ENGINEERING MECHANICS

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Unit-III

Moment of Inertia

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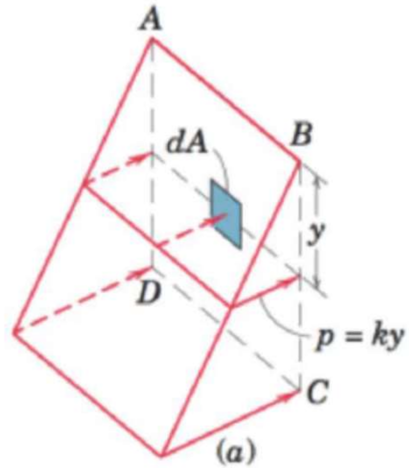
When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area.



Frequently the intensity of the force (pressure or stress) is proportional to the distance of the line of action of the force from the moment axis.
The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area.

We see, therefore, that the total moment involves an integral of form $\int(\text{distance})^2\, d\,(\text{area})$. This integral is called the moment of inertia or the second moment of the area. The integral is a function of the geometry of the area and occurs frequently in the applications of mechanics. Thus it is useful to develop its properties in some detail and to have these properties available for ready use when the integral arises.

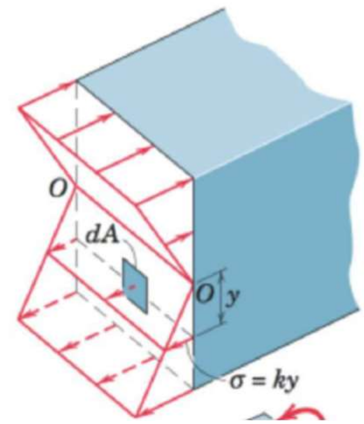
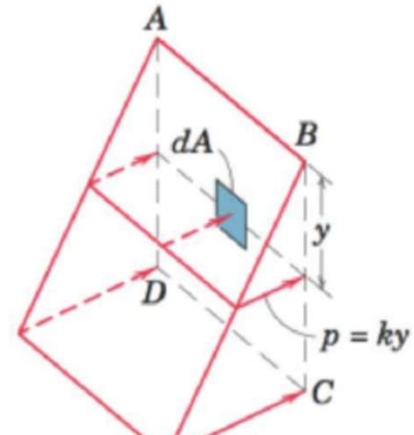
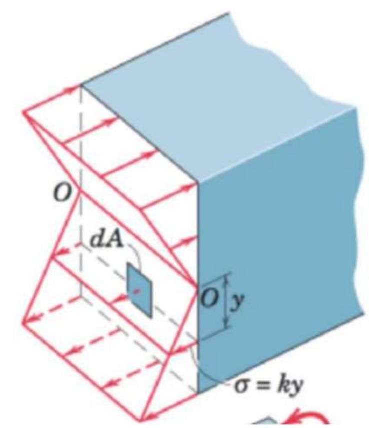


Figure (a) illustrates the physical origin of these integrals. In part a of the figure, the surface area ABCD is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis AB. The moment about AB due to the pressure on the element of area dA is py dA $=ky^2$ dA. Thus, the integral in question appears when the total moment $M = k \int y^2 \, dA$ is evaluated.

In Fig. (b) shows the distribution of stress acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends. At any section of the beam, a linear distribution of force intensity or stress σ, given by $\sigma = ky$, is present. The stress is positive (tensile) below the axis O-O and negative (compressive) above the axis. We see that the elemental moment about the axis 0-0 is $dM = y(\sigma \, dA) = ky^2 \, dA$. Thus, the same integral appears when the total moment $M = k \int y^2 \, dA$ is evaluated.



(a)



Moment of Inertia

The concept of inertia is provided by Newton’s first law of motion . The property of matter by virtue of which it resists any change in its state of rest or of uniform motion is called inertia.

The translator inertia is defined as mass Whereas the rotational inertia is termed as moment of inertia

In other words,
“ The moment of inertia is the rotational analogue of the mass , it plays the role of resisting a change in rotational motion in quite the same sense as mass plays the role of resisting a change in translator motion”

Rectangular and Polar Moments of Inertia

Consider the area A in the x-y plane, Fig. A/2. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively. The moments of inertia of A about the same axes are therefore

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \tag{A/1}$$

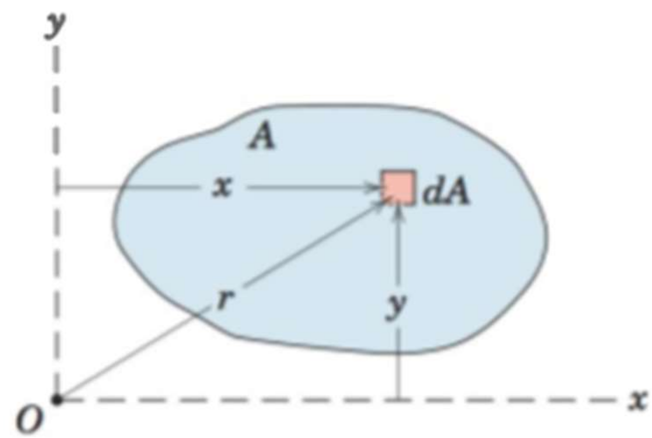


Figure A/2

where we carry out the integration over the entire area.

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2\,dA$. The moment of inertia of the entire area about O is

$$\boxed{I_z = \int r^2\,dA} \tag{A/2}$$

The expressions defined by Eqs. A/1 are called *rectangular* moments of inertia, whereas the expression of Eq. A/2 is called the *polar* moment of inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$\boxed{I_z = I_x + I_y} \tag{A/3}$$

For an area whose boundaries are more simply described in rectangular coordinates than in polar coordinates, its polar moment of inertia is easily calculated with the aid of Eq. A/3.

Radius of gyration

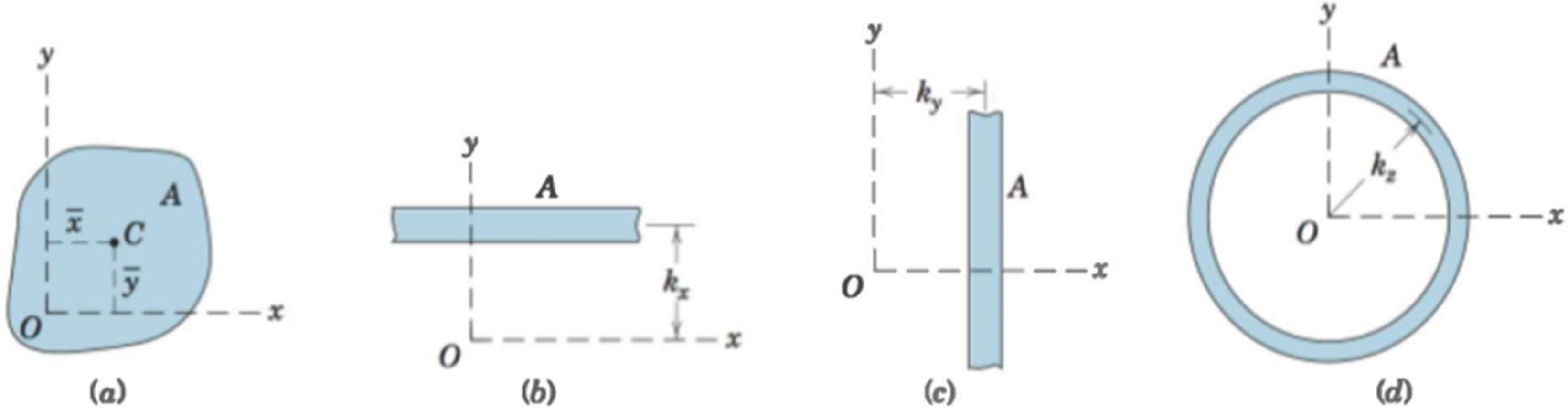


Figure A/3

Consider an area A, which has rectangular moments of inertia Ix and Iy and a polar moment of inertia Iz about 0. We now visualize this area as concentrated into a long narrow strip of area A at a distance Kx from the x-axis, Figb. By definition the moment of inertia of the strip about the x-axis will be the same as that of the original area if kx^2 A = Ix. The distance kx is called the **radius of gyration** of the area about the x-axis.

A similar relation for they-axis is written by considering the area as concentrated into a narrow strip parallel to they-axis as shown in Fig. A/3c.

Also, if we visualize the area as concentrated into a narrow ring of radius kz as shown in Fig. A/3d, we may express the polar moment of inertia as kz^2 A = Iz.

Radius of gyration

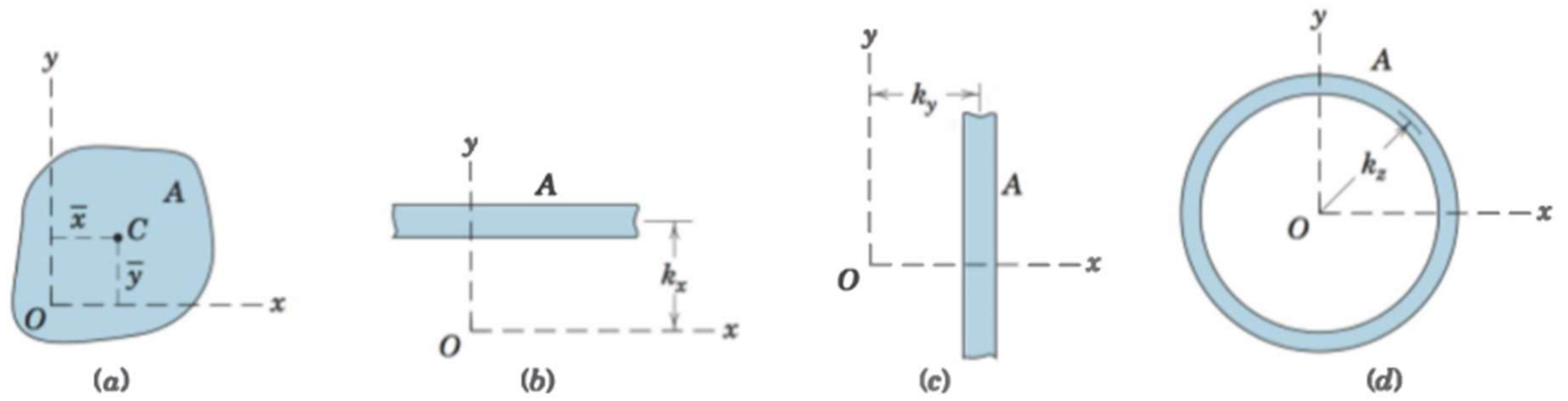


Figure A/3

In summary we write

$$I_x = k_x^2 A \qquad k_x = \sqrt{I_x/A}$$
$$I_y = k_y^2 A \quad \text{or} \quad k_y = \sqrt{I_y/A}$$
$$I_z = k_z^2 A \qquad k_z = \sqrt{I_z/A}$$

Radius of gyration

$$I_x = k_x^2 A \qquad k_x = \sqrt{I_x/A}$$
$$I_y = k_y^2 A \quad \text{or} \quad k_y = \sqrt{I_y/A}$$
$$I_z = k_z^2 A \qquad k_z = \sqrt{I_z/A}$$

A rectangular or polar moment of inertia may be expressed by specifying the radius of gyration and the area.

$$k_z^2 = k_x^2 + k_y^2$$

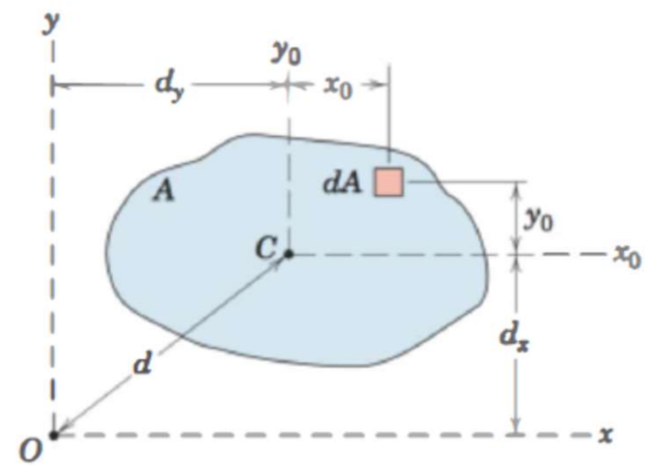
<u>Conditions for parallel axis theorem</u>

1. Two axis should be there and two axis must be parallel to each other
2. Between two axis, one axis has to pass through the centroidal axis

Transfer of Axes

The moment of inertia of an area about a non centroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig the x_0-y_0 axes pass through the centroid C of the area. Let us now determine the moments of inertia of the area about the parallel x-y axes. By definition, the moment of inertia of the element dA about the x-axis is

$$dI_x = (y_0 + d_x)^2\, dA$$



Expanding and integrating give us

$$I_x = \int y_0^2\, dA + 2d_x \int y_0\, dA + d_x^2 \int dA$$

We see that the first integral is by definition the moment of inertia Ix about the centroidal X_0-axis. The second integral is zero, since $\int y_0\, dA = AY_0$ and Y_0 is automatically zero with the centroid on the X_0-axis. The third term is simply Adx^2.

