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**Department of Science and Humanities** 



**UNIT 4: Partial Differential Equations** 

Session: 8

**Sub Topic: Solution of Homogeneous Linear Partial Differential** 

**Equations with Constant Coefficients** 

Department of Science and Humanities

# Solution of Homogeneous Linear Partial Differential Equations with constant coefficients



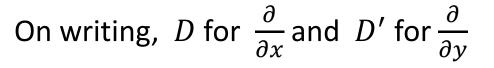
### An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \dots (1)$$

where  $a_{0_n}a_1,...,a_n$  are constants, is called **homogeneous linear**PDE of the  $n^{th}$  order with constants coefficients.

It is called **homogeneous** because all terms contain derivatives of the same order. Here, all the partial derivatives are of  $n^{th}$  order.

# Solution of Homogeneous Linear Partial Differential Equations with constant coefficients



(1) can be written as

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} \left(D'\right)^2 + \dots + a_n \left(D'\right)^n\right) z = F(x, y)$$

i.e., 
$$F(D, D')z = F(x, y) \dots \dots (2)$$



# Solution of Homogeneous Linear Partial Differential Equations with constant coefficients



As in the case of ordinary linear differential equations with constant coefficients the complete solution of equation (2) consists of two parts, namely

- (i) the complementary function (CF) which is the complete solution of the equation F(D,D')z=0. It must contain n arbitrary functions, where n is the order of the differential equation.
- (ii) the particular integral (PI) which is a particular solution of equation (2).

The complete solution of (2) is z = CF + PI

# Rules for finding complementary function

Consider the equation 
$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0.....(1)$$

which in symbolic form is

$$(D^{2} + a_{1}DD' + a_{2}(D')^{2})z = 0.....(2)$$

$$\div (D')^{2}$$

$$\Rightarrow \left(\left(\frac{D}{D'}\right)^2 + a_1\frac{D}{D'} + a_2\right)z = 0$$
 Auxiliary equation is  $m^2 + a_1m + a_2 = 0$  where  $m = \frac{D}{D'}$ 

Let its root be  $m_1, m_2$ 



# Rules for finding complementary function



Case(i): If the roots are real and distinct then equation (2) is equivalent to

$$(D - m_1 D')(D - m_2 D')z = 0....(3)$$

It will be satisfied by the solution of

$$(D - m_2 D')z = 0 \Rightarrow p - m_2 q = 0$$

This is a Lagrange's linear differential equation and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

From first and second  $\Rightarrow y + m_2 x = a$ 

From second and third  $\Rightarrow z = b$ 

Therefore its solution is 
$$\phi(y+m_2x,z)=0$$
 or  $z=\phi(y+m_2x)$ 

# **Rules for finding complementary function**

Similarly (2) will also be satisfied by the solution of

$$(D - m_1 D')z = 0 \Rightarrow z = f(y + m_1 x)$$

Hence the complete solution of (1) is  $z = f(y + m_1 x) + \phi(y + m_2 x)$ 



# Rules for finding complementary function



# **Example:**

**Solve**: 
$$(D^2 - D'^2)z = 0$$

## **Solution:**

Consider 
$$(D^2 - D'^2)z = 0$$
  
 $\div (D')^2$ 

$$\left(\left(\frac{D}{D'}\right)^2 - 1\right)z = 0$$

Its auxiliary equation is  $m^2 - 1 = 0$  where  $m = \frac{D}{D'}$ 

$$m = 1, -1$$

Here the complete solution is  $z = f_1(y + x) + f_2(y - x)$ 

# Rules for finding complementary function



Case(ii): If the roots are equal i.e.,  $m_1 = m_2$  then (2) is equivalent to

$$(D - m_1 D')^2 z = 0.....(3)$$

Putting 
$$(D - m_1 D')z = u ......$$
 (4)

(3) becomes  $(D - m_1 D')u = 0$ 

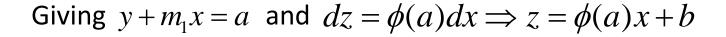
$$\Rightarrow u = \phi(y + m_1 x)$$

Therefore (4) takes the form  $(D - m_1 D')z = \phi(y + m_1 x)$ 

or 
$$(p - m_1 q) = \phi(y + m_1 x)$$

This is again Lagrange's linear and the subsidiary equations are  $\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y+m_1x)}$ 

# Rules for finding complementary function



$$\therefore z - x\phi(y + m_1 x) = f(y + m_1 x)$$

That is 
$$z = f(y + m_1x) + x\phi(y + m_1x)$$



# Rules for finding complementary function



# **Example:**

**Solve**: 
$$(4D^2 + 12DD' + 9D'^2)z = 0$$

## **Solution:**

Consider 
$$(4D^2 + 12DD' + 9D'^2)z = 0$$

$$\div (D')^2$$

$$\left(4\left(\frac{D}{D'}\right)^2 + 12\frac{D}{D'} + 9\right)z = 0$$

Its auxiliary equation is  $4m^2 + 12m + 9 = 0$  where  $m = \frac{D}{D'}$ 

$$m = -\frac{3}{2}, -\frac{3}{2}$$

Here the complete solution is  $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$ 

