

Linear Algebra

Q.B-3



Data _____

Page _____

$$1) A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left[\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] = 0$$

$$\left[\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right] = 0 \Rightarrow \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix} = 0 \xrightarrow{-ij} |\lambda^2 - 9| = 0 \quad \lambda = \pm 3$$

Eigen values of A are 3 & -3.

For $\lambda = 3$ (also same process to calculate value of V for $\lambda = -3$)

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3v_1 + 3v_2 \\ 3v_1 - 3v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3v_1 + 3v_2 = 0 \Rightarrow v_1 = v_2$$

$$3v_1 - 3v_2 = 0 \quad v_1 = v_2$$

$$\therefore V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence, V is an eigenvector of A

$$2) A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 6 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)[(1-\lambda)^2 - 0] = 0$$

$$(3-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda = 3, 1, 1$$

For $\lambda = 1$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2v_1 = 0$$

$$-2v_3 = 0$$

$$v_1 = 0$$

$$v = \begin{bmatrix} v_1 \\ 0 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$(A - \lambda I)v = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2v_2 - 2v_3 = 0 \Rightarrow v_2 = -v_3$$

$$v_1 - 2v_3 = 0 \Rightarrow v_1 = 2v_3$$

$$\therefore v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_3 \\ -v_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus $(2, -1, 1)$ is an eigen vector corresponding
to $\lambda = 3$

$$3) A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)(-1-\lambda) - 4 = 0$$

$$\Rightarrow -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda^2 - 3\lambda + 2\lambda - 6 = 0$$

$$\lambda(\lambda - 3) + 2(\lambda - 3) = 0$$

$$(\lambda + 2)(\lambda - 3) = 0$$

$$\lambda = -2, 3$$

Eigen vectors using $\lambda = 3$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-v_1 + 2v_2 = 0$$

$$2v_1 - 4v_2 = 0$$

$$v_1 = 2v_2$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$4) A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & 1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(1-\lambda)(1-\lambda) + 2(2\lambda - 2)$$

$$(1-\lambda)[(1-\lambda)(1-\lambda) - 4] = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda - 3) = 0$$

$$(1-\lambda)(\lambda+1)(\lambda-3) = 0$$

Eigen vectors using $\lambda = -1$ $\lambda = 1, -1, 3$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} +$$

$$\begin{aligned} 2V_1 + 2V_3 &= 0 \quad - (i) \\ -V_1 + 2V_2 + V_3 &= 0 \quad - (ii) \\ 2V_1 + 2V_3 &= 0 \quad - (iii) \end{aligned}$$

$$V_1 = V_3$$

From (i) $\Rightarrow -V_1 + 2V_2 + V_3 = 0$
 $-V_1 + 2V_2 + V_1 = 0$
 $V_2 = 0$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ by } \textcircled{d}$$

5) A characteristic polynomial of a square matrix is defined as a polynomial that contains the eigenvalues as roots and is invariant under matrix similarity.

6) The algebraic multiplicity of an eigenvalue e is the power to which $(\lambda - e)$ divides the characteristic polynomial. The geometric multiplicity of an eigenvalue is the number of linearly independent eigenvectors associated with it.

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(6-\lambda) + 6 = 0$$

$$\Rightarrow 6 - \lambda - 6\lambda + 6 + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 12 = 0$$

$$\lambda^2 - 4\lambda - 3\lambda + 12 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda = 4 \text{ or } 3$$

$$\begin{array}{l} 6 - \lambda - 6\lambda + \lambda^2 = 0 \\ \lambda^2 - 7\lambda + 6 = 0 \\ \lambda^2 - 6\lambda - \lambda + 6 = 0 \end{array}$$

(a) Characteristics polynomial is $(\lambda - 4)(\lambda - 3) = 0$

(b) ∵ eigen value of A is 4 and 3

(c) Basic

$$\text{For } \lambda = 4, \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow -3v_1 + 3v_2 = 0 \Rightarrow v_1 = v_2 - 1 \quad (i)$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$$\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow -2v_1 + 3v_2 = 0 \Rightarrow v_1 = \frac{3}{2}v_2 - 1 \quad (ii)$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

