Unit I: Mathematical Logic

Prerequisite

- Number System
- Basic Elementary Mathematics

Objective

Mathematical Logic and its Laws

Mathematical Logic - Introduction

Logic is the basis of all mathematical reasoning, and of all automated reasoning. The rules of mathematical logic specify methods of reasoning mathematical statements.

Logical reasoning provides the theoretical base for many areas of Mathematics and consequently Computer Science. It has many practical applications in Computer Science like design of Computing Machines, Artificial Intelligence, Programming languages, Computer Security etc.

Mathematical logics can be broadly categorized into three categories.

- Propositional Logic − Propositional Logic is concerned with statements to which the truth values, "true" and "false", can be assigned.
- Predicate Logic − Predicate Logic deals with predicates, which are propositions containing variables. A predicate represents an expression of one or more variables.
- Rules of Inference − Rules of Inference provide the templates or guidelines for constructing valid arguments from the known statements.

Proposition

A **proposition** is the basic building block of logic. Proposition is a statement or assertion that expresses a judgement or opinion. It is defined as a declarative sentence that is either True or False, but not both. The truth value of a proposition is **True** (denoted as T) if it is a true statement. And the truth value of a proposition is **False** (denoted as F) if it is a false statement. e.g., 1. The sun rises in the East and sets in the West.

2. $3 + 1 = 4$

3. ‘c' is a vowel.

all of the above sentences are propositions, where the first two are Valid (True) and the third one is Invalid (False).

Some sentences that do not have a truth value or may have more than one truth value are not propositions. e.g.,

1. What time is it?
2. Go to School and study.
3. $x + 2 = 5$.

The above sentences are not propositions as the first two do not have a truth value, and the third one may be true or false.

To represent propositions, propositional variables are used and are represented by small alphabets such as p, q, r, s….

The area of logic which deals with propositions is called propositional logic.

Truth Values

Truth values

Every proposition (simple or compound) will take one of the two values true or false and these values are called the truth values.

We denote the value true as 1 (T) and value false as 0 (F).

e.g.,

Consider a simple proposition, 2 is greater than 1

The **truth value** of the proposition is **TRUE.**

Consider another simple proposition,

The word Mango comes before the word Apple in Oxford Dictionary.

The **truth value** of the proposition is **FALSE** this is because M comes after A.

Truth Value of Compound Proposition

The truth value of a compound proposition can be figured out based on the truth values of its components.

e.g., consider a compound proposition,

March 1, 2022 was Tuesday and Tuesday is a holiday.

The given compound proposition is made up of two simple propositions,

p = March 1, 2022 was Tuesday

q = Tuesday is a holiday

Check the calendar, 1st March was Tuesday. So, truth value of the simple proposition p is TRUE.

Tuesday is a holiday. So, the truth value of the simple proposition q is TRUE.

So, p = TRUE and q = TRUE.

Note the word ’and’ in the statement, is joining the **two simple propositions** into a **compound proposition**.

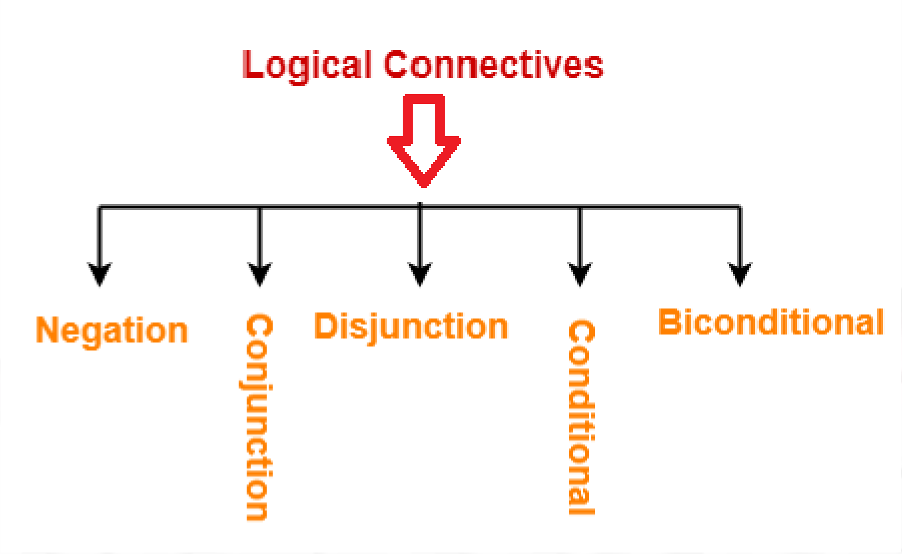
So, we can write x = p AND q

Connectives

Propositions constructed using one or more propositions are called compound propositions. The propositions are combined together using Logical Connectives or Logical Operators.

i.e., **Connectives** are the operators that are used to combine propositions.

