Round off the numbers 865250 and 37.46235 to four significant figures and compute E_a, E_r, E_p in each case.

Solution:

(*i*) Number rounded off to four significant figures = 865200

$$E_a = |X - X_1| = |865250 - 865200| = 50$$

$$E_r = \left|\frac{X - X_1}{X}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E^p = E_r \times 100 = 6.71 \times 10^{-3}$$

(*ii*) Number rounded off to four significant figures = 37.46

$$\therefore\ E_a = |X - X_1| = |37.46235 - 37.46000| = 0.00235$$

$$E_r = \left|\frac{X - X_1}{X}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_p = E_r \times 100 = 6.27 \times 10^{-3}$$

EXAMPLE 1.2

Find the absolute error if the number $X = 0.00545828$ is

(i) truncated to three decimal digits.
(ii) rounded off to three decimal digits.

Solution: We have $X = 0.00545828 = 0.545828 \times 10^{-2}$

(i) After truncating to three decimal places, its approximate value $X' = 0.545 \times 10^{-2}$

$\therefore$ Absolute error $= |X - X'| = 0.000828 \times 10^{-2}$

$= 0.828 \times 10\text{-}5 < 10^{-2-3}$

This proves rule (1).

(ii) After rounding off to three decimal places, its approximate value $X' = 0.546 \times 10^{-2}$

$\therefore$ Absolute error $= |X - X'|$

$= |0.545828 - 0.546| \times 10^{-2}$

$= 0.000172 \times 10^{-2} = 0.172 \times 10^{-5}$

which is $< 0.5 \times 10^{-2-3}$. This proves rule (2).

EXAMPLE 1.3

Find the relative error if the number $X = 0.004997$ is

(i) truncated to three decimal digits

(ii) rounded off to three decimal digits.

Solution: We have $X = 0.004997 = 0.4997 \times 10^{-2}$

(i) After truncating to three decimal places, its approximate value $X = 0.499 \times 10^{-2}$.

$$\therefore \text{ Relative error } = \left|\frac{X - X'}{X}\right| = \left|\frac{0.4997 \times 10^{-2} - 0.499 \times 10^{-2}}{0.4997 \times 10^{-2}}\right|$$

$$= 0.140 \times 10^{-2} < 10^{1-3}$$

This proves rule (3).

(ii) After rounding off to three decimal places, the approximate value of the given number

$X' = 0.500 \times 10^{-2}$

$$\therefore \text{ Relative error } = \left|\frac{X - X'}{X}\right| = \left|\frac{0.4997 \times 10^{-2} - 0.500 \times 10^{-2}}{0.4997 \times 10^{-2}}\right|$$

$$= 0.600 \times 10^{-3} = 0.06 \times 10^{-3+1}$$

which is less than $0.5 \times 10^{-3+1}$. This proves rule (4).

Exercises 1.1

1. Round off the following numbers correct to four significant figures: 3.26425, 35.46735, 4985561, 0.70035, 0.00032217, and 18.265101.

2. Round off the number 75462 to four significant digits and then calculate the absolute error and percentage error.

3. If 0.333 is the approximate value of 1/3, find the absolute and relative errors.

4. Find the percentage error if 625.483 is approximated to three significant

EXAMPLE 1.4

Find the absolute error and relative error in $\sqrt{6}+\sqrt{7}+\sqrt{8}$ correct to 4 significant digits.

Solution:

We have $\sqrt{6} = 2.449, \sqrt{7} = 2.646, \sqrt{8} = 2.828$

$$\therefore \qquad S = \sqrt{6} + \sqrt{7} + \sqrt{8} = 7.923.$$

Then the absolute error E_a in S, is

$$E_a = 0.0005 + 0.0007 + 0.0004 = 0.0016$$

This shows that S is correct to 3 significant digits only. Therefore, we take $S = 7.92$ Then the relative error E_r is

$$E_r = \frac{0.0016}{7.92} = 0.0002.$$

EXAMPLE 1.5

The area of cross-section of a rod is desired up to 0.2% error. How accurately should the diameter be measured?

Solution:

If A is the area and D is the diameter of the rod, then $A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D \times D.$

Now error in area A is 0.2%, i.e., 0.002 which is due to the error in the product $D \times D$.

We know that if E_a is the absolute error in the product of two numbers X and Y, then

$$E_a = X_{aY}E + YE_{aX}$$

Here, $X = Y = D$ and $E_{aX} = E_{aY} = E_D$, therefore

$$E_a = DE_D + DE_D \quad \text{or} \quad 0.002 = 2DE_D$$

Thus, $E_d = 0.001/D$, i.e., the error in the diameter should not exceed $0.001\ D^{-1}$.

EXAMPLE 1.6

Find the product of the numbers 3.7 and 52.378 both of which are correct to given significant digits.

Solution:

Since the absolute error is greatest in 3.7, therefore we round off the other number to 3 significant figures, i.e., 52.4.

