System of Linear DE with Constant Coefficients:

Till now, we had DE of one dependent and one independent variable.
Now, we will take two or more dependent variables and one independent variable, $x = a\cos t$
$y = a\sin t$.

If there is one DE with 2 D.V. and 1 I.V. then that can't be solved. So we take two DEs of Order One as

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y.$$

So, this is called system of Linear DE of Order One.
This system can be solved by various methods but we will take simple Elimination Method. And for this we will DEs in operator form.

Ex.
$$(D-2)y_1 + (2D-1)y_2 = e^{2t} \quad \text{———} \quad ①$$
$$y_1 + (D-2)y_2 = 0 \quad \text{———} \quad ②$$

This system is already in operator form.
Now, operate $(D-2)$ on ② and subtract from ①;

$$(D-2)y_1 + (2D-1)y_2 = e^{2t}$$
$$\underline{\underset{-}{}(D-2)y_1 \underset{-}{\pm} (D-2)^2 y_2 = 0}$$

$$(2D-1)y_2 - (D-2)^2 y_2 = e^{2t}$$

$$\Rightarrow (2D-1-D^2-4+4D)\,y_2 = e^{2t}$$

$$\Rightarrow (-D^2+6D-5)y_2 = e^{2t} \quad \therefore \boxed{(D^2-6D+5)y_2 = -e^{2t}}$$

Now, we have 2nd Order linear Non-homogeneous DE with constant coefficients. So

$$(D^2-6D+5)\,y_2 = -e^{2t}$$

A.E. is $m^2-6m+5=0$

$\Rightarrow m = 1, 5$

$\therefore$ C.F. $= Ae^{t} + Be^{5t}$

Now, P.I. $= \dfrac{1}{D^2-6D+5}(-e^{2t})$

$$= \frac{-1}{2^2-6\cdot 2+5}\,e^{2t}$$

$$= \frac{-1}{4-12+5}\,e^{2t} = \frac{-1e^{2t}}{-3} = \frac{1}{3}e^{2t}.$$

$\therefore\ y_2(t) = Ae^{t} + Be^{5t} + \frac{1}{3}e^{2t}$

Put $y_2(t)$ in (2),

$$y_1 + (D-2)\left(Ae^{t}+Be^{5t}+\tfrac{1}{3}e^{2t}\right) = 0$$

$$\Rightarrow y_1 = -D\left(Ae^{t}+Be^{5t}+\tfrac{1}{3}e^{2t}\right) + 2\left(Ae^{t}+Be^{5t}+\tfrac{1}{3}e^{2t}\right)$$

$$= -Ae^{t} - 5Be^{5t} - \tfrac{2}{3}e^{2t} + 2Ae^{t} + 2Be^{5t} + \tfrac{2}{3}e^{2t}$$

$$= Ae^{t} - 3Be^{5t} + 0$$

Hence,
$$\boxed{\begin{aligned} y_1(t) &= Ae^{t} - 3Be^{5t} \\ y_2(t) &= Ae^{t} + Be^{5t} + \tfrac{1}{3}e^{2t} \end{aligned}}$$

Ex. Find the solution of the system of equations

$$(3D+1)y_1 + 3Dy_2 = 3t+1 \quad \text{—— (1)}$$

$$(D-3)y_1 + Dy_2 = 2t \quad \text{—— (2)}$$

Multiply (3) in (2), subtract from (1),

$$(3D+1)y_1 - 3(D-3)y_1 = 3t+1-6t$$

$$\Rightarrow y_1 + 9y_1 = 1-3t$$

$$\Rightarrow 10y_1 = 1-3t$$

$$\therefore \boxed{y_1 = \frac{1}{10}(1-3t)}$$

Now put y_1 in (2),

$$(D-3)\left\{\frac{1}{10}(1-3t)\right\} + Dy_2 = 2t$$

$$\Rightarrow \frac{1}{10}(0-3) - \frac{3}{10}(1-3t) + Dy_2 = 2t$$

$$\Rightarrow -\frac{3}{10} - \frac{3}{10} + \frac{9}{10}t + Dy_2 = 2t$$

$$\Rightarrow Dy_2 = 2t + \frac{6}{10} - \frac{9}{10}t = \frac{11}{10}t + \frac{6}{10}$$

$$\therefore \boxed{y_2 = \frac{11}{20}t^2 + \frac{6}{10}t + A.}$$