Unit-5

1) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x}$

$E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

and $A = 1,\ B = 0,\ C = 3$

Now, find $B^2 - 4AC$

$= 0^2 - 4\times 1\times 3$

$= -12 < 0$

Hence the given PDE is Elliptic

2) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x\delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x} +$

$E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0,$

& $A = -2,\ B = 3,\ C = -4$

Now, find $B^2 - 4AC$

$= 3^2 - 4(-2)(-4)$

$= 9 - 32 = -23 < 0$

Hence the given PDE is Elliptic

3) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x\delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x} +$

$E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

& $A = -4,\ B = -2,\ C = -1$

Now, find $B^2 - 4AC$

$= (-2)^2 - 4(-4)(-1) = 4 - 16 = -12 < 0$

Hence the given PDE is Elliptic

4) $A(x,y)\dfrac{\delta^2 u}{\delta x^2} + B(x,y)\dfrac{\delta^2 u}{\delta x \delta y} + C(x,y)\dfrac{\delta^2 u}{\delta y^2} + D(x,y)\dfrac{\delta u}{\delta x}$

$+ E(x,y)\dfrac{\delta u}{\delta y} + F(x,y) = 0$

& $A = 2, B = 2, C = 2$

Now, find $B^2 - 4AC$

$= (2)^2 - 4(2)(2)$

$= 4 - 16 = -12 < 0$

Hence the given PDE is elliptic

5) $A(x,y)\dfrac{\delta^2 u}{\delta x^2} + B(x,y)\dfrac{\delta^2 u}{\delta x \delta y} + C(x,y)\dfrac{\delta^2 u}{\delta y^2} + D(x,y)\dfrac{\delta u}{\delta x}$

$+ E(x,y)\dfrac{\delta u}{\delta y} + F(x,y) = 0$

& $A = -5, B = -5, C = -5$

Now find $B^2 - 4AC$

$= (-5)^2 - 4(-5)(-5)$

$= 25 - 100 = -75 < 0$

Hence the given PDE is elliptic

6) $A(x,y)\dfrac{\delta^2 u}{\delta x^2} + B(x,y)\dfrac{\delta^2 u}{\delta x \delta y} + C(x,y)\dfrac{\delta^2 u}{\delta y^2} + D(x,y)\dfrac{\delta u}{\delta x}$

$+ E(x,y)\dfrac{\delta u}{\delta y} + F(x,y) = 0$

& $A = 100, B = 10, C = 10$

Now, find $B^2 - 4AC$

$= (10)^2 - 4(100)(10)$

$= 100 - 4000 = -3900 < 0$

Hence the given PDE is elliptic

7) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x} +$

$E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

& $A = x^2$, $B = x^2 y$, $C = x^3$

Now find $B^2 - 4AC$

$= (x^2 y)^2 - 4(x^2)(x^3)$

$= x^4 y^2 - 4x^5$

$= x^4 (y^2 - 4x)$

Case 1:- if $\cancel{x>0 \ \& \ y>0}$ $y^2 - 4x > 0$

then

$\cancel{x^4(y^2-4x)}$

$x^4 (y^2 - 4x) > 0$

$\Rightarrow$ then PDE is hyperbolic

Case 2:- if $y^2 - 4x = 0$

then

$x^4 (y^2 - 4x) = 0$

$\Rightarrow$ then PDE is Parabolic.

Case 3:- if $y^2 - 4x < 0$

then

$x^4 (y^2 - 4x) < 0$

$\Rightarrow$ then PDE is Elliptic.

8.) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x}$

$+ E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

& $A = x^2 y$, $B = y^2$, $C = x^5$

Now, find $B^2 - 4AC$

$= (y^2)^2 - 4(x^2y)(x^5)$

$= y^4 - 4x^7y$

$= y(y^3 - 4x^7)$

Case 1:- if $y^3 - 4x^7 > 0$ & $y > 0$

then $y(y^3 - 4x^7) > 0$

$\Rightarrow$ given PDE is hyperbolic.

