1 Compute : $\int_0^2\int_0^2 2x\,dy\,dx$

$$\Rightarrow \int_0^2 \left[2xy\right]_0^2 dx = \int_0^2 4x\,dx = 4\left[\frac{x^2}{2}\right]_0^2 = 4.2 = 8$$

2. Compute : $\int_0^3\int_0^2 (4-y^2)\,dy\,dx$

$$\Rightarrow \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^3 \left(8 - \frac{8}{3}\right) dx = \frac{16}{3}\left[x\right]_0^3 = 16$$

3. Compute : $\int_{-1}^{0}\int_{-1}^{1} (x+y+1)\,dy\,dx$

$$\Rightarrow \int_{-1}^{0} \left[xy + \frac{y^2}{2} + y\right]_{-1}^{1} dx = \int_{-1}^{0} \left\{x + \frac{1}{2} + 1 - \left(-x + \frac{1}{2} - 1\right)\right\} dx$$

$$= \int_{-1}^{0} \left(x + \frac{3}{2} + x + \frac{1}{2}\right) dx = 2\int_{-1}^{0} (x+1)\,dx = 2\left[\frac{x^2}{2} + x\right]_{-1}^{0}$$

$$= 2.-\left(\frac{1}{2} - 1\right) = 1.$$

4 Compute : $\int_1^2\int_0^4 2xy\,dy\,dx$

$$\Rightarrow 2\int_1^2 \left[x\frac{y^2}{2}\right]_0^4 dx = 2.\int_1^2 8x\,dx = 16\left(\frac{x^2}{2}\right)_1^2$$

$$= 16\left(2 - \frac{1}{2}\right) = 8(3) = 24$$

5. Compute double integral over the region $R: 0 \le x \le 1,\ 0 \le y \le 2$: $\iint_R (6y^2 - 2x)\,dA$

$$\Rightarrow \int_0^1\int_0^2 (6y^2 - 2x)\,dy\,dx = \int_0^1 \left[2y^3 - 2xy\right]_0^2 dx$$

$$= \int_0^1 (16 - 4x)\,dx = \left[16x - 2x^2\right]_0^1 = 16 - 2 = 14$$

$0 \le y \le \sin x$
$0 \le x \le 1$

6. Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ and

$R: 0 \le x \le 2 \quad , -1 \le y \le 1$

$$\Rightarrow \int_0^2 \int_{-1}^{1} (100 - 6x^2y)\,dy\,dx$$

$$= \int_0^2 \left[100y - 3x^2y^2\right]_{-1}^{1} dx$$

$$= \int_0^2 \{100 - 3x^2 - (-100 - 3x^2)\}\,dx$$

$$= \int_0^2 (100 - \cancel{3x^2} + 100 + \cancel{3x^2})\,dx = 200\int_0^2 dx = 400$$

7. Compute: $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$.

$$\Rightarrow \int_0^2 \left[4xy + 2y\right]_{x^2}^{2x} dx = \int_0^2 (8x^2 + 4x - 4x^3 - 2x^2)\,dx$$

$$= \int_0^2 (6x^2 - 4x^3 + 4x)\,dx = \left[2x^3 - x^4 + 2x^2\right]_0^2$$

$$= 16 - 16 + 8 = 8.$$

8. Compute: $\int_0^{\pi} \int_0^{\sin x} dy\,dx$.

$$\Rightarrow \int_0^{\pi} [y]_0^{\sin x} = \int_0^{\pi} \sin x\,dx = -[\cos x]_0^{\pi} = -\cos\pi + \cos 0$$
$$= 2.$$

9. Compute: $\int_0^{\pi} \int_0^{x} x \sin y\,dy\,dx$.

$$\Rightarrow \int_0^{\pi} [-x\cos y]_0^{x} = \int_0^{\pi} [-x(\cos x - \cos 0)]\,dx$$

$$= \int_0^{\pi} (-x\cos x + x)\,dx = \int_0^{\pi} x\,dx - \int_0^{\pi} x\cos x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^{\pi} - \left[x\int \cos x\,dx - \int \left(\int \cos x\,dx\right)dx\right]_0^{\pi}$$

$= \frac{\pi^2}{2} - \left[x \sin x - \int \sin x \, dx\right]_0^{\pi}$

$= \frac{\pi^2}{2} - \left[x \sin x + \cos x\right]_0^{\pi}$

$= \frac{\pi^2}{2} - \left[\pi \sin \pi + \cos \pi - 1\right].$

$= \frac{\pi^2}{2} - (-2) = \frac{\pi^2}{2} + 2$

10. Compute: $\int_1^2 \int_y^{y^2} dx\, dy$

$\Rightarrow \int_1^2 [x]_y^{y^2} dy = \int_1^2 (y^2 - y)\, dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2.$

$= \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$

11. Integrate $f(x,y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$

$\Rightarrow \iint_R f(x,y)\, dA. \qquad x \le y \le 2x\,; \quad 1 \le x \le 2.$

$= \int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx = \int_1^2 x\left[\ln y\right]_x^{2x} dx.$

$= \int_1^2 x(\ln 2x - \ln x)\, dx = \int_1^2 x \ln 2\, dx.$

$= \ln 2 \left[\frac{x^2}{2}\right]_1^2 = \left(2 - \frac{1}{2}\right)\ln 2 = \frac{3}{2}\ln 2$

12 Compute: $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

$\Rightarrow \int_0^1 \left(3y^3 \frac{e^{xy}}{y}\right)_0^{y^2} dy = \int_0^1 \left\{\left[3y^2(e^{y^3} - 1)\right]\right\} dy.$

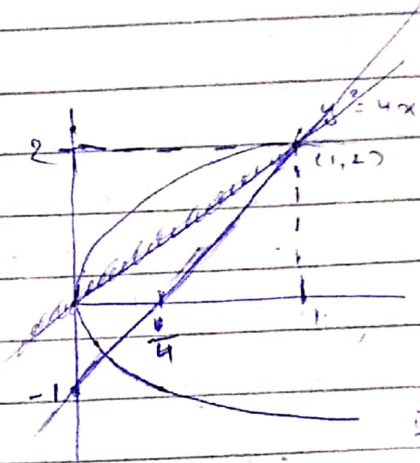
$= 3\int_0^1 y^2 e^{y^3} dy - \int_0^1 3y^2\, dy$

