Unit 1 : Error

① When performing computations with algebraic operations among approx. numbers, we naturally carry to some extent the errors of the original data into final result. Such errors are called inherent errors.

E.g. Let $x = 0.3333$ and $y = 3.1416$ be two approx. no. for $1/3$ and π. If we perform algebraic operation the error will come in final result.

② When the nos. like $1/3$, $22/7$, $5/9$ etc. whose decimal have infinite digits are given in calculations we only take few digits after decimal point and this is how round off error is involved.

E.g. The irrational no. π equals approximately to 3.14, rounded to two decimal places or three significant digits.

③ This error occurs when mathematical functions likes $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots \infty$ and

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \infty$ are used in calculations.

In the equations when we do not take remaining terms an error is introduced. It is called truncation error.

The magnitude of the difference between the true value of the quantity and individual measurement value is called the absolute error of the measurement. In other words the absolute value of the deviation is known as the absolute error.

eg.
$$|\Delta a_1| = |a_{mean} - a_1|.$$
$$|\Delta a_2| = |a_{mean} - a_2|.$$
$$\vdots$$
$$|\Delta a_n| = |a_{mean} - a_n|.$$

The absolute error is always positive.

The relative error is the ratio of the mean absolute error Δa_{mean} to the mean value a_{mean} of the quantity measured.

$$\text{Relative Error} = \frac{\Delta a_{mean}}{a_{mean}}.$$

When the relative error is multiplied by 100 it becomes percentage error.

$$\delta a = \Delta a_{mean} \times 100.$$

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(7) Round off 3.6252782 to 3 decimal places.

Ans. 3.625.

(8) No. of significant digits in 35.00537 is 7.

(9) Maximum absolute error is 2.62×10^{-4}.

(10) $\delta f \simeq \frac{dy}{dx_1} \delta x_1 + \frac{dy}{dx_2} \delta x_2$.

(11) Round off errors depend on the fact that practically each number in a numerical computation must be rounded to a certain no. of digits. e.g. the value of π is rounded off to 3.14 upto two decimal places.

Truncation Errors arise when an infinite process is replaced by a finite one.
e.g. $x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$.

(12) A root of the equation $f(x) = 0$ is that value of x where the graph of the function $y = f(x)$ intersects the x-axis.

(13) If $f(x)$ contains some other functions such as trignometric, logarithmic, exponential etc. then $f(x) = 0$ is called transcendental equation.

Elementary transcendental functions are exponential, logarithmic, trignometric function. If transcendental functions are considered as functions of a complex variable, then their characteristic feature is the presence of atleast one singularity in addition to poles and branch points of finite order. For example the functions e^z, $\cos z$, $\sin z$ have a significant singular point $z = \infty$ and also branch points of infinite order $z = 0$ and $z = \infty$.

The polynomial $f(x) = 0$ is called polynomial or algebraic equation of degree n.

e.g. – $3x^3 + x + 1 = 0$, $x^2 + 6x + 5 = 0$.

If $f(x) = 0$ contains functions like trigonometric, exponential, logarithmic then $f(x) = 0$ is called transcendental equation.

e.g. – $e^x + \sin x + x = 0$, $\log x + \tan x = 0$.

If function $f(x)$ is continuous in the interval $[a, b]$ and $f(a)$, $f(b)$ have different signs then the equation $f(x) = 0$ has atleast one root between $x = a$ and $x = b$.

We apply this to find the roots of

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(16) Let a equation be $f(x) = 0$

For two values (y, z), $f(y)$ is $-ve$ & $f(z)$ is $+ve$ (i.e., $f(y) \cdot f(z) < 0$) i.e., root lies between y and z. Bisect the interval (y, z) and find $x_0 = \frac{y+z}{2}$. So, we get three possibilities.

(i) $f(x) = 0$ $\Rightarrow$ root is at x_0.

(ii) $f(x) > 0$ $\Rightarrow$ root is between x_0 and y

Find next approximation $x_1 = \frac{x_0 + y}{2}$.

(iii) $f(x) < 0$ $\Rightarrow$ root is between x_0 and z.

Find next approximation $x_1 = \frac{x_0 + b}{2}$.

(17) For bisection method,

$$n \geq \frac{\log(b-a) - \log \epsilon}{\log 2}$$

where, ϵ is ~~[illegible]~~ accuracy, b and a are intervals, n is no. of iterations.

$f(x) = x^3 - 2x - 5.$

$f'(x) = 3x^2 - 2.$

For maxima or minima,
$f'(x) = 0.$

$\therefore \quad 3x^2 - 2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{\frac{2}{3}}.$

$f''(x) = 6x.$

$f''\left(\sqrt{\frac{2}{3}}\right) > 0, \quad x = \sqrt{\frac{2}{3}}$ is minima.

$f''\left(-\sqrt{\frac{2}{3}}\right) < 0, \quad x = -\sqrt{\frac{2}{3}}$ is maxima.

$f\left(\sqrt{\frac{2}{3}}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} - 5 < 0$

$f\left(-\sqrt{\frac{2}{3}}\right) = \frac{4}{3}\sqrt{\frac{2}{3}} - 5 < 0$

Hence, point of intersection lies below x-axis.

