UNIT - 3 - Random Variables and Probability distributions

Problems on discrete probability distribution

1) In a voice communication system with 50 lines the random variable is the number of lines in use at a particular time.

Ans: $X = \{0, 1, 2, 3, \dots 50\}$.

2) A die is tossed thrice. A success is "getting 1 or 6" on a toss. Find the mean and variance of the number of successes.

Solution:

Probability of a success $= \frac{2}{6} = \frac{1}{3}$.

$\therefore$ Probability of a failure $= 1 - \frac{1}{3} = \frac{2}{3}$.

Let X be the R.V which denotes the no. of success

X	0	1	2	3
$P(X=x)$	$8/27$	$4/9$	$2/9$	$1/27$.

Prob. of no success = Prob of all 3 failures

$$= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

Probability of one success and 2 failures

$$\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}.$$

Probability of two success and 1 failures.

$$\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

Probability of three successes.

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = 27.$$

Mean

$$\mu_x = \sum_x x \cdot P(X=x)$$

$$= 0 + \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

Also,

$$\sum_x x^2 \cdot P(X=x)$$

$$= 0 + \frac{4}{9} + \frac{8}{9} + \frac{9}{27} = \frac{5}{3}$$

Variance

$$\sigma^2 = \sum_x x^2 \cdot P(X=x) - \mu_X^2$$

$$= \frac{5}{3} - (1)^2 = \frac{2}{3}$$

3) A coin is tossed three times let X denote the number of heads showing up Find its mean and Variance.

Probability of success or getting a head $= \frac{1}{2}$

Probability of failure $= \frac{1}{2}$

Let X denote the no. of success

X	0	1	2	3.
$P(X=x)$.	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Probabity of getting a head all the three times

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Probability of one success

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{3}{8}$$

Probability of two success $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$$= \frac{3}{8}$$

Probability of failure in all three times

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Mean $= \sum_x x \cdot P(X=x)$

$$\mu_X = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$

Also

$$\sum_x x^2 \cdot P(X=x)$$

$$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = \frac{6}{2} = 3$$

Variance $\sigma_X^2 = \sum x^2 \cdot P(X=x) - \mu_X^2$

$$= 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4}$$

$$= \frac{3}{4}$$

4) The probability density function of a random Variable X is given as follows.

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11k	13 K

Find the Value of K. Evaluate

i) $P(X<4)$; $P(X \geq 5)$ and $P(3 < X \leq 6)$.

ii) What will be the minimum Value of k so that $P(X \leq 2) > 0.3$?

We know that

$$\sum P(x) = 1.$$

$$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$49k = 1 \Rightarrow \boxed{k = 1/49}$$

X	0	1	2	3	4	5	6
P(x)	1/49	3/49	5/49	7/49	9/49	11/49	13/49

1) $P(X < 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$.

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

$$P(X \geq 5) = P(X=5) + P(X=6)$$

$$= \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < X \leq 6) = P(X=4) + P(X=5) + P(X=6)$$

$$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$$

2) $P(X \leq 2) > 0.3$

$$P(X=0) + P(X=1) + P(X=2) > 0.3$$

$$k + 3k + 5k > 0.3$$

$$9k > 0.3 \Rightarrow k > 1/30$$

Thus minimum value of $k = 1/30$

5) The probability density function of a random Variable X is given as follows

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find the Value of k. Evaluate $P(X<6)$; $P(X\geq 6)$ and $P(0<x<5)$.

We know that $\Sigma P(X) = 1$

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0$$

$(10-K)(K+1) = 0$, $\boxed{K = 1/10.}$ or. $K = -1$

Not possible. As probability can't be -ve.

$$P(X<6) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}. = \frac{81}{100}.$$

$$P(X\geq 6) = \frac{2}{100} + \frac{7}{100} + \frac{1}{10} = \frac{19}{100}.$$

$$P(0<x<5) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$$

$$= \frac{8}{10} = \frac{4}{5}.$$

6). The Sample space of a random experiment is $\{a, b, c, d, e, f\}$ and each outcome is equally likely. A random Variable is defined as follows.

