L19 Cauchy-Euler Equation

A special type of Linear DE with variable coefficients

(1) — $x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = r(x)$

where $a_{n-1}, \dots, a_1, a_0$ are constants and $r(x)$ is a function of x only.

is called Cauchy-Euler Equation.

This DE is solved by transforming (1) into DE with constant coefficients.

For that, the independent variable x is changed into another I.V. (t) by the substitution $x = e^t$ — (2)

From (2), $t = \ln x$ and $\frac{dx}{dt} = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{e^t}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt}$$

$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dt}}$ or write it in operator form

or, $\boxed{xDy = \theta y}$ where $\theta = \frac{d}{dt}$ & $D = \frac{d}{dx}$

or, $\boxed{xD = \theta}$

Similarly, diff $\frac{dy}{dx} = \frac{1}{e^t}\frac{dy}{dt}$ w.r.t. x.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{-t}\frac{dy}{dt}\right] = \frac{d}{dt}\left[e^{-t}\frac{dy}{dt}\right]\frac{dt}{dx}$$

$$= -e^{-t}\frac{dy}{dt}\frac{dt}{dx} + e^{-t}\frac{d^2y}{dt^2}\frac{dt}{dx} = e^{-2t}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = x^{-2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

In Operator Notation,

$$x^2D^2y = \theta^2y - \theta y$$

$$\boxed{x^2D^2 = \theta(\theta-1)}$$

Similarly, differentiating again,

$$\boxed{x^3D^3 = \theta(\theta-1)(\theta-2)}$$

$$\boxed{x^4D^4 = \theta(\theta-1)(\theta-2)(\theta-3)}$$

and By Mathematical Induction

$$x^nD^n = \theta(\theta-1)(\theta-2)\ldots(\theta-(n-1))$$

Now Consider a Second order DE

$$x^2y'' + axy' + by = r(x)$$

Replace $x = e^t$ then

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a\frac{dy}{dt} + by = r(e^t)$$

$$\Rightarrow \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = r(e^t)$$

$$\theta(\theta-1)y + a\theta y + by = r(e^t)$$

$$\Rightarrow \theta^2y - \theta y + a\theta y + by = r(e^t)$$

$$\Rightarrow \theta^2y + (a-1)\theta y + by = r(e^t)$$

This is a 2nd Order DE (Linear) with constant coefficient So it can be solved by previous discussed Methods.

Ex. $x^2y'' - 3xy' + 3y = 0$ with $y(1) = 0, y'(1) = -2$

Put $x = e^t$ and its derivatives $xD = \theta$, $x^2D^2 = \theta^2 - \theta$ in DE

$\therefore (\theta^2 - \theta)y - 3\theta y + 3y = 0$

$\Rightarrow \theta^2 y - 4\theta y + 3y = 0$

The A.E. is $m^2 - 4m + 3 = 0 \Rightarrow m = 1, 3$

G.S. is $y = c_1 y_1(t) + c_2 y_2(t)$

$= c_1 e^t + c_2 e^{3t}$

$= c_1 x + c_2 x^3$

Conditions are $y(1) = 0$; $y'(1) = -2$

$\therefore 0 = c_1 + c_2$ & $y'(x) = c_1 + 3c_2 x^2$

$y'(1) = c_1 + 3c_2 = -2$

$\therefore c_1 = 1$ & $c_2 = -1$

Hence the particular solution $y = x - x^3$.

Ex. $x^2y'' - 2xy' + 2y = x^3 \sin x$

Put $x = e^t$ and its derivatives $xD = \theta$, $x^2D^2 = \theta^2 - \theta$ in DE

$\therefore (\theta^2 - \theta)y - 2\theta y + 2y = e^{3t}\sin(e^t)$

$\Rightarrow \theta^2 y - \theta y - 2\theta y + 2y = e^{3t}\sin(e^t)$

$\Rightarrow \theta^2 y - 3\theta y + 2y = e^{3t}\sin(e^t)$

The A.E. is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$

G.F. $= c_1 e^t + c_2 e^{2t}$

$$P.I. = u_1 e^t + u_2 e^{2t}$$

$$W = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t} = e^{3t}$$

$$u_1 = -\int \frac{r(t)\, y_2(t)}{W}\, dt$$

$$= -\int \frac{e^{3t}\sin(e^t)\, e^{2t}}{e^{3t}}\, dt$$

$$= -\int e^{2t}\sin(e^t)\, dt$$

Put $e^t = s \Rightarrow e^t dt = ds$

$$= -\int s\sin(s)\, ds$$

$$= -\left[s(-\cos s)\right] + \int 1\cdot(-\cos s)\, ds$$

$$= s\cos s - \sin s = e^t\cos e^t - \sin e^t$$

$$u_2 = \int \frac{r(t)\, y_1(t)}{W}\, dt = \int \frac{e^{3t}\sin(e^t)\, e^t}{e^{3t}}\, dt = \int e^t \sin(e^t)\, dt$$

$$= -\cos(e^t)$$

$$\therefore\ P.I. = e^t\left(e^t\cos(e^t) - \sin(e^t)\right) - e^{2t}\cos(e^t)$$

$$= -\sin(e^t)\, e^t$$

$\therefore$ The G.S. is $y = C.F. + P.I.$

$$= c_1 e^t + c_2 e^{2t} - e^t\sin(e^t)$$

$$= c_1 x + c_2 x^2 - x\sin x$$