Thus, the expression for Ix and the similar expression for Iy become

Transfer of Axes

$$dI_x = (y_0 + d_x)^2\,dA$$

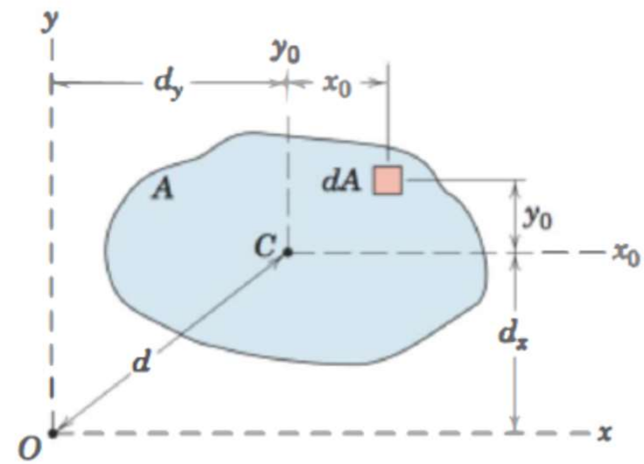
Expanding and integrating give us

$$I_x = \int y_0^2\,dA + 2d_x \int y_0\,dA + d_x^2 \int dA$$

The third term is simply Ad_x^2. Thus, the expression for Ix and the similar expression for Iy become

$$\boxed{\begin{aligned} I_x &= \bar{I}_x + Ad_x^2 \\ I_y &= \bar{I}_y + Ad_y^2 \end{aligned}}$$

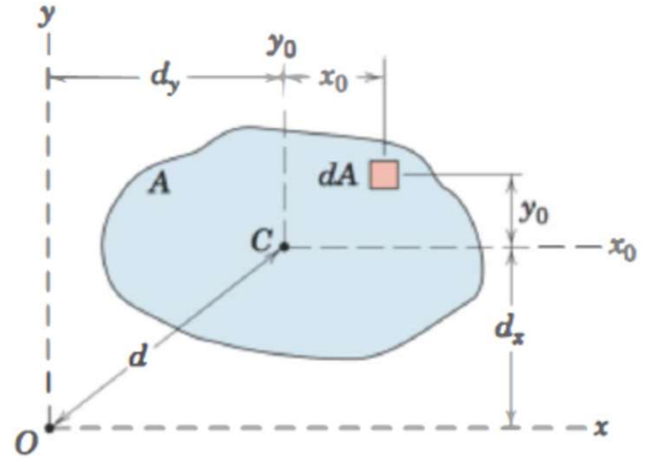
Eqn. A/6



Transfer of Axes

By Eq. A/3 the sum of these two equations gives

$$I_z = \bar{I}_z + Ad^2$$ Eqn. A/6a



Equations A/6 and A/6a are the so-called parallel-axis theorems. Two points in particular should be noted.

- First, the axes between which the transfer is made must be parallel, and
- second, one of the axes must pass through the centroid of the area.

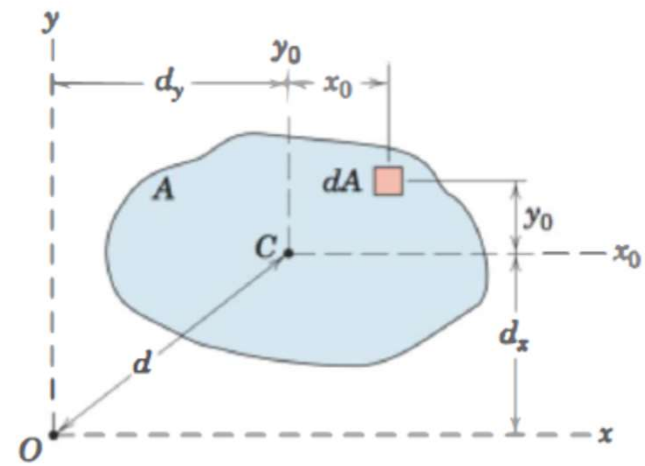
If a transfer is desired between two parallel axes neither of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis

- and then to transfer from the centroidal axis to the second axis.

Transfer of Axes

By Eq. A/3 the sum of these two equations gives

$$I_z = \bar{I}_z + Ad^2 \longrightarrow \text{Eqn. A/6a}$$



The parallel-axis theorems also hold for radii of gyration. With substitution of the definition of k into Eqs. A/6, the transfer relation becomes

$$k^2 = \bar{k}^2 + d^2$$

where k is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes. The axes may be either in the plane or normal to the plane of the area.

Determine the moments of inertia of the rectangular area about the centroidal X-o and yo-axes, the centroidal polar axis z0 through C, the x-axis, and the polar axis z through 0.

Solution: For the calculation of the moment of inertia Ix about the xo-axis, a horizontal strip of area b dy is chosen so that all elements of the strip have the same y-coordinate. Thus,

$[I_x = \int y^2\, dA]$ $\qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{1}{12} bh^3$ **Ans.**

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

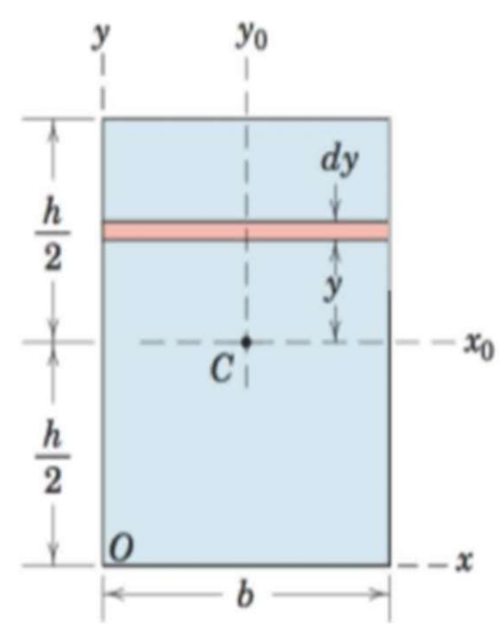
$$\bar{I}_y = \frac{1}{12} hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$[\bar{I}_z = \bar{I}_x + \bar{I}_y]$ $\qquad \bar{I}_z = \frac{1}{12}(bh^3 + hb^3) = \frac{1}{12} A(b^2 + h^2)$ **Ans.**

By the parallel-axis theorem, the moment of inertia about the x-axis is

$[I_x = \bar{I}_x + Ad_x{}^2]$ $\qquad I_x = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3 = \frac{1}{3} Ah^2$ **Ans.**



Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.

Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

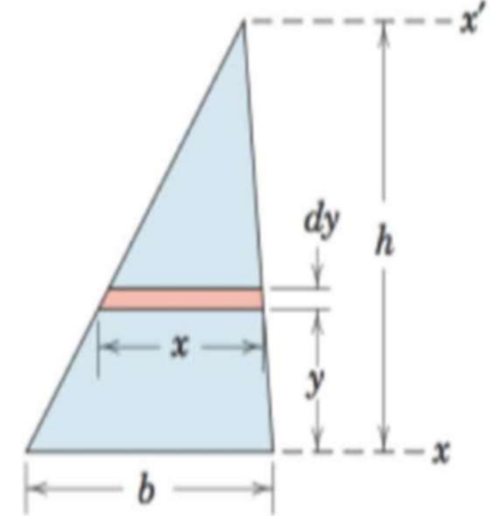
$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem, the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$



Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.

Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus,

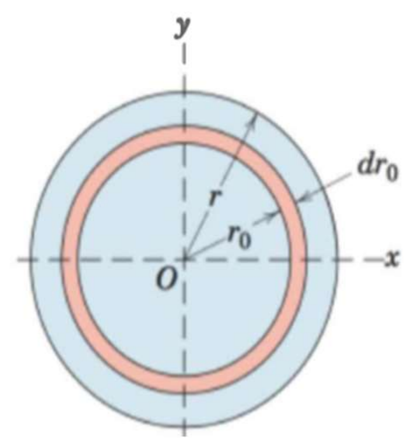
$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \qquad \textit{Ans.}$$



The radius of gyration about the diametral axis is

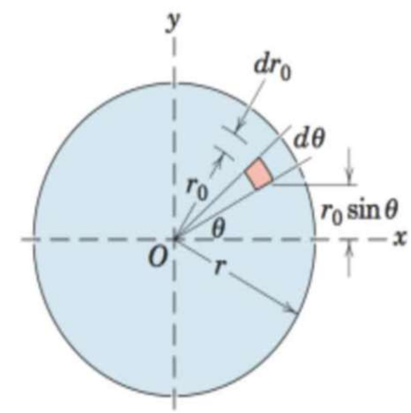
$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad \textit{Ans.}$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_0\, dr_0\, d\theta$ shown in the lower figure. By definition

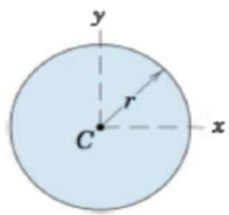
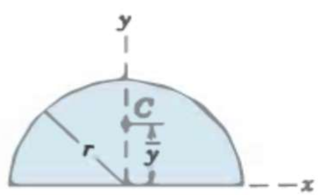
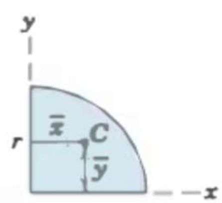
$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^{r} (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

$$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \qquad \textit{Ans.}$$

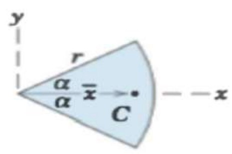
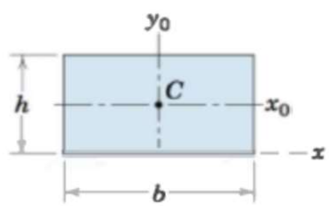
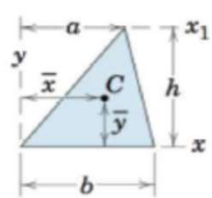


ENGINEERING MECHANICS
Moment of Inertia

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Circular Area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$

ENGINEERING MECHANICS

Moment of Inertia

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Area of Circular Sector 	$\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha}$	$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2}\sin 2\alpha)$ $I_y = \frac{r^4}{4}(\alpha + \frac{1}{2}\sin 2\alpha)$ $I_z = \frac{1}{2}r^4\alpha$
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

COMPOSITE AREAS

It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity.

The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A, its centroidal moment of inertia I, the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the productAd^2. For any one of the parts the moment of inertia about the desired axis by the transfer-of-axis the

the moment of inertia about the desired axis by the transfer-of-axis theorem is $\bar{I} + Ad^2$. Thus, for the entire section the desired moment of inertia becomes $I = \Sigma\bar{I} + \Sigma Ad^2$.