Case 2:- if $y^3 - 4x^7 = 0$

then $y(y^3 - 4x^7) = 0$

$\Rightarrow$ given PDE is parabolic.

Case 3:- if $y^3 - 4x^7 < 0$ & $y < 0$

then $y(y^3 - 4x^7) > 0$

$\Rightarrow$ given PDE is hyperbolic.

Case 4:- if $y^3 - 4x^7 < 0$ & $y > 0$

$y(y^3 - 4x^7) < 0$

$\Rightarrow$ given PDE is elliptic

Case 5:- if $y^3 - 4x^7 > 0$ & $y < 0$

then $y(y^3 - 4x^7) < 0$

$\Rightarrow$ given PDE is elliptic.

9) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} +$

$D(x,y)\frac{\delta u}{\delta y} + E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

& $A = y^3$, $B = x^2$, $C = y^2$

Now find $B^2 - 4AC$

$$= (x^2)^2 - 4(y^3)(y^2)$$
$$= x^4 - 4y^5$$

~~Case 1:- If $x > 0$ & $y > 1$ & $x > y$~~

~~$x^4 - 4y^5$~~

Case 1:- if $x^4 - 4y^5 < 0$

$\Rightarrow$ then given PDE is elliptic.

Case 2:- If $x^4 - 4y^5 = 0$

$\Rightarrow$ given PDE is Parabolic

Case 3:- if $x^4 - 4y^5 > 0$

$\Rightarrow$ given PDE is parabolic.

10) $A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + C(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x} + E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

$A = x^{-3}$, $B = (xy)^{-3/2}$, $C = y^{-3}$, $D = xy^2$, $E = y^2$

$F = y^2$

Now, find $B^2 - 4AC$

$$= ((xy)^{-3/2})^2 - 4(x^{-3})(y^{-3})$$

$$= (xy)^{-3} - 4x^{-3}y^{-3}$$
$$= x^{-3}y^{-3} - 4x^{-3}y^{-3} = -3x^{-3}y^{-3}$$

$B^2 - 4AC = \frac{-3}{x^3 y^3}$

Case 1:- if $x > 0$ & $y > 0$

$\Rightarrow \frac{-3}{x^3 y^3} < 0$

$\Rightarrow$ Hence given PDE is elliptic

Case 2:- if $x > 0$ & $y < 0$ or $x < 0$ & $y > 0$

$\Rightarrow \frac{-3}{x^3 y^3} > 0$

$\Rightarrow$ given PDE is Hyperbolic

Case 3:- if ~~xxxxx~~ $x = 0$ or $y = 0$
the Solution is not define.
Hence x & y Should not be equal to zero.

11. $A(x,y) \frac{\delta^2 u}{\delta x^2} + B(x,y) \frac{\delta^2 u}{\delta x \delta y} + C(x,y) \frac{\delta^2 u}{\delta y^2} + D(x,y) \frac{\delta u}{\delta x}$
$+ E(x,y) \frac{\delta u}{\delta y} + F(x,y) = 0$

& $A = x^3$, $B = x^2$, $C = y^2$, $D = 7^2$, $E = 8^{-1}$
$F = 3^{-1}$

Now, find $B^2 - 4AC$
$= (x^2)^2 - 4(x^3) y^2$
$= x^4 - 4x^3 y^2$
$= x^3 (x - 4y^2)$

Case 1:- if $x-4y^2>0$ & $x>0$

then $x^3(x-4y^2)>0$

$\Rightarrow$ given PDE is hyperbolic.

Case 2:- if $x-4y^2>0$ & $x<0$

then $x^3(x-4y^2)<0$

$\Rightarrow$ given PDE is elliptic

Case 3:- if $x-4y^2=0$

then $x^3(x-4y^2)=0$

$\Rightarrow$ given PDE is Parabolic.

Case 4:- if $y=0$ & ($x>0$ or $x<0$)

then $x^3(x-4y^2)>0$

$\Rightarrow$ given PDE is hyperbolic.