∴ Basics = (1, 1) & (3, 2)

(d) A.M of $\lambda = 4$ is 1

A.M of $\lambda = 3$ is 1

b.M

For $\lambda = 4$

$$[A - 4I] = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad (R_1 \rightarrow R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix} \quad (R_1 \rightarrow 2R_1, -R_2)$$

$$P(A - 4I) = 1$$

nullity = 1

$$\therefore b.M = n(A - 4I) = n - P(A - 4I) = 2 - 1 = 1$$

$$A.M = b.M = 1$$

For $\lambda = 3$

$$[A - 3I] = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \quad (R_1 \rightarrow R_1 - R_2) = \begin{bmatrix} 0 & 0 \\ -2 & 3 \end{bmatrix} \quad \therefore P(A - 3I) = 1$$

$$b.M = n(A - 3I) = n - P(A - 3I) = 2 - 1 = 1$$

$$A.M = b.M = 1$$

8)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda) + 1 = 0 \\ \lambda^2 - 2\lambda + 1 = 0 \\ (\lambda-1)^2 = 0$$

$$\lambda = 1, 1$$

a) characteristics polynomial is $(\lambda-1)^2 = 0$

b) eigen value ~~of~~ is 1

~~$[A - \lambda I] = [0]$~~

c) $[A - \lambda I][v] = [0] \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \text{Basis} = (1, -1)$$

d) A.M of $\lambda = 1$ is 2

For $\lambda = 1$, b.M

$$(A - I) \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} (R_1 \rightarrow R_1 + R_2) \Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \text{ rank} = 1$$

$$b.M = n(A - I) = n - r(A - I) = 2 - 1 = 1$$

9) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, -2, 3$$

a) characteristics polynomial is $(1-\lambda)(-2-\lambda)(3-\lambda) = 0$

b) eigen value is 1

c) Basis \rightarrow similar way as we calculate in

Q-no-7 & 8.

(d)

A.M of $\lambda = 1$ is 1A.M of $\lambda = -2$ is 1A.M of $\lambda = 3$ is 1

G.M

For $\lambda = 1$

$$[A - I] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{rank} = \rho(A-I) = 3$$

$b \cdot M = n - \rho(A-I) = 3-3=0$

For $\lambda = -2$

~~$[A+2I]$~~ $[A+2I] \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \text{rank} = \rho(A+2I) = 2$

$b \cdot M = n - \rho(A+2I) = 3-2=1$

For $\lambda = 3$

$$[A - 3I] = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = \rho(A-3I) = 2$$

$b \cdot M = n - \rho(A-3I) = 3-2=1$

* { ~~10, 11 & 12~~ questions are same as 7, 8 & 9
 solve by yourselves.

13)

17) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 1 = 0$$

$$-1 + \lambda + \lambda^2 + 1 - 1 = 0$$

$$\lambda^2 = 2, \lambda = \pm\sqrt{2}$$

\therefore The eigen values of A^{13} are $(\sqrt{19})^{13}$ & $(-\sqrt{2})^{13}$

$$\begin{aligned} (\sqrt{19})^{13} &= 512\sqrt{19} \\ (-\sqrt{2})^{13} &= -512\sqrt{2} \end{aligned}$$

18) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 4 \text{ & } 3$$

\therefore Hence, eigen values are 1, 4 & 3.

19) ~~Ques~~ $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ x & y-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(y-\lambda) - 3x = 0 \quad \text{--- (i)}$$

given eigen values $\lambda = 4 \text{ & } 8$, we get from eq(i)

$$(i) \rightarrow -2(y-4) - 3x = 0$$

$$-3x - 2y + 8 = 0 \quad \text{--- (ii)}$$

$$(ii) \rightarrow -8(y-8) - 3x = 0$$

$$-3x - 6y + 48 = 0 \quad \text{--- (iii)}$$

Solving eq (ii) & (iii), we get

~~$x = -4, y = 10$~~

22)

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

$$(R_1 \rightarrow R_1 + R_3)$$

$$\begin{bmatrix} A - \lambda I & = 0 \\ \hline 3 - \lambda & 5 & 2 \\ 5 & 12 - \lambda & 7 \\ 2 & 7 & 5 - \lambda \end{bmatrix}$$

