> ~~FFT - A method for efficiently calculating DFT~~

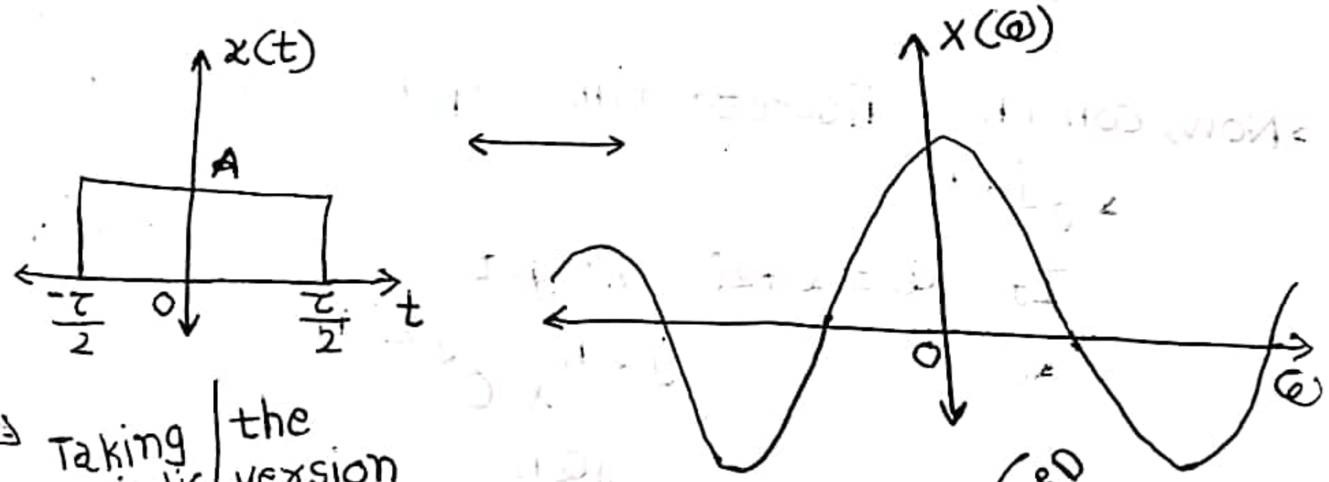
> DISCRETE TIME FOURIER TRANSFORM:

For computing fourier transform of discrete time signals.

> If a signal is discrete in time, its frequency domain representation will be periodic.
Also if signal is periodic in time, its frequency domain would be discrete.

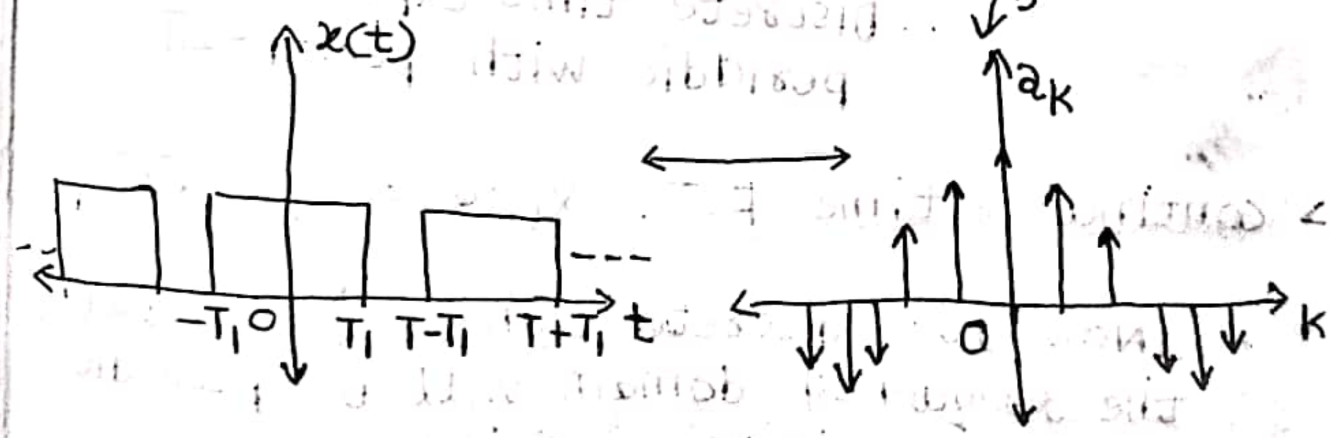
i.e. Periodicity in time domain results in sampling in frequency domain.

