Q bank - 3

1. Define function of two variables.

⇒ If for every x and y a unique value $f(x,y)$ is associated then f is said to be a function of the two independent variable x and y and is denoted by

$$z = f(x,y)$$

2. Define level curves.

⇒ The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a level curve of f.

3. Find the domain and range of the function

$$f(x,y) = \frac{2x}{y-x^2}.$$

⇒ Domain - $\{(x,y) \in R^2 : x^2 \neq y\}$

Range - R (all real numbers)

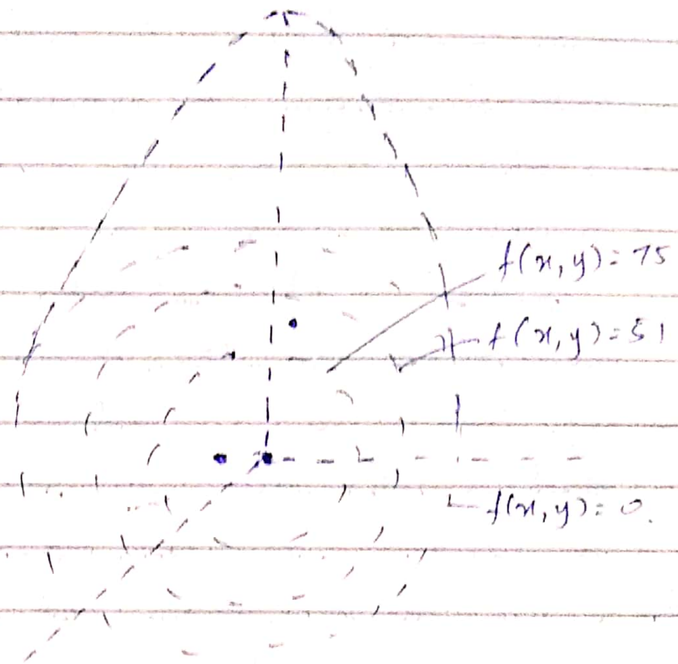
4. Find the domain and range of the function $f(x,y,z) = xy \ln z$.

⇒ Domain - $\{(x,y,z) \in R^3 : z > 0\}$

Range - R (all real numbers)

5. Plot the level curves $f(x,y) = 51$ and $f(x,y) = 75$ in the domain of the function $f(x,y) = 100 - x^2 - y^2$ in the plane.

Here domain is entire xy plane and range of f is the set of all real numbers less than or equal to 100
graph is paraboloid



Level curve, $f(x,y) = 0$

$$100 - x^2 - y^2 = 0$$

$$\Rightarrow x^2 + y^2 = 100$$

Level curve, $f(x,y) = 51$

$$100 - x^2 - y^2 = 51$$

$$\Rightarrow x^2 + y^2 = 49$$

Level curve, $f(x,y) = 75$

$$100 - x^2 - y^2 = 75$$

$$\Rightarrow x^2 + y^2 = 25$$

Level curve, $f(x,y) = 100$ consists of origin alone

6. Find the limit: $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$

$$\Rightarrow \lim_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2}.$$

7. Find the limit: $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \left[\frac{x(x-y)}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + y}\right].$$

$$\lim_{(x,y)\to(0,0)} \left[\frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{(x-y)}}\right].$$

$$\lim_{(x,y)\to(0,0)} \left[x(\sqrt{x} + \sqrt{y})\right] = 0$$

8. Find the limit: $\lim_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x^2y + 5xy - y^3}$

$\Rightarrow \lim_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x^2y + 5xy - y^3} = \dfrac{3}{-1} = -3$

9. Show that the limit does not exist of the function $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$.

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$

along $x = 0$; $y \to 0$.

$\lim_{y\to 0} \dfrac{2(0)y}{0 + y^2} = \dfrac{0}{0}$ form

along $y = mx$; $x \to 0$.

$\lim_{x\to 0} \dfrac{2x(mx)}{x^2 + m^2x^2} = \lim_{x\to 0} \dfrac{2\cancel{x^2} m}{\cancel{x^2}(1+m^2)} = \dfrac{2m}{m^2+1}$

along $y = x$; $x \to 0$

$\lim_{x\to 0} \dfrac{2x^2}{x^2+x^2} = \lim_{x\to 0} \dfrac{2\cancel{x^2}}{2\cancel{x^2}} = 1$.

Thus limit does not exists

10. Show that the limit does not exist of the function: $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$.

$\Rightarrow$ along $y = 0$; $x \to 0$

$\lim_{x\to 0} \dfrac{x(0)}{x^2 + (0)^4} = 0$

along $y = mx$; $x \to 0$.

$\lim_{x\to 0} \dfrac{x(mx)^2}{x^2 + (mx)^4} = \lim_{x\to 0} \dfrac{mx^3}{x^2(1+m^4x^2)} = \lim_{x\to 0} \dfrac{mx}{1+m^4} = \infty$.

along $x = y^2$; $y \to 0$.

$$\lim_{y \to 0} \frac{(y^2)(y^2)}{y^4 + y^4} = \lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Thus, limit does not exists.

11. Show that the limit does not exist of the function : $\lim_{(x,y) \to (1,-1)} \frac{xy+1}{x^2 - y^2}$

=> $$\lim_{(x,y) \to (1,-1)} \frac{xy+1}{x^2 - y^2} = \frac{-1+1}{1-1} = \frac{0}{0} \text{ form}$$

along $x = 1$; $y \to 0$.

$$\lim_{y \to 0} \frac{y+1}{1 - y^2} = 1$$

along $y = -1$; $x \to 0$.

$$\lim_{x \to 0} \frac{-x+1}{x^2 - 1} = -1$$

Thus, limit does not exists.

12 Define the continuity of a function $f(x,y)$ at a point (a,b).

=> A function $f(x,y)$ is continuous at point (a,b) if -

- f is defined at (a,b)
- $\lim_{(x,y) \to (a,b)} f(x,y)$ exists.
- $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

13. At what points (x,y) in the plane is the function continuous : $f(x,y) = \sin(x+y)$

$\Rightarrow$ $f(x,y) = \sin(x+y)$

Let $(x+y) = t$

$f(t) = \sin t$

It is a trigonometric function, hence it is continuous everywhere.

$$\{(x,y) \in R^2 \mid -1 \leq x+y \leq 1\}$$

but, x, y is negative neigh (-1)

so,

$$\{(x,y) \in R^2 \mid -1 < x+y \leq 1\}$$

14 At what points (x,y) in the plane is the function continuous : $f(x,y) = \ln(x^2+y^2)$.

$\Rightarrow$ $f(x,y) = \ln(x^2+y^2)$

$f(t) = \ln(t)$.

