Unit - 2

2D Transformations

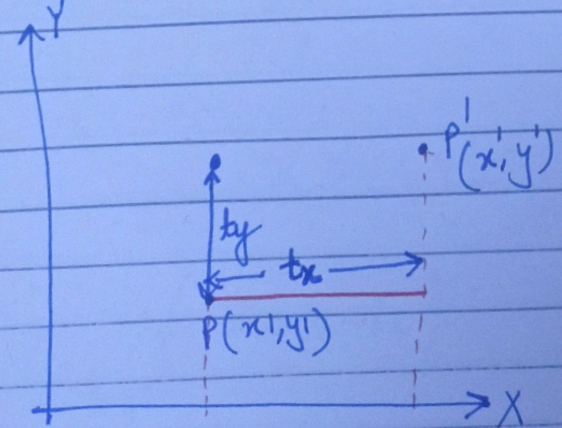
→ here, we will discuss how to apply some operations on 2-D.

2D
- ↳ x-axis
- ↳ y-axis

Basic Transformations, that can apply on a 2-D object are:-
1. Translation
2. Rotation
3. Scaling

Translation

→ It is a process of transforming / plotting a object from one co-ordinate pt to another coordinate pt.



Suppose, we have a pt P at (x_1, y_1). Now, we want to move P to a new position i.e P' - (x', y')

how to get this pt $P'(x', y')$

To do that we need to ~~travel to~~ add this x-value to a translation distance along x-axis, (t_x)
& you need to add y-value to a translation distance along y-axis. (t_y)

$$\boxed{x' = x_1 + t_x}$$
$$\boxed{y' = y_1 + t_y}$$

By doing this, we get new pixel value.

Matrix form

$$\underbrace{[x' \quad y']} = \underbrace{[x_1 \quad y_1]}_{P} + \underbrace{[t_x \quad t_y]}_{\text{Translation matrix}}$$

$$\boxed{P' = P + T}$$

Example Consider a polygon with 3 vertices

A → (2, 5)
B → (7, 10)
C → (10, 2)

$$\begin{cases} x' = x + t_x \\ y' = y + t_y \end{cases}$$

Suppose, $t_x = 2$, $t_y = 2$ } so by using t_x, t_y we will find new pixel values

$A' =$ $x' = 2+2 = 4$, $y' = 5+2 = 7$ (4, 7)

$\underline{\underline{B'}}$

$x' = 7+2 = 9$
$y' = 10+2 = 12$

B' $(9,12)$

$\underline{\underline{C'}}$

$x' = 10+2 = 12$
$y' = 2+2 = 4$

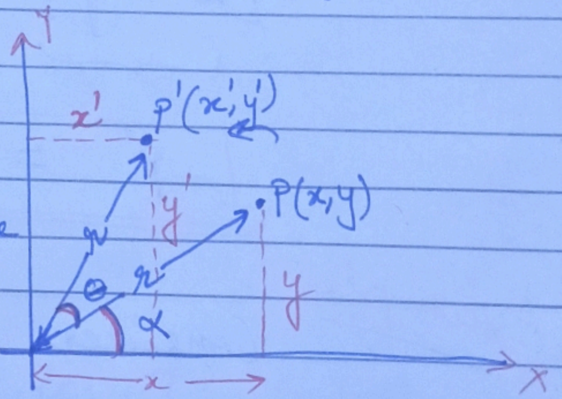
$\underline{C'}$ $(12,4)$

When you plot A', B', C', you will see that the original polygon is shifted from its prev. coordinates

Rotation

* You want to rotate an object from a given angle & distance of that object from the origin should be same.

