Basic Properties of an inner product:

(a) For each fixed $u \in V$, the inner product that takes v to $\langle v, u\rangle$ is a linear map from V to $\mathbb{F}$ (real or complex)

(b) $\langle 0, u\rangle = 0$ for every $u \in V$

(c) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$

(d) $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbb{F}$ and $u, v \in V$.

Norm (length): For $v \in V$, the norm of v, denoted by $\|v\|$, is defined by $\|v\| = \sqrt{\langle v, v\rangle}$.

Ex. If $v = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ (with the usual inner product) then

$$\|v\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\langle v, v\rangle}$$

Ex. If $f(x) \in C[0,1]$ then

$$\|f(x)\| = \sqrt{\int_0^1 (f(x))^2 dx} = \sqrt{\langle f, f\rangle}$$

Orthogonal Sets: A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ is called an orthogonal set if all pair of distinct vectors in the set are orthogonal, i.e,

$$\langle v_i, v_j\rangle = 0 \text{ whenever } i \neq j \text{ for } i,j = 1, 2, \ldots, k$$

Geometrically, the vectors in set are mutually perpendicular.

1. 0 is orthogonal to every vector in V
2. 0 is the only vector in V that is orthogonal to itself.

Ex. Show that $\{v_1, v_2, v_3\}$ is an orthogonal set in $\mathbb{R}^3$ if

$v_1 = (2, 1, -1)$, $v_2 = (0, 1, 1)$, $v_3 = (1, -1, 1)$

The inner product in $\mathbb{R}^n$ is usual dot product. so

$v_1 \cdot v_2 = 2\cdot 0 + 1\cdot 1 + (-1\cdot 1) = 0$, $v_2 \cdot v_3 = (0\cdot 1 + 1\cdot(-1)) + 1\cdot 1 = 0$, $v_3 \cdot v_1 = 2\cdot 1 + 1\cdot(-1) + (1\cdot 1)$ $= 0$

<u>Orthogonal basis</u>: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthogonal set.

$\{(1,0),(0,1)\}$ is an orthogonal basis of $\mathbb{R}^2$.

<u>Ex.</u> Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}.$$

$$x - y + 2z = 0 \Rightarrow x = y - 2z$$

$$\therefore W = \left\{ \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since $u = (1,1,0)$ & $v = (-2,0,1)$ are L.I. so they form a basis for W. But they are not orthogonal as $u \cdot v = -2 \neq 0$.

It suffices to find another nonzero vector in W that is orthogonal to either one of u and v.

Suppose $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a vector in W that is orthogonal to u.

Then $x - y + 2z = 0$ since w is in W. Since $u \cdot w = 0$, so

$$x + y = 0$$

Solving, these two equations, we find $x = -z$, $y = z$.

Thus, any nonzero vector for w of the form $w = \begin{bmatrix} -z \\ z \\ z \end{bmatrix}$ will $\perp^r$ to u. To be specific, we could take $w = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

$\therefore$ $\{u, w\}$ is an orthogonal set in W and hence an orthogonal basis for W, since $\dim W = 2$.

The advantage of working with an orthogonal basis is that the coordinates of a vector w.r.t. such a basis are easy to compute.

Orthonormal set and basis: A set of vectors in $\mathbb{R}^n$ is an orthonormal set if it is an orthogonal set of unit vectors.
An orthonormal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

Ex. Show that $S=\{q_1, q_2\}$ is an orthonormal set in $\mathbb{R}^3$ if

$$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Check that $q_1 \cdot q_2 = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$

$$\|q_1\| = q_1 \cdot q_1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\|q_2\| = q_2 \cdot q_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1.$$

→ If we have an orthogonal set, we can easily obtain an orthonormal set from it: simply normalize each vector i.e. divide each vector with its norm (length).

Orthogonal Matrix: An $n \times n$ matrix Q whose columns form an orthonormal set is called an orthogonal matrix.

Q. Determine the matrix $A = \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

If it is, find its inverse.

To check $AA^T = I$

AA^T

$$= \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\sin^2\theta + \cos^2\theta + \sin^4\theta & \cos^3\theta\sin\theta - \sin\theta\cos\theta + \cos\theta\sin^3\theta & \cos\theta\sin^2\theta - \cos\theta\sin^2\theta \\ \cos^3\theta\sin\theta - \sin\theta\cos\theta + \cos\theta\sin^3\theta & \cos^4\theta + \sin^2\theta + \cos^2\theta\sin^2\theta & \cos^2\theta\sin\theta - \cos^2\theta\sin\theta \\ \cos\theta\sin^2\theta - \cos\theta\sin^2\theta & \cos^2\theta\sin\theta - \cos^2\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$= I \quad \therefore A$ is orthogonal matrix $\therefore A^{-1} = A^T$.