Domain : $\{t \in R \mid t > 0\}$

While the function $g(x) = x^2+y^2$

g is a polynomial function of two variable which is continuous everywhere in R^2.

$$f(g(x,y)) = f(x^2+y^2) = \ln(x^2+y^2)$$
$$= f(x,y)$$

which is continuous in its domain.

$$\{(x,y) \in R^2 , x^2+y^2 > 0\}$$

15. Show that the function is continuous at every point except the origin

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$\Rightarrow$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

along $y = mx$; $x \to 0$.

$$\lim_{x\to 0} \frac{x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{1}{1+m^2} \text{ (depends on m)}$$

along $x = y$; $y \to 0$.

$$\lim_{y\to 0} \frac{y^2}{y^2+y^2} = \lim_{y\to 0} \frac{\cancel{y^2}}{2\cancel{y^2}} = \frac{1}{2}$$

Thus, f is not continuous on $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \neq f(0,0)$$

16. Show that the function is continuous at every point except the origin

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$\Rightarrow$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^4}$$

along $y = mx$; $x \to 0$.

$$\lim_{x\to 0} \frac{3x^2(mx)}{x^4+m^4x^4} = \lim_{x\to 0} \frac{3m\cancel{x^3}}{\cancel{x^4}(1+m^4)} = \lim_{x\to 0} \frac{3m}{x(1+m^4)} = \infty$$

along $y = x^2$, $x \to 0$.

$$\lim_{x\to 0} \frac{3x^2(x^2)}{x^4+x^8} = \lim_{x\to 0} \frac{3\cancel{x^4}}{\cancel{x^4}(1+x^4)} = 3$$

Function is not unique at every point.

Thus, f is not continuous on $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^4} \neq f(0,0)$$

17. Define the partial derivative of a function $f(x,y)$ with respect to x at a point (x_0, y_0)

$\Rightarrow$ $$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h\, y_0) - f(x_0, y_0)}{h}$$

$$\left(\frac{\partial f}{\partial x}\right) = y_0 + x_0 \frac{\partial y}{\partial x}$$

18. Define the partial derivative of a function $f(x,y)$ with respect to y at a point (x_0, y_0)

$\Rightarrow$ $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x_0, y_0)$$

$$\frac{\partial f}{\partial y} = x_0 + y_0 \frac{\partial x}{\partial y}$$

19. Find the partial derivative of the function with respect to each variable: $f(x,y) = 2x^2 - 3y - 4$

$\Rightarrow$ $$f(x,y) = 2x^2 - 3y - 4.$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x^2 - 3y - 4) = 4x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^2 - 3y - 4) = -3$$

20. Find the partial derivative of the function with respect to each variable: $f(x,y) = (x+y)/(xy-1)$

$\Rightarrow$ $$f(x,y) = \frac{x+y}{xy-1}$$

$$\frac{\partial f}{\partial x} = \frac{(xy-1)(1+0) - (x+y)(y)}{(xy-1)^2} = \frac{-(1+y^2)}{(xy-1)^2} \cdot \frac{y(x+y)}{(xy-1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(xy-1)(x+1) - (x+y)(x)}{(xy-1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{xy - 1 - x^2 - xy}{(xy-1)^2} = \frac{-(1+x^2)}{(xy-1)^2}.$$

21. Find the partial derivative of the function with respect to each variable: $f(x,y) = \sqrt{x^2+y^2}$.

$\Rightarrow$

$$\frac{\partial f}{\partial x} = \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

22. Find the partial derivative of the function with respect to each variable: $f(x,y) = \tan^{-1}(y/x)$.

$\Rightarrow$ $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2} = \frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{-y}{\cancel{x^2}}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$$

$$\boxed{\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}}$$

23. Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = x^2y - y^3 + \ln x$.

$\Rightarrow \quad f(x,y) = x^2y - y^3 + \ln x$

$$\frac{\partial f}{\partial y} = x^2 - 3y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x.$$

24 Find all the second order partial derivatives of the function: $f(x,y) = x + y + xy$.

$\Rightarrow \quad \frac{\partial f}{\partial x} = 1 + y \quad , \quad \frac{\partial f}{\partial y} = 1 + x.$

$\frac{\partial^2 f}{\partial x^2} = 0 \quad , \quad \frac{\partial^2 f}{\partial y^2} = 0$

$\frac{\partial f}{\partial y \partial x} = 1 \qquad \frac{\partial f}{\partial x \partial y} = 1.$

25 Find all the second-order partial derivatives of the function: $z = x^2 \tan(xy)$

$\Rightarrow \quad \frac{\partial z}{\partial x} = \tan(xy) \cdot (2x) + x^2 \sec^2(xy) \cdot (y)$

$$\frac{\partial z}{\partial x} = 2x \tan(xy) + x^2y \sec^2(xy)$$

$$\frac{\partial z}{\partial y} = \tan(xy)(0) + x^2 \sec^2(xy) \cdot (x)$$

$$\frac{\partial z}{\partial y} = x^3 \sec^2(xy)$$

$$\frac{\partial^2 z}{\partial x^2} = 2x \sec^2(xy)(y) + \tan(xy)(2) + \sec^2(xy)(2xy) + yx^2 \cdot 2\sec(xy)\sec(xy)\tan(xy) \cdot y$$

$$\frac{\partial^2 z}{\partial y^2} = x^3 \cdot 2\sec(xy) \cdot \sec(xy) \tan(xy) \cdot x.$$

$$\frac{\partial z}{\partial x \partial y} = 3x^2 \sec^2(xy) + x^3 . 2\sec(xy)\sec(xy)\tan(xy).y.$$

$$\frac{\partial z}{\partial x \partial y} = 3x^2 \sec^2(xy) + 2x^3 y \sec^2(xy)\tan(xy)$$

$$\frac{\partial z}{\partial y \partial x} = 2x \sec^2(xy).x + x^2 \sec^2(xy) + x^2 y . 2\sec(xy)\sec(xy) \tan(xy)$$

26. Find the value of $\frac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation $xy + z^3 x - 2yz = 0$ defines z as a function of the two independent variables x and y and the partial derivative exists.

=> $$xy + z^3 x - 2yz = 0$$

$$\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(z^3 x) - \frac{\partial}{\partial x}(2yz) = 0$$

$$y + \cancel{z}^3 + 3z^2 . x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$$

$$1 + 1 + 3\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -2$$

27. Define total differential of function of two variables.

=> $$dF = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

28. Define total differential of function of three variables

=> $$dF = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

29 Find the total differential of the function at the point (1,1): $f(x,y) = x^3y^4$.

$\Rightarrow$

$$f(x,y) = x^3y^4$$

$$dF = \frac{\partial}{\partial x}(x^3y^4)\,dx + \frac{\partial}{\partial y}(x^3y^4)\,dy$$

$$dF = 3x^2y^4\,dx + 4x^3y^3\,dy$$

$$(dF)_{(1,1)} = 3dx + 4dy$$

30 Find the total differential of the function at the point (1,0,0): $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$\Rightarrow$

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

$$dF = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}dx + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}dy + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}dz.$$

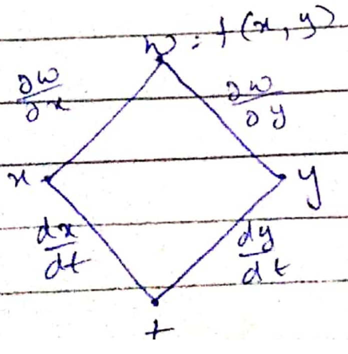
$$dF = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}dx + \frac{1}{2}\cdot\frac{2y}{\sqrt{x^2+y^2+z^2}}dy + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}dz.$$

$$dF = \frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz$$

$$(dF)_{(1,0,0)} = \frac{1}{\sqrt{1+0+0}}dx = dx.$$

31 Draw a branch diagram and write a chain rule for derivative of a function of 1 independent variable and 2 intermediate variables.

