Set Theory
Unit-1

Proposition (Statement) logic –

Definition – A proposition is a declarative sentence which is either true or false. But not both. These statements are called proposition. for example –

(a) I am a Doctor.
(b) Paris is in England.
(c) $2+2=4$
(d) Bangalore is in India.
(e) 5 is greater than 4.

Proposition (✓) (either true or false)

(a) How nice!
(b) Where are you going?
(c) Put the homework on the blackboard.
(d) What do you mean?
(e) This statement is false.

Proposition (X)

* The sentences which include Questions, exclaimation & expressions of opinions are not statements (proposition)

* We usually denote the statements or proposition by lower case letter like p, q, r, s ....

eg p: Paris is in England.

Here p is false.

Basic Connectives, Truth Table, Tautology, logical equivalance-

Basic Connectives-

(i) 'and' is denoted by $\wedge$ → conjective.

(ii) 'or' is denoted by $\vee$ → dis-conjective.

(iii) 'not' is denoted by $\sim$ → negation.

(iv) 'if ---- then' is denoted by $\rightarrow$ or $\Rightarrow$

(v) 'if and only if' is denoted by $\leftrightarrow$ or $\Leftrightarrow$

The implication of $p \rightarrow q$ can also read as-

(a) if p then q
(b) p is sufficient for q.
(c) q is necessary for p.
(d) q if p.
(e) p implies q.
(f) q is implied by p.

Q Translate the following Compound statements into symbolic form.

(1) I am a doctor and I am a professor.
p: I am a doctor. q: I am a professor.

$$\boxed{p \wedge q}$$

(2) He studies hard and does not understand logic.

q: He understand logic {not को हमेशा negation में लेंगे}

$\cancel{p \wedge q}$ $\boxed{p \wedge \sim q}$

(3) Ram and shyam went to college.

$\boxed{p \wedge q}$

(4) Govind is clever and handsome.

$\boxed{p \wedge q}$

(5) If all odd integers are primes and 4 divides 6 then it will not rain.

p: All odd integers are prime
q: 4 divides 6
r: it will rain.

$$\boxed{(p \wedge q) \rightarrow (\sim r)}$$

Truth Table –

(a) Conjuction ($\wedge$) –

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) Disjunction ($\vee$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(c) Negation ($\sim$)

p	$\sim p$
T	F
F	T

(d) Conditional ($\rightarrow$)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(e) Biconditional ($\leftrightarrow$)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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Tautology -

A proposition P is a tautology if it is true under all circumstances. It means it contains only 'T' in the final column of its truth table.

example

p	$p \vee p$	$p \vee p \leftrightarrow p$
T	T	T
F	F	T

Q Which of the following are tautology.

(a) $(p \to q) \leftrightarrow (\sim q \to \sim p) = X$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	X
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(b) $p \vee p \leftrightarrow p$

p	$p \vee p$	$p \vee p \leftrightarrow p$
T	T	T
F	F	T

(c) $(p \wedge \sim q) \vee (\sim p \wedge q) = X$ (Let)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	X
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

(d) $\sim(p \vee q) \vee (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p \vee \sim q)$	X
T	T	F	F	T	F	F	F
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	T	T

(e) $\underbrace{(p \to q) \wedge (q \to r)}_{X} \to \underbrace{(p \to r)}_{Y}$

p	q	r	$p \to q$	$q \to r$	$p \to r$	X	$X \to Y$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Date
Page

Contradiction-

A compound statement (proposition) is said to be contradiction if it is always False for all possible combination of truth values of its components.

That is the final coloumn in truth table are always False.

ex.

p	$\sim p$	$(p \wedge \sim p)$
T	F	F
F	T	F

→ Contradiction ✓

Q Determine whether the following one is a Tatology or Contradiction.

① $(p \vee q) \vee (\sim p \vee \sim q) = X$

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \vee \sim q)$	X
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	T

↓
Tautology

② $(q \wedge \sim p) \leftrightarrow r = X$

p	q	r	$\sim p$	$q \wedge \sim p$	X
T	T	T	F	F	F
T	T	F	F	F	T
T	F	T	F	F	F
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	F	F
F	F	F	T	F	T

neither Tautology nor Contradiction

Inverse, Converse & Contrapositive of a Statement -

Consider an implication $p \rightarrow q$

(i) The implication $\sim p \rightarrow \sim q$ is called the inverse of $p \rightarrow q$

(ii) The implication $q \rightarrow p$ is called the converse of $p \rightarrow q$

(iii) The implication $\sim q \rightarrow \sim p$ is called the contrapositive of $p \rightarrow q$.

ex. ① If n is an even integer then $2n$ is divisible by 4.

Inverse - If n is not an even integer then $2n$ is not divisible by 4.

Converse → If $2n$ is divisible by 4 then n is an even integer.

Contrapositive → If $2n$ is not divisible by 4 then n is not even integer.

Q If I do not go to cinema then I will study.

Inverse → If I go to cinema then I will not study.

Converse → If I will study then I do not go to cinema.

Contrapositive → If I will not study then I go to cinema.

Logical equivalance - In logical equivalance if & only if two statements have same logical truth values then those statements are logical equivalance. it is denoted by '$\equiv$'.

eg p : 8 is prime number.
q : $3^2 + 2^2 = 3^2$

So Both are false then $(p \equiv q)$

Or

$\left[\sim \Rightarrow \neg \text{ (negation)}\right]$

Date
Page

Any two simple statement (proposition) p & q are said to be logical equivalance if & only if they have same truth values.

eg

p : 2 is even number.

q : $3 + 2 = 5$

Both are true $\Rightarrow$ $p \equiv q$

eg p : 8 is prime number

q : $2 + 3 = 5$

Here $p \not\equiv q$ bcoz p is false & q is true.

Theorem ①

$$\boxed{\sim(\sim p) \equiv p}$$

Theorem ②

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q}$$

Theorem ③

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

Theorem ④

$$\boxed{\sim(p \rightarrow q) \equiv p \wedge \sim q}$$

The Law of Logic -

① De-Morgan's Law -

(a) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

(b) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

② Associative law -

(a) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

③ Distributive Law -

(a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

④ Absorption Law -

(a) $p \vee (q \wedge r) \Leftrightarrow p$

(b) $p \wedge (q \vee r) \Leftrightarrow p$

Theorem ⑤

Let $P(P_1, P_2, \ldots\ldots P_n)$ & $Q(Q_1, Q_2, \ldots Q_n)$ be two propositions. The following conditions are equivalent -

(1) $\sim P(P_1, P_2 \ldots) \vee q(q_1, q_2 \ldots)$ is a tautology.

(2) $P(P_1, P_2, \ldots) \wedge q(q_1, q_2 \ldots)$ is a contradiction.

(3) $P(P_1, P_2 \ldots) \rightarrow q(q_1, q_2 \ldots)$ is a tautology.

Q Write negation of -

(i) $q \vee \sim(p \wedge r)$

(ii) $(p \rightarrow r) \wedge (q \rightarrow p)$

Q Prove the logical equivalance -

(i) $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

(ii) $p \vee (p \wedge q) \equiv p$

Sol^n (i)

$\sim(q \vee \sim(p \wedge r))$ (use of Theorem ②)

$= \sim q \wedge p \wedge r$

(ii)

$\sim((p \rightarrow r) \wedge (q \rightarrow p))$

(use of theorem ③ & ④)

$= \sim(p \rightarrow r) \vee \sim(q \rightarrow p)$

$= p \wedge \sim r \vee q \wedge \sim p$

L.H.S.

(1) $(p \vee q) \wedge \sim p \equiv \sim p \wedge (p \vee q)$ {Commutative}

$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$ {Distributive}

$\{\sim p \wedge p \Rightarrow F\}$

$\equiv f \vee (\sim p \wedge q)$

$= \sim p \wedge q$ (Identity law)

$f \Rightarrow$ false

Identity Law ① $p \vee f \Leftrightarrow p$ ⑤ $p \wedge \sim p \Leftrightarrow f$
② $p \wedge t \Leftrightarrow p$
③ $p \vee t \Leftrightarrow t$
④ $p \wedge f \Leftrightarrow f$

(ii) $p \vee (p \wedge q) \equiv p$

L.H.S.

$$p \vee (p \wedge q) \equiv (p \wedge t) \vee (p \wedge q) \quad \{\text{Identity}\}$$
$$\equiv p \wedge (t \vee q) \quad \{\text{Distributive}\}$$
$$\equiv p \wedge t \quad \{\text{identity}\}$$
$$\equiv p = \text{R.H.S.}$$

Rule of Inference -

ⓐ Law of Detachment (or ~~Modds~~ Modus Ponens) -
(In German)

The form of the arguement is -

$$\begin{array}{c} p \to q \\ p \\ \hline q \end{array}$$

Here the premises are

P 1 : $p \to q$ ("p implies q")
P 2 : p ("p is assumed to be true")

Conclusion : q ("so q is true")

i.e;

$p \to q$ (premise) is true
$\therefore$ q (Conclusion) is true

Q Supposing the following propositions are true

p : two triangle are similar

$p \rightarrow q$: if two triangle are similar then their corresponding sides are proportional.

