Error:- The difference between exact value and an estimated value or approximated value is called error.

Exact Number:- 1, 2, $\frac{1}{2}$, $\frac{3}{4}$, ...

Approximate Number:- Approximate numbers are those numbers that represent the numbers to a certain degree of accuracy.

Example: π is approximated as 3.1416 or if we desire a better approximation i.e. 3.14159265.

We can not get the exact value.

Significant Digits:- The digits that are used to express a number are called significant digits or significant figure.

Example:- ① 3.1416 → Five significant digits

② 0.00023 → Two significant digits.

Type of errors:-

① **Inherent Error:-** Errors which exist in the problem either due to approximate given data or limitations of computing. It can be minimized by taking better data and using high ~~precision~~ precision computing.

Example :- π, e

$\pi = 3.14159\,2654$ to 3.1416

$e = 2.7182818$ to 2.718

$x = 0.3333$ to $1/3$

Hence $\frac{1}{3} \approx 0.30$, $\frac{1}{3} \approx 0.33$ & $\frac{1}{3} \approx 0.34$

Then

$\left|\frac{1}{3} - 0.30\right| = \frac{1}{30}$, $\left|\frac{1}{3} - 0.33\right| = \frac{1}{300}$

and

$\left|\frac{1}{3} - 0.34\right| = \frac{1}{150}$

Here $\left|\frac{1}{3} - 0.33\right| = 1/300$ is best

that mean $\frac{1}{3} \approx 0.33$ is best.

Round off error :- In numerical compatations, we come across numbers which has large of digits and it will be necessary to cut them to a usuable numbers of figures. This process is called rounding off and the error comes in this process are called round off errors.

Ex :- Suppose we have a numbers as 7.5846712 and I want rounding it upto 3 decimal points, then we have 7.585 (round off number).

Hence round off error = True - round off numbers.

Truncation Error :- This type of error ~~araised~~ arise due to use of approximation formula in computation or by truncating the infinite series to some approximate terms.

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

If we take

$$e^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \text{ (upto n terms)}$$

Abslule Error :- Abslute error is the abslute value of the difference between the ~~t~~ exact value of a quantity and its approximate value. Thus, if X is the exact value of a quantity and X_1 is its approximate value, then the abslute error E_A is given by

$$E_A = |X - X_1|, \quad \text{where } X - X_1 = \delta X$$

Relative Error:- The relative error E_R is defined by

$$E_R = \frac{E_A}{X} = \frac{|\delta X|}{X}.$$

Percentage Error:- The percentage error (E_p) is given by

$$E_p = 100\, E_R \quad \text{or} \quad E_p = \frac{E_A}{X} \times 100 .$$

Relative Accuracy:- If ΔX denotes a number such that

$|X - X_1| \leq \Delta X$, where X is true value of a quantity and X_1 denotes the approximate value, then ΔX represents the upper ~~limits~~ limit on the abslute error and is called measure of abslute accuracy, and $\frac{\Delta X}{|X|}$ measures the relative accuracy.

Example:- If π is given by $\frac{22}{7} = 3.1428571 = X_1$ and its true value is $3.1415926 = X$.

Then

Abslute error $= |X - X_1| = 0.0012645$

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<u>Algebraic equation</u> :- The eqn of the form $f(x)=0$, where $f(x)$ is purely a polynomial is called an algebraic eqn.

<u>Example</u> :- $x^6 - x^4 + 2x^3 + 1 = 0$

<u>Transcendental equation</u> :- The eqn of the form $f(x)=0$, where $f(x)$ involves trigonometric, arithmetic or exponential terms ~~in it~~ is called transcendental eqn.

Ex :- $x\log x - 1 = 0$, $xe^x + 10x - 1 = 0$

<u>Bolzano or Bisection Method</u>

Suppose we have an eqn $f(x)=0$, where

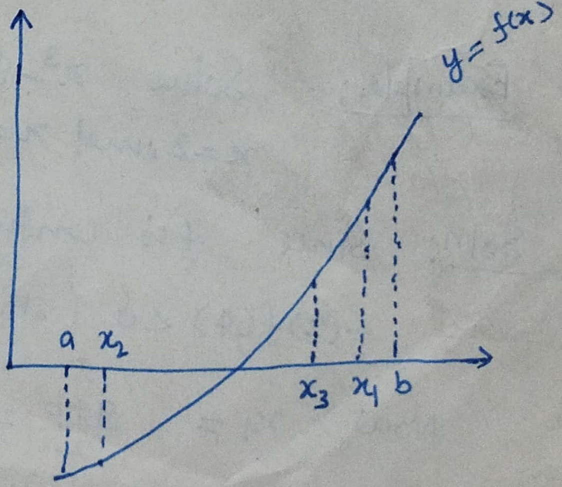
(I) f is continuous in $[a, b]$, and

(II) $f(a)f(b) < 0$

Then, by Bolzano theorem, there exist atleast one point $c \in (a,b)$ s.t $f(c)=0$.

