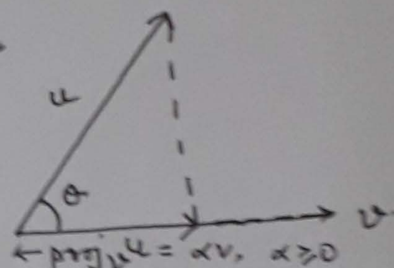
L-31.32

Orthogonal Projection :-

Let u and v be two vectors in an inner product space V. If $v \neq 0$ then the orthogonal projection of u onto v is given by

$$\text{proj}_v u = \frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

Similar as For dot product :→ The orthogonal projection for dot product in $\mathbb{R}^n$ is shown in adj figure.



$\text{proj}_v(u) = \alpha v$

Consider $\alpha \geq 0$, $\|\alpha v\| = \alpha \|v\|$

$$= \|u\| \cos\theta$$

$$= \frac{\|u\| \|v\| \cos\theta}{\|v\|} = \frac{\|u\|\|v\|}{\|v\|} \cdot \frac{u \cdot v}{\|u\|\|v\|}$$

$$= \frac{u \cdot v}{\|v\|}$$

$$\Rightarrow \alpha = \frac{u \cdot v}{\|v\|^2} = \frac{u \cdot v}{v \cdot v} \Rightarrow \boxed{\text{projection of } u \text{ on } v = \frac{u \cdot v}{v \cdot v} v}$$

Ex: Finding an orthogonal projection in $\mathbb{R}^3$.

use Euclidean inner product in $\mathbb{R}^3$ to find the orthogonal projection of $u = (6, 2, 4)$ onto $v = (1, 2, 0)$

Solⁿ: $\langle u, v\rangle = (6)(1) + (2)(2) + 4(0) = 10$

$\langle v, v\rangle = 1^2 + 2^2 + 0^2 = 5$

$$\text{proj}_v u = \frac{\langle u, v\rangle}{\langle v, v\rangle} v = \frac{u \cdot v}{v \cdot v} v = \frac{10}{5}(1, 2, 0) = (2, 4, 0)$$

The Gram - Schmidt Process :

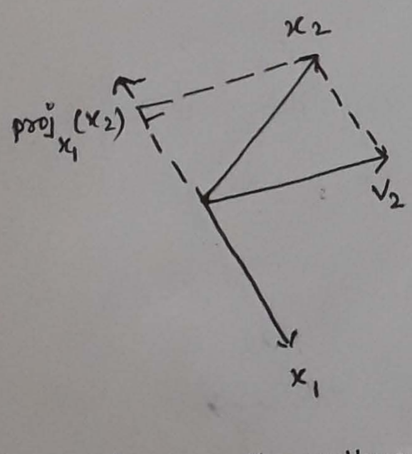
It is a simple method for constructing an orthogonal (or orthonormal) basis for any subspace of $\mathbb{R}^n$. The idea is to begin with an arbitrary basis $\{x_1, x_2, \dots, x_k\}$ for W and to 'orthogonalize' it one vector at a time. We will explain the process with the help of following example,

Ex: Let $W = \text{span}(x_1, x_2)$ then construct an orthogonal basis for W.

where $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Same question we did in ~~last~~ previous Lect.

Sol^n: starting with x_1, we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1



Constructing v_2 orthogonal to x_1.

Algebraically, we set $v_1 = x_1$ so

$$v_2 = \text{perp}_{x_1}(x_2) = x_2 - \text{proj}_{x_1}(x_2)$$

$$= x_2 - \left(\frac{x_1 \cdot x_2}{x_1 \cdot x_1}\right) x_1$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Thus $\{v_1, v_2\}$ is an orthogonal set of vectors in W. Hence $\{v_1, v_2\}$ is L.I and therefore a basis for W, since $\dim W = 2$.

Remark: observe that this method depends on the <u>order</u> of the original basis vectors. In Ex above, if we had taken $x_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then we would have obtained a different orthogonal basis for W.

The process to iteratively construct the component of subsequent vectors ortho to all vectors already constructed, is known as Gram - Schmidt ~~method~~ process.

Steps of Gram Schmidt Process:

Let $\{x_1, x_2, \ldots, x_K\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following

$$v_1 = x_1 \qquad W_1 = \operatorname{span}(x_1)$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 \qquad W_2 = \operatorname{span}(x_1, x_2)$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2 \qquad W_3 = \operatorname{span}(x_1, x_2, x_3)$$

$$\vdots$$

$$v_K = x_K - \left(\frac{v_1 \cdot x_K}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_K}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{v_{K-1} \cdot x_K}{v_{K-1} \cdot v_{K-1}}\right) v_{K-1}$$

$$W_K = \operatorname{span}(x_1, x_2, \ldots, x_K)$$

Then for each $i = 1, \ldots, K$, $\{v_1, v_2, \ldots, v_i\}$ is an orthogonal basis for W_i. In particular $\{v_1, v_2, \ldots, v_K\}$ is an orthogonal basis for W and $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|}\right\}$ is an orthonormal basis.

