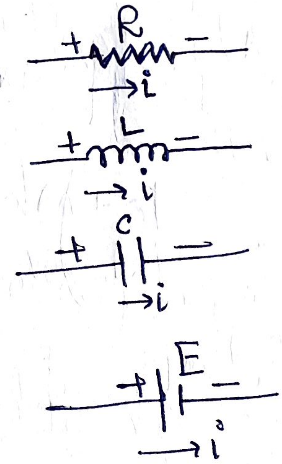
Application of Linear Differential Equations to Electric Circuits

When a word problem converted into mathematical terms such as operators or expression then this is called Mathematical modeling.

Electric circuit made up of Voltage source which may be a battery or a generator and Resistance, inductance and capacitance.

Element	Symbol
Charge	q
Current	i
Resistance	R
Inductance	L
Capacitance	C
Emf or Voltage	E



Some laws:

(i) $i = \dfrac{dq}{dt}$

ii) Voltage drop across resistance $= Ri$

iii) " inductance $= L\dfrac{di}{dt}$

iv) " capacitance $= \dfrac{q}{C}$

Kirchhoff's laws

(Emf)

Voltage law - The voltage impressed on a closed loop is equal to the sum of the voltage drops across all the other elements of the loop.

Current law - At any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.

~~To~~ Formulate the physical System as a mathematical ~~model~~ expression in terms of variables, functions and equations and such an expressions is known as a mathematical model of the problem.

The Process of setting up a model, solving it mathematically and interpreting the result in physical or other terms is called mathematical modeling.

Physical System

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Mathematical Model

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Mathematical Solution

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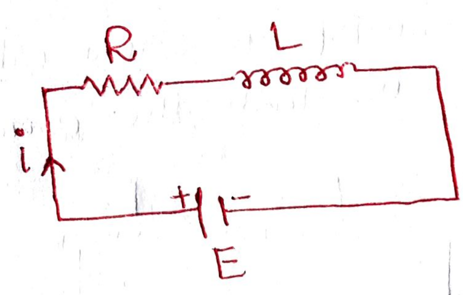
Physical Interpolation

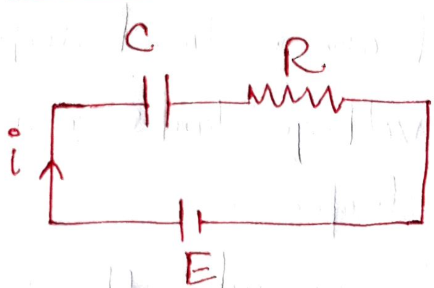
Simple LR circuits



$$Ri + L\frac{di}{dt} = E$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

CR circuit

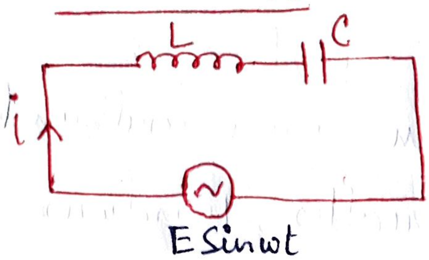


$$\frac{q}{c} + Ri = E$$

$$\Rightarrow R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = 0$$

$$\Rightarrow \frac{di}{dt} + \frac{1}{RC}i = 0.$$

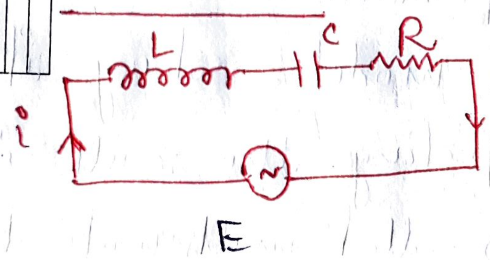
LC circuit



$$L\frac{di}{dt} + \frac{q}{c} = E\sin\omega t$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{1}{LC}q = E\sin\omega t$$

LCR circuit

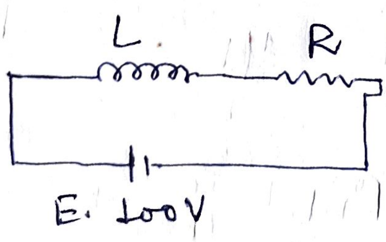


$$L\frac{di}{dt} + \frac{q}{c} + Ri = 0$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = 0$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c}i = 0$$

Q. An inductance of 2 henry and a resistance of 20 ohms are connected in series with an Emf. If the current is zero when $t=0$, find the current at end of 0.01 sec if emf is 100 volts.