Ans-

q : The corresponding sides are proportional is true.

* Let us the truth table for $[(p \rightarrow q) \wedge p] \rightarrow q$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(b) Law of Contraposition (or Modus Tollens)

The form of the arguement is

$$p \rightarrow q$$
$$\therefore \sim q$$
$$\therefore \sim p$$

The logical expression like as
$[(p \rightarrow q) \wedge (\sim q)] \rightarrow \sim p$

Truth table –

p	q	$\sim p$	$(p \to q)$	$(\sim q)$	$(p \to q) \wedge (\sim q)$	$[(p \to q) \wedge (\sim q) \to (\sim p)]$
T	T	F	T	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	F	T	T	T	T	T

eg ① If $\underbrace{\text{it snows today}}_{p}$ then $\underbrace{\text{we will go for skiing}}_{q}$.

② We $\underbrace{\text{will not go}}_{\sim q}$ for skiing.

① $p \to q$

② $\sim q$

© Disjunctive syllogism –

The form of the arguement is

$$p \vee q$$

$$\therefore \frac{\sim p}{q}$$

logical expression

$$[(p \vee q) \wedge (\sim p)] \to q$$

(Truth table से Proof हो जायेगा)

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Date
Page

eg 1: Either it is below freezing or raining now.
2: It is not below freezing.

Conclusion - It is raining now.

(d) Hypothetical syllogism -

The form of the arguement is

$$p \rightarrow q$$

$$\therefore \frac{q \rightarrow r}{p \rightarrow r}$$

logical expression $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

eg 1: if the weather is good then we will go for an outing.
2: if we go for an outing, we will play a game.

Conclusion -
if the weather is good, we will play a game.

Normal form -

(i) Disjunctive Normal form (DNF)

(ii) Conjunctive Normal form (CNF)

Q. If the advertisement is successfull then the sales of the product will grow up. Either the advertisement is successfull or ~~then~~ production of Product will stop. The sales of the product will not go up. therefore the production of the product will be stop. So check the validity of product.

Soln p : The advertisement is successfull
q : The sales of the product will go up
r : The production of the product will be stop.

$$\left[\begin{array}{ll} p \to q & \text{--- (1)} \\ p \vee r & \text{--- (2)} \\ \sim q & \text{--- (3)} \end{array}\right] \to \sim p \quad (4)$$

r

from (1)

$$p \to q \equiv \sim q \to \sim p \quad \text{--- (4)}$$

from (3) & (2)

$$\begin{array}{l} \sim q \to \sim p \\ \sim q \\ \hline \sim p \quad \text{--- (5)} \end{array}$$

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So

$\sim p$ is True

$\Rightarrow p$ is false

from $p \vee r \Rightarrow r$ is True.

① Disjunctive normal form (DNF) –

A conjunction of statement variables & (or) their negations is called as fundamental conjunction.

eg $p, \sim p, \sim p \wedge q, p \wedge q$ are called fundamental conjuction.

A statement form which consists of Disjunction of fundamental conjuctions This is called as Disjunctive normal form.

eg ① $(p \wedge q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$

② $(p \wedge \sim q) \vee (p \wedge r)$

③ $(p \wedge q \wedge r) \vee \sim r$

④ $(p \wedge q) \vee \sim q$

Disjunction मतलब OR ($\wedge$) लगाना चाहिये।

classmate

Date
Page

Q Obtain the DNF of the form $(p \to q) \wedge (\sim p \wedge q)$

$$(p \to q) \wedge (\sim p \wedge q)$$

$$(\sim p \vee q) \wedge (\sim p \wedge q)$$

$$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$$

$$(\sim p \wedge q) \vee (\sim p \wedge q)$$

Q Obtain the DNF of the form $\sim(p \to (q \wedge r))$

Q find the DNF of $(\sim p \to r) \wedge (p \leftrightarrow q)$ by truth table.

	p	q	r	$\sim p$	$(\sim p \to r)$	$(p \leftrightarrow q)$	$(\sim p \to r) \wedge (p \leftrightarrow q)$
① →	T	T	T	F	T	T	T
② →	T	T	F	F	T	T	T
	T	F	T	F	T	F	F
	T	F	F	F	T	F	F
	F	T	T	T	T	F	F
	F	T	F	T	F	F	F
③ →	F	F	T	T	T	T	T
	F	F	F	T	F	T	F

DNF is –

$$\underset{①}{(p \wedge q \wedge r)} \vee \underset{②}{(p \wedge q \wedge \sim r)} \vee \underset{③}{(\sim p \wedge \sim q \wedge r)}$$

or $(p \cdot q \cdot r) + (p \cdot q \cdot \bar{r}) + (\bar{p} \cdot \bar{q} \cdot r)$ = sum of products

(समझाने के लिये)

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CNF ⇒ False statement } ⇒ Conclusion ⇒ True
DNF ⇒ True statement

Date ____
Page ____

(2) Conjunctive normal form (CNF) -

A disjunction of statement variables and (or) their negation is called fundamental disjunction. for eg - $p, \sim p, \sim p \vee q, p \vee q, p \vee \sim p \vee q$ are fundamental disjunction.

The statement form, which consist of conjunction of fundamental disjunction is called conjunctive normal form.

Note - it is noted that CNF is a tautology if & only if every fundamental disjunction content if it is a tautology.

for eg -

Conjunctive Ficha and (∧)

(i) $p \wedge r$

(ii) $\sim p \wedge (p \vee r)$

(iii) $(p \vee q \vee r) \wedge (\sim p \vee r)$

Q obtain the CNF of $(p \wedge q) \vee (\sim p \wedge q \wedge r)$.

$$(p \wedge q) \vee (\sim p \wedge q \wedge r)$$

$\equiv [p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$ { By distributive law

$\equiv [(p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$

$\equiv [T \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge (q) \wedge (q \vee r)]$

$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge (q) \wedge (q \vee r)$

$$\left\{ \begin{array}{l} \text{or} \Rightarrow (+) \\ \text{and} \Rightarrow (\cdot) \end{array} \right\}$$

classmate
Date
Page

Q $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

$\equiv (\sim p \rightarrow r) \wedge [(p \rightarrow q) \wedge (q \rightarrow p)]$

$\equiv (\sim(\sim p) \vee r) \wedge [(\sim p \vee q) \wedge (\sim q \vee p)]$

$\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

Truth table of this example is in example of DNF & there all false statements are for CNF -

This negation is for ~~to make~~ ~~...~~change all statements from false to true

$\equiv \sim [(p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r)$
$\vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)]$

now by the demorgan's law all sign will be interchange.

$\equiv [(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge$
$(p \vee \sim q \vee r) \wedge (p \vee q \vee r)]$

समझने के लिये।

$\equiv [(\sim p + q + \sim r) \cdot (\sim p + q + r) \cdot (p + \sim q + \sim r) \cdot$
$(p + \sim q + r) \cdot (p + q + r)]$

$=$ Product of Sum.

Predicate logic -

Let A be a given set. A propotional function [or open sentence or condition or predicate logic] defined on A is an expression

Which has the property that $P(a)$ is true or false for each $a \in A$.

That is $P(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x.

In other words

T_p = collection of all elements of A then

$$T_p = \{x : x \in A,\ P(x) \text{ is true}\}$$

$$\text{Or } T_p = \{x : P(x)\}$$

Q find the truth set of each propositional function $P(x)$ defined on the set N of (+)ve integers

(a) Let $P(x) = x + 2 > 7$ Its truth set is

$$T_p = \{x ; x \in N,\ x+2 > 7\} = \{6, 7, 8 \text{------}\}$$

consisting of all integers greater than 5.

(b) Let $P(x) = x+5 < 3$ Its truth set is $T_P = \{x : x \in N, x+5<3\} = \phi$ the empty set.