Ex: we have,



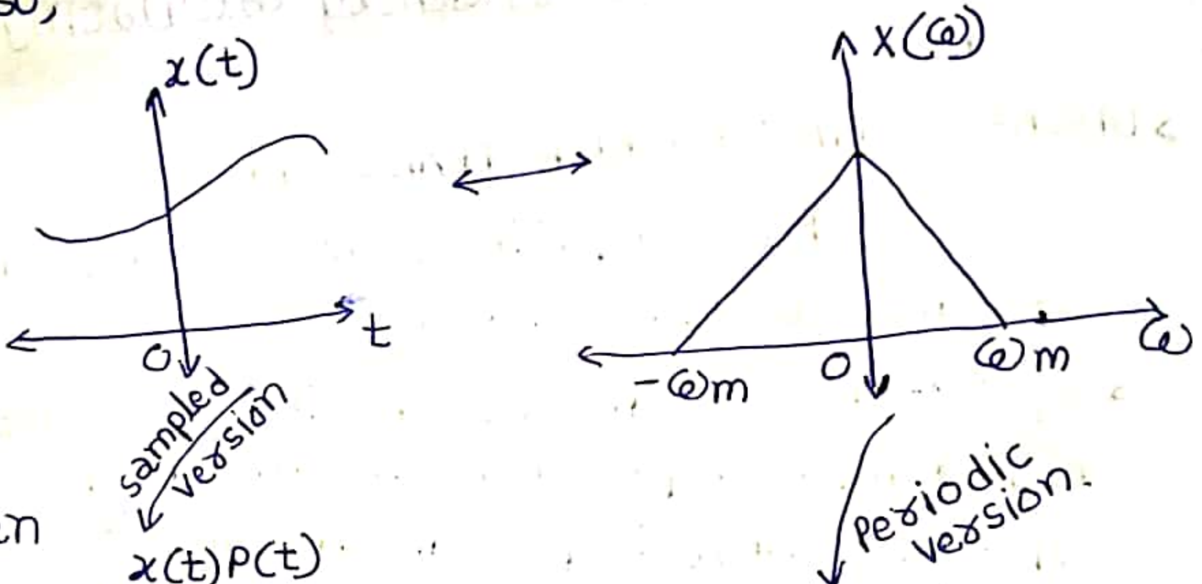
Taking the periodic version

SAMPLED

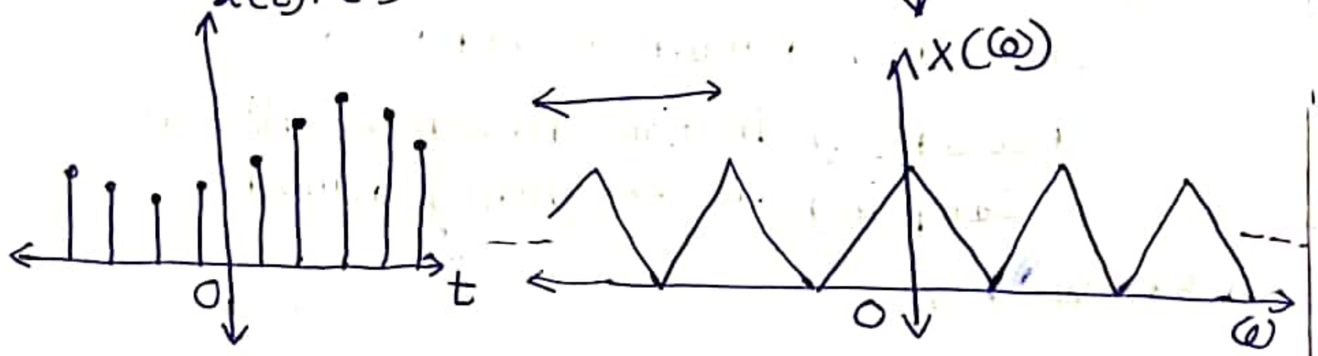


Also,

If



Then



> Now, consider discrete time exponentials: We have

> $e^{j\omega n}$

If $\omega = \omega + 2\pi$, we get

$$e^{j\omega n} \times e^{j2\pi n}$$

Even multiples of π

$$= e^{j\omega n}$$

$\therefore$ Discrete time exponentials are periodic with period $= 2\pi$

> continuous time F.T: $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

Now for discrete signals, $x(t) \to x[n]$ the frequency domain will be periodic. We represent it as $X(e^{j\omega})$

