L5

Equivalence Relation : A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive

Two elements a and b that are related by an equivalence relation are called equivalent. $(a \sim b)$

Ex. Let R be the relation on the set of real numbers such that aRb iff $a-b$ is an integer.

R is reflexive because $a-a=0$ is an integer $\forall a$, aRa $\forall a \in \mathbb{R}$.

Now suppose that aRb. Then $a-b$ is an integer, so $b-a$ is also an integer. Hence bRa. It follows that R is symmetric.

If aRb and bRc then $a-b$ and $b-c$ are integers. Sum of two integers is again an integer i.e. $a-b+b-c$ $= a-c$ is an integer. So, aRc. It follows that R is transitive.

Consequently, R is equivalence relation.

Ex. Let m be an integer with $m>1$. Show that the relation $R = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers.

{Congruence modulo m}

Recall that $a \equiv b \pmod{m} \Leftrightarrow m \mid a-b$.

For every integer a, $a \equiv a \pmod{m}$ holds because $a-a=0$ is divided by m. $\Rightarrow$ Reflexive

Suppose $aRb \Rightarrow a \equiv b \pmod{m} \Rightarrow m \mid a-b$

$\Rightarrow ml = a-b \Rightarrow ml = -(b-a) \Rightarrow m(-l) = (b-a)$

$\Rightarrow m \mid b-a \Rightarrow b \equiv a \pmod{m} \Rightarrow bRa$

$\Rightarrow$ R is symmetric.

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a.
The equivalence class of a w.r.t. R is denoted by $[a]_R$ or $[a]$. $[a]_R = \{x \mid (a, x) \in R\}$
If $b \in [a]_R$ then b is called a representative of this equivalence class.

Ex. What are the equivalence classes of 0 and 1 for congruence modulo 4?
The equivalence class $[0] = \{x \mid (0, x) \in R\}$

$$= \{x \mid 0 \equiv x \pmod 4\}$$

$$= \{0, \pm 4, \pm 8, \pm 12, \ldots\}$$

$$[1] = \{x \mid 1 \equiv x \pmod 4\}$$

$$= \{1, 5, 9, \ldots, -3, -7, \ldots\}$$

Thm: The equivalence classes of two elements of A are either identical or disjoint. — (1)

$[a] = [b] \Leftrightarrow aRb$

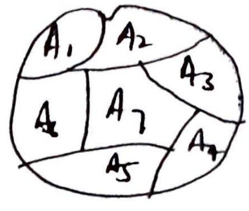
$[a] \cap [b] \Leftrightarrow a \not R b$

And $\bigcup_{a \in A} [a] = A$ i.e. the union of all equivalence classes of R is the set A. — (2)

These two observation show that the equivalence classes form a partition of A, because they split A into disjoint subsets.

The collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of A iff $A_i \neq \phi$ for $i \in I$, $A_i \cap A_j = \phi$ when $i \neq j$ and $\bigcup_{i \in I} A_i = A$.

Ex. Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of set $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$ and $A_3 = \{6\}$ forms a partition of S.



Ex. What are the sets in the partition of the integers arising from congruence modulo 4?

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$