Ex: Apply the Gram-Schmidt process to construct an orthonormal basis for the subspace $W = \operatorname{span}(x_1, x_2, x_3)$ of $\mathbb{R}^4$ where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Sol^n: First note that $\{x_1, x_2, x_3\}$ is a linearly independent set, so it forms a basis for W. We begin by setting $v_1 = x_1$. Next, we compute the component of x_2 orthogonal to $W_1 = \operatorname{span}(v_1)$.

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Tips
* For hand calculation, it is good to scale v_2 to eliminate fractions. When we finish the process, we can rescale the orthogonal set ~~to ortho~~ we are constructing to obtain an orthonormal set.

Let $v_2' = 2v_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

Now, find x_3 orthogonal to $W_2 = \text{span}(x_1, x_2) = \text{span}(v_1, v_2)$
$= \text{span}(v_1, v_2')$

using orthogonal basis $\{v_1, v_2'\}$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right) v_2'$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{1}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{15}{20}\right) \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Again, we rescale $v_3' = 2v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

we now have an orthogonal basis $\{v_1, v_2', v_3'\}$ for W {check to make sure that these vectors are orthogonal}.

To obtain an orthonormal basis, we normalize each vector

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$q_2 = \left(\frac{1}{\|v_2'\|}\right) v_2' = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}$$

$$q_3 = \left(\frac{1}{\|v_3'\|}\right) v_3' = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

Then $\{q_1, q_2, q_3\}$ is an orthonormal basis for W.

Ex: Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Sol^n:- We first find any basis for $\mathbb{R}^3$ containing v_1. If we take $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then $\{v_1, x_2, x_3\}$ is clearly a basis for $\mathbb{R}^3$, becoz all three v_1, x_2, x_3 are L.I. and span $\{v_1, x_2, x_3\} = \mathbb{R}^3$

We now apply Gram-Schmidt process to this basis to obtain

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) \cdot v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{2}{14}\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 5/7 \\ -3/7 \end{bmatrix}.$$

Let $v_2' = 7 v_2 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$

and finally,

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right) v_2'$$

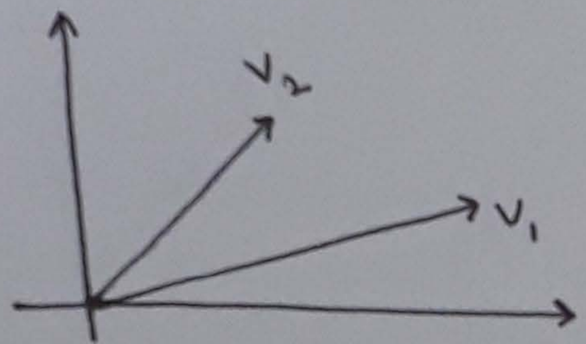
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{3}{14}\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{-3}{35}\right) \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -3/10 \\ 0 \\ 1/10 \end{bmatrix}$$

$$v_3' = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

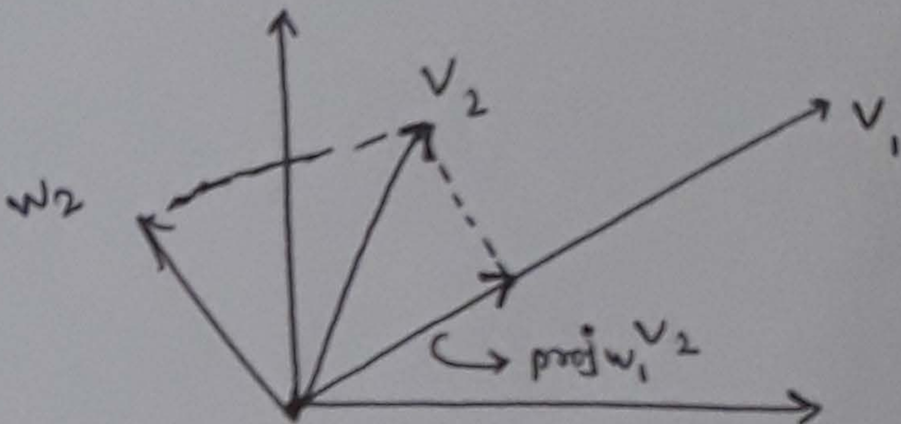
Thus $\{v_1, v_2', v_3'\}$ is an orthogonal basis for $\mathbb{R}^3$ that contains v_1.

<u>Remark</u> :- Given a unit vector, we can obtain orthonormal basis that contains it by using the preceding method and then normalizing the resulting orthogonal vectors.

The geometric intuition of the Gram - Schmidt Process to find an orthonormal basis in $\mathbb{R}^2$.



$\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$



$w_2 = v_2 - \text{proj}_{w_1} v_2$ is orthogonal to $w_1 = v_1$.

$\Rightarrow \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.