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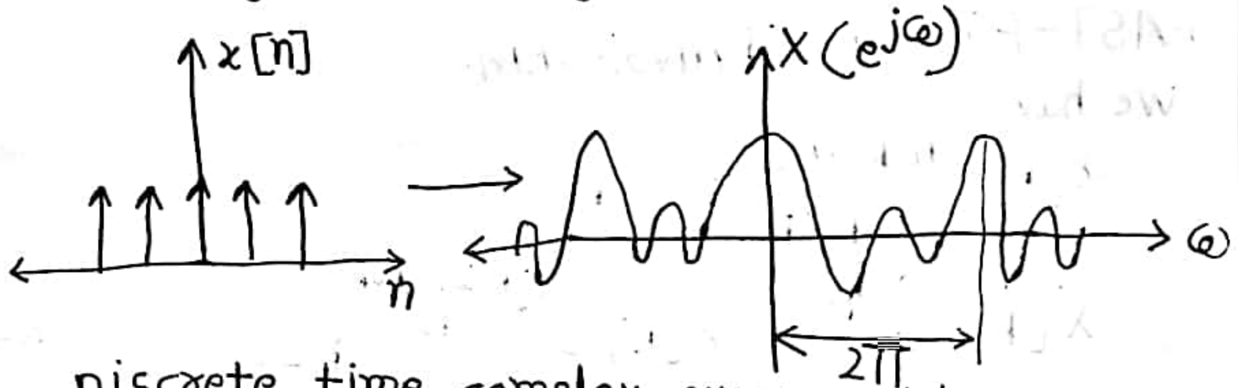
$$\therefore X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{— (1)}$$

showing that transform is periodic. (with period 2π)

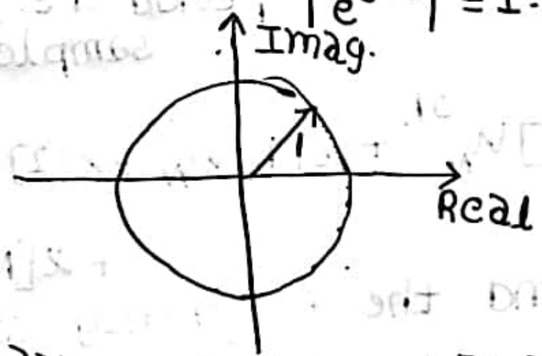
› Discretising the rectangular function :



Discrete time complex exponentials are identical at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, ----

Hence, we need not sum it up from $-\infty$ to ∞ but only for an interval of 2π since the transform is periodic. in eqn (1). This is because, $e^{j\omega} = \cos\omega + j\sin\omega$

$$|e^{j\omega}| = 1.$$



i.e. we are evaluating DTFT on a unit circle unlike the case of CTFT which is evaluated on $j\omega$ axis. Actually, $j\omega$ axis is a circle with infinite radius and hence the transform would not be periodic since the periodicity on an infinite radius circle is ∞.

Hence we put $X(e^{j\omega})$ but not $X(j\omega)$ since the transform is periodic.

$$\therefore x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$2\pi \rightarrow 0$ to 2π or $-\pi$ to π etc.

FAST-FOURIER TRANSFORM:

We have

$$x[n] \xrightarrow[\text{DFT}]{\text{n-point}} X[k]$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-jk2\pi n}{N}} \qquad k = 0, 1, 2, ---, N-1$$

N = Number of samples.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

where $W_N = e^{-j2\pi/N}$ (Twiddle factor)