Outcome	a	b	c	d	e	f
X	0	0	1.5	1.5	2	3

Determine the probability mass function of X.
Use the probability mass function to determine the following probabilities.

a) $P(X = 1.5)$ b) $P(0.5 < X < 2.7)$ c) $P(X > 3)$
d) $P(0 \leq X < 2)$ e) $P(X = 0 \text{ or } X = 2)$

Solution

Each outcome is equally likely. Hence.
Probability of each outcome $= 1/6$.
Therefore Probability mass function X is

X	$P(X=x)$
0	$2/6$
1.5	$2/6$
2	$1/6$.
3	$1/6$.

a) $P(X = 1.5) = 2/6$

b) $P(0.5 < X < 2.7) = P(X = 1.5) + P(X = 2)$

$$= \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = 1/2.$$

c) $P(X > 3) = 0.$

d) $P(0 \leq X < 2) = P(X = 0) + P(X = 1.5)$

$$= \frac{2}{6} + \frac{2}{6} = \frac{4}{6} = \frac{2}{3}.$$

e) $P(X = 0 \text{ or } X = 2)$

$$= P(X = 0) + P(X = 2)$$

$$= \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

7) The space shuttle flight control system called PASS (primary Avionics Software set) uses four independent computers working in parallel. At each critical step, the computers 'vote' to determine the appropriate step. The probability that a computer will ask for roll to the left when a roll to the right is appropriate is 0.0001. Let X denote the number of computers that vote for a left roll when a right roll is appropriate. What is the probability mass function of X? What is mean Variance of X?

Solution :

Let X be a random Variable can Take any integer from 0 to 4.

The probability that a computer will ask for a roll to the left when a roll to the right is appropriate $= 0.0001$

This means that the probability that a computer will ask for a right roll when a right roll is appropriate will be $1 - 0.0001$
$= 0.9999$.

$$X \sim Bin(n, p).$$

The probability mass function function of X is

$$P(X = x) = {}^nC_x \, p^x \cdot q^{n-x}.$$

X	$P(X=x)$
0	${}^4C_0 \cdot (0.0001)^0 \cdot (0.9999)^{4-0}$ $= 0.9996$.
1	${}^4C_1 (0.0001)^1 \cdot (0.9999)^{4-1}$ $= 0.0003999$
2	${}^4C_2 (0.0001)^2 (0.9999)^{4-2}$ $= 5.999 \times 10^{-8}$
3	${}^4C_3 (0.0001)^3 \cdot (0.9999)^{4-3}$ $= 3.9996 \times 10^{-12}$
4	${}^4C_4 (0.0001)^4 \cdot (0.9999)^{4-4}$ $= 10^{-16}$

Mean $= n \cdot p$

$= 4 \times 0.0001 = \underline{0.0004}$.

Variance $= n p \cdot q$

$= 4 \times 0.0001 \times 0.9999$

$= \underline{4 \times 10^{-4}}$.

Problems on Continuous probability Function.

8) Is the function defined as follows a density function ?

$$f(x) = e^{-x} \quad (x \geq 0)$$
$$= 0 \quad (x < 0).$$

If so, determine the probability that random Variable X having this density with fall in The interval (1, 2).

Solution :

The density function f(x) is always positive and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

clearly $f(x) \geq 0$.

and $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} e^{-x} \cdot dx$

$= \left.\frac{e^{-x}}{-1}\right]_0^{\infty} = -1\left[e^{-\infty} - e^{0}\right]$

$= -1\left[0 - 1\right] = 1$

Hence The function $f(x)$ Satisfies the requirements for a density function.

$$P(1 \leq x \leq 2) = \int_1^2 e^{-x} \cdot dx = e^{-1} - e^{-2} = \underline{0.233}$$

9) Let X be a continuous random Variable with probability density function given by

$$f(x) = kx \quad (0 \leq x \leq 2)$$
$$= 2k \quad (2 \leq x < 4)$$
$$= -kx + 6k \quad (4 \leq x < 6)$$

Find the Value of k and the mean value of X.