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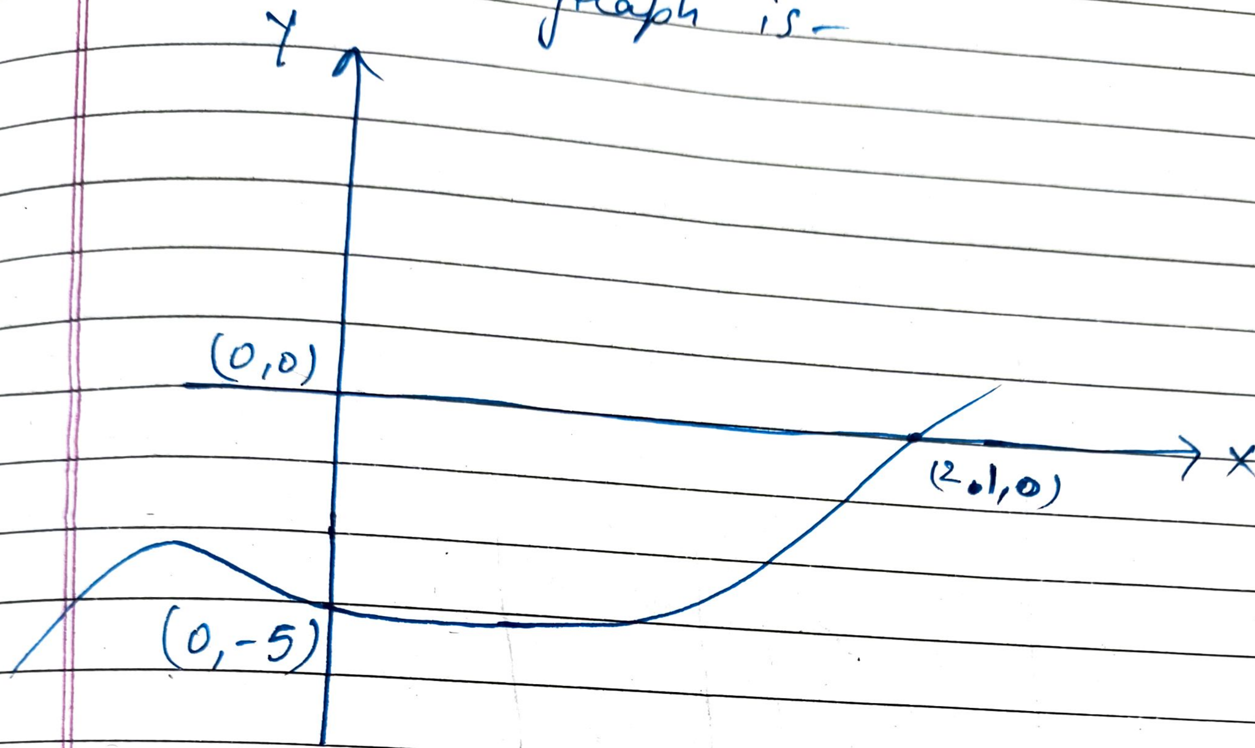
Hence, the graph is-



Hence, the root lies in (2, 3).

(19) Iterative Method on the basis of convergence on descending order.

⇒ Newton-Raphson Method → Order - 2.

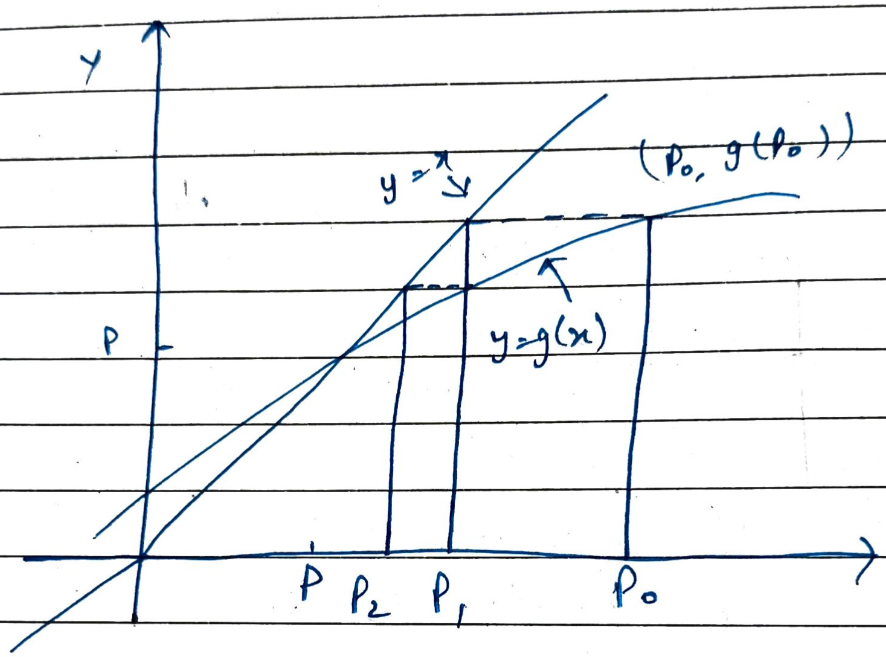
Secant Method → Order - 1.62.

