UNIT- IV Numerical solution of Ordinary differential equations.

CBNT NOTES

Solution of a d.e.:

The solution of a ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x.

Taylor's series Method:

Consider the first order d.e. $\frac{dy}{dx} = f(x,y)$.

The Taylor's series expansion of the d.e. at $x = x_0$ and $y = y(0)$ is given by

$$y = y_0 + (x-x_0)(y')_{x=0} + \frac{(x-x_0)^2}{\lfloor 2}(y'')_0 + \frac{(x-x_0)^3}{\lfloor 3}(y''')_0 + \cdots$$

Ex: Solve $\frac{dy}{dx} = x+y$, $y(0)=1$ by Taylor's series method. Hence find the values of y at $x=0.1$ and $x=0.2$

Soln: We have

$\frac{dy}{dx} = y' = x+y$ $\qquad y'(0) = x_0 + y_0$.

$y'(0) = 0+1 = 1$ $\qquad [x_0=0, y(0)=1]$

$\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = y''$

$y''(0) = 1+1 = 2$

$\frac{d^3y}{dx^3} = y'''$

$y'''(0) = 2$

$\frac{d^4y}{dx^4} = y^{iv}$

$y^{iv}(0) = 2$ etc.

Taylor's series is

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{\lfloor 2}(y'')_0 + \frac{(x-x_0)^3}{\lfloor 3}(y''')_0 + \cdots$$

Put $x_0 = 0$, $y_0 = 1$

$$y = 1 + x(1) + \frac{x^2}{2}\cdot 2 + \frac{x^3}{\lfloor 3}2 + \frac{x^4}{\lfloor 4}\times 2 + \cdots$$

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \cdots$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{6} + \cdots$$

$y(0.1) = 1.1103$

$$y(0.2) = 1+0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$

$$y(0.2) = 1.2427$$

Ex: Find by Taylor's series Method, the Value of y at $x=0.1$, and $x=0.2$ to five decimal places from $\frac{dy}{dx} = x^2y-1$, $y(0)=1$.

Soln: We have $\frac{dy}{dx} = y' = x^2y-1$

and $x_0 = 0$, $y_0 = 1$

$$\frac{dy}{dx} = y' = x^2y-1 \qquad y'(0) = (x_0^2)y_0 - 1 = 0\times1-1 = -1$$

$$\frac{d^2y}{dx^2} = y'' = 2xy + x^2y' \qquad y''(0) = 2\times0\times1 + 0\times(-1) = 0$$

$$\frac{d^3y}{dx^3} = y''' = (2y + 2xy') + (2xy' + x^2y'')$$

$$= 2y + 4xy' + x^2y''$$

$$\frac{d^3y}{dx^3} = y'''(0) = 2y_0 + 4x_0(y')_0 + x_0^2(y'')_0 = 2 + 4\times0 + 0 = 2$$

$$\frac{d^4y}{dx^4} = 2y' + 4[y' + xy''] + [2xy'' + x^2y''']$$

$$= 6y' + 6xy'' + x^2y'''$$

$$y^{iv}(0) = 6(y')_0 + 6x_0(y'')_0 + x_0^2(y''')_0$$

$$y^{iv} = -6 + 0 + 0 = -6$$

By Taylor's series

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{\lfloor 2}(y'')_0 + \frac{(x-x_0)^3}{\lfloor 3}(y''')_0 + \cdots$$

$$y = 1 + x(-1) + \frac{x^2}{2}\times0 + \frac{x^3}{\lfloor 3}\times2 + \frac{x^4}{\lfloor 4}(-6) + \cdots$$

At $x=0.1$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}\times0 + \frac{(0.1)^3}{6}2 - \frac{6(0.1)^4}{24} + \cdots$$

$$y(0.1) = 0.90033$$

$$y(0.2) = 1 + (0.2)(-1) + \frac{(0.2)^2}{2}\times0 + \frac{(0.2)^3}{6}\times2 - \frac{6(0.2)^4}{24} + \cdots$$

$$y(0.2) = 0.80227$$

Euler's Method

Employ Taylor's series method to obtain approximate value of y at $x=0.2$ for the differential equations $\frac{dy}{dx} = 2y+3e^x$. Given $y(0)=0$.

Soln: We the the d.e.

$$y' = \frac{dy}{dx} = 2y+3e^x \Rightarrow (y')_0 = 2\times 0+3e^0 = 3$$

also $x_0=0, y_0=0$

$$\frac{d^2y}{dx^2} = y'' = 2y'+3e^x \Rightarrow (y'')_0 = 2\times 3+3\cdot e^0 = 9$$

$$\frac{d^3y}{dx^3} = y''' = 2y''+3e^x \Rightarrow (y''')_0 = 2\times 9+3e^0 = 21$$

$$\frac{d^4y}{dx^4} = y^{iv} = 2y'''+3e^x \Rightarrow (y^{iv})_0 = 2\times 21+3e^0 = 45 \text{ etc}$$

By Taylor's Series Method

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{\lfloor 2}(y'')_0 + \frac{(x-x_0)^3}{\lfloor 3}(y''')_0 + \frac{(x-x_0)^4}{\lfloor 4}(y^{iv})_0 - $$

$$y = 0 + x(3) + \frac{x^2}{2}(9) + \frac{x^3}{6}(21) + \frac{x^4}{24}\times 45 - - -$$

$$y = 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 - - \quad (1)$$

Put $x=0.2$ in above eqn.

$$y(0.2) = 3\times(0.2) + \frac{9}{2}(0.2)^2 + \frac{21}{6}(0.2)^3 + \frac{45}{24}(0.2)^4 -$$

$$y(0.2) = 0.8110$$

Ex: Solve by Taylor's series method the equation $\frac{dy}{dx} = \log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1)=2$.

Soln: We have $\frac{dy}{dx} = y' = \log(xy) \Rightarrow \log x + \log y = y'$

$\cancel{x_0=0}$, $x_0 = 1$, $y_0 = 2$.

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{xy}[y + xy'] = \frac{1}{x} + \frac{y'}{y};$$

$$\frac{d^3y}{dx^3} = y''' = -\frac{1}{x^2} + \frac{y\cdot y'' - y'y'}{y^2} \Rightarrow -\frac{1}{x^2} + \frac{yy''-(y')^2}{y^2} \Rightarrow -\frac{1}{x^2} + \frac{y''}{y} - \frac{(y')^2}{y^2}$$

$(y')_1 = y'(1) = \log 1 + \log 2 = \log 2$

$(y'')_1 = y''(1) = \frac{1}{1} + \frac{1}{2}\log 2 = 1+\frac{1}{2}\log 2$

$(y''')_1 = y'''(1) = -1 + \frac{1}{2}\left(1+\frac{1}{2}\log 2\right) + \frac{1}{4}(\log 2)^2$

By Taylor's series Method.

$$y_1 = y_{x_0} + (x-x_0)(y')_{x_0} + \frac{(x-x_0)^2}{\lfloor 2}(y'')_{x_0} + \left(\frac{x-x_0}{\lfloor 3}\right)^3 (y''')_{x_0} + \cdots$$

$$y = 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left[1+\frac{1}{2}\log 2\right] + \frac{1}{6}(x-1)^3\left[-\frac{1}{2}+\frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] + \cdots \quad \text{(A)}$$

Put $x=1.1$ in equation (A)

$$y(1.1) = 2+(1.1-1)\log 2 + \frac{1}{2}(1.1-1)^2\left(1+\frac{1}{2}\log 2\right) + \frac{1}{6}(1.1-1)^3\left(-\frac{1}{2}+\frac{1}{4}\log 2-\frac{1}{4}(\log 2)^2\right)$$

$$y(1.1) = 2.036$$

and $y(1.2) = 2+(1.2-1)\log 2 + \frac{1}{2}(1.2-1)^2\left(1+\frac{1}{2}\log 2\right) + \frac{1}{6}(1.2-1)^3\left(-\frac{1}{2}+\frac{1}{4}\log 2+\frac{1}{4}(\log 2)^2\right)$

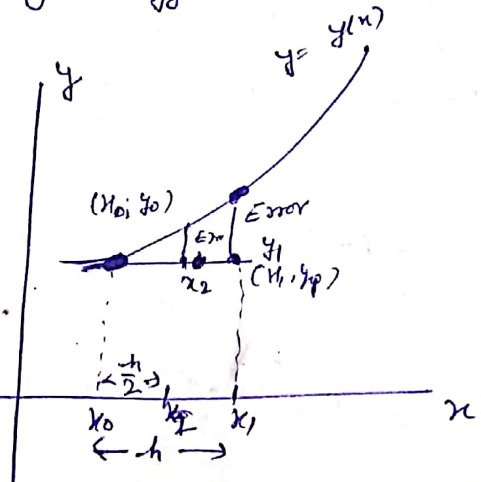
$y(1.2) = 2.081$

Euler's Method

Let $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

be given d.e.

Let $y = y(x) = f(x)$ be the solution of given d.e.



Equation of tangent at (x_0, y_0)

$$y - y_0 = (y')(x - x_0)$$

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

Let x_1 be any point on x-axis and let y_1 be corresponding value.

$$y_1 = y_0 + f(x_0, y_0) \cdot (x_1 - x_0)$$

$$\boxed{y_1 = y_0 + h\, f(x_0, y_0)}$$

Similarly $y_2 = y_1 + h\, f(x_1, y_1)$

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

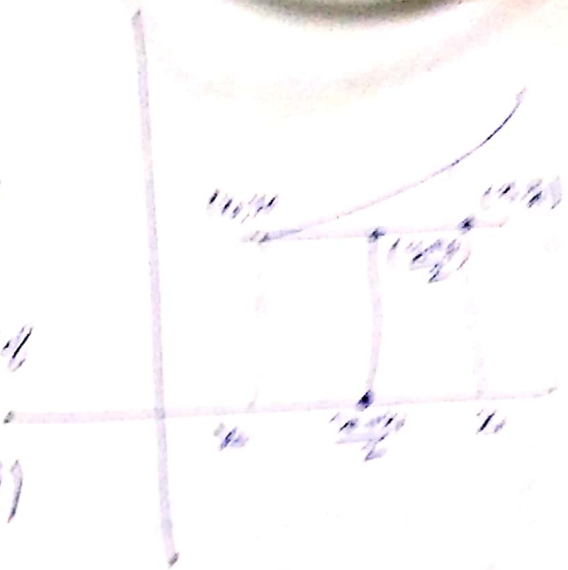
This is called Euler's Method.