(c) Let $P(x) = x+5 > 1$ Its truth set is $T_P = \{x : x \in N, x+5>1\} = N$ is true for every element in $\underline{\underline{N}}$

Quantifiers -

Certain proposition involving a specified number of objects. for example -

(a) All squares are rectangle.
$= \forall x \; [x \text{ is rectangle}]$

(b) Some men are short.
$= \exists y \; [y \rightarrow \text{men are short}]$

(c) for every real no. of x, $x^2 \geq 0$
$= \forall x \; [x^2 \geq 0]$

(d) At least one student is interested in logic.
$= (\exists)$

(e) Ram or Mohan none of there stood first
$= (\exists)$

(f) There exists a function whose derivative is t^3
$= (\exists)$

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The words 'all', 'some', 'for every', 'or' 'at least one', 'exists' indicate quantity. The words expression is called quantifiers.

There are two type of quantifiers-

(1) Universal

⇓

'for all'
'for every'
'for ~~some~~ each'

These all are denoted by ($\forall$).

(2) Existential

⇓

'for some'
'there exists'
'or'
'at least'

It is denoted by ($\exists$)

Skolemization-

The procedure for systematic elemination of the existential quantifiers is logic [prenex form] is called skolemization by introducing new constant & function symbols.

That new constant is called skolem constant.

That new function is called skolem function.

Date ______
Page ______

① for simple case -

The result of skolenization of the formula

$\exists x \forall y \forall z$ (A) → prenex form

$= \forall y \forall z (c/x)$ where c is new (skolen) Constant

② for instance -

The result of skolenization of the formula

$\exists x \forall y \forall z (P(x,y) \rightarrow Q(x,z))$ is

$= \forall y \forall z ((P(c,y) \rightarrow Q(c,y))$

Therefore the general formula result is -

$\exists x_1 ---\exists x_k \forall y_1 ---- \forall y_n \; A$

$= \forall y_1 ----- \forall y_n \; A [c_1/x_1, ----- c_n/x_n]$

Where $c_1, ------ c_n$ are skolen constant.

Note- It is noted that resulting formula is not equivalent to the original statement, But ~~this~~ is equally satified by it.

Q every philosopher writes at least one book.

$= \forall x\,[\text{Philosopher}(x) \rightarrow \exists y\,[\text{Book}(y) \wedge \text{Write}(x,y)]]$

now eliminate implication

$= \forall x\,[\sim \text{philosopher}(x) \vee \exists y\,[\text{Book}(y) \wedge \text{Write}(x,y)]$

Skolemization ⇒ substitute y by g(x)

$= \forall x\,[\sim \text{philosopher}(x) \vee [\text{Book}(g(x)) \wedge \text{Write}(x, g(x))]$

* Proof By Contraposition

Indirect proof is known as proof by contraposition.

Proofs By contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$

This means the conditional statement $p \rightarrow q$ showing that its contrapositive by $\sim q \rightarrow \sim p$

Date
Page

Q Proove that if n is integer & $3n+2$ is odd then n is odd.

Soln Direct –
Let $(3n+2) = (2k+1)$ for some integer k

or

$$3n+1 = 2k$$

$3n$ → odd
क्योंकि 1 में
add करके even
बनाना है।

$2k$ → even

by contraposition method –

$p = (3n+2)$ is odd

$q = n$ is odd

Assume –
$q = n$ is not odd

i.e;
n is even $\Rightarrow \boxed{n = 2k}$

$$(3n+2) = (3 \times 2k) + 2$$
$$= 6k + 2$$
$$= 2(3k+1) \rightarrow \text{even}$$

Hence
$\sim q = n$ is not odd
$(3n+2) =$ even no. $= \sim(3n+2) = \sim p$

Therefore it satisfies

$$\boxed{\sim q \rightarrow \sim p} \equiv \boxed{p \rightarrow q}$$

* Proof by Contradiction -

Suppose we want to proove that a statement p is true. further more, suppose that we can find a contradiction q such that $\sim p \rightarrow q$ is true. Because q is false but $\sim p \rightarrow q$ is true, we can find conclusion of that statement $\sim p$ is false. which means that p is true.

Q Proove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

p : $\sqrt{2}$ is irrational

We suppose
$\sim p$ is true
i.e;
$\sim p = \sqrt{2}$ is not irrational
or $\sqrt{2}$ is rational.

if $\sqrt{2}$ is rational no. then

$$\sqrt{2} = \frac{a}{b} \quad \left(\begin{array}{l} b \neq 0, \ a \ \& \ b \text{ have} \\ \text{not common factor} \end{array} \right)$$

$$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$

$$2 = \frac{a^2}{b^2}$$

$$\boxed{\underbrace{2b^2}_{\text{even}} = a^2} \text{ —— } ①$$

Here a^2 is an even number (By the definition of even integer)

$a^2 \rightarrow$ even

So $a \rightarrow$ even

$a = 2c \qquad c \rightarrow$ integer

from ①

$$2b^2 = 4c^2$$

$$b^2 = 2c^2 \rightarrow \text{even}$$

$b^2 \rightarrow$ even

$b \rightarrow$ even

So both are even (a & b) then a & b have common factor

So up leads to the equation $\sqrt{2} = \frac{a}{b}$ where (a & b have no common factor)

But both a & b are even that is 2 divides both a & b.

Therefore our assumption ~p must be false. Hence statement p "$\sqrt{2}$ is irrational" is true.

Q Proof that if $(3n+2)$ is odd then n is odd by proof by contradiction.

Unit - 3

Set -

A set is any well defined ~~function~~ collection of distinct objects or entities of any kind.

for eg -

(1) A is set of six letter alphabets a, u, v, c, y, z.

(2) A is set of cities

(3) A is set of capital cities of India.

(4) (i) N = the set of natural numbers
= 1, 2, 3, - - - - - - - - - - - -

(ii) W = the set of whole numbers
= 0, 1, 2, 3, - - - - - - - - -

(iii) I or Z = the set of integers, includes 'zero' and all positive & negative numbers

(iv) Q = the set of Rational numbers
= P/q ($q \neq 0$)

(v) R = the set of real numbers $\{-\infty$ to $\infty\}$

(vi) C = the set of complex number $(x+iy)$

Date
Page

Representation of set -

(a) Rule Method (Builder Method)

(b) Roster Method (Tabulation Method)

(a) Rule Method -

eg (1) If P = set of all prime no's then
$P = \{x : x \text{ is prime number}\}$

(2) if A = set of natural numbers b/w 10 and 100
$A = \{x : x \in N \text{ and } 10 < x < 100\}$

(b) Roster Method -

eg (1) A = set of letters of word 'MATHEMATICS' then
$A = \{M, A, T, H, E, I, C, S\}$

★ Null set or Empty set = $\{\}$, ϕ

eg (1) $A = \{x : x^2 + 1 = 0 \text{ and } x \in Z\}$
$x^2 + 1 = 0$
$x = \pm i \notin Z$
$A = \phi$ or $\{\}$

Scanned by CamScanner

Date
Page

(2) The set of all even numbers between 2 & 4

(3) The set of all even prime number greater than 2

finite & infinite set - A set is said to be finite if it has finite number of elements otherwise it is said to be infinite.

for eg-

(1) The set of vowels in the english alphabet is a finite set.

(2) The set of natural number is an infinite set.

Cardinality of Set - The number of elements in a finite set A is called its cardinal number of set & it is denoted by $n(A)$ or $|A|$.

Subset - A set A is a subset of the set B if & only if every element of A is also an element of B. it is denoted by $A \subseteq B$

Symbollically if $x \in A \Rightarrow x \in B$ then

$$A \subseteq B$$

Proper Subset - A set A is called proper subset of B if

(i) A is subset of B.
(ii) B is not subset of A.
& It is denoted by $\boxed{A \subset B}$

Super Set -

If A is a subset of B then B is called Super set of A.

eg

(i) if $A = \{0, 2, 9\}$

$B = \{0, 2, 7, 9, 11\}$

then $A \subset B$
(A is proper set of B)

(ii) if $A = \{1, 2, 4\}$

$B = \{2, 4, 6, 8\}$

A is proper subset of B &
B is super set of A.

classmate
Date
Page

Power Set - The set of all subset of S is called power set of S. It is denoted by P(S) or 2^S

eg
$$A = \{1, 2, 3\}$$

$$S = 3$$
$$P(A) = 2^3 = 8$$

$$B = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

Note - Every Empty Set ϕ is the subset of every set.

Note - Every set is also its own subset.

Q find the power set of the following

(i) $\{a, \{b\}\} = B$

$\Downarrow$

$2^2 = 4$

$\Downarrow$

$P(B) = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$

(ii) $\{1, \phi, \{\phi\}\} = A$

$\Downarrow$

$2^3 = 8$

$\Downarrow$

$P(A) = \{\phi, \{1\}, \{\phi\}, \{\{\phi\}\}, \{1, \phi\}, \{\phi, \{\phi\}\}, \{1, \{\phi\}\}, A\}$

Universal Set-

A set U which contains all the sets under consideration as subset is called a Universal Set.

