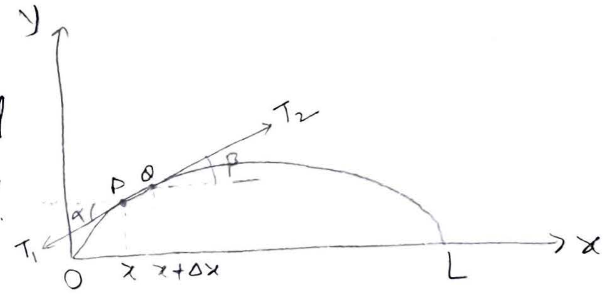
Modeling : Vibrating String, Wave Equation

Consider a Vibrating (Elastic) String along the x-axis, stretch it to length L, and fasten it at the ends $x=0$ and $x=L$.
We then distort the string and allow it to vibrate.
The problem is to determine the vibrations of the string, that is, to find its deflection (displacement) $u(x,t)$ at any point x and any time $t>0$.



If there is no resistance to bending, the tension is tangential to the curve of the string at each point. There is no motion in the horizontal direction. So, the horizontal components of the tension is constant.

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{const.}$$

In the vertical direction, there two forces, $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2; here the minus sign appears because the component at P is directed downward.
By Newton's second law, the resultant of these two forces is equal to the mass $\rho\Delta x$ of the portion times the acceleration $\partial^2 u/\partial t^2$.

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_Q - \left.\frac{\partial u}{\partial x}\right|_P = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_Q - \left.\frac{\partial u}{\partial x}\right|_P\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

If Δx approaches zero, $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$ One-D Wave Equation

$c^2 = \frac{T}{\rho}$

Solution of 1-D Wave Eqn

The model of a vibrating elastic string consists of the 1D wave equation

$$(1) \text{ — } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \boxed{c^2 = \frac{T}{\rho}}$$

for the unknown deflection $u(x,t)$ of the string.

Some conditions:

Since the string is fastened at ends $x=0$ and $x=L$, at these point, there is no deflection, so two boundary conditions :-

$$\boxed{u(0,t)=0 \; ; \; u(L,t)=0} \quad \text{for all } t \geq 0$$

Furthermore, the form of the motion of the string will depend on its initial deflection $(t=0)$, call it $f(x)$, and on its initial velocity $(t=0)$, call it $g(x)$.

$$\boxed{u(x,0)=f(x) \; ; \; \frac{\partial u}{\partial t}(x,0) = g(x)} \quad 0 \leq x \leq L$$

Step I: Assume that $u(x,t)$ is separable i.e.

$$u(x,t) = X(x)\,T(t)$$

Diff.,

$$\frac{\partial^2 u}{\partial t^2} = X(x)\,\ddot{T}(t) \; ; \; \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

Now, (1) becomes

$$X\ddot{T} = c^2 X'' T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = k$$

$$\therefore X'' - kX = 0 \;\&\; \ddot{T} - c^2 k T = 0$$

Step II:
$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$
$$u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$$

The solution of $X'' + p^2 X = 0$ is $\quad (k = -p^2)$

$$X(x) = A\cos px + B\sin px$$

When $X(0) = 0$ then

$$\boxed{0 = A}$$

When $X(L) = 0$ then

$$0 = B \sin pL \Rightarrow \sin pL = \sin n\pi \Rightarrow pL = n\pi$$

[Since $B \neq 0$] $\Rightarrow p_n = \frac{n\pi}{L}$, ($n$ integer)

This results in infinitly many solutions

$$X_n(x) = \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots)$$

Now, $\ddot{T} - kc^2 T = 0 \Rightarrow \ddot{T} + p^2 c^2 T = 0$

A General Solution is

$$T_n(t) = B_n \cos pct + B_n^* \sin pct$$

Hence solutions of ① satisfying BC are

$$u_n(x,t) = X_n(x) T_n(t)$$

$$= (B_n \cos p_n ct + B_n^* \sin p_n ct) \sin \frac{n\pi}{L} x$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} (B_n \cos p_n ct + B_n^* \sin p_n ct) \sin \frac{n\pi}{L} x$$

Now, Initial conditions, $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \qquad n = 1, 2, \cdots$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-B_n p_n c \sin p_n ct + B_n^* p_n c \cos p_n ct) \sin \frac{n\pi}{L} x$$

$$g(x) = \sum_{n=1}^{\infty} B_n^* p_n c \sin \frac{n\pi}{L} x$$

$$B_n^* p_n c = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$\boxed{B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx} \quad n=1,2,\ldots$$

Q An elastic string of length π which is fastened at ends $x=0$ and $x=\pi$ is released from its horizontal position with initial velocity $\sin x$. Find the displacement (deflection) of the string at any ~~string~~ instant of time.