# Rules for finding the Particular Integral



# Consider the equation

$$(D^2 + a_1 DD' + a_2 (D')^2)z = F(x, y)$$

i.e., 
$$F(D, D')z = F(x, y)$$

$$\therefore P.I. = \frac{1}{F(D, D')} F(x, y)$$

# Case (i): When $F(x, y) = e^{ax+by}$

P.I. = 
$$\frac{1}{F(D,D')}e^{ax+by}$$
$$= \frac{1}{F(a,b)}e^{ax+by}; \quad F(a,b) \neq 0$$

# Rules for finding the Particular Integral

# Case (ii): When F(x, y) = sin(ax + by) or cos(ax + by)

P. I. = 
$$\frac{1}{F(D^2, DD', (D')^2)} \sin(ax + by)$$

P. I. = 
$$\frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by)$$
;  $F(-a^2, -ab, -b^2) \neq 0$ 



# Rules for finding the Particular Integral

# Case (iii): When $F(x, y) = x^m y^n$ , m and n being constants

P. I. = 
$$\frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} x^m y^n$$

(a) If  $n < m, \frac{1}{F(D,D')}$  is expanded in powers of  $\frac{D'}{D}$ 

- (b) If  $m < n, \frac{1}{F(D,D')}$  is expanded in powers of  $\frac{D}{D'}$

# Rules for finding the Particular Integral



# Case (iv): Exponential shift

When 
$$F(x, y) = e^{ax+by}V(x, y)$$
, where  $V(x, y)$  is any function of  $x$  and  $y$ 

P. I. = 
$$\frac{1}{F(D,D')}e^{ax+by}V(x,y) = e^{ax+by}\frac{1}{F(D+a,D'+b)}V(x,y)$$

# Rules for finding the Particular Integral



# Case (v): When F(x, y) is any function of x and y

$$P.I. = \frac{1}{F(D, D')}V(x, y)$$

Resolve  $\frac{1}{F(D,D')}$  into partial fractions considering F(D,D') as a function of

D alone and operate each partial fractions on V(x, y) remembering that

$$\frac{1}{D-mD'}V(x,y) = \int V(x,c-mx)dx$$

where c is replaced by y + mx after integration.

# Rules for finding the Particular Integral

# Case (i): When $F(x, y) = e^{ax+by}$

P.I. = 
$$\frac{1}{F(D,D')}e^{ax+by}$$
$$= \frac{1}{F(a,b)}e^{ax+by}; \quad F(a,b) \neq 0$$



#### **Problems**



1. Find the general solution of the partial differential equation  $(D^2 + DD' - 2(D')^2)z = 5e^{x+2y}$ .

#### Solution:

Consider 
$$(D^2 + DD' - 2(D')^2)z = 0$$
  
 $\div (D')^2$ 

$$\left(\left(\frac{D}{D'}\right)^2 + \frac{D}{D'} - 2\right)z = 0$$

Auxiliary equation is  $m^2 + m - 2 = 0$ 

$$m = 1, -2$$

$$: CF = f_1(y + x) + f_2(y - 2x)$$

### **Problems**



$$PI = \frac{1}{D^2 + DD' - 2(D')^2} 5e^{x + 2y}$$

$$F(a,b) = F(1,2) = 1^2 + 1 * 2 - 2(2)^2 = -5$$
 (Replacing D by 1 and D' by 2)

$$\therefore PI = \frac{1}{-5} \, 5e^{x+2y}$$

$$PI = -e^{x+2y}$$

Therefore, the general solution of the given differential equation is given by

$$z = f_1(y+x) + f_2(y-2x) - e^{x+2y}$$

#### **Problems**



# 2. Solve: $(D^2 + 5DD' + 6(D')^2)z = e^{x-y}$ .

#### Solution:

Consider 
$$(D^2 + 5DD' + 6(D')^2)z = 0$$
  
 $\div (D')^2$ 

$$\left(\left(\frac{D}{D'}\right)^2 + 5\frac{D}{D'} + 6\right)z = 0$$

Auxiliary equation is  $m^2 + 5m + 6 = 0$ 

$$m = -2, -3$$

$$: CF = f_1(y - 2x) + f_2(y - 3x)$$

## **Problems**



$$PI = \frac{1}{D^2 + 5DD' + 6(D')^2} e^{x - y}$$

$$F(a,b) = F(1,-1) = 1^2 + 5(1)(-1) + 6(-1)^2 = 2$$
 (Replacing D by 1 and D' by -1)

$$\therefore PI = \frac{1}{2} e^{x-y}$$

Therefore, the general solution of the given differential equation is given by

$$z = f_1(y - 2x) + f_2(y - 3x) + \frac{1}{2}e^{x - y}$$



# **THANK YOU**

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