Operations on Sets-

① Union of sets-

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Rules-

$$A \cup \phi = A$$
$$A \cup A = A$$
$$A \cup B = B \cup A$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

② Intersection of Sets-

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Rules-

$$A \cap \phi = \phi$$
$$A \cap A = A$$
$$A \cap B = B \cap A$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

③ Difference of Sets-

If A & B be any two sets, the difference of B & A is written as A - B is set consisting of all elements of A which are not element of B.

$$A - B = \{x : x \in A \cap B, x \notin B\}$$

eg

$A = \{a, b, c\}$

$B = \{b, c, d, e, f, g\}$

$A - B = \{a\}$

Symmetric Difference - The symmetric difference of two sets A & B is defined as the smallest set containing elements that are either in A or in B. But not in both.

It is denoted by $\boxed{A \oplus B}$ or $\boxed{A \Delta B}$ & sometimes $\boxed{A + B}$

eg

$A = \{a, b, c, d, e\}$

$B = \{c, d, e, f, g\}$

$A - B = \{a, b\}$

$B - A = \{f, g\}$

$$A \oplus B = (A - B) \cup (B - A)$$

$$= \{a, b, f, g\}$$

Rules on difference of sets-

(i) $A^c = U - A$
(ii) $A - B = A \cap B^c$
(iii) $A - A = \phi$
(iv) $A - \phi = A$
(v) $A - B = B - A \Leftrightarrow A = B$
(vi) $A - B = A \Leftrightarrow A \cap B = \phi$
(vii) $A - B = \phi \Leftrightarrow A \subseteq B$

Rules on symmetric difference-

(i) $A \oplus \phi = A$
(ii) $A \oplus A = \phi$
(iii) $A \oplus B = B \oplus A$
(iv) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
(v) $A \oplus B = (A - B) \cup (B - A)$

* Law of set theory -

① Independent law-

ⓐ $A \cup A = A$
ⓑ $A \cap A = A$

② Associative law -

ⓐ $(A \cup B) \cup C = (A) \cup (B \cup C)$
ⓑ $(A \cap B) \cap C = A \cap (B \cap C)$

③ Commutative law -

ⓐ $A \cup B = B \cup A$

ⓑ $A \cap B = B \cap A$

④ Distributive law -

ⓐ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ⓑ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

⑤ Identity law -

ⓐ $A \cap U = A$

ⓑ $A \cap \phi = \phi$

ⓒ $A \cup \phi = A$

ⓓ $A \cup U = U$ (U - universal set)

⑥ Involution law -

ⓐ $(A^c)^c = A$

ⓑ $A \cup A^c = U \rightarrow$ universal set

ⓒ $U^c = \phi$

⑦ Complement law -

ⓐ $A \cap A^c = \phi$

ⓑ $\phi^c = U$

⑧ Demorgan's law -

ⓐ $(A \cup B)^c = A^c \cap B^c$

ⓑ $(A \cap B)^c = A^c \cup B^c$

Relations

Cartesian Product of two set – Let A & B be two sets. The set of all ordered pair (a,b) where $a \in A$ & $b \in B$ is called cartesian Product of A & B.

for eg –

$A = \{1, 2, 3\}$ $B = \{a, b\}$

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

But $\Rightarrow \boxed{A \times B \neq B \times A}$

Relation –

Let A & B be two non-empty sets then any subset of $A \times B$ is called "a relation from A into B".

i.e

Let $a \in A$ & $b \in B$ then Pair (a,b) is called ordered pair of $(a,b) \in R$ then we write aRb which is read as "a" is related to "b".

classmate
Date
Page

for example-

① Let $A = \{3, 6, 9\}$, $B = \{4, 8, 12\}$

then

$R = \{(3,4), (3,8), (4,12)\}$ is relation A to B.

Domain & Range of a relation-

Domain ⇒ If R is a relation from A & B then set of all first elements of the ordered Pair (x, y), which belongs to R is called domain of R and written as $D(R)$ or $Dom(R)$.

Range ⇒ The set of all objects y such that for some x, $(x,y) \in R$

Types of Relation-

① Reflexive Relation-

A relation R on a set A is reflexive if $a R a$ for every $a \in A$.
i.e;
if $(a,a) \in R$ for every $a \in A$ / Thus R is not reflexive if there exist an $a \in A$ such that $(a,a) \notin R$.

Page

eg

Set $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_1 \Rightarrow$ reflexive $\Rightarrow$ ✗

$R_2 \Rightarrow$ reflexive $\Rightarrow$ ✓

* $R_3 = \emptyset \Rightarrow$ reflexive ✗

* $R_4 = A \times A$, the universal relation $\Rightarrow$ reflexive ✓

Q Determine the reflexive relation for following example.

(1) Relation $\leq$ (less than or equal) on the set Z of integers

(2) Set inclusion $\leq$ on a collection C of sets

(3) Relation $\perp$ on the set L of lines in the plane

(4) Relation // on the set L of lines in the plane.

Soln R_3 is not reflexive since no lines is $\perp$ to itself. Similarly R_4 is not reflexive

Since no lines are parallel to itself.
& R_1 & R_2 are reflexive.

Parallel & Perpendicular होने के लिए दो different lines होना compulsory है।

② Irreflexive relation -

A relation R on a set A is irreflexive if (a, a) does not comes to R for every a ∈ A.
i.e.
R is not irreflexive if there exist atleast one a ∈ A such that (a, a) ∈ A

eg
Let R on set A such that $R = \{(a,b) \mid a \neq b\}$

③ Symmetric relation -

A relation R on a set A is symmetric if whenever a R b then b R a.

Thus R is not symmetric if there exist a, b ∈ A such that (a, b) ∈ R but (b, a) ∉ R

eg Set $A = \{1, 2, 3, 4\}$
$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$
$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_4 = \phi$

$R_5 = A \times A$, the universal set

$R_1 \Rightarrow$ Not symmetric
$R_2 \Rightarrow$ symmetric
$R_3 \Rightarrow$ Not symmetric
$R_4 \Rightarrow$ symmetric
$R_5 \Rightarrow$ symmetric

Q ① The relation $\perp$ on the set L of lines in the plane $\rightarrow$ Symmetric

② The relation $//$ on the set L of lines in the plan $\rightarrow$ Symmetric

③ The relation inequalities, subsets are not symmetric.

④ Anti Symmetric-

A relation R on a set A is anti-symmetric if whenever aRb and bRa then $a = b$
i.e.

if whenever $(a,b), (b,a) \in R$ then $a = b$

Thus R is not antisymmetric if there exist $a, b \in A$ such that (a,b) and $(b,a) \in R$ but $a \neq b$

{ ϕ के लिये ⇒ Reflexive को छोड़कर ϕ सारे relation follow करेगा }

classmate
Date
Page

eg ⇒

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$

$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

$R_3 = \{(1,3), (2,1)\}$

$R_1 \Rightarrow$ anti symmetric ✓
$R_2 \Rightarrow$ not anti symmetric
$R_3 \Rightarrow$ anti symmetric

$R_4 = \phi \Rightarrow$ Antisymmetric
$R_5 = A \times A \Rightarrow$ universal set ⇒ Not antisymmetric

★ The properties of being symmetric & being anti symmetric are not negative of each other.

eg ⇒ Let * $R = \{(1,3), (3,1), (2,3)\}$

R is neither symmetric nor anti symmetric.

* $R' = \{(1,1), (2,2)\}$

R' is both symmetric & anti symmetric.

classmate

Date

Page

(5) Transitive relation-

A relation R on set A is transitive if whenever aRb & bRc the aRc.

i.e.

if whenever $(a,b),(b,c) \in R$ then $(a,c) \in R$

Thus R is not transitive if there exist $a,b,c \in A$ such that $(a,b),(b,c) \in R$ but $(a,c) \notin R$

eg in last example-

$R_1 \Rightarrow$ transitive

$R_2 \Rightarrow$ transitive

$R_3 \Rightarrow$ not transitive

(6) Equivalance relation-

A relation R on set S is an equivalance relation if R is reflexive, symmetric & transitive.

i.e.

(i) for every $a \in S$, aRa

(ii) if aRb then bRa

(iii) if aRb, bRc then aRc

eg ① in last example

$R_2 \Rightarrow$ symmetric ✓, reflexive ✓, transitive ✓

$\Downarrow$

$\boxed{\text{equivalance ✓}}$

② equality of numbers ~~of~~ on a set of real number

③ equality of subsets of universal sets.

④ Similarity of trigle on the set of triangles.