For such an area in the x-y plane, for example, and with the notation of Fig. A/4, where $\bar{I}_x$ is the same as I_{x_0} and $\bar{I}_y$ is the same as I_{y_0} the tabulation would include

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^2$$
$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^2$$

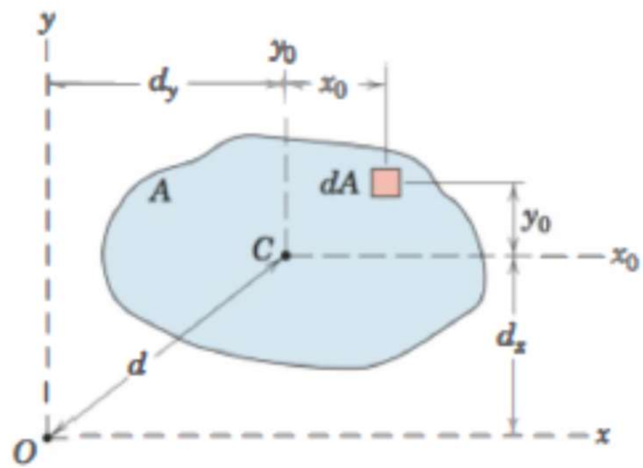
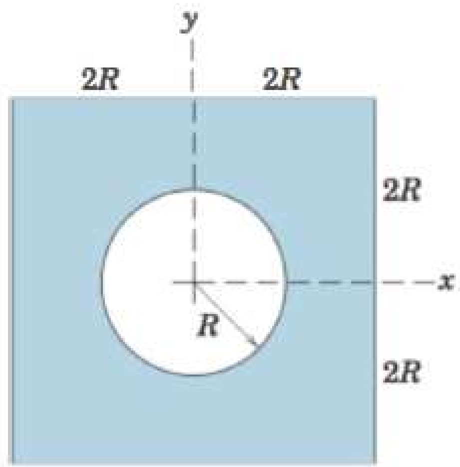
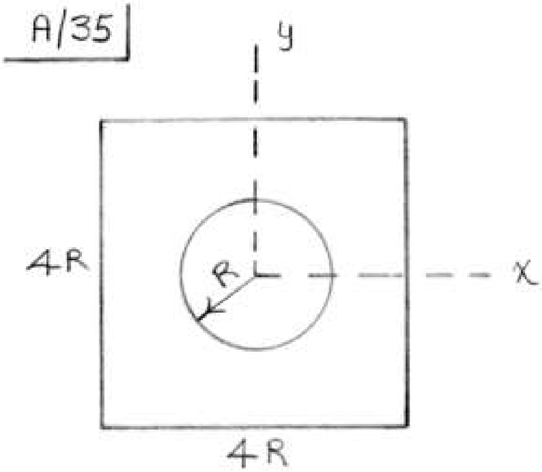


Figure A/4

A/35 Determine the moment of inertia about the x-axis of the square area without and with the central circular hole.



A/35



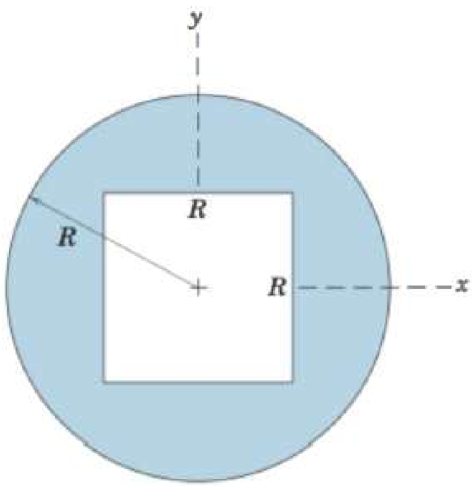
Without hole, $I_x = \frac{1}{12}(4R)(4R)^3 = \frac{64}{3}$

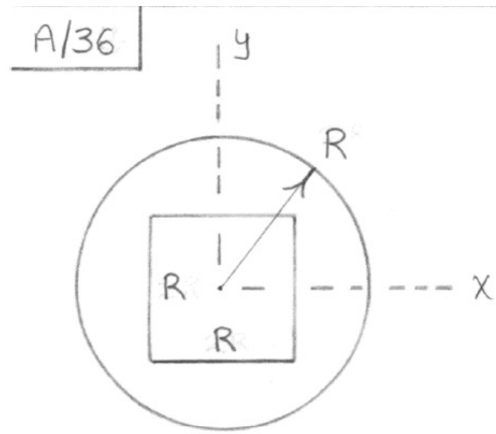
$(21.3R^4)$

With hole, $I_x = \frac{64}{3}R^4 - \frac{1}{4}(\pi R^2)R^2$

$= \underline{20.5R^4}$

A/36 Determine the polar moment of inertia of the circular area without and with the central square hole.





Without square hole:

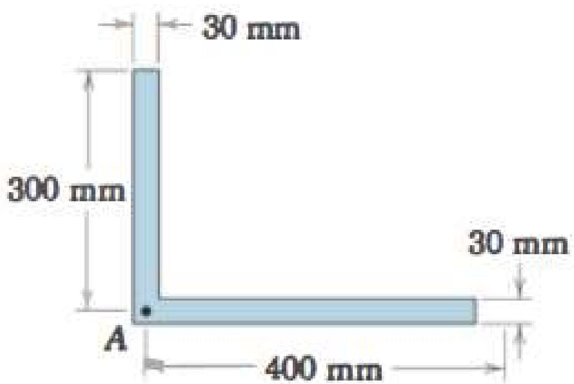
$$I_z = 2I_x = 2\left(\frac{1}{4}\pi R^2 \cdot R^2\right) = \underline{1.571 R^4}$$

With hole:

$$I_z = 1.571R^4 - 2\left(\frac{1}{12} R \cdot R^3\right) = \underline{1.404 R^4}$$

(a reduction of 10.61%)

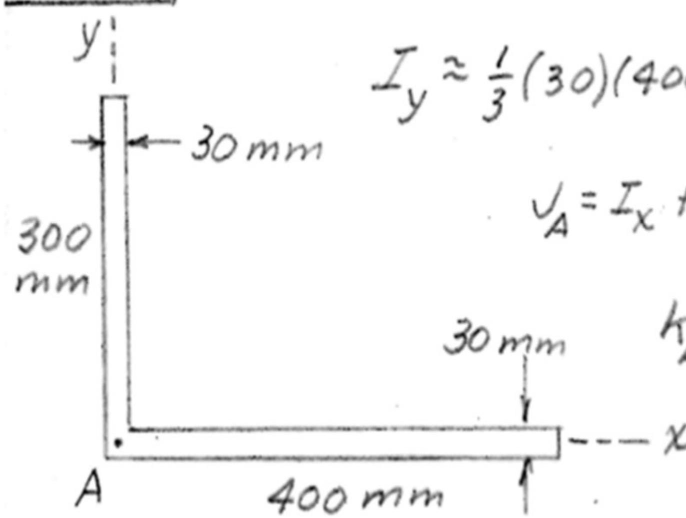
A/37 Calculate the polar radius of gyration of the area of the angle section about point A. Note that the width of the legs is small compared with the length of each leg. Neglect the width effect.



A/37

$$I_x \approx \frac{1}{3}(30)(300)^3 + 0 = 270(10)^6 \text{ mm}^4$$

$$I_y \approx \frac{1}{3}(30)(400)^3 + 0 = 640(10)^6 \text{ mm}^4$$

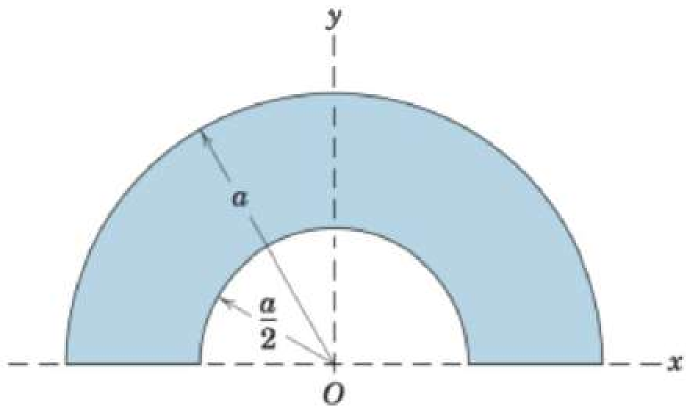


$$J_A = I_x + I_y = 910(10)^6 \text{ mm}^4$$

$$k_A = \sqrt{J_A/A} = \sqrt{\frac{910(10)^6}{30(300+400)}}$$

$$k_A = \underline{208 \text{ mm}}$$

A/38 Determine the rectangular and polar radii of gyration of the shaded area.



A/38

$$I_z = \frac{1}{2}\left[\frac{\pi a^4}{2} - \frac{\pi\left(\frac{a}{2}\right)^4}{2}\right] = \frac{15}{64}\pi a^4$$

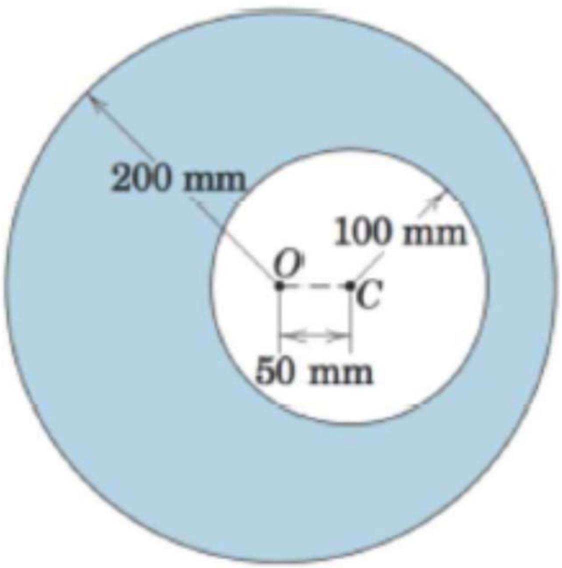
$$k_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{\frac{15}{64}\pi a^4}{\frac{3}{8}\pi a^2}} = \underline{\frac{\sqrt{10}}{4}a}$$

From $k_x^2 + k_y^2 = k_z^2$ and the fact that

$k_x = k_y$ for the present case,

$$2k_x^2 = \left(\frac{\sqrt{10}}{4}a\right)^2, \quad k_x = k_y = \underline{\frac{\sqrt{5}}{4}a}$$

A/39 Calculate the polar radius of gyration of the shaded area about the center O of the larger circle.



Area $A = A_1 - A_2 = \pi(200^2 - 100^2) = 3(10^4)\pi \text{ mm}^2$

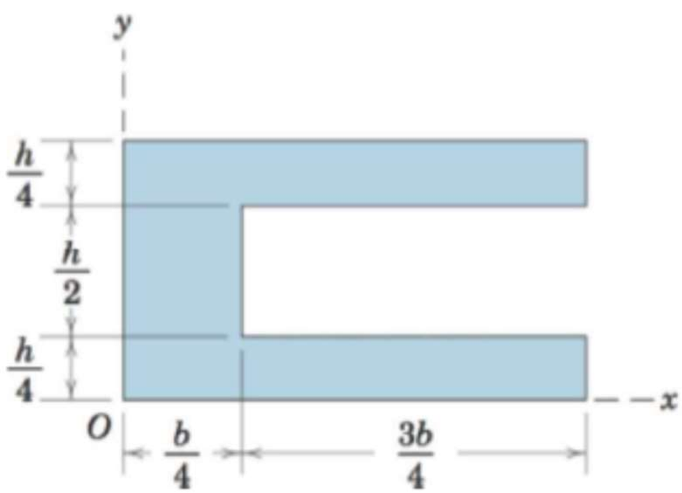
① $I_{O_1} = \frac{1}{2}(\pi \cdot 200^2)(200^2) = 8(10^8)\pi \text{ mm}^4$

② $I_{O_2} = \frac{1}{2}(\pi \cdot 100^2)(100^2) + \pi(100^2)(50^2) = 0.75(10^8)\pi \text{ mm}^4$

So $I_O = I_{O_1} - I_{O_2} = 7.25(10^8)\pi \text{ mm}^4$

$k_O = \sqrt{I_O/A} = \sqrt{\frac{7.25(10^8)\pi}{3(10^4)\pi}} = \underline{155.5 \text{ mm}}$

A/40 Determine the percent reduction in both area and area moment of inertia about the y-axis caused by removal of the rectangular cutout from the rectangular plate of base b and height h.