Euler Modified

By Euler Method

$y_1 = y_0 + h f(x_0, y_0)$

y_1 = old value + step size × slope



$y_1 = y_0 + h \times$ slope at mid point of (x_0, y_0) & (x_1, y_1)

$y_1 = y_0 + h f\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$

$y_1 = y_0 + h f\left(\frac{x_0+x_0+h}{2}, \frac{y_0+y_0+hf(x_0,y_0)}{2}\right)$

$= y_0 + h f\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}hf(x_0,y_0)\right)$

$\boxed{y_1 = y_0 + K_2}$ $\Rightarrow y_0 + h f\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$ $\boxed{y_1 = y_0 + K_2}$

$K_1 = h f(x_0, y_0)$

$K_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$$\boxed{y_1 = y_0 + h f\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)}$$

or

$y_1 = y_0 + K_2$

where $K_1 = h f(x_0, y_0)$

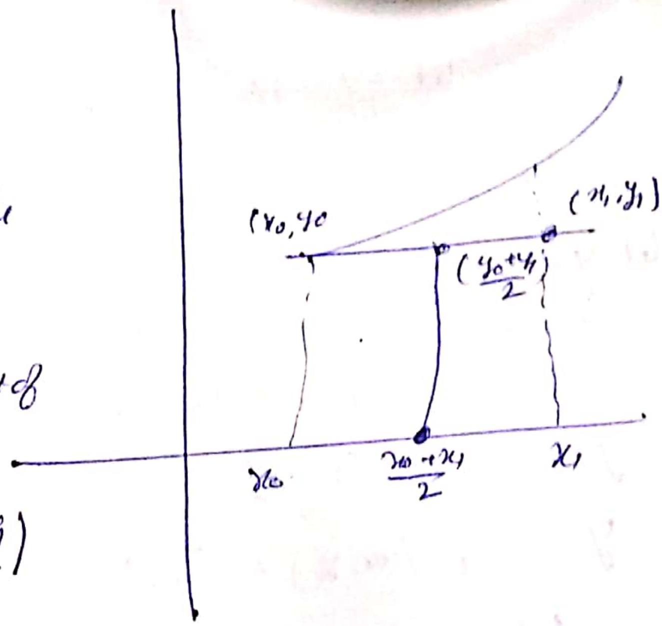
$K_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

Euler modified:

By Euler method

$y_1 = y_0 + h f(x_0, y_0)$

$y =$ old value + stepsize × slope



$y_1 = y_0 + h \times$ slope at mid point of (x_0, y_0) & (x_1, y_1)

$$y_1 = y_0 + h\, f\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$$

$$y_1 = y_0 + h\, f\left(\frac{x_0 + x_0 + h}{2}, \frac{y_0 + y_0 + hf(x_0, y_0)}{2}\right)$$

$$= y_0 + h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2} h f(x_0, y_0)\right)$$

$\boxed{y_1 = y_0 + K_2}$ $\Rightarrow y_0 + h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_1 = h\, f(x_0, y_0)$ $\boxed{y_1 = y_0 + h K_2}$

$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$$\boxed{y_1 = y_0 + h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)}$$

or

$$y_1 = y_0 + K_2$$

where $K_1 = h\, f(x_0, y_0)$

$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

Q. Use Euler's modified method to compute $y(1)$ $y_{x=5}$. Given that $\frac{dy}{dx} = x+y$ with initial condition $x_0 = 0$, $y_0 = 0$. ~~find $y(1)$~~ in 5 steps.

Soln. We have $\frac{dy}{dx} = f(x,y) = x+y \Rightarrow f(x,y) = x+y$

$x_0 = 0, \; y_0 = 1, \; h = 0.2$

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	0.2	.4	0.6	.8	1.0
y	.0	~~.0884~~ .02	.0884			

$f(x,y) = x+y$

$$y_1 = y_0 + h\, f\left(x_0 + \frac{h}{2},\; y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

$$= 0 + h\, f\left(0+0.1,\; 0 + \frac{0.2}{2}[x_0 + y_0]\right)$$

$$= 0 + h\, f(0.1,\; 0 + 0.1[0+0])$$

$$= 0 + h\, f(0.1, 0)$$

$$= 0 + 0.2[0.1 + 0]$$

$$y_1 = 0.02$$

$$y_2 = y_1 + h\, f\left(x_1 + \frac{h}{2},\; y_1 + \frac{h}{2} f(x_1, y_1)\right)$$

$$= 0.02 + 0.2 f(0.2 + 0.1,\; 0.02 + 0.1 f(0.2, 0.02))$$

$$= 0.02 + 0.2 f(0.3,\; 0.02 + 0.1[0.2 + 0.02])$$

$$= 0.02 + 0.2 f(0.3,\; 0.042)$$

$$= 0.02 + 0.2[0.3 + 0.042]$$

$$= 0.0884$$

$$y_3 = y_2 + h\, f\left(x_2 + \frac{h}{2},\; y_2 + \frac{h}{2} f(x_2, y_2)\right)$$

$$= 0.0884 + 0.2 f(0.4 + 0.1,\; 0.0884 + 0.1 f(0.4, 0.0884))$$

$$= 0.0884 + 0.2 f(0.5,\; 0.0884 + 0.1(0.4 + 0.0884))$$

$$= 0.0884 + 0.2 f(0.5,\; 0.\cancel{09328}\ .13724)$$

$$= 0.0884 + 0.2[0.5 + 0.\cancel{09328}]\ .13724$$

$$= 0.0884 + \cancel{0.1186568}\ .127448$$

$$= \cancel{0.2070568} = 0.215848$$

$$y_4 = y_3 + h\, f\left(x_3 + \frac{h}{2},\; y_3 + \frac{h}{2} f(x_3, y_3)\right)$$

$$= 0.215848 + 0.2 f(0.6 + 0.1,\; 0.215848 + 0.1(0.6 + 0.215848))$$

$$= 0.215848 + 0.2 f(0.7,\; 0.2974328)$$

$$= 0.215848 + 0.2[0.7 + 0.2974328]$$

$$.215848 + 0.199487$$

$$= 0.41533456$$

$$= 0.415335$$

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$$y_5 = y_4 + h f\left(x_4 + \frac{h}{2},\ y_4 + \frac{h}{2} f(x_4, y_4)\right)$$

$$= 0.415335 + 0.2 f(0.8 + 0.1,\ 0.415335 + 0.1 f(0.8, 0.4153356))$$

$$= 0.415335 + 0.2 f(0.9,\ 0.415335 + 0.1 [0.8 + 0.415335])$$

$$= 0.415335 + 0.2 f(0.9,\ 0.53687)$$

$$= 0.415335 + 0.2 [0.9 + 0.53687]$$

$$= 0.415335 + 0.287374$$

$$= \underline{0.702709}$$

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Ex: Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's Method.

Soln: We have

$$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$$

$x_0 = 0,\ y_0 = 1$

Let $n = 5$

$$h = \frac{x_n - x_0}{n} = \frac{0.1-0}{5} = 0.02$$

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	0.02	0.04	0.06	0.08	.1
y	1					
	y_0	y_1	y_2	y_3	y_4	y_5

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.02\left[\frac{0.1-0}{1+0}\right] = 1.02$$

$$y_2 = y_1 + hf(x_1, y_1) = \cancel{1+0}\ 1.02 + 0.02\left[\frac{1.02 - 0.02}{1.02 + 0.02}\right] = 1.0392$$

$$y_3 = y_2 + hf(x_2, y_2) = 1.0392 + 0.02\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right] = 1.0572$$

$$y_4 = y_3 + hf(x_3, y_3) = 1.0572 + 0.02\left[\frac{1.0572 - 0.06}{1.0572 + 0.06}\right] = 1.0756$$

$$y_5 = y_4 + hf(x_4, y_4) = 1.0756 + 0.02\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right] = 1.0928$$

Hence $y_5 = y(0.1) = 1.0928$ is the required result.

Ex: Using Euler's method, find approximate value of y when $x = 0.6$ of $\frac{dy}{dx} = 1 - 2xy$ given that $y = 0$ when $x = 0$ (take $h = 0.2$)

ANS $= 0.4748$

Soln: We have $\frac{dy}{dx} = 1 - 2xy = f(x,y)$

$\Rightarrow f(x,y) = 1 - 2xy$

$x_0 = 0,\ y_0,\quad h = 0.2$

	x_0	x_1	x_2	x_3
x	0	0.2	0.4	
y	0			
	y_0	y_1	y_2	y_3

$y_1 = y_0 + h\, f(x_0, y_0) = 0 + 0.02[1 - 0\times 0] = 0.2$

$y_2 = y_1 + h\, f(x_1, y_1) = 0.2 + 0.2\,[1 - 0.2\times 0.2\times 2]$

$= \cancel{0.2 + 0.2\,[1 - 0.9} = 0.384$

$f(x,y) = 1 - 2xy$

$x_1 = 0.2$

$x_2 = 0.4$

$y = 0.2$

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 0.384 + 0.2\,[1 - 2\times 0.4\times 0.384] = \cancel{0.52}$

$= 0.384 + 0.13856 = \underline{0.52256}$

Runge-Kutta Method of 4th order: Fourth Order

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{with } y(x_0) = y_0$$

Formula is given by

$$y = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where $K_1 = hf(x_0, y_0)$, $K_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$, $K_4 = h\,f(x_0 + h, y_0 + K_3)$

Ex: Estimate $y(1)$ if $2yy' = x^2$ and $y(0) = 2$ using Runge-Kutta method of fourth order by taking $h = 0.5$. Also compare the result with exact value.