Ex: $A = \{1,2,3,4\}$

$R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\} \rightarrow$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\} \rightarrow$

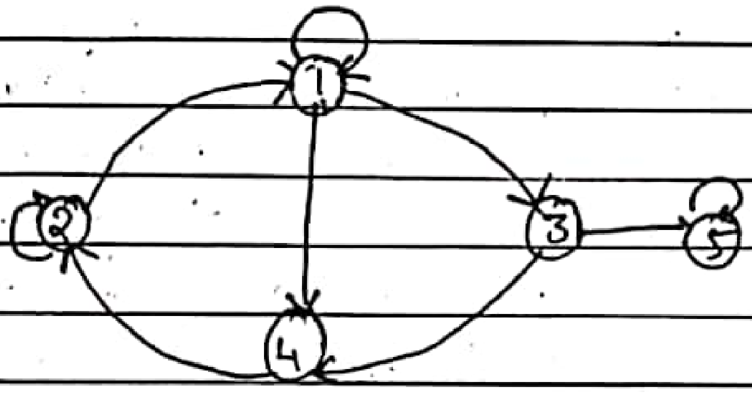
$R_3 = \{(1,3), (2,1)\}$

$R_4 = A \times A$, the Universal Relation

Soln :⇒ R_2, R_4 is Equivalance Relation

Pictorial Representation Of Relation :⇒

Que The find Relation determined by following figure

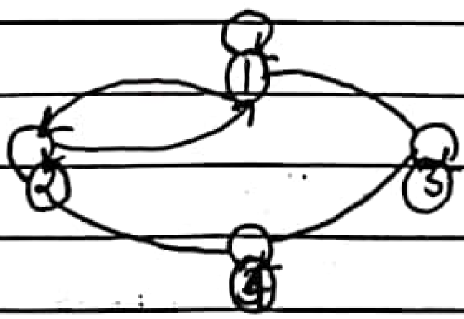


Let $A = \{1,2,3,4,5\}$

$R = \{(1,1), (1,3), (3,5), (5,5), (3,4), (1,4), (4,2), (2,2), (2,1)\}$

Ex :⇒ $A = \{1,2,3,4\}$

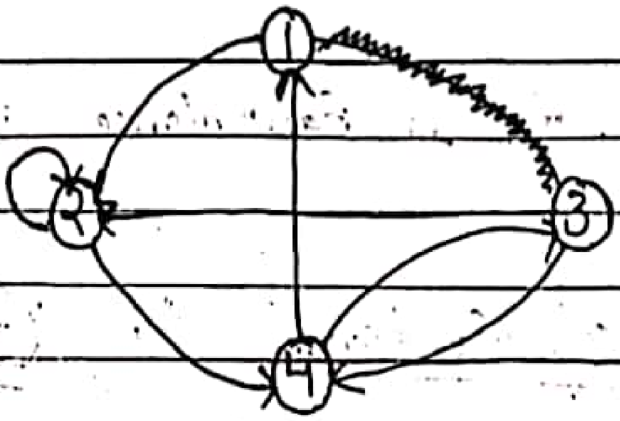
$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$



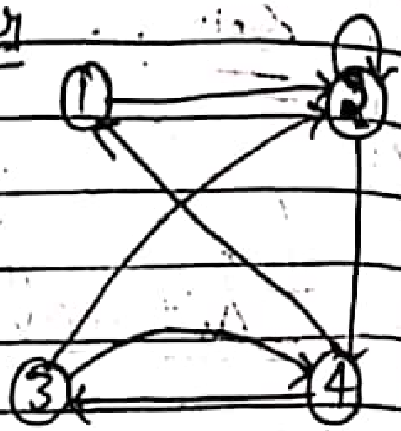
Qu. Find the directed graph from the following set.

$A = \{1, 2, 3, 4\}$

$R = \{(1,2); (2,2), (2,4); (3,2), (3,4), (4,1), (4,3)\}$



or

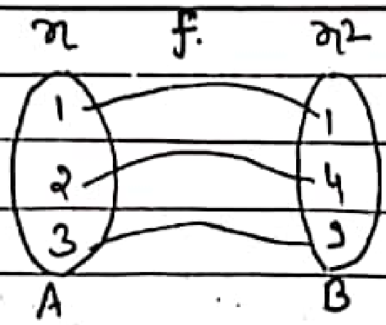


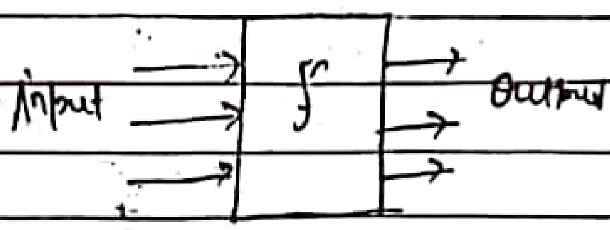
Function :⇒ function is a mapping from set A to B under certain Rules. In another word the each element of set A each is assigned with unic element of set B, the collection of such assignments is called function.

i.e. $f: A \rightarrow B$

which is Read as function f is from set A into B.

$f(x) = x^2$





Types of function

① **One - one function (Surjective) :⇒**

or

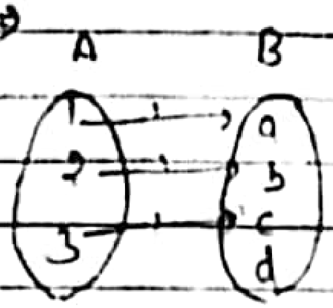
one - to - one function (1 - 1) function :⇒

A function $f : A \to B$ is said to be one - to - one function if different elements in the domain A have distinct different images in set B. in another words. f is one - one function

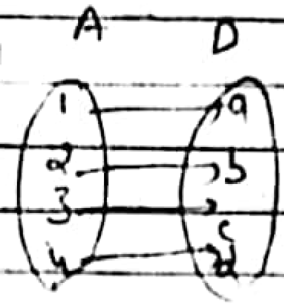
$f(a) = f(a')$ implies $a = a'$ Codomain f Range

For Ex :⇒

(i)

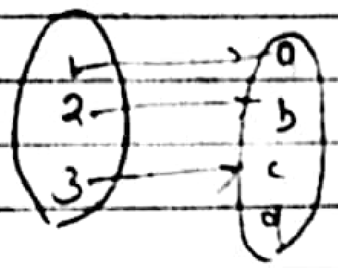


(ii)



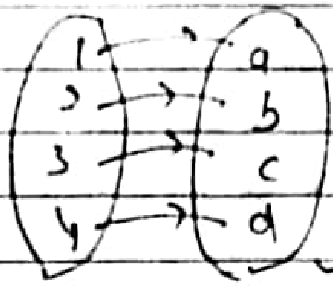
② **on to function :⇒** A function $f : A \to B$ is said to be one to f^n if each element of B is the image of some elements of A. in other words f^n f is said to be onto f^n if the image of f is the entire Codomain.

(i)



not onto f^n

(2)



or one - one function

onto function

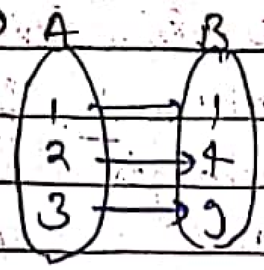
having range = Codomain.

Page:
Date:

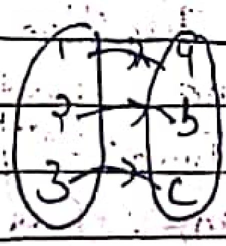
(3) **One-one onto function (Bijective f^n):-**

A function $f : A \to B$ is said to be one-one onto f^n (Bijective) if function is one-one & onto.

For ex :→ (1)



(2)



(4) (one-one onto हो तभी) **Inverse function or Invertiable f^n:-**

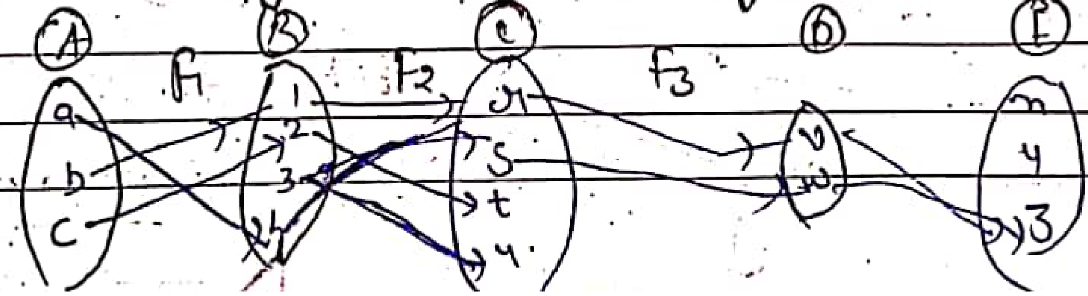
A function $f : A \to B$ is said to be invertiable if its inverse relations f^{-1} is a function from B to A.