(Rows 1 and 3 are crossed out)

$$\begin{bmatrix} 5 & 12 & 7 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

Two rows are same, so $|A| = 0$

From the properties of eigen values, product of eigen values is equal to determinant

Hence, product of eigen values is zero. That implies one of the eigen values is zero

Hence, minimum value of eigen value is zero

24) Similar matrix - Similar matrix represents the same linear map under two (possibly) different bases with P being the change of basis matrix a transformation $A \leftrightarrow P^{-1}AP$ is called similar matrix transformation

$$25) A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$$

$$, B = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2 - \lambda & 1 \\ -4 & 6 - \lambda \end{vmatrix} = 0$$

$$(2 - \lambda)(6 - \lambda) + 4 = 0$$

$$12 - 2\lambda - 6\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

$$\lambda = 4 \text{ or } 4$$

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 3 - \lambda & -1 \\ -5 & 7 - \lambda \end{vmatrix} = 0$$

$$(3 - \lambda)(7 - \lambda) - 5 = 0$$

$$21 - 3\lambda - 7\lambda + \lambda^2 - 5 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2 \text{ or } 8$$

\therefore eigen values of $[A] \neq [B]$

We know that if eigenvectors of two matrix is not equal than matrix are not similar.

26) Similar as 25

27) $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 4 = 0 \Rightarrow (5-\lambda)^2 = 4$$

$$\Rightarrow 5-\lambda = \pm 2$$

$$\Rightarrow \lambda = 3 \text{ or } 7$$

A has different eigen value hence matrix A is diagonalisable.

$$D = P^{-1}AP$$

Let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

For $\lambda = 3$

$$[A - 3I][V] = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \Rightarrow 2V_1 + 2V_2 = 0 \Rightarrow V_2 = -V_1$$

$$V(\text{at } \lambda=3) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 7$

$$[A - 7I][V] = 0$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2V_1 + 2V_2 = 0 \Rightarrow V_1 = V_2$$

$$V(\text{at } \lambda=7) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad AP = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} V_{\lambda=3} & V_{\lambda=7} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 7 \end{bmatrix}$$

$$\therefore AP = PD \Rightarrow \underline{P^{-1}AP = D}$$

Proved

28) $A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(1-\lambda) + 4 = 0$$

$$-3 + 3\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda + 1)(\lambda + 1) = 0, \lambda = -1, -1$$

eigen value are not distinct Hence, matrix A is not diagonalisable.

29) Same as Q. no-27

~~30) Same as Q. no-27~~

31) Same as Q. no-27

32) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 1) - 2(\lambda - 1) + 1(-1+\lambda) = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda - 2\lambda + 2 - 1 + \lambda = 0$$

$$\lambda^2 - \lambda^3 = 0$$

$$\lambda^2(1-\lambda) = 0$$

$$\lambda = 0, 0 \& 1$$

→ eigen value are not distinct. Hence, matrix A is not diagonalisable.

33) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[-\lambda(1-\lambda)-1] + 1(\lambda-1) = 0$$

$$(1-\lambda)(-\lambda+\lambda^2 - 2) = 0$$

$$(1-\lambda)(\lambda^2 - \lambda - 2) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + \lambda - 2) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = 1, -1, 2$$

Since eigen values are different.

Hence, matrix A is diagonalisable, $D = P^{-1}AP$

$$\text{Let } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For $\lambda_1 = 1$

$$[A - I][V] = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = 0, v_1 + v_2 = 0, v_1 = -v_2$$

$$\therefore V_{\lambda=1} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ -v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = -1$

$$[A + I][V] = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2v_1 + v_3 &= 0 \\ v_3 &= -2v_1 \\ 2v_2 + v_3 &= 0 \Rightarrow v_3 = -2v_2 \\ v_1 + v_2 + v_3 &= 0, v_2 = -v_1 + 2v_2 \\ &= -v_1 \end{aligned}$$

$$V_{\lambda=-1} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \\ -2v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