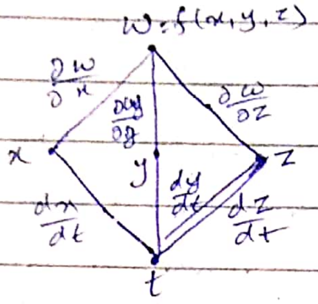
$\Rightarrow$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

32. Draw a branch diagram and write chain rule for the derivative of a function of 1 independent variable and 3 intermediate variables.

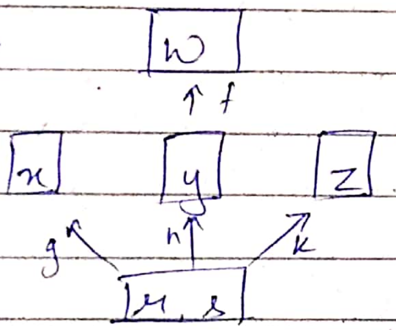
$\Rightarrow$

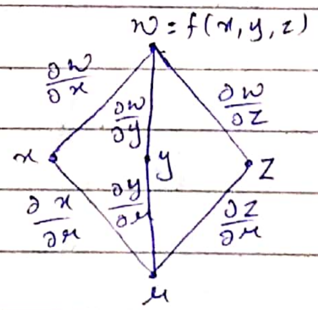


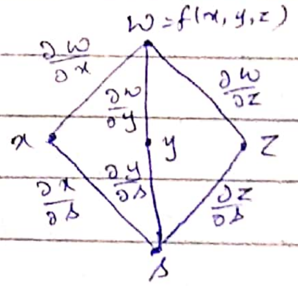
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

33. Draw a branch diagram and write chain rule for the derivative of a function of 2 independent variable and 3 intermediate variables.

$\Rightarrow$







$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

34. Express $\frac{dw}{dt}$ as a function of t, both by using the chain rule and by expressing w in terms of t and differentiating directly with respect to t. Then evaluate $\frac{dw}{dt}$ at $t=\pi$: $w = x^2+y^2$, $x = \cos t$, $y = \sin t$, $t=\pi$.

$\Rightarrow \quad w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t.$

Using chain rule:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial w}{\partial t} = 2x(-\sin t) + 2y.\cos t$$

$$\frac{\partial w}{\partial t} = -2x\sin t + 2y\cos t$$

$$\frac{\partial w}{\partial t} = -2\cancel{\sin t\cos t} + 2\cancel{\sin t\cos t} = 0$$

$$\boxed{\frac{\partial w}{\partial t} = 0}.$$

By direct differentiation:

$$w = x^2 + y^2 = \cos^2 t + \sin^2 t.$$

$$w = 1.$$

$$\boxed{\frac{dw}{dt} = 0.}$$

$$\boxed{\left(\frac{dw}{dt}\right)_{\pi} = 0}.$$

55 Evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the point (u, v):

$w = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v; (uv) = (2, \pi/4)$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial u} = 4e^x \ln y.\frac{\cos v}{u\cos v} + \frac{4e^x}{y}.\sin v.$$

$$\frac{\partial w}{\partial u} = 4e^{\ln(u\cos v)}\ln(u\sin v) + \frac{4e^{\ln(u\cos v)}}{u\cancel{\sin v}}.\cancel{\sin v}$$

$$\frac{\partial w}{\partial u} = 4u\cos v.\ln(u\sin v) + 4\cos v$$

$$\left(\frac{\partial w}{\partial u}\right)_{(2,\pi/4)} = 8.\frac{1}{\sqrt{2}}\ln\sqrt{2} + 4.\frac{1}{\sqrt{2}} = \sqrt{2}(\ln 2 + 2)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \cancel{+ \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}}$$

$$\frac{\partial w}{\partial v} = (4e^x \ln y)\cdot\left(\frac{-u\sin v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)\cdot u\cos v$$

$$\frac{\partial w}{\partial v} = -4e^x \ln y \cdot \frac{\sin v}{\cos v} + \frac{4e^x}{y}\cdot u\cos v$$

$$\frac{\partial w}{\partial v} = -4\,u\cancel{\cos v}\ln(u\sin v)\cdot\frac{\sin v}{\cancel{\cos v}} + \frac{4\cancel{u}\cos v}{\cancel{u}\sin v}\cdot u\cos v$$

$$\frac{\partial w}{\partial v} = -4u\sin v\ln(u\sin v) + 4u\cot v\cos v$$

$$\left(\frac{\partial w}{\partial v}\right)_{(2,\pi/4)} = -8\cdot\frac{1}{\sqrt{2}}\ln(\sqrt{2}) + 8\cdot\frac{1}{\sqrt{2}} = 4\sqrt{2}\left(1-\ln(\sqrt{2})\right)$$

36 Evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the point $(u,v) = (1/2, 1)$

$w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$.

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial u} = (y+z)(1) + (x+z)(1) + (x+y)(v)$$

$$\frac{\partial w}{\partial u} = y+z+x+z+(2uv)$$

$$\frac{\partial w}{\partial u} = u - \cancel{v} + 2uv + u + \cancel{v} + 2uv = 2u + 4uv$$

$$\left(\frac{\partial w}{\partial u}\right)_{(1/2,1)} = 2(1/2) + 4(1/2)(1) = 3$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial v} = (y+z)(1) + (x+z)(-1) + (x+y)(u)$$

$$\frac{\partial w}{\partial v} = y + \cancel{z} - x - \cancel{z} + 2u^2 = \cancel{u} - v - \cancel{u} - v + 2u^2 = 2u^2 - 2v$$

$$\left(\frac{\partial w}{\partial v}\right)_{(1/2,1)} = 2(1/4) - 2 = -3/2$$

37. Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if $w = x + 2y + z^2$, $x = r/s$, $y = r^2 + \ln s$, $z = 2r$.

$\Rightarrow$ $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$\frac{\partial w}{\partial r} = 1 . \frac{1}{s} + 2 . 2r + 2z . 2 . = \frac{1}{s} + 4r + 4z$

$\frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$

$\frac{\partial w}{\partial s} = 1 . \frac{-r}{s^2} + 2 . \frac{1}{s} + 2z(0) = \frac{-r}{s^2} + \frac{2}{s}$

38. Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if $w = x^2 + y^2$, $x = r - s$, $y = r - s$

$\Rightarrow$ $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$

$\frac{\partial w}{\partial r} = 2x . 1 + 2y . 1 = 2(r - s + r - s) = 4(r - s)$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial w}{\partial s} = 2x . 1 + 2y(-1) = 2(r - s - r + s) = 0$

39. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

$\Rightarrow$ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$ —(i)

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$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \quad \text{- (ii)}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \quad \text{- (iii)}$$

Adding eq (i), (ii) & (iii).

