Properties of Eigenvalues and Eigenvectors :

1. If A is a real matrix, its eigenvalues are real or complex conjugate in pairs.

2. The trace of the matrix A is equal to sum of the eigenvalues of the matrix A.

3. The determinant of the matrix A is equal to the product of the eigenvalues of the matrix A.

4. The matrix A and its transpose A' have the same eigenvalues. but different eigenvectors.

Since $\det(A-\lambda I) = \det(A-\lambda I)^T$
$\Rightarrow \det(A-\lambda I) = \det(A^T - \lambda I^T)$
$= \det(A^T - \lambda I)$
$\Rightarrow$ charact. poly of $(A-\lambda I)$ is equal to charact. poly of $(A^T - \lambda I)$
$\Rightarrow$ roots of $(A-\lambda I)$ = roots of $(A^T - \lambda I)$
$\Rightarrow$ eigenvalues of A = eigenvalues of A^T.

[Determinant of A and its transpose A^T are equal because determinant can be found by row or column and transpose is nothing but about changing of rows and columns]

5. If all the eigenvalues are non-zero, then $\det(A) \neq 0$ [By 3.]
$\Rightarrow$ A is invertible (Non-singular)

6. Let λ be an eigenvalue of A and x be its corresponding then corresponding eigenvector is x.
(i) αA has eigenvalue $\alpha\lambda$ and corresponding eigenvector is x.
$$Ax = \lambda x \Rightarrow \alpha A x = (\alpha\lambda)x$$
(ii) A^m has eigenvalues λ^m and corresponding eigenvector is x for any positive integer m.
$$Ax = \lambda x \Rightarrow AAx = A\lambda x \Rightarrow A^2x = \lambda Ax \Rightarrow A^2x = \lambda(\lambda x)$$
$\Rightarrow A^2x = \lambda^2 x \Rightarrow A^2$ has eigenvalue λ^2 and corresponding eigenvector is x.

Premultiplying successively m times, we obtain $A^m x = \lambda^m x$.

(iii) $A-kI$ has the eigenvalue $\lambda-k$, for any scalar k and the corresponding eigenvector is x.

$$Ax=\lambda x \Rightarrow Ax-kIx=\lambda x-kIx$$
$$\Rightarrow (A-kI)x=(\lambda-k)x$$

(iv) A^{-1} (if it exists) has the eigenvalue $1/\lambda$ and the corresponding eigenvector is x. $Ax=\lambda x \Rightarrow A^{-1}Ax=A^{-1}\lambda x$

$$\Rightarrow Ix=\lambda A^{-1}x$$
$$\Rightarrow \frac{1}{\lambda}x=A^{-1}x$$

7. Eigenvector cannot correspond to two distinct eigenvalues.
8. Eigenvalues of diagonal matrix, triangular matrix (lower and upper) are the diagonal elements since the characteristic polynomial becomes the product of factors made by diagonal elements.

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det(A-\lambda I)=(\lambda-1)(\lambda-4)(\lambda-6).$$

[redacted]

Ex. Let $A=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then find the eigenvalues of the matrix A^{19}.

First find eigenvalues of A and then λ^{19} will be eigenvalues of A^{19}.

$\therefore \det(A-\lambda I)=0 \Rightarrow (1-\lambda)(-1-\lambda)-1=0 \Rightarrow -1-\lambda+\lambda+\lambda^2-1=0$

$\Rightarrow \lambda^2-2=0 \Rightarrow \lambda=\pm\sqrt{2} \Rightarrow \lambda_1=\sqrt{2}, \lambda_2=-\sqrt{2}$.

$\therefore$ The eigenvalues of A^{19} are $(\sqrt{2})^{19}$ and $(-\sqrt{2})^{19}$ which are $512\sqrt{2}$ and $-512\sqrt{2}$.