Mathematical Model: Suppose $u(x,t)$ be the displacement (deflection) of the string at any instant of time t. Then the deflection of string is governed by PDE (Wave equation):

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

$$u(0,t)=0=u(\pi,t)\ ;\ u(x,0)=0\ ,\ \frac{\partial u}{\partial t}(x,0)=\sin x$$

Assume $u(x,t)=X(x)T(t)$

The general solution of (1) is

$$u(x,t)=(A\cos px+B\sin px)(C\cos pct+D\sin pct)$$

Use first Bound. Condi: $u(0,t)=0$

$$0 = A(C\cos pct+D\sin pct)$$

$\Rightarrow A=0$ [$\because$ T(t) is zero then $u(x,y)=0$ which is trivial]

Use second Bound. Condi: $u(\pi,t)=0$

$$0 = B\sin p\pi\,(C\cos pct+D\sin pct)$$

$\Rightarrow \sin p\pi = 0$ [$\because B\neq 0,\ T(t)\neq 0$]

$\Rightarrow p\pi = n\pi \Rightarrow \underbrace{p_n = n}_{\text{eigenvalues}},\ n\in\mathbb{Z},\ n=1,2,\ldots$

Therefore, $\underbrace{u_n(x,t) = B\sin nx\,(C\cos nct+D\sin nct)}_{\text{eigenfunctions}}$

By Superposition Principle,

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}(B_n\cos nct+B_n^*\sin nct)\sin nx$$

Use first Initial Condi. $u(x,0)=0$

$$0=\sum_{n=1}^{\infty}B_n\sin nx \Rightarrow B_n=0$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} (B_n^* \sin nct) \sin nx$$

Use last I.C. $\frac{\partial u}{\partial t}(x,0) = \sin x$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (B_n^* nc \cos nct) \sin nx$$

$$\sin x = \sum_{n=1}^{\infty} ncB_n^* \sin nx$$

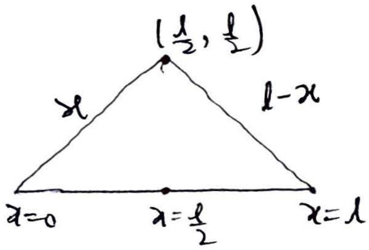
$$\sin x = cB_1^* \sin x + 2cB_2^* \sin 2x + \dots$$

$$B_1^* = 1/c, \quad B_2^* = 0, \quad B_3^* = 0 \dots$$

$$\therefore \boxed{u(x,t) = \frac{1}{c} \sin ct \sin x}$$

2. An elastic string of length l ~~with~~ which is fastened at the ends $x=0$ and $x=l$ is ~~released from its~~ picked up at its centre point $x = \frac{l}{2}$ to a height of $\frac{l}{2}$ and released from rest. Find the displacement of the string at any instant of time.

The physical system is governed by



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

$$u(0,t) = 0 = u(l,t)$$

$$u(x,0) = \begin{cases} x & ; 0 \le x \le \frac{l}{2} \\ l-x & ; \frac{l}{2} \le x \le l \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

Assume $u(x,t) = X(x)\,T(t)$

The G.S. of (1) is

$$u(x,t) = (A \cos px + B \sin px)(C \cos pct + D \sin pct)$$

Use $u(0,t) = 0$ $\Rightarrow A = 0$

, Use $u(l,t) = 0$ $\Rightarrow \sin pl = 0 \Rightarrow p_n = \frac{n\pi}{l}, n = 1, 2, \dots$

The most G.S. of ① with Bound. Condi.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos\frac{n\pi}{l}ct + B_n^* \sin\frac{n\pi}{l}ct\right)\sin$$

Use first Initial Condi $u(x,0) = \left\{\begin{array}{ll} x & ; \\ l-x & ; \end{array}\right\} = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{l}\int_0^{l/2} x\sin\frac{n\pi}{l}x\,dx + \frac{2}{l}\int_{l/2}^{l}(l-x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{l}\left[\frac{-x\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right]_0^{l/2} + \frac{2}{l}\left[-(l-x)\frac{\cos\frac{n\pi}{l}}{\frac{n\pi}{l}} + (-1)\frac{\sin\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi}\cdot\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] - \frac{2}{l}\left[0 - \frac{l}{2}\cdot\frac{l}{n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Use $\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow B_n^* = 0$

$\therefore$ The soln is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi}{l}ct\sin\frac{n\pi}{l}x\right]$$