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \cancel{\frac{\partial f}{\partial u}} - \cancel{\frac{\partial f}{\partial w}} - \cancel{\frac{\partial f}{\partial u}} + \cancel{\frac{\partial f}{\partial v}} - \cancel{\frac{\partial f}{\partial v}} + \cancel{\frac{\partial f}{\partial w}}$$

$$\boxed{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0}$$

40. Show that if $w = f(u,v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2-y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

$\Rightarrow$

$$w = f(u,v)$$

$$f_{uu} + f_{vv} = 0$$

$$u = (x^2-y^2)/2 \quad , \quad v = xy$$

$$w_{xx} + w_{yy} = 0$$

$$u_x = \frac{2x}{2} = x \quad , \quad u_y = -y$$

$$u_{xx} = 1 \quad , \quad u_{yy} = -1$$

$$u_{xy} = 0 \quad , \quad u_{yx} = 0$$

41. Find dy/dx if $y^2 - x^2 - \sin xy = 0$

$$\Rightarrow \quad 2y\frac{dy}{dx} - 2x - \cos xy\left(x\frac{dy}{dx} + y\right) = 0.$$

$$2y\frac{dy}{dx} - x\cos xy\frac{dy}{dx} - y\cos xy - 2x = 0$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy.$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

42. Find dy/dx if $xe^y + \sin xy + y - \ln 2 = 0$.

$$\Rightarrow \quad e^y + xe^y\frac{dy}{dx} + \cos xy\left(x\frac{dy}{dx} + y\right) + \frac{dy}{dx} = 0$$

$$e^y + xe^y\frac{dy}{dx} + x\cos xy\frac{dy}{dx} + y\cos xy + \frac{dy}{dx} = 0$$

$$(xe^y + x\cos xy + 1)\frac{dy}{dx} = -(e^y + y\cos xy)$$

$$\boxed{\frac{dy}{dx} = \frac{-(e^y + y\cos xy)}{xe^y + x\cos xy + 1}}$$

43. Find Taylor series upto 2^{nd} degree of $f(x,y) = x^2y + y^3$ at the point $(1, 3)$.

$\Rightarrow$ Taylor series $- f(x,y) + (hf_x + Kf_y) + \frac{1}{2!}(h^2f_{xx} + K^2f_{yy} + 2hKf_{xy})$.

$f(x,y) = x^2y + y^3 \Rightarrow f(1,3) = 1(3) + (27) = 30.$

$f_x = 2xy \Rightarrow f_x(1,3) = 2(1)(3) = 6.$

$f_y = x^2 + 3y^2 \Rightarrow f_y(1,3) = 1 + 3(9) = 28.$

$f_{xx} = 2y \Rightarrow f_{xx}(1,3) = 2(3) = 6$ | $f_{yy} = 6y \Rightarrow f_{yy}(1,3) = 6(3) = 18.$

$f_{xy} = 2x \Rightarrow f_{xy}(1,3) = 2(1) = 2$

Taylor series $\Rightarrow 30 + [(x-1)6 + (y-3)28] + \frac{1}{2!}[(x-1)^2 6 + (y-3)^2 18 + 2(x-1)(y-3)2]$

$= 30 + 6x - 6 + 28y - 84 + \frac{1}{2}[6x^2 - 12x + 6 + 18y^2 - 108y + 162 + 4(xy - 3x - y + 3)]$

$= \cancel{6x} + 28y - 60 + 3x^2 - \cancel{6x} + 3 + 9y^2 - 54y + 81 + 2xy - 6x - 2y + 6$

$= 3x^2 + 9y^2 - 6x - 28y + 2xy + 30$

44. Find the Taylor series upto 2nd degree of $f(x,y) = xe^y$ at the point $(0,0)$.

$\Rightarrow$ Taylor series $- f(x,y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + k^2 f_{yy} + 2hk f_{xy})$

$f(x,y) = xe^y \Rightarrow f(0,0) = 0$

$f_x = e^y \Rightarrow f_x(0,0) = e^0 = 1$

$f_y = xe^y \Rightarrow f_y(0,0) = 0.$

$f_{xx} = 0 \Rightarrow f_{xx}(0,0) = 0.$

$f_{yy} = xe^y \Rightarrow f_{yy}(0,0) = 0.$

$f_{xy} = e^y \Rightarrow f_{xy}(0,0) = e^0 = 1$

Taylor series $\Rightarrow 0 + (x-0)(1) + (y-0)(0) + \frac{1}{2}[(x-0)^2(0) + (y-0)^2(0) + 2xy(1)]$.

$= x + \frac{1}{2}(2xy) = x + xy$

45. Find the Taylor series upto 2nd degree of $f(x,y) = y\sin x$ at the point $(0,0)$.

$\Rightarrow$ Taylor series $- f(x,y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + k^2 f_{yy} + 2hk f_{xy})$

$f(x,y) = y\sin x \Rightarrow f(0,0) = 0$

$f_x = y\cos x \Rightarrow f_x(0,0) = 0$

$f_y = \sin x \Rightarrow f_y(0,0) = 0.$

$f_{xx} = -y\sin x \Rightarrow f_{xx}(0,0) = 0.$

$f_{yy} = 0 \Rightarrow f_{yy}(0,0) = 0.$

$f_{xy} = \cos x \Rightarrow f_{xy}(0,0) = 1.$

Taylor series => $0 + (x-0)(0) + (y-0)(0) + \frac{1}{2}\left[x^2(0) + y^2(0) + 2xy(1)\right]$.

$= 0 + \frac{1}{2}(2xy) = xy.$

46 Find the Taylor series upto 3rd degree of $f(x,y) = e^x \cos y$. at the point $(0,0)$

=> Taylor series {3rd degree} - $f(x,y) + (hf_x + Kf_y) + \frac{1}{2!}(h^2 f_{xx} + K^2 f_{yy} + 2hK f_{xy})$

$+ \frac{1}{3!}(h^3 f_{xxx} + K^3 f_{yyy} + 3h^2K f_{xxy} + 3hK^2 f_{xyy})$

$f(x,y) = e^x \cos y \Rightarrow f(0,0) = e^0 \cos 0 = 1$

$f_x = e^x \cos y \Rightarrow f_x(0,0) = 1$

$f_y = -e^x \sin y \Rightarrow f_y(0,0) = 0$

$f_{xy} = -e^x \sin y \Rightarrow f_{xy}(0,0) = 0$

$f_{xx} = e^x \cos y \Rightarrow f_{xx}(0,0) = 1$

$f_{yy} = -e^x \cos y \Rightarrow f_{yy}(0,0) = -1$

$f_{xyy} = -e^x \cos y \Rightarrow f_{xyy}(0,0) = -1$

$f_{xxy} = -e^x \sin y \Rightarrow f_{xxy}(0,0) = 0.$

$f_{xxx} = e^x \cos y \Rightarrow f_{xxx}(0,0) = 1$

$f_{yyy} = e^x \sin y \Rightarrow f_{yyy}(0,0) = 0$

Taylor series => $1 + (x)(1) + y(0) + \frac{1}{2}\left[x^2(1) + y^2(-1) + 2xy(0)\right]$

$+ \frac{1}{6}\left[x^3(1) + y^3(0) + 3x^2y(0) + 3xy^2(-1)\right].$

$= 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2)$

47. Find the critical points of the function ~~$f(x,y) = xy + x^2 + y^2$~~
$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum occurs.