Since the total probability is Unity.

$$\int_0^6 f(x) dx = 1.$$

$$\int_0^2 kx \cdot dx + \int_2^4 2k \, dx + \int_4^6 (-kx + 6k) dx = 1$$

$$k \cdot \left[\frac{x^2}{2}\right]_0^2 + 2k \left[x\right]_2^4 + -k \left[\frac{x^2}{2}\right]_4^6 + 6k\left[x\right]_4^6 = 1$$

$$2k + 4k + (-10k + 12k) = 1.$$

$$\boxed{k = 1/8.}$$

Mean Value of X $= \int_0^6 x \cdot f(x)\,dx$

$$= \int_0^2 k \cdot x^2\,dx + \int_2^4 2k \cdot x\,dx + \int_4^6 (-kx^2 + 6kx)\,dx$$

$$= k \cdot \left[\frac{x^3}{3}\right]_0^2 + 2k \cdot \left[\frac{x^2}{2}\right]_2^4 - k \cdot \left[\frac{x^3}{3}\right]_4^6 + 6k \cdot \left[\frac{x^2}{2}\right]_4^6$$

$$= k \cdot \left(\frac{8}{3}\right) + k \cdot (12) - k \cdot \left(\frac{152}{3}\right) + 3k \cdot (20)$$

$$= 24k = 24 \cdot \frac{1}{8} = 3$$

Problems on Bernoulli Distribution.

10) A coin has a probability of 0.5 of landing heads when tossed. Let $X = 1$ if the coin comes up heads and $X = 0$ if the coin comes up tails. What is the distribution X.

Solution:

Since $X = 1$ when head comes up, heads is the success outcome. The success probability

$P(X = 1) = 0.5$.

Therefore $X \sim$ Bernoulli (0.5)

11) A die has probability $1/6$ of coming up 6 when rolled. Let $X=1$ if the die comes up 6 and $X=0$ otherwise. What is the distribution of X?

Solution:

The success probability is $p = P(X=1) = 1/6$.

Therefore $X \sim \text{Bernoulli}(1/6)$.

12) Ten percent of the components manufactured by a certain process are defective. A component is chosen at random. Let $X=1$ if the component is defective and $X=0$ otherwise. What is the distribution of X?

Solution:

The success probability is $p = P(X=1) = 0.1$.

Therefore $X \sim \text{Bernoulli}(0.1)$.

Problems on Binomial distribution

13) The phone lines to an airline reservation system are occupied 40% of the time. Assume that the events that the lines are occupied on successive calls are independent. Assume that 10 calls are placed to the airline.

i) What is the probability that for exactly three calls the lines are occupied?

ii) What is the probability that for atleast one call the lines are not occupied?

iii) What is the expected number of calls in which the lines are all occupied.

Let X be the random variable of number of calls during which the phone lines were occupied.

$X \sim Bin(n, p)$. Here $n = 10$

$p = 0.4$.

$\therefore q = 1 - p = 0.6$.

Probability mass function is

$nC_x \, p^x \, q^{n-x} = 10C_x \, (0.4)^x \, (0.6)^{n-x}$.

1) $P(X = 3) = 10C_3 \, (0.4)^3 \, (0.6)^{10-3}$.

$= 0.2149$.

2) P(atleast one call is not occupied)

$P(X \leq 9) = 1 - P(X=10)$.

$= 1 - 10C_{10} \, (0.4)^{10} \, (0.6)^{10-10}$.

$= 1 - 0.0001 = 0.99989$

The probability that atleast one call unoccupied is same as the probability that atmost 9 calls occupied.

iii) Expected number of calls $= n \cdot p$

$= 10 \times 0.4$

$= 4$

14) Heart failure is due to either natural occurance (87%) or outside factors (13%). Outside factors are related to induce substances or foreign objects. Natural occurence are caused by arterial blockage, disease and infection. Suppose that 20 patients will visit an emergency room with heart failure. Assume that cause of heart failure between individuals are independent.

a) What is The probability that three individuals have conditions caused by outside factors?

b) What is the probability that ~~for atleast one~~ conditions ~~catt~~ three or more individuals have conditions caused by outside factors?

c) What is the mean and standard deviation of the number of individuals with conditions caused by outside factors?

