# B-Tech- II

Department of Science and Humanities

# PROBLEMS ON LAPLACE TRANSFORMS

# Problems on Linearity property

1. Find $L[2+5t^3+4e^{-3t}+10e^t+\sin 2t]$

Let $f(t)=2+5t^3+4e^{-3t}+10e^t+\sin 2t$

$$\mathrm{L}[f(t)]=L[2+5t^3+4e^{-3t}+10e^t+\sin 2t]$$

$$\mathrm{L}[f(t)]=L[2]+5L[t^3]+4L[e^{-3t}]+10L[e^t]+L[\sin 2t]$$

$$L[f(t)]=\frac{2}{s}+5\frac{6}{s^4}+4\frac{1}{s+3}+10\frac{1}{s-1}+\frac{2}{s^2+4}$$

# Problems on Linearity property…

2. $\sin 2t \cos 3t$

   Let $f(t) = \sin 2t \cos 3t$

   $$L[f(t)] = L[\sin 2t \cos 3t]$$

   $$= \frac{1}{2} L[\sin 5t - \sin t] = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

3. $L[\sin t \sin 3t \sin 5t] = \frac{1}{4}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1} - \frac{9}{s^2+81} + \frac{7}{s^2+49}\right]$

4. $L[\cos^2 4t] = \frac{1}{2}[1 + \cos 8t]$

   $$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$$

# Problems on Linearity property...

5. Find the Laplace transform of $\left(t^{5/4} - t^{-5/4}\right)^2$

Let $f(t) = \left(t^{5/4} - t^{-5/4}\right)^2 = t^{5/2} + t^{-5/2} - 2$

$$L[f(t)] = L\left[t^{5/2}\right] + L\left[t^{-5/2}\right] - L[2]$$

$$L[f(t)] = \frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{7/2}} + \frac{\Gamma\left(-\frac{5}{2}+1\right)}{s^{-3/2}} - \frac{2}{s}$$

# Problems on Linearity property…

Consider

$$\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)=\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{15}{4}\Gamma\left(\frac{1}{2}+1\right)=\frac{15}{4}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}+1\right)=\Gamma\left(-\frac{3}{2}\right)=\frac{\Gamma\left(-\frac{3}{2}+1\right)}{(-3/2)}=\frac{-2}{3}\Gamma\left(-\frac{1}{2}\right)=\frac{-2}{3}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{(-1/2)}=\frac{4}{3}\sqrt{\pi}$$

Therefore

$$L[f(t)]=\frac{15}{8s^{7/2}}\sqrt{\pi}+\frac{4}{3s^{-3/2}}\sqrt{\pi}-\frac{2}{s}$$