Full rectangle: $A = bh$, $I_y = \frac{1}{3}hb^3$

With cutout: $A = bh - \frac{3b}{4}\left(\frac{h}{2}\right) = \frac{5}{8}bh$

$$I_y = \frac{1}{3}hb^3 - \left[\frac{1}{12}\frac{h}{2}\left(\frac{3b}{4}\right)^3 + \frac{3}{8}bh\left(\frac{b}{4}+\frac{3b}{8}\right)^2\right]$$

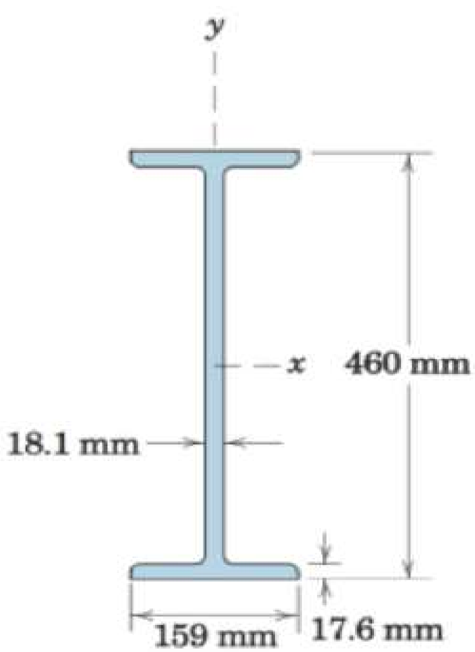
$$= \frac{65}{384}hb^3$$

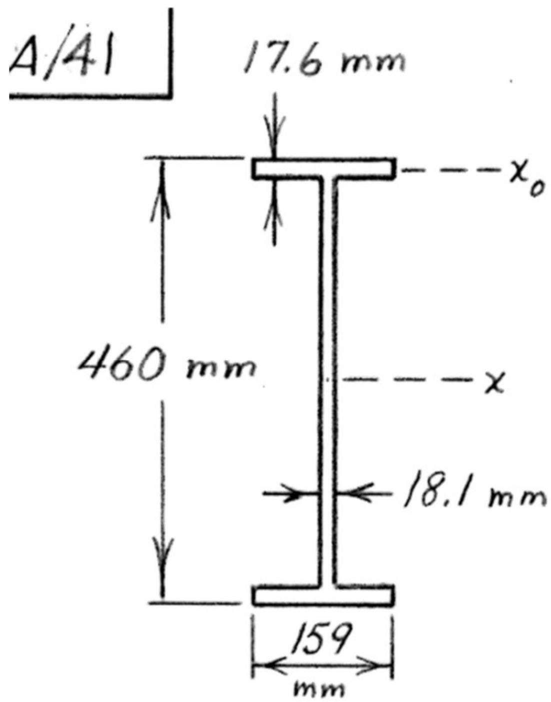
Percent reductions:

$$n_A = \frac{bh - \frac{5}{8}bh}{bh}(100\%) = \underline{37.5\%}$$

$$n_{I_y} = \frac{\frac{1}{3}hb^3 - \frac{65}{384}hb^3}{\frac{1}{3}hb^3} = \underline{49.2\%}$$

A/41 The cross-sectional area of an I-beam has the dimensions shown. Obtain a close approximation to the handbook value of $I_x = 385 \times 10^6$ mm^4 by treating the section as being composed of three rectangles.





Flanges: $\bar{I}_x = I_{x_0} + Ad^2$

$$= 2\left\{\frac{1}{12}(159)(17.6^3) + 159(17.6)\left(230 - \frac{17.6}{2}\right)^2\right\}$$

$$= 2\{7.22(10^4) + 1.369(10^8)\} \text{ mm}^4$$

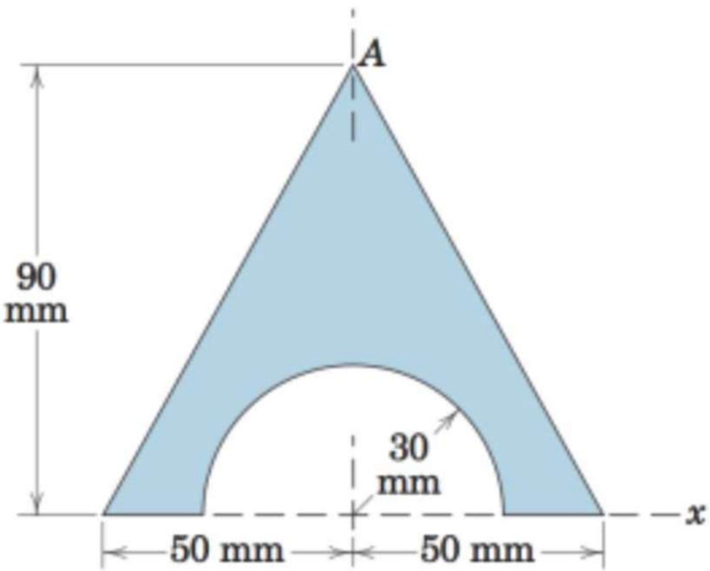
$$= 2.74(10^8) \text{ mm}^4$$

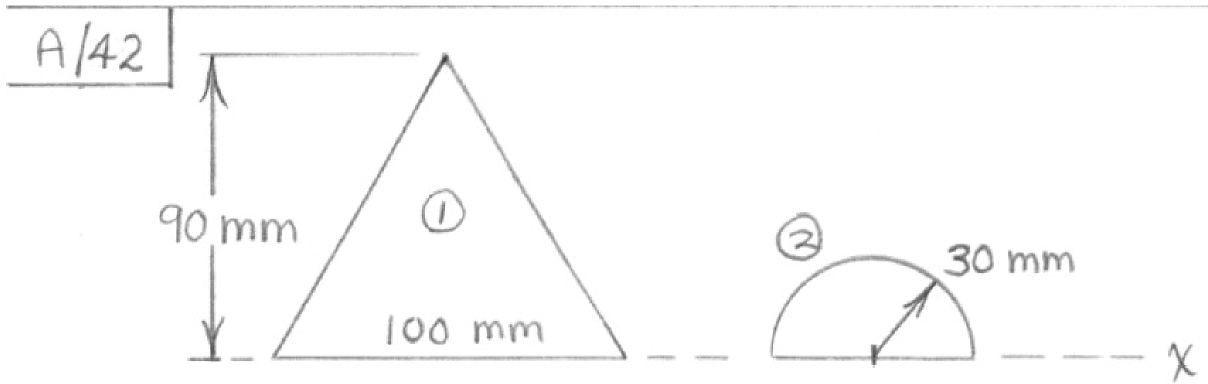
Web: $\bar{I}_x = \frac{1}{12}(18.1)(460 - 2[17.6])^3$

$$= 1.156(10^8) \text{ mm}^4$$

Total $\underline{\bar{I}_x = 3.90(10^8) \text{ mm}^4}$

A/42 Calculate the moment of inertia of the shaded area about the x-axis.



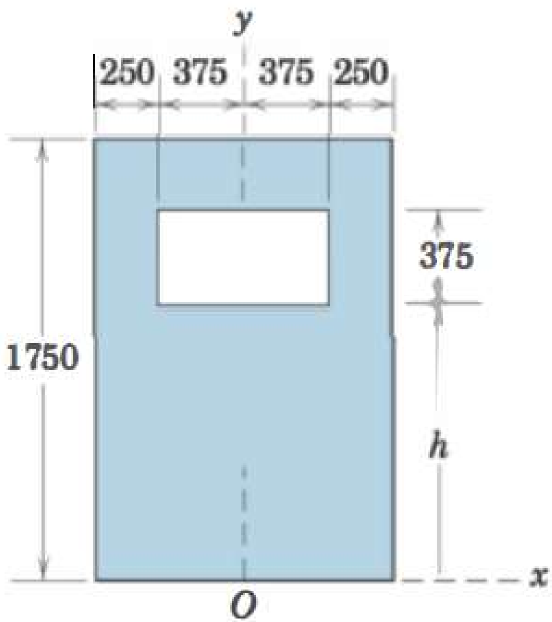


$$I_{x_1} = \frac{1}{12}(100)(90^3) = 6.08(10^6)\ \text{mm}^4$$

$$I_{x_2} = -\frac{\pi(30^4)}{8} = -0.318(10^6)\ \text{mm}^4$$

$$\text{So } I_x = (6.08 - 0.318)10^6 = \underline{5.76(10^6)\ \text{mm}^4}$$

A/43 The variable h designates the arbitrary vertical location of the bottom of the rectangular cutout within the rectangular area. Determine the area moment of inertia about the x-axis for (a) h = 1000 mm and (b) h = 1500 mm.



Dimensions in millimeters

(a) $h = 1000$ mm (hole complete)

$$I_x = \frac{1}{3}(1250)(1750^3) - \left[\frac{1}{12}(750)(375)^3 + 750(375)\left(1000 + \frac{375}{2}\right)^2\right]$$

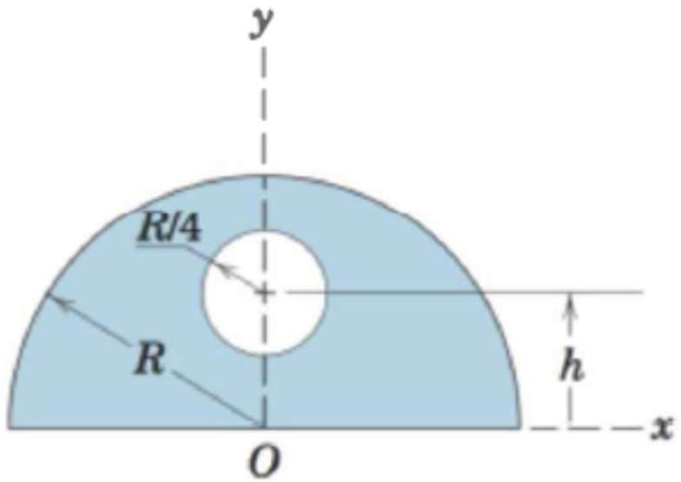
$$= \underline{1.833(10^{12})\ \text{mm}^4} \text{ or } \underline{1.833\ \text{m}^4}$$

(b) $h = 1500$ mm (250 mm of hole in play)

$$I_x = \frac{1}{3}(1250)(1750^3) - \left[\frac{1}{12}(750)(250)^3 + 750(250)\left(1500 + \frac{250}{2}\right)^2\right]$$

$$= \underline{1.737(10^{12})\ \text{mm}^4 \text{ or } 1.737\ \text{m}^4}$$

A/44 The variable h designates the arbitrary vertical location of the center cutout within the semicircular area. Determine the area moment of inertia about the x-axis for (a) h = 0 and (b) h = R/2.



(a) $h = 0$ (One-half of hole considered)

$$I_x = \frac{\pi R^4}{8} - \frac{\pi (R/4)^4}{8} = \frac{255}{2048}\pi R^4$$

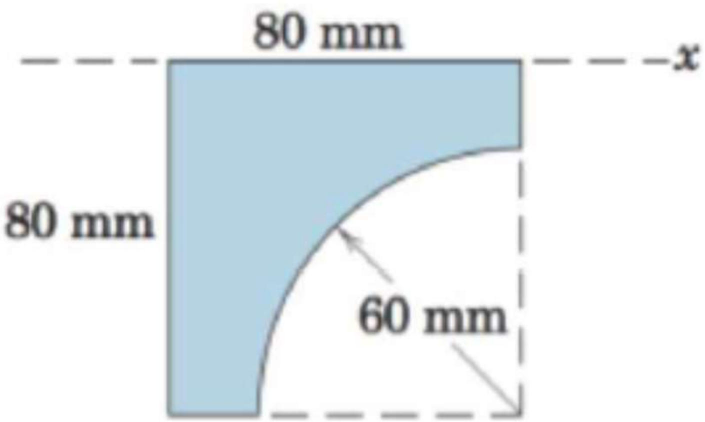
$$(0.391 R^4)$$

(b) $h = \frac{R}{2}$ (Entire hole now in play)

$$I_x = \frac{\pi R^4}{8} - \left[\frac{\pi (R/4)^4}{4} + \pi\left(\frac{R}{4}\right)^2\left(\frac{R}{2}\right)^2\right]$$

$$= \frac{111}{1024}\pi R^4 \quad (0.341 R^4)$$

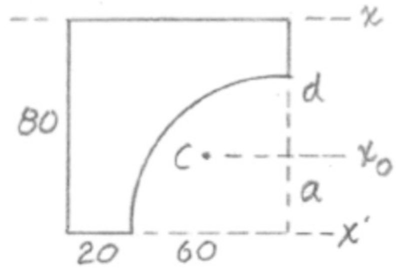
A/45 Calculate the moment of inertia of the shaded area about the x-axis.