Regula - Falsi Method → Order - 1.

(20) Under fairly general conditions, it can be shown that if the initial guess is close to the given solution, then the Newton-Raphson method converges quadratically

(21) If we do not have a good starting point or interval then the secant method fails. If $f(a_n)f(b_n) \geq 0$ at any point in the iteration then also secant method fails.

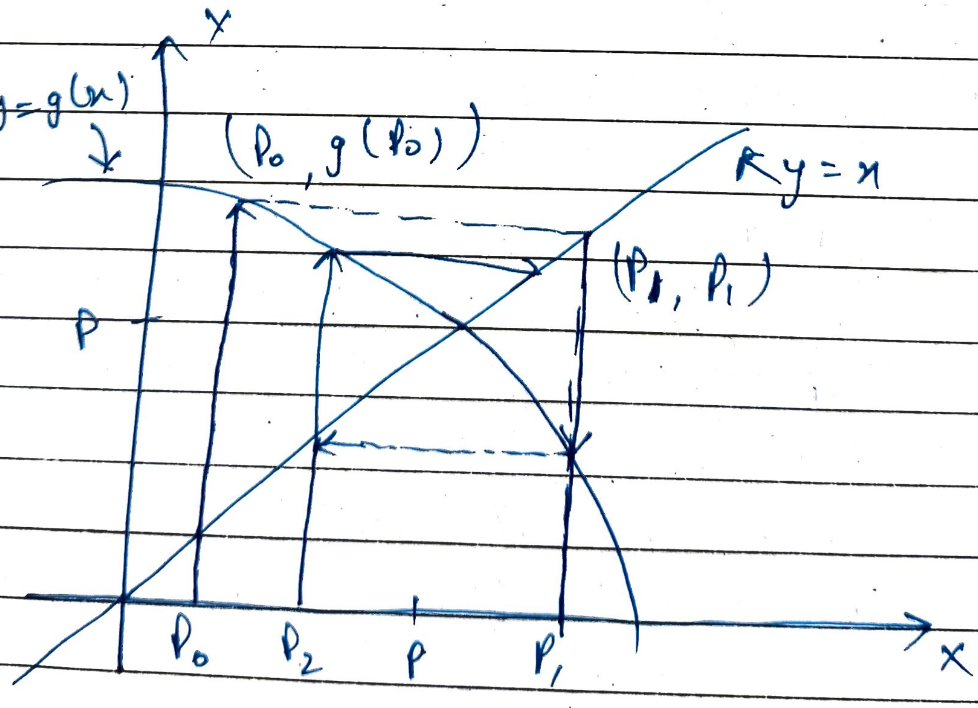
(22) (a)



~~Solving of non-linear equations $f(x)=0$~~

Monotone Convergence when $0 < g'(P) < 1$.

(b)



Oscillating Convergence when $-1 < g'(P) < 0$.

In fig.(a), if we draw a tangent at the point of intersection for the curve $g(x)$, the angle θ_1 (let) made between tangent and x-axis is less than 45°, so slope $< \tan 45$. Hence, $g'(p) < 1$.

In fig:(b), if we draw a tangent at the point of intersection for curve $g(x)$, the angle θ_2 (let) made between tangent and x-axis is greater than 90°. So,

slope $\tan(\theta_2) = \tan(90+\phi) = -\cot\phi, 0<\phi<$
But this does not imply $\cot\phi < 1$ for $0 < \phi < 90^\circ$. So, $\tan(\theta_2) > -1$ is not true always.

So, in fig (b) we $\theta_2 > 135^\circ$, so that $-1 < \tan\theta_2 < 0$.

Hence, $-1 < g'(p) < 1$.

23 Sufficient condition for convergence of iterations.

It is not sure whether the sequence of approximations $x_1, x_2, \ldots\ldots x_n$ always converge to the same no. which is a root of the given equation or not. We have to choose the initial approx. x_0 suitably so that the successive approximations $x_1, x_2, \ldots\ldots x_n$ converge to the root α. The given theorem helps in choosing x_0.

If, (i) α be a root of $f(x)=0$ which is equivalent to $x = \phi x$.

(ii) I, be any interval containing point $x=\alpha$.

(iii) $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2, \ldots\ldots x_n$ will converge to the root α provided the initial approx. x_0 is chosen in I.

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(24) Direct Methods	Iterative Methods
① Direct method always use a formula to solve.	Iterative methods iterate until convergence for the solution.
② In direct method the solution comes in a finite no. of steps.	In iterative method are not expected to terminate in a no. of steps. These methods form approximations that converge to the solution only in limit.

Direct method — e.x.:-

$x^2+29x+100=0$

$\Rightarrow x^2+25x+4x+100=0$

$\Rightarrow x(x+25)+4(x+25=0)$

$\Rightarrow (x+4)(x+25)=0$

$\Rightarrow x=-4,-25$. Ans.