Case 5:- if $x-4y^2<0$ & $x>0$

then $x^3(x-4y^2)<0$

$\Rightarrow$ given PDE is elliptic

Case 6:- if $x-4y^2<0$ & $x<0$

then $x^3(x-4y^2)>0$

$\Rightarrow$ given PDE is hyperbolic.

12. $A(x,y)\frac{\delta^2 u}{\delta x^2}+B(x,y)\frac{\delta^2 u}{\delta x\delta y}+C(x,y)\frac{\delta^2 u}{\delta y^2}+D(x,y)\frac{\delta u}{\delta x}+$

$E(x,y)\frac{\delta u}{\delta y}+F(x,y)=0$

& $A=x^2$, $B=x^2$, $C=x^2$, $D=x^2$, $E=x^2$, $F=x^2$

Now, find $B^2 - 4AC$

$= (x^2)^2 - 4(x^2)(x^2)$

$= x^4 - 4x^4$

$= -3x^4$

Case 1:- if $x > 0$

$\Rightarrow -3x^4 < 0$

$\Rightarrow$ given PDE is elliptic

Case 2:- if $x < 0$

$\Rightarrow -3x^4 < 0$

$\Rightarrow$ given PDE is elliptic

Case 3:- if $x = 0$

$\Rightarrow -3x^4 = 0$

$\Rightarrow$ given PDE is parabolic.

13) ① $\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2} \Rightarrow \frac{\delta^2 u}{\delta t^2} - c^2 \frac{\delta^2 u}{\delta x^2} = 0$ — ①

eqn ① Compare with

$A(x,y)\frac{\delta^2 u}{\delta x^2} + B(x,y)\frac{\delta^2 u}{\delta x \delta y} + P(x,y)\frac{\delta^2 u}{\delta y^2} + D(x,y)\frac{\delta u}{\delta x} +$

$E(x,y)\frac{\delta u}{\delta y} + F(x,y) = 0$

Now, $A = 1$, $B = 0$, $P = -c^2$

find $B^2 - 4AP$

$= 0 - 4 \times 1(-c^2)$

$= +4c^2$

Case 1:- if $c < 0$

then $4c^2 > 0$

$\Rightarrow$ given PDE is hyperbolic

Case 2:- if $c > 0$

$\Rightarrow 4c^2 > 0$

$\Rightarrow$ given PDE is hyperbolic

Case 3:- if $c = 0$

$\Rightarrow 4c^2 = 0$

$\Rightarrow$ given PDE is parabolic.

(ii) $\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2} \Rightarrow c^2 \frac{\delta^2 u}{\delta x^2} - \frac{\delta u}{\delta t} = 0$

after compare with general eqn

$A = c^2, \ B = 0, \ C = 0$

$B^2 - 4AC$

$= 0 - 4 \times c^2 \times 0$

$= 0$

$\Rightarrow$ given PDE is parabolic.

(iii) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$

after compare with general eqn

$A = 1, \ B = 0, \ C = 1$

Now, find $B^2 - 4AC$

$= 0 - 4 \times 1 \times 1$

$= -4 < 0$

$\Rightarrow$ given PDE is elliptic

14) $(x^2-y^2)u_{xx} + 2(x^2+y^2)u_{xy} + (x^2-y^2)u_{yy} = 0$

for region $x>0,\ y>0$

after compare above equation with general eqn

$A = x^2-y^2$, $B = 2(x^2+y^2)$, $C = x^2-y^2$

Now, find B^2-4AC

$= (2(x^2+y^2))^2 - 4(x^2-y^2)(x^2-y^2)$

$= 4(x^2+y^2)^2 - 4(x^2-y^2)^2$

$= 4[(x^2+y^2)^2 - (x^2-y^2)^2]$

$= 4[(\cancel{x^2}+y^2-\cancel{x^2}+y^2)(x^2+\cancel{y^2}+x^2-\cancel{y^2})]$

$= 4[2y^2 \cdot 2x^2]$

$= 16x^2y^2$

for region $x>0,\ y>0$

$\Rightarrow 16x^2y^2 > 0$

$\Rightarrow$ given PDE is ~~parabolic~~ hyperbolic.