Suppose, you have an original pixel as $P(x,y)$. Now you want to rotate/move it anticlockwise to a new pt $P'(x',y')$, then distance of P' from origin should remain same as that of the distance of P from origin i.e (r).



$$\cos\alpha = \frac{x}{r} \quad \text{or} \quad x = r\cos\alpha$$
$$\sin\alpha = \frac{y}{r} \quad \text{or} \quad y = r\sin\alpha$$
(I)

$\cos(\theta+\alpha) = \frac{x'}{r}$ or $x' = r\cos(\theta+\alpha)$

$x' = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$= \underline{r}\cos\theta\underline{\cos\alpha} - \underline{r}\sin\theta\underline{\sin\alpha}$

$= x\cos\theta - y\sin\theta$

from (I) $x = r\cos\alpha$, $y = r\sin\alpha$

Hence, $\boxed{x' = x\cos\theta - y\sin\theta}$ — (II)

||ly

$\sin(\theta+\alpha) = \frac{y'}{r}$ or $y' = r\sin(\theta+\alpha)$

$y' = r(\cos\theta\sin\alpha + \cos\alpha\sin\theta)$

$= \underline{r}\cos\theta\underline{\sin\alpha} + r\cos\alpha\sin\theta$

$y' = y\cos\theta + x\sin\theta$

from (I)

Hence, $\boxed{y' = y\cos\theta + x\sin\theta}$ — (III)

So, by using (II) & (III) new pixel co-ordinates can be found.

Matrix form

$$[x' \quad y'] = [x \quad y] \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{P' = P \cdot R}$$

$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ = Rotational matrix

Example $P(4,3)$ angle 45°

$\theta = 45^\circ$

$$x' = x\cos\theta - y\sin\theta$$
$$= 4\cos(45^\circ) - 3\sin(45^\circ)$$
$$= \frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$y' = y\cos\theta + x\sin\theta$$
$$= 3\cos(45^\circ) + 4\sin(45^\circ)$$
$$= \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

so, $P' \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

Scaling

→ you can increase/decrease the size of an object.

S_x, S_y are the scaling parameters

S_x: scaling along x-axis

S_y: scaling along y-axis

$$P = (x, y)$$
$$P' = (x', y')$$

$$\boxed{x' = x \cdot S_x}$$
$$\boxed{y' = y \cdot S_y}$$

if

$S_x > 1$ & $S_y > 1$ } Increasing the size

$S_x < 1$ & $S_y < 1$ } Decreasing the size

$S_x = S_y$ } uniform scaling

$S_x \neq S_y$ } Non-uniform scaling

Matrix Representations and homogenous Co-ordinates

Till now, we have considered in 2D — Two coordinates i.e x & y.

While in homogenous co-ordinates — we will consider triplets i.e $\boxed{(x_w, y_w, w)}$

Purpose of homogeneous coordinates

The different transformations can be represented in the form of matrix multiplication

So, we need to convert all x, y values in the form of triplets.

$$(x, y) \Leftrightarrow (x_w, y_w, w)$$

$x \Rightarrow \frac{x_w}{w}$ where, w = non-zero value.

$y \Rightarrow \frac{y_w}{w}$ let say $w = 1$

Homogenous Co-ordinates for basic transformations

(1) **Translation**

$$x' = x + t_x$$
$$y' = y + t_y$$

Homogenous Coordinate representation

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$\boxed{P' = P \cdot T}$ → Matrix Multiplication

② Rotation

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

Homogeneous co-ordinates representation

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{P' = \underbrace{P \cdot R_\theta}_{\text{Matrix multiplication}}}$$

③ Scaling

$$x' = x \cdot S_x$$

$$y' = y \cdot S_y$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{P' = \underbrace{P \cdot S}_{\text{matrix multiplication}}}$$

Example

① Translation 2 units right

[** right = +ve
left = -ve
up = +ve
down = -ve]

$t_x = 2$
$t_y = 0$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

② Translation 2 units right & 5 units up.

$t_x = 2$
$t_y = +5$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$

③ Translation 2 units ~~right~~ left & $\frac{3}{4}$ down.