Iterative method — ex.:- Let $f(x)=x^3-x-1$

$\Rightarrow x=\sqrt[3]{x+1}$

$\Rightarrow \phi(x)=\sqrt[3]{x+1}$.

x	0	1	2
$f(x)$	-1	-1	5

$f(1)=-1<0,\ f(2)=5>0$

Root between 1 & 2.

$x_0=\frac{1+2}{2}=1.5$

$x_1=\phi(1.5)=1.35721$

$x_2=\phi(1.35721)=1.33086$

$x_3=\phi(1.33086)=1.32588$

$x_4=\phi(1.32588)=1.32494$

$x_5=\phi(1.32494)=1.32476$

Approximation root is

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The condition for convergence of Gauss-Seidel method is that the co-efficients matrix should be diagonally dominant. A diagonally dominant matrix is one in which the magnitude of the diagonal term in each row is greater than the sum of the other elements in that row.

If the co-efficients matrix is not diagonally dominant, rearrange the rows in such a way so as to make it diagonally dominant. Diagonally dominant matrices will have a faster rate of convergence.

(26) $0.77729 = 0.777$

$0.0022218 = 0.002$

(27) $\sqrt{3} = 1.732050$
$\sqrt{5} = 2.236067$
$\sqrt{7} = 2.645751$

Hence, Sum $= 6.614$

Absolute Error $= \frac{1}{2} \times 10^{-3} = 0.0005.$

Total Absolute Error $= 0.0005 + 0.0005 + 0.0005$
$= 0.0015.$

The total absolute error shows that the sum is correct to 3 significant figures only.

We take $S = 6.61$

Relative Error $= \frac{0.0015}{6.61} = 0.0002.$

(28) (a) Round off to four significant figures $= 865300.$
$\therefore X = 865250, \; X' = 865300.$

Error $= X - X' = 865250 - 865300 = -50.$
$E_a = |X - X'| = |50| = 50.$
$E_R = \frac{|X - X'|}{|X|} = \frac{50}{865250} = 5.77 \times 10^{-5}$

$10^{-5} \times 100 = 5.77 \times 10^{-3}$

No. round off to four significant digits = 37.46

$X = 37.46235$, $X' = 37.46$

$E = X - X' = 0.00235$
$E_a = |X - X'| = 0.00235$
$E_r = \left| \frac{X - X'}{X} \right| = \frac{0.00235}{37.46235} = 6.2729 \times 10$
$E_p = 6.2729 \times 10^{-3}$.

$X = 0.00545828 = 0.545828 \times 10^{-2}$

After truncating $X' = 0.545 \times 10^{-2}$.

$\therefore E_a = |X - X'| = 0.000828 \times 10^{-2}$
$= 0.828 \times 10^{-5}$.

After rounding off, $X' = 0.546 \times 10^{-2}$.

$\therefore E_a = |X - X'| = |0.545828 - 0.546| \times$
$= 0.000172 \times 10^{-2}$
$= 0.172 \times 10^{-5}$.

(30) $u = \dfrac{4x^2y^3}{z^4}$. Error in u = 0.001

$x = y = z = 1.$

Solving,

$\dfrac{du}{dx} = \dfrac{8xy^3}{z^4}, \quad \dfrac{du}{dy} = \dfrac{12x^2y^2}{z^4}, \quad \dfrac{du}{dz} = -\dfrac{16x^2y^3}{z^5}$

$\therefore \delta u = \left|\dfrac{8xy^3}{z^4}\delta x\right| + \left|\dfrac{12x^2y^2}{z^4}\delta y\right| + \left|\dfrac{-16x^2y^3}{z^5}\delta z\right|$

$(\delta u)_{max} = 8(0.001) + 12(0.001) + 16(0.001)$

$= 0.036.$

$(E_r)_{max} = \dfrac{(\delta u)_{max}}{u} = \dfrac{0.036}{4} = 0.009.$

(31) $A = \pi r^2 \quad \because r = d/2.$

$A = \pi\left(\dfrac{d}{2}\right)^2 = \dfrac{\pi d^2}{4}.$

Taking log both sides,

$\log A = \log \pi + 2\log d - \log 4.$

$\dfrac{\delta A}{A} = 0 + \dfrac{2\,\delta d}{d} - 0$

$\dfrac{\delta A}{A} \times 100 = \dfrac{2\,\delta d}{d} \times 100.$

$0.1 = 2\dfrac{\delta d}{d} \times 100 \Rightarrow \dfrac{\delta d}{d} \times 100 = 0.05.$

$\delta A = \dfrac{\delta A}{\delta d}\cdot \delta d$

$= \dfrac{\pi 2d}{4}\cdot \delta d$

$\delta A = \dfrac{\pi d\,\delta d}{2}$

$\dfrac{\delta A}{A} = \dfrac{2\,\delta d}{d}.$

(32) $\therefore T = 2\pi\sqrt{\frac{l}{g}}$

Taking log both sides,

$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g.$

$\Rightarrow \frac{dT}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - 0.$

$\Rightarrow \frac{dT}{T}\times 100 = \frac{1}{2}\frac{dl}{l} - 0.$

$\Rightarrow \frac{dT}{T}\times 100 = \frac{1}{2}\times 0.01\times 100$

$\Rightarrow \frac{dT}{T}\times 100 = 0.5\%$. Ans.