Soln: We have $\frac{dy}{dx} = \frac{x^2}{2y}$, $x_0 = 0$, $y_0 = 2$, $h = 0.5$

$$f(x, y) = \frac{x^2}{2y}$$

$$K_1 = hf(x_0, y_0) = 0.5\left[\frac{0}{4}\right] = 0 \quad \frac{x_0^2}{2y_0}$$

$$K_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.5\,f(0 + 0.25, 2 + 0) = 0.5\,f(.25, 2)$$

$$= 0.5\,\frac{(.25)^2}{2 \times 2} = .0078$$

$$K_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.5\,f(0.25, 2 + \frac{.0078}{2}) = 0.5\,f(.25, 2.0039) \quad \Big| \frac{x^2}{2y}$$

$$= 0.5\,\frac{(.25)^2}{2.0039 \times 2} = .0078 \quad \Big| f(x,y) = \frac{x^2}{2y}$$

$$K_4 = h\,f(x_0 + h, y_0 + K_3) = 0.5\,f(0.5, 2 + .0078) = 0.5\,f(0.5, 2.0078)$$

$$K_4 = 0.5\,\frac{(.5)^2}{2 \times 2.0078} = 0.0311 \quad \Big| f(x,y) = \frac{x^2}{2y}$$

By Runge Kutta Method

$$y = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 2 + \frac{1}{6}(0 + 2(.0078) + 2(.0078) + 0.0311)$$

$$y_{0.5} = 2.0104$$

For Second Step $\quad x_1 = 0.5, \quad y_1 = 2.0104$

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2(K_2 + K_3) + K_4)$$

$k_1 = h f(x_1, y_1) = 0.5 f(0.5, 2.0104) = 0.5 \frac{(.5)^2}{2 \times 2.0104} = .0311$ | $f(x,y) = \frac{x^2}{2y}$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.5 f(.5 + .25, 2.0104 + \frac{.0311}{2}) = 0.5 f(.75, 2.0156)$

$= 0.5 \frac{(.75)^2}{2 \times 2.0156} = 0.698$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.5 f(.75, 2.0104 + \frac{.698}{2}) = 0.5 f(.75, 2.0349)$

$= 0.5 \frac{(.75)^2}{2 \times 2.0349} = 0.0691$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.5 f(1, 2.0104 + 0.0691) = 0.5 f(1, 2.0691)$

$= 0.5 \frac{(1)^2}{2 \times 2.0691} = 0.1208$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = 2.0104 + \frac{1}{6}[0.0311 + 2(.0698 + 0.0691) + .1208]$

$$\boxed{y_2 = 2.0104 + 0.0716 = 2.082}$$

Exact

$2yy' = x^2$

$d(y^2) = x^2$

on integrating

$y^2 = \frac{x^3}{3} + C$ —— (A)

~~Put x=1 in (A)~~ Put $x = 0, y = 2$ in (A)

$\cancel{[y(1)]^2 = \frac{1}{3} + 4}$ $C = 4$

$y^2 = \frac{x^3}{3} + 4$ —— (B)

Put $x = 1$ in (B)

$y(1) \sqrt{\frac{1}{3} + 4} = \sqrt{\frac{13}{3}} = 2.08166$

Ex: Apply Runge-Kutta Method of fourth order to solve $5\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ $\Rightarrow y' = \frac{x^2 + y^2}{5} = f(x,y)$ and find y in the interval $0 \le x \le 0.2$ taking $h = 0.1$

$y_{0.1} = 1.020476$, $y_{0.2} = 1.042212$

Que: Apply Runge-Kutta Method of fourth order, to find an approximate value of y at $x=0.2$, given that $\frac{dy}{dx} = x+y^2$ and $y=1$ when $x=0$, ~~taking $h=0.1$~~.

Soln: let $h=0.1$ we have $\frac{dy}{dx} = x+y^2$

Here $x_0=0$, $y_0=0$, $f(x,y) = x+y^2$

Now $K_1 = hf(x_0, y_0) = h[x_0+y_0^2] = 0.1\,(0+1^2) = \underline{0.1}$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = hf(0+0.05 + 1+0.05) = h\,f(0.05, (1.05))$

$= 0.1\,[0.05+(1.05)^2] = \underline{0.11525}$

$K_3 = h\,f(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = h\,f(0+0.05+1+0.05763) = h\,f(0.05, 1.0576$

$= 0.1\,[0.05+(1.05763)^2] = \underline{0.11686}$

$K_4 = h\,f(x_0+h, y_0+K_3) = h\,f(0+0.1, 1+0.11686) = h\,f(0.1, 1.11686)$

$= 0.1\,[0.1+(1.11686)^2] = \underline{0.13474}$

$y_1 = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 1+\frac{1}{6}[0.1+2(0.11525+0.11686)+0.13474])$

$y_1 = 1+0.11649 = 1.11649$

For Second Step: $x_1 = 0.1$, $y_1 = 1.11649$

$K_1 = hf(x_1, y_1) = h\,f(0.1, 1.11649) = 0.1\,[0.1+(1.11649)^2]$

$K_1 = 0.13466$

$K_2 = hf(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}) = \cancel{0.1}\,h\,f(0.1+0.05, 1.11649+\frac{0.13466}{2})$

$= h\,f(0.15, 1.18382) = 0.1\,[0.15+(1.18382)^2] = 0.15514$

$K_3 = h\,f(x_1+\frac{h}{2}, y_1+\frac{K_2}{2}) = hf(0.15, 1.11649+\frac{0.15514}{2})$

$= h\,f(0.15, 0.\frac{15514}{2}) = 0.1f[0.15, 1.194061$

Que: Apply Runge-Kutta Method of fourth order, to find an approximate value of y at $x=0.2$, given that $\frac{dy}{dx} = x+y^2$ and $y=1$ when $x=0$, ~~taking~~ $h=0.1$.

Soln: let $h=0.1$ we have $\frac{dy}{dx} = x+y^2$

Here $x_0 = 0$, $y_0 = 0$, $f(x,y) = x+y^2$

Now $K_1 = hf(x_0, y_0) = h[x_0+y_0^2] = 0.1\,[0+1^2] = \underline{0.1}$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = hf(0+0.05 + 1+0.05) = hf(0.05, (1.05))$

$= 0.1\,[0.05 + (1.05)^2] = \underline{0.11525}$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = hf(0+0.05 + 1+0.05763) = hf(0.05, 1.05763)$

$= 0.1\,[0.05 + (1.05763)^2] = \underline{0.11686}$

$K_4 = hf(x_0+h, y_0+K_3) = hf(0+0.1, 1+0.11686) = hf(0.1, 1.11686)$

$= 0.1\,[0.1 + (1.11686)^2] = \underline{0.13474}$

$y_1 = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 1 + \frac{1}{6}[0.1 + 2(0.11525+0.11686) + 0.13474]$

$y_1 = 1 + 0.11649 = 1.11649$

For Second step: $x_1 = 0.1$, $y_1 = 1.11649$

$K_1 = hf(x_1, y_1) = hf(0.1, 1.11649) = 0.1\,[0.1 + (1.11649)^2]$

$K_1 = 0.13466$

$K_2 = hf(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}) = \cancel{0.1}\, hf(0.1+0.05, 1.11649 + \frac{0.13466}{2})$

$= hf(0.15, 1.18382) = 0.1\,[0.15 + (1.18382)^2] = 0.15514$

$K_3 = hf(x_1+\frac{h}{2}, y_1+\frac{K_2}{2}) = hf(0.15, 1.11649 + \frac{0.15514}{2})$

$= hf(0.15, \frac{0.15514}{2}) = 0.1\,[0.15, 1.19406]$

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$K_3 = 0.1[0.15 + (1.19406)^2] = 0.15758$

$K_4 = hf(x_1+h, y_1+K_3) = hf(0.1+0.1;\ 1.11649+0.15758)$

$= hf(0.2, 1.27407) = 0.1[0.2 + (1.27407)^2] = 0.18233$

$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1.11649 + \frac{1}{6}[0.13466 + 2(0.15574 + 0.15758) + 0.18233]$

$= 1.11649 + \frac{1}{6}(0.94243) = 1.11649 + 0.15707 = \cancel{1.23}\ 1.27356$

$y_1 = y(0.1) =$

y_2

	x_0	x_1	x_2
x	0	0.1	0.2
y	1	1.11649	1.27356
	y_0	$y_1 = y(0.1)$	$y_2 = y(0.2)$

Ex Using Runge-Kutta Method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$, and $x = 0.4$

Ans = 1.19600, 1.37527

Ex: Find the solution $y(0.1)$ of the initial value problem $y' = -2ty^2$, with $y(0) = 1$, with $h = 0.1$ using Runge-Kutta Method of fourth order.

ANS = 0.99009893

$K_1 = 0,\ K_2 = -0.01,\ K_3 = -0.00990025,\ K_4 = -0.01960595$

Ex Apply Runge-Kutta Method of fourth order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x+y$ and $y = 1$ when $x = 0$.

Soln: We have $\frac{dy}{dx} = x+y = f(x,y)$

Here $x_0 = 0$, $y_0 = 1$ let $h = 0.2$

$K_1 = h f(x_0, y_0) = 0.2[0+1] = 0.2$

$K_2 = h f(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = 0.2[0+0.1, 1+0.1] = 0.2[0.1+1.1] = .2400$

$K_3 = h f(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = 0.2[0+0.1+1+0.12] = 0.2(0.1+1.12) = .2440$

$K_4 = h f(x_0+h, y_0+K_3) = 0.2 \times f(0.2, 1.244) = 0.2[0.2+1.244] = 0.2888$

$y_1 = y(0.2) = y_0 + \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$

$= 1 + \frac{1}{6}(0.2 + 2(.24 + .244) + .2888) = 1 + 0.2428$

$y(0.2) = 1.2428$

Ex: Apply Runge-Kutta Method to solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$ $h = 0.2$

x	x_0: 0	x_1: 0.2	x_2: 0.4
y	y_0	y_1	y_2

Soln: We have

$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} = f(x,y)$

$x_0 = 0$, $y_0 = 1$.

let $h = 0.2$

To find $y(0.2)$

$K_1 = h f(x_0, y_0) = 0.2 f(0,1) = 0.2000$

$K_2 = h f(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = 0.2 f(0.1, 1.1) = 0.19672$

$K_3 = h f(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = 0.2 f(0.1, 1.09836) = 0.1967$

$K_4 = h f(x_0+h, y_0+K_3) = 0.2 f(0.2, 1.1967) = 0.1891$

$y = y_0 + \frac{1}{6}(K_1 + 2(K_2+K_3) + K_4)$

$= 1 + \frac{1}{6}[0.2 + 2(0.19672 + 0.1967) + 0.1891) = 1 + 0.19599$

$y_1 = y(0.2) = 1.19599$

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To find $y(0.4)$

$x_1 = 0.2,\ y_1 = 1.196,\ h = 0.2$

$k_1 = hf(x_1, y_1) = 0.2[2, 1.196] =$ 0.1891

$k_2 = hf(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}) = 0.2 f(0.3, 1.2906) =$ 0.1795

$k_3 = hf(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}) = 0.2 f(0.3, 1.2858) =$ 0.1793

$k_4 = hf(x_1+h, y_1+k_3) = 0.2 f(0.4, 1.3753) =$ 0.1688

$y_2 = y_1 + \frac{1}{6}(k_1 + 2(k_2+k_3) + k_4)$

$= 1.196 + \frac{1}{6}[0.1891 + 2(0.1795 + 0.1793) + 0.1688]$

$= 1.196 + 0.1792$

$y(0.4) = y_2 = 1.3752$

Ex: Using Runge-Kutta method of fourth order, solve for y at $x = 1.2, 1.4$ from $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1, y_0 = 0$

ANS: 0.1402, 0.2705

MILNE'S Predictor - corrector Method:

To find an appro. value of y for $x = x_0 + nh$ by Milne's

Method

Consider $\frac{dy}{dx} = f(x,y)$

Step I: Starting from $y_0 = y(x_0)$ we have to estimate successively

$y_1 = y(x_0+h)$ $\quad y_1 = y(x_1)$

$y_2 = y(x_0+2h)$ $\quad y_2 = y(x_2)$

$y_3 = y(x_0+3h)$ $\quad y_3 = y(x_3)$

By Picard or Taylor series method.

