Consider the n^{th} order Homo linear DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{--- (1)}$$

The A.E. of (1) is obtained by replacing y by 1, y' by m, y'' by m^2, ..., $y^{(n)}$ by m^n in (1).

So, the A.E. of (1) is

$$m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

This is a n^{th} degree polynomial in m whose m roots.

The general solution of (1) containing n arbitrary constants is obtained according as the nature of the roots of A.E. :-

Nature of the n roots of A.E.	General solution of (1)
1. n distinct and real roots $m_1, m_2, m_3, \ldots$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots$
2. Two equal and real roots $m_1, m_1, m_3, m_4, \ldots$	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots$
3. Three equal roots and real roots $m_1, m_1, m_1, m_4, m_5, \ldots$	$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \cdots$
4. Two complex roots, $(n-2)$ distinct real roots, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3, m_4, \ldots$	$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots$
5. Two equal complex roots, $(n-4)$ distinct real roots, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, $m_5, m_6, \ldots$	$y = e^{\alpha x}((c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x) + c_5 e^{m_5 x} + \cdots$

Ex. $y^{(5)} - y^{(3)} = 0$

A.E. $m^5 - m^3 = 0 \Rightarrow m^3(m^2-1) = 0$

So $m = 0$ is a triple root and $m = -1, 1$ are distinct real roots.

Therefore, the G.S. is

$$y = c_1 e^{-x} + c_2 e^{x} + (c_3 + c_4 x + c_5 x^2) e^{0 \cdot x}$$

$$= c_1 e^{-x} + c_2 e^{x} + c_3 + c_4 x + c_5 x^2$$

Ex. $y^{(4)} - 4y^{(3)} + 14y'' - 20y' + 25y = 0$

The A.E. is $m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$

Every 4th order polynomial can be written as

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = (m^2 + am + b)(m^2 + cm + d)$$

$$= m^4 + (a+c)m^3 + (b+ac+d)m^2 + (ad+bc)m + bd$$

Equate the coefficients of ~~m, m~~ power of m,

$a + c = -4$, $b + ac + d = 14$, $ad + bc = -20$, $bd = 25$

$bd = 25 \Rightarrow b = 5, d = 5$ or $b = 1, d = 25$

When $b = 5, d = 5$ then $b + ac + d = 14 \Rightarrow ac = 4$

$\Rightarrow a = -2, c = -2.$

Because when $a = -2, c = -2$ then $a + c = -4$.

And $a = -2, b = 5, c = -2, d = 5$ verifies $ad + bc = -20$

$$\therefore m^4 - 4m^3 + 14m^2 - 20m + 25 = (m^2 - 2m + 5)(m^2 - 2m + 5)$$

$$= (m^2 - 2m + 5)^2$$

The roots of $m^2 - 2m + 5$ is $1 \pm 2i$.

Therefore, roots of A.E. is $1+2i, 1+2i, 1-2i, 1-2i$

Hence, the G.S. is $y = e^{x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$