**Coset**: Let $G$ be a group and $H$ a subset of $G$. For any $a \in G$, the set $\{ah \mid h \in H\}$ is denoted by $aH$.

$Ha = \{ha \mid h \in H\}$

$aHa^{-1} = \{aha^{-1} \mid h \in H\}$

When $H$ is a subgroup of $G$, the set $aH$ is called the left coset of $H$ in $G$.

And $Ha$ is called right coset in $G$.

Ex1. Let $G = S_3$ and $H = \{I, (13)\}$. Then the left cosets of $H$ in $G$ are

$[(1) = I]$

$(1)H = \{(1)(1), (1)(13)\} = \{(1), (13)\} = H$

$(12)H = \{(12)I, (12)(13)\} = \{(12), (132)\} = (132)H$

$(13)H = \{(13)I, (13)(13)\} = \{I, (13)\} = H$

$(23)H = \{(23), (123)\} = (123)H$

Ex2. Let $G = Z_9 = \{0, 1, 2, \ldots, 8\}$ & $H = \{0, 3, 6\}$, the left cosets are

$0H = \{0+0, 0+3, 0+6\} = \{0, 3, 6\} = H$

$1H = \{1+0, 1+3, 1+6\} = \{1, 4, 7\} = 4H = 7H$

$2H = \{2+0, 2+3, 2+6\} = \{2, 5, 8\} = 5H = 8H$

**Properties of Cosets**: Let $H$ be a subgroup of $G$ and let $a$ and $b$ belong to $G$. Then,

1. $a \in aH$ 2. $aH = H$ iff $a \in H$
3. $aH = bH$ or $aH \cap bH = \phi$
4. $aH = bH$ iff $a^{-1}b \in H$
5. $|aH| = |bH|$
6. $aH$ is a subgroup of $G$ iff $a \in H$.

These properties are true for right coset also.

Thm:- Two diff (right) cosets are identical or disjoint. (Property 3)

Pf:- Let $H$ be a subgroup of $G$ and $a, b \in G$. The two left cosets are $aH$ and $bH$. To show $aH = bH$ or $aH \cap bH = \phi$.

Let. $aH \cap bH \neq \phi$ then to show $aH = bH$.

Let $x \in aH \cap bH$. Then there exist $h_1, h_2$ in $H$ such that

$$x = ah_1 \ \& \ x = bh_2$$

Thus, $ah_1 = bh_2 \Rightarrow a = bh_2h_1^{-1}$

$$\therefore aH = bh_2h_1^{-1}H = bH \quad [\because h_2h_1^{-1} \in H$$

as $h_1, h_2 \in H$ and $H$ is a subgroup.

Lagrange's theorem: If $G$ is a finite group and $H$ is a subgroup of $G$, then order of $H$ divides order of $G$.

Pf:- Let $a_1H, a_2H, \ldots, a_rH$ denote the distinct left cosets of $H$ in $G$. Then for each $a$ in $G$, we have $aH = a_iH$ for some $i$. Also by property of coset, $a \in aH$. Thus, each member of $G$ belongs to one of the cosets $a_iH$. In symbols,

$$G = a_1H \cup a_2H \cup \ldots \cup a_rH$$

Since, cosets are distinct so union of distinct cosets is disjoint

$$\therefore |G| = |a_1H \cup a_2H \cup \ldots \cup a_rH|$$

$$= |a_1H| + |a_2H| + \ldots + |a_rH|$$

Finally, since $|a_iH| = |H|$ for each $i$,

$$\therefore |G| = |H| + |H| + \ldots + |H|$$

$$= r \cdot |H|$$

$$\Rightarrow |H| \,\big|\, |G|$$

Hence, proved.

The converse of Lagrange's theorem is false.
For exmaple, $A_4$ don't have subgroup of order 6.
But the converse of Lagrange's thm is true for cyclic group and abelian group.

<u>Corollary</u>: In a finite group, the order of each element of the group divides the order of the group.

Q Show that union of two subgroups may not be a subgroup.

<u>Soln</u>:- Let $(\mathbb{Z}, +)$ be a group of integers with addition.
$H_2 = \{2n \mid n \in \mathbb{Z}\}$, $H_3 = \{3m \mid m \in \mathbb{Z}\}$ are two subgroups of $(\mathbb{Z}, +)$.
But $H_2 \cup H_3 = \{2n, 3m\}$ is not subgroup as it does not hold closure ($3 \in H_2 \cup H_3$, $2 \in H_2 \cup H_3$ but $3+2 \notin H_2 \cup H_3$).

<u>Thm</u>:- Intersection of two subgroups is a subgroup.

<u>Pf</u>:- Let $G$ be group and $H_1$ & $H_2$ are its two subgroups.
To show $H_1 \cap H_2$ is a subgroup of $G$.
Since $H_1$ and $H_2$ are subgroups so they must contain identity.
$\therefore$ $H_1 \cap H_2 \neq \phi$. and $H_1 \subseteq G$ & $H_2 \subseteq G \Rightarrow H_1 \cap H_2 \subseteq G$.
Therefore, $H_1 \cap H_2$ is non empty subset of $G$.

Suppose $a, b \in H_1 \cap H_2$
$\Rightarrow a, b \in H_1$ & $a, b \in H_2$
($\because$ $H_1$ & $H_2$ are subgroups)
$\Rightarrow ab \in H_1$ & $ab \in H_2$
$\Rightarrow ab \in H_1 \cap H_2$
$\therefore$ Closure hold.

suppose, $a \in H_1 \cap H_2$
$\Rightarrow a \in H_1$ & $a \in H_2$
$\because$ $H_1$ & $H_2$ are subgroups they must have inverse of their elements.)
$\Rightarrow a^{-1} \in H_1$ & $a^{-1} \in H_2$
$\Rightarrow a^{-1} \in H_1 \cap H_2$
$\therefore$ Inverse property hold.

Hence proved