Let $e^{y^3} = t \Rightarrow 3y^2 e^{y^3}\,dy = dt$

$$= \int_1^e dt - \left[y^3\right]_0^1$$

$$= \left[t\right]_1^e - 1 = e - 1 - 1 = e - 2$$

13. Compute : $\int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy$



$x = \frac{y^2}{4} \Rightarrow y^2 = 4x$

$x = \frac{(y+2)}{4} \Rightarrow y = 4x - 2$

$(4x-2)^2 = 4x$

$16x^2 - 16x + 4 = 4x$

$4x^2 - 4x - x + 1 = 0$

$4x^2 - 5x + 1 = 0$

$4x^2 - 4x - x + 1 = 0$

$4x(x-1) - (x-1) = 0$

$(4x-1)(x-1) = 0$

$x = \frac{1}{4}, \; x = 1$

$y = -1, \; y = 2$

2 | 192, 12
2 | 96, 6
2 | 48, 3
2 | 24, 3
2 | 12, 3
2 | 6, 3
3 | 3

$$\int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{12} - \frac{y^2(y+2)}{4} - \left\{4y^2 - \frac{y^6}{192} - \frac{y^4}{4}\right\}\right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{(y^3 + 8 + 6y^2 + 12y)}{12} - \frac{(y^3 + 2y^2)}{4} - 4y^2 + \frac{y^6}{192} + \frac{y^4}{4}\right] dy$$

$$= \int_0^2 \left(\frac{768y + 1536 - 16y^3 - 128 - 96y^2 - 192y - 48y^3 - 96y^2 - 768y^2 + y^6 + 48y^4}{192}\right) dy$$

$$= \int_0^2 \left(\frac{y^6 + 48y^4 - 64y^3 - 960y^2 + 576y + 1408}{192}\right) dy$$

$$= \frac{1}{192}\left[\frac{y^7}{7} + \frac{48}{5}y^5 - 16y^4 - 320y^3 + 988y^2 + 1408y\right]_0^2$$

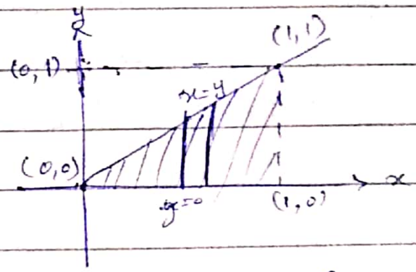
$$= \frac{1}{192}\left[\frac{128}{7} + \frac{48(32)}{5} - 256 - 2560 + 1152 + 2816\right]$$

$$= \frac{1}{192}\left[\frac{128}{7} + \frac{1536}{5} + 1152\right] = \frac{128}{1344} + 8 + 6 = 14.095$$

14/15. Plot the region, reverse the order of integration of the integral: $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$. (Calculate integral)

$\Rightarrow$ $0 \le y \le 1$, $y \le x \le 1$

$y = 0$ to $y = 1$ & $x = y$ to $x = 1$.



$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx.$$

$\therefore$ $x = 0$ to $x = 1$ &

$y = 0$ to $y = x$.

$$= \int_0^1 \frac{\sin x}{x}\left[y\right]_0^x dx$$

$$= \int_0^1 \sin x\, dx = -\left[\cos x\right]_0^1 = -\left[\cos(1) - \cos(0)\right] = 1 - \cos(1)$$

16. Sketch the region of integration and write an equivalent integral with the order of integration reversed for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$.

$\Rightarrow$ $y = x^2$ to $y = 2x$. / $x = 0$ to $x = 2$.

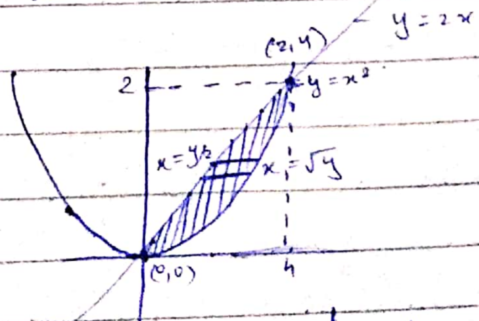
$x^2 = 2x$.

$x^2 - 2x = 0$.

$x(x-2) = 0$.

$x = 0$, $x = 2$.

$y = 0$, $y = 4$.



$\therefore$ $y = 0$ to $y = 4$ / $x = y/2$ to $x = \sqrt{y}$.

$$\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx = \int_0^4\int_{y/2}^{\sqrt{y}}(4x+2)\,dx\,dy$$

$$= \int_0^4 \left[2x^2+2x\right]_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(2y+2\sqrt{y}-\frac{y^2}{2}-y\right) dy$$

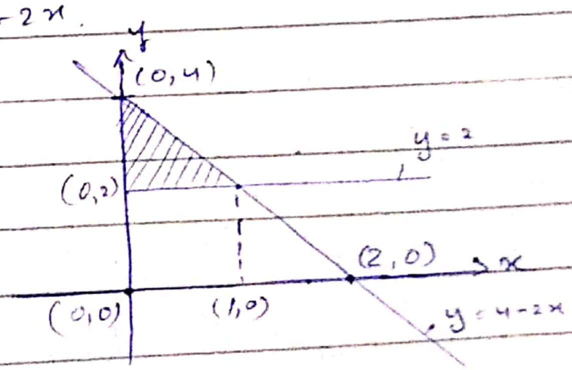
$$= \int_0^4 \left(y+2\sqrt{y}-\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2}+\frac{4}{3}y^{3/2}-\frac{y^3}{6}\right]_0^4$$

$$= 8+\frac{4}{3}\cdot 8-\frac{32}{3} = \frac{24+32-32}{3} = 8.$$

17 Sketch the region of integration and write an equivalent integral with the order of integration reversed for the integral $\int_0^1\int_2^{4-2x} dy\,dx$.