Solution:

Let X = The number of patients with heart failure who enters the emergency room has the condition due to outside factors

$X \sim Bin(n, p)$

Here $n = 20$, $p = 0.13$

$\therefore q = 0.87$.

a) $P(X=3) = {}^{20}C_3 \cdot (0.13)^3 \cdot (0.87)^{20-3}$

$= \underline{0.235}$

b) $P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - (0.06171 + 0.18443 + 0.26181)$

$= \underline{0.492}$.

c) Mean $= n \cdot p$

$= 20 \times 0.13 = \underline{2.6}$

Vareance $= npq = 20 \times 0.13 \times 0.87$

$= 2.262$

Standard deviation $= \sqrt{npq} = \underline{1.504}$.

15) In eight throws of a fair die, 5 or 6 is considered a success. Find the mean of the number of success and the standard deviation.

$n = 8, \quad p = \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

Mean : $n \cdot p$ $\qquad$ S.D $= \sqrt{npq}$

$= 8 \cdot \frac{1}{3} = \frac{8}{3}$ $\qquad$ $= \sqrt{8 \times \frac{1}{3} \times \frac{2}{3}}$

$= \sqrt{\frac{16}{9}} = \frac{4}{3}$

16) The probability that a man hits a target is $\frac{1}{3}$ how many times must be fire so that the probability of hitting the target atleast once is more than 90% ?

Given

$p = \frac{1}{3}$

$P(X \geq 1) > 0.9$

$1 - P(X=0) > 0.9$

$1 - {}^nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n > 0.9$

$1 - \left(\frac{2}{3}\right)^n > 0.9$ equivalently $\left(\frac{2}{3}\right)^n < 0.1$

Compute. $\left(\frac{2}{3}\right)^4 = 0.1975$ $\left(\frac{2}{3}\right)^5 = 0.1316$

$n > 5.6$. $\qquad \left(\frac{2}{3}\right)^6 = 0.08779$

So, he must fire 6 times so that the probability of his hitting target atleast once is more than 90%

Problems on Poisson Distribution

17) Suppose that X has Poisson distribution with a mean of 4. Determine the following probabilities

a) $P(X=0)$ b) $P(X \leq 2)$ c) $P(X=4)$ d) $P(X=8)$

Probability mass function is $\dfrac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$X \sim \text{Poisson}(4)$

Here $\lambda = 4$.

a) $P(X=0) = \dfrac{e^{-4} \cdot 4^{0}}{0!} = 0.01832$

b) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \dfrac{e^{-4} \cdot 4^{0}}{0!} + \dfrac{e^{-4} \cdot 4^{1}}{1!} + \dfrac{e^{-4} \cdot 4^{2}}{2!}$

$= 0.01832 + 0.07326 + 0.14652$

$= 0.2375$

c) $P(X=4) = \dfrac{e^{-4} \cdot 4^{4}}{4!} = 0.1954$

d) $P(X=8) = \dfrac{e^{-4} * 4^{8}}{8!} = 0.0298$

18) The number of telephone calls that arrive at a phone exchange is often modelled as a Poisson random Variable Assume that on the average there are 10 calls per hour

a) What is the probability that there are exactly 5 calls in one hour?

b) What is the probability that there are 3 or fewer calls in one hour?

c) What is the probability that there are exactly 15 calls in two hour?

d) What is the probability that there are exactly 5 calls in 30 minutes?

