L35: Heat Equation (One Dimensional)

Consider a long thin metal bar or wire or rod of constant cross section and homogeneous material, which is oriented along the x-axis and is perfectly insulated laterally, so that heat flows in the x-direction only. So, the temperature u of bar depends only x and t (time). And length of bar is L.

This physical system is described by the one-dimensional heat equation

$$① \text{ — } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad c^2 = \frac{K}{\rho\sigma}$$

The constant c^2 is thermal diffusivity ; K is thermal conductivity, σ is specific heat and ρ the density of the material of the bar.

Also assume that the temperature at ends of bar are zero and initial temperature distribution is $f(x)$.

$u(0,t) = u(L,t) = 0$ — Boundary Conditions

$u(x,0) = f(x)$ — Initial condition

Step I: Assume that the solution $u(x,t)$ is separable i.e.

$$u(x,t) = X(x)\,T(t)$$

Diff.,

$$u_t = X\frac{dT}{dt} \quad , \quad u_{xx} = \frac{d^2X}{dx^2}T$$

The eqn ① becomes,

$$X\frac{dT}{dt} = c^2\frac{d^2X}{dx^2}T \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{dT}{dt}\frac{1}{T} = k$$

So, we have two ODEs,

$$\frac{d^2X}{dx^2} - kX = 0 \; ; \; \frac{dT}{dt} - c^2kT = 0$$

for $k=0$ & $k>0$ the only soln satisfying Boundary Condition is $u=0$. So, $k = -p^2$

$\therefore \frac{d^2X}{dx^2} + p^2 X = 0$; $\frac{dT}{dt} + p^2 c^2 T = 0$

$\Rightarrow X(x) = A\cos px + B\sin px$; $T(t) = Ce^{-p^2c^2t}$

$\therefore$ The G.S. of ① is $\boxed{u(x,t) = (A\cos px + B\sin px)\, Ce^{-p^2c^2t}}$

Use first Bound. Cond. $u(0,t) = 0$

$\Rightarrow 0 = A\,T(t) \Rightarrow A = 0.$

Use $u(L,t) = 0$

$\Rightarrow 0 = B\sin Lp\, T(t) \Rightarrow \sin pL = 0 \Rightarrow pL = n\pi \Rightarrow p_n = \frac{n\pi}{L}$; $n = 1, 2, \ldots$

The G.S. of ① with Bound. Condi. is

$u(x,t) = B\sin p_n x\, Ce^{-p_n^2c^2t}$

$u_n(x,t) = B_n \sin p_n x\, e^{-p_n^2c^2t}$

The most G.S. of ① with BC is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin p_n x\, e^{-p_n^2c^2t} \quad \text{— (A)}$$

Use $u(x,0) = f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x$$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x\, dx, \quad n = 1, 2, \ldots \quad \text{— (B)}$$

Hence, the temperature distribution in the bar is given by (A) where B_n is given by (B).

Steady-state condition: A condition is said to be steady-state if the dependent variables are free of time t.

Ex. Find the temperature $u(x,t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $\left(100 \sin \frac{\pi x}{80}\right)$ and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C. Physical data for copper: density 8.92 g/cm³, specific heat 0.092 cal/g°c, thermal conductivity 0.95 cal/(cm sec°c).

This system gives $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ — ①, $u(0,t) = 0 = u(80,t)$

$u(x,0) = 100 \sin\left(\frac{\pi x}{80}\right)$

The soln of ① with BC is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80} e^{-p_n^2 c^2 t}$$

Use $u(x,0) = 100 \sin \frac{\pi x}{80}$

$$100 \sin \frac{\pi x}{80} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{80} x = B_1 \sin \frac{\pi x}{80} + B_2 \sin \frac{2\pi x}{80} + \text{---}$$