Mathematical model of this circuit is

$$L\frac{di}{dt} + Ri = E$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

$$\Rightarrow \frac{di}{dt} + 10i = 50$$

Initial condition

$i(0) = 0$

$i(0.01) = ?$

This is a Linear DE of order 1

$I.F. = e^{+\int 10 dt} = e^{10t}$

Soln is $i e^{10t} = \int 50 e^{10t} dt + A$

$i e^{10t} = 5e^{10t} + A$

$\Rightarrow i = 5 + Ae^{-10t}$

When $t=0,\ i=0$

$\Rightarrow 0 = 5 + Ae^{0} \Rightarrow A = -5$

$\therefore i = 5(1 - e^{-10t})$

Now, $t = 0.01$ then

$i = 5(1 - e^{-10\times 0.01}) = 5(1 - e^{-0.1})$

$= 5(1 - e^{-0.1})$

Q: A 20 ohms resistor is connected in series with a capacitor of 0.01 farad and emf $E = 40e^{-3t} + 20e^{-6t}$. If $q=0$ at $t=0$, find charge at time t.

By Voltage law

$Ri + \frac{q}{C} = E$

$\Rightarrow \frac{dq}{dt} + \frac{1}{RC} q = \frac{E}{R}$

$\Rightarrow \frac{dq}{dt} + 5q = \frac{1}{20}(40e^{-3t} + 20e^{-6t})$

$\therefore \frac{dq}{dt} + 5q = 2e^{-3t} + e^{-6t}$

$\therefore I.F. = e^{+5t}$

Soln is

$q e^{5t} = \int e^{5t}(2e^{-3t} + e^{-6t}) dt + A$

$= 2\int e^{2t} dt + \int e^{-t} dt + A$

$= e^{2t} - e^{-t} + A$.

$\therefore q = e^{-3t} - e^{-6t} + Ae^{-5t}$

When $t=0,\ q=0$

$\Rightarrow A = 0$

$$\therefore \boxed{q(t) = e^{-3t} - e^{-5t}}$$

L22.

Q. Find current when $L = 0.5H$, $C = 0.005F$. and $E = \sin t\ V$, assuming zero initial current and charge.

By Voltage law,

$$L\frac{di}{dt} + \frac{q}{C} = E$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{1}{LC} i = \frac{1}{L} E'$$

$$\Rightarrow \frac{d^2 i}{dt^2} + 400 i = 2\cos t$$

$$C.F. = m^2 + 400 = 0 \Rightarrow m^2 = -400 \Rightarrow m = \pm 20i$$

$$i_h(t) = A\cos 20t + B\sin 20t$$

$$i_p(t) = \frac{2}{D^2+400}\cos t = \frac{2}{399}\cos t$$

$$\therefore i(t) = A\cos 20t + B\sin 20t + \frac{2}{399}\cos t$$

When $i = 0$, $t = 0 \Rightarrow 0 = A + \frac{2}{399} \Rightarrow A = -\frac{2}{399}$

And $\frac{dq}{dt} = A\cos 20t + B\sin 20t + \frac{2}{399}\cos t$

$$q(t) = \frac{A}{20}\sin 20t - \frac{B}{20}\cos 20t + \frac{2}{399}\sin t$$

when $t=0$, $q = 0 \Rightarrow 0 = -\frac{B}{20} \Rightarrow B = 0$

$$\therefore \boxed{i(t) = \frac{2}{399}(\cos t - \cos 20t)}$$