A/45

Square: $I_x = \frac{1}{3}b^4 = \frac{1}{3}(80)^4 = 13.65(10^6)\ mm^4$

Dimen. in mm



Quarter-circle: $a = \frac{4r}{3\pi} = \frac{4(60)}{3\pi}$

$= 25.46\ mm$

$d = 80 - 25.46 = 54.54\ mm$

$$I_x = I_{x_0} + Ad^2 = I_{x'} - Aa^2 + Ad^2$$

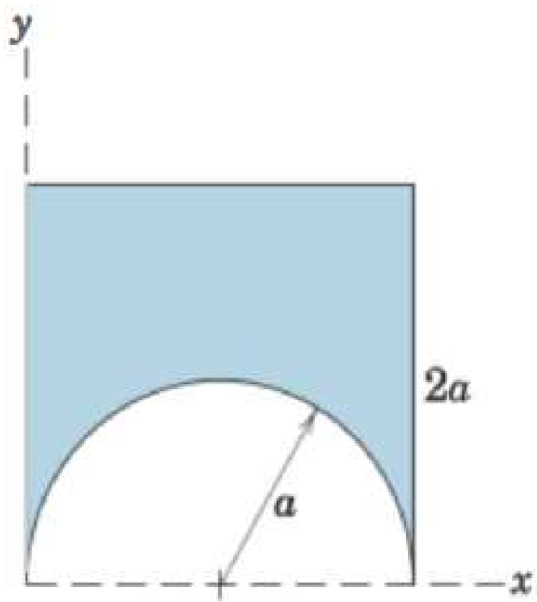
$$= \frac{-1}{4}\frac{\pi r^4}{4} - \frac{\pi r^2}{4}(d^2 - a^2) = -\frac{\pi r^2}{4}\left(\frac{r^2}{4} + d^2 - a^2\right)$$

$$= -\frac{\pi(60)^2}{4}\left[\frac{60^2}{4} + (54.54)^2 - (25.46)^2\right]$$

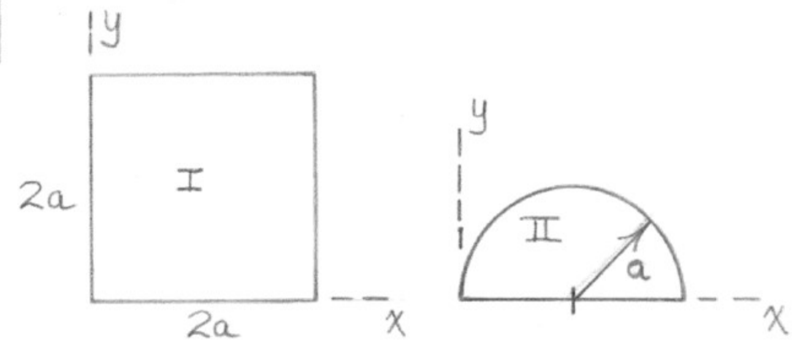
$$= -9.120(10^6)\ mm^4$$

Total $I_x = (13.65 - 9.120)(10^6) = \underline{4.53(10^6)\ mm^4}$

A/46 Determine the moments of inertia of the shaded area about the x and y axes.



A/46



I. Square $I_x = I_y = \frac{1}{3}(4a^2)(2a)^2 = \frac{16}{3}a^4$

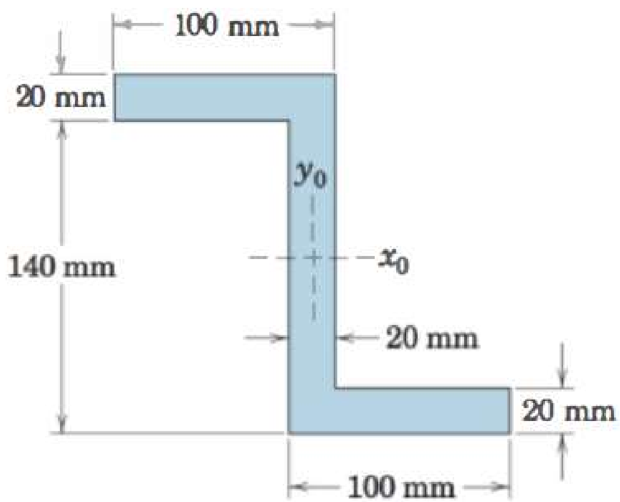
II. Semicircle $I_x = \frac{1}{8}\pi a^4$

$$I_y = \frac{1}{8}\pi a^4 + \frac{1}{2}\pi a^2 (a^2) = \frac{5}{8}\pi a^4$$

Combined: $I_x = \frac{16}{3}a^4 - \frac{\pi}{8}a^4 = \underline{4.94a^4}$

$$I_y = \frac{16}{3}a^4 - \frac{5}{8}\pi a^4 = \underline{3.37a^4}$$

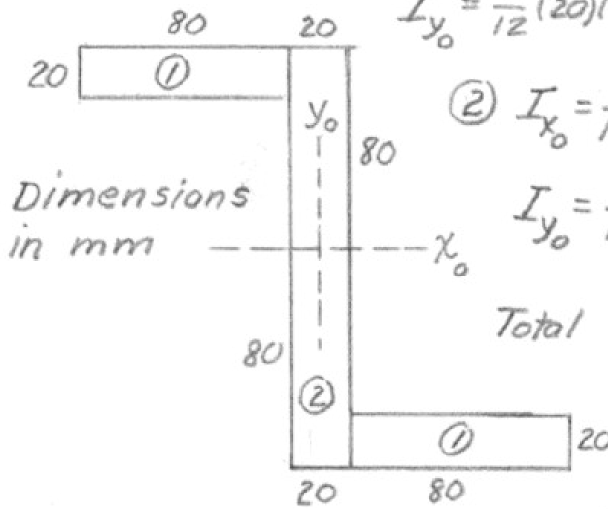
A/48 Determine the moments of inertia of the Z-section about its centroidal X_o and Y_o axes.



A/48

① $I_{x_o} = \frac{1}{12}(80)(20)^3 + (80)(20)(70)^2 = 7.89(10^6)\ mm^4$

$I_{y_o} = \frac{1}{12}(20)(80)^3 + (20)(80)(50)^2 = 4.85(10^6)\ mm^4$



② $I_{x_o} = \frac{1}{12}(20)(160)^3 = 6.83(10^6)\ mm^4$

$I_{y_o} = \frac{1}{12}(160)(20)^3 = 0.1067(10^6)\ mm^4$

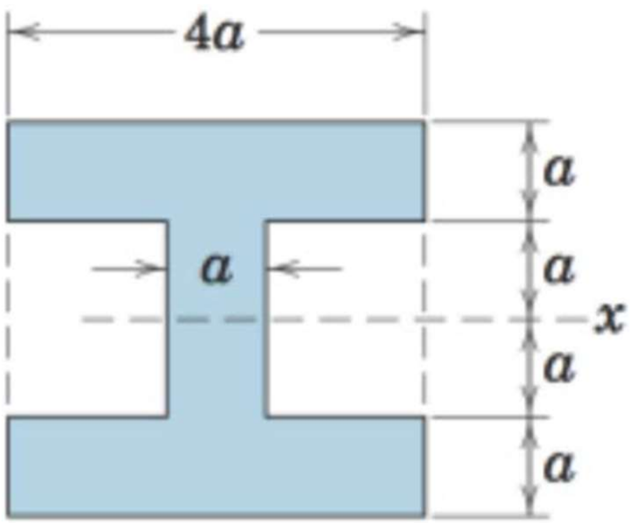
Total $\bar{I}_x = [2(7.89) + 6.83](10^6)$

$= \underline{22.6(10^6)\ mm^4}$

$\bar{I}_y = [2(4.85) + 0.1067](10^6)$

$= 9.81(10^6)\ mm^4$

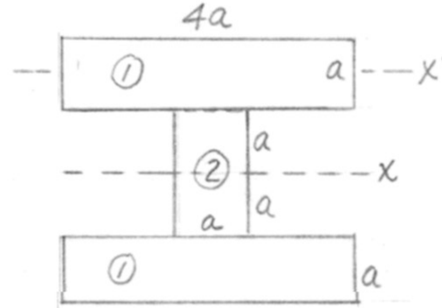
A/49 Determine the moment of inertia of the shaded area about the x-axis in two different ways.



A/49

Sol. I

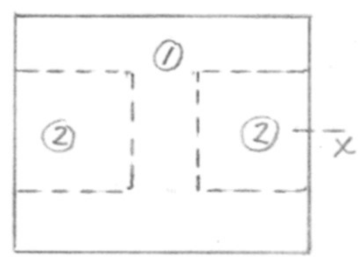
$$I_{1} = 2\left[\frac{1}{12}4a(a^3) + 4a^2\left(\frac{3a}{2}\right)^2\right] = \frac{56}{3}a^4$$



$$I_{2} = \frac{1}{12}a(2a)^3 = \frac{2}{3}a^4$$

Total $\underline{I_x = \frac{58}{3}a^4}$

Sol. II

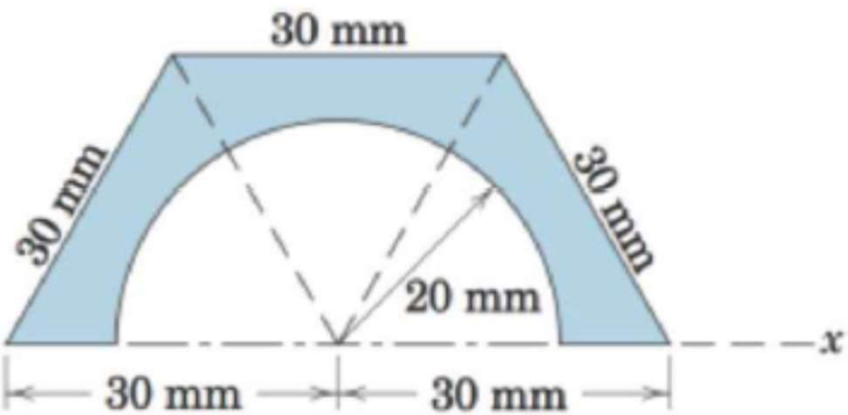


$$I_{1} = \frac{1}{12}(4a)(4a)^3 = \frac{64}{3}a^4$$

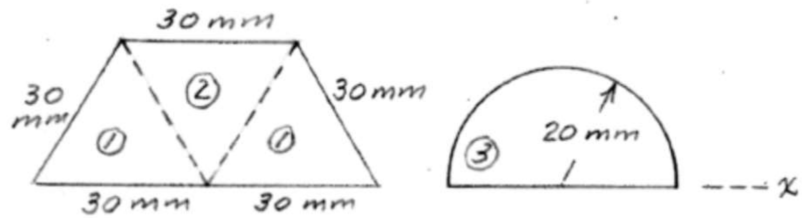
$$I_{2} = -\frac{1}{12}(3a)(2a)^3 = -2a^4$$

$$\text{Total} = \left(\frac{64}{3} - \frac{6}{3}\right)a^4 = \underline{\frac{58}{3}a^4}$$

A/51 Calculate the moment of inertia of the shaded area about the x-axis.



A/51



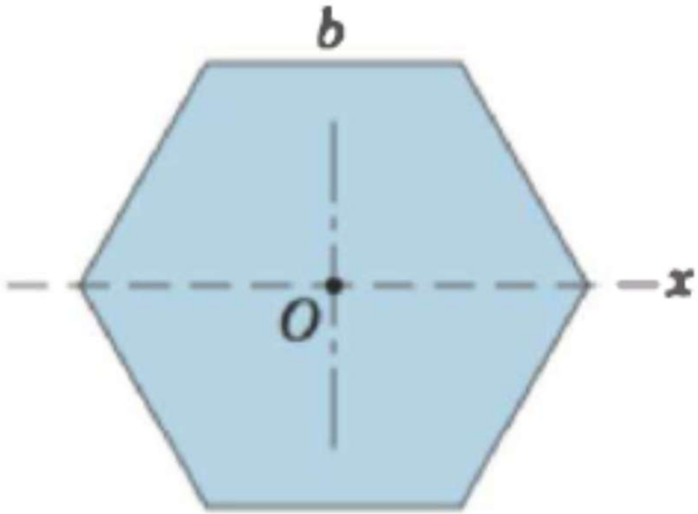
① $I_x = 2\left(\frac{1}{12}\right)(30)\left(30\frac{\sqrt{3}}{2}\right)^3 = \frac{81}{16}\sqrt{3}(10^4)$ mm^4

② $I_x = \frac{1}{4}(30)\left(30\frac{\sqrt{3}}{2}\right)^3 = \frac{243}{32}\sqrt{3}(10^4)$ mm^4

③ $I_x = -\frac{1}{2}\left(\frac{1}{4}\pi[20]^4\right) = -2\pi(10^4)$ mm^4

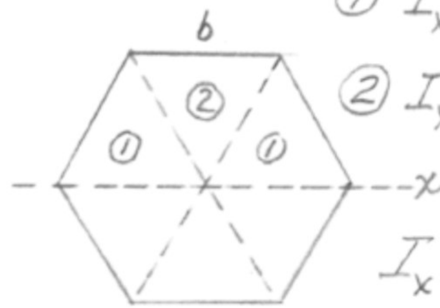
Total $I_x = \underline{15.64(10^4)\text{ mm}^4}$

A/53 Develop a formula for the moment of inertia of the regular hexagonal area of side b about its central x-axis.