Let X be the number of telephone calls that arrive at a phone exchange.

$$X \sim \text{Poisson}(10).$$

$$\text{Probability mass function} = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

Here $\lambda = 10$.

a) $P(X=5) = \dfrac{e^{-10}\, 10^{5}}{5!} = 0.0378$

b) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.01033$

c) $P(X = x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

P(15 calls in 2 hrs)

Here $\lambda = 20$ $\left[\begin{aligned} &2 \times 10 \\ &= 20 \end{aligned} \right)$

$x = 15$

$P(X = 15) = \dfrac{e^{-20} \cdot 20^{15}}{15!} = \underline{0.0516}$

P(15 calls in 2 hrs)

d) $P(X = x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

P(5 calls in 30 minutes) $\Rightarrow$ Here

$\lambda = 0.5 \times 10$
$= 5$

[On the average there are 10 call in one hr)]

$\therefore P(X = 5) = \dfrac{e^{-5} \cdot 5^5}{5!} = \underline{0.1755}$

19) Fit a Poisson distribution for the following data and calculate the theoretical frequencies.

X	0	1	2	3	4
f	122	60	15	2	1

Solution

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{60+36+6+4}{200} = 0.5 = \lambda$$

Mean of Poisson distribution

$$\lambda = 0.5.$$

Hence the theoretical frequency for x success is

$$N \cdot P(X=x) = N \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$ where $x = 0, 1, 2, 3, 4$

$$N = 200.$$

$\therefore$ The theoretical frequencies are.

X	0	1	2	3	4
f	121	61	15	2	0.

20) In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing.

(i) no defective. (ii) one defective

(iii) two defective blades respectively in a consignment of 10000 packets.

Solution

We know that $\lambda = np$

$$= 10 \times \frac{1}{500} = 0.02.$$

$P(\text{no defective}) = P(X=0)$

$$= \frac{e^{-0.02} \cdot (0.02)^0}{0!} = 0.9802$$

No. of packets containing no defective blade $= 10,000 \times 0.9802$
$= 9802$

$P(\text{one defective})$

$= P(X=1)$

$$= \frac{e^{-0.02}(0.02)^1}{1!} = 0.0196$$

No. of packets containing one defective blade

$= 10,000 \times 0.0196 = 196.$

$P(\text{two defective}) = P(X=2)$

$$= \frac{e^{-0.02} (0.02)^2}{2!} = 0.000196.$$

No. of packets containing two defective blade

$= 0.000196 \times 10,000$

$\simeq 2.$

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large ($n \to \infty$) and p the probability of success is very small ($p \to 0$) so that np tends to a fixed finite constant say λ.

We have in the case of binomial distribution the probability of x successes out of n trials

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-\overline{x-1})}{x!} p^x q^{n-x}$$

$$= \frac{n \cdot n\left(1-\frac{1}{n}\right) n \cdot \left(1-\frac{2}{n}\right) \cdots n\left(1-\frac{x-1}{n}\right)}{x!} p^x \cdot q^{n-x}$$

$$= \frac{n^x \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!} p^x \cdot q^{n-x}$$

$$= \frac{(np)^x \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!\, q^x} \cdot q^n$$

But

$$np = \lambda \quad ; \quad q^n = (1-p)^n$$

$$= \left(1-\frac{\lambda}{n}\right)^n = \left[\left(1-\frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right]^{-\lambda}$$

Denoting $-\frac{\lambda}{n} = k$ we have

$$q^n = \left\{(1+k)^{1/k}\right\}^{-\lambda}$$

$$\rightarrow e^{-\lambda} \quad \text{as } n \rightarrow \infty \text{ or } k \rightarrow 0.$$

NOTE: $\lim_{k \rightarrow 0} (1+k)^{1/k} = e.$

Further $q^x = (1-p)^x$

$\rightarrow 1$ for a fixed x as $p \rightarrow 0$.

Also the factors $\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)$

will all tend to 1 as $n \rightarrow \infty$

Thus we get

$$\boxed{P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$$

$P(x)$ is also called Poisson probability function.

Problems on Normal distribution.

21) If X is normally distributed with mean 6 and standard deviation 5. Find

i) $P(0 \leq X \leq 8)$

Given $\mu = 6$, $\sigma = 5$

Standard Normal Variate, the Z score is

$$Z = \frac{x-\mu}{\sigma}$$

If $x = 0$, $Z = \frac{0-6}{5} = -1.2$

If $x = 8$ $Z = \frac{8-6}{5} = 0.4$

$\therefore P(0 \leq X \leq 8) = P(-1.2 \leq Z \leq 0.4)$.