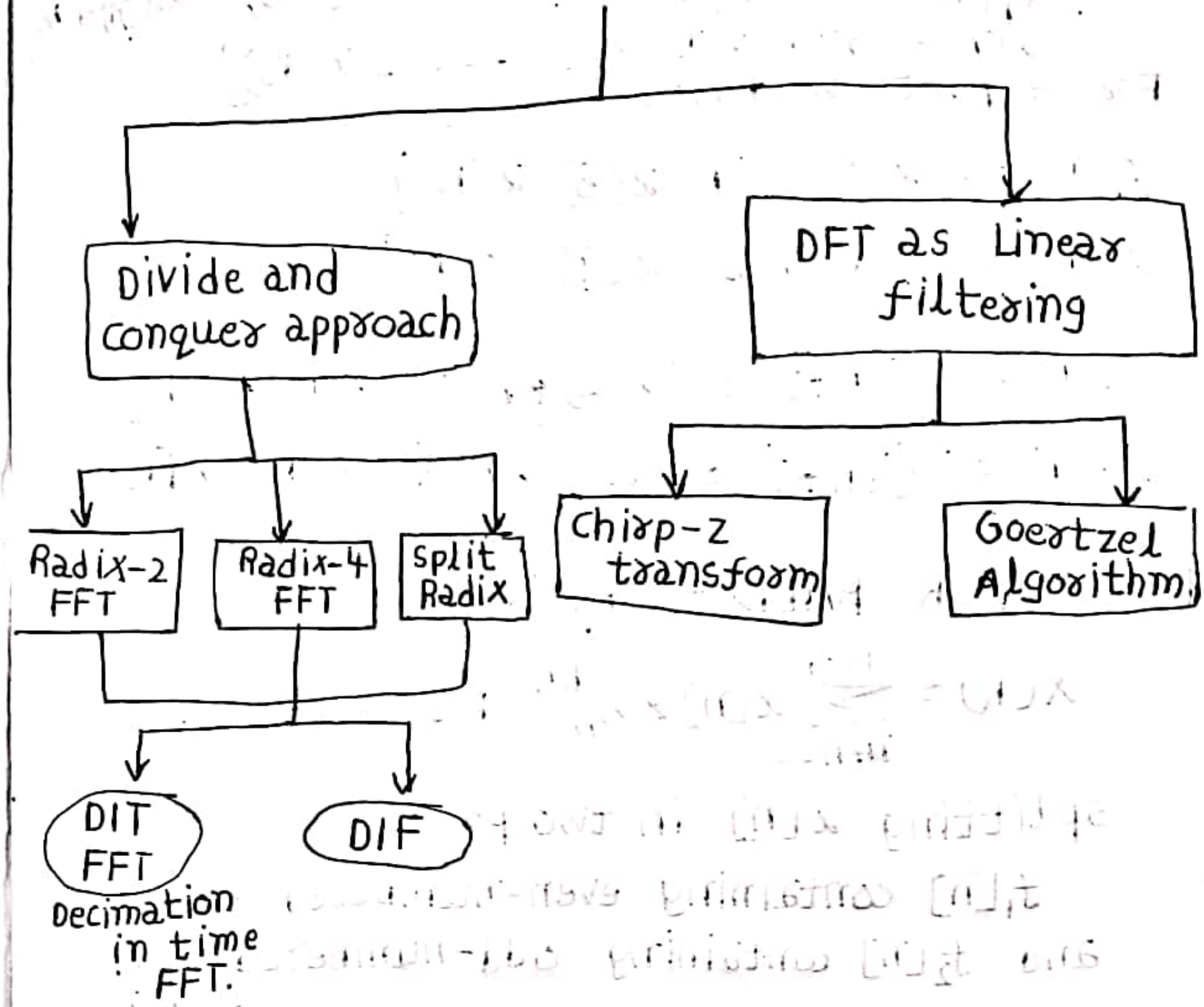
we are dividing the whole period i.e. 2π into 'N' samples equally.

$$X[k] = x[0]W_N^{0k} + x[1]W_N^{1k} + x[2]W_N^{2k} ----- + x[N-1]\, W_{N-1}^{(n-1)k}$$

- Hence, to find the frequency component at one sample, we are performing $n-1$ additions and n multiplications.

- For all values of k i.e. at all samples, we have n^2 multiplications and $n(n-1)$ additions.

COMPUTATION OF DFT



RADIX-2 DIT FFT:

Repeatedly divide the time-domain sequence into two-parts.

we have $x[n] = \{x(0), x(1), \text{----} \; x(N-1)\}$

N-point sequence.

Ex:

To divide $x[n]$ in m-parts:

$$
\left.\begin{aligned}
f_1[n] &= x[mn] \\
f_2[n] &= x[mn+1] \\
f_3[n] &= x[mn+2]
\end{aligned}\right\} \quad n = 0, 1, 2, \text{---} \frac{N}{m} - 1
$$

$$\vdots$$

$$f_m[n] = x[mn + (m-1)]$$

$\therefore$ ~~$f_1[n] = \{x(0), x(m), x(2m), x(3m), \ldots x((\frac{N}{m}-1)m)\}$~~

$x[n] = \{x(0), x(1), x(2), \text{-----} x(15)\}$

For 4 parts we have,

$$f_1[n] = \{x(0), x(4), x(8), x(12)\}$$

$$f_2[n] = \{x(1), x(5), x(9), x(13)\}$$

$$f_3[n] = x[4n+2] = \{x(2), x(6), x(10), x(14)\}$$

$$f_4[n] = x[4n+3] = \{x(3), x(7), x(11), x(15)\}$$

NOW FOR RADIX-2 FFT:

$$X(k) = \sum_{n=0}^{N-1} x[n] \, W_N^{kn} \quad k = 0, 1, 2, \text{----}, N-1.$$

splitting $x[n]$ in two parts,

$f_1[n]$ containing even-numbered samples and $f_2[n]$ containing odd-numbered samples of $x[n]$.

$$f_1[n] = x[2n] \quad , \quad f_2[n] = x[2n+1]$$

$$n = 0, 1, 2, \ldots \frac{N}{2} - 1 \quad \text{~~ie. 0, 1~~}$$

$$X(k) = \sum_{\text{Even}} x[n] \, W_N^{kn} + \sum_{\text{odd}} x[n] \, W_N^{kn}$$

i.e $$X(k) = \sum_{m=0}^{N/2-1} x[2m] \, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1] \, W_N^{2k(m+1)}$$