Now, we will take first approximation as $x_1 = \dfrac{a+b}{2}$.

If $f(x_1)=0$, then x_1 is root of $f(x)=0$, otherwise root lies in either $[a, x_1]$ or $[x_1, b]$ according as $f(a)f(x_1) < 0$ or $f(x_1)f(b) \leq 0$. Similarly, we will bisect the interval as before and continue the process to improve the resut.



From figure, we can see that $f(a)f(x_1) < 0$, then we will take next approximation as

$$x_2 = \frac{a+x_1}{2}.$$

If $f(x_2)f(x_1) < 0$, then the next approximation will be $x_3 = \frac{x_1+x_2}{2}$.

Similarly, we will find the next approximation

Working Rule:-

Step 1:- Firstly, we will choose two points $x_0 (x=a)$ and $x_n (x=b)$ such that $f(x_0)f(x_n) < 0$.

Step 2: We will take the approximate soln of $f(x)=0$ as $x_r = \frac{x_0+x_n}{2}$

Step 3:- We will check that in which subintervals, the root lies.

(a) If $f(x_0)f(x_r) < 0$, then root lies in $[x_0, x_r]$ and repeat the process from step 1 to 3.

(b) If $f(x_r)f(x_n) < 0$, then root lies in $[x_r, x_n]$ and repeat the process from step 1 to 3.

(c) If ~~f(x0) f~~ $f(x_r)=0$, then x_r is root of $f(x)$ and stop the process.

Example:- Solve $x^3 - 9x + 1 = 0$ for the root between $x=2$ and $x=4$ by using Bisection method.

Soln: Since f is continuous in $[2, 4]$ and $f(2)f(4) < 0$ (since $f(2) = -9$ and $f(4) = 29$).

Now $x_1 = \frac{a+b}{2} = \frac{2+4}{3} = 3$.

Since $f(3) = 1$, therefore $f(2)f(3) < 0$. ~~and~~ and hence

$$x_2 = \frac{2+x_1}{2} = \frac{2+3}{2} = 2.5.$$

Since $f(2.5) = -5.87$, therefore $f(x_2)f(x_1) < 0$ i.e, $f(2.5)\,f(3) < 0$, and hence

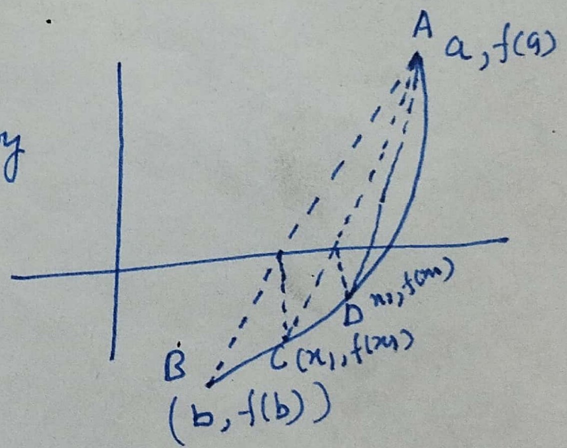
$$x_3 = \frac{2.5+3}{2} = 2.75.$$

False Position Method or Regula Falsi Method

Suppose we have an equation $f(x) = 0$, where

(i) f is continuous in $[a, b]$

(ii) $f(a)\,f(b) < 0$, then by Bolzano theorem there exist a point $c \in (a, b)$ s.t $f(c) = 0$.



Now, equation of the chord joining $A(a, f(a))$ and $B(b, f(b))$ is given by

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

Let $y = 0$ be the point of the intersection of the chord eqn with the x-axis. Then,

$$-f(a) = \frac{f(b)-f(a)}{(b-a)}(x-a)$$

$$\Rightarrow \frac{-f(a)(b-a)}{f(b)-f(a)} = x-a$$

$$\Rightarrow \frac{af(a)-bf(a)}{f(b)-f(a)} + a = x$$

$$\Rightarrow \quad x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Thus, the first approximation is

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

If $f(x_1) = 0$, then x_1 is root of $f(x) = 0$.

