Eigenvector corresponding to $\lambda=-5$;

$$(A+5I)v=0 \Rightarrow \begin{bmatrix} 8 & -4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 8v_1-4v_2=0\ ;\ 2v_1-v_2=0$$

$\therefore$ $v=(1,2)$ is an eigenvector corresponding to $\lambda=-5$.

Verification:

$$Av=\begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}=\begin{bmatrix} 8 \\ 2 \end{bmatrix}=2\begin{bmatrix} 4 \\ 1 \end{bmatrix}=2v$$

$$Av=\begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} -5 \\ -10 \end{bmatrix}=-5\begin{bmatrix} 1 \\ 2 \end{bmatrix}=-5v$$

Example: Find all eigenvalues and corresponding eigenvectors of

$$A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}=(1-\lambda)\{(-\lambda)(1-\lambda)-1\}+1(0-1+\lambda)$$

$$=(1-\lambda)\{-\lambda+\lambda^2-1-1\}$$

$$=(1-\lambda)(\lambda^2-\lambda-2)=(1-\lambda)(\lambda+1)(\lambda-2)$$

$\therefore$ $|A-\lambda I|=0 \Rightarrow \lambda=1,-1,2$

For $\lambda_1=1$,

$$(A-1I)v=0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3=0,\ v_1+v_2-v_3=0$$

$\Rightarrow v_1+v_2=0 \Rightarrow v_1=-v_2$

$\therefore$ $v=(1,-1,0)$ is an eigenvector corresponding $\lambda_1=1$.

For $\lambda_2=-1$,

$$(A+1I)v=0 \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xRightarrow{R_3-2R_1} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=0$$

$\Rightarrow 2v_1+v_3=0\ ;\ 2v_2+v_3=0 \Rightarrow v_3=-2v_1=-2v_2$

$\therefore$ $v=(1,1,-2)$ is an eigenvector corresponding $\lambda_2=-1$.

For $\lambda_3 = 2$, $(A-2I)v=0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3+R_1 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v_1 = v_3\ ;\ v_2 = v_3$

$\therefore v(1,1,1)$ is an eigenvector corresponding $\lambda = 2$.

When eigenvalue repeats :-

Example: Find all eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

soln:- $|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = -\lambda(-4\lambda+\lambda^2+5) - 1(0-2)$

$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2$

$\therefore |A-\lambda I| = 0 \Rightarrow -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0 \Rightarrow -(\lambda-1)^2(\lambda-2) = 0$

$\Rightarrow \lambda = 1, 1, 2$

For $\lambda_1 = \lambda_2 = 1$, $(A-I)v = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$R_3+2R_1 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v_1 = v_2,\ v_2 = v_3$

$\therefore v = (1,1,1)$ is an eigenvector corresponding $\lambda_1 = \lambda_2 = 1$.

For $\lambda_3 = 2$; $(A-2I)v = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{bmatrix} v = 0$

$R_3+R_1 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow 2v_1 = v_2,\ 2v_2 = v_3$

$\therefore v(1,2,4)$ is an eigenvector corresponding to $\lambda_3 = 2$.