(33) $f(0) = -1.$

$f(1) = 1.123189156$

$\therefore$ Root lies b/w 0 and 1

$a = 0, \quad f(a) = -1$

$b = 1, \quad f(b) = 1.123189156$

Using,

$$x = \frac{b\,f(a) - a\,f(b)}{f(a) - f(b)}$$

$$x_1 = \frac{-1}{-1-1.123189156} = 0.470989594$$

$f(x_1) = 0.265158816$

...omes b to find the next point.

$$X_2 = \frac{-0.470989594}{-1-0.26515816} = 0.37227051$$

$$f(X_2) = 0.02953368$$

Now, x_2 becomes b to find the next point

$$X_3 = \frac{-0.02953368}{-0.02953368-1} = 0.36159743$$

$$f(X_3) = 2.940998193 \times 10^{-3}$$

Now, X_3 becomes b to find the next point.

$$X_4 = \frac{-0.36159743}{-1-(2.940998193 \times 10^{-3})} = 0.360537403$$

$$f(X_4) = 2.89448416 \times 10^{-3}$$

Now, X_4 becomes b to find the next point.

$$X_5 = \frac{-0.360537403}{-1(-2.89448416 \times 10^{-3})}$$ ~~= 0.360433076~~

$$= 0.360433076$$

$$f(X_5) = 2.84536596 \times 10^{-5}.$$

Now, X_5 becomes b to find next point.

$$X_6 = \frac{-0.36043307 6}{-1-(2.84536596 \times 10^{-5})} = 0.36042282.$$

$\therefore$ The true root corrected to 4 decimal

$f(x) = x \cdot \log_{10} x - 1.9$

$x_1 = 4 \quad f(4) = 0.5082 \quad f(x_1) = 0.5082$

$x_0 = 3 \quad f(3) = -0.4686 \quad f(x_0) = -0.468$

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$$x_2 = x_0 - \frac{(x_1 - x_0)\, f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = 3 - \frac{(4-3) \times (-0.4682)}{(0.5082 + 0.4682)}$$

$$= 3 + \frac{0.4682}{0.9068}$$

$$= 3 + 0.4745$$

$x_2 = 3.4745$

$f(x_2) = -0.02066$

$$x_3 = x_1 - \frac{(x_2 - x_1)\, f(x_1)}{f(x_2) - f(x_1)}$$

$$= 4 - \frac{(3.4745 - 4) \cdot (0.5082)}{-0.02066 - 0.5082}$$

$$= 4 - \frac{(-0.2670)}{-0.52886}$$

$$= 4 - 0.5050$$

$x_3 = 3.49495$

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(35) $f(x) = x^3 - x^2 - 1 = 0$

$f(0) = 0^3 - 0^2 - 1 = -1.$

$f(1) = 1^3 - 1^2 - 1 = 1.$

So, root is between 0 & 1.

$x^3 - x^2 - 1 = 0.$

$\Rightarrow x^2(x-1) - 1 = 0.$

$\Rightarrow x^2 = \frac{1}{x-1} \Rightarrow x = \sqrt{\frac{1}{x-1}}.$

Let $x = 0.5,$

$x_1 = f(x_0) = \frac{1}{\sqrt{0.5-1}} = -\frac{1}{\sqrt{0.5}}.$

$= -\frac{1}{0.7071}$

$= 1.4142$

$x_2 = f(x_1) = \frac{1}{\sqrt{1.4142-1}}$

$= \frac{1}{\sqrt{0.4142}} = \frac{1}{0.6435}$

$= 1.554$

$$x_3 = f(x_2) = \frac{1}{\sqrt{x_2 + 1}}$$

$$= \frac{1}{\sqrt{1.554 + 1}}$$

$$= \frac{1}{0.74416} = 1.2467$$

$$x_4 = f(x_3) = \frac{1}{\sqrt{x_3 + 1}}$$

$$= \frac{1}{\sqrt{1.2467 - 1}} = 1.2468$$

$\therefore$ The root is 1.246.

$C = 4e^{-2t} + e^{-0.1t}$

Here, bacteria conc. = 0.5

$0.5 = 4e^{-2t} + e^{-0.1t}$

$f(t) = 4e^{-2t} + e^{-0.1t} - 0.5$

$f'(t) = -8e^{-2t} - 0.1e^{-0.1t}$.

Let $t1 = 1$.

$t1$	$f(t1)$	$f'(t1)$	$t2 = t1 - \frac{f(t_1)}{f'(t1)}$
1	0.946179	-1.17317	1.806517189
1.806517189	0.442606	-0.29923	3.285657224
3.285657	0.225555	-0.0832	5.996816144
5.996816144	0.049011	-0.05495	6.888769184
6.888769	0.002144	-0.05022	6.931458438
6.931456	4.58 E-06	-0.05001	6.931548088

$x^3 - 9x + 1 = 0$.

$f(x) = x^3 - 9x + 1$

$f(0) = 0 - 0 + 1 = 1 > 0$.

$f(1) = 1 - 9 + 1 = -7 < 0$.

$\therefore$ Root lies between 0 & 1.