(*) In general the inverse Relation f^{-1} may not be a function.

Theorem :- A function $f : A \to B$ is said to be invertiable if and only if is both one to one and onto.

Classify the following function from following diagram / figure :-

(1)



$f_1 \rightarrow$ one-one

$f_2 \rightarrow$ one-one ~~onto~~ (invertible)

$f_3 \rightarrow$ Not one-one & not onto.

$f_4 \rightarrow$ Niether one-one Nor onto.

Composition of function :⇒

Consider the function $f: A \rightarrow B$ and $g: B \rightarrow C$ that is. where the co-Domain of f is the domain of g. the we may define a new function from A to C. called the Composition of f and g. which is written as $g o f$. as follows $(gof)(a) = g(f(a))$.

For Ex :⇒ let f, g, h are function on X = {1, 2, 3} or

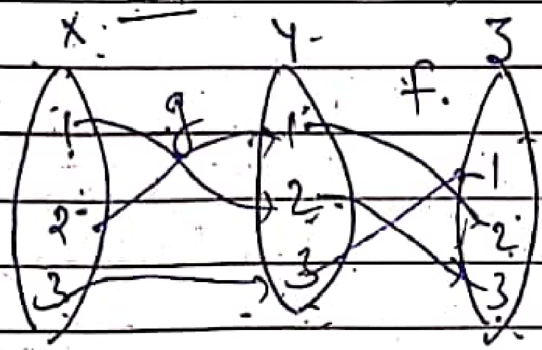
(a) $f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,3)\}$

$h = \{(1,1), (2,2), (3,1)\}$

Comute fog, gof, $fogoh$ and $fohog$

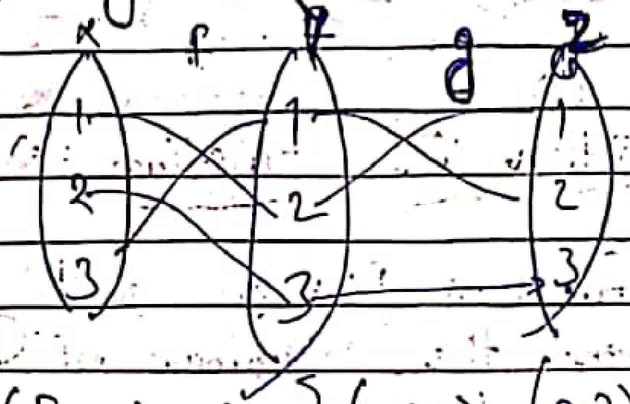
Sol^n :⇒ $fog = f(g(x))$



Page:
Date:

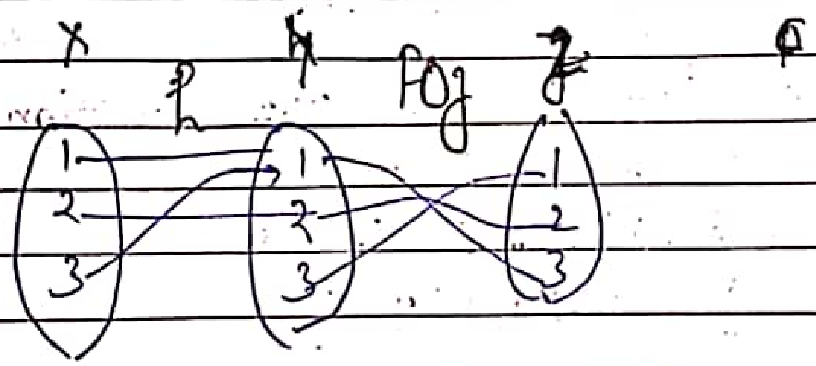
① $fog = f(g(x))$
$= \{(1,3), (2,2), (3,1)\}$

② $gof :=$



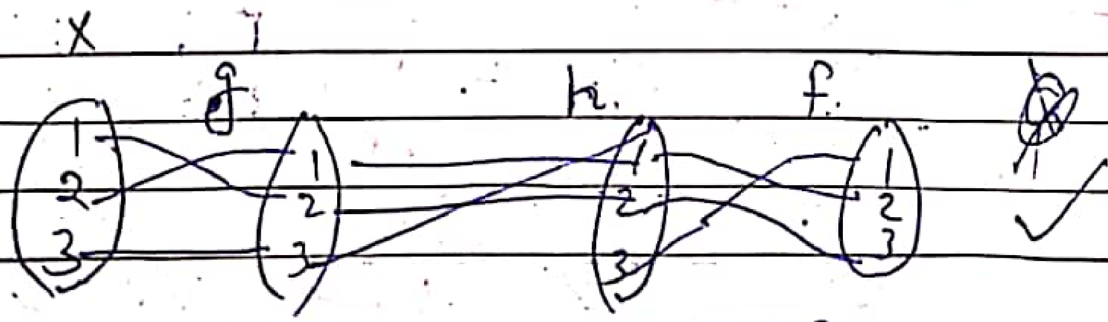
$g(f(x)) \Rightarrow \{(1,1), (2,3) (3,2)\}$

③ $fogoh$



$fogoh = (fog)(h(x)) = \{(1,3), (2,2), (3,3)\}$

④ $fohog$



$fohog = (foh)(g(x)) \Rightarrow \{(1,3), (2,2), (3,2)\}$

Que :-)

Let f, g, h are function from N to N where N is set of natural no. so that

$f(n) = n+1$ $\quad g(n) = 2n$

$h(n) = 0$ if n is even

$h(n) = 1$ if n is odd

Determine (a) $f \circ f$ (b) $f \circ g$ (c) $g \circ f$ (d) $g \circ h$ (e) $(f \circ g) h$.

Soln. :-) (1) $f \circ f = f(f(n))$

$= f(n+1)$

$(n+1)+1$

$f \circ f = n+2$

(2) $f \circ g$

$f(g(n))$

$f(2n)$

$\cancel{2(n+1)}$

$\cancel{= 2n+2}$

$2n+1$

(3) $g \circ f = g(f(n))$

$g(n+1)$

$2(n+1) = 2n+2$

(4) $g \circ h = g(h(n))$

$g(0)$

$= 0$ if even

(4) $g(h(n))$

$g(1)$

$2 \times 1 = 2$ if odd.

Page:
Date:

(c) $(f \circ g) h$ $= \cancel{(f \circ g)(h(x))}$

$\cancel{(f \circ g)(0)}$

$f \circ (\cancel{g}(0))$

$(f \circ g)(h(x)) = (f \circ g)(1)$

$f \circ (g(1))$

$f \circ (2x)$

$\underline{\underline{2x+1}}$ if ~~even~~ odd

$(f \circ g)(h(x)) = (f \circ g)(0)$

$f \circ (g(0))$

$f \circ (0)$

$\Rightarrow \underline{\underline{1}}$ if even

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Date
Page

Mathematical function-

Floor & Ceiling function-

Let x be any real number

$\lfloor x \rfloor$, called the floor of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, called ceiling of x, denotes the least integer that is not less than x.

* If x is itself an integer, then

$\lfloor x \rfloor = \lceil x \rceil$ other wise $\lfloor x \rfloor + 1 = \lceil x \rceil$

for eg -

$\lfloor 3.14 \rfloor = 3$ $\lfloor \sqrt{5} \rfloor = 2$

$\lfloor -8.5 \rfloor = -9$ $\lceil 3.14 \rceil = 4$

$\lceil \sqrt{5} \rceil = 3$ $\lceil -8.5 \rceil = -8$

$\lfloor 7 \rfloor = 7$ $\lfloor -4 \rfloor = -4$

$\lceil 7 \rceil = 7$ $\lceil -4 \rceil = -4$

Recursively Defined function -

A function is said to be recursively defined if the function definition refers to itself. The function definition must have the following two property -

(1) There must be certain arguements, called base values, for which the function does not refer to itself.

(2) Each time the function does refers to itself, the arguement of the function must be closer to a base value.

eg factorial function

(a) if $n = 0$ then $n! = 1$ or $0! = 1$

(b) if $n > 0$ then $n! = n(n-1)!$

fibonacci Series

(a) if $n = 0$ or $n = 1$ then $F_n = n$

(b) if $n > 1$ then $F_n = F_{n-2} + F_{n-1}$

Cardinality -

Two sets A & B are said to be same cardinality, written as $A \cong B$, if there exists a one to one correspondence $f: A \to B$.

Finite - A set A is finite if A is empty or if A has same cardinality as the set $\{1, 2, --- n\}$ for some positive integer n.

Infinite - A set is infinite if it is not finite.