Step 2: Now we calculate

$f_0 = f(x_0, y_0)$

$f_1 = f(x_1, y_1)$ $\quad x_1 = x_0 + h$

$f_2 = f(x_2, y_2)$ $\quad x_2 = x_0 + 2h$

$f_3 = f(x_3, y_3)$. $\quad x_3 = x_0 + 3h$.

To find $y_4 = y(x_0+4h)$ we use Newton's forward interpolation formula

$$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{L2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \text{---}$$

Picards method

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)\,dx$$

$$= y_0 + \int_{x_0}^{x_0+4h} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \text{--}\right] dx$$

$$= y_0 + h\left(\int_0^4 f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \text{-}\right] dn$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 \text{--}\right]$$

$x = x_0 + nh$

$dx = h\,dn$

limit 0 to 4

$$= y_0 + h\left[4f_0 + [8f_1 - 8f_0] + \frac{20}{3}[f_2 - 2f_1 + f_0] + \frac{8}{3}[f_3 - 3f_2 + 3f_1 - f_0]\right]$$

$$y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

Neglecting 4th order and higher order differ.

this is called predictor method

Step 3: Find $f_4 = f(x_4, y_4)$

Step 4: We Apply correter formula

$$y_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

to find better approximate value of y_4.

Step 5: ~~Repeatedy~~ Repeadely we apply correcter formula to get better approximation of y_4.

Ex: Apply Milne's method to find the solution of the d.e. $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$

Soln: We have $\frac{dy}{dx} = x - y^2$ ——— (1) $\quad x = 0, y = 0$

On integrating (1) w. r. to x; we get.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} (x - y^2)\,dx \qquad \text{[Picard method]}$$

$$y = y_0 + \int_{x_0}^{x} (x - y^2)\,dx \quad \text{——— (2)}$$

Putting $x_0 = 0$, $y_0 = 0$ in (2) ; we get

$$y_1 = 0 + \int_0^x (x - 0^2)\,dx \Rightarrow y_1 = \frac{x^2}{2}$$

$$y_1 = \frac{(0.2)^2}{2} = .02000 \quad \text{at } (x = 0.2)$$

$$\boxed{f_1 = (x - y)^2 = (0.2 - (0.02)^2) = 0.1996}$$

$$y_2 = y_0 + \int_0^x (x - y_1^2)\,dx$$

$$= \int_0^x \left(x - \left(\frac{x^2}{2}\right)^2\right) dx = \int_0^x \left(x - \frac{x^4}{4}\right) dx =$$

$$y_2 = \frac{x^2}{2} - \frac{x^5}{20}$$

$$y_2 = \frac{(0.4)^2}{2} - \frac{(0.4)^5}{20} = .0795 \quad \text{at } (x = 0.4)$$

$$\boxed{f_2 = (x - y^2) = \{0.4 - (.0795)^2\} = 0.3937}$$

$$y_3 = \int_0^x (x - y_2^2)\,dx$$

$$= \int_0^x \left(x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right) dx$$

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Find $y(0.2)$ by Milne's Method for the d.e. $\frac{dy}{dx} = x+y$ with initial condition: $x_0 = 0,\ y_0 = 1$.

	x_0	x_1	x_2	x_3	x_4
x	0	0.05	0.1	.15	.2
y	1	1.0525	1.1103	1.1737	
	y_0	y_1	y_2	y_3	y_4

We have $\frac{dy}{dx} = x+y \Rightarrow f(x,y) = x+y$

$x_0 = 0,\ y_0 = 1$

$x_0 = 0,\ x_n = 0.2,\ n = 4$

$h = \frac{x_n - x_0}{n} = \frac{0.2-0}{4} = 0.05$

We have $y' = x+y \quad (y')_0 = x_0 + y_0 = 0+1 = 1$

$y'' = 1+y' \quad (y'')_0 = 1+(y')_0 = 2$

$y''' = y'' \Rightarrow (y''')_0 = (y'')_0 = 2$

$y^{iv} = y''' \Rightarrow (y^{iv})_0 = (y''')_0 = 2$

By Taylor series method

$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{\lfloor 2}(y'')_0 + \frac{(x-x_0)^3}{\lfloor 3}(y''')_0 + \frac{(x-x_0)^4}{\lfloor 4}(y^{iv})_0$

$y = 1 + x + \frac{2x^2}{2} + \frac{2x^3}{\lfloor 3} + \frac{2x^4}{\lfloor 4} + \cdots$

$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \cdots$

$y(0.05) = 1 + (0.05) + (0.05)^2 + \frac{(0.05)^3}{3} + \frac{(0.05)^4}{6} + \cdots$

$y_1 = \cancel{1.0500} = \underline{1.0525}$

$y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{6}$

$y_2 = 1.11035$

$y(0.15) = 1 + (0.15) + (0.15)^2 + \frac{(0.15)^3}{3} + \frac{(0.15)^4}{6} + \cdots$

$y_3 = 1.1737$

Now

$y(0.2) = y_4 =$

$f_1 = f(x_1, y_1) = [x_1 + y_1] = [.05 + 1.0525] = \cancel{1.5525} = \underline{1.1025}$

$f_2 = f(x_2, y_2) = [x_2 + y_2] = [.1 + 1.11035] = \underline{1.21035}$

$f_3 = f(x_3, y_3) = [x_3 + y_3] = [.15 + 1.1737] = \underline{1.3237}$

By Predictor Formula:

$y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] = 1 + \frac{4 \times 0.05}{3}[2 \times 1.1025 - 1.21035 + 2 \times 1.3237]$

$= 1 + 0.06667[3.64205]$

$= 1 + .2428 = 1.2428$

corrector formula

find $f_4 = f(x_4, y_4) = [x_4 + y_4] = [1.2 + 1.2428] = \underline{1.4428}$

$$y_4^c = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 1.11035 + \frac{0.05}{3}[1.21035 + 4 \times 1.3237 + 1.4428]$$

$$= 1.11035 + [7.94795] \times 0.016666$$

$$= 1.11035 + 0.132465 = \underline{1.242815}$$

$f_4 = [x_4 + y_4] =$

$$y_5(1.25) = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4] = 1.3180$$

$$y_5^c = 1.3180$$

Numerical Solution of Partial Differential Equations:

The general second order partial differential equation is defined as,

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + C(x,y)\frac{\partial^2 u}{\partial y^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

The above equation is

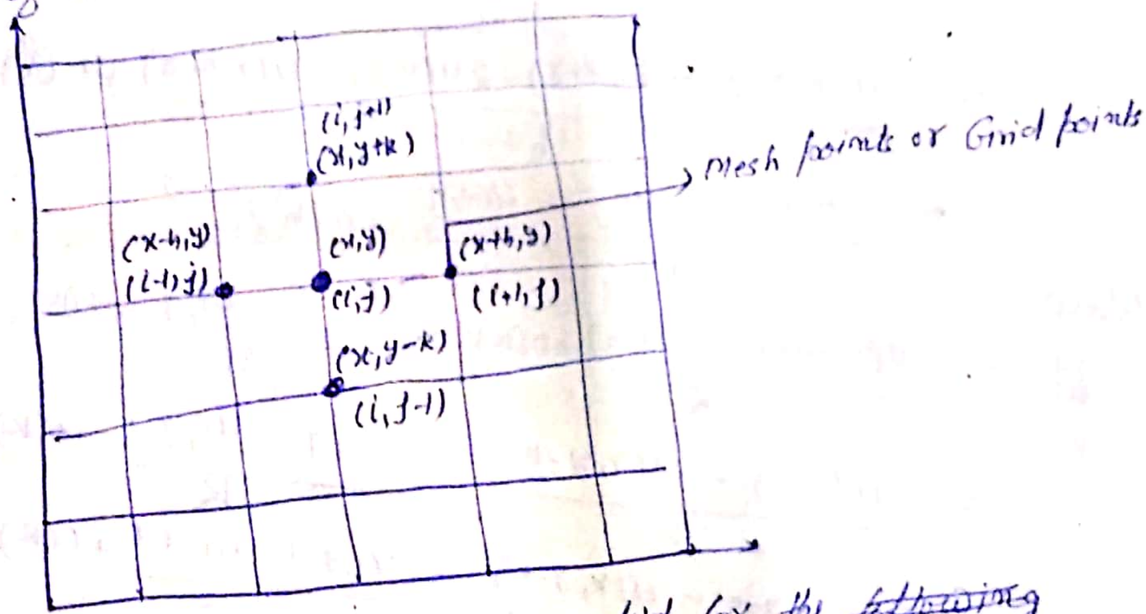
(i) Elliptic if $B^2 - 4AC < 0$

(ii) Parabolic if $B^2 - 4AC = 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Finite Difference approximation to Partial Derivatives:

Divide the rectangular region (x-y plane) into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$.