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1[m] \, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} f_2[m] \, W_N^{2km} \, W_N^{k} \quad \text{— (1)}$$

we have, $> W_N^{k+N} = W_N^{k}$

$> W_N^{k+\frac{N}{2}} = -W_N^{k}$ By putting $W_N = e^{\frac{-2\pi j}{N}}$

$> W_{\frac{N}{2}} = W_N^{2}$

$$\therefore X(k) = \underbrace{\sum_{m=0}^{\frac{N}{2}-1} f_1[m]\, W_{\frac{N}{2}}^{km}}_{\substack{\frac{N}{2} \text{ point} \\ \text{DFT of } \frac{N}{2} \text{ point} \\ \text{sequence } f_1[m]}} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} W_{\frac{N}{2}}^{km} f_2[m]$$

i.e. $\underbrace{X(k)}_{\text{N-point DFT}} = \underbrace{F_1(k)}_{\frac{N}{2}\text{ pt. DFT}} + W_N^k \left.F_2(k)\right\} \frac{N}{2}$ pt. DFT

$k = 0, 1, 2, \ldots N-1$ —— (2)

We have,

$X(k+N) = X(k)$ ∵ Transform is periodic by N samples which are taken over 2π.

$\therefore F_1(k+\frac{N}{2}) = F_1(k)$ ∵ $F_1(k)$ is $\frac{N}{2}$ point DFT.

$F_2(k+\frac{N}{2}) = F_2(k)$

Replacing k by $k+\frac{N}{2}$ in (3).

$$X(k+\tfrac{N}{2}) = F_1(k+\tfrac{N}{2}) + W_N^{(k+\frac{N}{2})} F_2(k+\tfrac{N}{2})$$

$$X(k+\tfrac{N}{2}) = F_1(k) - W_N^k F_2(k) \quad \text{—— (3)}$$

$$\therefore \left.\begin{aligned} X(k) &= F_1(k) + W_N^k F_2(k) \\ X(k+\tfrac{N}{2}) &= F_1(k) - W_N^k F_2(k) \end{aligned}\right\} (4) \quad k = 0, 1, 2, \ldots \tfrac{N}{2}-1$$

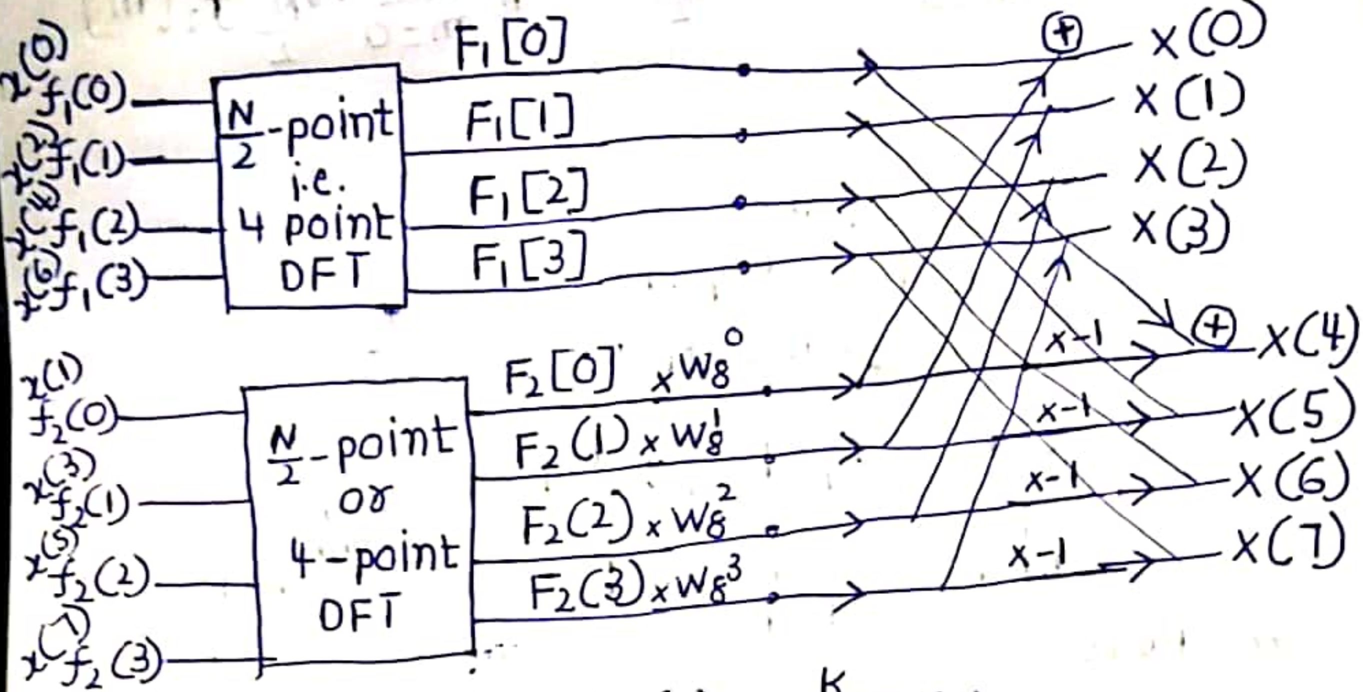
Considering an example of $N = 8$.