In propositional logic, there are 5 basic connectives:



Types of Connectives

Name of Connective	Connective Word	Symbol
Negation	Not	⌉ or ~ or – or ¬
Conjunction	And	∧
Disjunction	Or	∨
Conditional	If-then	→
Biconditional	If and only if	↔

Truth Table

A **Truth Table** is a complete list of possible truth values of a given proposition. So, if we have a proposition say p, then its possible truth values are TRUE and FALSE, we can draw the truth table for p as under:

p
FALSE
TRUE

or

p
F
T

or

p
0
1

Negation

1. Negation

If p is a proposition, then negation of p is a proposition which is $\begin{cases} \text{True when p is false} \\ \text{False when p is true} \end{cases}$

Truth Table

p	**~p**
F	T
T	F

Example:

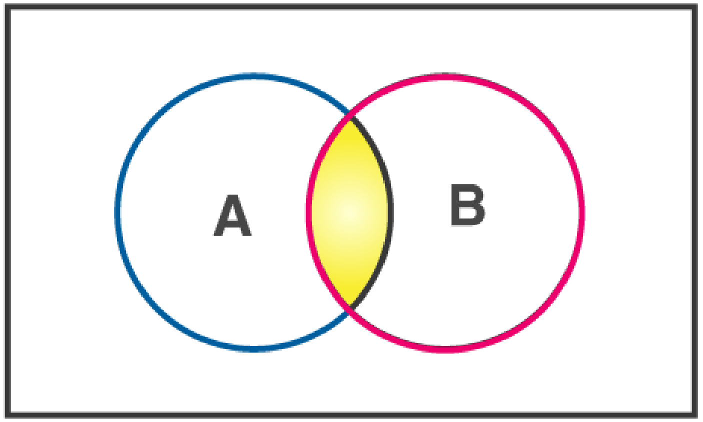
If p : It is raining outside.

Then, Negation of p is ~p : It is not raining outside.

2. Conjunction

If p and q are two propositions, then conjunction of p and q is a proposition

which is $\begin{cases} \text{True when both p and q are true} \\ \text{False} \qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$



Truth Table

p	q	p ∧ q
F	F	F
F	T	F
T	F	F
T	T	T

Example:

If p and q are two propositions where-
p : John likes cold coffee
q : Jac likes milkshake.

p	**q**	**p ∧ q**
T	T	T

Then, conjunction of p and q is-

p ∧ q : John likes cold coffee and Jac likes milkshake

The two statements must be true for the compound statement.

Question 1: Let r: 5 be a rational number and s: 15 be a prime number.
What is the truth value of $r \wedge s$?

Solution: Given that

r: 5 is a rational number. This proposition is true.

s: 15 is a prime number. This proposition is false as 15 is a <u>composite number</u>.

Therefore, as per the truth table, r **and** s is a false statement.

So, $r \wedge s = F$ i.e., truth value of $r \wedge s$ is False.

Question 2: Let a: x be greater than 9 and b: x be a prime number. What is its conjunction?

Solution:

Since x is a variable whose value we don’t know. Let us define a range for a and b.

To find the range let us take certain values for x;

When x= 6: a and b is false. Hence, a ∧ b is false.

When x= 3: a is false but b is true. But still, a ∧ b is false.

When x= 10: a is true but b is false. But still, a ∧ b is false.

When x= 11: a is true and b is true. Hence, a ∧ b is true.

Hence the conjunction a and b is only true when x is a prime number greater than 9.

Disjunction

3. Disjunction

If p and q are two propositions, then disjunction of p and q is a proposition

$$\text{which is } \begin{cases} \text{True when either of p or q or both p and q are true} \\ \text{False } when\ both\ p\ and\ q\ are\ false \end{cases}$$

Truth Table

p	**q**	**p ∨ q**
F	F	F
F	T	T
T	F	T
T	T	T

Example: If p and q are two propositions where

p : A square is a quadrilateral.

q : Aamir Khan is a Film Actor. Write the truth value for the disjunction “p or q”

Solution:

p	**q**	**p ∨ q**
T	T	T

The truth value of p or q is True

Example: Complete a truth table for each disjunction below.

1. a or b
2. a or not b
3. not a or b

a	b	a or b
T	T	T
T	F	T
F	T	T
F	F	F

a	b	~b	a ∨ ~b
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

a	b	~a	~a ∨ b
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

4. Conditional Proposition

If p and q are two propositions, then the compound proposition "if p then q" denoted by $p \rightarrow q$ is called a conditional proposition or implication proposition,

which is $\begin{cases} \text{True when both p \& q are true or both are false or p is false but q is true} \\ \text{False } when\ p\ is\ true\ and\ q\ is\ false \end{cases}$

Truth Table

p	**q**	**$p \rightarrow q$**
F	F	T
F	T	T
T	F	F
T	T	T

Note: in conditional proposition $p \rightarrow q$,

p is antecedent (hypothesis)

q is consequent (conclusion)

e.g., if it rains then I will carry an umbrella

Here, p: It rains (is antecedent)

q: I will carry an umbrella (is consequent)

In logic, the antecedent and the consequent in a conditional proposition are not required to refer to the same subject matter. e.g., "If I get the money then this book is red" does not make sense but in logics, the statement is perfectly acceptable and has a truth-value.