Cardinal number -

The cardinal number of a set A is denoted by $|A|$, $n(A)$, or card(A)

The cardinal number of the infinite set N of ~~post~~ positive integer is $\aleph_0$ (Alph-Naugh)

$(\aleph_0)$

eg (a) $A = |\{x, y, z\}| = 3$

$B = |\{1, 2, 3, 4, 6\}| = 5$

Date
Page

(b) Let $E = \{2, 4, 6, \ldots\}$ the set of even positive integer

The function $f : N \to E$ defined by $f(n) = 2n$ is one-to-one correspondence between the positive integer N & E.

Thus E has the same cardinality as N and so we write $|E| = \aleph_0$

* A set with cardinality $\aleph_0$ is said to be countably infinite.

find the Cardinility of the following set; ⇒

1. $A = \{a, b, c, \dots, y, z\} = |A| = 26$
2. $B = \{1, -3, 5, 11, -28\} = |B| = 5$
3. $C = \{x : x \in N, x^2 = 5\} = |C| = 0$
4. $D = \{10, 20, 30, 40, \dots\} = |D| = N_0$ because $f : N \to D$ defined by $f(n) = 10n$, is one-one correspondence b/w N and D.
5. $E = \{6, 7, 8, 9 \dots\} = |E| = N_0$ because $f : N \to E$, defined by $f(n) = n + 5$, is a one-one correspondence b/w N and E.

Que. Show that Set Z of Integers has Cardinility N_0

$$
\begin{array}{cccccccccc}
N = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots\dots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
Z = & 0 & -1 & 1 & 2 & -2 & 3 & -3 & \dots\dots
\end{array}
$$

The function $f : N \to Z$ is one to one-onto and onto

$$
f(n) = \begin{cases} n/2 & \text{if } n \text{ is even.} \\ (1-n)/2 & \text{if } n \text{ is odd.} \end{cases}
$$

Therefore $|Z| = |N| = N_0$

Case ①

Proof :=> let A be any non-empty set and $P(A)$ be its power set.

(Let $B = \{\{x\} \mid x \in A\}$, the set of all single element subsets of A

Let $f: B \to A$ such that $f\{\{x\}\} = x$

Obviously f is one-one and onto.

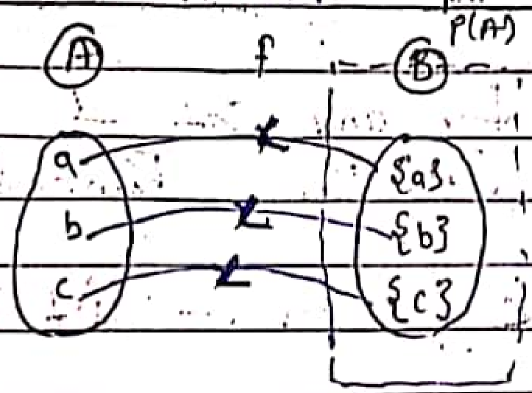
So $B \sim A \Rightarrow |B| = |A|$ — ①

Clearly

$$B \subseteq P(A)$$

$$\Rightarrow |B| \leq |P(A)| \quad \text{— ②}$$

$$|A| \leq |P(A)| \quad \text{— ③}$$



Case ② We have show that $|A| \neq |P(A)|$

let $|A| = |P(A)|$ (contradiction) then there exist a one-one and onto from A to $P(A)$.

But any map which exists from A to $P(A)$ can not be onto as follows:—

To show that map b/w A and $P(A)$ is not onto :->

Consider the set $C = \{x \in A : x \notin g(x)\}$, the set of all those elements x in A whose image under f does not include itself.

Clearly $C \subseteq A$ (by defination of C)

Since $A \in P(A)$ (by Power set def^n)

So, $C \in P(A)$

if f is surjective then it must map something onto C i.e. $\exists\, \alpha \in A$.

such that $g(\alpha) = C$ — ④

:⇒ There are two Cases :⇒

i) if $\alpha \in C$ then by defination C, $\alpha \notin g(\alpha)$

But $g(\alpha) = C$ [by eq^n (④)]

So $\alpha \in g(\alpha)$ — Contradiction.

ii) if $\alpha \notin C$, then α must fail the membership Condition for C, which is $(\alpha \notin g(\alpha))$. So in the Case $\alpha \in g(\alpha)$. But $g(\alpha) = C$, so α has to be in B if $\alpha \in g(\alpha)$. This is Contradiction.

So it can not be onto on our assumption must be wrong but g was any arbitrary function so there is no one one and onto b/w A and $P(A)$.

So $|A| \neq P(A)$ — ⑤

Combined with the result from ①, this Concludes the Proof.

Page:
Date:

Que Prove that $|(0,1)| = |(0,1]|$

Sol^n :=) Since $(0,1) \subset (0,1]$
$f(x) = x$ is one-one from $(0,1) \subset (0,1]$
So $|(0,1)| \leq |(0,1]|$

$g(x) = x/2$ is clearly one-one map $(0,1]$ to
$(0,1/2] \subset (0,1)$

Hence we have one-one function from $(0,1)$ to
$(0,1]$ and $(0,1]$ to $(0,1)$

Hence by Schroder - Bernstein theorem
$|(0,1)| = |(0,1]|$ Proof

classmate
Date
Page

Principle of Mathematical Induction-

Let P be proposition defined on the integer $n \geq 1$ such ~~th~~ that -
(i) $P(1)$ is true
(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every integer $n \geq 1$

Q Let P be the proposition that the sum of the first n odd number is n^2.

i.e;

$$1 + 3 + 5 + 7 + ----+(2n-1) = n^2$$

by mathematical induction-

$$P(1) = 1^2 = 1$$

Suppose $P(n)$ is true Adding eq^n ① $(2n+1)$ to both side of $P(n)$, we get -

$$1 + 3 + 5 + ---------+(2n-1) + (2n+1) = n^2 + (2n+1)$$

$$\text{So } P(n+1) = (n+1)^2$$

We have shown that $P(n+1)$ is true whenever $P(n)$ is true. By PMI P is true for all n.

Q Proof the proposition that sum of the first n (+)ve integer is $\frac{n(n+1)}{2}$.

$$1+2+3+ \cdots + n = \frac{n(n+1)}{2}$$

$$P(1) = 1 \times \frac{2}{2} = 1$$

$$1+2+3+ \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$P(n+1) = \frac{(n+1)(n+2)}{2}$$

We have shown that $P(n+1)$ is true whenever $p(n)$ is true.

Q Proof the proposition P that sum of the square of first n (+)ve integers is $\frac{n(n+1)(2n+1)}{6}$

$$P(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$P(1) = 1 \times \frac{6}{6} = 1$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \cdots$$

$$P(n+1)$$

$$P(n+1) = (n+1)\left[\frac{2n^2+n+6n+6}{6}\right]$$

$$P(n+1) = (n+1)\left(\frac{2n^2+7n+6}{6}\right)$$

$$= (n+1)\left(\frac{2n^2+4n+3n+6}{6}\right)$$

$$P(n+1) = \frac{(n+1)(n+2)(2n+3)}{6}$$

We have shown that $P(n+1)$ is true whenever $P(n)$ is true.

Second form of MI (Mathematical Induction)

Let P be a proposition which is defined on the integer $n \geq 1$ such that -

(1) $P(1)$ is true.
(2) $P(n)$ is true whenever $P(K)$ is true for all $1 \leq K \leq n$

Then P is true for every integer $n \geq 1$

Q Suppose $a \neq 1$ Let P is the proposition on $n \geq 1$ defined by

$$P(n) = 1 + a + a^2 + \text{-----} + a^n = \frac{a^{n+1}-1}{a-1}$$

P(1) is true since

$$1+a = \frac{a^2-1}{a-1}$$

$$1+a+a^2+\dots+a^n+a^{n+1} = \frac{a^{n+1}-1}{a-1}+a^{n+1}$$

$$P(n+1) = \frac{a^{n+2}-1}{a-1}$$

Which is true for P(n+1)

Well Ordering Principle-

Theorem- Let S be a non-empty set of positive integer. Then S ~~contin~~ contains a least element that is, S contain an element a such that $a \leq s$ for every s in S.

In other words-

An ordered set S is said to be well ordered if every subset of S contains a first element.

Q Prove that if $n \in Z$ & n is positive integer then $n \geq 1$ i.e. if $P(n)$ is the statement that $n \geq 1$ then $P(n)$ is true for every $n \in N$.

Sol^n Suppose that there does not exist a (+)ve integer less than 1.
By the well ordering principle there exist a (+)ve integer a such that $0 < a < 1$
Multipling the inequality by the (+)ve integer a we obtain $0 < a^2 < a$
Therefore, a^2 is a (+)ve integer less than a which is also less than a which is also less than 1.

