Laplace's Equation (Steady Two-Dimensional Heat Problem)

The two-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

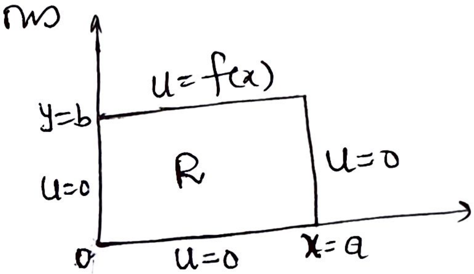
reduces to Laplace's equation or potential equation in two dimensions given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

When the heat flow is in the steady-state (i.e. $\frac{\partial u}{\partial t} = 0$)
The solution $u(x,y)$ of (1) in a rectangular region can be obtained with help of boundary conditions

$u(0,y) = u(a,y) = u(x,0) = 0$

$u(x,b) = f(x)$



Assume that u is separable;

$$u(x,y) = X(x)\,Y(y) \quad \text{--- (2)}$$

Substitute (2) in (1),

$$X''Y + X\ddot{Y} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{\ddot{Y}}{Y} = k$$

$\therefore\ X'' - kX = 0 \ \&\ \ddot{Y} + kY = 0$

If $k \geq 0$ then (1) with Bound. condi. have only trivial soln.
So assume $k = -p^2 < 0$. Then

$$X'' + p^2 X = 0 \ \&\ \ddot{Y} - p^2 Y = 0$$

$$X(x) = (A\cos px + B\sin px) \ \&\ Y(y) = (Ce^{py} + De^{-py})$$

$\therefore$ The G.S. of (1) is $u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

Use $u(0,y) = 0$

$\Rightarrow 0 = A\,(Ce^{py} + De^{-py}) \Rightarrow A = 0$

Use $u(a,y) = 0$

$\Rightarrow 0 = (B\sin pa)\,Y(y) \Rightarrow \sin pa = 0 \Rightarrow p = \frac{n\pi}{a},\ n = 1, 2, \ldots$

Use $u(x,0)=0$

$$\Rightarrow 0 = (B\sin px)(C+D) \Rightarrow C+D=0 \Rightarrow C=-D$$

$\therefore$ The soln is

$$u(x,y) = B\sin\frac{n\pi}{a}x\, C\left(e^{py}-e^{-py}\right)$$

$$= BC\sin\frac{n\pi}{a}x\left(e^{\frac{n\pi}{a}y}-e^{-\frac{n\pi}{a}y}\right)$$

The most G.S. of ① with zero Bound. condi is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$

Use $u(x,b) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}b$$

$$= \sum_{n=1}^{\infty}\left(B_n\sinh\frac{n\pi}{a}b\right)\sin\frac{n\pi}{a}x$$

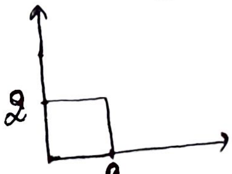
$$\Rightarrow B_n\sinh\frac{n\pi}{a}b = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

$$\Rightarrow B_n = \frac{2}{a\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx.$$

Q Find the steady-state solution (temparature) in the square plate $0\le x\le 2,\ 0\le y\le 2$ if the upper side is kept at the temperature $1000\sin\frac{\pi}{2}x$ and the other sides are at $0^\circ C$.

Consider $u(x,y)$ is the steady-state solution (temperature)

So, the model of given problem is



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

And Boundary condition are

$u(x,0)=0 \qquad u(x,2)=1000\sin\frac{\pi}{2}x$

$u(0,y)=0$

$u(2,y)=0$

Assume that u is separable,

$$u(x,y) = X(x)Y(y)$$

The G.S. of (1) is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$$

Use $u(x,0) = 0$

$\Rightarrow C + D = 0 \Rightarrow D = -C$

Use $u(0,y) = 0$

$\Rightarrow A = 0$

Use $u(a,y) = 0$

$\Rightarrow \sin pa = 0 \Rightarrow p = \frac{n\pi}{a}, \quad n = 1, 2, \dots$

$\therefore$ The soln is $u(x,y) = B\sin\frac{n\pi}{a}x\,(Ce^{py} - Ce^{-py})$

$$u_n(x,y) = B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{a}y \sin\frac{n\pi}{a}x$$

Use $u(x,a) = 1000\sin\frac{\pi}{a}x$

$$\Rightarrow 1000\sin\frac{\pi}{a}x = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin\frac{n\pi}{a}x$$

$$= B_1 \sinh\pi \sin\frac{\pi}{a}x + B_2 \sinh 2\pi \sin\pi x + \dots$$

On Comparing

$\Rightarrow B_1 \sinh\pi = 1000, \quad B_2 = 0, \; B_3 = 0, \dots$

$$\Rightarrow B_1 = \frac{1000}{\sinh\pi}$$

$$\therefore u(x,y) = \frac{1000}{\sinh\pi} \sinh\frac{\pi}{a}y \sin\frac{\pi}{a}x$$

Q. The boundary value problem governing the steady state temperature distribution in a flat, thin, rectangular plate is given by

① $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$ $\qquad u(0,y) = f(y), \; u(x,0) = 0$

$u(a,y) = g(y) \quad u(x,b) = 0$

Assume the temperature $u(x,y)$ is separable i.e.

$$u(x,y) = X(x)\,Y(y)$$

This system has non-trivial solution for $k > 0$.

