Unit II: Counting Techniques

Prerequisite

- Number System
- Basic Knowledge of Mathematics

Objective

Counting Principles of Addition and Multiplication

Permutations and Combinations

Pigeonhole Principle

Inclusion and Exclusion Principle

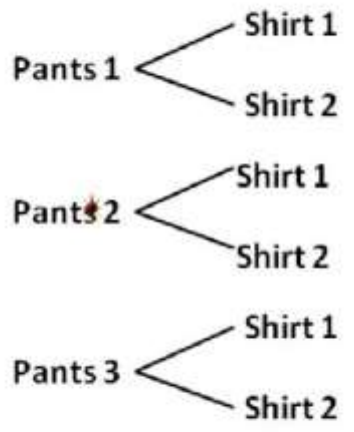
Introduction

There are some basic counting techniques which will be useful in determining the number of different ways of arranging or selecting objects and helpful to calculate the probability of an event to happen.

There are two basic counting principles as under:

1. Fundamental Principle of Counting - Multiplication principle

If we have 2 events: one event can occur in ‘m’ ways and another event can occur in ‘n’ ways, then the total number of ways that both can occur in $m \times n$ ways.



e.g., If Mickel has 3 paints and 2 shirts, then there are

$3 \times 2 = 6$ ways, by which he can be dressed up.

2. Addition Principle

If an event E can occur in ‘m’ ways and another event F can occur in ‘n’ ways, and suppose that both can not occur together, then E or F can occur in m + n ways.

e.g., If you are going to have ice cream or coffee for dessert, there are 3 choices for ice cream (Vanilla, Butter Schoch or Chocolate) and 2 choices for coffee(hot or cold), then by the principle of addition, there are 3 + 2 = 5 choices for dessert.

The number of ways this activity can occur would be $=$?

Example: How many numbers are there in between 99 and 1000 having 7 in the unit place?

Solution: First note that all these numbers have three digits. 7 is in the unit's place.

The middle digit can be any one of the 10 digits from 0 to 9.

The digit in hundred's place can be any one of the 9 digits from 1 to 9.

Therefore, by the fundamental principle of counting, there are $10 \times 9 = 90$ numbers between 99 and 1000 having 7 in the unit's place.

Factorial

Factorial is used to find how many ways we can arrange or order a set number of things and is defined as under:

Factorial of a natural number is the product of all natural numbers from 1 to that number. Let n be a natural number, then n factorial, written by n! or by $\lfloor n$,

i.e., n! = $\lfloor n$ = 1 × 2 × 3 × ... (n - 1) × n

e.g., 1! = 1 2! = 2 × 1 = 2

3! = 3 × 2 ×1 = 6 4! = 4 × 3 × 2 × 1 = 24

For integer $n \geq 1$, this may be written in Pi product notation as $\mathrm{n!} = \prod_{i=1}^{n} i$

This leas to the recurrence relation $\mathrm{n! = n.(n - i)!}$ $\mathrm{e.g.,5! = 5\,.\,4!\,,6! = 6\,.\,5!}$ and so on.

Note: The factorial of 0 is 1, or in symbols, 0! = 1

Question: Evaluate the followings:

i) $6! - 5!$ ii) $4! - 0!$ iii) $2 \times 5!$ iv) $\frac{9!}{6! \times 3!}$ v) $\frac{10!}{5!}$

Solution:

i) $6! - 5! = 6 . 5 . 4 . 3 . 2 . 1 - 5 . 4 . 3 . 2 . 1 = 720 - 120 = 600$

ii) $4! - 0! = 4 . 3 . 2 . 1 - 1 = 24 - 1 = 23$

iii) $2 \times 5! = 2 \text{ x } (5 .4 . 3 . 2 . 1) = 2 \text{ x } 120 = 240$

iv) $\frac{9!}{6! \times 3!} = \frac{9 . 8 . 7 . 6!}{6! \times 3!} = \frac{9 . 8 . 7}{3 . 2 . 1} = 84$ v) $\frac{10!}{5!} = \frac{10 . 9 . 8 . 7 . 6 . 5!}{5!} = 10 . 9 . 8 . 7 . 6 = 30240$

Permutations and Combinations

The study of permutations and combinations is concerned with determining the number of different ways of arranging and selecting objects out of a given number of objects, without actually listing them.

A **permutation** is a mathematical technique that determines the number of possible arrangements in a definite order of a number of things taking some or all at a time.