Converse, Inverse & Contrapositive Statements

1. To form the **converse of the conditional statement**, interchange the hypothesis and the conclusion.

e.g., the converse of "If it rains, then they cancel school" is "If they cancel school, then it rains.“

2. To form the **inverse of the conditional statement**, take the negation of both the hypothesis & the conclusion.

e.g., the inverse of "If it rains, then they cancel school" is "If it does not rain, then they do not cancel school."

3. To form the **contrapositive of the conditional statement**, interchange the hypothesis and the conclusion of the inverse statement. e.g., the contrapositive of "If it rains, then they cancel school" is "If they do not cancel school, then it does not rain.“

Converse, inverse and contrapositive

of the conditional statement can be defined as per the adjoining table:

Statement	If p , then q .
Converse	If q , then p .
Inverse	If not p , then not q .
Contrapositive	If not q , then not p .

Converse, Inverse & Contrapositive Statements

Example

Statement	If two angles are congruent, then they have the same measure.
Converse	If two angles have the same measure, then they are congruent.
Inverse	If two angles are not congruent, then they do not have the same measure.
Contrapositive	If two angles do not have the same measure, then they are not congruent.

Example: Find the truth values of the following propositions:

(i) If the Earth is round then the Earth travels round the Sun

(ii) If Graham Bell invented telephone then tiger have wings

(iii) If tigers have wings then RDX is dangerous

Solution:

(i) p: The Earth is round

q: The Earth travels round the Sun

Here, ‘p’ is true and ‘q’ is true and hence the truth value of $p \rightarrow q$ is True

(ii) p: Graham Bell invented telephone

q: tiger have wings

Here, ‘p’ is true and ‘q’ is false and hence the truth value of $p \rightarrow q$ is False

(iii) p: tigers have wings

q: RDX is dangerous

Here, ‘p’ is false and ‘q’ is true and hence the truth value of $p \rightarrow q$ is True

Example: Construct the truth table for $(\sim(p \wedge q) \vee r) \longrightarrow \sim p$

Solution: Truth Table

p	q	r	$p \wedge q$	$\sim(p \wedge q)$	$\sim(p \wedge q) \vee r$	$\sim p$	$(\sim(p \wedge q) \vee r) \longrightarrow \sim p$
T	T	T	T	F	T	F	F
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T
T	F	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	F	F	F	T	T	T	T

5. Biconditional Proposition

If p and q are two propositions, then the compound proposition ‘ p if and only if q’ denoted by $p \leftrightarrow q$ is called biconditional or bi-implication proposition,

which is $\begin{cases} \text{True when both p \& q are true or both are false} \\ \text{False } otherwise \end{cases}$

Truth Table

p	**q**	**$p \leftrightarrow q$**
F	F	T
F	T	F
T	F	F
T	T	T

Example: p and q are two propositions, defined as under:

p: A new car will be acquired

q: Additional funding is available

Clearly, $p \leftrightarrow q$ as a new car will be acquired if and only if additional funding is available.

Example: Show that $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

Solution:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

From above truth table, the truth values in columns 3 and 6 are identical. Hence the result.

Algebra of Propositions (Laws of Logic)

1. Idempotent Laws

(i) $p \vee p \equiv p$ $\quad$ (ii) $p \wedge p \equiv p$

2. Associative Laws

(i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. Commutative Laws

(i) $p \vee q \equiv q \vee p$

(ii) $p \wedge q \equiv q \wedge p$

4. Distributive Laws

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. Absorption Laws

(i) $p \vee (p \wedge q) \equiv p$ $\quad$ (ii) $p \wedge (p \vee q) \equiv p$

6. De Morgan's Laws

(i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\quad$ (ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

7. Involution Law $\quad$ $\sim\sim p \equiv p$

Note: All of above laws can be proved with the help of Truth Table.

The Hierarchy Rule for Logical Connectives

Each logical connective has some priority while solving the problems. The order of priority will be considered as under:

order of priority

(i) ( )

(ii) ~ or ¬

(iii) ∧

(iv) ∨

(v) →

(vi) ↔

Example: Construct the truth table for the compound proposition $p \wedge (\neg q)$.

Truth Table

p	q	$\neg q$	$p \wedge (\neg q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Example: Construct the truth table for the compound proposition $\neg p \vee q \leftrightarrow p \longrightarrow q$.