$x^3 - 9x + 1 = 0 \Rightarrow 9x = x^3 + 1$.

$\Rightarrow x = x^3 + 1$.

$$x_1 = f(x_0) = \frac{(0.5)^3+1}{9} = 0.125$$

$$x_2 = f(x_1) = \frac{(0.125)^3+1}{9} = 0.1113.$$

$$x_3 = f(x_2) = \frac{(0.111)^3+1}{9} = 0.1112.$$

So, the x_2 and x_3 are same.

Upto two decimal point root is 0.11.

(38) If (i) α be the root $f(x)=0$ which is equivalent to $x = \phi x$.

(ii) I be any interval containing $x=\alpha$.

(iii) $|\phi'(x)| < 1$ for all x in I, then sequence of approximations $x_0, x_1, x_2, \dots x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

(39) $2x+2y+z=6, \quad 4x+2y+3z=4, \quad x-y+z=0.$

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}.$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}.$$

$R_2 \rightarrow R_2 - \frac{1}{2} R_1 \quad , \quad R_3 \rightarrow R_3 - \frac{1}{4} R_1$

$$\begin{bmatrix} 4 & 2 & 3 \\ 0 & 1 & -1/2 \\ 0 & -3/2 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 4 & 2 & 3 \\ 0 & -3/2 & 1/4 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$

$R_3 \rightarrow R_3 + \frac{2}{3} R_2.$

$$\begin{bmatrix} 4 & 2 & 3 \\ 0 & -3/2 & 1/4 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 10/3 \end{bmatrix}.$$

By Back Substitution Method,

$$4x + 2y + 3z = 4$$

$$-\frac{3}{2} y + \frac{1}{4} z = -1$$

$$-\frac{1}{3} z = \frac{10}{3}$$

$z = -10 \quad , \quad y = -1, \quad x = 9.$ Ans.

(40) Let,

$f(x) = \cos x - xe^{x}$.

$f(0) = 1$

$f(1) = -2.18$.

$\therefore$ Root is between ~~0~~ of 1.

First approximation $= \frac{0+1}{2} = 0.5$.

$f(0.5) = 0.5322$

$\therefore$ Root is between 0.5 & 1

Second approximation $= \frac{0.5+1}{2} = 0.75$.

$f(0.75) = -0.85606$

$\therefore$ Root is between 0.5 & 0.75.

$x_3 = \frac{0.5+0.75}{2} = 0.625$

$f(0.625) = -0.35669$.

$\therefore$ Root is between 0.5 & 0.625

$x_4 = \frac{0.5+0.625}{2} = 0.5625$.

$f(0.5625) = -0.1412$

$\therefore$ Root is between 0.5 & 0.5625.

$x_5 = \frac{0.5 + 0.5625}{2} = 0.53125.$

$f(0.53125) = -0.04151.$

$\therefore$ Root is between 0.5 & 0.53125

$x_6 = \frac{0.5 + 0.53125}{2} = 0.51563.$ ~~$f(0.5156) = -0.04151.$~~

~~$\therefore$ Root is between 0.5 & 0.53125.~~

$f(0.5156) = 0.00640.$

$\therefore$ Root is between 0.51563 & 0.53125.

$x_7 = \frac{0.51563 + 0.53125}{2} = 0.52344.$

$f(0.5234) = -0.472$

$\therefore$ Root is between 0.5156 & 0.5234.

$x_8 = \frac{0.5156 + 0.5234}{2} = 0.5196.$

$f(0.5195) = 0.00564$

$\therefore$ Root is between 0.5195 & 0.5234.

$x_9 = \frac{0.5195 + 0.5234}{2} = 0.51965.$

$\because$ $x_8 \approx x_9$ and are correct to three dekimal places.

$\therefore$ Root $= 0.519.$ Ans.

$x^3 - 4x + 9 = 0.$

$4x = x^3 + 9$

$x = \frac{x^3+9}{4}.$

$f(0) = \frac{0^3+9}{4} = \frac{9}{4} = 2.25.$

$f(-3) = \frac{(-3)^3+9}{4} = -\frac{18}{4} = -4.5.$

$\therefore$ Root is between 2.25 & -4.5.

first approximation $= \frac{2.25 - 4.5}{2}$

$= -1.125.$

$f(-1.125) = 1.894$

$\therefore$ Root is between -1.125 & 2.25

Second approximation $= \frac{-1.125 + 2.25}{2}$

$= 0.5625$

$f(0.5625) = 2.294.$

$\therefore$ Root is between 0.5625 & -1.125.

$x_3 = \frac{0.5625 - 1.125}{2} = -0.2812.$

$f(-0.2812) = 2.244.$

$\therefore$ Root is between -1.125 & -0.2812.

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(41) $x_4 = \frac{-1.125 - 0.2812}{2} = -0.421$

$f(-0.421) = 2.248$

$\therefore$ Root is between -1.125 & -0.421.

$x_5 = \frac{-1.125 - 0.421}{2} = -0.773$. Ans

$f(-0.773) = 2.134$.

$\therefore$ Root is between -1.125 & 2.134.

$x_6 = \frac{-1.125 + 2.134}{2} = 0.504$

$f(0.504) = 2.282$.