$\Rightarrow$ $x=0$ to $x=1$ / $y=2$ to $y=4-2x$.



$$\int_0^1\int_2^{4-2x} dy\,dx = \int_2^4\int_0^{\frac{4-y}{2}} dx\,dy$$

$$= \int_2^4 \left[x\right]_0^{\frac{4-y}{2}} dy$$

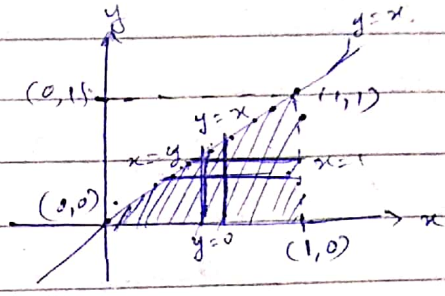
$$= \int_2^4 \left(\frac{4-y}{2}-0\right) dy = \int_2^4 \left(2-\frac{y}{2}\right) dy$$

$$= \int_2^4 \cancel{\frac{y}{2}}\,dy = \left[2y-\frac{y^2}{4}\right]_2^4 = 8-4-4+1 = 1$$

18. Sketch the region of integration, reverse the order of integration and evaluate the integral $\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy$.

$\Rightarrow$ $y=0$ to $y=1$ & $x=y$ to $x=1$

$$\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy = \int_0^1\int_0^x x^2 e^{xy}\,dy\,dx$$



$$= \int_0^1 \left[\frac{x^2 e^{xy}}{x}\right]_0^x dx$$

$$= \int_0^1 \left(x e^{x^2}\right) dx - \int_0^1 x\,dx$$

$\therefore$ $y=0$ to $y=x$

$x=0$ to $x=1$

Let $e^{x^2} = t \Rightarrow 2x e^{x^2}\,dx = dt$

$$= \frac{1}{2}\int_1^e dt - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left[t\right]_1^e - \frac{1}{2}$$

$$= \frac{1}{2}(e-1) - \frac{1}{2} = \frac{e}{2} - 1$$

19. Compute: $\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta$.

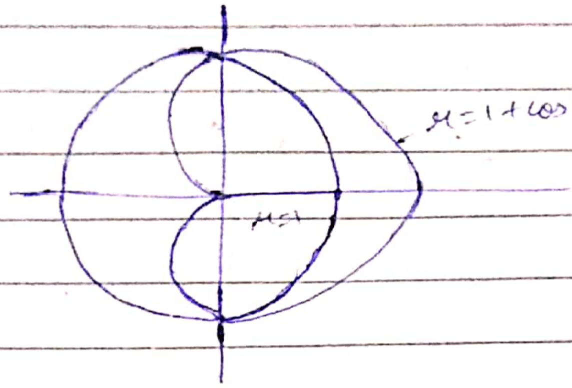
$\Rightarrow$ $\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{1}{4}\left[\theta\right]_0^{\pi/2} = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$

20. Compute: $\int_0^{\pi}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta$.

$\Rightarrow$ $\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_0^{\pi}\frac{\cos 2\theta}{2}\,d\theta = \left[\frac{\sin 2\theta}{4}\right]_0^{\pi} = 0$

21 Find the limits of integration for $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$

$\Rightarrow$



$r = 1$ to $r = 1 + \cos\theta$

$\theta = -\pi/2$ to $\theta = \pi/2$

22 Change the Cartesian integral into polar integral and then compute the polar integral : $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\, dx$

$\Rightarrow \quad \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\, dx$

$x = r\cos\theta$, $y = r\sin\theta$, $dy\, dx = r\, dr\, d\theta$.

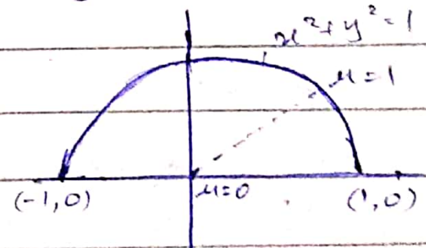
Limits: $x = -1$ to $x = 1$ and $y = 0$ to $y = \sqrt{1-x^2}$

$y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$.

$\therefore \int_{0}^{\pi}\int_{0}^{1} r\, dr\, d\theta$

$= \int_{0}^{\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta = \int_{0}^{\pi} \frac{1}{2} d\theta$

$= \frac{1}{2} [\theta]_0^{\pi} = \frac{\pi}{2}$



$\therefore r = 0$ to $r = 1$.

$\theta = 0$ to $\theta = \pi$.

23. Change the Cartesian integral into polar integral and then compute the polar integral : $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$

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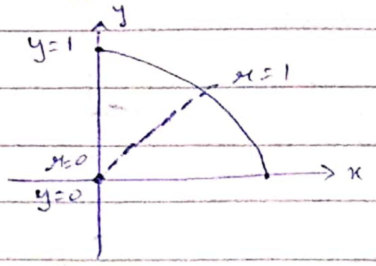
$\Rightarrow$ $y=0$ to $y=1$ & $x=0$ to $x=\sqrt{1-y^2}$.

$x = r\cos\theta$, $y = r\sin\theta$, $dy\,dx = r\,d\theta\,dr$.

$x^2+y^2 = r^2$.

$$\int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy.$$



$$= \int_0^{\pi/2}\int_0^1 r^2 . r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta$$

$\therefore$ $\theta = 0$ to $\theta = \pi/2$.

$r = 0$ to $r = 1$

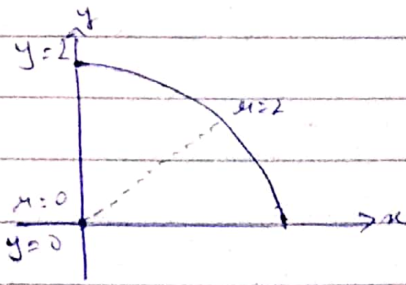
$$= \frac{1}{4}\left[\theta\right]_0^{\pi/2} = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

24. Change the cartesian integral into polar integral and then compute the polar integral: $\int_0^2\int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$.

$\Rightarrow$ $y=0$ to $y=2$ and $x=0$ to $x=\sqrt{4-y^2}$

$\Rightarrow x^2 = 4-y^2 \Rightarrow x^2+y^2 = 4$

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$.



$$\int_0^2\int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^2 r^2 . r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{\pi/2} 4\,d\theta.$$

$\therefore$ $\theta = 0$ to $\theta = \pi/2$.

$r = 0$ to $r = 2$.

$$= 4\left[\theta\right]_0^{\pi/2} = 4\left(\frac{\pi}{2}-0\right) = 2\pi$$

25 Change the Cartesian integral into polar integral and then compute the polar integral

$$\int_1^{\sqrt3}\int_1^{x} dy\,dx.$$

$\Rightarrow$ $x = 1$ to $x = \sqrt{3}$ and $y = 1$ to $y = x$.