$\Rightarrow$ $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2$, $f_y = x - 2y - 2$.

Now, $f_x = 0 \Rightarrow y - 2x - 2 = 0 \Rightarrow 2x - y + 2 = 0$.

$f_y = 0 \Rightarrow x - 2y - 2 = 0$

$2x - y + 2 = 0$.

$(-)\ 2x \underset{(+)}{-} 4y \underset{(+)}{-} 4 = 0$.

$3y + 6 = 0 \Rightarrow \boxed{y = -2}$

$2x + 2 + 2 = 0 \Rightarrow \boxed{x = -2}$

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-2)(-2) - 1 = 4 - 1 = 3 > 0$.

$D(x,y) > 0$, $r < 0$.

$\therefore$ Local maximum at $(-2,-2)$

48. Find the critical points of the function $f(x,y) = -3x^2 + 3y^2 + 6xy - 2y^3$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum occurs.

$\Rightarrow$ $f(x,y) = -3x^2 + 3y^2 + 6xy - 2y^3$

$f_x = -6x + 6y$, $f_y = 6y + 6x - 6y^2$.

Now, $f_x = 0 \Rightarrow -6x + 6y = 0 \Rightarrow -x + y = 0$.

$f_y = 0 \Rightarrow 6y + 6x - 6y^2 = 0 \Rightarrow y(1-y) + y = 0$

$\Rightarrow y(2-y) = 0 \Rightarrow y = 0 \,/\, y = 2$.

$\therefore (x,y) = (0,0)$ or $(2,2)$

$f_{xx} = -6$, $f_{yy} = 6 - 12y$, $f_{xy} = 6$

$D(x,y) = (-6)(6-12y) - 36 = 72y - 72 = 72(y-1)$

at $(0,0)$: $D(0,0) = 72(0-1) = -72 < 0$

at $(2,2)$: $D(2,2) = 72(2-1) = 72 > 0$.

Saddle at $(0,0)$

also, $D(x,y) > 0$ and $f_{xx} < 0$

$\therefore$ Local maximum at $(2,2)$.

4a. Find the critical points of the function $f(x,y) = 10xye^{-(x^2+y^2)}$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum.

$\Rightarrow$
$$f(x,y) = 10xye^{-(x^2+y^2)}$$

$$f_x = 10y\left[x e^{-(x^2+y^2)} \cdot -2x + e^{-(x^2+y^2)}\right].$$

$$f_x = 10y\left[-2x^2 \cdot e^{-(x^2+y^2)} + e^{-(x^2+y^2)}\right]$$

$$f_x = -20x^2y\, e^{-(x^2+y^2)} + 10y\, e^{-(x^2+y^2)}$$

$$f_{xx} = -20y\left[x^2 e^{-(x^2+y^2)} \cdot -2x + e^{-(x^2+y^2)} \cdot 2x\right] + 10y \cdot e^{-(x^2+y^2)} \cdot -2x$$

$$f_{xx} = 40x^3y\, e^{-(x^2+y^2)} - 40xy\, e^{-(x^2+y^2)} - 20xy\, e^{-(x^2+y^2)}$$

$$f_y = 10x\left[y e^{-(x^2+y^2)} \cdot -2y + e^{-(x^2+y^2)}\right].$$

$$f_y = -20xy^2 e^{-(x^2+y^2)} + 10x e^{-(x^2+y^2)}$$

$$f_{yy} = -20x\left[y^2 e^{-(x^2+y^2)} \cdot -2y + e^{-(x^2+y^2)} \cdot 2y\right] + 10x \cdot e^{-(x^2+y^2)} \cdot -2y$$

$$f_{yy} = 40xy^3 e^{-(x^2+y^2)} - 40xy\, e^{-(x^2+y^2)} - 20xy\, e^{-(x^2+y^2)}$$

$$f_{xy} = 40x^2y^2 e^{-(x^2+y^2)} - 20x^2 e^{-(x^2+y^2)} - 20y^2 e^{-(x^2+y^2)} + 10e^{-(x^2+y^2)}$$

Now, $f_x = 0 \Rightarrow y - 20x^2 y e^{-(x^2+y^2)} + 10y e^{-(x^2+y^2)} = 0$

$\cancel{\Rightarrow -2x^2 + y = 0 \Rightarrow}$

$\Rightarrow 10 y e^{-(x^2+y^2)} (-2x^2+1) = 0.$

$y = 0, \quad x = \pm \frac{1}{\sqrt{2}}.$

$f_y = 0 \Rightarrow -20xy^2 e^{-(x^2+y^2)} + 10x e^{-(x^2+y^2)} = 0.$

$\Rightarrow 10x e^{-(x^2+y^2)} (-2y^2+1) = 0.$

$x = 0, \quad y = \pm \frac{1}{\sqrt{2}}$

Critical points: $(0,0), \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \left(0, \pm\frac{1}{\sqrt{2}}\right), \left(\pm\frac{1}{\sqrt{2}}, 0\right).$

$f_{xx}(0,0) = 0, \; f_{yy}(0,0) = 0,$

$f_{xx}(1/\sqrt{2}, 1/\sqrt{2}) = -\frac{30}{e} + \frac{10}{e} = -\frac{20}{e} = f_{xx}(-1/\sqrt{2}, -1/\sqrt{2})$

$f_{yy}(1/\sqrt{2}, 1/\sqrt{2}) = -\frac{30}{e} + \frac{10}{e} = -\frac{20}{e} = f_{yy}(-1/\sqrt{2}, -1/\sqrt{2})$

$f_{xx}(0, \pm 1/\sqrt{2}) = 0, \quad f_{yy}(0, \pm 1/\sqrt{2}) = 0.$

$f_{xx}(\pm 1/\sqrt{2}, 0) = 0, \quad f_{yy}(\pm 1/\sqrt{2}, 0) = 0.$

$f_{xy}(0,0) = 10$

$f_{xy}(\pm 1/\sqrt{2}, 0) = 10 e^{-1/2}(1-1) = 0.$

$f_{xy}(0, \pm 1/\sqrt{2}) = 0$

$f_{xy}(1/\sqrt{2}, 1/\sqrt{2}) = 0$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2.$

at $(0,0) = 0 - (10)^2 = -100 < 0$

Saddle point at $(0,0)$

at $(\pm 1/\sqrt{2}, 0) = 0 - 0 = 0$

at $(0, \pm 1/\sqrt{2}) = 0 - 0 = 0.$

at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = -\frac{20}{e} \cdot -\frac{20}{e} - 0 = \frac{40}{e^2} > 0$, {local maximum}

50 Find the critical points of the function $f(x,y) = x^3 - y^3 - 2xy + 6$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum occurs.