$x[n] = \{x(0), x(1), \ldots, x(7)\}$

$f_1[n] = \{x(0), x(2), \ldots x(6)\} = x[2n]$

$f_2[n] = x[2n+1] = \{x(1), x(3), x(5), x(7)\}$

$f_1[n] \xrightarrow{DFT} F_1(K) \quad f_2[n] \xrightarrow{DFT} F_2(K)$



$$X(K) = F_1(K) + W_8^K F_2(K)$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - W_8^K F_2(K)$$

$$K = 0, 1, 2, 3$$

SECOND STAGE OF DECIMATION:

i.e. Decimating $f_1[n]$ and $f_2[n]$ into even and odd parts for each. calculating 4-point DFT by taking two 2-point DFTs.

$f_1[n]$ → $V_{11}[n]$, $V_{12}[n]$

$f_2[n]$ → $V_{21}[n]$, $V_{22}[n]$

$V_{11}[n]$	$V_{12}[n]$	$V_{21}[n]$	$V_{22}[n]$
$= f_1[2n]$	$= f_1[2n+1]$	$= f_2[n]$	$= f_2[2n+1]$
$f_1(0)$, $f_1(2)$.	$f_1(1)$, $f_1(3)$.	$f_2(0)$, $f_2(2)$.	$f_2(1)$, $f_2(3)$
i.e. $\{x(0), x(4)\}$	$\{x(2), x(6)\}$	$\{x(1), x(5)\}$	$\{x(3), x(7)\}$

$$X(K) = F_1(K) + W_N^K F_2(K)$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - W_N^K F_2(K)$$

Now we get $F_1(K) = V_{11}(K) + W_{\frac{N}{2}}^K V_{12}(K)$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}(K) - W_{\frac{N}{2}}^K V_{12}(K) \qquad k=0,1,2,\ldots \frac{N}{4}-1$$

$\because F_{21}(k)$ is $\frac{N}{2}$ point sequence

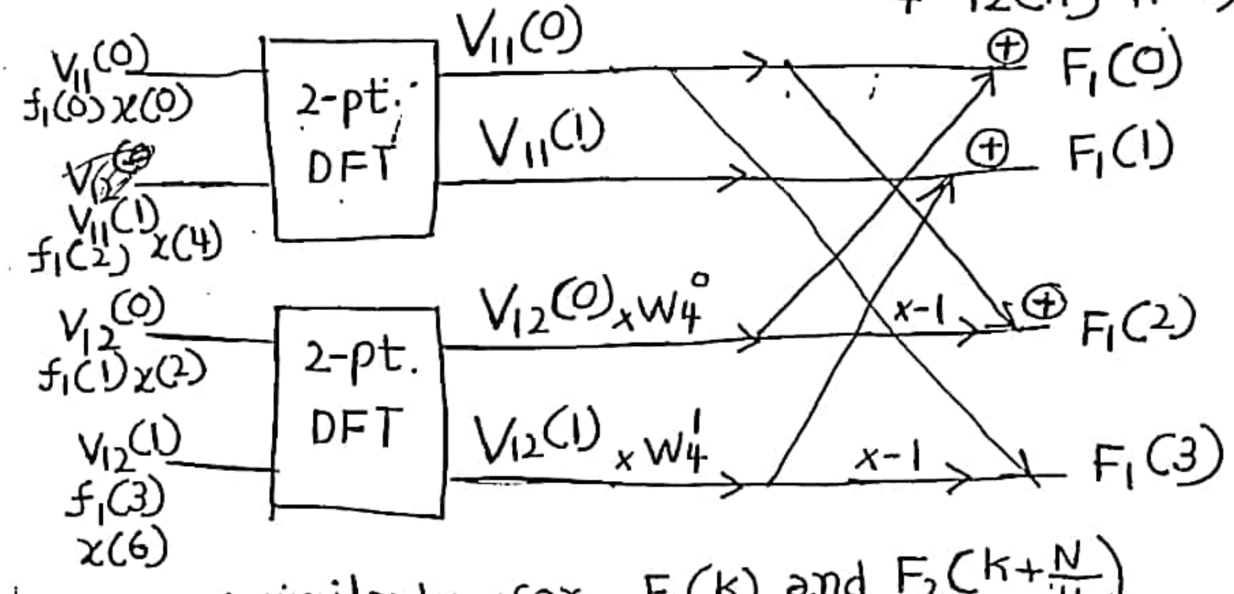
$$F_2(K) = V_{21}(K) + W_{\frac{N}{2}}^K V_{22}(K)$$