86.20%

EXAMPLE 1.7

If $u = 4x^2y^3/z^4$ and errors in x, y, z are 0.001, compute the relative maximum error in u when $x = y = z = 1$.

Solution:

Since $\frac{\partial u}{\partial x} = \frac{8xy^3}{z^4}, \frac{\partial u}{\partial y} = \frac{12x^2y^2}{z^4}, \frac{\partial u}{\partial z} = -\frac{16x^2y^3}{z^5}$

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$$\therefore\ \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z = \frac{8xy^3}{z^4}\delta x + \frac{12x^2y^2}{z^4}\delta y - \frac{16x^2y^3}{z^5}\delta z.$$

Since the errors δx, δy, δz may be positive or negative, we take the absolute values of the terms on the right side, giving

$$(\delta u)_{\max} = \left|\frac{8xy^3}{z^4}\delta x\right| + \left|\frac{12x^2y^2}{z^4}\delta y\right| - \left|\frac{16x^2y^3}{z^5}\delta z\right|$$

$= 8(0.001) + 12(0.001) + 16(0.001) = 0.036$

Hence the maximum relative error $= (\delta u)_{\max}/u = 0.036/4 = 0.009.$

linear.

EXAMPLE 2.15

(*a*) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.

(*b*) Using bisection method, find the negative root of the equation $x^3 - 4x + 9 = 0$.

Solution:

(*a*) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is – ve and $f(3)$ is + ve, a root lies between 2 and 3.

∴ First approximation to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5.$$

Thus $\quad f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ *i.e.*, –ve.

∴ The root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75.$$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ *i.e.*, +ve.

∴ The root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625.$$

Then $\quad f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.*, –ve.

The root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875.$$

Repeating this process, the successive approximations are

$x_5 = 2.71875$, $\quad x_6 = 2.70313$, $\quad x_7 = 2.71094$
$x_8 = 2.70703$, $\quad x_9 = 2.70508$, $\quad x_{10} = 2.70605$
$x_{11} = 2.70654$, $\quad x_{12} = 2.70642$

Hence the root is 2.7064.

Rate of Convergence. This method has linear rate of convergence which is faster than that of the bisection method.

EXAMPLE 2.19

Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.

Solution:

Let $\quad f(x) = x^3 - 2x - 5$

so that $\quad f(2) = -1$ and $f(3) = 16$,

i.e., A root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -1$, $f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \qquad (i)$$

Now $f(x^2) = f(2.0588) = -0.3908$

i.e., The root lies between 2.0588 and 3.

$\therefore$ Taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3908$, $f(x_1) = 16$, in (*i*), we get

$$x_3 = 2.0588 - \frac{0.9412}{19.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$x_4 = 2.0862$, $\quad x_5 = 2.0915$, $\quad x_6 = 2.0934$,

$x_7 = 2.0941$, $\quad x_8 = 2.0943$ etc.

Hence the root is 2.094 correct to three decimal places.

EXAMPLE 2.20

Find the root of the equation $\cos x = xe^x$ using the regula-falsi method

$$\therefore \quad x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.081263$$

and $\quad f(x_3) = -0.147204$

$$\therefore \quad x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.094824$$

and $\quad f(x_4) = 0.003042$

$$\therefore \quad x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.094549$$

Hence the root is 2.094 correct to three decimal places

EXAMPLE 2.24

Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places.

Solution:

Let $\quad f(x) = \cos x - xe^x = 0.$

Taking the initial approximations $x_0 = 0, x_1 = 1$

so that $f(x_0) = 1, f(x_1) = \cos 1 - e = -2.17798$

Then by the secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1 + \frac{1}{3.17798}(-2.17798) = 0.31467$$

Now $\quad f(x_2) = 0.51987$

$$\therefore x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.44673 \textit{ and } f(x_3) = 0.20354$$

$$\therefore \quad x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 0.53171$$

Repeating this process, the successive approximations are $x_5 = 0.51690$, $x_6 = 0.51775$, $x_7 = 0.51776$ etc.

Hence the root is 0.5177 correct to four decimal places.

EXAMPLE 2.30

Find the positive root of $x^4 - x = 10$ correct to three decimal places, using the Newton-Raphson method.

Solution:

Let $f(x) = x4 - x - 10$

so that $f(1) = -10 = -$ ve, $f(2) = 16 - 2 - 10 = 4 = +$ ve.

$\therefore$ A root of $f(x) = 0$ lies between 1 and 2.

Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (*i*), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (*ii*), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} = 1.856$$

Here $x_2 = x_3$. Hence the desired root is 1.856 correct to three decimal places.

2.13 Some Deductions From Newton-Raphson Formula

We can derive the following useful results from the Newton's iteration formula:

(1) Iterative formula to find $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

(2) Iterative formula to find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

(3) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)$

(4) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x-2$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n - 2} = x_n + \left(\frac{1}{x_n} - N\right)x_n 2.$$

$$= x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$.

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n N / x_n)$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$