Algebraic multiplicity (AM) - If λ is an eigenvalue of a matrix A, then the algebraic multiplicity of λ is defined to be the multiplicity of λ as a root of the characteristic polynomial of A.

Geometric multiplicity (GM) - The geometric multiplicity of λ eigenvalue is no of linearly independent eigenvectors corresponding to λ.

In previous example, AM of $\lambda=1$ is 2 and GM of $\lambda=1$ is 1.

Ex Find all eigenvalues and the corresponding eigenvectors of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Soln:- The characteristic equation is $0 = |A-\lambda I| \Rightarrow$

$$0 = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = -\lambda(1+\lambda^2+2\lambda-1)$$

$$= -\lambda(\lambda^2+2\lambda) = -\lambda^2(\lambda+2)$$

$\therefore$ The eigenvalues are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -2$.

AM of eigenvalue 0 is 2 and eigenvalue 2 is 1.

Eigenvector for $\lambda_1 = \lambda_2 = 0$; $(A - 0I)v = 0 \Rightarrow Av = 0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} v = 0 \xRightarrow[R_3+R_1]{R_2+3R_1} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v_1 = v_3$$

$\therefore v = (v_1, v_2, v_3) = (v_1, v_2, v_1) = v_1(1,0,1) + v_2(0,1,0)$

$\therefore$ The eigenvectors are $(1,0,1)$ & $(0,1,0)$ for $\lambda_1 = \lambda_2 = 0$.

Here, GM of eigenvalue 0 is 2.

Eigenvector for $\lambda_3 = -2$; $(A+2I)v = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix} v = 0 \xrightarrow[R_2 - 3R_1]{R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} v_1 + v_3 = 0 \\ 2v_2 - 6v_3 = 0 \end{matrix}$$

$\Rightarrow v_1 = -v_3$; $v_2 = 3v_3$

$\therefore$ $v = (-1, 3, 1)$ is an eigenvector of $\lambda_3 = -2$.

(Note: the AM equals the GM for each eigenvalue.)

Ex: Find all eigenvalues and eigenvectors corresponding to them of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Soln:- The characteristic equation is $0 = |A - \lambda I|$

$$\Rightarrow 0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)^3$$

$\therefore$ The eigenvalue is $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and so AM of eigenvalue 1 is 3.

Eigenvector for $\lambda_1 = \lambda_2 = \lambda_3 = 1$; $(A - I)v = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

$\therefore$ $v = (v_1, v_2, v_3) = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)$

$\therefore$ The eigenvectors are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ corresponding to $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and GM of eigenvalue 1 is 3.