This Contradict a property of being the least possitive integer less than 1. Thus there exist no (+)ve integer less than 1.

Divisibility–

Let a & b integer with $a \neq 0$. Suppose $ac = b$ for some integer c. Then we ~~can~~ say that a divides b or b is divisible by a write $\boxed{a \mid b}$

eg. (a) $3 \mid 6 = 2$

(b) The divisiors ;–
(i) of 1 are ± 1
(ii) of 2 are $\pm 1, \pm 2$
(iii) of 4 are $\pm 1, \pm 2, \pm 4$

classmate

Date
Page

Primes -

A (+)ve integer $p > 1$ is called prime number or a prime if its only division are ± 1 & $\pm p$.

i.e.

if P only has trivial ~~solution~~ divisior.

if $n > 1$ is not prime then n is said to be composite.

Greatest common divisior (GCD)

Suppose a & b are integers, not both zero & integer d is called common divisior of a & b if d divides both a & b. that is ~~[illegible]~~ $d|a$ & $d|b$.

The largest common division of a & b is denoted by $gcd(a, b)$

eg $gcd(12, -18) = 6$

$gcd(12, -16) = 4$

(i) for any integer a, $gcd(1, a) = 1$

(ii) for prime number P, $gcd(P, a) = P$

or

$gcd(P, a) = 1$

(iii) Suppose a is (+)ve then
$a|b$ if & only if $gcd(a,b) = a$

* **Euclidean Algorithm -**

Let a, b be are integer & let $d = gcd(a,b)$
the euclidean algorithm is find x & y
such that [$d = ax + by$]

Q Let $a = 540$
$b = 168$

$d = ax + by$

$gcd(540, 168) = 12$

$$168 \overline{)540} \; 3 \quad \frac{504}{36}$$

$$36 \overline{)168} \; 4 \quad \frac{144}{24}$$

$$24 \overline{)36} \; 1 \quad \frac{24}{12}$$

$$12 \overline{)24} \; 2 \quad \frac{24}{\times\times}$$

$$12 = 540x + 168y$$

The first three quotient yield the equation

eqn ① $540 = 3 \times 168 + 36$ or $36 = 540 - 3(168)$

eqn ② $168 = 4(36) + 24$ or $24 = 168 - 4(36)$

eqn ③ $36 = 1(24) + 12$ or $12 = 36 - 1(24)$

Page

eqⁿ ③ tells us that 12 is the linear Combination 36 & 24. We replace 24 by use of eqⁿ ②

eqⁿ ④

$$12 = 36 - 1\,[168 - 4(36)]$$

$$12 = 5(36) - 2(168)$$

36 is replaced by eqⁿ ①

$$12 = 5[540 - 3(168)] - 1\,(168)$$

$$12 = 5(540) - 16(168)$$

Therefore

$$\boxed{x = 5 \quad \& \quad y = 16}$$

~~Fundamental~~

Q Let $a = 8316$ $b = 10920$

(1) find $d = \gcd(a, b)$

(2) find integer m & n such that $d = ma + nb$

(3) find $LCM(a, b)$

$$LCM = \frac{|a \cdot b|}{\gcd(a,b)} = \frac{(8316)(10920)}{84} = 1081080$$

fundamental theorem of Airthmatic

Every integer $n>1$ can be expressed as uniquely (except for order) as a product of primes.

The primes is the ~~function~~ factorization of n need not to be distinct. n is expressed as uniquely

$$n = P_1^{M_1} \cdot P_2^{M_2} \text{ - - - - } P_n^{M_n}$$

Where M_i are positive and $P_1 < P_2 < \text{- - -} \text{- - -} P_n$. This is called cannonical factorization of n.

example -

Q Let $a = 2^4 \cdot 3^3 \cdot 7 \cdot 11 \cdot 13$
$b = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17$

find $d = \gcd(a, b) \Rightarrow d = 2^3 \cdot 3^2 \cdot 11$

$M = \text{lcm}(a, b) \Rightarrow M = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

Unit - 2

Counting

Law of addition - Suppose some elements E can occur in m ways & event F can ocur in n ways & suppose both events can not simontaneously. Then E or F can occur in $(m+n)$ ways.

eg Suppose there are 8 ~~ma~~ male proffexors & 5 female proffesors teaching a calcular class. Then a student can choose a calculas proffesor in $(8+5) = 13$ ways.

Law of multiplication - Suppose there is an event E which can occur in m ways & independent of this event, there is second event F which can occur in n ways. then combination of E & F can occur in $(m \times n)$ ways.

eg Suppose a licsense plate contains two letters followed ~~by~~ by three digit with the first digit not zero. How many different licsenses plates can be printed.

Soln each letter can be printed in 26 different ways, the first digit in 9

Ways & each of the other two digit in 10 ways

letter digit

26 26 9 10 10

$$26 \times 26 \times 9 \times 10 \times 10$$

$$= 608400$$

PERMUTATION -

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

n = Total no. of events
r = required no. of events.

Q Suppose repitition are not permitted.

(a) How many three digits number can be formed from six digit (2, 3, 5, 6, 7, 9)

6 5 4

$$6 \times 5 \times 4 = 120$$

(b) How many of three number are less than 400.

3 9 8

$$3 \times 9 \times 8 = 216$$

(c) In above question (a) How many are even.

		2, 6
4	5	2

$$5 \times 4 \times 2 = 40$$

* Permutation with repitition -

$$P(n, n_1, n_2 - - - -) = \frac{n!}{n_1!\, n_2! - - - - n_n!}$$

Q How many seven letter words can be formed using the letters of word BENZENE.

$$\Rightarrow \frac{7!}{3!\, 2!} = 10 \times 6 \times 7 = 420$$

Q find the number of ways that Party of 7 Persons can arrange themselves
(a) in a row of 7 chairs $= 7!$
(b) around a circular table $= 6!$

Combination-

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

Q How many Comittee of 3 can be form from the 8 people.

Q A farmer buys 3 cows, 2 pigs & 4 hens from a man who has 6 cows, 5 Pigs & 8 hens. How many choices the farmer a have.

Sol^n (2)

$${}^{6}C_{3} \times {}^{5}C_{2} \times {}^{8}C_{4}$$

$$20 \times 10 \times 70$$
$$= 14000$$

(1)

$${}^{8}C_{3} \Rightarrow 56$$

* The pigeonhole principle -

if n pigeonhole are occupied by $(n+1)$ or more pigeons then at least one pigeonhole is occupied by more than one pigeon.

Generalized pigeonhole principle -

If n pigeonhole are occupied by $kn+1$ or more pigeons when k is an integer then at least one pigeonhole is occupied by $(k+1)$ or more pigeons.

Q find the minimum number of students in a class to be sure that 3 of them are borned in same month.

Here

$n = 12$ months is the pigeonhole

$k+1 = 3$

or $\boxed{k=2}$

So $kn+1$

$2 \times 12 + 1 = \boxed{25}$

Q Suppose the department contains 13 professors then 2 of the proffesors born in the same months.

Sol^n 2 of the professor ⇒ pigeon

Same month ⇒ pigeon hole

Q Suppose a laundary bag contains many red, white, & blue shocks then one need only grab 4 shocks to be sure of getting a pair with same colour.

pair of same colour ⇒ pigeon hole

4 shocks ⇒ pigeon

Q Suppose a laundary bag contains many red, white & blue shocks. find the minimum number of shocks. Then one needs to choose in order to get two pair (4 shocks) of the same colour.

$n = 3$ colour (pigeon hole)

$K+1 = 4$

$\boxed{K=3}$

$Kn+1 \Rightarrow 3\times3+1 = 10$

Date
Page

Q find the minimum number of the student needed to gurantee that 5 of them belongs to the same class. (freshman, Sophomore, Junior, Senior)

Here

$n = 4$ classes are pigeonhole

& $K+1 = 5$

$K = 4$

$Kn+1 \Rightarrow 4\times4+1 = 17$

Q find the minimum number n of integers to be selected from $S = \{1, 2, 3, \ldots 9\}$

(a) The sum of two of n integers is even.

(b) The difference of two of n integers is 5.

Soln (a)

{sum of even numbers}, {sum of odd numbers}, {sum of same numbers}.

Total set of numbers = 3

(b) $\{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}$ $\{5\}$

$\Rightarrow 5$

Q find the minimum number of elements that one need to take from the set S.

$S = \{1, 2, 3, \ldots 9\}$ to be sure that two of the numbers added up to 10.

$\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5, 5\} \Rightarrow 5$

pigeon holes are $\Rightarrow 5 + 1 = 6$