If $f(a) f(x_1) < 0$, then second approximation will be as

$$x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

If $f(x_1) f(b) < 0$, then second approximation will be as

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

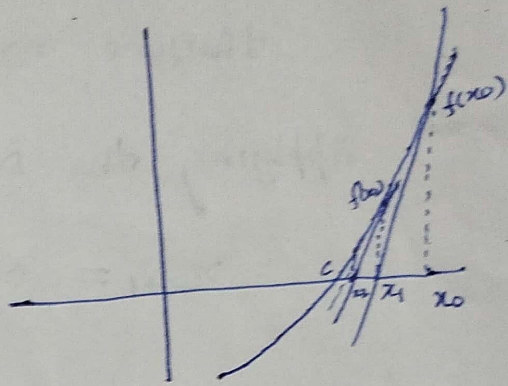
Similarly, we can find x_3, x_4, x_5 and so on.

Newton - Raphson Method :-

Let we have an eqn $f(x)=0$, and x_0 be an initial approximation to the root of $f(x)=0$.

Then $P(x_0, f(x_0))$ ~~be where~~ is a point on the curve f.

~~Beca~~



Now, the ~~tangent line~~ eqn of tangent line to the curve $y=f(x)$ at the point $P(x_0, f(x_0))$ is given by

$$y - f(x_0) = (x-x_0)\, f'(x_0)$$

where $f'(x_0)$ is the slope of the tangent line to the curve at P.

$$\left[\text{Since } m = \left.\frac{dy}{dx}\right|_{x_0} = f'(x)\big|_{x_0} = f'(x_0) \right.$$

For $y=0$, we have

$$0 - f(x_0) = (x-x_0)\, f'(x_0)$$

$$\Rightarrow \quad -\frac{f(x_0)}{f'(x_0)} = x - x_0$$

$$\Rightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad f'(x_0) \neq 0$$

Hence, Next approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad f'(x_0) \neq 0.$$

We repeat the procedure, we have

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Exam :- Perform four iteration of the ~~Net~~ Newton's method to find the smallest positive root of the eqn $f(x) = x^3 - 5x + 1 = 0$

Soln :- Since $f(0) = 1$, $f(1) = -3$ and $f(0)f(1) < 0$, therefore root lies in $(0, 1)$.

Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5}$$

$$= \frac{2x_k^3 - 1}{3x_k^2 - 5}, \quad \text{for } k = 0, 1, 2 \ldots$$

let $x_0 = 0.5$, then

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5}$$

$$= \frac{2(0.5)^3 - 1}{3(0.5)^2 - 5}$$

$$= 0.176471$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5}$$

$$= \frac{2(0.176471)^3 - 1}{3(0.176471)^2 - 5}$$

$$= 0.201568$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 5} = \cancel{2(0.1764}$$

$$= \frac{2(0.201568)^3 - 1}{3(0.201568)^2 - 5}$$

$$= 0.201640$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 5}$$

$$= \frac{2(0.201640)^3 - 1}{3(0.201640)^2 - 5}$$

$$= 0.201640$$

Therefore, the root correct to six decimal place is $x \simeq 0.201640$

Sec-23 8004 (2:30-3:30)
Sec-19 (3:30-4:30) 8001

Iteration Method or fixed point Iteration method

Suppose we have an eq^n as $f(x)=0$, then firstly, we will arrange this eq^n as

$$x = \phi(x). \quad \text{—— (1)}$$

Now, finding a root of $f(x)=0$ is same as finding a number α such that $\alpha = \phi(\alpha)$ i.e a fixed point of $\phi(x)$.

Using eq^n (1), the iteration method is written as

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, 2, \dots$$

The function ϕ is called the iteration function.

~~Initial with~~

Starting with the initial approximation x_0, we compute next approximation as

$$x_1 = \phi(x_0), \quad x_2 = \phi(x_1), \quad x_3 = \phi(x_2), \dots$$

The stopping criterion is same as used in previous method

Remark 1:- There are many ways of rewriting $f(x)=0$ as $x = \phi(x)$.

~~for example~~:-

Example:- For $f(x) = x^3 - 5x + 1 = 0$, we can be rewritten it in the following forms.

$x = \frac{x^3+1}{5}$, $x = (5x-1)^{1/3}$, $x = \sqrt{\frac{5x-1}{x}}$, etc. — (2)

Since there are many ways of ~~writ~~ writing $f(x)=0$ as $x = \phi(x)$, it is important to know wether all or at least ~~on~~ one of then iteration method converges.

Convergence of an iteration method $x_{k+1} = \phi(x_k)$, $k=0,1,2..$ to root, depends on the choice of the iteration function ϕ, and suitable ~~of~~ initial approximation x_0.