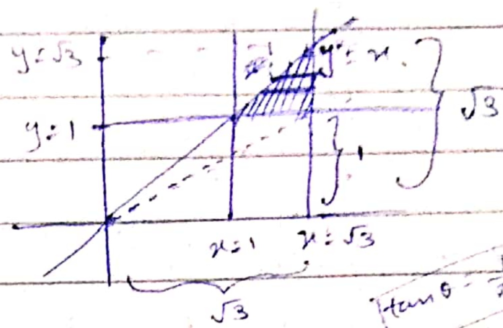
$x = r\cos\theta$, $y = r\sin\theta$

$r\sin\theta = 1$

$r = \operatorname{cosec}\theta$.

$r\cos\theta = \sqrt{3} \Rightarrow r = \sqrt{3}\sec\theta$.

$r \to \operatorname{cosec}\theta$ to $\sqrt{3}\sec\theta$.



$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\theta = \frac{\pi}{6}}$ $\therefore$ $\theta \to \pi/6$ to $\pi/4$.

$\tan\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \frac{\sqrt{3}}{\sqrt{3}} = 1$

$\tan\theta = 1 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$.

$$\int_1^{\sqrt{3}}\int_1^{x} dy\, dx = \int_{\pi/6}^{\pi/4}\int_{\operatorname{cosec}\theta}^{\sqrt{3}\sec\theta} r\, dr\, d\theta.$$

$\frac{216}{3} = 72$

$$= \int_{\pi/6}^{\pi/4} \left[\frac{r^2}{2}\right]_{\operatorname{cosec}\theta}^{\sqrt{3}\sec\theta} d\theta$$

$$= \int_{\pi/6}^{\pi/4} \frac{3\sec^2\theta - \operatorname{cosec}^2\theta}{2}\, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/4} \frac{3}{\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$= \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4}$$

$$= \frac{3}{2} + \frac{1}{2} - \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}} = 2 - \sqrt{3}.$$

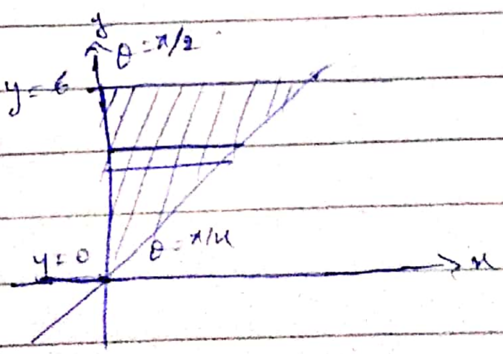
26\. Change the cartesian integral into polar integral and then compute the polar integral.

$$\int_0^6\int_0^y x\, dx\, dy.$$

$\Rightarrow$ $y = 0$ to $y = 6$ and $x = 0$ to $x = y$

$x = r\cos\theta$, $y = r\sin\theta$.

$r = 6\operatorname{cosec}\theta$.



$$\int_{\pi/4}^{\pi/2}\int_0^{6\operatorname{cosec}\theta} r^2\cos\theta\, dr\, d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{6\operatorname{cosec}\theta}(\cos\theta)$$

$$= 72\int_{\pi/4}^{\pi/2} \operatorname{cosec}^3\theta\, \cos\theta\, d\theta$$

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$= 72 \int_{\pi/4}^{\pi/2} \cot\theta \, \operatorname{cosec}^2\theta \, d\theta$

$\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta \, d\theta = dt$

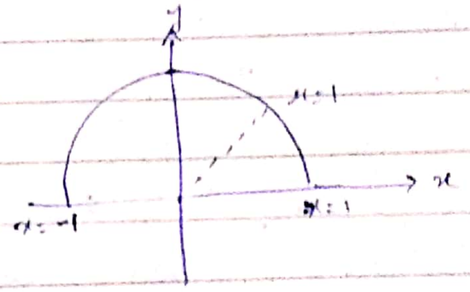
$= 72 \int_1^0 -t \, dt = -72 \left[\frac{t^2}{2}\right]_1^0 = -72\left(-\frac{1}{2}\right) = 36$

Q7 Evaluate $\iint_R e^{x^2+y^2} \, dy \, dx$ where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

$\Rightarrow \quad y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1.$

$x = r\cos\theta, \; y = r\sin\theta.$

$\int_0^{\pi}\int_0^1 e^{r^2} r \, dr \, d\theta.$



Let $r^2 = t \Rightarrow 2r\,dr = dt$

$= \frac{1}{2}\int_0^{\pi}\int_0^1 e^t \, dt \, d\theta.$

$= \frac{1}{2}\int_0^{\pi} \left[e^t\right]_0^1 d\theta.$

$= \frac{1}{2}\int_0^{\pi} (e-1) \, d\theta = \frac{e-1}{2}\left[\theta\right]_0^{\pi} = \frac{\pi}{2}(e-1)$

Q8 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, \; 0 \le y \le 2$

$\Rightarrow \int_0^1\int_0^2 (10 + x^2 + 3y^2) \, dy \, dx.$

$= \int_0^1 \left[10y + x^2y + y^3\right]_0^2 dx$

$= \int_0^1 (20 + 2x^2 + 8) \, dx = \left[28x + \frac{2x^3}{3}\right]_0^1 = 28 + \frac{2}{3} = \frac{86}{3}$

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29 Find the volume of the region bounded above by the paraboloid $z = x^2+y^2$ and below by the square R
$-1 \le x \le 1$, $-1 \le y \le 1$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} (x^2+y^2)\,dy\,dx = \int_{-1}^{1} \left[x^2y + \frac{y^3}{3}\right]_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left(x^2 + \frac{1}{3} + x^2 + \frac{1}{3}\right) dx$$

$$= 2\int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) dx = \frac{2}{3}\left[\frac{x^3}{8} + \frac{x}{8}\right]_{-1}^{1}$$

$$= \frac{2}{3}(1+1+1+1) = \frac{8}{3}.$$

30 Find the volume of the region bounded above by the plane $z = 2-x-y$ and below by the square:
R : $0 \le x \le 1$, $0 \le y \le 1$

$$\Rightarrow \int_{0}^{1}\int_{0}^{1} (2-x-y)\,dy\,dx = \int_{0}^{1} \left[2y - xy - \frac{y^2}{2}\right]_{0}^{1} dx.$$

$$= \int_{0}^{1} \left(2 - x - \frac{1}{2}\right) dx.$$

$$= \int_{0}^{1} \left(\frac{3}{2} - x\right) dx = \left[\frac{3}{2}x - \frac{x^2}{2}\right]_{0}^{1} = \frac{3}{2} - \frac{1}{2} = 1.$$