The G.S. of ① is

$$u(x,y) = (A\cos py + B\sin py)(Ce^{px} + De^{-px})$$

Use $u(x,0) = 0 \Rightarrow A = 0$

$\Rightarrow 0 = C + D \Rightarrow D = -C$

Use $u(x,b) = 0 \Rightarrow B\sin pb\, X(x) = 0 \Rightarrow \sin pb = 0$

$\Rightarrow p = \frac{n\pi}{b}, \; n = 1, 2$

$\Rightarrow 0 = X(x)(Ce^{pb} + De^{-pb})$

$\Rightarrow e^{pb} - e^{-pb} = 0 \Rightarrow e^{pb} = \frac{1}{e^{pb}} \Rightarrow e^{2pb} = 1 = e^{0}$

$\Rightarrow 2pb = 0 \Rightarrow p = 0$

$\therefore u(x,t) = \sum_{n=1}^{\infty} (B_n e^{px} + C_n e^{-px}) \sin \frac{n\pi}{b} y$

Use $u(0,y) = f(y)$

$\Rightarrow f(y) = \sum_{n=1}^{\infty} (B_n + C_n) \sin \frac{n\pi}{b} y$

$\Rightarrow (B_n + C_n) = \frac{2}{b}\int_0^b f(y) \sin \frac{n\pi}{b} y \, dy$

Use $u(a,y) = g(y)$

$\Rightarrow g(y) = \sum_{n=1} (B_n e^{pa} + C_n e^{-pa}) \sin \frac{n\pi}{b} y$

$\Rightarrow B_n e^{pa} + C_n e^{-pa} = \frac{2}{b}\int_0^b g(y) \sin \frac{n\pi}{b} y \, dy$

$$X'' - p^2 X = 0 \quad \& \quad Y'' + p^2 Y = 0$$

$$\Rightarrow X(x) = (Ae^{px} + Be^{-px}) \quad \& \quad Y(y) = (C_3 \cos py + C_4 \sin py)$$

$$= A\cosh px + A\sinh px + B\cosh px - B\sinh px$$

$$= (A+B)\cosh px + (A-B)\sinh px$$

$$= (C_1 \cosh px + C_2 \sinh px)$$

$$\boxed{\begin{aligned} \frac{e^x + e^{-x}}{2} &= \cosh x \\ \frac{e^x - e^{-x}}{2} &= \sinh x \\ e^x &= \cosh x + \sinh x \\ e^{-x} &= \cosh x - \sinh x \end{aligned}}$$

$\therefore$ The G.S. of (1) is

$$u(x,y) = (C_1 \cosh px + C_2 \sinh px)(C_3 \cos py + C_4 \sin py)$$

Use $u(x,0) = 0$

$$\Rightarrow 0 = X(x)\, C_3 \Rightarrow C_3 = 0$$

Use $u(x,b) = 0$

$$\Rightarrow 0 = X(x)\, C_4 \sin pb \Rightarrow \sin pb = 0 \Rightarrow p = \frac{n\pi}{b}, \; n = 1, 2, \ldots$$

$\therefore$ The G.S. becomes

$$u_n(x,y) = (A_n \cosh px + B_n \sinh px) \sin py$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b}x + B_n \sinh \frac{n\pi}{b}x\right) \sin \frac{n\pi}{b}y \quad \text{— (A)}$$

Use $u(0,y) = f(y)$

$$\Rightarrow f(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b}y \Rightarrow A_n = \frac{2}{b}\int_0^b f(y) \sin \frac{n\pi}{b}y \, dy \quad \text{— (B)}$$

Use $u(a,y) = g(y)$

$$\Rightarrow g(y) = \sum_{n=1}^{\infty} \underbrace{\left(A_n \cosh \frac{n\pi}{b}a + B_n \sinh \frac{n\pi}{b}a\right)}_{D_n} \sin \frac{n\pi}{b}y$$

$$\Rightarrow D_n = \frac{2}{b}\int_0^b g(y) \sin \frac{n\pi}{b}y \, dy$$

$$\Rightarrow B_n = \frac{\left(D_n - A_n \cosh \frac{n\pi a}{b}\right)}{\sinh \frac{n\pi}{b}a} \quad \text{— (C)}$$

Hence, the required solution is given by (A) ~~and~~, where A_n & B_n are given by (B) & (C).