The following approximations are valid for the ~~following~~ derivative of $f(x)$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \quad \text{[Forward difference approximation]}$$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \quad \text{[Backward difference approximation]}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad \text{[Central difference approximation]}$$

$$f''(x) = \frac{d}{dx}\left[\frac{f(x+h) - f(x)}{h}\right] = \frac{[f(x+2h) - f(x+h)] - [f(x+h) - f(x)]}{h^2}$$

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} + O(h^2)$$

(Forward diff appr)

Let $u(x,y)$ be any function. Then:

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y)-u(x,y)}{h} + O(h) = \frac{u_{i+1,j}-u_{i,j}}{h} + O(h)$$

$$= \frac{u(x,y)-u(x-h,y)}{h} + O(h) = \frac{u_{i,j}-u_{i-1,j}}{h} + O(h)$$

$$= \frac{u(x+h,y)-u(x-h,y)}{2h} + O(h^2) = \frac{u_{i+1,j}-u_{i-1,j}}{2h} + O(h^2)$$

Similary

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{u(x+h,y)-u(x,y)}{h}\right]_{\text{Forward}} + O(h) = \left[\frac{[u(x+h,y)-u(x,y)] - \{u(x,y)-u(x-h,y)\}}{h^2}\right]_{\text{Backward}}$$

$$= \frac{u(x+h,y) - \cancel{2(x,y)} - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2)$$

$$= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

Now

$$\frac{\partial u}{\partial y} = \frac{u(x,y+k)-u(x,y)}{k} + O(k) = \frac{u_{i,j+1}-u_{i,j}}{h} + O(k)$$

$$= \frac{u(x,y)-u(x,y-k)}{k} = \frac{u_{i,j}-u_{i,j-1}}{k} + O(k)$$

$$= \frac{u(x,y+k)-u(x,y-k)}{k} = \frac{u_{i,j+1}-u_{i,j-1}}{k} + O(k)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x,y+k) - 2u(x,y) + u(x,y+k)}{k^2} + O(k^2)$$

$$= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(k^2)$$

To solve Boundary Value Problem:

Solution of Elliptic equation [Laplace equation]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Consider a rectangular region R for which $u(x,y)$ is known at boundary

Divide this region into a network of square mesh of side h $(\Delta x = \Delta y = h)$

From eqn (1) we get.

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

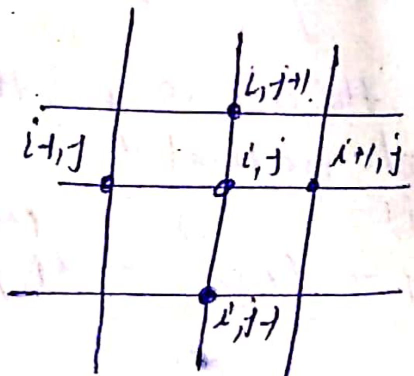
$(h = k)$

$$\Rightarrow 4u_{i,j} = [u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}]$$

Since $(h = k)$

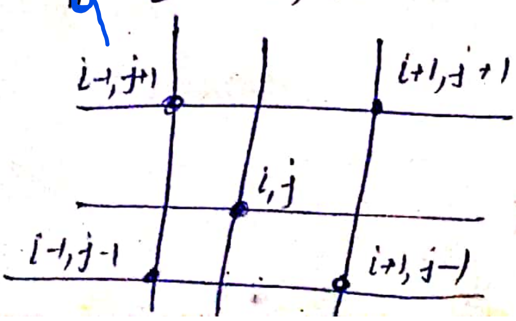
$$\therefore u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] \quad \text{--- (2)}$$

This shows that value of u at any interior mesh point is the average of its value at four neighbouring points. The formula is called standard 5-point formula.



Some times a similar formula is used which is called diagonal five point formula.

$$u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$$



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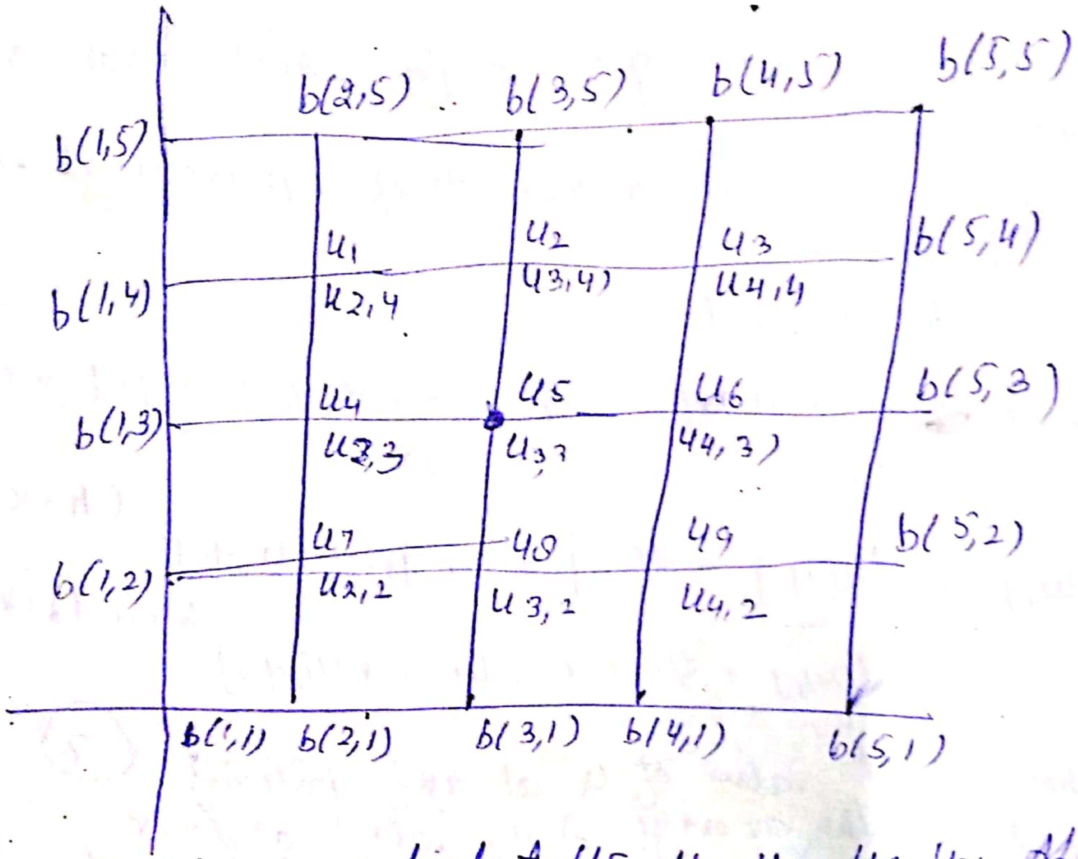
Rectangular Region:



First we calculate u_5, u_1, u_3, u_9, u_7. After this we calculate u_2, u_6, u_8, u_4,

[We need to use the diagonal formula less no. of times]

Now, we improve the accuracy of these values by using Gauss-Seidel method

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right]$$

or

Jacobi's Method:-

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left\{u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

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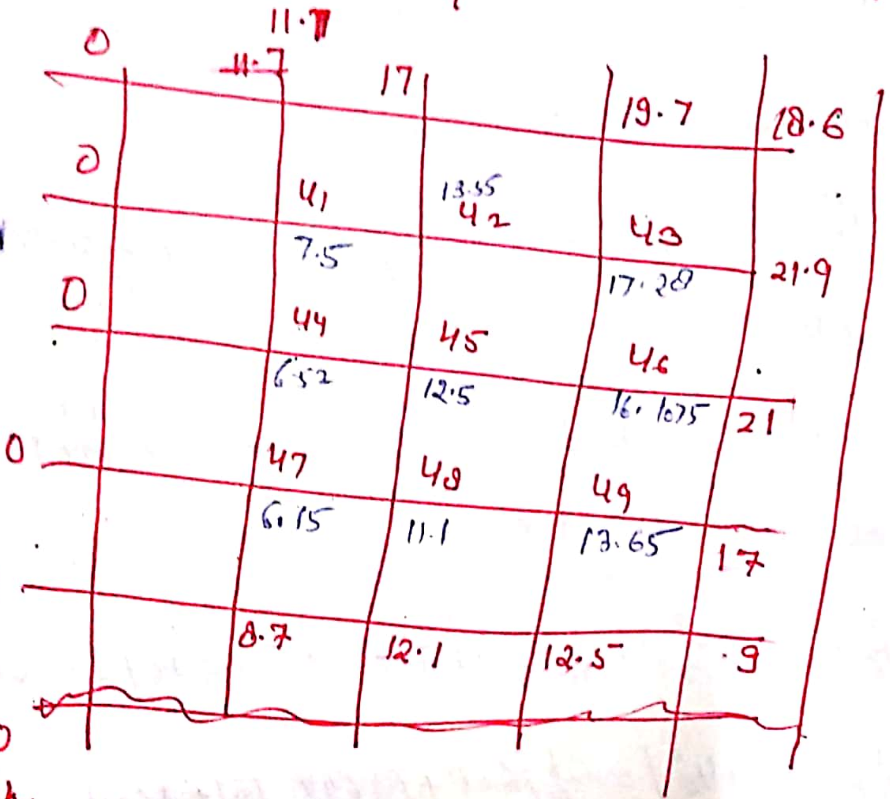
Solve Laplace equation $u_{xx} + u_{yy} = 0$, given that



Soln:

$$u_5 = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right]$$

$$u_5 = \frac{1}{4}\left[17 + 12.1 + 0 + 21\right] = 12.5 \quad \text{[Standard]}$$

$$u_1 = \frac{1}{4}\left[0 + 12.5 + 0 + 17\right] = 7.4 \quad \text{[Diagonal]}$$

$$u_3 = \frac{1}{4}\left[12.5 + 18.6 + 17 + 21.9\right] = 17.28 \quad \text{(diagonal)}$$

$$u_9 = \frac{1}{4}\left[9 + 12.5 + 21 + 12.1\right] = 13.65 \quad \text{(diagonal)}$$

$$u_7 = \frac{1}{4}\left[0 + 12.5 + 0 + 12.1\right] = 6.15 \quad \text{(diagonal)}$$

Now

$$u_2 = \frac{1}{4}\left[17 + 12.5 + 7.5 + 17.28\right] = 13.55 \quad \text{(Standard)}$$

$$u_4 = \frac{1}{4}\left[0 + 12.5 + 7.5 + 6.15\right] = 6.52$$

$$u_6 = \frac{1}{4}\left[12.5 + 21 + 17.28 + 13.65\right] = 16.1075$$

$$u_8 = \frac{1}{4}\left[12.1 + 12.5 + 6.15 + 13.65\right] = 11.1$$

Solve

How improve accuracy by Gauss-Seidel Method.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)}\right]$$

