Similar matrices :

Let A and B be $n\times n$ matrices. A is similar to B if there is an invertible $n\times n$ matrix P such that $P^{-1}AP = B$.

$[AP = PB]$.

Ex. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then A is similar to B ($A \sim B$), since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus, $AP = PB$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ [But how to get this P]

Thm :- Let A and B be $n\times n$ matrices with $A \sim B$. Then

(i). $\det A = \det B$ & $tr(A) = tr(B)$

(ii) A is invertible iff B is invertible

(iii) A and B have the same rank

(iv) A and B have same characteristic polynomial

(v) A and B have the same eigenvalues

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are not similar, since $\det A = -3$ but $\det(B) = 3$.

Ex. $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. They have same det and rank. But characteristic poly are $\lambda^2 - 3\lambda - 4$ and $\lambda^2 - 4$ which are diffrent. So A is not similar B.

Diagonalization:

An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D, that is, if there exists an invertible $n \times n$ matrix P such that

$$P^{-1}AP = D.$$

Necessary and Sufficient condition for a matrix to be diagonalizable

An n-square matrix A is diagonalizable iff A has n linearly independent eigenvectors.

This means, there exist an invertible matrix P and a digonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Thm:- If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Thm:- If A is an $n \times n$ matrix then A is diagonalizable iff the algebraic multiplicity of each eigenvalues equals to its geometric multiplicity.

The Big use of diagonal factorization:

$$D = P^{-1}AP \Leftrightarrow A = PDP^{-1}$$

$$A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = PD^3P^{-1} \text{ and so } A^m = PD^mP^{-1}.$$