Truth Table

p	q	$\neg p$	$\neg p \vee q$	$p \longrightarrow q$	$\neg p \vee q \leftrightarrow p \longrightarrow q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Tautologies

Definition: An expression (compound statement) involving logical variables, which is true for all cases of its truth table is called tautology.

e.g. $(\neg \mathrm{p} \vee q) \leftrightarrow (p \longrightarrow q)$ is tautology as shown in truth table:

p	q	$\neg p$	$\neg \mathrm{p} \vee q$	$p \longrightarrow q$	$\neg \mathrm{p} \vee q \leftrightarrow p \longrightarrow q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Contradiction

Definition: An expression (compound statement) involving logical variables, which is false for all cases of its truth table is called a Contradiction.

e.g. $p \land \neg p$ is a contradiction.

Example: Prove that $(p \lor q) \land (\neg p \land \neg q)$ is a contradiction

Truth Table

p	q	$p \lor q$	$\neg p$	$\neg q$	$\neg p \land \neg q$	$(p \lor q) \land (\neg p \land \neg q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Since the truth values for all cases of the truth table are ‘False’ therefore the expression is Contradiction.

Contingency

Definition: An expression (compound statement) involving logical variables, which is neither a tautology nor a contradiction is called a Contingency.

Note: Tautologies and Contradictions are symbolized as ‘T’ and ‘F’ respectively.

$(\mathrm{i}) \neg T \equiv F, \; (ii) \neg F \equiv T$

Example: Prove that $p \wedge q$ is a contingency.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Solution: In the adjoining Truth Table,

truth values are not all true and not all false.

Therefore the given expression is Contingency.

Practice Questions

Check the following expressions for tautology, contradiction or contingency:

1. $\neg(p \vee q) \wedge (p \wedge q)$
2. $p \wedge \neg p$
3. $(p \rightarrow q) \wedge (q \rightarrow p)$
4. $p \rightarrow (p \vee q)$
5. $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
6. $(p \vee q) \leftrightarrow (q \vee p)$
7. $[p \rightarrow (q \vee r)] \wedge (\neg q) \rightarrow (p \rightarrow r)$

Argument – Implication (Inferences)

Definition: If from a given set of propositions $P_1, P_2, P_3, \ldots . P_n$ (called premises), some another propositions (say P) are derived (called conclusion), then this assertion is called an **Argument**. And it is denoted by $P_1, P_2, P_3, \ldots . P_n \vdash P$

<u>Valid Argument</u>

An argument $P_1, P_2, P_3, \ldots . P_n \vdash P$ is said to be valid if whenever all the premises $P_1, P_2, P_3, \ldots . P_n$ are true then the conclusion P is likewise true.

An argument which is not valid is called a <u>Fallacy</u>

Theorem: The argument $P_1, P_2, P_3, \ldots . P_n \vdash P$ is valid iff the proposition

$$P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \rightarrow P \text{ is a tautology.}$$

Proof: The propositions $P_1, P_2, P_3, \ldots . P_n$ are true simultaneously if and only if the proposition $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n$ is true

Thus the argument $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \vdash P$ is valid if and only if P is true.

i.e., $P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n \rightarrow P$ is a tautology

Example: Show that the argument $p, \ p \rightarrow q \ \vdash q$ is valid.

Solution: Truth Table

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the truth table, all truth values are true, thus the given argument is tautology and hence it is valid argument.

Theorem: $If\ p\ implies\ q\ and\ q\ implies\ r, then\ p\ implies\ r$

$$\text{i.e.}, the\ argument\ p \to q, q \to r \vdash p \to r\ is\ valid$$

Verification: Truth Table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	T	T	F	T	F	F	T
T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

From the truth table, all truth values are true, thus the given argument is tautology and hence it is valid argument.

Example: Show that the argument $p \to q,\ \neg p\ \vdash \neg q$ is a Fallacy.

Solution: Truth Table

p	q	$p \to q$	$\neg p$	$(p \to q) \land (\neg p)$	$\neg q$	$(p \to q) \land (\neg p) \to (\neg q)$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

From the truth table, all truth values are not true, thus the given argument is not valid argument and hence it is Fallacy.

Rules of Inference

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism

Rules of Inference Contd..

Rule of Inference	*Tautology*	*Name*
$\dfrac{p}{\therefore\ p \vee q}$	$p \to (p \vee q)$	Addition
$\dfrac{p \wedge q}{\therefore\ p}$	$(p \wedge q) \to p$	Simplification
$\dfrac{\begin{array}{l} p \\ q \end{array}}{\therefore\ p \wedge q}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$\dfrac{\begin{array}{l} p \vee q \\ \neg p \vee r \end{array}}{\therefore\ q \vee r}$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Validity and Satisfiability

A formula is **valid** if all truth values in the truth table are true.

