Diagonalization Algorithm:

1. Find all eigenvalues of the matrix A ($n\times n$ order)
2. Find all eigenvectors corresponding all eigenvalues
3. If all L.I. eigenvectors are n then matrix is diagonalizable otherwise not diagonalizable.
4. If diagonalizable then let P be the matrix whose columns are the eigenvectors. Then
$$P^{-1}AP = D \text{ (diagonal matrix)}$$

Ex. Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D s.t. $P^{-1}AP = D$. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

Soln:- First find eigenvalues of A as

$$|A-\lambda I| = 0 \Rightarrow \lambda^2 - 10\lambda + 21 = 0 \Rightarrow (\lambda-7)(\lambda-3) = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 3$$

Since, both eigenvalues are distinct so matrix A is diagonalizable, So to find matrix P, find eigenvectors:

Eigenvectors for $\lambda_1 = 7$,

$$(A-7I)v = 0 \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$$

$\therefore$ $(1,1)$ is an eigenvector for $\lambda_1 = 7$.

Eigenvector for $\lambda_2 = 3$:

$$(A-3I)v = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2$$

$\therefore$ $(1,-1)$ is an eigenvector for $\lambda_2 = 3$.

$\therefore$ The matrix P is $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So, $$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 7 & 7 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} = D.$$

Ex. Determine whether $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is diagonalizable and if so find an invertible matrix P and a diagonal matrix D s.t. $P^{-1}AP = D$.

Soln:- As discussed in previous example. This matrix has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$. All are distinct. So A is diagonalizables. And its eigenvectors are $(1,-1,0)$, $(1,1,-2)$, $(1,1,1)$.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$\therefore$ $AP = PD \Rightarrow P^{-1}AP = D.$