Example: How many relations are there on a set with n elements?

A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, ~~and a set~~ there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements.
So, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{1,2,3\}$

Properties of Relation

→ A relation R on a set A is called 'reflexive' if $(a, a) \in R$ for every element $a \in A$.

Ex. $A = \{1, 2, 3, 4\}$ $\quad R = \{(1,1), (2,2)\}$ - Not Reflexive

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

Here, R_1 is reflexive because it contains all pairs $(1,1), (2,2), (3,3), (4,4)$.

Ex. Is the "divides" relation on the set of positive integers reflexive?

Because $a | a$ whenever a is a positive integer, the divides relation is reflexive.

→ A relation R on a set A is called Symmetric if $(b, a) \in R$ whenever $(a, b) \in R$.

A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$ is called antisymmetric.

$R_2 = \{(1,1), (1,2), (2,1), (2,2)\}$ - Symmetric
- not antisymmetric

$R_3 = \{(1,1), (2,2), (3,3)\}$ – symmetric, antisymmetric
$R_4 = \{(1,2), (2,1), (2,3)\}$ – Neither Symmetric nor antisymmetric

Note:- The term symmetric and antisymmetric are not opposites.

Ex. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

→ A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for all $a, b, c \in A$.

e.g. $R_5 = \{(1,2), (2,3), (1,3), (1,1)\}$ transitive

$R_6 = \{(1,2), (2,3), (1,3), (3,1)\}$ not transitive

$R_7 = \{(1,2), (3,3)\}$ transitive

Ex. Is the "divides" relation on the set of positive integers transitive?

⇒ Counting of reflexive relations on a set of n elements –

All reflexive relations must contain n ordered pairs (a, a).
Rest (n^2-n) pairs, will give $2^{(n^2-n)}$ subsets (Relations).
All these relations become reflexive if we put all n ordered pairs (a, a) in. So, there are $\boxed{2^{(n^2-n)}}$ reflexive relations.

⇒ Counting of symmetric relations on a set of n elements

Transitive
$T(4) = 3994$

Antisymmetric
$2^n \cdot 3^{\frac{1}{2}(n^2-n)}$

There are n orderd pairs (a, a), these will give 2^n sym. Rel.
There are (n^2-n) " (a, b), these will give $2^{\left(\frac{n^2-n}{2}\right)}$ "
So, total Sym. Rels $= 2^n \times 2^{\frac{n^2-n}{2}} = 2^{\left(\frac{n^2+n}{2}\right)} = 2^{\Sigma(n+1)}$