$$F_2\left(K+\frac{N}{4}\right) = V_{21}(K) - W_{\frac{N}{2}}^K V_{22}(K)$$

$$k=0,1,2,\ldots \frac{N}{4}-1$$

i.e. $F_1(K) = V_{11}(K) + W_4^K V_{12}(K). \quad K=0,1$

$$F_1(K+2) = V_{11}(K) - W_4^K V_{12}(K) \quad K=0,1$$



> similarly for $F_2(K)$ and $F_2\left(K+\frac{N}{4}\right)$

CALCULATION OF 2-point DFT:

$$X(K) = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$V_{11}[n] \xrightarrow[\text{DFT}]{\text{2-point}} V_{11}(K)$$

9 $$V_{11}(k) = \sum_{n=0}^{1} v_{11}[n]\, W_2^{kn} \qquad k = 0, 1.$$

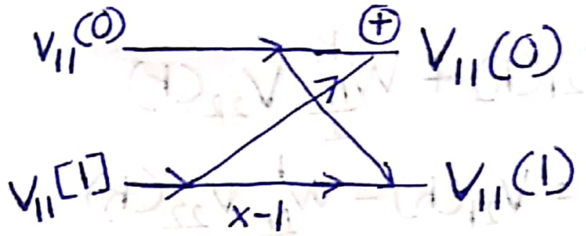
$$= v_{11}[0] + v_{11}[1]\, W_2^{k}$$

$$V_{11}(0) = v_{11}[0] + v_{11}[1] \quad \text{—— (1)}$$

$$V_{11}(1) = v_{11}[0] + v_{11}[1]\, W_2$$

$$W_2 = -1$$

$$\therefore V_{11}(1) = v_{11}[0] - v_{11}[1] \quad \text{—— (2)}$$



$v_{11}[n]$ $\quad$ $v_{12}[n]$

$$f_1[n] = \{x(0), x(2), x(4), x(6)\}$$

$$f_2[n] = \{x(1), x(3), x(5), x(7)\}$$

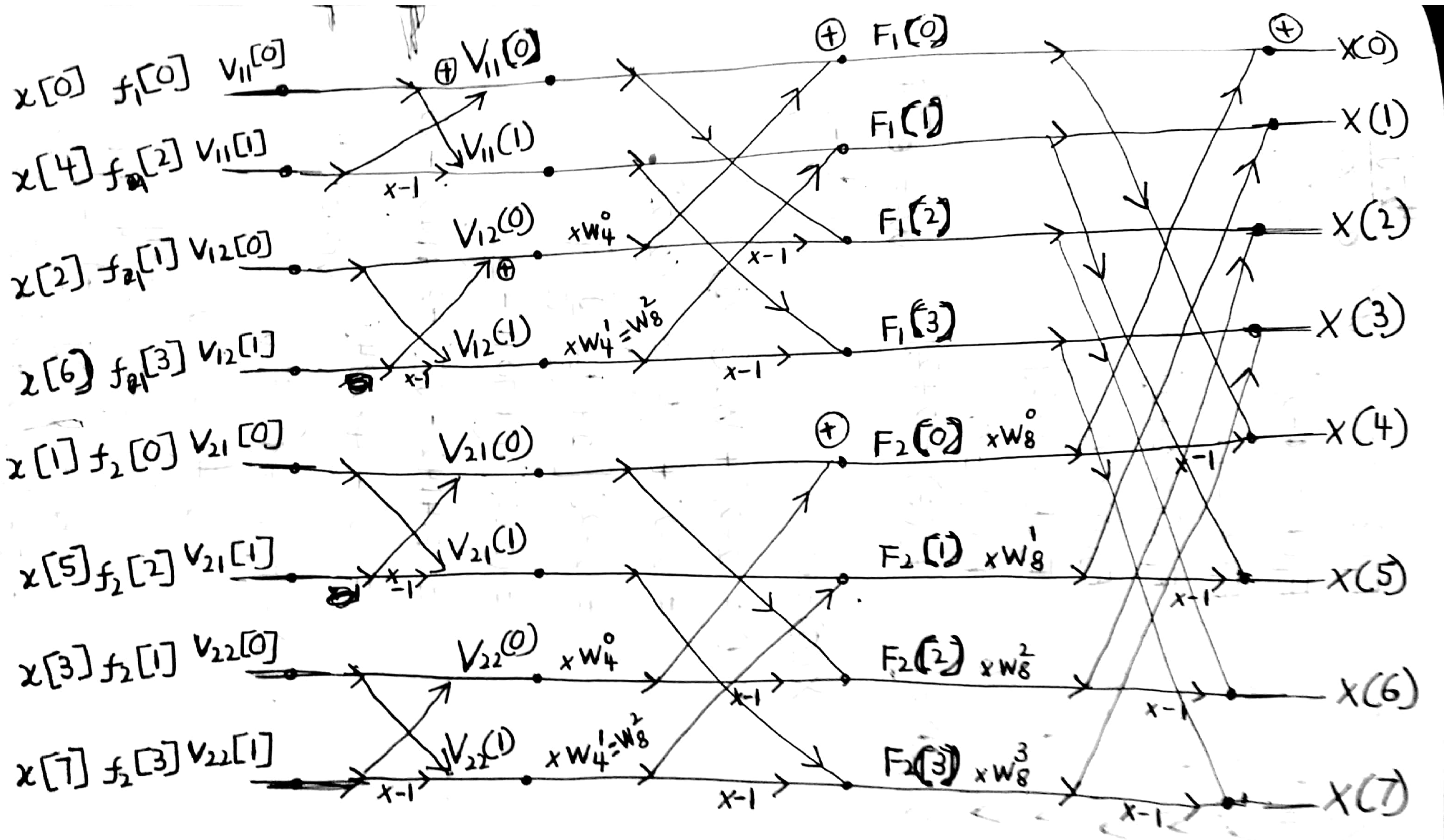
$v_{21}[n]$ $\quad$ $v_{22}[n]$

PROGRAM **OUTPUT**

Data: 1 1 1 1 -1 -1 -1 -1

FFT: 0, (2, -4.82843), 0, (2, -0.828427), 0, (2, 0.828427), 0, (2, 4.82843).

Execution Time: 258833.891249 usec



x[0] f1[0] V11[0]
x[4] f1[2] V11[1]
x[2] f21[1] V12[0]
x[6] f21[3] V12[1]
x[1] f2[0] V21[0]
x[5] f2[2] V21[1]
x[3] f2[1] V22[0]
