Predicates and Quantifiers :

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part - the predicate, "is greater than 3".
We can denote the statement by $P(x)$.
Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

e.g. let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(4)$ and $P(2)$?

When the variables in a propositional function ($P(x)$) are assigned values, the resulting statement a proposition with a certain truth values.
However, there is another important way to create a proposition from a propositional function. Using some quantify words like all, some, many, none and few etc. The words are called quantifiers and process is called quantification.
We will focus on two types of quantification:

1. universal quantification, which tells us that a predicate is true for all element under consideration.
2. existential quantification, which tells us that there is one or more element under consideration for which the predicate is T.

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of discourse.
The universal quantification of $P(x)$ is the statement
"$P(x)$ for all values of x in the domain."
The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.
for all x $P(x)$ or for every x $P(x)$

An element for which $P(x)$ is false is called a <u>Counterexample</u> of $\forall x\, P(x)$.

e.g: Let $P(x)$ be the statement "$x+1 > x$". What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Soln:- $\forall x\, P(x)$: for all x, $x+1 > x$.

$\therefore$ the quantification $\forall x\, P(x)$ is true.

Note:- Besides "for all" and "for every", universal quantification can be expressed in many other ways, including "all of", "for each", "given any", "for arbitrary", "for each" and "for any".

e.g: let $Q(x)$: "$x < 2$". What is truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Sol:- $Q(x)$ is not true for every real number x, because, for instance $Q(3)$ is false. That is, $x=3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

When all the elements in the domain can be listed – say $x_1, x_2, ---, x_n$ – it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ because this conjunction is true iff $P(x_1)$, $P(x_2)$, ---, $P(x_n)$ are all true.

→ The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$"

Notation is $\underbrace{\exists}_{\text{existential quantifier}} x\, P(x)$

A domain must be specified when a statement $\exists x P(x)$ is used.

Some other words for existential Qut. is for some, for at least one, there is

e.g. $P(x)$: "$x > 3$". The existential quantification $\exists x P(x)$ is there is an element x for which $x > 3$. So, this is a true statement.

When all elements in the domain can be listed — say $x_1, x_2, \ldots, x_n$ — the existential quantification $\exists x P(x)$ is same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$
because this disjunction is true iff at least one of $P(x_i)$ is T.

Quantifiers with Restricted Domain:

$$\forall x < 0 \, (x^2 > 0), \quad \forall y \neq 0 \, (y^3 \neq 0), \quad \exists z > 0 \, (z^2 = 2)$$

$$\forall x \, (x < 0 \rightarrow x^2 > 0) \quad \forall y \, (y \neq 0 \rightarrow y^3 \neq 0) \quad \exists z \, (z > 0 \wedge z^2 = 2)$$

Logically equivalent: $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Negation of Quantified Expressions:

"Every student in your class has taken a course in calculus."

$\forall x P(x)$, $P(x)$: x has taken a course in calculus. The domain consists of the student in your class.

The negation is $\neg \forall x P(x)$: It is not the case that every student in your class has taken a course in calculus.

Or, There is a student in your class who has not taken a course in calculus.

$\exists x \neg P(x)$

$\therefore$ $\boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$ On Same way $\boxed{\neg \exists x Q(x) \equiv \forall x \neg Q(x)}$

Q Translate each of these statements into logical expressions using predicates, quantifiers and logical connectives -

(a) No one is perfect (b) Not everyone is perfect

(c) All your friends are perfect. (d) At least one of your friends is perfect.

(e) Everyone is your friend and is perfect. $\equiv \boxed{\forall x(Q(x) \land P(x))}$

(f) Not everybody is your friend or someone is not perfect.

Soln:- Let $P(x)$ is " x is perfect " and
$Q(x)$ is " x is your friend " and x is domain consists of all people.

(a) No one is perfect.
This can be written as Everyone is not perfect.
So, $\forall x \neg P(x)$

(b) Not $\underbrace{\text{everyone is perfect}}_{\forall x\, P(x)}$.

$\therefore \neg \forall x P(x)$

(c) All your friends are perfect.

All x, if x is your friend then x is perfect

$\therefore \forall x(Q(x) \rightarrow P(x))$

(d) At least one of your friends is perfect.

At least one x,

$\therefore \exists x(Q(x) \land P(x))$

(f) $\underbrace{\text{Not}}_{\neg}$ $\underbrace{\text{everybody is your friend}}_{\forall x\, Q(x)}$ $\underbrace{\text{or}}_{\lor}$ $\underbrace{\text{someone}}_{\exists x}$ $\underbrace{\text{is not perfect}}_{\neg P(x)}$

$\therefore (\neg \forall x\, Q(x)) \lor (\exists x \neg P(x))$