Case I. Two distinct real-roots m_1 and m_2 :-

If $a^2 - 4b > 0$ then the auxiliary equation gives two distinct roots m_1 and m_2.
Then there exists two solutions of (1) as

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}$$

Since m_1 & m_2 are distinct so $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent.
Therefore, the general solution of (1) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

or $\quad y = c_1 y_1 + c_2 y_2.$

e.g. $y'' - y' - 12y = 0$

The A.E. is $m^2 - m - 12 = 0$

The solution is $m_1 = -3$ and $m_2 = 4$.

So, there are two L.I. solutions $y_1 = e^{-3x}$ and $y_2 = e^{4x}$.

Thus, the general solution of ODE is

$$y = c_1 e^{-3x} + c_2 e^{4x}.$$

Case-II If the discriminant $a^2-4b=0$ then there is only one root m. So, there is only one L.I. solution

$$y_1 = e^{mx}$$

To obtain a second independent solution y_2, use the method of reduction of order, set $y_2 = uy_1$.

Substitute y_2 and its derivaties in ①,

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0$$

$$\Rightarrow u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$$

Since y_1 is sol of ①, so $y_1'' + ay_1' + by_1 = 0$, also

$2y_1' = -ay_1$.

$$\therefore u''y_1 = 0 \Rightarrow u'' = 0 \Rightarrow u = cx + d.$$

let $c=1$, $d=0$ then $u = x$.

Therefore, $y_2 = xy_1 = xe^{mx}$. (Second L.I. Solution)

Thus, the general solution is $\boxed{\begin{array}{l} y = c_1 e^{mx} + c_2 x e^{mx} \\ \hline y = (c_1 + c_2 x) e^{mx}. \end{array}}$

e.g. $y'' - 6y' + 9y = 0$

The A.E. is $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3$

$\therefore$ The G.S. is $y = (c_1 + c_2 x) e^{3x}$.

Case III: If the discriminant a^2-4b of the A.E. is negative then roots of A.E. are complex numbers i.e., $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

So, two solutions (L.I.) are

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

$$y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

Thus, the G.S. is

$$y = c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= e^{\alpha x}\{c_1\cos\beta x + c_2\cos\beta x + (c_1 - ic_2)\sin\beta x\}$$

$$= e^{\alpha x}\{\underbrace{(c_1+c_2)}_{c_3}\cos\beta x + \underbrace{(c_1-ic_2)}_{c_4}\sin\beta x\}$$

$$\boxed{y = e^{\alpha x}(c_3\cos\beta x + c_4\sin\beta x)}$$

Euler Formula: $\boxed{e^{ix} = \cos x + i\sin x}$

e.g. $y'' + 9y = 0$

The A.E. is $m^2 + 9 = 0 \Rightarrow m = 3i$

So, $\alpha = 0$, $\beta = 3$

The G.S. is

$$y = e^{0x}(A\cos 3x + B\sin 3x)$$

$$= A\cos 3x + B\sin 3x.$$