From (2), we have

(i) $x_{k+1} = \frac{x_k^3+1}{5}$, $k=0,1,2,...$

(ii) $x_{k+1} = (5x_k-1)^{1/3}$, $k=0,1,2..$

(iii) $x_{k+1} = \sqrt{\frac{5x_k-1}{x_k}}$, $k=0,1,2..$

Since $f(0)=1$, $f(1)=-3$ and $f(0)f(1)<0$, hence at ~~root~~ least one root lies in $(0,1)$.

let $x_0=1$, then from (i),

$x_1 = 0.4$, $x_2 = 0.2128$, $x_3 = 0.20193$,

$x_4 = 0.20165$, $x_5 = 0.20164$.

Hence

$|x_5 - x_4| = |0.20164 - 0.20193| = 0.00001 < 0.0005$.

Hence

$x \simeq x_5 = 0.20164$ is taken as the required approximation to the root.

From (ii), we have

$x_1 = 1.5874$, $x_2 = 1.9072$, $x_3 = 2.0437$,

which does not converge to the root in $(0,1)$

From (iii), we have

$x_1 = 2.0$, $x_2 = 2.1213$, $x_3 = 2.1280, \ldots$

which does not converge to the root in $(0,1)$.

Now, we ~~der~~ derive the condition that the iteration function ϕ should satisfy ~~the condition of convergence.~~ in order that method converges.

Condition of convergence:-

The iteration method for finding a root of $f(x)=0$, is written as

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, 2 \ldots \quad — (3)$$

~~let α is root~~

Let α be the exact root. Then

$$\alpha = \phi(\alpha). \quad — (4)$$

We define the error of approximation at kth iterations as

$$\epsilon_k = x_k - \alpha, \quad k = 0, 1, 2 \ldots \quad — (5)$$

Substracting (3) and (4), we have

$$x_{k+1} - \alpha = \phi(x_k) - \phi(\alpha)$$

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$= (x_k - \alpha)\, \phi'(t_k), \quad x_k < t_k < \alpha.$

(by the mean value theorem)

$$\Rightarrow \quad \epsilon_{k+1} = \phi'(t_k)\, \epsilon_k, \quad x_k < t_k < \alpha.$$

Setting $k = k-1$, we get

$$\epsilon_k = \phi'(t_{k-1})\, \epsilon_{k-1}, \quad x_{k-1} < t_{k-1} < \alpha.$$

Hence

$$\epsilon_{k+1} = \phi'(t_k)\, \phi'(t_{k-1})\, \epsilon_{k-1}.$$

Similarly, we get

$$\epsilon_{k+1} = \phi'(t_k)\, \phi'(t_{k-1}) \cdots \phi'(t_0)\, \epsilon_0.$$

The initial error ϵ_0 is known and is constant.

We have

$$|\epsilon_{k+1}| = |\phi'(t_k)|\, |\phi'(t_{k-1})| \cdots |\phi'(t_0)|\, |\epsilon_0|.$$

let $|\phi'(t)| \leq c$, $k = 0, 1, 2 \ldots$, then

$$|\epsilon_{k+1}| < c^{k+1} |\epsilon_0|$$

For convergence, we require that $|\epsilon_{k+1}| \to 0$ as $k \to \infty$.

This ~~or~~ result is possible, if and only if $c < 1$.

Therefore, the iteration method converges ~~iff~~ if and only if

$$|\phi'(x_k)| \leq c < 1, \quad k = 0, 1, 2 \ldots$$

$$\Rightarrow \quad |\phi'(x)| < 1, \text{ for all } x \text{ in the interval } (a, b).$$

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Example :- Find smallest positive root of the eqn $x^3-x-10=0$, using iteration method or fixed point iteration method.

Soln: Since $f(0)=-10$, $f(1)=-10$, $f(2)=-4$, $f(3)=14$,

and $f(2)f(3)<0$, the smallest positive ~~rott~~ root lies in the interval $(2,3)$.

Write $x^3-x-10=0$ as

$x^3 = x+10 \Rightarrow x=(x+10)^{1/3} \Rightarrow x=\phi(x)$

where $\phi(x)=(x+10)^{1/3}$

Now

$$\phi'(x) = \frac{1}{3(x+10)^{2/3}}$$

Since $|\phi'(x)| = \left|\frac{1}{3(x+10)^{2/3}}\right| < 1$ for all $x \in (2,3)$.

Hence, ~~we have~~

$x_{k+1} = (x_k+10)^{1/3}$ converges.

let $x_0 = 2.5$, then

$x_1 = (2.5+10)^{1/3} = 2.3208$,

$x_2 = (2.3208+10)^{1/3} = 2.3097$

$x_3 = (2.3097+10)^{1/3} = 2.3090$

$x_4 = (2.3090+10)^{1/3} = 2.3089$

Hence

$|x_4-x_3| = 0.0001 < 0.0005$

Hence the required root as $x \approx 2.3089$