$\Rightarrow B_1 = 100,\ B_2 = 0,\ B_3 = 0, \ldots$

$$\therefore\ u(x,t) = 100 \sin \frac{\pi x}{80} e^{-\frac{\pi^2}{80^2} c^2 t}$$

$\frac{\pi^2}{80^2}\left[\frac{k}{\rho\sigma}\right] = 1.158\ \mathrm{cm^2/sec}$

$$= 100 \sin \frac{\pi}{80} x e^{-0.001785 t}$$

The $u(x,t)$ is maximum $100 e^{-0.001785t}$ when $\sin \frac{\pi}{80} x = 1$.

$$\therefore\ 100 e^{-0.001785 t} = 50$$

$$\Rightarrow t = \log(0.5)/0.001785 = 388 \text{ sec} = 6.5 \text{ min}$$

Q Find the temperature in a laterally insulated bar of length L whose ends are kept at temp. 0, assuming that the initial temperature is

$$f(x) = \begin{cases} x, & 0 < x < \frac{L}{2} \\ L-x, & \frac{L}{2} < x < L \end{cases}$$

The system is modeled by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $u(0,t) = 0 = u(L,t)$ and $u(x,0) = f(x)$.

The G.S. of Heat eqn with BC is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \, e^{-\frac{n^2\pi^2}{L^2} c^2 t}$$

Use $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

$$\Rightarrow B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L} \int_0^{L/2} x \sin \frac{n\pi}{L} x \, dx + \frac{2}{L} \int_{L/2}^{L} (L-x) \sin \frac{n\pi}{L} x \, dx$$

$$= 0, \quad n = 2, 4, 6, \ldots$$

$$\frac{4L}{n^2\pi^2}, \quad n = 1, 5, 9, \ldots$$

$$-\frac{4L}{n^2\pi^2}, \quad n = 3, 7, 11, \ldots$$

Hence, the solution is

$$u(x,t) = \frac{4L}{\pi^2} \left[\sin \frac{\pi x}{L} e^{-\frac{c^2\pi^2}{L^2} t} - \frac{1}{9} \sin \frac{3\pi}{L} x \, e^{-\frac{9c^2\pi^2}{L^2} t} + \ldots \right]$$

Q Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ — ① , $0 < x < \pi$, $t > 0$, $u(0,t) = u(\pi,t) = 0$

$u(x,0) = T$ (constant)

Assume $u(x,t)$ is separable i.e.

$$u(x,t) = X(x)\, T(t) \quad -②$$

Substitute ② in ①

$$X \frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = -p^2$$

$$\therefore \frac{dT}{dt} + p^2c^2 T = 0 \quad \& \quad \frac{d^2X}{dx^2} + p^2 X = 0$$

$$\Rightarrow T(t) = ce^{-p^2c^2t} \quad \& \quad X(x) = A\cos px + B\sin px$$

The G.S. of ① is

$$u(x,t) = (A\cos px + B\sin px)\, ce^{-p^2c^2t}$$

Use $u(0,t) = 0$

$$\Rightarrow 0 = Ace^{-p^2c^2t} \Rightarrow A = 0$$

Use $u(\pi,t) = 0$

$$\Rightarrow 0 = B\sin p\pi \, ce^{-p^2c^2t} \Rightarrow \sin p\pi = 0 \Rightarrow p = n,\ n = 1,2,\ldots$$

$\therefore$ The most G.S. of ① with Bound. Condi. is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum B_n \sin nx \, e^{-n^2c^2t}$$

Use $u(x,0) = T$ (constant)

$$\Rightarrow T = \sum B_n \sin nx \Rightarrow B_n = \frac{2}{\pi}\int_0^{\pi} T \sin nx \, dx$$

$$= \frac{2}{\pi} T \left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2T}{\pi n}\left[1 - \cos n\pi\right]$$

$$= \frac{2T}{n\pi}\left[1-(-1)^n\right]$$

$\therefore$ The req. solution is

$$u(x,t) = \frac{4T}{\pi} \sin x \, e^{-c^2t} + \frac{4T}{3\pi} \sin 3x \, e^{-9c^2t} + \frac{4T}{5\pi} \sin 5x \, e^{-25c^2t} + \ldots$$