Non-Homogeneous linear differential equation with constant coefficients :-

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x) \quad \text{--- (1)}$$

A general solution (G.S.) of the non-homogeneous DE (1) is of the form

$$y(x) = y_h(x) + y_p(x)$$

Here, $y_h(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ is a general solution of the corresponding homogeneous ODE $y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$.

→ $y_h(x)$ is called complementary function (C.F.)

And we know how to find G.S. ($y_h(x)$) of homogeneous DE.

$y_p(x)$ is any solution corresponding to $r(x)$ without any arbitrary constant. $y_p(x)$ is also particular integral (P.I.).

Therefore, the G.S. of (1) is = C.F. + P.I.

The method of finding particular integral (P.I.) is called undetermined coefficients.

There is another method of finding P.I. which will be followed on Page 19.

Method of Undetermined Coefficients

Term in $r(x)$	choice of $y_p(x)$
ke^{px}	Ce^{px}
$kx^n \; (n=0,1,\ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \ldots + K_1 x + K_0$
$k\cos\omega x$, $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$, $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Choice Rules for the Method of Undetermined Coefficients

(a) **Basic Rule** – If $r(x)$ in (1) is one of the functions in the first column in above table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (1).

(b) **Modification Rule** – If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (1), multiply this term by x (or by x^2 if this solution corresponds to a double root of the A.E. of the homogeneous ODE.

(c) **Sum Rule** – If $r(x)$ is a sum of functions in the first column of above table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Method to find P. I.
When $r(x) = e^{ax}$

Consider n^{th} order linear non-homogeneous DE with constant coefficients

$$D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 Dy + a_0 y = r(x) = e^{ax}$$

$$\Rightarrow (D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)\, y = e^{ax}$$

$$\Rightarrow f(D) y = e^{ax}$$

where $f(D) = D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

So, the particular integral (P. I.) is

$$y_p = \frac{1}{f(D)} e^{ax}$$

Case I: When $f(a) \neq 0$ then

$$\boxed{P.I. = y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

Case II: When $f(a) = 0$ then

$f(D) = (D-a)\phi(D)$ such that $\phi(a) \neq 0$.

$$\therefore P.I. = y_p = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{(D-a)\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} \frac{1}{(D-a)} e^{ax}$$

$$\boxed{y_p = \frac{1}{\phi(a)} x e^{ax}}$$

When $f(a) = 0$ then

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$$

When $f'(a) = 0$ then

$$\frac{1}{f(D)} e^{ax} = \frac{x^2}{f''(a)} e^{ax}$$