# Problems on Frequency shift property

1. $L\left[e^{-at}\cos wt\right]$

In this case $f(t)=\cos wt$

$$F(s)=\frac{s}{s^2+w^2}$$

$$and\ \ F(s+a)=\frac{(s+a)}{(s+a)^2+w^2}$$

$$L[e^{-at}\cos(wt)]=\frac{(s+a)}{(s+a)^2+(w)^2}$$

# Illustration of Frequency shift property

1. $$L\left[e^{at}t^{n}\right]=\frac{n!}{(s-a)^{n+1}}$$

2. $$L\left[e^{at}\cosh bt\right]=\frac{s-a}{(s-a)^{2}-b^{2}}$$

3. $$L\left[e^{at}\sinh bt\right]=\frac{b}{(s-a)^{2}-b^{2}}$$

4. $$L\left[e^{at}\cos bt\right]=\frac{s-a}{(s-a)^{2}+b^{2}}$$

5. $$L\left[e^{at}\sin bt\right]=\frac{b}{(s-a)^{2}+b^{2}}$$

# Laplace transform of derivatives…

1. Find the Laplace transform of $y'' - 10y' + 9y = 5t$

$$L[y''] - 10L[y'] + 9L[y] = L[5t]$$

$$s^2Y(s) - sy(0) - y'(0) - 10\big(sY(s) - y(0)\big) + 9Y(s) = \frac{5}{s^2}$$

2. Find the Laplace transform of $x'' - 3x' + 2x = e^{-4t}$

$$L[x''] - 3L[x'] + 2L[x] = L[e^{-4t}]$$

$$s^2X(s) - sx(0) - x'(0) - 3\big(sX(s) - x(0)\big) + 2X(s) = \frac{1}{s+4}$$

# Problem on Transform of integral of a function

Find $$L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right]$$

$$L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

$$L\left[\int_0^t t\sin 3t\,dt\right] = \frac{6}{(s^2+9)^2}$$

$$L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \frac{6}{\left[(s+4)^2+9\right]^2}$$

i) $$L\left[2+5t^3+4e^{-3t}+10e^{t}+\sin 2t\right]$$

$$= L[2] + 5L[t^3] + 4L[e^{-3t}] + 10L[e^{t}] + L[\sin 2t]$$

$$= \frac{2}{s} + 5.\frac{3!}{s^{3+1}} + 4\frac{1}{s+3} + 10.\frac{1}{s-1} + \frac{2}{s^2+2^2}$$

$$= \frac{2}{s} + \frac{30}{s^4} + \frac{4}{s+3} + \frac{10}{s-1} + \frac{2}{s^2+4} \quad //$$

$$L[\sqrt{t}] = L[t^{1/2}] = \frac{\Gamma[1/2+1]}{s^{\frac{1}{2}+1}} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}$$

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$$

$$L[\sqrt{t}] \therefore = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$L[\cos^2 4t]$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

$$f(t) = \cos^2 4t = \frac{1+\cos 8t}{2}$$

$$L[f(t)] = \frac{1}{2} L[1+\cos 8t]$$

$$= \frac{1}{2}\left[L(1) + L[\cos 8t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+8^2}\right]$$

Q) $L[f(t)]$ given $f(t) = \sin t \cdot \sin 3t \sin 5t$

Solution

$$\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin t \cdot (\underline{\sin 5t \cdot \sin 3t}) = \frac{1}{2}[\cos(2)t - \cos 8t] \cdot \sin t$$

$$= \frac{1}{2}[\cos 2t \cdot \sin t - \cos 8t \cdot \sin t]$$

$$= \frac{1}{2}\left[\frac{1}{2}[\sin 3t - \sin t] - \frac{1}{2}\{\sin 9t - \sin 7t\}\right]$$

$$[f(t)] = \frac{1}{4}[\sin 3t - \sin t - \sin 9t + \sin 7t]$$

$$L[f(t)] = \frac{1}{4}[L(\sin 3t) - L[\sin t] - L(\sin 9t) + L[\sin 7t]]$$

$$= \frac{1}{4}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1^2} - \frac{9}{s^2+81} + \frac{7}{s^2+49}\right]$$

Q) find $L[f(t)]$ given $f(t) = \begin{cases} 0 & \text{for } 0<t<2 \\ 4 & \text{for } t>2 \end{cases}$

$$\Rightarrow \quad L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 \cancelto{0}{e^{-st}\, 0}\, dt + \int_2^\infty e^{-st} \cdot 4\, dt$$

$$= 0 + 4\left[\frac{e^{-st}}{-s}\right]_2^\infty = -\frac{4}{s}\left[0 - e^{-2s}\right]$$

$$= \frac{4 \cdot e^{-2s}}{s} \;//$$

Q) find $L[\cos\sqrt{t}]$

$$\cos[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$

$$\cos(\sqrt{t}) = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots\cdots$$

$$L[\cos\sqrt{t}] = L\left[1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots\cdots\right]$$

$$= \frac{1}{s} - \frac{1}{2!}\frac{1!}{s^2} + \frac{1}{4!}\frac{2!}{s^3} - \frac{1}{6!}\frac{3!}{s^4} + \cdots\cdots$$

$$\therefore L[\cos\sqrt{t}] = \frac{1}{s} - \frac{1}{2s^2} + \frac{1}{12s^3} - \frac{1}{120s^4} \cdots\cdots$$

# THANK YOU

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