$t_x = -2$
$t_y = -3/4$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -3/4 & 1 \end{bmatrix}$$

Composite Transformations

→ when we are performing some sequence of transformations, than what would be the final matrix representation

① Translation

Applying 2 successive translations

$T(tx_1, ty_1)$

$T(tx_2, ty_2)$

$$P' = \{P \cdot T(tx_1, ty_1)\} \cdot T(tx_2, ty_2)$$

$$= P\{T(tx_1, ty_1) \cdot T(tx_2, ty_2)\}$$

Now: $T(tx_1, ty_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx_1 & ty_1 & 1 \end{bmatrix}$

$T(tx_2, ty_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx_2 & ty_2 & 1 \end{bmatrix}$

then

$\therefore T(tx_1, ty_1) \cdot T(tx_2, ty_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ tx_1 + tx_2 & ty_1 + ty_2 & 1 \end{bmatrix}$

$$= T(tx_1 + tx_2, ty_1 + ty_2)$$

i.e

2 successive translations $\Rightarrow$ add the two translations.

$$\boxed{P' = P\{T(tx_1 + tx_2, ty_1 + ty_2)\}}$$

② Rotations

2 Successive Rotations with angle θ_1 & θ_2.

$$P' = P\{R(\theta_1) \cdot R(\theta_2)\}$$

$$R(\theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_2) = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_1) \cdot R(\theta_2) = \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 & 0 \\ -\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R(\theta_1+\theta_2)$$

$$\boxed{P' = P \cdot R(\theta_1+\theta_2)}$$

③ Scaling

2 Successive scalings

$$S(S_{x_1}, S_{y_1})$$

$$S(S_{x_2}, S_{y_2})$$

$$P' = P \cdot S(S_{x_1}, S_{y_1}) \cdot S(S_{x_2}, S_{y_2})$$

$$S(S_{x_1}, S_{y_1}) \cdot S(S_{x_2}, S_{y_2}) = \begin{bmatrix} S_{x_1} & 0 & 0 \\ 0 & S_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{x_2} & 0 & 0 \\ 0 & S_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

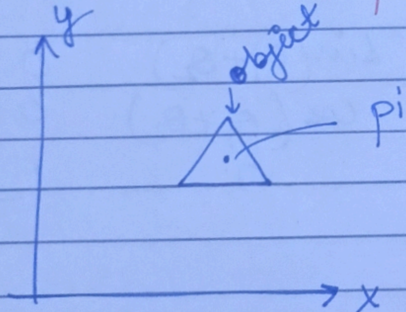
$$= \begin{bmatrix} S_{x_1} S_{x_2} & 0 & 0 \\ 0 & S_{y_1} S_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= S(S_{x_1} S_{x_2}, \; S_{y_1} S_{y_2})$$

$$\boxed{P' = P \cdot S(S_{x_1} S_{x_2}, \; S_{y_1} S_{y_2})}$$ Multiplicative

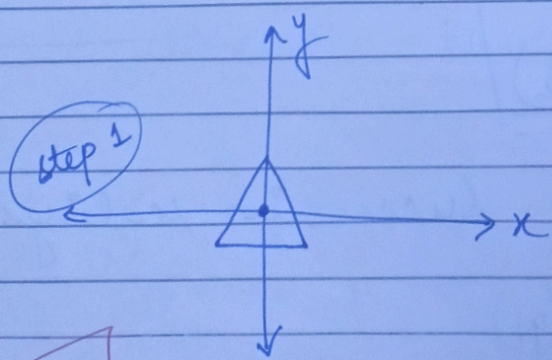
4

Suppose you want to apply rotation w.r.t one pivot value.



