# Linear Independence and Dependence

Two functions $y_1(x)$ and $y_2(x)$ are called linearly independent on an interval I where they are defined if Ⓐ $-$ $k_1 y_1(x) + k_2 y_2(x) = 0$ everywhere on I

implies $K_1 = 0$ and $k_2 = 0$.

And $y_1$ and $y_2$ are called linearly dependent on I if Ⓐ holds for some constants $k_1$, $k_2$ not both zero.
Then, if $k_1 \neq 0$ and $k_2 \neq 0$ then $y_1$ and $y_2$ are propotional

$$y_1 = -\frac{k_2}{k_2} y_2 \quad \text{or} \quad y_2 = -\frac{k_1}{k_2} y_1$$

## To check L.I or not

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

i) $W \neq 0$ then $y_1$ & $y_2$ are linearly Independent
ii) $W = 0$ on whole Interval I, then $y_1$ & $y_2$ are L.D.

e.g. 1) $y_1 = \cos ax$, $y_2 = \sin ax$ $a \neq 0$ on $\mathbb{R}$ L.I.

2) $y_1 = \ln x$, $y_2 = \ln x^n$ n is non-negative integer L.D.

Ex. $y_1 = e^{ax}$, $y_2 = e^{bx}$

$$W = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = be^{ax}e^{bx} - ae^{ax}e^{bx} = e^{ax+bx}(b-a)$$

It is known that $e^{ax+bx} \neq 0$ so when $a = b$ then $W = 0$.
$\therefore$ $y_1$ & $y_2$ are L.I if $a \neq b$.

## Homogeneous linear DE with constant coefficients

It is known that the first order linear DE with constant coefficients can be written as

$$\frac{dy}{dx} + ay = b \quad \text{—} \quad (1)$$

When $b=0$, then (1) called Homogeneous linear DE with constant coefficients otherwise Non-Homogeneous.

Linear DE means the dependent variable $y$ and its derivatives are present at most once in each term. i.e.,

(1) $\frac{dy}{dx} + ay^2 = b$ is not linear because second term $(ay^2)$ consists two $y$.

(2) $yy' + ay = b$ is not linear because first term $yy'$ consists two $y$ and $y'$.

Therefore, $n$th order Homogeneous linear DE with constant coefficients is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{(n-1)}y}{dx^{n-1}} + \dots + a_1\frac{dy}{dx} + a_0 y = 0$$

It can be written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

or

$$(D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0)y = 0 \quad \text{where } \boxed{\frac{d^n}{dx^n} = D^n}$$

The above DE is in standard form because coefficients of haighest order is 1.

## Solution of 2nd Order Homo Linear DE with constant coefficients —

Standard form

$$y'' + ay' + by = 0 \quad \text{—— (1)}$$

Suppose the solution is $y = e^{mx}$ —— (2)

Its derivatives are $y' = me^{mx}$ —— (3)

$$y'' = m^2 e^{mx} \quad \text{—— (4)}$$

Put (2), (3) and (4) in (1),

$$m^2 e^{mx} + ame^{mx} + be^{mx} = 0$$

$$\Rightarrow e^{mx}(m^2 + am + b) = 0$$

Since $e^{mx} \neq 0$,

$\therefore m^2 + am + b = 0$ (auxiliary equation)

Thus, if $m$ is a solution of above auxiliary equation then the exponential function (2) is a solution of the ODE (1).

Since auxiliary equation is a quadratic equation, so it has three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$.