The total no. of permutations of n different objects taking all at a time, denoted by the symbol n_{P_n}, and is given by $n_{P_n} = \frac{n!}{(n-n)!} = n!$, ... (1).

The number of permutations of n objects taken r at a time, where 0 < r ≤ n, denoted by n_{P_r} , is given by $n_{P_r} = \frac{n!}{(n-r)!}$ ….(2)

Illustration

The permutations of 3 things a, b, c taking 2 at a time are:

a b	b c	c a
b a	c b	a c

i.e., $3_{P_2} = \frac{3!}{(3-2)!} = \frac{3\,.\,2\,.\,1}{1!} = 6$

Some Results: 1. $n_{P_0} = 1$

2. $n_{P_1} = n$

3. $n_{P_n} = n!$

Question

Evaluate the followings:

i) 5_{P_3} ii) 7_{P_2} iii) 6_{P_6}

Solution: i) $5_{P_3} = \frac{5!}{(5-3)!} = \frac{5\,.\,4\,.\,3\,.\,2\,.1}{2\,.\,1} = 5\,.\,4\,.\,3 = 60$

ii) $7_{P_2} = \frac{7!}{(7-2)!} = \frac{7\,.\,6\,.\,5\,!}{5\,!} = 7\,.\,6 = 42$

iii) $6_{P_6} = \frac{6!}{(6-6)!} = \frac{6\,!}{0\,!} = \frac{6\,.\,5\,.\,4\,.\,3\,.\,2\,.1}{1} = 720$

Question for practice: Find n, if $5.\, 9_{P_4} + 9_{P_5} = 10_{P_n}$

Example: Using all the letters of the word 'POWER', how many different words can be formed?

Solution: The total number of letters in the word 'POWER' = 5 and all are different.

Therefore, Using these 5 letters, the number of 5 letters words formed

$$= 5_{P_5}$$

$$= 5! \ = 120$$

Example: How many three letter words are formed using the letters of the word TIME?

Solution: The number of letters in the given word is four.

The number of three letter words that can be formed using these four letters is

$$^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$$

Example: How many words can be made with the letters of the word MATHEMATICS. And in how many of them the vowels occur together?

Solution: In the word MATHEMATICS, there are 11 letters.

M occurs 2 times

A occurs 2 times

T occurs 2 times and the rest are all different.

Therefore, required number of words $= \frac{11!}{2!\,.2!\,.\,2!} = 4989600$

Solution Continued

The vowels are A, E, A and I. we treat these 4 letters as one object.
Therefore, we have to find the arrangements of 7 consonants and one object (AEAI)
i.e., there are 8 objects, where

M occurs 2 times

T occurs 2 times and the rest are all different.

Therefore, number of such words are $= \frac{8\,!}{2!\,.2!} = 10080.$

But the four vowels in single object (AEAI) can also be arranged in $= \frac{4\,!}{2!} = \frac{24}{2} = 12\ ways$

Hence, number of words when all vowels occur together = 10080 x 12 = 120960

Combinations

Here, we shall discuss about the no. of ways of selections of certain things taking all or some of them at a time. The selections are different from arrangements(permutations).

In permutation, the order of things are taken into consideration whereas in case of selections(combinations), the order of things is immaterial.

e.g., **a b** and **b a** are two distinct permutations but same selection.

Thus. a **combination**, is a selection of a number of things taking some or all at a time.

The total number of combinations of n distinct objects taking r at a time, where 0 ≤ r ≤ n, is denoted by n_{C_r} , is given by $n_{C_r} = \frac{n!}{(n-r)!\,.r\,!}$

Illustration: The combinations of 3 things a, b, c taking 2 at a time are:

a b b c c a i.e., $3_{C_2} = \frac{3!}{(3-2)!\,.\,2\,!} = \frac{3\,.\,2\,.\,1}{1\,.2\,.1} = 3$

Some Results:

1. $n_{C_0} = 1 = n_{C_n}$

2. $n_{C_1} = n = n_{C_{n-1}}$

3. $n_{C_2} = \frac{n\,(n-1)}{2}$

4. If $n_{C_\mathrm{a}} = \mathrm{n}_{C_\mathrm{b}}$ $then\ either\ a = b\ \ or\ a + b = n$

5. $n_{P_r} = n_{C_r}\,.\,r\,!$

6. $n_{C_r} + n_{C_{r+1}} = n + 1_{C_{r+1}}$

Example: If there are 10 persons in a party and each two of them shakes hands with each other, how many handshakes happen in the party?