Ist iteration

$$u_1^{(0+1)} = \frac{1}{4}\left[0 + 11.1 + 13.55 + 6.52\right] = 7.79$$

$$u_2^{(0+1)} = \frac{1}{4}\left[u_1^{(n+1)} + 17 + u_3^{n} + u_5^{(n)}\right] = \frac{1}{4}\left[7.79 + 17 + 17.28 + \right.$$

$$u_2^{(1)} = 13.64$$

$$u_3^{(1)} = \frac{1}{4}\left[u_2^{1} + u_2^{0} + 21 + 19.7\right] = \frac{1}{4}\left[\cancel{13.64}^{16.107} + 13.55 + \cancel{21}^{21.9} + 19.7\right]$$

$$u_3^{(1)} = \cancel{16.972}\ 17.014 = 17.82$$

$$u_4^{(1)} = \frac{1}{4}\left[u_1' + u_7^{0} + u_0^{5} + 0\right] = \frac{1}{4}\left[7.79 + 0 + 12.5 + 6.15\right] = 6.61$$

$$u_5^{(1)} = \frac{1}{4}\left[u_4' + u_2' + u_8^{0} + u_6^{0}\right] = \frac{1}{4}\left[6.61 + 13.64 + 11.1 + 16.11\right] = 11.87$$

$$u_6^{(1)} = \frac{1}{4}\left[u_5' + u_3' + u_9 + 21\right] = \frac{1}{4}\left[11.87 + 17.82 + 13.65 + 21\right] = 16.085$$

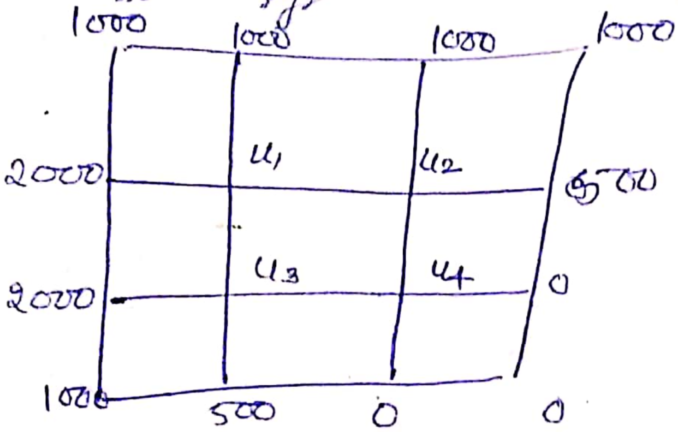
$$= 16.09$$

$$u_7^{(1)} = \frac{1}{4}\left[0 + u_4' + u_8 + 8.7\right] = \frac{1}{4}\left[0 + 6.61 + 11.1 + 8.7\right] = 6.61$$

$$u_8^{(1)} = \frac{1}{4}\left(u_7^{(1)} + u_5^{(1)} + u_9^{0} + 12.1\right] = \frac{1}{4}\left[6.61 + 11.87 + 13.65 + 12.1\right] = 11.06$$

$$u_9^{(1)} = \frac{1}{4}\left(17 + 12.5 + u_8^{(1)} + u_6^{(1)}\right. = \frac{1}{2}\left[17 + 12.5 + 11.06 + 16.09\right] = 14.16$$

Q: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the pivotal points.



To get the initial value u_1, u_2, u_3, we assume $u_4 = 0$,

then $u_1 = \frac{1}{4}[1000 + 2000 + 1000 + \overset{u_4}{0}] = 1000$ (Diag. Formula)

$u_2 = \frac{1}{4}[1000 + \overset{500}{\cancel{0}} + 1000 + \overset{u_4}{0}] = 625$ (Stand.)

$u_3 = \frac{1}{4}[u_1 + u_4 + 2000 + 500] = \frac{1}{4}[1000 + 2000 + 500 + 0]$.

$= 875$. (Stand)

$u_4 = \frac{1}{4}[0 + 0 + u_3 + u_2] = \frac{1}{4}[0 + 0 + 875 + 625] = 375$ (Stand

Now improve accuracy

$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{n}]$

$u_2^{(n+1)} = \frac{1}{4}[500 + 1000 + u_1^{n+1} + u_4^{n}]$

$u_3^{(n+1)} = \frac{1}{4}[2000 + 500 + u_4^{n} + u_1^{(n+1)}]$

$u_4^{(n+1)} = \frac{1}{4}[0 + 0 + u_3^{(n+1)} + u_2^{(n+1)}]$

For first iteration

$u_1^1 = \frac{1}{4}[2000 + 625 + 1000 + 875] = 1125$

$u_2^1 = \frac{1}{4}[1125 + 500 + 1000 + 375] = 750$

$u_3^1 = \frac{1}{4}[2000 + 375 + 1125 + 500] = 1000$

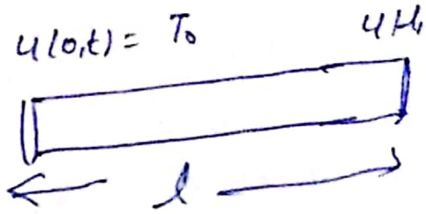
$u_4^1 = \frac{1}{4}[0 + 0 + 1000 + 750] = 438$

Alt Ans $\cancel{u_1^4 = 45}$ $u_1^5 = 1208$, $u_2^5 = 792$, $u_3^5 = 1042$, $u_4^5 = 458$

$u_1^4 = 1207$, $u_2^4 = 791$, $u_3^4 = 1041$, $u_4^4 = 458$

Solution of one-Dimension heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \ (1)$$

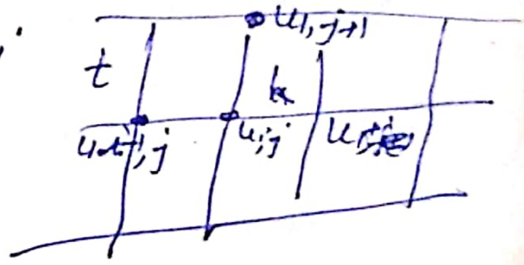


$u(x,0) = f(x)$, $0 \le x \le l$

$u(0,t) = T_0$ and $u(l,t) = T_1$

Bender Smith Method:

Considering a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. Denoting the mesh point $(x,t) = (i,j)$, we have



$$\frac{\partial u}{\partial t} = \frac{u(x,t+k) - u(x,t)}{k} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

$$= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

From (1)

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{[u_{i+1,j} - 2u_{i,j} + u_{i,j+1})}{h^2}$$

$$u_{i,j+1} - u_{i,j} = \frac{c^2 k}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$u_{i,j+1} = u_{i,j} + \lambda\,(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

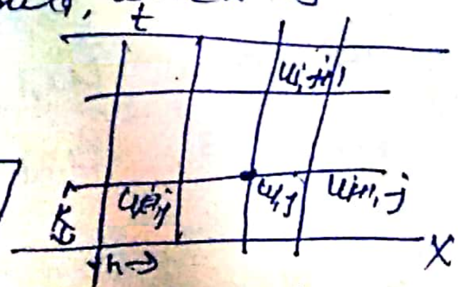
$$\boxed{u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}}$$

where $\boxed{\lambda = \frac{c^2 k}{h^2}}$

This is called Bender smith formula, which is only valid for $0 \le \lambda \le 1/2$.

In particular $\lambda = 1/2$, then

$$\boxed{u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]}$$



Boundary Conditions can be written in difference form.

$u_{0,j} = T_0$ and $u_{l,j} = T_1$, $u_{i,0} = f(i)$ $0 \le i \le l$

Que! Solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0,\ u(4,t) = 0,\ u(x,0) = x(4-x)$

Taking $h=1$, $k=1$ and employing Bender Schmidt recurrence relation. Continue to the solution for two time steps.

Put $j=1$

$u_{i,2}$

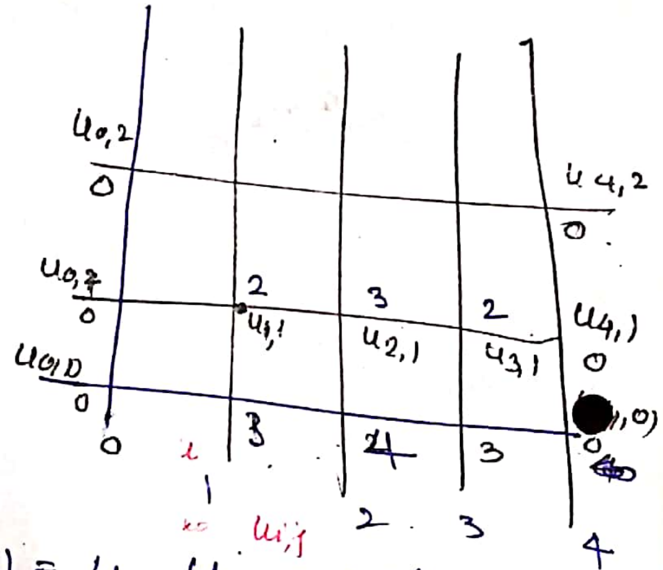
For i

Soln: $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

$u_{0,j} = 0$, and $u_{4,j} = 0$

$u_{0,1} = 0, = u_{0,2}$

$u_{4,1} = 0,\quad u_{4,2} = 0$



Now $u_{i,0} = i(4-i)$.

$u_{0,0} = 0$

$u_{1,0} = 1\times 3 = 3,\quad u_{2,0} = 2(4-2) = 4,\quad u_{3,0} = 3\times(4-3) = 3$

$u_{4,0} = 4(4-0) = 0$

From Bender Schmidt Formula.