(42) Suppose x_r is a root of $f(x)=0$ and x_n is an estimate of x_r, $|x_r - x_n| = \delta << 1$

By Taylor Series,

$$0 = f(x_r) = f(x_n + \delta) = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(\xi)}{2}(x_r - x_n)^2. \quad -(1)$$

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$f(x_n) = f'(x_n)(x_n - x_{n+1}). \quad -(2)$$

Putting (2) in (1),

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(\xi)}{2}(x_r - x_n)^2$$

Let $e_n = (x_r - x_n)$, $e_{n+1} = x_r - x_{n+1}$.

where e_n & e_{n+1} are errors,

$$\therefore \quad e_{n+1} = \frac{f''(\xi)}{2f'(x_n)} \sim e_n^2$$

$$\Rightarrow e_{n+1} \propto e_n^2 .$$

$\therefore$ Newton - Raphson Method is said to have quadratic convergence.

Hence, Proved

$\cancel{x = e^{-x}}$

$\cancel{x_0 = 0.5}$

$\cancel{x_1 = f(x_0) \Rightarrow 0.5 = e^{-0.5}}$

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(43) $f(x) = x - e^{-x}$.

$f(0) = -1$.
$f(1) = 0.632$.

$\therefore$ Root lies between 0 & 1.

Rewriting the equation,

$x = e^{-x} = \phi(x)$.

$\phi'(x) = -e^{-x}$.

$|\phi'(x)| < 1$, when $0 < x < 1$.

Hence,

$x_0 = 0.5$

$x_1 = e^{-0.5} = 0.60653$.

$x_2 = e^{-0.6065} = 0.54525$.

$x_3 = e^{-0.5452} = 0.57972$.

$x_4 = e^{-0.5797} = 0.57006$.

$x_5 = e^{-0.5600} = 0.57125$

$x_6 = e^{-0.5712} = 0.57187$

$x_7 = e^{-0.5718} = 0.57189$.

$\therefore$ Positive root is 0.57189.

Rate of convergence is a quantity that represent how quickly the sequence approaches its limit.

Let f be continuous on $[a,b]$ and f' be continuous on (a,b). Let $k<1$ so that $|f'(x)| \leq k$ for all x in (a,b).

So, if $f'(r) \neq 0$, the sequence converges linearly to the fixed point.

Now;

$$|x_1 - r| = |f(x_0) - f(r)| \leq |f'(x)|\,|x_0 - r| \leq k\,|x_0 - r|.$$

where x is between x_0 & r.

$$|x_n - r| \leq |f'(x)|\,|x_{n-1} - r| \leq k\,|x_{n-1} - r| \leq k^n |x_0 - r|$$

with $0 \leq k < 1$ so that $k^n \to 0$ as $n \to \infty$.

Therefore, $|x_n - r| \to 0$ an $n \to \infty$. ~~An important~~

Now, we show that sequence converges linearly if $f'(r) \neq 0$:

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = \lim_{n\to\infty} \frac{|f(x_n) - f(r)|}{|x_n - r|}$$

$$= \lim_{n\to\infty} |f'(c_n)|.$$

where c_n is between x_n & r

Note that, because f' is continuous on our interval that contain all the c_n's and r.

$$\lim_{n\to\infty} |f'(c_n)| = \left| f'\left(\lim_{n\to\infty} c_n\right)\right| = |f'(r)|.$$

Hence, Proved.

$x^4 - x - 10 = 0 \Rightarrow x^4 = x + 10.$

$x^4 \approx 10 \Rightarrow x \approx \pm\sqrt[4]{10} \approx \pm 1.75.$

The equation has two real roots.

Using N-R method,

Let $f(x) = x^4 - x - 10 \Rightarrow f'(x) = 4x^3 - 1.$
We want to determine the value of x for which $f(x) = 0$.

Let first approx. of x be x_1.

$x_1 = 1.75.$

x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1.75	-2.3710938	20.4375	1.8660168
[illegible]60168	0.2584379	24.990022	1.8556752
[illegible]6752	0.0022262	24.560295	1.8555845
[illegible]845	1.697E-07	24.55655	1.8555845
[illegible]	0	24.555655	1.8555845

Let $f(x) = 3x - \cos x - 1$.

$\therefore f'(x) = 3 + \sin x - 0$.

when $x = 0$, $f(0) = 3(0) - \cos(0) - 1 = -2$.

when $x = 1$, $f(1) = 3(1) - \cos(1) - 1 = 1.459$

$\therefore$ Root is between 0 & 1.