Solution: In the process of hand shaking, participation of two persons at a time is required

and it can be done in $10_{C_2} ways$

$$= \frac{10\,(10-1)}{2} = \frac{10\ \times 9}{2} = 45$$

Question: Evaluate i) 9_{C_4} ii) $51_{C_{49}}$

Example: Find n if

(i) $25_{C_{n+5}} = 25_{C_{2n-1}}$ (ii) $n_{C_{n-4}} = 5$ (iii) $2n_{C_3} : n_{C_2} = 12:1$

Solution: (i) As $25_{C_{n+5}} = 25_{C_{2n-1}}$

$\Rightarrow Either\ n + 5 = 2n - 1\ or\ (n+5) + (2n-1) = 25 \quad \Rightarrow\ n = 6, 7$

(ii) As $n_{C_{n-4}} = 5 \quad \Rightarrow \frac{n!}{(n-4)!\,4!} = 5 \quad \Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!} = 5$

$\Rightarrow n(n-1)(n-2)(n-3) = 5\,.\,4\,.\,3\,.\,2 \quad \Rightarrow\ n = 5$ (product of 4 consecutive numbers on both sides, so equate the highest number on each side)

(iii) Try yourself Ans. n = 5

Example: 16 players of football go to England for football match. In how many ways can the team of 11 be selected?

Solution: Number of ways of selecting 11 players out of 16 players

$$= 16_{c_{11}}$$

$$= \frac{16\,!}{(16-11)\,!\,.\,11\,!}$$

$$= \frac{16\,!}{(5)\,!\,.\,11\,!}$$

$$= \frac{16\,.\,15\,.\,14\,.\,13\,.\,12\,.\,11!}{5\,.\,4\,.\,3\,.\,2\,.\,1\,.\,11\,!}$$

$$= 16\,.\;3\,.\;7\,.\;13 = 4368\; ways$$ Ans.

Example: In a class, there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make this selection?

Solution: Here the teacher is to perform two operations:

(i) Selecting a boy from among the 27 boys, it can be done in $27_{C_1}\ ways$

$$= \frac{27!}{(27-1)!\,.\,1!} = \frac{27!}{26!} = 27 \text{ ways}$$

and (ii) Selecting a girl from among 14 girls, it can be performed in $14_{C_1}\ ways$

$$= \frac{14!}{(14-1)!\,.\,1\,!} = \frac{14!}{13!} = 14 \text{ ways}$$

Now, by the fundamental principle of counting,

the required number of ways by which 1 boy and 1 girl is elected = 27 × 14 = 378 ways

Example: In how many ways can a student, choose a program of 5 courses out of 9 courses, if 2 specific courses are compulsory for every student?

Solution: 5 courses to be selected from 9 but 2 courses are mandatory to be taken.

So, a student can select 5 -2 = 3 courses out of 9 – 2 = 7

which can be done in 7_{C_3} ways

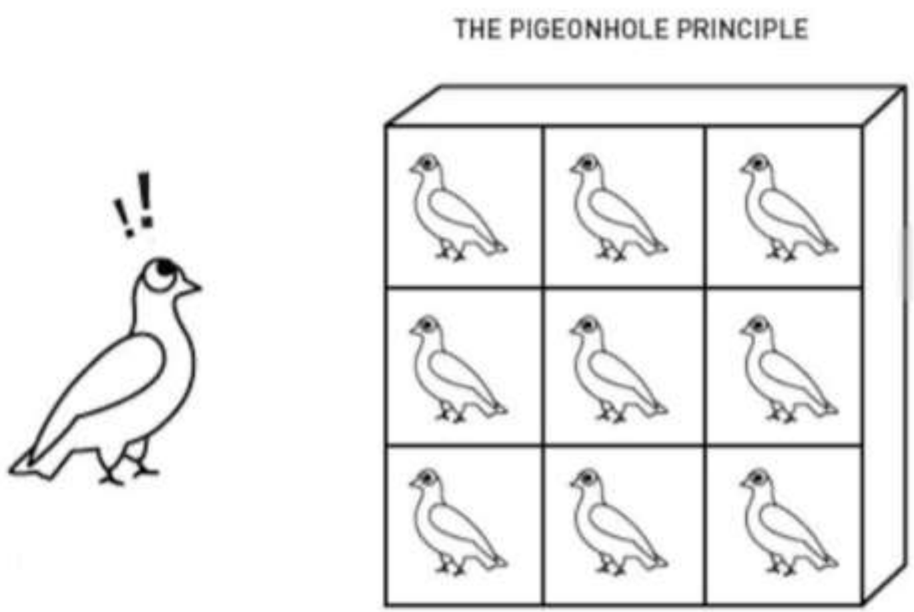
$$= \frac{7\,!}{(7-3)\,!\,.\,3\,!} = \frac{7.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.1}{4\,.\,3\,.\,2\,.1\,.\,3\,.\,2\,.1}$$