$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$

$\lambda = \frac{kc^2}{h^2} = \frac{1\cdot 1/2}{1} = 1/2$

$$\boxed{u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]} \quad (A)$$

Put $j = 0$ in eqn (A)

$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right]$

For $i=1$

$u_{1,1} = \frac{1}{2}\left[u_{2,0} + u_{0,0}\right] = \frac{1}{2}[4+0] = 2$

For $i=2$, $u_{2,1} = \frac{1}{2}\left[u_{3,0} + u_{1,0}\right] = \frac{1}{2}[3+3] = 3$

For $i=3$, $u_{3,1} = \frac{1}{2}\left[u_{4,0} + u_{2,0}\right] = \frac{1}{2}[0+4] = 2$

at $j=1$ in eq4 (1A)

$u_{i,2} = \frac{1}{2}[u_{i+1,1} + u_{i-1,1}]$

For $i=1$

$u_{1,2} = \frac{1}{2}[u_{2,1} + u_{0,1}] = \frac{1}{2}[3,0] = 1.5$

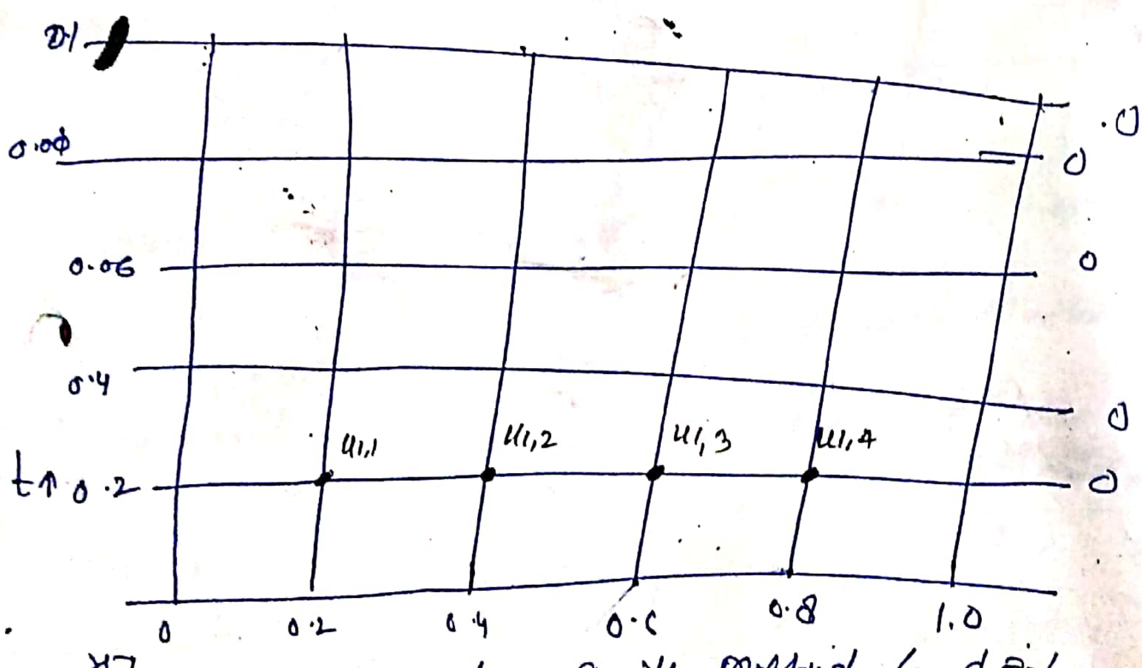
$u_{2,2} = \frac{1}{2}[u_{3,1} + u_{1,1}] = \frac{1}{2}[2+2] = 2$

$u_{3,2} = \frac{1}{2}[u_{4,1} + u_{2,1}] = \frac{1}{2}[0+3] = 1.5$

Ex. Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x$ for $0 \le x \le 1$ using Schmidt method (Take $h=0.2$, and $\lambda = 1/2$)

Soln: Since $h = 0.2$, $\lambda = \frac{1}{2}$ the $c=1$

$\lambda = \frac{c^2 k}{h^2} \Rightarrow \frac{1}{2} = \frac{1 \cdot k}{0.04}$, $k = 0.02$



By Bender Smith method for $\lambda = \frac{1}{2}$

$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$ ——— (1)

$u_{0,j} = 0$, $u_{1,0} = 0$

$u_{i,0} = \sin \pi i$, $u_{0,0} = 0$, $u_{0.2,0} = \sin\frac{\pi}{5} = 0.5875$

$u_{0.4,0} = \sin\frac{2\pi}{5} = 0.9511$, $u_{0.6,0} = \sin\frac{3\pi}{5} = 0.9511$

$u_{0.8,0} = \sin$ 5

$u_{0.8,0} = \sin\frac{4\pi}{5} = 0.5875, \; u(1,0) = \sin\pi = 0$

Crank-Nicholson Method:

$$u_{i,j+1} = \frac{\lambda u_{i-1,j+1} + \lambda u_{i+1,j+1} + \lambda u_{i-1,j} + (2-2\lambda) u_{i,j} + \lambda u_{i+1,j}}{(2+2\lambda)}$$

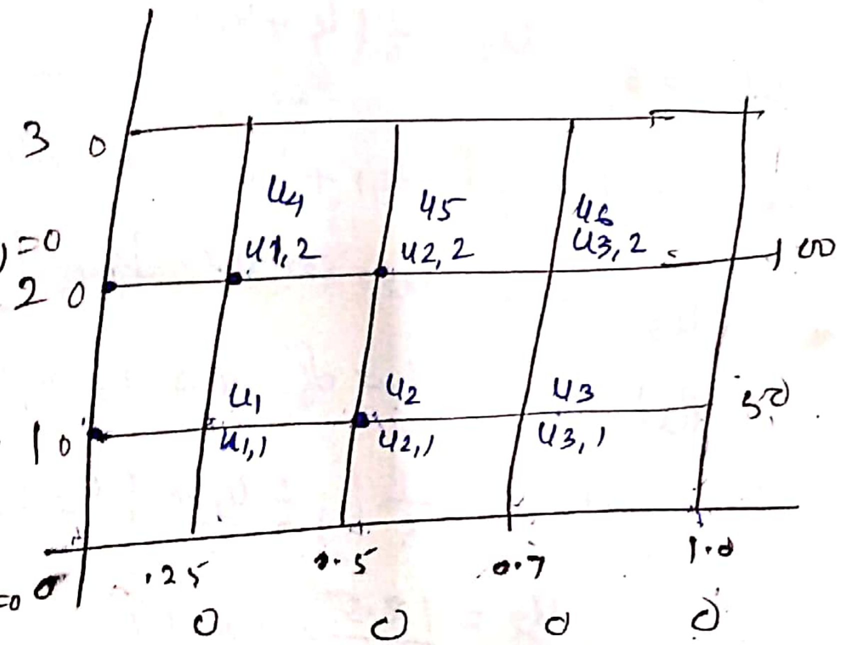
Here $\lambda = \frac{c^2 k}{h^2}$,

Ex: using Crank-Nicholson's Method. Solve $\frac{\partial^2 u}{\partial x^2} = 16 \frac{\partial u}{\partial t}$, $0 < x < 1$, $t > 0$ given $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 50t$. Compute u for two steps in t direction taking $h = \frac{1}{4}$, $k = 1$

Soln: $h = 0.25$

$u_{i,0} = 0$
$u(0.25,0) = 0$
$u(0.5,0) = 0$
$u(0.75,0) = 0$
$u(1,0) = 0$
$u(0,0) = 0$
due to $u(x,0) = 0$

$u(0,t) = 0$
$u(0,i) = 0$
$u(0,1) = 0$
$u(0,2) = 0$
$u(0,0) = 0$
due to $u(0,t) = 0$



$u(1,0) = 0$
$u(1,1) = 50$
$u(2,1) = 100$

By Crank Nicholson Method.

$$u_{i,j+1} = \frac{\lambda u_{i-1,j+1} + \lambda u_{i+1,j+1} + \lambda u_{i-1,j} + (2-2\lambda) u_{i,j} + \lambda u_{i+1,j}}{(2+2\lambda)}$$

Here $\lambda = \frac{kc^2}{h^2}$, $k = 1$, $c^2 = \frac{1}{16}$, $h = \frac{1}{4}$

$\lambda = 1$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

for $j = 0$

$$u_{i,1} = \frac{1}{4}\left[u_{i-1,1} + u_{i+1,1} + u_{i-1,0} + u_{i+1,0}\right]$$

u_6

Put $i=1$

$u_{1,1} = \frac{1}{4}[u_{0,1} + u_{2,1} + u_{0,0} + u_{2,0}]$

$= \frac{1}{4}[0 + u_{2,1} + 0 + 0]$

$u_1 = u_{1,1} = \frac{1}{4} u_{2,1}$ —— (1) $\left(u_1 = \frac{1}{4} u_2\right)$ —— (1)

Put $i=2$

$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{3,1} + u_{1,0} + u_{3,0}]$

$u_2 = \frac{1}{4}[u_1 + u_3 + 0 + 0] = \frac{1}{4}(u_1 + u_3)$ —— (2)

$u_2 = \frac{1}{4}(u_1 + u_3)$ —— (2)

Put $i=3$

$u_3 = u_{3,1} = \frac{1}{4}[u_{2,1} + u_{4,1} + u_{2,0} + u_{4,0}]$

$u_3 = \frac{1}{4}[u_2 + 50 + 0 + 0] = \frac{1}{4}[u_2 + 50]$ —— (3)

Put the value of u_1 & u_3 in eqn (2) we have

$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}u_2 + \frac{50}{4}\right]$

$u_2 = \frac{12.5}{3.5} = 3.5714$

From (1) From (3)

$u_1 = 0.89285$, $u_3 = 13.39285$

For $j=1$

$u_{i,2} = \frac{1}{4}[u_{i-1,2} + u_{i+1,2} + u_{i-1,1} + u_{i+1,1}]$

Put $i=1$

$u_4 = u_{1,2} = \frac{1}{4}[u_{0,2} + u_{2,2} + u_{0,1} + u_{2,1}] = \frac{1}{4}[0 + u_5 + 0 + u_2]$ $(u_0 = 0!)$

$u_4 = \frac{1}{4} u_5$ —— (4) $4u_4 = u_5 + 3.5714$

$\Rightarrow 4u_4 - u_5 = 3.5714$ —— (4)

Put $i=2$

$u_5 = u_{2,2} = \frac{1}{4}[u_{1,2} + u_{3,2} + u_{1,1} + u_{3,1}] = \frac{1}{4}[u_4 + u_6 + u_1 + u_3]$

$4u_5 = u_4 + u_6 + \cancel{14.2857} + 0.89285 + 13.39285$

$4u_5 - u_4 - u_6 = 14.2857$ —— (5)