Let $x_0 = 0$,

$$x_{n+1} \quad - \frac{f(x)}{f'(x)}.$$

$$\frac{x_n - 3x_n - \cos x_n - 1}{3 + \sin x_n}.$$

$$\frac{x_n(3 + \sin x_n) - (3x_n - \cos x_n - 1)}{3 + \sin x_n}.$$

$$\frac{3x^n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n}$$

$$\therefore x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}.$$

1st iteration, $n = 0$,

$x_1 = \underline{0 + \cos 0 + 1} \quad = 0.66$

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2^{nd} iteration, $n=1$,

$$x_2 = \frac{0.6667\sin(0.6667)+\cos(0.6667)+1}{3+\sin(0.6667)}$$

$= 0.6075.$

3^{rd} iteration, $n=2$,

$$x_3 = \frac{0.6075\sin(0.6075)+\cos(0.6075)+1}{3+\sin(0.6075)}.$$

$= 0.6071.$

4^{th} iteration, $n=3$,

$$x_4 = \frac{0.6071\sin(0.6071)+\cos(0.6071)+1}{3+\sin(0.6071)}$$

$= 0.6071.$

Hence, Root of $3x-\cos x-1=0$ is 0.6071

The eqn. is $x^3-2x-5=0$.

Let $f(x)= x^3-2x-5$.

Now,

x	0	1	2	3
$f(x)$	-5	-6	-1	16

first iteration,

$f(2) = -1 < 0$ and $f(3)=16>0$.

$\therefore$ Root lies between 2 and 3.

$x_0 = \frac{2+3}{2} = 2.5$

$f(x_0) = f(2.5) = (2.5)^3 - 2\times 2.5 - 5$

$= 5.625 > 0$.

~~Second iteration,~~

~~$f(2) = -1 < 0$ & $f(2.25) = 1.890625 >$~~

~~& Root lies~~

$\therefore$ Root is 2.0625 (upto 4 decimal

Second iteration,

$f(2) = -1 < 0$, $f(2.5) = 5.625 > 0$.

$\therefore$ Root is between 2 & 2.5.

$x_1 = \frac{2+2.5}{2} = 2.25$.

$f(x_1) = f(2.25) = 1.89062 > 0$.

Third iteration,

$f(2) = -1 < 0$, $f(2.25) = 1.89062 > 0$.

$\therefore$ Root is between 2 & 2.25.

$x_2 = \frac{2+2.25}{2} = 2.125$.

$f(x_2) = f(2.125) = (2.125)^3 - 2 \times 2.125 - 5$
$= 0.34577 > 0$.

fourth Iteration,

$f(2) = -1 < 0$, $f(2.125) = 0.3457 > 0$

$\therefore$ Root is between 2 & 2.125

$x_3 = \frac{2+2.125}{2} = 2.0625$.

$f(x_3) = (2.0625)^5 - 2 \times 2.0625 - 5$
$= -0.3513 < 0$.

$x = (32)^{1/4} \Rightarrow x^4 = 32$

$\Rightarrow x^4 - 32 = 0.$

Let $f(x) = x^4 - 32$

$f(0) = -32$

$f(1) = -31$

$f(2) = -16 < 0$

$f(3) = 49 > 0.$

$\therefore$ Root is between 2 and 3.

$\therefore$ By Regula Falsi, $x = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

Let $a = 2$, $f(a) = -16$, $f(b) = 49.$

Iteration 1:

$$x_1 = \frac{2(49) - 3(-16)}{49 - (-16)} = 2.2462.$$

$\therefore f(x_1) = f(2.2462) = -6.5438 < 0$

$\therefore$ Root is between 2.2462 & 3.

Now, $a = 2.2462$, $f(a) = -6.5438$,

$b = 3$, $f(b) = 49.$

Iteration 2 :

$$x_2 = \frac{2.2462(49) - 3(-6.5438)}{49-(-6.5438)}$$

$$= 2.335$$

$\therefore f(x_2) = f(2.335) = -2.2732 < 0$

$\therefore$ Root is between 2.335 & 3.

Now, $a = 2.335$, $f(a) = -2.2732$

$b = 3$, $f(b) = 49$.

Iteration 3 :

$$x_3 = \frac{2.335(49) - 3(-2.2732)}{49-(-2.2732)} = 2.3645.$$

$\therefore f(x_3) = f(2.3645) = -0.7422 < 0$

$\therefore$ Root is between 2.3645 and 3.

$\therefore$ Taking $a = 2.3645$, $f(a) = -0.7422$

$b = 3$, $f(b) = 49$

Iteration 4 :

$$x_4 = \frac{2.3645(49) - 3(-0.7422)}{49-(-0.7422)} = 2.3770.$$

$\therefore f(x_4) = f(2.3770) = -0.0760 < 0.$

$\therefore$ The root is between 2.3770 & 3.

Now, $a = 2.3770$, $f(a) = -0.0760$.

$f(b) = 49.$

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Iteration 5 :

$$x_5 = \frac{2.3770(49) - 3(-0.0760)}{49-(-0.0760)} = 2.3779.$$

$\therefore f(x_5) = f(2.3779) = -0.0276$

$\therefore$ The approximate root is 2.378.

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(50) $x+y+z=9,\quad 2x-3y+4z=13,\quad 3x+4y+5z=40.$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}.$$

$R_2 \to R_2 - 2R_1,\quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}.$$

$R_3 \to R_3 + \frac{1}{5} R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & +5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}.$$

$R_2 \to R_2 + \frac{1}{6} R_3,\quad R_3 \to \frac{1}{12} R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}.$$

$R_2 \to \frac{1}{5} R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}.$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

$\therefore \; x = 1, \; y = 3, \; z = 5.$