$$= 35 \text{ ways}$$

Pigeonhole Principle

The word ‘pigeonhole’ refers to the shelves in the form of square boxes or holes that were utilized to place pigeons. Here is simply a concept, inspired by such pigeonholes, known as pigeonhole principle which was introduced in 1834 by a German Mathematician Peter Gustav L. Dirichlet.

On his name, this principle is also termed as Dirichlet’s box principle states that if there are few boxes available; also, there are few objects that are greater than the total number of boxes as well as one needs to place objects in the given boxes, then at least one box must contain more than one such objects.



Pigeonhole Principle

Statement: If n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon.

Example: If there are 7 pairs of different colours of socks in a drawer. How many socks to be drawn out in order to guarantee that it is grabbed at least one pair?

Solution: After grabbing 7 socks, worst case scenario, it is grabbed a sock of each colour.

Thus, after grabbing one more sock, it has to match up with one of the previous socks.

Hence after grabbing 8 socks, it is guaranteed to have a pair of same colour.

Example: A bag contains 25 balls such as 10 balls are red, 7 are white and 8 are blue. What is the number of balls that must be picked up from the bag blindfolded without replacing any of it, to be assured of picking at least one ball of each colour?

Solution: Consider three buckets red, white and blue and it is required that each bucket contain at least one ball. Now consider the state of picking up a ball, (in the worst scenario) starting 10 balls all are red and thus goes to bucket named red.

Now again picking up the balls gives 8 ball which are of same colour blue and put them in a bucket named blue. The next pick will definitely be of different colour thus: we picked

$10 + 8 + 1 = 19$

Example: Find the minimum number of elements that one needs to take from the set S = {1, 2, 3, . . . , 9} to be sure that two of the numbers add up to 10.

Answer: The minimum number of integers to select from 1 thru 9 to ensure that one or more pairs of the selected integers add to 10, **is six**.

Explanation - There are four possible pairs of integers which sum to 10:

1 and 9,
2 and 8,
3 and 7,
4 and 6.

If it happen to select one integer from each of these pairs and also select 5, it is to have selected 5 another integer, but we won't have any pairs that sum to 10.

However, our next selection, **the sixth integer**, guarantees that we will complete one of the pairs that sum to 10.

Example: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Explanation: As there are 12 months in a year.

Let us consider, initially 2 students born in each month, so there are 12 * 2 = 24 students

Now 25th student will make the month in which he is born the month having 3 students born in same month.

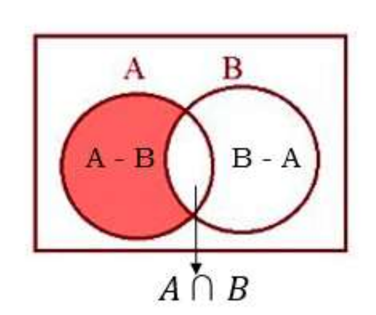
Thus, 25 students in a class make sure that three of them are born in the same month.

Ans.

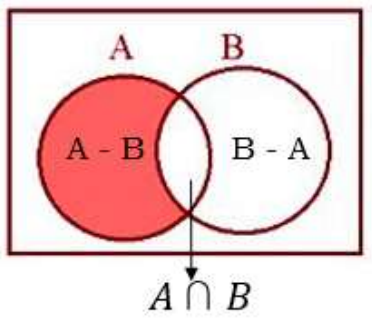
Principle of Inclusion and Exclusion (PIE)

Principle of inclusion and exclusion, is a counting technique that computes the number of elements that satisfy at least one of several properties and the elements satisfying more than one property are not counted twice. i.e., the principle of inclusion and exclusion is summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevent double counting.