$= 0.38493 + 0.15542$

$= \underline{0.54035}$

2) $P(|X-6| < 10)$

$P(|X-6| < 10) = P(-10 < X-6 < 10)$

$= P\left(-\frac{10}{5} < \frac{X-6}{5} < \frac{10}{5}\right)$

$= P(-2 < Z < 2)$

$= \underline{0.9545}$

22) The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months

Solution

Mean, $\mu = 8$

S.D $\sigma = 2$.

No. of pairs of shoes $= 5000$.

Total months $(x) = 12$

$$Z = \frac{x-\mu}{\sigma} = \frac{12-8}{2} = 2.$$

$$P(X > 12) \Rightarrow P(Z \geq 2)$$

$$= 0.0228$$

No. of pairs of shoes whose life is more than 12 months is $= 5000 \times 0.0228$

$= 114$.

Replacement after 12 months $= 5000 - 114$

$= 4886$ pairs of shoes.

23) If the height of 300 students is normally distributed with 64.5 inches and standard deviation 3.3 inches. Find the height below which 99% of the students lie.

Solution

Let X be the random variable, the height of a student

$$X \sim N(64.5, (3.3)^2)$$

Here $\mu = 64.5$

$\sigma = 3.3$.

To find the height below which 99% of the students lie. Let x_1 be the height.

$$P(X < x_1) = 0.99.$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{x_1-\mu}{\sigma}\right) = 0.99.$$

$$P(Z < z_1) = 0.99$$

$$\Rightarrow z_1 = 2.33.$$

$$\therefore \quad z_1 = \frac{x_1-\mu}{\sigma}$$

$$2.33 = \frac{x_1 - 64.5}{3.3.}$$

$$\Rightarrow x_1 = 72.25 \text{ inches}.$$

24) In a normal distribution 30.85% of the items are over 64 and 8% are under 45. Find the mean and S.D.

Let μ and σ be the mean and S.D of the normal distribution.

By data

$P(x < 45) = 0.08$ $\quad$ $P(x > 64) = 0.3085$

When $x = 45$, $z = \frac{45-\mu}{\sigma} = z_1$ (say).

$x = 64$, $z = \frac{64-\mu}{\sigma} = z_2$ (say).

Thus we have

$P(z < z_1) = 0.08$ $\quad$ $P(z > z_2) = 0.3085$

$z_1 = -1.4051$. $\quad$ $z_2 = 0.5001$

$\Rightarrow \frac{45-\mu}{\sigma} = -1.4051$ $\quad$ $\frac{64-\mu}{\sigma} = 0.5001$

$45-\mu = -1.4051\sigma$. $\rightarrow (1)$ $\quad$ $64-\mu = 0.5001\sigma$ $\rightarrow (2)$

Solving (1) & (2) we get

$$\boxed{\begin{aligned}\mu &= 59.01\\ \sigma &= 9.97\end{aligned}}$$

Problems on Exponential distribution

25) If X is an exponential Variate with mean 3 find (i) $P(X>1)$ (ii) $P(X<3)$

$X \sim Exp(\lambda)$ Given $\mu_X = \frac{1}{\lambda} = 3.$

$\Rightarrow \lambda = 1/3$

Probability mass fn.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & x \le 0 \end{cases}$$

i) $P(X>1) = \int_1^\infty \frac{1}{3} \cdot e^{-1/3 x} dx$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_1^\infty$$

$$= -\left[e^{-\infty} - e^{-1/3}\right]$$

$$= +e^{-1/3} = 0.71653$$

(OR)

$F(x) = P(X \le x)$

$= 1 - e^{-\lambda x}.$

$P(X>1) = 1 - P(X \le 1)$

$= 1 - F(1)$

$= 1 - [1 - e^{-1/3}]$

$= e^{-1/3} = 0.71653.$

(ii) $P(X<3) = \int_0^3 1/3 \cdot e^{-1/3 x} dx$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_0^3 = -1 \cdot \left[e^{-1} - 1\right]$$

$= 1 - e^{-1}$

$= 0.6321.$

(OR)

$P(X<3) = F(3)$

$= 1 - e^{-\frac{1}{3} \cdot 3} = 1 - e^{-1}$

$= \underline{0.63212}$

26) The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth

1) ends less than 5 minutes
2) ends between 5 and 10 minutes.