A formula/proposition is **satisfiable** if there is at least one true result in its truth table.

Also, a propositional statement is **satisfiable** if and only if, its truth table is not contradiction.

Not contradiction means, it could be a tautology also.

Hence, every tautology is also Satisfiable. However, Satisfiability doesn't imply Tautology.

Note: If a propositional statement is Tautology, then its always valid.

Thus, Tautology implies (Satisfiability + Validity)

Logical Equivalences Involving Conditional and Biconditional Statements

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Duality

The **dual** of a statement formula is obtained by replacing $\vee$ by $\wedge$, $\wedge$ by $\vee$, T by F & F by T.

Note:

(1) The symbol $\neg$ is not changed while finding the dual.

(2) Dual of a dual is the statement itself.

(3) The special statements T (tautology) and F (contradiction) are duals of each other.

For example

(i) The dual of $(p \vee q) \wedge (r \wedge s) \vee F$ is $(p \wedge q) \vee (r \vee s) \wedge T$.

(ii) The dual of $p \wedge [\neg q \vee (p \wedge q) \vee \neg r]$ is $p \vee [\neg q \wedge (p \vee q) \wedge \neg r]$.

Normal Form (CNF & DNF)

Let $A(P_1, P_2, P_3, \ldots, P_n)$ be a statement formula where $P_1, P_2, P_3, \ldots, P_n$ are the propositions/variables. We may use the word "product" in place of "conjunction" and "sum" in place of "disjunction". The given formula transformed /reduced to **product of sums form** or **sum of products form** is called **normal form**.

There are two types of normal forms:

1. Conjunctive Normal Form (CNF), is a formula which is equivalent to a given formula and it consists of **Product**(and) **of Sums**(ors of one or more literals)
2. Disjunctive Normal Form (DNF), is a formula which is equivalent to a given formula and it consists of **Sum**(or) **of Products**(ands of one or more literals)

CNF

Any Boolean function can be represented in CNF & DNF. One way to obtain CNF & DNF formulas is based upon the truth table for the function.

•A CNF formula is the pessimistic approach, focusing on the rows where the function is false. For each row where the function is false, create a disjunction of the propositions. (e.*g*., if in this row p is **false** and q is **true**, form (p ∨¬*q*). Now, form the conjunction of all those disjunctions.

$p \lor q \lor r = F \iff p = F \text{ and } q = F \text{ and } r = F$

$p \lor q \lor \neg r = F \iff p = F \text{ and } q = F \text{ and } r = T$

$\neg p \lor q \lor \neg r = F \iff p = T \text{ and } q = F \text{ and } r = T$

CNF

p	q	r	$G(p,q,r)$
T	T	T	T
T	T	F	T
$\boxed{\text{T}}$	$\boxed{\text{F}}$	$\boxed{\text{T}}$	$\boxed{\text{F}}$
T	F	F	T
F	T	T	T
$\boxed{\text{F}}$	$\boxed{\text{T}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$
F	F	T	T
$\boxed{\text{F}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$	$\boxed{\text{F}}$

Any given formula

$\Rightarrow \neg p \lor q \lor \neg r$

$\land$

$\Rightarrow p \lor \neg q \lor r$

$\land$

$\Rightarrow p \lor q \lor r$

$$G(p,q,r) \equiv \underbrace{\underbrace{(\neg p \lor q \lor \neg r)}_{\text{Sun}} \land \underbrace{(p \lor \neg q \lor r)}_{\text{Sum}} \land \underbrace{(p \lor q \lor r)}_{\text{Sum}}}_{\text{Product}}$$

CNF (Example)

Example: Obtain CNF for the given formula, $\neg(p \leftrightarrow q)$

Solution: Construct the Truth Table for the given formula, $\neg(p \leftrightarrow q)$ as under:

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Focus on False for CNF

Focus on False for CNF

For each row where the function is false, create a disjunction of the propositions, i.e.,
in row 1, corresponding to F, values of propositions p & q are ‘T’, it should have been false, so take disjunction of $(\neg p)$ & $(\neg q)$ i.e., $(\neg p \vee \neg q)$.
And in row 4, corresponding to F, values of propositions p & q are ‘F’, matching with false, so take disjunction of p & q i.e., $(p \vee q)$.
Now, form the conjunction of these disjunctions. i.e., $(\neg p \vee \neg q) \wedge (p \vee q)$
Thus $\neg(p \leftrightarrow q) \equiv (\neg p \vee \neg q) \wedge (p \vee q)$ is the CNF for the given formula. Ans.

DNF

A DNF formula results from looking at a truth table, and focusing on the rows where the function is true. For each row where the function is true, form a conjunction of the propositions. (e.g., for the row where p is false, and q is true, form (¬p ∧ q). Now, form the disjunction of all those conjunctions.