For $\lambda = 2$

$$[A - 2I][V] = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -v_1 + v_3 &= 0 \Rightarrow v_1 = v_3 \\ -v_2 + v_3 &= 0 \Rightarrow v_2 = v_3 \\ v_1 = v_2 &= v_3 \end{aligned}$$

$$V_{\lambda=2} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Q} \quad d.i.P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\therefore AP = PD \Rightarrow D = P^{-1}AP \quad \text{proved}$$

33) $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^3$

$$\boxed{\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} &= 0 \Rightarrow (-4-\lambda)(5-\lambda) + 18 = 0 \\ -20 + 4\lambda - 5\lambda + \lambda^2 + 18 &= 0 \\ \lambda^2 + \lambda + 2 &= 0 \end{aligned}}$$

$$A^2 = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 30 \\ -15 & 31 \end{bmatrix}$$

$$A^8 = A^4 \cdot A^4 = \begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix} \begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix} = \begin{bmatrix} -254 & -510 \\ -255 & 511 \end{bmatrix}$$

$$A^9 = A^8 \cdot A = \begin{bmatrix} -254 & -510 \\ -255 & 511 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2546 & -4074 \\ 513 & 1025 \end{bmatrix}$$

34)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given To find $\rightarrow [A]^{2018}$

$$[A]^{2018} = [A^2]^{1009} = [I]^{1009} = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{d}$$

35)

$$\text{adj } A = \begin{bmatrix} 1 & 1 \\ 0 & K \end{bmatrix}$$

$$|A - KA| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & K-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(K-\lambda) = 0$$

~~$\lambda = 1 \text{ or } \lambda = K$~~

Hence, value of λ is any real number except 1
 $\lambda = R - \underline{\underline{[1]}}$



41) A norm is a function from a real or complex vector space to the non-negative real numbers that behaves in certain ways like the distance from the origin.

42) Orthogonally in \mathbb{R}^n means a vectors $(v_1, v_2, v_3, \dots, v_k)$ in \mathbb{R}^n is called an orthogonal set if all the vectors in the set are pairwise orthogonal.

43) $v_1 = (2, 1, -1), v_2 = (0, 1, 1), v_3 = (1, -1, 1)$

$$\langle v_1, v_2 \rangle = 0 + 1 - 1 = 0$$

$$\langle v_2, v_3 \rangle = 0 - 1 + 1 = 0$$

$$\langle v_1, v_3 \rangle = 2 - 1 - 1 = 0$$

Hence, $\{v_1, v_2, v_3\}$ is an orthogonal set in \mathbb{R}^3 .

44) Orthogonal sets :- basic :- An orthogonal basis for a subspace W of \mathbb{R}^n is a basis of W that is an orthogonal set.
 $\{(0, 1), (1, 0)\}$ is an orthogonal basis of \mathbb{R}^2

45) Orthonormal sets :- A set of vectors in \mathbb{R}^2 is an orthonormal set if it is an orthogonal set of unit vectors.

46) $q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ & $q_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

$$\langle q_1, q_2 \rangle = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

$$\|q_1\| = \langle q_1, q_1 \rangle = \sqrt{\left(\frac{1}{3}\right)^2} = \frac{1}{3}$$

$$\|q_1\| = q_1 \cdot q_1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$\|q_2\| = q_2 \cdot q_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\text{Now } \langle q_1, q_2 \rangle = \frac{1}{3} + \frac{1}{6} + \left(-\frac{1}{3}\right) \left(\frac{1}{6}\right) + \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{3} - \frac{1}{18} = \frac{5}{18}$$

$\therefore \{q_1, q_2\}$ is an orthogonal set in \mathbb{R}^3 .

48) ~~Orthogonal basis~~ :- An ~~orthogonal~~ ^{norm} basis for a subspace W that is an ~~orthogonal~~ ^{norm} set.

48) Orthonormal basis :- An orthonormal basis for a subspace W of \mathbb{R}^n is a basis of W that is an ~~orthogonal~~ normal set.