Let X be the random variable which denotes the length of telephone conversation in a booth.

$X \sim Exp(\lambda)$ Given Average = 5 min

$$\frac{1}{\lambda} = 5$$

$$\boxed{\lambda = 1/5}$$

i) $P(X<5) = \int_0^5 \frac{1}{5} \cdot e^{-\frac{1}{5}x} dx = \underline{0.6321}$

(OR) $P(X<5) = F(5)$

$= 1 - e^{-\frac{1}{5} \cdot 5}$

$= \underline{0.6321}$

ii) $P(5<X<10) = \int_5^{10} \frac{1}{5} \cdot e^{-\frac{1}{5}x} dx = \underline{0.2325}$

(OR) $P(5<X<10) = F(10) - F(5)$

$= 1 - e^{-\frac{1}{5}(10)} - [1 - e^{-\frac{1}{5} \cdot 5}]$

$= [1 - e^{-2}] - [1 - e^{-1}] = \underline{0.2325}$

27) In a certain town the duration of a shower is exponential distribution with mean 5 minutes. What is the probability that a shower will last for

(i) 10 minutes or more

(ii) less than 10 minutes

(iii) between 10 and 12 minutes.

Given mean $= \frac{1}{\lambda} = 5$

$\Rightarrow \lambda = 1/5$

Let X be a random variable which denote the duration of a shower.

i) $P(X \geq 10) = \int_{10}^{\infty} \lambda . e^{-\lambda x} dx$

$= \int_{10}^{\infty} 1/5 \, e^{-1/5 x} dx.$

$= \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_{10}^{\infty} = -1\left[e^{-\infty} - e^{-1/5 \cdot 10}\right]$

$= -1\left[0 - e^{-2}\right] = e^{-2}$

$= 0.13534$

(ii) $P(X < 10) = \int_{0}^{10} \lambda e^{-\lambda x} dx$

$= \int_{0}^{10} \frac{1}{5} . e^{-1/5 x} dx = \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_{0}^{10}$

$= 0.8647$

(iii) $P(10 < X < 12) = F(12) - F(10)$

$= \left[1 - e^{-1/5(12)}\right] - \left[1 - e^{-1/5(10)}\right]$

$= 0.0446$

Problems on Normal Approximation to binomial dist

28) The manufacturing of semiconductor chips produces 2% defective chips. Assume the chips are independent and that a lot contains 1000 chips

a) Approximate the probability that more Than 25 chips are defective

b) Approximate the probability that between 20 and 30 chips are defective.

Solution:

Given parameters are.

$p = 2\% = 0.02$

$n = 1000$

Mean $= np = 1000 \times 0.02 = 20.$

S.D $= \sqrt{npq} = \sqrt{np(1-p)}$

$= \sqrt{1000 \times 0.02 \times}$

$= 4.43$

a) The probability That more than 25 chips are defective.

using Continuity Correction, the probability is represented as

$P(X > 25) = P(X > 25.5)$

$= P\left(\frac{X-\mu}{\sigma} > \frac{25.5-20}{4.43}\right)$

$= P(Z > 1.241) = 0.1073$

b) The probability that between 20 and 30 chips are defective

Using continuity correction, the probability is represented as

$$P(20 < X < 30) = P(20.5 < X < 29.5)$$

$$= P\left(\frac{20.5 - 20}{4.43} < \frac{X-\mu}{\sigma} < \frac{29.5 - 20}{4.43}\right)$$

$$= P(0.113 < Z < 2.144)$$

$$= \underline{\underline{0.4390}}.$$

29) There were 49.7 million people with some type of long-lasting condition or disability living in the United States in 2000. This represented 19.3 percent of the majority of civilians aged five and over. A sample of 1000 persons is selected at random.