$p \land q \land r = \mathrm{T} \Longleftrightarrow p = \mathrm{T}$ and $q = \mathrm{T}$ and $r = \mathrm{T}$

$p \land q \land \neg r = \mathrm{T} \Longleftrightarrow p = \mathrm{T}$ and $q = \mathrm{T}$ and $r = \mathrm{F}$

$\neg p \land q \land \neg r = \mathrm{T} \Longleftrightarrow p = \mathrm{F}$ and $q = \mathrm{T}$ and $r = \mathrm{F}$

DNF

Write $p \Rightarrow q$ in DNF (sum of products)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \land q$

$\lor$

$\neg p \land q$

$\lor$

$\neg p \land \neg q$

Ans.

$$\boxed{p \Rightarrow q \equiv (p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q)}$$

p	q	r	$G(p,q,r)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

$G(p,q,r)$ → Any given formula

$p \wedge \neg q \wedge r$

$\vee$

$\neg p \wedge q \wedge \neg r$

$\vee$

$\neg p \wedge \neg q \wedge \neg r$

$$G(p,q,r) \equiv \underbrace{\underbrace{(p \wedge \neg q \wedge r)}_{\text{Product}} \vee \underbrace{(\neg p \wedge q \wedge \neg r)}_{\text{Product}} \vee \underbrace{(\neg p \wedge \neg q \wedge \neg r)}_{\text{Product}}}_{\text{Sum}}$$

DNF (Example)

Example: Obtain DNF for the given formula, $\neg(p \leftrightarrow q)$

Solution: Construct the Truth Table for the given formula, $\neg(p \leftrightarrow q)$ as under:

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$
T	T	T	F
T	F	F	**T**
F	T	F	**T**
F	F	T	F

Focus on True for DNF

Focus on True for DNF

For each row where the function is True, create a conjunction of the propositions, i.e., in row 2, corresponding to T, values of propositions p & q are 'T'& 'F' respectively, these should have been true for both, so take conjunction of (p) & $(\neg q)$ i.e., $(p \wedge \neg q)$.
And in row 3, corresponding to T, values of propositions p & q are 'F' & 'T' respectively, these should have been true for both, so take conjunction of $(\neg p)$ & (q) i.e., $(\neg p \wedge q)$.
Now, form the disjunction of these conjunctions. i.e.,$(p \wedge \neg q) \vee (\neg p \wedge q)$
Thus $\neg(p \leftrightarrow q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ is the DNF for the given formula. Ans.

Practice Questions

Obtain CNF and DNF for the following formulas:

1. $\neg(p \lor q)$
2. $p \land \neg p$
3. $p \rightarrow (p \lor q)$
4. $[p \rightarrow (q \rightarrow r)]$
5. $[p \rightarrow (q \lor r)]$

Predicate Calculus

Statements involving variables, such as “x > 3,” “x = y + 3” and “computer x is functioning properly,” are neither true nor false when the values of the variables are not specified are called **Predicates**. Let us denote the statement “x is greater than 3” by P (x), where P denotes the predicate “is greater than 3” and x is the variable. Once a value has been assigned to the variable x, the statement P (x) will become a proposition and will have a truth value.

e.g., if x = 4, then 4 > 3, is true and if x = 2, then 2 > 3, is false.

Quantifiers are words that refer to quantities such as ‘**all**’, ‘**some**’, ‘**many**’, ‘**none**’, **&** ‘**few**’ and tell us, for how many elements, a given predicate, is true or false.

Broadly two kinds of quantifiers are there: **Universal Quantifiers** & **Existential Quantifier**

The area of logic that deals with predicates and quantifiers is called the **predicate calculus.**

Quantifiers

Quantifier	Statement	When True?	When False?
Universal Quantifier	$\forall x \, P(x)$	P(x) is True for every x	There is an x for which P(x) is False.
Existential Quantifier	$\exists x \, P(x)$	There is an x for which P(x) is True.	P(x) is False for every x.

Universal & Existential Quantifier

The symbol $\forall$ is called a **Universal Quantifier**, and the statement $\forall x\, P(x)$ is called a universally quantified statement.

In $\forall x\, P(x)$, the notation $\forall$ states that all the values in the domain of x will yield a true statement. We read $\forall x\, P(x)$ as "for all x P(x)" or "for every x P(x)."

The symbol $\exists$ is called the **Existential Quantifier**, and the statement $\exists x\, P(x)$ is called a Existential Quantified statement.

In $\exists x\, P(x)$, the notation $\exists$ states that the statement is true for at least one value of x in the domain. We read $\exists x\, P(x)$ as "There exists an element x in the domain such that P (x)."

Note: Domain specifies the possible values of the variable x.

When a value in the domain of x proves the universal quantified statement false, the x value is called a **counterexample**.