$\Rightarrow \quad f(x,y) = x^3 - y^3 - 2xy + 6.$

$f_x = 3x^2 - 2y \quad , \quad f_y = -3y^2 - 2x$

Now, $f_x = 0 \Rightarrow 3x^2 - 2y = 0 \Rightarrow 3x^2 = 2y$

$f_y = 0 \Rightarrow -2x - 3y^2 = 0 \Rightarrow 2x + 3y^2 = 0.$

$\Rightarrow \quad 2x + 3\left(\frac{3x^2}{2}\right)^2 = 0.$

$2x + \cancel{1}\,.\,\frac{9x^4}{4} = 0..$

$x\left(2 + \frac{27x^3}{4}\right) = 0 \Rightarrow x = 0 \,/\, x = \pm\frac{2}{3}.$

$\therefore \; y = 0 \quad , \quad y = \pm\frac{2}{3}$

Critical points : $(0,0)$, $\left(\pm\frac{2}{3}, \pm\frac{2}{3}\right)$, $\left(0, \pm\frac{2}{3}\right)$, $\left(\pm\frac{2}{3}, 0\right)$

$f_{xx} = 6x \quad , \quad f_{yy} = -6y$

$f_{xy} = -2,$

$f_{xx}(0,0) = 0 \quad , \quad f_{yy}(0,0) = 0.$

$f_{xx}\left(\pm\frac{2}{3}, \pm\frac{2}{3}\right) = 4 \quad , \quad f_{yy}\left(\frac{2}{3}, \frac{2}{3}\right) = -4.$

$f_{xx}\left(0, \pm\frac{2}{3}\right) = 0 \quad , \quad f_{yy}\left(0, \pm\frac{2}{3}\right) = -4.$

$f_{xx}\left(\pm\frac{2}{3}, 0\right) = 4 \quad , \quad f_{yy}\left(\pm\frac{2}{3}, 0\right) = 0.$

$D(x,y) = f_{xx}.f_{yy} - (f_{xy})^2$

~~D(0,0)~~ at $(0,0)$ = $0 - (-2)^2 = -4 < 0.$

saddle point at $(0,0)$

at $\left(\frac{2}{3}, \frac{2}{3}\right)$ = $4(-4) - (-2)^2 = -16 - 4 = -20 < 0.$

Saddle point at $\left(\pm\frac{2}{3}, \pm\frac{2}{3}\right)$

at $\left(0, \pm\frac{2}{3}\right)$: $0 - (-2)^2 = -4 < 0$

at $\left(\pm\frac{2}{3}, 0\right)$: $4(0) - (-2)^2 = -4 < 0.$

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51. Find the critical points of the function $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum occurs.

$\Rightarrow$ $$f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$$

$$f_x = 12x - 6x^2 + 6y, \quad f_y = 6y + 6x.$$

Now, $f_x = 0 \Rightarrow 12x - 6x^2 + 6y = 0 \Rightarrow \cancel{x(2-x)=0}.$

$$2x - x^2 + y = 0.$$

$$f_y = 0 \Rightarrow 6y + 6x = 0 \Rightarrow x = -y.$$

$$2x - x^2 - x = 0.$$

$$x(1-x) = 0 \Rightarrow x = 0 \,/\, x = 1.$$

$$\therefore y = 0 \,/\, y = -1$$

Critical points : $(0,0)$, $(1,-1)$

$$f_{xx} = 12 - 12x, \quad f_{yy} = 6.$$

$$f_{xy} = 6.$$

$$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

at $(0,0)$ $= 12(6) - 36 = 36 > 0$; $r > 0$.

Local minimum at $(0,0)$

at $(1,-1)$: $12(0) - 36 = -36 < 0$.

saddle point at $(1,-1)$

52. Find the critical points of the function $f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$ and use second derivative test to classify each point as one where a saddle, local minimum or local maximum occurs.

$\Rightarrow$ $$f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$$

$f_x = 3x^2 + 3y^2 - 15$, $f_y = 6xy + 3y^2 - 15$

Now, $f_x = 0 \Rightarrow 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5$

$f_y = 0 \Rightarrow 6xy + 3y^2 - 15 = 0 \Rightarrow 2xy + y^2 = 5$

$\Rightarrow 2xy + 5 - x^2 = 8$ | $4y^2 + y^2 = 5$

$\Rightarrow x(2y - x) = 0$ | $y^2 = 1$

$x = 0$ / $x = 2y$ | $y = 1$ / $y = 0$

$\therefore x = 0$ / $x = 2$

Critical points : $(0,0)$, $(2,1)$, $(-2,-1)$

$f_{xx} = 6x$, $f_{yy} = 6x + 6y$

$f_{xy} = 6y$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$

at $(0,0)$: $0 - 0 = 0$ {No conclusion}

at $(2,1)$: $12(18) - 36 = 216 - 36 = 180 > 0$; $r > 0$

Local minimum at $(2,1)$

at $(-2,-1)$: $(-12)(-18) - (-6)^2 = 216 - 36 = 180 > 0$; $r < 0$

Local maximum at $(-2,-1)$

53. A delivery company accepts only rectangular boxes, the sum of whose length and girth (perimeter of a cross-section) does not exceed 108. Find the dimensions of an acceptable box of largest volume

$\Rightarrow$ Let the dimensions be, x, y, z respectively and Volume be V

$x + 2(y+z) = 108 \Rightarrow x + 2y + 2z = 108$

$x = 108 - (2y + 2z)$

$V = xyz$

$V = (108 - 2y - 2z)yz = 108yz - 2y^2z - 2yz^2$

$V_y = 108z - 4yz - 2z^2$

$V_z = 108y - 2y^2 - 4yz$

Now, $V_y = 0 \Rightarrow 108z - 4yz - 2z^2 = 0$

$\Rightarrow z(108 - 4y - 2z) = 0 \Rightarrow 108 - 4y - 2z = 0$

$V_z = 0 \Rightarrow 108y - 2y^2 - 4yz = 0$

$\Rightarrow y(108 - 2y - 4z) = 0 \Rightarrow 108 - 2y - 4z = 0$

$108 - \cancel{4y} - 2z = 0$

$\underset{(-)}{216} \; \underset{(+)}{-} \; \cancel{4y} \; \underset{(+)}{-} \; 8z = 0$