31 Find the volume of the region bounded above by the surface $z = 2\sin x \cos y$ and below by the rectangle R: $0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

$$\Rightarrow \int_{0}^{\pi/2}\int_{0}^{\pi/4} (2\sin x \cos y)\,dy\,dx$$

$$= 2\int_0^{\pi/2} \sin x \,[\sin y]_0^{\pi/4}\, dx$$

$$= 2\int_0^{\pi/2} \sin x \cdot \frac{1}{\sqrt{2}}\, dx.$$

$$= -\sqrt{2}\,[\cos x]_0^{\pi/2}$$

$$= -\sqrt{2}\,(0-1) = \sqrt{2}.$$

32. Find the value of the constant k so that

$$\int_1^2\int_0^3 k x^2 y \, dx\, dy = 1.$$

$$\Rightarrow k\int_1^2\int_0^3 x^2 y\, dx\, dy = 1.$$

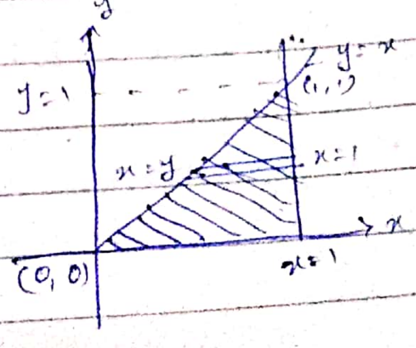
$$= k\int_1^2 \left[\frac{x^3}{3} y\right]_0^3 dy = 1$$

$$= k\int_1^2 9y\, dy = 1.$$

$$= k\,\frac{9}{2}\left[y^2\right]_1^2 = 1$$

$$= \frac{9k}{2}(4-1) = 1. \Rightarrow \boxed{k = \frac{2}{27}}$$

33. Find the volume of the prism whose base is the triangle in the xy plane bounded by the x-axis and the lines $y = x$, $x = 1$ and whose top lies in the plane $z = f(x,y) = 3 - x - y$.



$$\Rightarrow \int_0^1\int_y^1 (3-x-y)\, dx\, dy.$$

$$= \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_y^1 dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y\right) dy - \left(3y - \frac{y^2}{2} - y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 3y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2} y - 2y^2 - \frac{y^2}{2}\right]_0^1 = \frac{5}{2} + \frac{1}{2} - 2 = 1$$

34 Find the Volume of the solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$ and the x-axis.

$\Rightarrow$

$$y = 2\sqrt{x} \Rightarrow y^2 = 4x.$$

$$y = y^2 - 2.$$

$$y^2 - y - 2 = 0$$

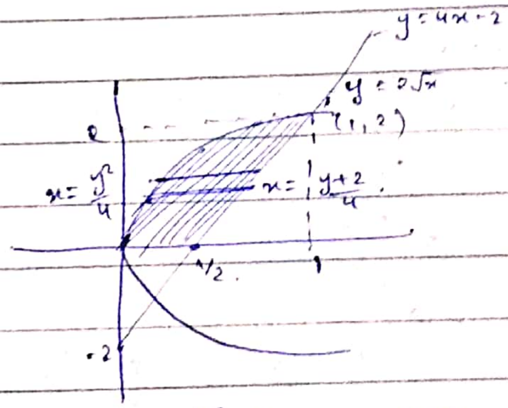
$$y^2 - 2y + y - 2 = 0.$$

$$y(y-2) + (y-2) = 0.$$

$$(y+1)(y-2) = 0.$$

$$y = -1, \; y = 2.$$

$$x = \frac{1}{4}, \; x = 1$$



$\therefore x = \frac{y^2}{4}$ to $x = \frac{y+2}{4}$

$y = 0$ to $y = 2$.

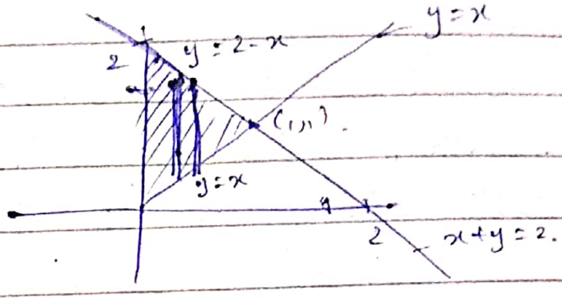
$$\int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy$$

$$= 4(y+2) - \frac{(y+2)^3}{192} - \frac{y^2(y+2)}{4} - \left[4y^2 - \frac{y^6}{192} - \frac{y^4}{4}\right] dy$$

$$= 14.095.$$ (Q 13)

35. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$ in the xy plane.



$$\Rightarrow \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx$$

$y = x$ to $y = 2-x$.

$x = 0$ to $x = 1$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx.$$

$$= \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx$$

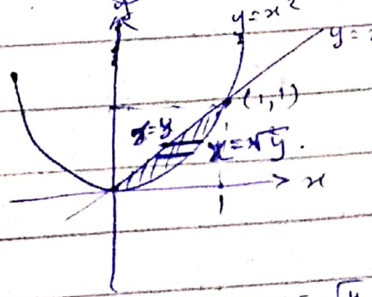
$$= \int_0^1 \left[2x^2 - x^3 + \frac{(2-x)^3}{3} - \frac{4}{3}x^3\right] dx$$

$$= \int_0^1 \left[\frac{6x^2 - 3x^3 - x^3 + 6x^2 - 12x + 8 - 4x^3}{3}\right] dx$$

$$= \frac{1}{3}\left[4x^3 - 2x^4 - 6x^2 + 8x\right]_0^1$$

$$= \frac{1}{3}\left[4 - 2 - 6 + 8\right] = \frac{4}{3}.$$

36. Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant using double integral.



$\Rightarrow$ $\therefore x^2 = x \Rightarrow x(x-1) = 0.$

$\Rightarrow x = 0,\ x = 1.$

$y = 0,\quad y = 1$

$x = y$ to $x = \sqrt{y}$.

