Unit-1 class-12

1. Using double integrals find moment of inertia about the x-axis of the area enclosed by the lines $x=0, y=0, \frac{x}{a}+\frac{y}{b}=1$ Ans: $\frac{ab^3}{12}$

2. Find the moment of inertia of an octant of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ about x-axis. Ans: $\frac{abc\left(b^2+c^2\right)\pi}{30}$

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1. $\Rightarrow \frac{x}{a}+\frac{y}{b}=1$

$$\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} y^2\,dy\,dx = \int_0^a \left[\frac{y^3}{3}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$\frac{1}{3}\int_0^a b^3\left(1-\frac{x}{a}\right)^3 dx = \frac{1}{3}\int_0^a \frac{b^3(a-x)^3}{a^3} = \frac{b^3}{3a^3}\int_0^a (a-x)^3 dx$$

$$= \frac{b^3}{3a^3}\int_0^a \left(a^3-x^3-3a^2x+3ax^2\right)dx = \frac{b^3}{3a^3}\left[a^3x-\frac{x^4}{4}-\frac{3a^2x^2}{2}+\frac{3ax^3}{3}\right]$$

$$= \frac{b^3}{3a^3}\left[a^4-\frac{a^4}{4}-\frac{3a^4}{2}+\frac{\cancel{3}a^4}{\cancel{3}}\right] = \frac{a^4b^3}{12a^3} = \frac{ab^3}{12}$$

2. $$\text{MOI} = \iiint \rho(x,y,z)\left(z^2+y^2\right) dx\,dy\,dz$$

$x = ar\sin\theta\cos\phi$

$y = br\sin\theta\sin\phi$

$z = cr\cos\theta$

$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$

$r = 0$ to 1

$\theta = 0$ to $\pi/2$

$\phi = 0$ to $\pi/2$

Now, $$\text{MOI} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(b^2r^2\sin^2\theta\sin^2\phi + c^2r^2\cos^2\theta\right) abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= abc\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 b^2r^4\sin^3\theta\sin^2\phi + c^2r^4\sin\theta\cos^2\theta)\,dr\,d\theta\,d\phi$$

$$= \frac{abc}{5}\int_0^{\pi/2}\int_0^{\pi/2}\left(b^2\sin^3\theta\sin^2\phi + c^2\sin\theta\cos^2\theta\right) d\theta\,d\phi$$

$$= \frac{abc}{5}\left[\frac{b^2\pi}{6}+\frac{c^2\pi}{6}\right] = \frac{abc\pi\left(b^2+c^2\right)}{30}$$