Partial Differential Equation

L31: Basic Concepts.

A partial differential equation (PDE) is an equation involving one or more partial derivatives of a function (u) that depends on two or more variables, often time t and one or several variables in space.

A PDE is linear if it is of the first degree in the unknown function u and its partial derivatives.
A linear PDE is homogeneous if each of its terms contains either u or its partial derivatives.

The general form of a Second-order PDE in the function u of the two independent variables x, y is given by

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} +$$
$$D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y) = 0 \quad \text{—(1)}$$

The PDE (1) is called Elliptic if $B^2 - 4AC < 0$

e.g. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Laplace's equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{Poisson's equation}$$

Parabolic if $B^2 - 4AC = 0$

e.g. $$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Hyperbolic if $B^2 - 4AC > 0$, e.g. $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ 1-D Wave Eqn

Q. Classify PDE : $\frac{\partial^2 u}{\partial x^2} = 5\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

The Separation of Variables Method (SOV)

For a PDE in the function u of two independent variables x and y, assume that the required solution is separable i.e. $u(x, y) = X(x)\, Y(y)$ — ①

Then substitution of u from ① and its derivatives reduces the PDE to the form

$$f(X, X', X'', \ldots) = g(Y, \dot{Y}, \ddot{Y}, \ldots) \quad — ②$$

which is separable in X and Y. Since the L.H.S. of ② is a function of x alone and RHS of ② is a function of y alone, ② must be equal to a common constant say k. Thus,

$$f(X, X', X'', \ldots) = k$$

$$g(Y, \dot{Y}, \ddot{Y}, \ldots) = k$$

Therefore, the determination of solution to PDE reduces to the determination of solutions to two ODE.

Ex Solve $2u_{xx} - u_y = 0$ by separation of variables—

Assume $u(x, y) = X(x)\, Y(y)$

Diff. w.r.t. x and y,

$$u_{xx} = X''(x)\, Y(y) \; ; \; u_y = X(x)\, \dot{Y}(y)$$

So, PDE becomes $2X''(x)\,Y(y) - X(x)\,\dot{Y}(y) = 0$

$$\Rightarrow \frac{2X''}{X} = \frac{\dot{Y}}{Y} = 2k \text{ (constant)}$$

So, $\frac{2X''}{X} = 2k$; $\frac{\dot{Y}}{Y} = 2k$

$\Rightarrow 2X'' - kX = 0$; $\frac{du}{dy} - 2ku = 0$

$\Rightarrow \frac{d^2u}{dx^2} - ku = 0$

A.E. $m - 2k = 0 \Rightarrow m = 2k$

A.E. $m^2 - k = 0$

$Y(y) = c_3 e^{2ky}$

$\Rightarrow m^2 = k$

$\Rightarrow m = \pm\sqrt{k}$

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

$\therefore$ The required solution $u(x,y) = X(x)\,Y(y)$

$= (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})\, c_3 e^{2ky}$

$= (Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}) e^{2ky}$

Q. Solve $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$

Assume $u(x,y) = X(x)Y(y)$

Diff w.r.t x and y

$u_x = X'(x)Y(y)$ & $u_y = X(x)\dot{Y}(y)$

So, PDE becomes $3X'Y + 2X\dot{Y} = 0$

$\Rightarrow \frac{X'}{X} = -\frac{2}{3}\frac{\dot{Y}}{Y} = k$

$\therefore X' - kX = 0$; $\dot{Y} + \frac{3}{2}kY = 0$

$X(x) = c_1 e^{kx}$

$Y(y) = c_2 e^{-\frac{3}{2}ky}$

$\therefore u(x,y) = c_1 e^{kx} c_2 e^{-\frac{3}{2}ky} = Ae^{kx} e^{-\frac{3}{2}ky}$

Given $u(x,0) = 4e^{-x}$

$\Rightarrow 4e^{-x} = Ae^{kx} \Rightarrow \frac{4}{A} e^{-x-kx} = e^0 \Rightarrow \boxed{4 = A}$ & $\boxed{k = -1}$

$\therefore \boxed{u(x,y) = 4e^{(-x+\frac{3}{2}y)}}$

Q Use separation of variables method to solve:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \; ; \; u(0,x) = 2e^{-3x}$$

Assume $u(t,x) = T(t)X(x)$

Diff. u w.r.t. t and x and put in PDE

$$\dot{T}X = TX' \Rightarrow \frac{\dot{T}}{T} = \frac{X'}{X} = k$$

$$\Rightarrow \dot{T} = kT \quad \& \quad X' = kX$$

$$\Rightarrow \dot{T} - kT = 0 \quad \& \quad X' - kX = 0$$

Both are First Order Linear Homogeneous ODE. So, their soln contain c.f only.

$$T(t) = Ae^{kt} \quad \& \quad X(x) = Be^{kx}$$

$$\therefore u(t,x) = Ae^{kt}Be^{kx} = Ce^{k(t+x)} \quad ; \text{ where } C = AB$$

Use $u(0,x) = 2e^{-3x}$

$$\Rightarrow 2e^{-3x} = Ce^{kx} \Rightarrow C = 2 \; ; \; k = -3.$$

Hence, the solution of given PDE is $\boxed{u(t,x) = 2e^{-3(t+x)}}$

Q $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \; ; \; u(x,0) = 2e^{-3x} - 3e^{2x}$

Assume $u(x,y) = X(x)Y(y)$

$$4X'Y + X\dot{Y} = 0 \Rightarrow \frac{X'}{X} = -\frac{1}{4}\frac{\dot{Y}}{Y} = k$$

$$\Rightarrow X' - kX = 0 \;\&\; \dot{Y} + 4kY = 0 \Rightarrow X(x) = Ae^{kx} \;\&\; Y(y) = Be^{-4ky}$$

$$\therefore u(x,y) = Ae^{kx}Be^{-4ky} = ABe^{k(x-4y)} = Ce^{k(x-4y)}$$

Use $u(x,0) = 2e^{-3x} - 3e^{2x}$

$$\Rightarrow Ce^{k(x)} = 2e^{-3x} - 3e^{2x}$$