$y = 0$ to $y = 1$.

$$\int_0^1 \int_y^{\sqrt{y}} dx\, dy = \int_0^1 \left[x\right]_y^{\sqrt{y}} dy.$$

$$= \int_0^1 \left[\sqrt{y} - y\right] dy = \left[\frac{2}{3} y^{3/2} - \frac{y^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2}$$

$$= \frac{4-3}{6} = \frac{1}{6}.$$

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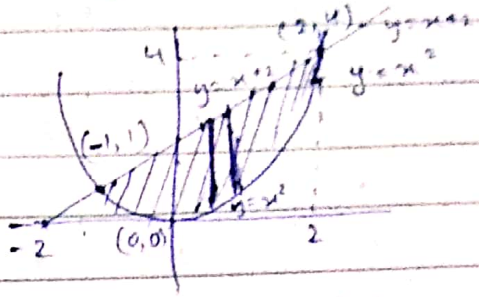
$2x \quad 4-x^2 + (y-1)^2$

$x^2 + (y-1)^2 = 4$

37. Find the area of the Region R enclosed by the parabola $y = x^2$ and the line $y = x+2$ using double integral.



$\Rightarrow \quad x^2 = x+2 \Rightarrow x^2 - x - 2 = 0.$

$\Rightarrow x^2 - 2x + x - 2 = 0.$

$\Rightarrow x(x-2) + (x-2) = 0.$

$\Rightarrow (x+1)(x-2) = 0.$

$x = -1, \; x = 2.$

$y = 1, \; y = 4.$

$y = x^2$ to $y = x+2$

$x = -1$ to $x = 2$.

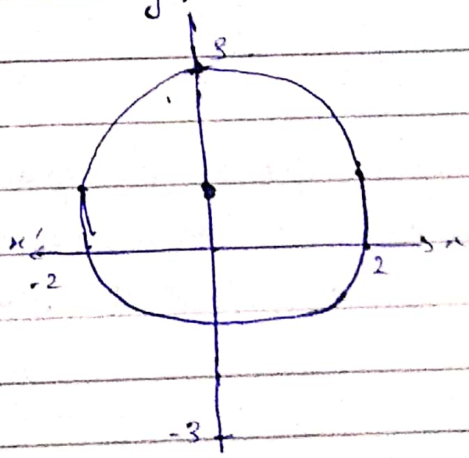
$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} [y]_{x^2}^{x+2}\,dx.$$

$$= \int_{-1}^{2} (x+2-x^2)\,dx.$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = 2+4-\frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 6 - \frac{8}{3} + \frac{3}{2} - \frac{1}{3} = \frac{15}{2} - 3 = \frac{9}{2}.$$

38. Find the area of the playing field described by $R: -2 \le x \le 2,\; -1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}$, using Fubini's theorem and simple geometry.



$$\Rightarrow \int_{-2}^{2}\int_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}} dy\,dx = \int_{-2}^{2} [y]_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}}\,dx$$

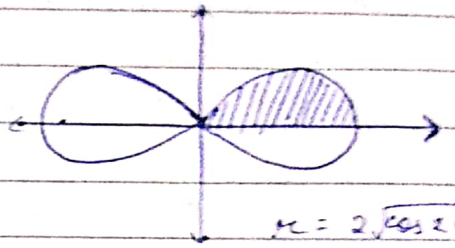
$$= \int_{-2}^{2} \left(1+\sqrt{4-x^2} + 1 + \sqrt{4-x^2}\right) dx$$

$$= 2\int_{-2}^{2} \left(1+\sqrt{4-x^2}\right) dx$$

$$= 2\left[x + \frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = 2\left[(2+\pi) + (2+\pi)\right]$$

$$= 4(2+\pi)$$

39 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$



$\Rightarrow 4\int_0^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta$

$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{2\sqrt{\cos 2\theta}} d\theta$

$= 2\int_0^{\pi/4} 4\cos 2\theta\, d\theta$

$= 8\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = 4(\sin \pi/2 - \sin 0) = 4.$

40 Find the volume of the solid region bounded above the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

$x^2+y^2 = 1 \Rightarrow z = 8$

$x^2+y^2 = 0 \Rightarrow z = 9$

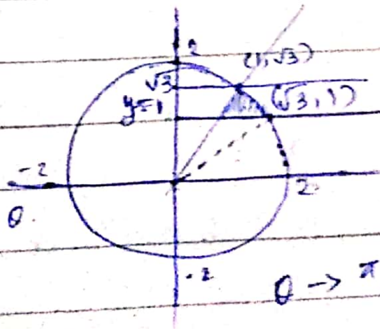
$\Rightarrow \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (9r - r^3)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{9}{2} - \frac{1}{4}\right) d\theta$

$= \frac{17}{4}\left[\theta\right]_0^{2\pi} = \frac{17}{4_2}(2\pi) = \frac{17\pi}{2}$

41. Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2+y^2=4$, above the line $y=1$ and below the line $y=\sqrt{3}x$.



$\tan\theta = \frac{1}{\sqrt{3}}$ $\boxed{\theta = \frac{\pi}{6}}$

$\tan\theta = \frac{\sqrt{3}}{1}$ $\boxed{\theta = \frac{\pi}{3}}$

$\theta \to \pi/6$ to $\pi/3$

$\Rightarrow x^2 + 3x^2 = 4$

$4x^2 = 4 \Rightarrow x = \pm 1$

$x = r\cos\theta, \quad y = r\sin\theta$

r enters at $y = 1$: $1 = r\sin\theta \Rightarrow r = \operatorname{cosec}\theta$.

r leaves at $r = 2$

$\therefore r \to \operatorname{cosec}\theta$ to 2.

$$\begin{matrix} 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & 1 \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \end{matrix}$$

$$\int_{\pi/6}^{\pi/3}\int_{\operatorname{cosec}\theta}^{2} r\,dr\,d\theta = \int_{\pi/6}^{\pi/3}\left[\frac{r^2}{2}\right]_{\operatorname{cosec}\theta}^{2} d\theta$$

$$= \int_{\pi/6}^{\pi/3}\left(2 - \frac{\operatorname{cosec}^2\theta}{2}\right)d\theta = \left[2\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{2\pi}{3} + \cot\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \cot\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} + \frac{1}{\sqrt{3}} - \sqrt{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3} = \frac{\pi + \sqrt{3} - 3\sqrt{3}}{3}$$

$$= \frac{\pi - 2\sqrt{3}}{3}$$

42. Evaluate: $\int_0^1\int_0^1\int_0^1 dz\,dy\,dx$

$$\Rightarrow \int_0^1\int_0^1 [z]_0^1\,dy\,dx = \int_0^1\int_0^1 dy\,dx = \int_0^1 [y]_0^1\,dx = \int_0^1 dx = 1.$$