A/53 | From Sample Problem A2

① $I_x = \frac{1}{12} b (b\sqrt{3}/2)^3 = \frac{\sqrt{3}}{32} b^4$

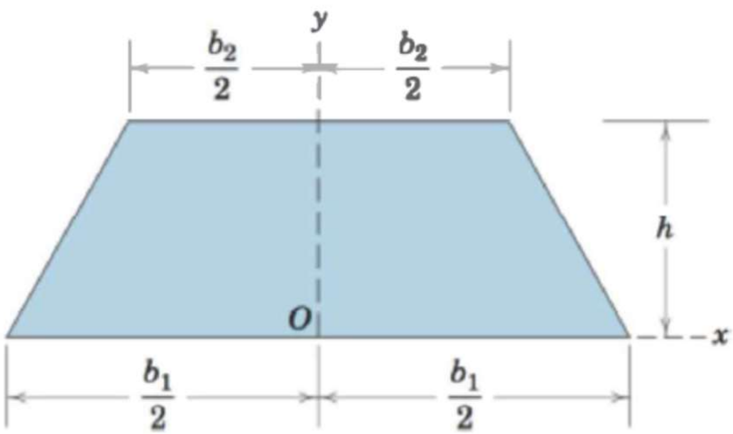


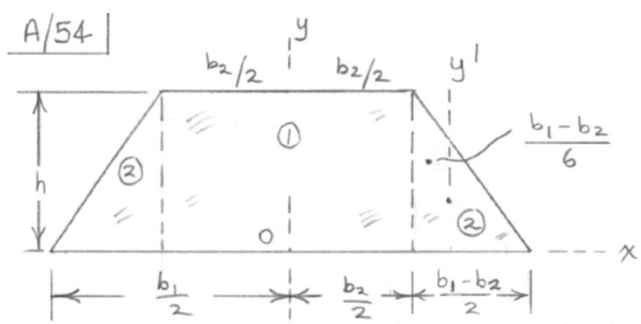
② $I_x = \frac{1}{4} b h^3 = \frac{1}{4} b (b\sqrt{3}/2)^3 = \frac{3\sqrt{3}}{32} b^4$

$$I_x = 4 I_{①} + 2 I_{②} = \frac{\sqrt{3}}{8} b^4 + \frac{3\sqrt{3}}{16} b^4$$

$$= \underline{\frac{5\sqrt{3}}{16} b^4}$$

A/54 By the method of the article, determine the moments of inertia about the x-axis and y-axis of the trapezoidal area.





$$I_x = I_{x_1} + 2I_{x_2} = \frac{1}{3}b_2h^3 + 2\left[\frac{1}{12}\frac{b_1-b_2}{2}h^3\right]$$

$$= \underline{h^3\left(\frac{b_1}{12}+\frac{b_2}{4}\right)}$$

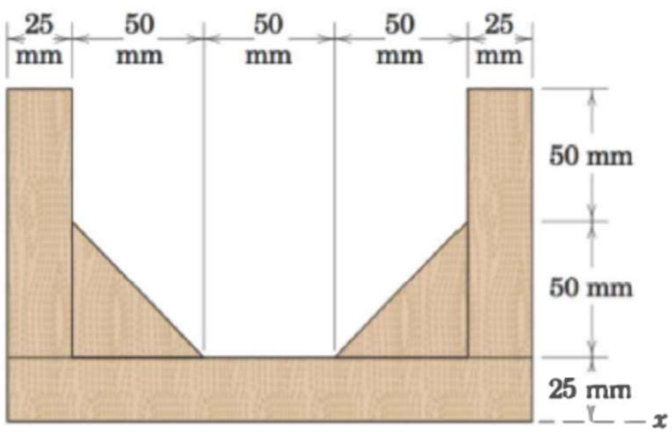
$$I_y = I_{y_1} + 2I_{y_2}$$

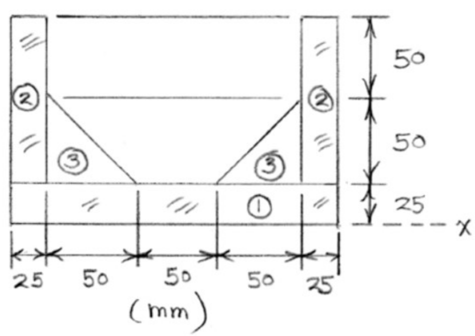
$$= \frac{1}{12}hb_2^3 + 2\left[\frac{1}{36}h\left(\frac{b_1-b_2}{2}\right)^3 + \frac{1}{2}h\left(\frac{b_1-b_2}{2}\right)\left(\frac{b_2}{2}+\frac{b_1-b_2}{6}\right)^2\right]$$

$$\vdots$$

$$= \underline{\frac{h}{48}\left(b_1^3 + b_1^2b_2 + b_1b_2^2 + b_2^3\right)}$$

A/55 Determine the moment of inertia of the cross sectional area of the reinforced channel about the x-axis.





(mm)

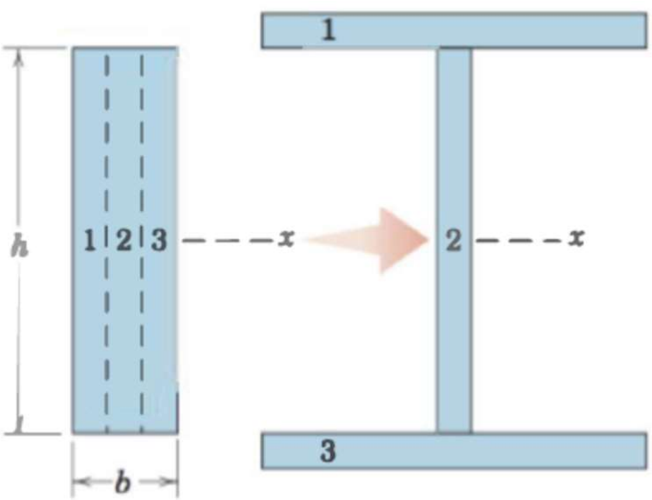
Comp.	A	d_x	$\bar{I}_x$	Ad_x^2
1	$200(25)$	$25/2$	$\frac{1}{12}(200)(25)^3$	781 250
2	$2[100(25)]$	75	$2[\frac{1}{12}(25)(100^3)]$	28 125 000
3	$2[\frac{1}{2}(50)(50)]$	$(25+\frac{50}{3})$	$2[\frac{1}{36}(50)(50^3)]$	+ 340 278

$$\Sigma \bar{I}_x = 4\ 774\ 306$$
$$\Sigma A d_x^2 = 33\ 246\ 528 \text{ mm}^4$$

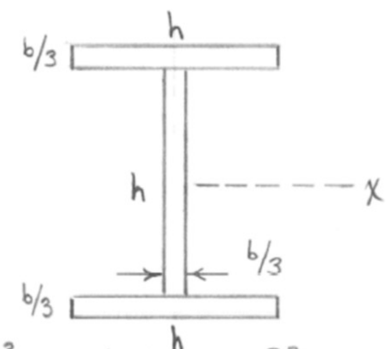
$$I_x = \Sigma \bar{I}_x + \Sigma A d_x^2 = 38{,}020{,}833 \text{ mm}^4$$

or $\underline{38.0(10^6) \text{ mm}^4}$

A/56 The rectangular area shown in part of the figure is split into three equal area which are then arranged as shown in part b of the figure. Determine an expression for the moment of inertia of the area in part b about the centroid x-axis. What percentage increase n over the moment of inertia for area a does this represent if h= 200 mm and b = 60 mm.



A/56



For area (a),

$$I_x = \frac{1}{12} b h^3$$

For area (b),

$$I_x = \frac{1}{12}\frac{b}{3}h^3 + 2\left[\frac{1}{12}h\left(\frac{b}{3}\right)^3 + h\frac{b}{3}\left(\frac{h}{2}+\frac{b}{6}\right)^2\right]$$

$$= \frac{hb}{9}\left(\frac{7}{4}h^2 + \frac{2}{9}b^2 + hb\right)$$

If $h = 200$ mm and $b = 60$ mm, we have

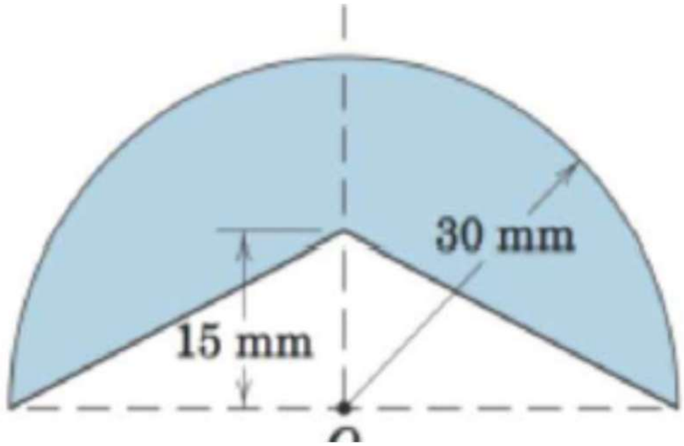
(a) $I_x = \frac{1}{12}(60)(200)^3 = 40(10^6)\ \text{mm}^4$

(b) $I_x = \frac{200(60)}{9}\left(\frac{7}{4}(200)^2 + \frac{2}{9}(60)^2 + 200(60)\right)$

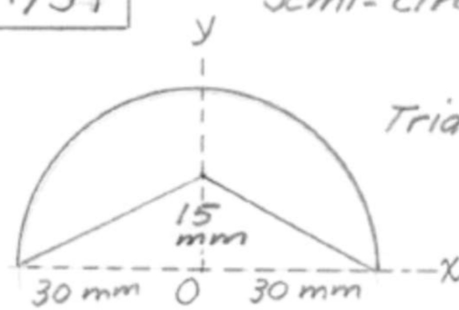
$= 110.4(10^6)\ \text{mm}^4$

Percent increase $n = \frac{110.4-40}{40}(100\%) = 176.0\%$

A/57 Calculate the polar moment of inertia of the shaded area about point O.



A/57



Semi-circle: $I_z = \frac{1}{4}\pi r^4 = \frac{1}{4}\pi(30)^4$

$= 0.6362(10^6)\ mm^4$

Triangle: $I_x = \frac{-1}{12}bh^3 = \frac{-1}{12}(60)(15)^3$

$= -0.01688(10^6)\ mm^4$

$I_y = -\frac{2}{12}(15)(30)^3$

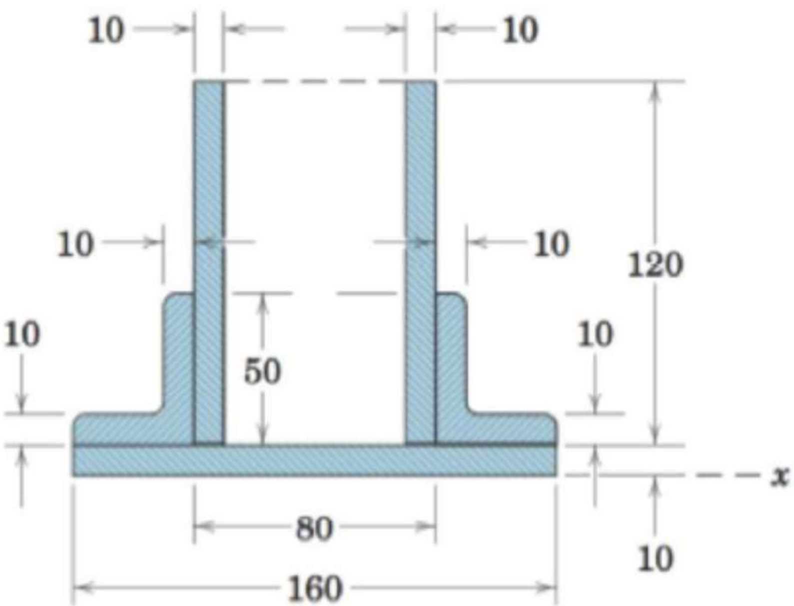
$= -0.06750(10^6)\ mm^4$

$I_z = I_x + I_y = -(0.01688 + 0.06750)10^6$

$= -0.0844(10^6)\ mm^4$

Total $I_z = (0.6362 - 0.0844)(10^6) = \underline{0.552(10^6)\ mm^4}$

A/58 Calculate the area moment of inertia about the x-axis for the built-up structure section shown.