15) $(x^2)u_{xx} - x(y^2-1)u_{xy} + y(x^2-y^2)u_{yy} = 0$

after comparing above eqn with general eqn

$A = x^2$, $B = x(y^2-1)$, $C = y(x^2-y^2)$

Now find B^2-4AC

$= x^2(y^2-1)^2 - 4x^2y(x^2-y^2)$

$= x^2(y-1)^2(y+1)^2 - 4x^2y(x-y)(x+y)$

$= \cancel{x^2(y^2+1-2y)(y}$

$= x^2(y^4+1-2y^2) - 4x^4y + 4x^2y^3$

$= x^2y^4 + x^2 - 2x^2y^2 - 4x^4y + 4x^2y^3$

$= x^2[y^4+1-2y^2-4x^2y+4y^3]$

$= x^2[(y^2-1)^2 - 4y(x^2-y^2)]$

for hyperbolic

$$x^2[(y^2-1)^2-4y(x^2-y^2)]>0$$

$$[(y^2-1)^2-4y(x^2-y^2)]>0$$

$$(y^2-1)^2 > 4y(x^2-y^2)$$

$$y^4+1-2y^2 > 4yx^2-4y^3$$

$$y^4+1-2y^2+4y^3 > 4yx^2$$

$$\frac{y^4+1-2y^2+4y^3}{4y} > x^2$$

$$\Rightarrow \frac{y^3}{4} + y^2 - \frac{y}{2} + \frac{1}{4y} > x^2$$

16) $\frac{\delta u}{\delta t} = \frac{c^2 \delta^2 u}{\delta x^2}$

$u(x,t) = X(x) \cdot T(t)$

Diff. w.r.t x & t

$u_t = XT'$

$u_{xx} = X''T$

So, PDE becomes

$XT' = c^2 X''T$

$\frac{T'}{T} = c^2 \frac{X''}{X} = K$ (constant)

$T' - KT = 0$ & $c^2 X'' - KX = 0$

A.E, $m - K = 0$ & $c^2m^2 - K = 0$

$M = K$

$c^2m^2 = K \Rightarrow m^2 = \frac{K}{c^2}$

$\Rightarrow T(t) = Ae^{Kt}$

$\Rightarrow m = \pm \frac{\sqrt{K}}{c}$

$X(x) = (Be^{\sqrt{K}/c\, x} + De^{-\sqrt{K}/c\, x})$

$\therefore$ the required Sol^n $u(x,t) = X(x)T(t)$

$= \left(Be^{\frac{\sqrt{K}x}{c}} + De^{\frac{-\sqrt{K}x}{c}}\right) Ae^{Kt}$

17) $\frac{\delta^2 u}{\delta t^2} = c\frac{\delta^2 u}{\delta x^2}$

$u(x,t) = X(x)T(t)$

Diff. w.r.t x & t

$\frac{\delta^2 u}{\delta t^2} = XT''$ & $\frac{\delta^2 u}{\delta x^2} = X''T$

So PDE become

$XT'' = cX''T$

$\frac{T''}{T} = \frac{cX''}{X} = K$ (constant)

$T'' - KT = 0$

A.E: $m^2 - K = 0$

$m^2 = K$

$m = \pm\sqrt{K}$

$T(t) = Ae^{-\sqrt{K}t} + Be^{\sqrt{K}t}$

$cX'' - KX = 0$

A.E:- $cm^2 - K = 0$

$cm^2 = K \Rightarrow m^2 = K/c$

$m = \pm\sqrt{\frac{K}{c}}$

$X(x) = De^{\sqrt{K/c}\,x} + Ee^{\sqrt{K/c}\,x}$

Now, the require solⁿ $u(x,t) = X(x)\,T(t)$

$= \left(De^{\sqrt{K/c}\,x} + Ee^{-\sqrt{K/c}\,x}\right)\left(Ae^{-\sqrt{K}t} + Be^{\sqrt{K}t}\right)$