43. Evaluate: $\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz.$

$$\Rightarrow \int_0^2\int_0^2 \left[\frac{x^2}{2}\right]_0^2 dy\,dz = \int_0^2\int_0^2 2\,dy\,dz = 2\int_0^2 [y]_0^2\,dz$$

$$= 2\int_0^2 2\,dz = 4\int_0^2 dz = 4[z]_0^2 = 4(2-0) = 8$$

44. Evaluate: $\int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx$

$$\Rightarrow \int_0^1\int_x^1 [z]_0^{y-x}\,dy\,dx = \int_0^1\int_x^1 (y-x)\,dy\,dx$$

$$= \int_0^1\left[\frac{y^2}{2} - xy\right]_x^1 dx = \int_0^1\left(\frac{1}{2} - x - \frac{x^2}{2} + x^2\right)dx$$

$$= \int_0^1\left(\frac{1}{2} - x + \frac{x^2}{2}\right)dx = \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$

45 Evaluate: $\int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\, dz\, dx$

$\Rightarrow \int_0^1 \int_0^{1-x} [y]_{x+z}^{1}\, dz\, dx = \int_0^1 \int_0^{1-x} (1-x-z)\, dz\, dx$

$= \int_0^1 \left[z - xz - \frac{z^2}{2} \right]_0^{1-x} dx$

$= \int_0^1 \left[1-x - x(1-x) - \frac{(1-x)^2}{2} \right] dx$

$= \int_0^1 \frac{2(1-x-x+x^2) - (1+x^2-2x)}{2}\, dx$

$= \int_0^1 \frac{2-4x+2x^2-1-x^2+2x}{2}\, dx$

$= \frac{1}{2}\int_0^1 (1-2x+x^2)\, dx = \frac{1}{2}\left[x - x^2 + \frac{x^3}{3} \right]_0^1$

$= \frac{1}{2}\left(1 - 1 + \frac{1}{3}\right) = \frac{1}{6}.$

46. Evaluate the triple integral $\iiint_Q 2xe^y \sin z\, dv$, where Q is the rectangular box defined by $Q = \{(x,y,z) \mid 1 \le x \le 2, 0 \le y \le 1, 0 \le z \le \pi\}$.

$\Rightarrow \int_1^2 \int_0^1 \int_0^{\pi} 2xe^y \sin z\, dz\, dy\, dx$

$= 2\int_1^2 \int_0^1 -xe^y [\cos z]_0^{\pi}\, dy\, dx$

$= -2\int_1^2 \int_0^1 xe^y(-2)\, dy\, dx.$

$= 4\int_1^2 x[e^y]_0^1\, dx.$

$= 4\int_1^2 (xe - x)\, dx = 4\left[\frac{x^2}{2}e - \frac{x^2}{2} \right]_1^2$

$= 4\left[2e - 2 - \frac{e}{2} + \frac{1}{2} \right] = 4\left(\frac{3e}{2} - \frac{3}{2} \right) = 6(e-1)$

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47. Find the volume of the cube of side 2 unit using triple integral.

$$\Rightarrow \int_0^2\int_0^2\int_0^2 dz\,dy\,dx = \int_0^2\int_0^2 [z]_0^2\,dy\,dx = 2\int_0^2\int_0^2 dy\,dx.$$

$$= 2\int_0^2 [y]_0^2\,dx = 4\int_0^2 dx = 4[x]_0^2 = 8.$$

48. Find the volume of the region enclosed by the surface $z = x^2+3y^2$ and $z = 8 - x^2 - y^2$.

$$\Rightarrow \quad x^2+3y^2 = 8-x^2-y^2$$

$$2x^2+4y^2 = 8.$$

$$\frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow \frac{y^2}{2} = \frac{4-x^2}{4} \Rightarrow y = \pm\sqrt{\frac{4-x^2}{2}}.$$

$$z \rightarrow (x^2+3y^2) \text{ to } (8-x^2-y^2)$$

$$y \rightarrow -\sqrt{\frac{4-x^2}{2}} \text{ to } \sqrt{\frac{4-x^2}{2}}.$$

$$2x^2+4(0) = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$$x \rightarrow -2 \text{ to } 2.$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx.$$

$$= 4\int_0^2\int_0^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{\frac{4-x^2}{2}}} [z]_{x^2+3y^2}^{8-x^2-y^2}\,dy\,dx.$$

$$= 4\int_0^2\int_0^{\sqrt{\frac{4-x^2}{2}}} (8-x^2-y^2-x^2-3y^2)\,dy\,dx = 4\int_0^2\int_0^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\,dy\,dx$$

$$= 4\int_0^2 \left[8y - 2x^2y - \frac{4}{3}y^3\right]_0^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= 4\int_0^2 8\sqrt{\frac{4-x^2}{2}} - 2x^2\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= 4\int_0^2 2(4-x^2)\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} dx.$$

$$= 4\int_0^2 (4-x^2)^{3/2} \cdot \frac{2\sqrt{2}}{3}\, dx$$

$$= \frac{8\sqrt{2}}{3}\int_0^2 (4-x^2)^{3/2}\, dx$$

$$= \frac{8\sqrt{2}}{3}\left[(4-x^2)^{3/2}\int dx - \int\left(\frac{3}{2}\cdot 2x\,(4-x^2)^{1/2}\int dx\right)dx\right]_0^2$$

$$= \frac{8\sqrt{2}}{3}\Big[x(4-x^2)^{3/2} - \underbrace{\int -3x^2(4-x^2)^{1/2}\,dx}_{I_1}\Big]_0^2$$

$$I_1 = -3\int_0^2 x^2(4-x^2)^{1/2}\,dx$$

Put $x = 2\sin t \Rightarrow dx = 2\cos t\, dt$

$x=0;\ t=0\ /\ x=2;\ t=\pi/2.$

$$= \frac{8\sqrt{2}}{3}\int_0^{\pi/2} (4-4\sin^2 t)^{3/2} \cdot 2\cos t\, dt$$

$$= \frac{8\sqrt{2}}{3}\int_0^{\pi/2} (4\cos^2 t)^{3/2} \cdot 2\cos t\, dt$$

$$= \frac{8\sqrt{2}}{3}\int_0^{\pi/2} 8\cos^3 t \cdot 2\cos t\, dt$$

$$= \frac{8\sqrt{2}}{3}\int_0^{\pi/2} 16\cos^4 t\, dt = \frac{8\sqrt{2}}{3}\cdot 16\left[\frac{\sin^4 t}{4}\right]_0^{\pi/2}$$

$$= \frac{8\sqrt{2}}{3}\cdot 4(1-0) = \frac{32\sqrt{2}}{3}$$

2/3

15=5(23)

49 Find the volume of the region enclosed by the cylinder $z = y^2$ and by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$.