Comp.	A	d_x	Ad_x^2	$\bar{I}_x$
1	$160(10)$	5	40 000	$\frac{1}{12}(160)(10^3)$
2	$2(120)(10)$	70	11 760 000	$2(\frac{1}{12})(10)(120^3)$
3	$2(50)(10)$	35	1 225 000	$2(\frac{1}{12})(10)(50^3)$
4	$2(30)(10)$	15	135 000	$2(\frac{1}{12})(30)(10^3)$

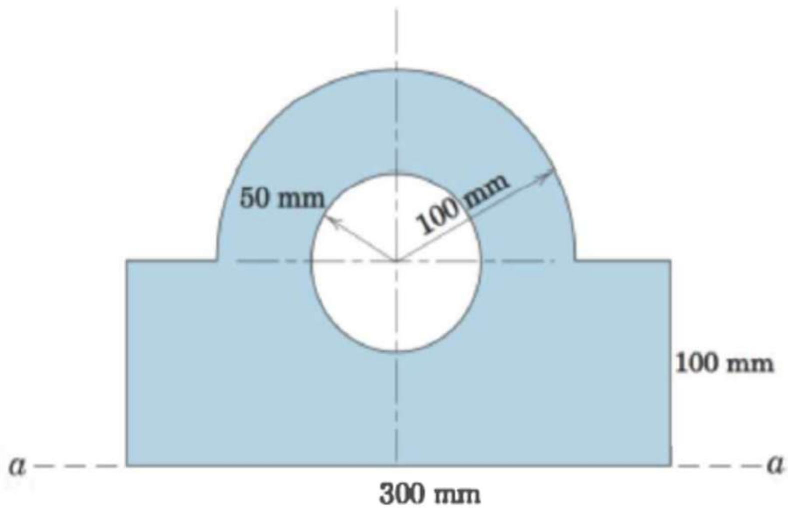
$\Sigma A d_x^2 = 13.16(10^6)$ mm^4

$\Sigma \bar{I}_x = 3.11(10^6)$ mm^4

$$I_x = \Sigma \bar{I}_x + \Sigma A d_x^2$$

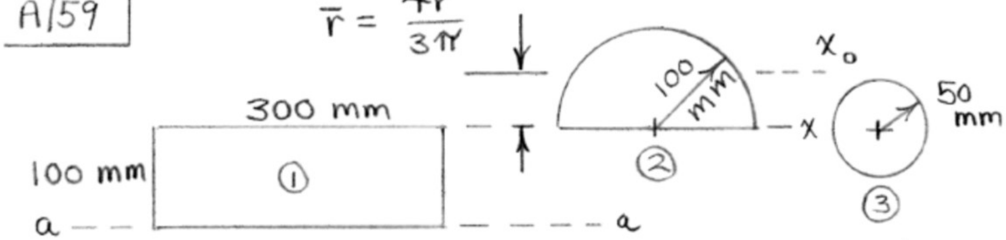
$$= \underline{16.27(10^6) \text{ mm}^4}$$

A/59 The cross section of a bearing blocks is shown in the figure by the shaded area. Calculate the moment of inertia of the section about its base a-a.



A/59

$\bar r = \frac{4r}{3\pi}$



Part ① : $I_{a-a} = \frac{1}{3}(300)(100)^3 = 10^8\ mm^4$

Part ② : $I_{a-a} = I_{x_0} + A\left(100 + \frac{4(100)}{3\pi}\right)^2$

where $I_{x_0} = I_x - A\bar r^2 = \frac{1}{8}\pi(100)^4 - \pi\frac{100^2}{2}\left(\frac{4\cdot 100}{3\pi}\right)^2$

$= 10.98(10^6)\ mm^4$

So $I_{a-a} = 10.98(10^6) + \frac{\pi(100)^2}{2}(142.4)^2 = 330(10^6)\ mm^4$

Part ③: $I_{a-a} = I_x + A(100)^2 = \frac{1}{4}\pi(50)^4 + \pi(50)^2(100)^2$

$= 83.4(10^6)\ mm^4$

Combined : $I_{a-a} = (100+330-83.4)10^6 = \underline{346(10^6)\ mm^4}$

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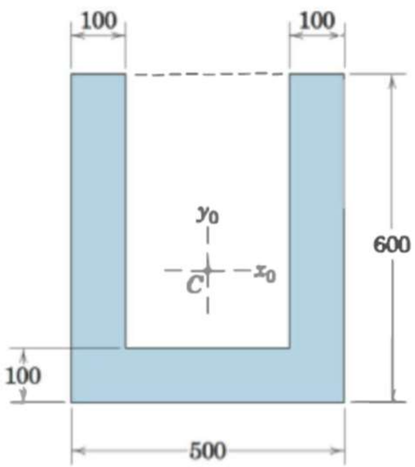
A/60 Calculate the polar radius of gyration of the shaded area about its centroid.



A/60

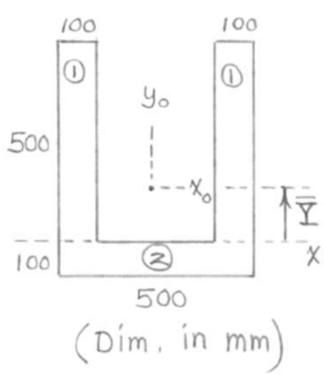
$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}$

$= \frac{2[(100)(500)(250)] + 500(100)(-50)}{2(100)(500) + 100(500)}$

$= 150$ mm

$A = 2(100)(500) + 100(500)$

$= 15(10^4)$ mm^2



① + ① $I_{x_0} = 2\left[\frac{1}{12}(100)(500)^3 + 100(500)(250-150)^2\right]$

$= 30.8(10^8)$ mm^4

$I_{y_0} = 2\left[\frac{1}{12}(500)(100)^3 + 100(500)(150+50)^2\right] = 40.8(10^8)$ mm^4

② $I_{x_0} = \frac{1}{12}(500)(100)^3 + 100(500)(50+150)^2 = 20.4(10^8)$ mm^4

$I_{y_0} = \frac{1}{12}(100)(500)^3 = 10.42(10^8)$ mm^4

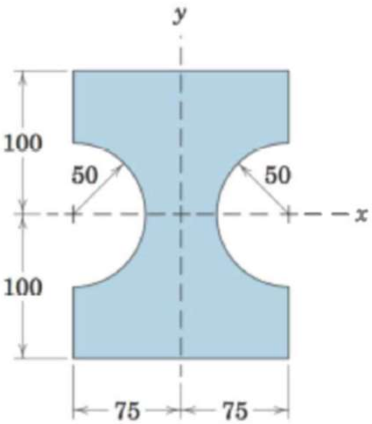
Totals ① + ① + ② : $I_{x_0} = 51.2(10^8)$ mm^4

$I_{y_0} = 51.2(10^8)$ mm^4

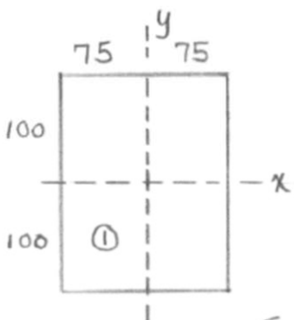
$I_C = I_{x_0} + I_{y_0} = 102.5(10^8)$ mm^4

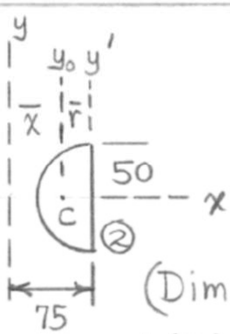
$k_C = \sqrt{I_C/A} = \sqrt{\frac{102.5(10^8)}{15(10^4)}} = \underline{261 \text{ mm}}$

A/62 Calculate the moments of inertia of the shaded area about the x- and y-axes.



A/62





(Dim. in mm)

$$\bar{r} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2\text{ mm}$$

$$\bar{x} = 75 - 21.2 = 53.8\text{ mm}$$

Part I: $I_x = \frac{1}{12}(150)(200)^3 = 100(10^6)\text{ mm}^4$

$I_y = \frac{1}{12}(200)(150)^3 = 56.2(10^6)\text{ mm}^4$

Parts II: $I_x = \frac{1}{4}\pi(50)^4 = 4.91(10^6)\text{ mm}^4$ (for both together)

$$I_y = I_{y_0} + A\bar{x}^2 = I_{y'} - A\bar{r}^2 + A\bar{x}^2 = I_{y'} + A(\bar{x}^2 - \bar{r}^2)$$

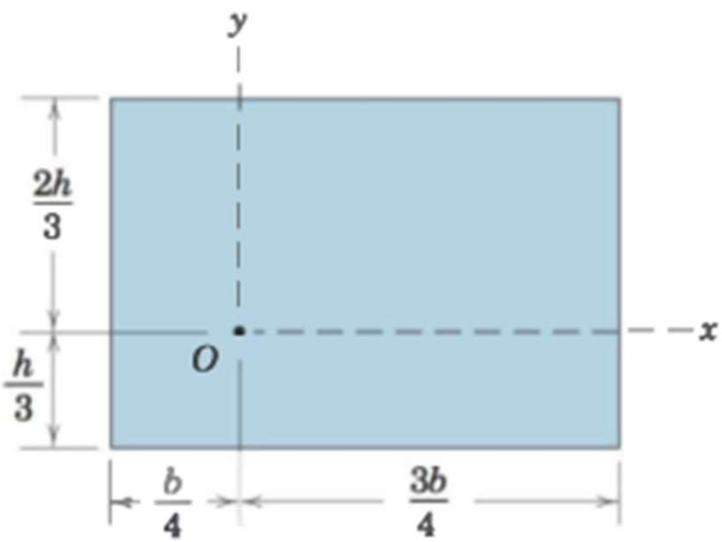
$$= \frac{1}{2}\left(\frac{1}{4}\pi 50^4\right) + \frac{\pi(50)^2}{2}(53.8^2 - 21.2^2)$$

$= 12.04(10^6)\text{ mm}^4$ for each, $24.1(10^6)\text{ mm}^4$ for both

Combined: $I_x = 100(10^6) - 4.91(10^6) = \underline{95.1(10^6)\text{ mm}^4}$

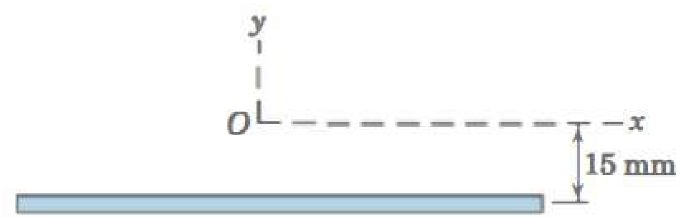
$I_y = 56.2(10^6) - 24.1(10^6) = \underline{32.2(10^6)\text{ mm}^4}$

A/1 Determine the moments of inertia of the rectangular area about the x- and y-axes and find the polar moment of inertia about point O.



$$I_x = \bar{I}_x + Ad_x^2 = \frac{1}{12} bh^3 + bh\left(\frac{h}{6}\right)^2$$
$$= \underline{\frac{1}{9} bh^3}$$
$$I_y = \bar{I}_y + Ad_y^2 = \frac{1}{12} hb^3 + bh\left(\frac{b}{4}\right)^2$$
$$= \underline{\frac{7}{48} hb^3}$$
$$I_z = I_x + I_y = \underline{bh\left(\frac{h^2}{9} + \frac{7b^2}{48}\right)}$$

A/3 The narrow rectangular strip has an area of 300 mm^2, and its moment of inertia about the y-axis is 35(10^3) mm^4. Obtain a close approximation to the polar radius of gyration about point 0.

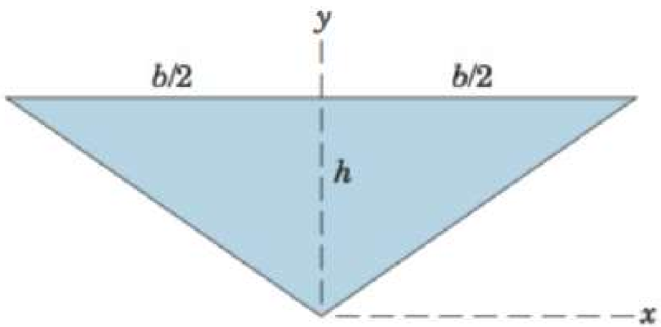


$$I_x \approx Ad^2 = 300(15)^2 = 67.5(10^3)\ \text{mm}^4$$

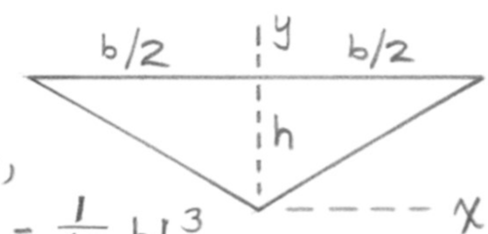
$$J_O = I_x + I_y = 67.5(10^3) + 35(10^3) = 102.5(10^3)\ \text{mm}^4$$

$$k_O = \sqrt{J_O/A} = \sqrt{\frac{102.5(10^3)}{300}} = \underline{18.48\ \text{mm}}$$

A/4 Determine the ratio b/h such that Ix = Iy for the area of the isosceles triangle.