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Put $i = 3$

$$u_6 = u_{3,2} = \frac{1}{4}\left[u_{2,2} + u_{4,2} + u_{2,1} + u_{4,1}\right]$$

$$\cancel{4u_6 =} \quad \cancel{u_6 = \frac{1}{4}\left[u_2 + \overset{50}{100} + u_5 + 100\right]}$$

$3.5714 \quad 3.5714$

$$u_6 = \frac{1}{4}\left[u_5 + 100 + u_2 + 50\right] = 4u_6 = u_5 + 153.5714$$

$$4u_6 - u_5 = 153.5714$$

$$\cancel{4u_6 - u_5 - u_2 = 150} \qquad (6)$$

From eqn (4), (5) & (6)

$$4u_4 - u_5 + 0 \cdot u_6 = 3.5714$$

$$-u_4 + 0 + 4u_5 - u_6 = 14.2857$$

$$0 \cdot \; -u_5 - 4u_6 = 153.5714$$

On Solving

$$\cancel{u_4 = +0.55804}$$
$$\cancel{u_5 = 5.803}$$
$$\cancel{u_6 = +36.9419}$$

$u_4 = 4.71935$

$u_5 = 15.306$

$u_6 = 42.1935$

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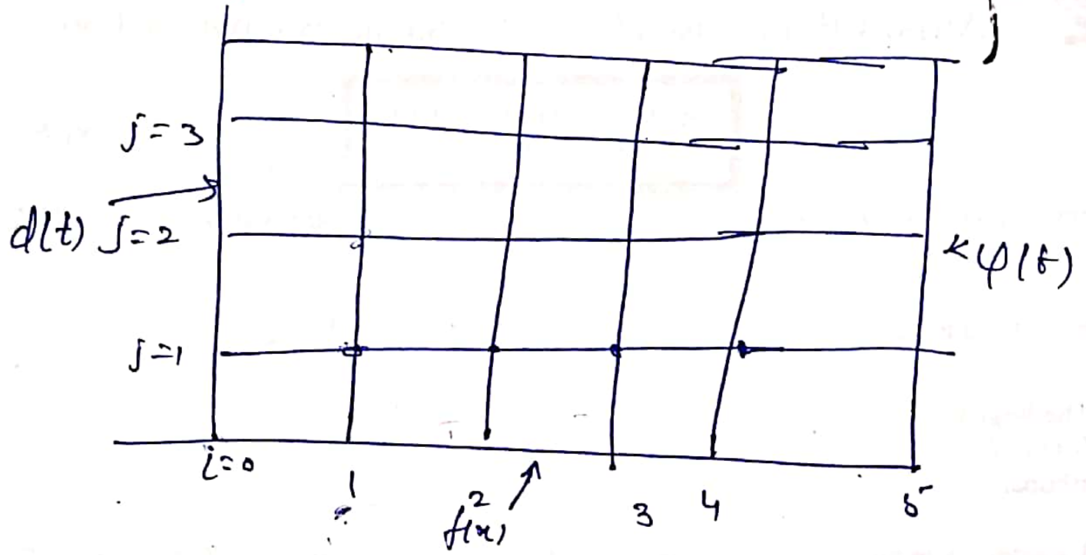
Solution of wave Equation (Hyperbolic equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

subjects to initial conditions $u = f(x) = u(x,0)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$ --- (2)
$0 < x < 1$ at $t = 0$

and boundary conditions $u(0,t) = \phi(t)$, $u(1,t) = \psi(t)$



Consider a rectangular mesh in the x-t plane spacing h along x-direction and k along time t direction.

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \quad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \quad \text{--- (3)}$$

From (1)

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{c^2 k^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$\Rightarrow \boxed{u_{i,j+1} = 2(1-\alpha^2 c^2)u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}} \quad \text{--- (4)}$$

~~and condition~~ $\frac{\partial u}{\partial t} = g(x)$ $\alpha^2 = \frac{k^2}{h^2}$

~~for~~ Now replacing derivative of (2) by its central differ approximation.

$$\frac{u(x,t+k) - u(x,t-k)}{2k}$$

$$\frac{\partial u}{\partial t} = g(x) \text{ at } t = 0$$

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = g(x) \quad \cancel{\Rightarrow u_{i,j+1} = g(x)+}$$

or $\boxed{u_{i,j+1} = u_{i,j-1} + 2k\,g(x)}$

A special case: Coeff of $u_{i,j}$ in (4) will vanish if $\alpha = \frac{1}{c}$ or $k = h/c$. Then (4) reduces to simple form.

$\alpha = \frac{k}{h}$ $\boxed{c = \frac{h}{k}}$

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

Note: For $\alpha = \frac{1}{c}$ the solution of (4) is stable and coincides with the solution of (1)

For $\alpha < 1/c$, the solution is stable but inaccurate

For $\alpha > 1/c$, Solution is unstable.

Note 2: Formula (4) converges for $\alpha \le 1$ i.e for $k \le h$.

Example: Evaluate the pivotal values of the equation $u_{tt} = 16 u_{xx}$ taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0,t) = 0$, $u(5,t) = 0$, $u_t(x,0) = 0$ and $u(x,0) = x^2(5-x)$

Soln: Here $c^2 = 16$

$\alpha^2 = \frac{1}{16} = \frac{k^2}{h^2}$

$\frac{1}{16} = \frac{k^2}{1^2}$ $k = \frac{1}{4}$

$k = \frac{1}{4} = 0.25$

The difference equation for the d.e. is

$$u_{i,j+1} = 2(1-16\alpha^2)u_{i,j} + 16\alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{(1)}$$

where $\alpha = k/h$

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$ i.e. $k = h/4 \Rightarrow k = \frac{1}{4} = 0.25$

From (1) $\Rightarrow u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

which is convergent solution (since $\frac{k}{h} \le 1$). (11)

Now

$$u_{0,j} = 0, \quad u_{5,j} = 0 \quad \text{for all } j$$

i.e.

$u(x,0) = x^2(5-x)$

$u_{i,0} = i^2(5-i)$

$u_{1,0} = 4$, $u_{2,0} = 12$, $u_{3,0} = 18$, $u_{4,0} = 16$, at $t = 0$

these are the entries of first row.

Now for 1st initial condition $\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \Rightarrow u_{i,j+1} = u_{i,j-1} \text{ for } j = 0$$

$$\boxed{u_{i,1} = u_{i,-1}} \quad (3)$$

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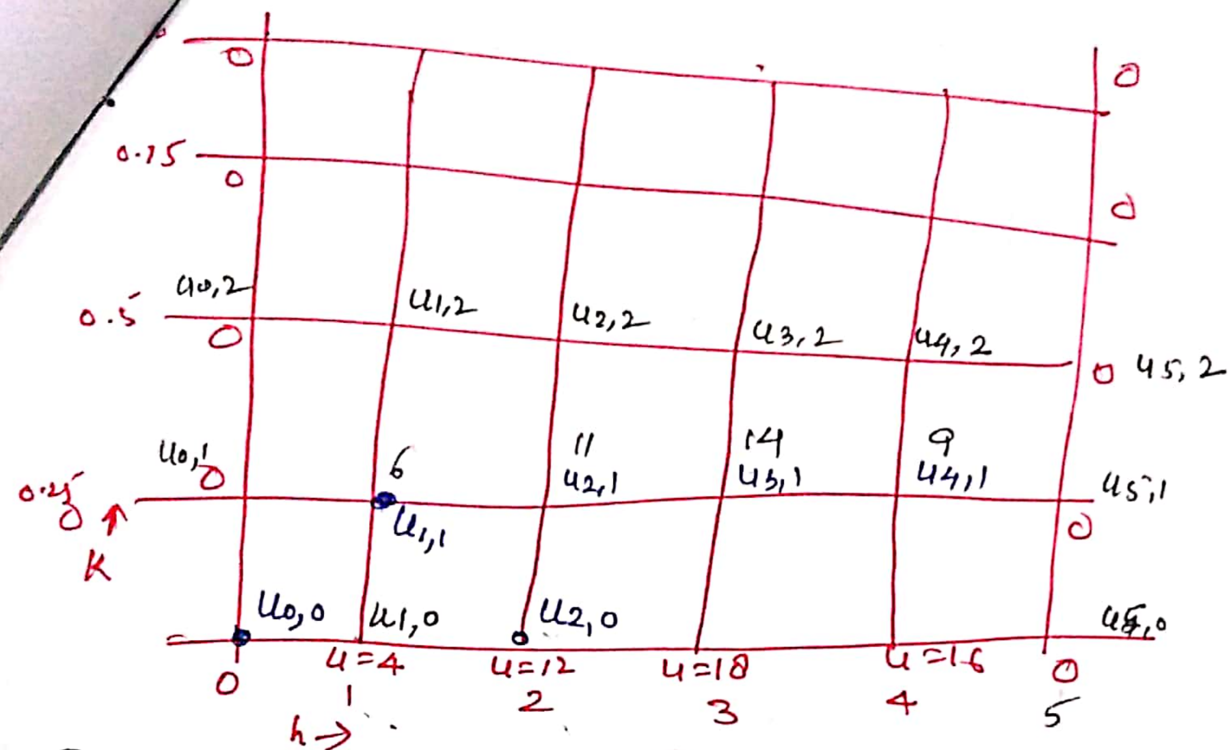

Putting $j=0$ in eqn (2)

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} + 2k \text{ (using (3))}$$

$$\Rightarrow u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0} + 2k\right)$$

Put $i = 1, 2, 3, 4$

$$u_{1,1} = \frac{1}{2}\left(u_{0,0} + u_{2,0}\right) = \frac{1}{2}(0+12) = 6$$

$$u_{2,1} = \frac{1}{2}\left(u_{1,0} + u_{3,0}\right) = \frac{1}{2}\cdot(4+18) = 11$$

$$u_{3,1} = \frac{1}{2}\left(u_{2,0} + u_{4,0}\right) = \frac{1}{2}(12+16) = 14$$

$$u_{4,1} = \frac{1}{2}\left(u_{3,0} + u_{4,0}\right) = \frac{1}{2}(18+0) = 9.$$

These are the entries for IInd low.

Putting $j=1$ in eqn (2);

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

Put $i = 1, 2, 3, 4$

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries for IIIrd ~~to~~ row

Similarly putting $j = 2, 3, 4$ successively in eq (2), the entries of fourth, fifth, sixth row can be obtained.