Example: If the domain of x is all positive integers e.g., {1, 2, 3, 4, …}
and $\forall x\, F(x)$: $x - 1 > 0$ (x minus 1 is greater than 0)
Here, all the numbers in the domain prove the statement true except for the number $x = 1$. Thus $x = 1$ is the counterexample.

Example: Let P (x) be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution: Because P (x) is true for all real numbers x, the quantification $\forall x\, P(x)$ is True.

Example: What is the truth value of ∀x P(x), where P (x) is the statement " $x^2 < 10$ " and the domain consists of the positive integers not exceeding 4?

Solution: The statement ∀x P(x) is the same as the conjunction

P (1) ∧ P (2) ∧ P (3) ∧ P (4) [∵ domain consists of the integers 1, 2, 3, & 4]

But P (4), which is the statement "$4^2 < 10$," is false,

Thus, ∀x P(x) is false.

Example: What is the truth value of ∃x P(x), where P (x) is the statement " $x^2 < 10$ " and the domain consists of the positive integers not exceeding 4?

Solution: Because the domain is {1, 2, 3, 4}, the statement ∃x P (x) is the same as the disjunction P (1) ∨ P (2) ∨ P (3) ∨ P (4).

Now as P(1), P(2), P(3) are true and P(4) is false, therefore the disjunction P (1) ∨ P (2) ∨ P (3) ∨ P (4) is true, it follows that ∃x P(x) is true.

Example: Let P (x) denote the statement “$x > 3$.” What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution: Because “$x > 3$” is sometimes true, for instance, when $x = 4$, the existential quantification of P (x), which is $\exists x\, P(x)$, is True.

Example: Let Q(x) denote the statement “$x = x + 1$.” What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution: Because Q(x), $x = x + 1$ is false for every real number x, the existential quantification of Q(x), which is $\exists x\, Q(x)$, is false.

Properties of Quantifiers

1. Universal Quantifier follows distributive property over a conjunction
i.e., $\forall x(P(x) \wedge Q(x)) \equiv \forall xP(x) \wedge \forall xQ(x)$ where the same domain is used throughout.

2. Existential Quantifier follows distributive property over a disjunction.
i.e., $\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$

However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction

De Morgan’s Laws for Quantifier

Negating Quantified Expressions [De Morgan’s Laws for Quantifier]

3. $\neg \forall x P(x) \equiv \exists x \neg P(x)$.

For instance, consider the negation of the statement “Every student in a class has taken a course in calculus.” This statement is a universal quantification, $\forall x P(x)$, where $P(x)$ is the statement “x has taken a course in calculus” and the domain consists of the students in a class. The negation of this statement is equivalent to “There is a student in a class who has not taken a course in calculus” and this is simply the Existential Quantification of the negation of the original propositional function, namely, $\exists x \neg P(x)$.

4. $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$

Example: What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$,
which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$.

The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$,
which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$.

Nested Quantifiers

A predicate has nested quantifiers if there is more than one quantifier in the statement.
Each quantifier can only bind to one variable, such as $\forall x\ \exists y\ E(x, y)$.
The first quantifier is bound to x ($\forall x$), and the second quantifier is bound to y ($\exists y$).
[The $\exists$ asserts that at least one value will make the statement true. If no value makes the statement true, the statement is false.
The $\forall$ asserts that all the values will make the statement true. The statement becomes false if at least one value does not meet the statement's assertion]

Example: Let x = {0, 1, 2, 3, 4, 5, 6} domain of x
and y = {0, 1, 2, 3, 4, 5, 6} domain of y,
$\exists x\ \forall y\ E(x + y = 5)$ reads as "At least one value of x plus any value of y equals 5."

The statement is false because no value of x plus any value of y equals 5.

Instead of saying “reads as,” use the biconditional symbol ↔ to indicate that the nested quantifier example and its English translation have the same truth value.

$\forall x\ \exists y\ E(x + y = 5) \leftrightarrow$ “Any value of x plus at least one value of y will equal 5.”

The statement is true.

$\forall x\ \forall y\ E(x + y = 5) \leftrightarrow$ “Any value of x plus any value of y will equal 5.”

The statement is false.

$\exists x\ \exists y\ E(x + y = 5) \leftrightarrow$ “At least one value of x plus at least any value of y will equal 5.”

The statement is true.

De Morgan's law also applies to nested quantifiers.