A/4

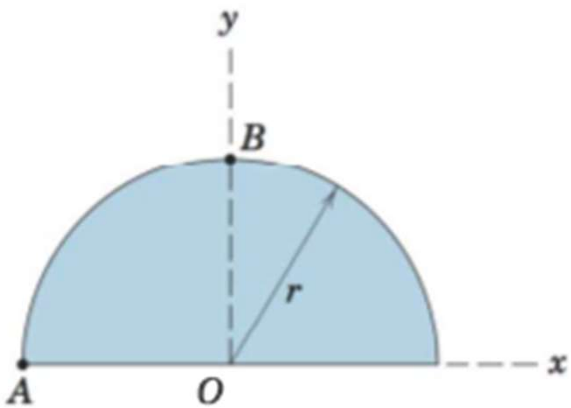


From Sample Problem A/2,

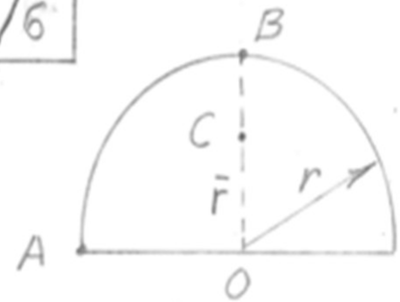
$I_x = \frac{1}{4} b h^3$, $I_y = 2\left\{\frac{1}{12} h \left(\frac{b}{2}\right)^3\right\} = \frac{1}{48} h b^3$

$I_x = I_y$ if $\frac{1}{4} b h^3 = \frac{1}{48} h b^3$, $\underline{\frac{b}{h} = 2\sqrt{3}}$

A/6 Determine the polar moments of inertia of the semicircular area about points A and B.



$\boxed{A/6}$



$\bar{r} = 4r/3\pi$

For complete circle

$$I_A = I_O + Ar^2 = \frac{1}{2}Ar^2 + Ar^2$$
$$= \frac{3}{2}Ar^2$$

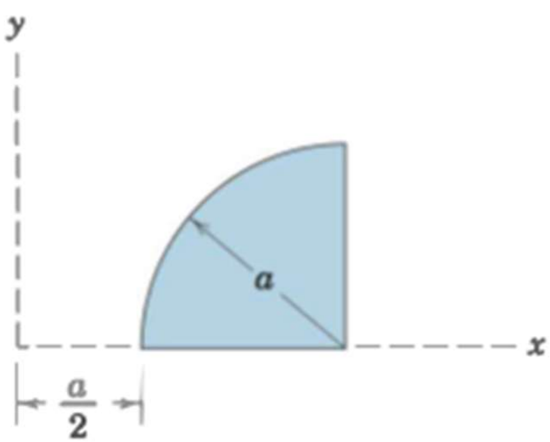
For half circle

$$I_A = \frac{1}{2}\left(\frac{3}{2}\pi r^4\right) = \underline{\frac{3}{4}\pi r^4}$$

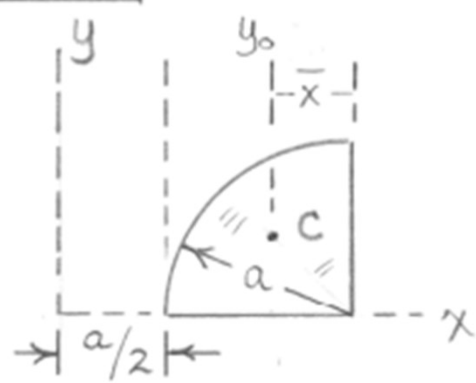
For half circle, $I_O = \frac{1}{4}\pi r^4$

$$I_B = I_C + A(r-\bar{r})^2 = I_O - A\bar{r}^2 + A(r-\bar{r})^2$$
$$= I_O + A(r^2 - 2r\bar{r})$$
$$= \frac{1}{4}\pi r^4 + \frac{\pi r^4}{2}\left(1 - \frac{8}{3\pi}\right) = \underline{r^4\left(\frac{3\pi}{4} - \frac{4}{3}\right)}$$

A/7 Determine the moments of inertia of the quarter-circular area about the y-axis.



A/7



From Table D/3:

$$\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^4$$

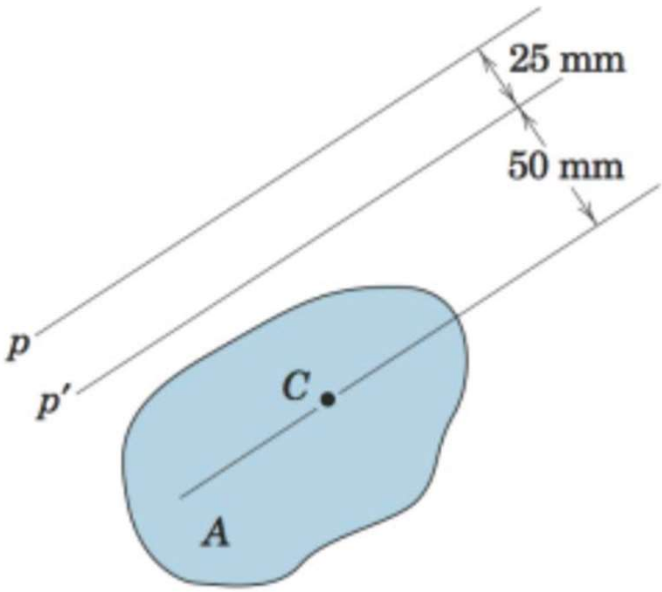
$$\bar{x} = \frac{4a}{3\pi}$$

$$I_y = \bar{I}_y + Ad_y^2$$

$$= \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^4 + \frac{\pi a^2}{4}\left[\frac{a}{2} + \left(a - \frac{4a}{3\pi}\right)\right]^2$$

$$= \underline{\left[\frac{5\pi}{8} - 1\right]a^4}$$

A/9 The moments of inertia of the area A about the parallel p- and p'-axes differ by $15(10^6)$ mm^4. Compute the area A, which has its centroid at C.



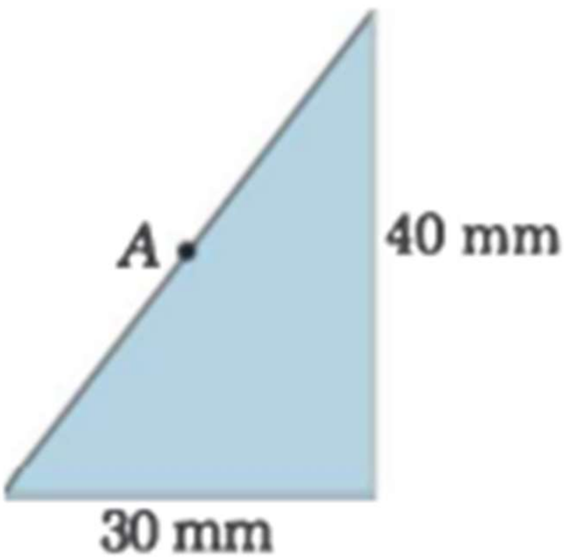
A/9

$$I_p = I_C + A(75)^2, \quad I_{p'} = I_C + A(50)^2$$

$$I_p - I_{p'} = 15(10^6) = A[(75)^2 - (50)^2]$$

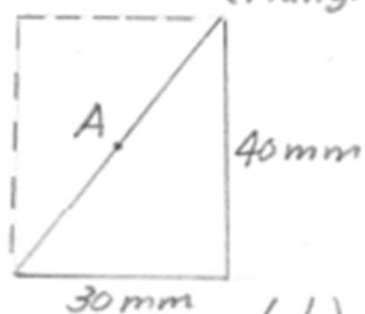
$$\underline{A = 4800 \text{ mm}^2}$$

A/16 Determine the radius of gyration about a polar axis through the midpoint A of the hypotenuse of the right-triangular area.



A/16

$$(J_A)_{triangle} = \frac{1}{2}(J_A)_{rectangle}$$



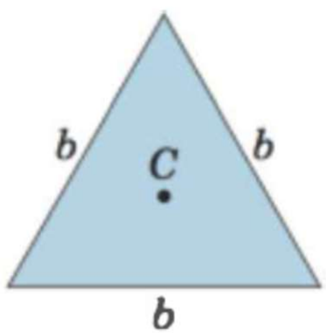
$$= \frac{1}{2}\left[\frac{1}{12}A(b^2+h^2)\right]$$ from Sample Prob. A/11

$$= \frac{1}{24}(30)(40)(\overline{30}^2+\overline{40}^2) = 12.5(10^4) \text{ mm}^4$$

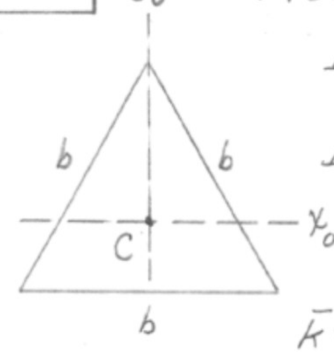
$$(J_A)_{triangle} = k_A^2 A$$

So $k_A = \sqrt{\frac{12.5(10^4)}{30(40)/2}} = \sqrt{208.4} = \underline{14.43 \text{ mm}}$

A/18 Determine the polar radius of gyration of the area of the equilateral triangle of side b about its centroid C.



A/18



From Sample Problem A/2

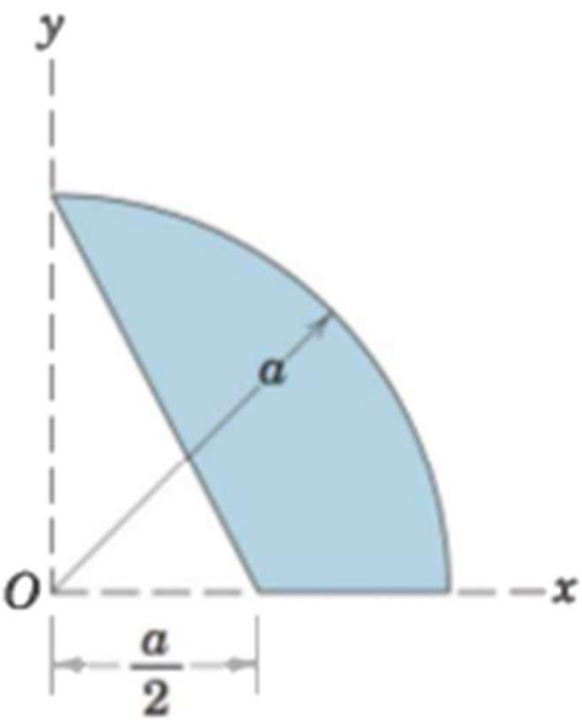
$$\bar{I}_x = \frac{1}{36} b\left(b\frac{\sqrt{3}}{2}\right)^3 = \frac{b^4}{96}\sqrt{3}$$

$$\bar{I}_y = 2\left(\frac{1}{12} b\frac{\sqrt{3}}{2}\left[\frac{b}{2}\right]^3\right) = \frac{b^4}{96}\sqrt{3}$$

$$\bar{J} = \bar{I}_x + \bar{I}_y = \frac{b^4}{48}\sqrt{3}$$

$$\bar{k} = \sqrt{\bar{J}/A} = \sqrt{\frac{b^4\sqrt{3}}{48}\Big/\left(\frac{b^2}{2}\frac{\sqrt{3}}{2}\right)} = \underline{\frac{b}{2\sqrt{3}}}$$

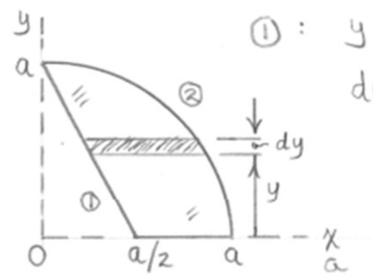
A/19 Determine the moment of inertia of the shaded area about the x-axis.



A/19

②: $x^2+y^2=a^2$

①: $y=-2x+a$



$$dA = (x_2-x_1)\,dy = \left[\sqrt{a^2-y^2} - \frac{a-y}{2}\right]dy$$

$$I_x = \int y^2\,dA = \int_0^a y^2\left[\sqrt{a^2-y^2} - \frac{a-y}{2}\right]dy$$

$$= \int_0^a \left[y^2\sqrt{a^2-y^2} - \frac{ay^2-y^3}{2}\right]dy$$

$$= \left[-\frac{y}{4}\sqrt{(a^2-y^2)^3} + \frac{a^2}{8}\left(y\sqrt{a^2-y^2} + a^2\sin^{-1}\frac{y}{a}\right) - \frac{ay^3}{6} + \frac{y^4}{8}\right]_0^a = a^4\left[\frac{\pi}{16} - \frac{1}{24}\right]$$

$$= \frac{a^4}{8}\left[\frac{\pi}{2} - \frac{1}{3}\right]$$

THANK YOU

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