$\Rightarrow$ $$\int_0^1 \int_{-1}^1 \int_0^{y^2} dz\, dy\, dx = \int_0^1 \int_{-1}^1 [z]_0^{y^2} dy\, dx$$

$$= \int_0^1 \int_{-1}^1 y^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^3}{3}\right]_{-1}^1 dx = \int_0^1 \left(\frac{1}{3} + \frac{1}{3}\right) dx$$

$$= \frac{2}{3} \int_0^1 dx = \frac{2}{3} [x]_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$$

50. Find the volume of the region in the first octant bounded by the coordinate planes, the plane $y + z = 2$ and the cylinder $x = 4 - y^2$.

$\Rightarrow$ Region is in first octant

$\therefore$ $x \geq 0$, $y \geq 0$, $z \geq 0$.

$\therefore$ $y + z = 2 \Rightarrow z = 2 - y$.

$x = 4 - y^2 \Rightarrow y = \sqrt{4 - x}$.

$x = 4 - (0) \Rightarrow x = 4$

Limits :-

$z \to 0$ to $2 - y$.

$y \to 0$ to $\sqrt{4 - x}$.

$x \to 0$ to 4.

$$\int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} [z]_0^{2-y} dy\, dx$$

$$= \int_0^4 \int_0^{\sqrt{4-x}} (2 - y)\, dy\, dx$$

$$= \int_0^4 \left[2y - \frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx = \int_0^4 \left(2\sqrt{4-x} - \frac{4-x}{2}\right) dx$$

$$= \left[-2 \cdot \frac{2}{3} (4-x)^{3/2}\right]_0^4 - \frac{1}{2}\left[4x - \frac{x^2}{2}\right]_0^4$$

$$= \frac{32}{3} - 4 = \frac{20}{3}$$

5) Find the volume of the region in the first ~~quadrant~~ octant bounded by the coordinate planes, and the surface $z = 4 - x^2 - y$.

$\Rightarrow$ Region lies in ~~first~~ first octant.

$\therefore x \geq 0, \quad y \geq 0, \quad z \geq 0.$

$z = 4 - x^2 - y.$

$z = 0 \Rightarrow 0 = 4 - x^2 - y.$

$\Rightarrow y = 4 - x^2$

$z = 0 \;\&\; y = 0.$

$4 - x^2 = 0 \Rightarrow x = 2.$

Limits.

$z \to 0$ to $(4 - x^2 - y)$

$y \to 0$ to $4 - x^2$

$x \to 0$ to 2

$$\int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dy\, dx = \int_0^2 \int_0^{4-x^2} [z]_0^{4-x^2-y}\, dy\, dx.$$

$$= \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\, dy\, dx$$

$$= \int_0^2 \left[4y - x^2 y - \frac{y^2}{2}\right]_0^{4-x^2} dx.$$

$$= \int_0^2 \left[4(4-x^2) - x^2(4-x^2) - \frac{(4-x^2)^2}{2}\right] dx$$

$$= \int_0^2 \left[16 - 4x^2 - 4x^2 + x^4 - \frac{(16 + x^4 - 8x^2)}{2}\right] dx$$

$$= \int_0^2 \frac{32 - 16x^2 + 2x^4 - 16 - x^4 + 8x^2}{2}\, dx$$

$$= \frac{1}{2}\int_0^2 \left(8 - \cancel{4}x^2 + \frac{x^4}{2}\right) dx$$

$$= \left[8x - \frac{\cancel{4}x^3}{3} + \frac{x^5}{10}\right]_0^2$$

$$= 16 - \frac{32}{3} + \frac{16}{5} = \frac{240 - 160 + 48}{15} = \frac{128}{15}$$

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5) Find the volume of the region in the first octant bounded by the coordinate planes, the plane $x+y=2$, and the cylinder $y^2+4z^2=16$.

→ Region lies in first octant.

$\therefore x \geq 0, y \geq 0, z \geq 0$

$y^2+4z^2=16$

$\Rightarrow z^2 = \frac{16-y^2}{4} \Rightarrow z = \frac{\sqrt{16-y^2}}{2}$

$z \rightarrow 0$ to $\frac{\sqrt{16-y^2}}{2}$ $\qquad y^2+4(0)=16$

$x = 2-y$ $\qquad y=4$

$\therefore x \rightarrow 0$ to $2-y$.

$y \rightarrow 0$ to 4

$$\int_0^4\int_0^{2-y}\int_0^{\frac{\sqrt{16-y^2}}{2}} dz\,dy\,dx = \int_0^4\int_0^{2-y} [z]_0^{\frac{\sqrt{16-y^2}}{2}} dz\,dx\,dy.$$

$$= \int_0^4\int_0^{2-y} \frac{\sqrt{16-y^2}}{2}\,dx\,dy = \int_0^4 \frac{16-y^2}{2}[x]_0^{2-y}\,dy.$$

$$= \int_0^4 \left[\frac{y}{2}\sqrt{16-y^2} + 8\sin^{-1}\left(\frac{y}{4}\right)\right]_0^{2-y} dx$$

$$= \int_0^4 \frac{16-y^2}{2}\cdot(2-y)\,dy.$$

$$= \int_0^4 \frac{32-16y-2y^2+y^3}{2}\,dy$$

$$= \frac{1}{2}\left[32y-8y^2-\frac{2}{3}y^3+\frac{y^4}{4}\right]_0^4$$

$$= \frac{1}{2}\left[128-128-\frac{128}{3}+64\right] = \frac{64}{3\cdot 2} = \frac{32}{3}$$