sin\theta + J_3\sin 3\theta + .J_5\sin 5\theta.........\right)\sin n\theta d\theta$$

Using important result: $\int_0^{\pi} \cos n\theta\cos m\theta\, d\theta = \begin{pmatrix} \frac{\pi}{2} when\ n = m, \\ otherwise = 0 \end{pmatrix}$

$$\int_0^{\pi} \cos n\theta \sin m\theta\, d\theta = 0\ always$$

Therefore, $RHS = J_0.0 + 2J_n.\frac{\pi}{2} + 0 = J_n\pi$

Equating LHS and RHS, we get, $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$

Orthogonality of Bessel Functions:

If α and β are the roots of the equation $J_n(ax) = 0$, then prove that

$$\int_0^{a} J_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0\ \ when\ \alpha \neq \beta \\ \frac{a^2}{2}J_{n+1}^2(a\alpha)\ \ when\ \alpha = \beta \end{cases}$$

Proof: Since α and β are the roots of the equation $J_n(ax) = 0$, $J_n(\alpha a) = 0$ and $J_n(\beta a) = 0$ ---- (1)

Consider the Bessel's equations,

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2)u = 0 ----(2)$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2)v = 0 ----(3)$$

The solutions of equations (2) and (3) are,

$u = J_n(\alpha x)\ and\ v = J_n(\beta x)$ (Equations reducible to Bessel form)

Differentiating wrt x,

$$u' = J_n'(\alpha x).\alpha \ and \ v' = J_n'(\beta x).\beta$$
$$u'' = J_n''(\alpha x).\alpha^2 \ and \ v'' = J_n''(\beta x).\beta^2$$

Multiplying equation (2) by $\frac{v}{x}$ and equation (3) by $\frac{u}{x}$ we get,

$$x u'' v + u'v + (\alpha^2 x^2 - n^2)\frac{uv}{x} = 0 \ ----(4)$$
$$x v'' u + uv' + (\beta^2 x^2 - n^2)\frac{uv}{x} = 0 \ ----(5)$$

$$(4)-(5) \ \Rightarrow x\left(u''v - uv''\right) + (u'v - uv') + (\alpha^2 - \beta^2)uvx = 0$$
$$\Rightarrow \frac{d}{dx}\left(x(u'v - uv')\right) = (\beta^2 - \alpha^2)uvx$$
$$\Rightarrow uvx = \frac{1}{(\beta^2 - \alpha^2)}\frac{d}{dx}\left(x(u'v - uv')\right)$$

Integrating wrt x between the limits $0 \ to \ a$, we get,

$$\int_0^a xJ_n(\alpha x)\ J_n(\beta x)\,dx = \frac{1}{(\beta^2 - \alpha^2)}\int_0^a \frac{d}{dx}\left(x(u'v - uv')\right)dx$$
$$= \frac{1}{(\beta^2 - \alpha^2)}\left(x(u'v - uv')\right)_0^a$$
$$= \frac{a(J_n'(\alpha a).\alpha.J_n(\beta a) - J_n'(\beta a).J_n(\alpha a).\beta)}{(\beta^2 - \alpha^2)}$$
$$= 0 \qquad (when \ \alpha \neq \beta)\left(because \ J_n(\alpha a) = 0 \, and \ \ J_n(\beta a) = 0\right)$$

However, when $\alpha = \beta$, the value of the integral is $\frac{0}{0}$ which is indeterminate.

To evaluate the integral, consider α as constant and β as a variable which approaches α

Applying limits on both sides

$$\lim_{\beta \to \alpha}\int_0^a xJ_n(\alpha x)\ J_n(\beta x)\,dx = \lim_{\beta \to \alpha}\left(\frac{a}{(\beta^2 - \alpha^2)}\right)\left(J_n'(\alpha a).J_n(\beta a)\alpha - 0\right)$$

Applying L'Hospital rule on RHS

$$\int_0^a xJ_n^2(\alpha x)\ dx = \lim_{\beta\to\alpha} \frac{a^2(J_n'(a\alpha).\alpha.J_n(\beta a)}{2\beta}$$

$$= \frac{a^2(J_n'(a\alpha))^2}{2} \text{ ---------(2)}$$

But $x\frac{d}{dx}J_n(x) = nJ_n(x) - xJ_{n+1}(x)$

Therefore, $a\alpha J_n'(a\alpha) = n\,J_n(a\alpha) - a\alpha\, J_{n+1}(a\alpha)$