x[7] f2[3] V22[1]
V11(0)
V11(1)
V12(0)
V12(1)
V21(0)
V21(1)
V22(0)
V22(1)
x-1
xW4^0
xW4^1=W8^2
F1[0]
F1[1]
F1[2]
F1[3]
F2[0] xW8^0
F2[1] xW8^1
F2[2] xW8^2
F2[3] xW8^3
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)

$V_{11}(0) = 0$ $\qquad F_1(0) = 0$ $\qquad W_8^2 = e^{-\frac{4\pi j}{8}}$

$V_{11}(1) = 2$ $\qquad F_1(1) = 2-2j$ $\qquad = -j$

$V_{12}(0) = 0$ $\qquad F_1(2) = 0$

$V_{12}(1) = 2$ $\qquad F_1(3) = 2+2j$

$V_{21}(0) = 0$ $\qquad F_2(0) = 0$

$V_{21}(1) = 2$ $\qquad F_2(1) = 2-2j$

$V_{22}(0) = 0$ $\qquad F_2(2) = 0$

$V_{22}(1) = 2$ $\qquad F_2(3) = 2+2j$

$$W_8 = e^{\frac{-\pi j}{4}} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$$

$$X(0) = 0$$

$$X(1) = 2-2j + W_8(2-2j)$$
$$= 2-2j + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)(2-2j)$$
$$= 2-2j\left[1 + \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right]$$
$$= (2-2j)\left(1.7071 - \frac{j}{\sqrt{2}}\right)$$
$$= 3.4142 - \sqrt{2}j - 3.4142j - \sqrt{2}$$
$$= 2 - 4.8284j$$

// verified.

$$X(3) = F_1(3) + F_2(3)\,W_8^3$$
$$= 2+2j + (2+2j)\,W_8^3$$
$$= 2+2j + (2+2j)\,e^{\frac{-3\pi j}{4}}$$
$$= 2+2j + (2+2j)\left(\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$
$$= 2-\sqrt{2}+\sqrt{2}+2j-\sqrt{2}j-\sqrt{2}j$$
$$X(3) = 2 - j0.828427$$