$\neg\forall x\ \forall y\ P(x, y) \equiv \exists x\ \exists y\ \neg P(x, y)$

$\neg\forall x\ \exists y\ P(x, y) \equiv \exists x\ \forall y\ \neg P(x, y)$

$\neg\exists x\ \forall y\ P(x, y) \equiv \forall x\ \exists y\ \neg P(x, y)$

$\neg\exists x\ \exists y\ P(x, y) \equiv \forall x\ \forall y\ \neg P(x, y)$

Example: Translate these statements into English, where C(x) is “x is a comedian” and F(x) is “ x is funny” and the domain consists of all people

a) $\forall x(C(x) \rightarrow F(x))$

b) $\exists x(C(x) \rightarrow F(x))$

c) $\forall x(C(x) \wedge F(x))$

d) $\exists x(C(x) \wedge F(x))$

Solution:

a) for all people if a person is comedian then he is funny.
b) there exists some persons if they are comedian then they are funny.
c) for all people person is comedian and person is funny
d) there exists some person which are comedian and funny.

Example: Consider the statement:“ Not all that glitters is gold”

Predicate glitters(x) is true if x glitters and predicate gold(x) is true if x is gold.

Which one of the following logical formula represents the above statement?

(a) $\forall x$:glitters(x) $\rightarrow \neg$gold(x)

(b) $\forall x$: gold(x) $\rightarrow$ glitters(x)

(c) $\exists x$: gold(x) $\wedge \neg$glitters(x)

(d) $\exists x$: (glitters (x) $\wedge \neg$gold (x))

Answer: “Not all that glitters is gold” means “There exist a glitter and it is not gold”

which can be expressed as

$\exists x$: (glitters (x) $\wedge \neg$gold (x)) [option (d)]

Methods of Proving Theorems

Proving mathematical theorems, the methods provide the overall approach and strategy of proofs. Here, we discuss 2 methods, **direct proof method** & **proof by contradiction**

The first method as **direct proof method**, in which axioms, definitions of terms, previously proved results, and rules of inference are to be used to complete the proof. Direct proofs of many results are quite straightforward, with a fairly obvious sequence of steps leading from the hypothesis to the conclusion.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Note that this theorem states ∀nP ((n) → Q(n)), where P (n) is "n is an odd integer" and Q(n) is "n^2 is odd."

To begin with, assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer.

Squaring on both sides, we have

$$n^2 = (2k+1)^2$$

$$\Rightarrow n^2 = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$ [which is the form of, one more than twice an integer]

Thus, by the definition of an odd integer, we can conclude that n^2 is an odd integer.

Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Proof by Contradiction

Proof by Contradiction is the method which does not prove a result directly.

Suppose we want to prove that a statement p is true, then we will begin with, assuming that p is not true and proceeding further as per the facts to reach at a contradiction (invalid results).

Example: Prove that $\sqrt{2}$ is an irrational number.

Solution: Let p be the proposition "$\sqrt{2}$ is an irrational."

To start a proof by contradiction, we suppose that $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that $\sqrt{2}$ is irrational," which says that $\sqrt{2}$ is rational. We will show that assuming that $\neg p$ is true leads to a contradiction.

Let $\sqrt{2}$ is rational, so there exist integers a and b with $\sqrt{2} = \frac{a}{b}$, where b ≠ 0 and a & b have no common factors, i.e., a & b are co-prime to each other.

Now because $\sqrt{2} = \frac{a}{b}$, squaring on both sides, we get

$$2 = \frac{a^2}{b^2}$$

$\Rightarrow \quad 2\,b^2 = a^2$ …. (*) , it follows that a^2 is even and thus ‘a’ must also be even, thus by the definition of an even integer, $a = 2c \; for \; some \; integer \; c.$

Squaring on both sides implies, $a^2 = 4\,c^2$ …. (**)

Comparing (*) & (**), we get $2\,b^2 = 4\,c^2$

$\Rightarrow \quad b^2 = 2\,c^2$ it follows that b^2 is even and thus ‘b’ must also be even.

Thus it is concluded that both a & b are even, i.e., they are not co-prime of each other as we assumed. Which is the contradiction as per our assumption.

Thus, the statement p, “√2 is irrational,” is true. Hence √2 is an irrational number.

Practice Questions

Q 1. Use a direct proof to show that the sum of two odd integers is even.

Q 2. Give a proof by contradiction of the theorem "If 3n + 2 is odd, then n is odd."

Q 3. Give a direct proof of the theorem "If n^2 is an odd integer, then n is odd."

Q 4. Determine the truth value of each of these statements if the domain consists of all integers **a)** $\exists n(n = -n)$ **b)** $\forall n(3n \leq 4n)$

Q 5. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values of the following statements?

a) $P(1)$ **b)** $\exists x P(x)$

References

1. Seymour lipschutz, Marc Lars Lipson, Theory and Problems of Discrete Mathematics Third Edition, Schaum's Outline Series McGRAW-HILL
2. J.L. Mott, A. Kandelad T.P. Baker, Discrete Mathematics for Computer Scientists and Mathematicians, PHI, 2nd Edition, 1999
3. Kenneth H. Rosen, Discrete Mathematics and Its Applications, McGraw-Hill

Thanks