L18 Variation of parameter method

Consider a linear Non-homogeneous Second order DE with constant coefficients –

$$y'' + ay' + by = r(x) \quad \text{—} \; ①$$

And suppose its complementary function is

$$y_h = c_1 y_1 + c_2 y_2 \quad \text{—} \; ②$$

where y_1 and y_2 are L.I. Solutions of corresponding homogeneous DE ①.

In method of Variation of parameters, the arbitrary constants c_1 and c_2 in ② are replaced by two unknown functions $u_1(x)$ and $u_2(x)$.

So, the particular integral y_p of ① is

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where $u_1(x) = -\int \frac{r(x)\, y_2(x)}{W} dx$

$$u_2(x) = \int \frac{r(x)\, y_1(x)}{W} dx$$

Here, W = Wronskian of y_1, y_2

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0$$

Hence, the G.S. of ① is $y = y_h + y_p$.

Ex. Solve the nonhomogeneous ODE

$$y'' + y = \sec x$$

The A.E. of corresponding homogeneous is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complimentary function is

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

So, $y_1 = \cos x$, $y_2 = \sin x$

$$\therefore W = y_1 y_2' - y_1' y_2$$
$$= \cos x \cdot \cos x + \sin x \cdot \sin x$$
$$= 1$$

The P.I. is $y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$

where $u_1(x) = -\int \frac{r(x) y_2(x)}{W} dx = -\int \frac{\sec x \cdot \sin x \, dx}{1}$

$$= -\int \frac{\sin x}{\cos x} dx = -\ln|\cos x|$$

$$u_2(x) = \int \frac{r(x) y_1(x)}{W} dx = \int \frac{\sec x \cdot \cos x \, dx}{1}$$

$$= \int dx = x.$$

$$\therefore y_p = -\cos x \ln|\cos x| + x \sin x$$

Hence, the G.S. of given DE is

$$y = c_1 \cos x + c_2 \sin x - \cos \ln|\cos x| + x \sin x.$$