$-108 + 6z = 0 \Rightarrow \boxed{z = 18}$

$108 - 4y - 36 = 0$

$4y = 72$

$\boxed{y = 18}$

$z = 0, y = 54 \;|\; y = 0, z = 54$

Critical points : $(18,18)$, $(54,0)$, $(0,54)$

$V_{yy} = -4z \quad , \quad V_{zz} = -4y$

$V_{yz} = 108 - 4y - 4z$

$D(y,z) = f_{yy} \cdot f_{zz} - (f_{yz})^2$

at $(18,18) = -4(18) \cdot -4(18) - (108 - 4(18) - 4(18))^2$

$= 5184 - 1296 = 3888 > 0 \quad , \; r < 0$

Local maximum at $(18,18)$

at $(0,54) = -4(54) \cdot -4(0) - 108 = -108 < 0$

at $(54,0) = -4(0) \cdot -4(54) - 108 = -108 < 0$

$\therefore$ V_{max} at $(18,18)$

$x + 2(y+z) = 108$

$x + 2(36) = 108$

$x = 108 - 72$

$\boxed{x = 36}$

Therefore dimensions are $36 \times 18 \times 18$.

54 Use Lagrange Multiplier method to find the maximum and minimum values of $f(x,y) = xy$ on the curve $3x^2 + y^2 = 6$.

$\Rightarrow$ $f(x,y) = xy$, $\phi = 3x^2 + y^2 - 6$

Consider Lagrange's function

$F(x,y) = f(x,y) + \lambda \phi$

where λ is Lagrange's Multiplier.

$F(x,y) = xy + \lambda(3x^2 + y^2 - 6)$

For maximum and minimum, $df = 0$.

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right)dy = 0$$

$$(y + \lambda.6x)dx + (x + \lambda 2y)dy = 0.$$

$\therefore$ $y + 6x\lambda = 0 \Rightarrow x = \frac{-y}{6\lambda} \Rightarrow \lambda = \frac{-y}{6x}$.

$x + 2y\lambda = 0 \Rightarrow y = \frac{-x}{2\lambda} \Rightarrow \lambda = \frac{-x}{2y}$.

as, $3x^2 + y^2 = 6$

$3\left(\frac{-y}{6\lambda}\right)^2 + \left(\frac{-x}{2\lambda}\right)^2 = 6$

$\frac{3y^2}{12\lambda^2} + \frac{1}{4\lambda^2}\frac{y^2}{36\lambda^2} = 6$

$\frac{y^2}{12\lambda^2} + \frac{y^2}{144\lambda^4} = 6$

$\frac{y}{6x} = \frac{x}{2y}$

$y^2 = 3x^2$.

$3x^2 + y^2 = 6$.

$y^2 + y^2 = 6$.

$y^2 = 3$.

$\boxed{y = \pm\sqrt{3}}$

$3x^2 = 3$.

$x^2 = 1$.

$\boxed{x = \pm 1}$.

$\boxed{\lambda = \frac{\sqrt{3}}{6}}$

Critical points : $(1, \sqrt{3})$ & $(-1, -\sqrt{3})$ and $(1, -\sqrt{3})$ & $(-1, \sqrt{3})$

$$\left.\begin{aligned} f(1,\sqrt{3}) &= (1)(\sqrt{3}) = \sqrt{3}. \\ f(-1,-\sqrt{3}) &= (-1)(-\sqrt{3}) = \sqrt{3} \end{aligned}\right\} \text{Maximum}$$

$$\left.\begin{aligned} f(-1,\sqrt{3}) &= (-1)(\sqrt{3}) = -\sqrt{3} \\ f(1,-\sqrt{3}) &= (1)(-\sqrt{3}) = -\sqrt{3} \end{aligned}\right\} \text{Minimum}$$

Maximum at $(1,\sqrt{3})$ & $(-1,-\sqrt{3})$ i.e., $\sqrt{3}$.

Minimum at $(-1,\sqrt{3})$ & $(1,-\sqrt{3})$ i.e., $-\sqrt{3}$.

55 Find the extreme values of the function $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2=1$ using Lagrange Multiplier method.

=>

$$f(x,y) = x^2+2y^2,$$
$$\phi(x,y) = x^2+y^2-1$$
$$F(x,y) = f(x,y) + \lambda\,\phi(x,y).$$
$$F(x,y) = x^2+2y^2 + \lambda(x^2+y^2-1)$$
$$dF = 0.$$
$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy = 0$$
$$(2x+\lambda.2x)dx + (4y+\lambda.2y)dy = 0.$$
$$\therefore \quad 2x+\lambda 2x = 0 \Rightarrow x(1+\lambda) = 0 \Rightarrow x=0 / \lambda=-1$$
$$4y+\lambda 2y = 0 \Rightarrow y(2+\lambda) = 0 \Rightarrow y=0 / \lambda=-2.$$

$$x=0; \quad y^2-1=0 \Rightarrow y=\pm 1.$$
$$y=0; \quad x^2-1=0 \Rightarrow x=\pm 1.$$

Critical points : $(0,1)$, $(0,-1)$, $(1,0)$ & $(-1,0)$.

$$\left.\begin{aligned} f(0,1) &= 0+2(1)^2 = 2. \\ f(0,-1) &= 0+2(-1)^2 = 2. \end{aligned}\right\} \text{Maximum}$$
$$\left.\begin{aligned} f(1,0) &= 1+2(0) = 1 \\ f(-1,0) &= (-1)^2+2(0) = 1 \end{aligned}\right\} \text{Minimum.}$$

Maximum at $(0,1)$ & $(0,-1)$ i.e., 2.
Minimum at $(1,0)$ & $(-1,0)$ i.e., 1.

56. Use Lagrange Multiplier method to find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

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$\Rightarrow \quad f(x,y) = xy,$

$\phi(x,y) = \frac{x^2}{8} + \frac{y^2}{2} = 1 = x^2 + 4y^2 - 8 = 0.$

$F(x,y) = f(x,y) + \lambda\, \phi(x,y)$

$F(x,y) = xy + \lambda(x^2 + 4y^2 - 8)$

$dF = 0.$

$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right)dy = 0$

$\left(y + \lambda(2x)\right)dx + \left(x + \lambda(8y)\right)dy = 0$

$\therefore \quad y + 2x\lambda = 0 \Rightarrow x = \frac{-y}{2\lambda}.$

$x + 8y\lambda = 0 \Rightarrow y = \frac{-x}{8\lambda}.$

$x = \frac{-(-x)}{2\lambda \cdot 8\lambda} \Rightarrow \cancel{x} = \frac{\cancel{x}}{16\lambda^2} \Rightarrow \boxed{\lambda = \pm 1/4}$

$\boxed{x = \frac{-2y}{\cancel{8}}} \Rightarrow \cancel{y = -8x}.$ | $\cancel{y = \frac{-x}{32}} \Rightarrow \cancel{x = -32y}$ $\boxed{x = 2y}.$

$\cancel{y = -8x(-32y)}$

$\cancel{x^2 + 4(64x^2) - 8 = 0}.$

$4y^2 + 4y^2 - 8 = 0.$

$y^2 = 1 \Rightarrow \boxed{y = \pm 1}.$

$y = -1; \quad x = 2 \quad \& \quad y = 1; \quad x = -2.$

$4y^2 + 4y^2 - 8 = 0.$

$\boxed{y = \pm 1}.$

$y = -1; \quad x = -2. \quad \&$

$y = 1; \quad x = 2.$

Critical points : $(2,-1)$ & $(-2,1)$ and $(2,1)$ & $(-2,-1)$.