49) Orthogonal matrix :- An $n \times n$ matrix A whose column form an orthonormal set is called an orthogonal matrix.

$$50) A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \\ -\frac{1}{3} & 0 & \frac{2}{5} \end{bmatrix}, \Rightarrow A^T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

For orthogonal matrix, $A A^T = I$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \\ -\frac{1}{3} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{25}\right), \left(\frac{1}{3} - \frac{1}{4} + \frac{1}{25}\right), \left(\frac{1}{3} + 0 + \frac{2}{25}\right) \\ \left(\frac{1}{2} - \frac{1}{4} + 0\right), \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{25}\right), \left(0 + 0 + \frac{2}{25}\right) \\ \left(\frac{1}{5} + 0 + \frac{2}{25}\right), \left(\frac{1}{5} + 0 + \frac{2}{25}\right), \left(\frac{1}{5} + \frac{4}{25}\right) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{3} - \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} + \frac{1}{25} & \frac{1}{4} + \frac{1}{25} & \frac{2}{25} \end{bmatrix} \quad R_1 \rightarrow (R_1 + R_2 - R_3) \\ R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1$$

$= I \quad \therefore A$ is orthogonal matrix $\therefore A^{-1} = A^T$

5)

$$A = \begin{bmatrix} \cos \theta \sin \theta & -\cos \theta & -\sin^2 \theta \\ \cos^2 \theta & \sin \theta & -\cos \theta \sin \theta \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos \theta \sin \theta & \cos^2 \theta & \sin \theta \\ -\cos \theta & \sin \theta & 0 \\ -\sin^2 \theta & -\cos \theta \sin \theta & \cos \theta \end{bmatrix}$$

For orthogonal matrix, $A A^T = I$

$$\Rightarrow \begin{bmatrix} \cos \theta \sin \theta & -\cos \theta & -\sin^2 \theta \\ \cos^2 \theta & \dots & \sin \theta \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta \sin \theta & \cos^2 \theta & \sin \theta \\ -\cos \theta & \sin \theta & 0 \\ -\sin^2 \theta & -\cos \theta \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos^2 \theta \sin^2 \theta + \cos^2 \theta + \sin^4 \theta & \cos^3 \theta \sin \theta - \sin \theta \cos \theta + \sin^3 \theta & \cos \theta \sin \theta & \cos \theta \sin^3 \theta \\ \cos^3 \theta \sin \theta - \sin \theta \cos \theta + \cos \theta \sin^3 \theta & \cos^4 \theta + \sin^2 \theta & -\cos \theta \sin \theta & -\cos \theta \sin^3 \theta \\ \cos \theta \sin^2 \theta - \cos \theta \sin^2 \theta & \cos^2 \theta \sin \theta - \cos^2 \theta \sin \theta & \cos^2 \theta \sin \theta - \cos^2 \theta \sin \theta & \sin^2 \theta \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

 $= I$

$\therefore A$ is orthogonal matrix $\therefore A^{-1} = A^T$

52) Orthogonal projection :- The projection of a vector on a plane is its orthogonal projection on that plane.

62) $A = \begin{bmatrix} 1 & 4 \\ b & a \end{bmatrix}$ eigen values are -1 & 7

$$|A - \lambda I|$$

$$\begin{vmatrix} 1-\lambda & 4 \\ b & a-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(a-\lambda) - 4b = 0 \quad \text{--- (i)}$$

$$\text{when } \lambda = -1 \Rightarrow 2(a+1) - 4b = 0 \quad \text{--- (i)}$$

$$\text{when } \lambda = 7 \Rightarrow -6(a-7) - 4b = 0 \quad \text{--- (ii)}$$

Solving eqn (ii) & (iii)

$$2a+2 - 4b = 0$$

$$-6a + 4b = 48 = 0$$

$$8a - 40 = 0 \Rightarrow a = 5, b = 3$$

$$\therefore A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

61) Determinant of a matrix is the product of its eigen values.

Determinant of inverse of matrix is reciprocal to the inverse of value of determinant.

Eigenvalue of matrix A are 1, 2 & 4.

Determinant of matrix $A = 1 \times 2 \times 4 = 8$

Determinant of inverse of $A = \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{8} = 0.125$

$$\det(A^{-1}) = \det(A^{-1})^T = 0.125$$