18. $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$

$u(x,y) = X(x)Y(y)$

Diff. w.r.t x & y

$\frac{\delta^2 u}{\delta x^2} = X''Y$ & $\frac{\delta^2 u}{\delta y^2} = XY''$

So PDE becomes

$X''Y + XY'' = 0$

$X''Y = -XY''$

$\frac{X''}{X} = -\frac{Y''}{Y} = k$ (constant)

$\frac{X''}{X} = k$

$\Rightarrow X'' - kX = 0$

$m^2 - k = 0$

$m = \pm\sqrt{k}$

A.E :- $X(x) = Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x}$

$-\frac{Y''}{Y} = k$

$\Rightarrow Y'' + kY = 0$

$m^2 + k = 0$

$m^2 = -k \Rightarrow m = \pm\sqrt{k}i$

~~A.E :- $Y(y) = (Ce^{-\sqrt{k}y} + De^{\sqrt{k}y})$~~

A.E :- $Y(y) = (C\cos\sqrt{k}y + D\sin\sqrt{k}y)$

The require solution $u(x,y) = X(x)Y(y)$

~~$= (Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x})(Ce^{-\sqrt{k}y} + De^{\sqrt{k}y})$~~

$= (Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x})(C\cos\sqrt{k}y + D\sin\sqrt{k}y)$

19) $\frac{\delta^2 u}{\delta x^2} = 5\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y}$

$u(x,y) = X(x)\,Y(y)$

diff. w.r.t x & y

$\frac{\delta^2 u}{\delta x^2} = X''Y$, $\frac{\delta u}{\delta x} = X'Y$, $\frac{\delta u}{\delta y} = XY'$

So PDE becomes

$X''Y = 5X'Y + XY'$

$\Rightarrow X''Y - 5X'Y - XY' = 0$

$\Rightarrow (X'' - 5X')Y = XY'$

$$\frac{(X''-5X')}{X} = \frac{Y'}{Y} = K \text{ (constant)}$$

$$\frac{X''-5X'}{X} = K$$

$$X''-5X'-KX=0$$

A.E:- $m^2-5m-K=0$

$$m = \frac{5 \pm \sqrt{25+4K}}{2}$$

$$m = \frac{5 \pm \sqrt{25+4K}}{2}$$

$$\frac{Y'}{Y} = K$$

$$Y'-KY=0$$

A.E:- $m-K=0$

$$m=K$$

$$Y(y) = Ce^{Ky}$$

$$X(x) = Ae^{\left(\frac{5+\sqrt{25+4K}}{2}\right)x} + Be^{\left(\frac{5-\sqrt{25+4K}}{2}\right)x}$$

The require solution $u(x,y) = X(x)Y(y)$

$$= \left(Ae^{\left(\frac{5+\sqrt{25+4K}}{2}\right)x} + Be^{\left(\frac{5-\sqrt{25+4K}}{2}\right)x}\right) Ce^{Ky}$$

20) $\frac{\delta u}{\delta t} = \frac{\delta u}{\delta x}$, $u(0,x) = 2e^{-3x}$

$u(t,x) = T(t)\,X(x)$

diff. w.r.t x & t

$\frac{\delta u}{\delta t} = XT'$, $\frac{\delta u}{\delta x} = X'T$

So PDE become

$XT' = X'T$

$\frac{T'}{T} = \frac{X'}{X} = k$ (constant)

$\frac{T'}{T} = k$

$T' - kT = 0$

A.E :- $m - k = 0$

$m = k$

$T(t) = Ae^{kt}$

$\frac{X'}{X} = k$

$X' - kX = 0$

$m - k = 0$

$m = k$

$X(x) = Be^{kx}$

The required solution $u(t, x) = T(t)\,X(x)$

$u(t,x) = (Ae^{kt})(Be^{kx})$

$= Ce^{kt} \cdot e^{kx}$, $\boxed{C = A \cdot B}$

Now

$u(0,x) = 2e^{-3x}$

$Ce^{0} \cdot e^{kx} = 2e^{-3x}$

$Ce^{kx} = 2e^{-3x}$

Compare both side

$C = 2$ & $k = -3$

Now $u(t,x) = 2e^{-3t} \cdot e^{-3x}$

$= 2e^{-3(t+x)}$