$f(2,-1) = 2(-1) = -2$
$f(-2,1) = (-2)(1) = -2.$ } Minimum

$f(2,1) = 1(2) = 2.$
$f(-2,-1) = (-1)(-2) = 2$ } Maximum

Minimum at $(2,-1)$ & $(-2,1)$ i.e, $-2.$

Maximum at $(2,1)$ & $(-2,-1)$ i.e., $2.$

57. Find the extreme values of the function $f(x,y) = 3x+4y$ on the circle $x^2+y^2=1$ using Lagrange Multiplier method

$\Rightarrow$ $f(x,y) = 3x+4y$.

$\phi(x,y) = x^2+y^2-1$.

$F(x,y) = f(x,y) + \lambda\phi(x,y)$

$F(x,y) = (3x+4y) + \lambda(x^2+y^2-1)$

$dF = 0$.

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy = 0$$

$$(3 + \lambda(2x))\,dx + (4 + \lambda(2y))\,dy = 0.$$

$$3 + 2x\lambda = 0 \Rightarrow x = \frac{-3}{2\lambda}.$$

$$4 + 2y\lambda = 0 \Rightarrow y = \frac{-4}{2\lambda}.$$

$$\left(\frac{-3}{2\lambda}\right)^2 + \left(\frac{-4}{2\lambda}\right)^2 - 1 = 0.$$

$$\frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} = 1 \Rightarrow \frac{25}{4\lambda^2} = 1 \Rightarrow \boxed{\lambda = \pm\frac{5}{2}}.$$

Put $\lambda = \frac{5}{2}$: $x = \frac{-3}{2}\times\frac{2}{5} \Rightarrow \boxed{x = \frac{-3}{5}}$ | $y = \frac{-4}{2}\times\frac{2}{5} \Rightarrow \boxed{y = \frac{-4}{5}}$.

Put $\lambda = -\frac{5}{2}$: $\boxed{x = \frac{3}{5}}$ & $\boxed{y = \frac{4}{5}}$

Critical points : $\left(\frac{3}{5}, \frac{4}{5}\right)$ & $\left(-\frac{3}{5}, \frac{-4}{5}\right)$

$f(3/5, 4/5) = 3(3/5) + 4(4/5) = 9/5 + 16/5 = 5.$

$f(-3/5, -4/5) = -3(3/5) - 4(4/5) = -9/5 - 16/5 = -5.$

Maximum at $(3/5, 4/5)$ i.e., 5

Minimum at $(-3/5, -4/5)$ i.e., -5.

58 Find the extreme values of the function $f(x,y,z) = x + y + 2z$ on the surface $x^2+y^2+z^2 = 3$ using Lagrange Multiplier method.

→ $f(x,y,z) = x + y + 2z$

$\phi(x,y,z) = x^2 + y^2 + z^2 - 3$

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$dF = 0$

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right)dz = 0$$

$$(1 + 2x\lambda)dx + (1 + 2y\lambda)dy + (1 + 2z\lambda)dz = 0$$

$\therefore\ 1 + 2x\lambda = 0 \Rightarrow \lambda = \frac{-1}{2x}$

$1 + 2y\lambda = 0 \Rightarrow \lambda = \frac{-1}{2y}$

$2 + 2z\lambda = 0 \Rightarrow \lambda = \frac{-2}{2z} = \frac{-1}{z}$

Now, $\frac{1}{2x} = \frac{1}{2y} \Rightarrow x = y$

and $\frac{1}{2x} = \frac{1}{z} \Rightarrow z = 2x \quad \therefore\ x = y = \frac{z}{2}$

$x^2 + x^2 + 4x^2 = 3$

$6x^2 = 3 \Rightarrow \boxed{x = \pm\frac{1}{\sqrt{2}}} \quad \therefore \boxed{y = \pm\frac{1}{\sqrt{2}}}\ \&\ \boxed{z = \pm\sqrt{2}}$

Critical points : $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ & $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 2\sqrt{2} = \frac{1+1+4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 3\sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 2\sqrt{2} = -3\sqrt{2}$

Maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ i.e., $3\sqrt{2}$

Minimum at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ i.e., $-3\sqrt{2}$

59. Find the point on the line $y = 3 - 2x$ which is nearest to the origin using Lagrange multiplier method.

$\Rightarrow$ The distance of any point (x, y) to the origin is

$d(x, y) = \sqrt{x^2 + y^2}$

Hence, we take function as:

$f(x, y) = d^2 = x^2 + y^2$.

Given that, $y = 3 - 2x$.

$\therefore \phi(x, y) = 2x + y - 3$.

$\therefore F(x, y) = f(x, y) + \lambda \phi(x, y)$

$F(x, y) = x^2 + y^2 + \lambda(2x + y - 3)$

$dF = 0$.

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy = 0$$

$$(2x + \lambda(2))\, dx + (2y + \lambda)\, dy = 0$$

$\therefore 2x + 2\lambda = 0 \Rightarrow x = -\lambda$

$2y + \lambda = 0 \Rightarrow y = -\lambda/2$.

Now, $2(-\lambda) - \frac{\lambda}{2} - 3 = 0$.

$-5\lambda/2 = 3 \Rightarrow \boxed{\lambda = -6/5}$

$\therefore \boxed{x = \frac{6}{5}}$ & $\boxed{y = \frac{3}{5}}$.

60. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

$\Rightarrow$ Let the three real numbers be x, y, z respectively.

$f(x, y, z) = x^2 + y^2 + z^2$.

$\phi(x, y, z) = x + y + z - 9$.

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(x + y + z - 9)$

$dF = 0$ (For minimum)

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

$$(2x + \lambda) dx + (2y + \lambda) dy + (2z + \lambda) dz = 0$$

$\therefore$ $2x + \lambda = 0 \Rightarrow x = -\lambda/2$.

$2y + \lambda = 0 \Rightarrow y = -\lambda/2$.

$2z + \lambda = 0 \Rightarrow z = -\lambda/2$.

$\therefore x = y = z$

$x + y + z - 9 = 0$.

$$\left(\frac{-\lambda}{2}\right) + \left(\frac{-\lambda}{2}\right) + \left(\frac{-\lambda}{2}\right) - 9 = 0$$

$$-3\lambda/2 = 9$$

$$\boxed{\lambda = -6}$$

$\therefore$ $x = \frac{-(-6)}{2} \Rightarrow \boxed{x = 3}$

$\therefore \boxed{x = y = z = 3}$