L8: Numbers:

Principle of mathematical induction To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function,

Basis step: Verify that $P(1)$ is true

Inductive step: Assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ is true

Ex. $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

Let $P(n)$ be the proposition that the sum of the first n positive integers is $n(n+1)/2$.

Basis step: $P(1)$ is true because $1 = \frac{1(1+1)}{2} = 1$

Inductive step: Assume that $P(k)$ holds for an arbitrary positive integer k. That is,

$$1+2+\ldots+k = \frac{k(k+1)}{2} \quad \text{—— (1)}$$

Under this assumption, to show that $P(k+1)$ is true, that is

$$1+2+\ldots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

Take (1), and $(k+1)$ both sides,

$$1+2+\ldots+k+(k+1) = \frac{k(k+1)}{2} + k+1$$

$$= (k+1)\left[\frac{k}{2}+1\right]$$

$$= (k+1)\frac{k+2}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2}$$

Ex: $2^n < n!$ $\forall n, n \geq 4$.

Ex $6^{n+2} + 7^{2n+1}$ is divisible by 43.

* Strong Mathematical Induction

To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, complete two steps

Basis step: Verify that the proposition P(1) is true.

Inductive step: Assume that P(1), P(2), ..., P(K) is true then show P(K+1) is true.

Ex Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher.

Ex Show that $U_n = 3^n - 2^n$ $\forall n \in \mathbb{N}$ where $U_1 = 1$, $U_2 = 5$ and $U_{n+1} = 5U_n - 6U_{n-1}$

Basis step: For n=1

$P(1)$: LHS $= U_1 = 3^1 - 2^1 = 1$

Inductive step: Suppose P(n) is true for $1 \leq n \leq k$

To show P(n) is true for $n = k+1$ i.e. $U_{k+1} = 3^{k+1} - 2^{k+1}$

$$\begin{aligned}
\therefore U_{k+1} &= 5U_k - 6U_{k-1} \\
&= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\
&= (3+2)(3^k - 2^k) - 6\cdot 3^{k-1} + 2\cdot 3\cdot 2^{k-1} \\
&= 3^{k+1} + 2\cdot 3^k - 3\cdot 2^k - 2^{k+1} - 2\cdot 3^k + 3\cdot 2^k \\
&= 3^{k+1} - 2^{k+1}
\end{aligned}$$