Step 1

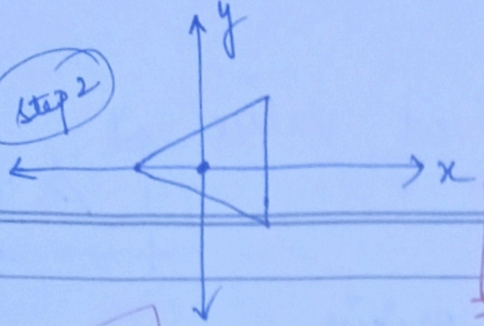
we need to move pivot value to origin. (known as translation)



so, matrix will be

$\boxed{T(-x_c, -y_c)}$ as we moved it left

Step 2 Perform rotation as per the requirement step 2

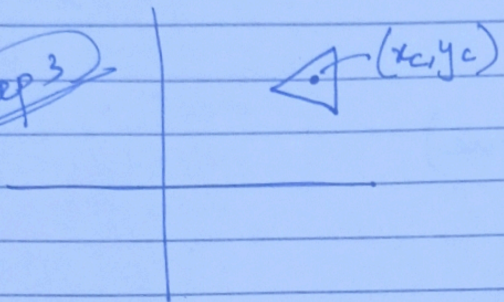


$\boxed{R(\theta)}$

Step 3

move the pivot pt from (0,0) to its original point (known as Inverse Translation)

step 3



$\boxed{T^{-1}(x_c, y_c)}$

So, $$T(-x_c, -y_c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_c & -y_c & 1 \end{bmatrix}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

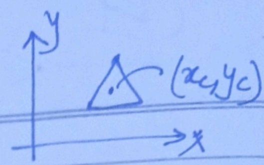
$$T^{-1}(x_c, y_c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_c & y_c & 1 \end{bmatrix}$$

So, $$R_M = T(-x_c, -y_c) \cdot R(\theta) \cdot T^{-1}(x_c, y_c)$$

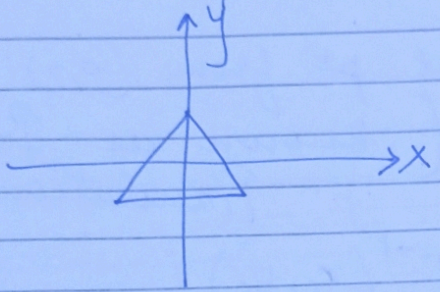
Resultant matrix, R_M $$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_c & -y_c & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_c & y_c & 1 \end{bmatrix}$$

after solving, you will get a resultant matrix.

(5) Scaling w.r.t a pivot value

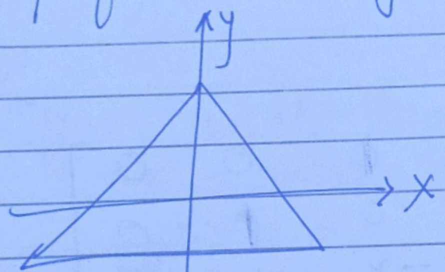


Step 1 move pivot value to origin



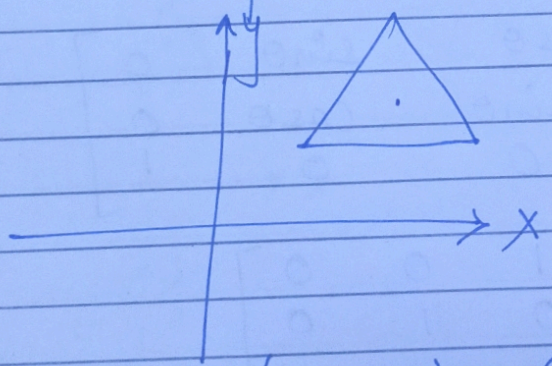
$\Rightarrow T(-x_c, -y_c)$

Step 2 perform scaling (+ve)



$\Rightarrow S$

Step 3



$\Rightarrow T^{-1}(x_c, y_c)$

$$R_M = T(-x_c, -y_c) \cdot S \cdot T^{-1}(x_c, y_c)$$

R_M, Resultant matrix

$$R_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_c & -y_c & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_c & y_c & 1 \end{bmatrix}$$