But $J_n(a\alpha) = 0 \Rightarrow J_n'(a\alpha) = -J_{n+1}(a\alpha)$

Using in equation (2)

$$\int_0^a xJ_n^2(\alpha x)\ dx = \frac{a^2(J_{n+1}(a\alpha))^2}{2} \text{ when } \alpha = \beta$$

Combining both the results, we get

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0 & when\alpha \neq \beta \\ \frac{a^2}{2}J_{n+1}^2(a\alpha) when\alpha = \beta \end{cases}$$

PROBLEMS

1. Using the values of $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ find $J_{\frac{3}{2}}(x)$ and $J_{-\frac{3}{2}}(x)$

. Ans: We know that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$ and $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$

We have recurrence property, $2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$ ---------- (1)

Put $n = \frac{1}{2}$ in equation (1), we get, $J_{\frac{1}{2}}(x) = x\left(J_{\frac{-1}{2}}(x) + J_{\frac{3}{2}}(x)\right)$

Using the expressions for $J_{\frac{1}{2}}(x)$ and $J_{\frac{-1}{2}}(x)$, we get

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Similarly, put $n=-\frac{1}{2}$ in equation (1), we get, $J_{\frac{-3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\sin x+\frac{\cos x}{x}\right)$

2. Express J_5 *in terms of* J_0 *and* J_1.

Ans: We have recurrence property, $2nJ_n(x)=x\left(J_{n-1}(x)+J_{n+1}(x)\right)$ ---------- (1)

Put $n=1$ in equation (1), we get $J_2(x)=\frac{1}{x}\left(2J_1(x)-xJ_0(x)\right)$ ---------------- (2)

Put $n=2$ in equation (1), we get $J_3(x)=\frac{1}{x}\left(4J_2(x)-xJ_1(x)\right)$ ----------------- (3)

Put n=3 in eqn (1), we get $J_4(x)=\frac{1}{x}\left(6J_3(x)-xJ_2(x)\right)$ --------------- (4)

Put n=4 in eqn (1), we get $xJ_5(x)=\left(8J_4(x)-xJ_3(x)\right)$------------------ (5)

Using eqn (4) in eqn (5), we get, $xJ_5(x)=\left(8\frac{1}{x}\left(6J_3(x)-xJ_2(x)\right)-xJ_3(x)\right)$

Therefore, $J_5(x)=\left(\frac{48}{x}-x\right)J_3(x)-8J_2$

Using eqn (3),

$$J_5(x)=\left(\frac{48}{x}-x\right)\frac{1}{x}\left(4J_2(x)-xJ_1(x)\right)-8J_2$$

Using eqn (2) and simplifying, we get $J_5(x)=\left(\frac{384}{x^4}-\frac{72}{x^2}-1\right)J_1+\left(\frac{12}{x}-\frac{192}{x^3}\right)J_0$

3. Prove that, $\int J_3(x)dx = c-J_2(x)-\frac{2}{x}J_1(x)$

Ans: We know that, $\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)$

$$\therefore \int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$$

$$\int J_3(x)dx=\int x^2x^{-2}J_3(x)dx+c$$

$$= x^2 \int x^{-2} J_3(x)dx - \int 2x\left[\int x^{-2} J_3(x)dx\right]dx + c$$

$$= x^2\left(-x^{-2} J_2(x)\right) - \int 2x\left[-x^{-2} J_2(x)\right]dx + c$$

$$= c - J_2(x) + \int \frac{2}{x} J_2(x)dx$$

$$= c - J_2(x) - \frac{2}{x} J_1(x)$$

4. Prove that, $\int x\, J_0{}^2(x)dx = \frac{1}{2}x^2\left(J_0{}^2(x) + J_1{}^2(x)\right)$

Ans: $\int xJ_0{}^2(x)dx = J_0{}^2(x)\frac{1}{2}x^2 - \int 2J_0(x)J_0{}'(x)\frac{1}{2}x^2 dx$

$$= \frac{1}{2}x^2 J_0{}^2(x) - \int x^2 J_0(x) J_0{}'(x)dx$$

$$= \frac{1}{2}x^2 J_0{}^2(x) + \int x^2 J_0(x) J_1(x)dx$$

$$= \frac{1}{2}x^2 J_0{}^2(x) + \int xJ_0(x)xJ_1(x)dx$$
$$= \frac{1}{2}x^2 J_0{}^2(x) + \int \frac{d}{dx}(xJ_1(x))xJ_1(x)dx$$
$$= \frac{1}{2}x^2 J_0{}^2(x) + \frac{1}{2}\left(xJ_1(x)\right)^2$$
$$= \frac{1}{2}x^2\left(J_0{}^2(x) + J_1{}^2(x)\right)$$

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