(a) Approximate the probability that more than 200 persons in the sample have a disability.

(b) Approximate the probability that between 180 and 300 people in the sample have a disability.

Given

$$p = 19.3\% = 0.193$$

$$n = 1000$$

$$\bar{x} = np = 1000 \times 0.193 = 193.$$

$$S.D = \sqrt{npq}$$

$$= \sqrt{1000 \times 0.193 \times 0.807}$$

$$= \underline{\underline{12.48}}$$

a) $P(x > 200) = P(x > 200 + 0.5)$

$$= P(x > 200.5)$$

$$= P\left(\frac{x-\mu}{\sigma} > \frac{200.5 - 193}{12.48}\right)$$

$$= P(Z > 0.6010)$$

$$= 0.2743 \simeq \underline{\underline{0.2739}}$$

b) $P(180 < X < 300)$

$$= P(180.5 < X < 299.5)$$

$$= P\left(\frac{180.5 - 193}{12.48} < \frac{X-\mu}{\sigma} < \frac{299.5 - 193}{12.48}\right)$$

$$= P(-1.0016 < Z < 8.5336)$$

$$= 0.84134$$

$$\simeq \underline{\underline{0.84173}}$$

Problems on Normal Approximation to Poisson distribution

30) Suppose that X is a Poisson random Variable with $\lambda = 6$

a) Compute the exact probability that X is less than 4

b) Approximate the probability that X is less than 4 and Compare to the result in part (a).

c) Approximate the probability that $8 < X < 12$.

Solution.

$X \sim \text{Poisson}(6)$.

a) $P(X<4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$

P.d.f of Poisson: $\dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$= \frac{e^{-6} \cdot 6^0}{0!} + \frac{e^{-6} \cdot 6^1}{1!} + \frac{e^{-6} \cdot 6^2}{2!} + \frac{e^{-6} \cdot 6^3}{3!}$$

$$= 0.1512.$$

b) $P(X<4)$

Now, let's use the normal approximation to the Poisson to calculate an approximate probability. First, we have to make a continuity correction

$$\therefore P(X<4) = P(X < 3.5)$$

$$= P\left(\frac{X-\lambda}{\sqrt{\lambda}} < \frac{3.5-6}{\sqrt{6}}\right)$$

$$= P(Z < -1.0206)$$

$$= 0.15372$$

c) $P(8 < X < 12)$

$\Rightarrow P(8.5 < X < 11.5)$

$= P\left(\frac{8.5-6}{\sqrt{6}} < Z < \frac{11.5-6}{\sqrt{6}}\right)$

$= P(1.0206 < Z < 2.2454)$

$= \underline{0.14135}$.

31) Hits to a high-volume website are assumed to follow a Poisson distribution with a mean of 10,000 per day. Approximate each of the following

a) the probability of more than 20,000 hits in a day.

b) The probability of less than 9900 hits in a day.

c) The value such that the probability that the number of hits in a day exceed the value is 0.01.

Solution.

$X \sim \text{Poisson}(10{,}000)$.

a) $P(X > 20000) = P(X > 20000.5)$

$= P\left(Z > \frac{20000.5 - 10000}{\sqrt{10000}}\right)$

$= P(Z > 100.005)$

$= \underline{0}$

b) $P(X < 9900)$

$$= P(X < 9899.5)$$

$$= P\left(Z < \frac{9899.5 - 10{,}000}{\sqrt{10{,}000}}\right)$$

$$= P(Z < -1.005)$$

$$= \underline{0.15745}$$

c) $P(X > X_1) = 0.01$

$$P(X > X_1 + 0.5) = 0.01)$$

$$P(Z > Z_1) = 0.01$$ where $$Z_1 = \frac{(X_1 + 0.5) - 10{,}000}{\sqrt{10{,}000}}.$$

$$Z_1 = 2.326348$$

$$2.3263 = \frac{(X_1 + 0.5) - 10{,}000}{100}$$

$$\boxed{10232.13 = X_1}$$