Mathematically, this principle provides a formula to find the number of elements in the union of all given sets, the size of each set, and the size of all possible intersections among the sets.



Consider two finite sets A and B, the principle of inclusion and exclusion formula is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



Where, $n(A)$ is no. of elements in A

or also called cardinality of set A,

$n(B)$ is no. of elements in B

or also called cardinality of set B,

$n(A \cap B)$ is the cardinality of $A \cap B$

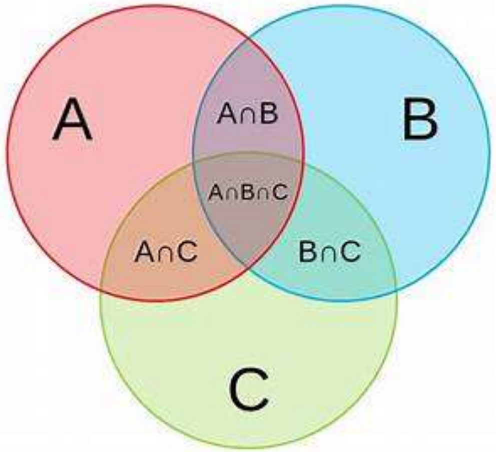
Here A & B are included, but excluded their common elements.

Similarly for 3 sets A, B , C, the principle of inclusion and exclusion is

$n\ (A \cup B \cup C)$

$= n\ (A) + n\ (B) + n\ (C)\ - n\ (A \cap B) - n\ (A \cap C) - n\ (B \cap C)$

$\quad + n\ (A \cap B \cap C)$



Example: Among a group of students, 49 study Physics, 37 study English and 21 study Biology. If 9 of these students study Physics & English, 5 study English & Biology, 4 study Physics & Biology and 3 study Physics, English & Biology. Find the number of students in the group.

Solution:

Let $n(P)$ = No. of students who study Physics, $n(E)$ = No. of students who study English,

$n(B)$ = No. of students who study Biology, $n(P \cap E)$ = No. of studts who study Phy & Eng,

$n(P \cap B)$ = No. of studts who study Phy & Bio, $n(B \cap E)$ = No. of studts who study Bio & Eng,

$n(P \cap E \cap B)$ = No. of students who study Physics, English and Biology,

$n(P \cup E \cup B)$ = Number of students in the group

Therefore, by the principle of inclusion and exclusion,

$$n(P \cup E \cup B) = n(P) + n(E) + n(B) - n(P \cap E) - n(P \cap B) - n(B \cap E) + n(P \cap E \cap B)$$

$$= 49 + 37 + 21 - 9 - 4 - 5 + 3 = 92 \quad \text{Ans.}$$

Example: How many integers from 1 to 100 are multiple of 2 or 3.

Solution: Let A be the set of integers from 1 to 100 that are multiple of 2,

therefore, $n(A) = 50$

and B, set of integers from 1 to 100 that are multiple of 3, so $n(B) = 33$

$A \cap B$, set of integers from 1 to 100 that are multiple of 2 & 3 both, i.e., multiple of 6

so $n(A \cap B) = 16$

Therefore, by the principle of inclusion and exclusion,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 50 + 33 - 16 = 67$ Ans.

Practice Questions

Q 1. A customer forgets a four-digit code for an Automatic Teller Machine(ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.

Q 2. A hall has three entrances and four exits. In how many ways can a man enter and exit from the hall?

Q 3. How many permutations can be made out of the letters of the word 'TRIANGLE'? How many begin with T and end with E?

Q 4. How many triangles can be formed by joining 10 non-collinear points.

Q 5. Find the number of diagonals that can be drawn by joining the angular points of octagon.

Q 6. A polygon has 44 diagonals. Find the number of its sides

Practice Questions

Q 7. In a class of 50 students, each playing at least one game, there are 38 students who play football and 25 play badminton, find

i) How many students play both games ?
ii) How many students play Football only ?
iii) How many students play badminton only ?

Q 8. There are 35 students in art class and 57 students in dance class.

Find the number of students who are either in art class or in dance class, When

i) two classes meet at different hours and 12 students are enrolled in both activities.
ii) two classes meet at the same hour.

Q 9. If $n(A - B) = 18$, $n(A \cup B) = 70$ and $n(A \cap B) = 25$, then find $n(B)$.

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Thanks