formulas

Fourier Series

→ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

where, $a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$

Dirichlet Conditions

i) $f(x)$ is periodic & single valued

ii) $f(x)$ is continuous / has finite no. of discontinuities in period of 2π

iii) $f(x)$ has finite no. of minima & maxima in given period

Important results

→ $\cos n\pi = (-1)^n$

→ $\sin n\pi = 0$

→ $e^{-\infty} = 0$

→ $\sin \frac{n\pi}{2} = \begin{cases} 0, & n = \text{even} \\ 1, & n = 1, 5, 9, 13, \dots \\ -1, & n = 3, 7, 11, 15, \dots \end{cases}$

→ $\cos \frac{n\pi}{2} = \begin{cases} 0, & n = \text{odd} \\ 1, & n = 0, 4, 8, 12, \dots \\ -1, & n = 2, 6, 10, 14, \dots \end{cases}$

→ $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

→ $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

→ $\int u^* v\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

$\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

$\cos a \cdot \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$

$\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

$\sin a \cdot \sin b = \frac{-1}{2}\left[\cos(a+b) - \cos(a-b)\right]$

Fourier Series of Even & Odd

→ For even functions,

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$; $a_n = \int_0^{\pi} f(x)\cos nx\,dx$

$b_n = 0$

→ For odd functions,

$a_0 = 0$; $a_n = 0$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Fourier Series for an arbitrary Interval

→ $(-l, l)$ & $(0, 2l)$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Where, $a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

Half Range $(0, \pi)$

→ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx$

→ Cosine Series

→ $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx$

→ Sine Series

Half Range $(0, l)$

→ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where, $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

→ Cosine Series

→ $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Where, $b_n = \frac{2}{l}\int_0 f(x)\sin\frac{n\pi x}{l}$

→ Sine Series

Parseval's Identity

→ For $(-l, l)$

$\int_{-l}^{l}[f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right]$

→ For $(0, 2l)$

$\int_{0}^{2l}[f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right]$

→ for even functions / H.R cosine

$2\int_0^{l}[f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

→ for odd functions / H.R sine

$2\int_0^{l}[f(x)]^2\,dx = l\left[\sum_{n=1}^{\infty} b_n^2\right]$

Complex Form of Fourier Series

→ $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$

Where, $C_n = \frac{1}{2l}\int_{-\infty}^{\infty} f(x) \cdot e^{\frac{-in\pi x}{l}}\,dx$

→ $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

→ $e^{in\pi} = \cos n\pi + i\sin n\pi$

$e^{-in\pi} = \cos n\pi - i\sin n\pi$

$\frac{e^{in\pi} + e^{-in\pi}}{2} = \cos n\pi$

$\frac{e^{in\pi} - e^{-in\pi}}{2i} = \sin n\pi$

Fourier Transform

→ $F[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t}\,dt$

Inverse Fourier Transform

→ $F^{-1}\left[F[f(t)]\right] = F^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\,d\omega$

Fourier Sine Transform

→ $F_s[f(t)] = F_s(\omega) = \int_0^{\infty} f(t)\sin\omega t\,dt$

→ $f(t) = F^{-1}[F_s(\omega)] = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega t\,d\omega$

Fourier Cosine Transform

→ $F_c[f(t)] = F_c(\omega) = \int_0^{\infty} f(t)\cos\omega t\,dt$

→ $f(t) = F^{-1}[F_c(\omega)] = \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t\,d\omega$

Properties of Fourier Transform

1) $F[af(t) + bg(t)] = aF[f(t)] + bF[g(t)]$ # Linearity

2) If $F[f(t)] = F(\omega)$ # Change of scale

then, $F[f(at)] = \frac{1}{a}F\left(\frac{\omega}{a}\right)$

3) Shifting Property

a) $F[f(t-a)] = e^{-i\omega a}F(\omega)$ # Time shifting

b) $F[e^{iat}f(t)] = F(\omega - a)$ # Frequency shifting

4. If $F[f(t)] = F[\omega]$ then $F[f(-t)] = F[-\omega]$ # Time reversal

5. $F[f^n(t)] = (i\omega)^n F[f(t)]$

Fourier Transform of basic functions

1) Unit impulse function, $f(t) = \delta(t)$

Then, $F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-i\omega t}\,dt = \left[e^{-i\omega t}\right]_{t=0} = 1$

Impulse in time ⇒ Constant in frequency

2) Constant function, $f(t) = K$

Then, $F[K] = 2\pi K \cdot \delta(\omega)$

Constant in time ⇒ Impulse in frequency

3) $f(t) = e^{iat}$

$F[f(t)] = 2\pi\,\delta(\omega - a)$

4) $f(t) = \cos at$

$F[f(t)] = \pi\{\delta(\omega + a) + \delta(\omega - a)\}$

$f(t) = \sin at$

$F[f(t)] = i\pi\{\delta(\omega + a) - \delta(\omega - a)\}$

5) Unit step function, $f(t) = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 1 \end{cases}$

$F[f(t)] = \frac{1}{i\omega}$

Finite Fourier Sine Transform

→ $F_s(n) = \int_0^{c} f(x) \cdot \sin\frac{n\pi x}{c}\,dx$

$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c}$

Finite Fourier Cosine Transform

→ $F_c(n) = \int_0^{c} f(x) \cdot \cos\frac{n\pi x}{c}\,dx$

$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\cos\frac{n\pi x}{c}$