1 : Exact differential equation

If a function $u(x, y)$ has continuous partial derivatives then its differential (also called total differential) is $\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$.

for example, $u(x, y) = x^2y + x = c$ then

$$\frac{du}{dx} = 2xy + x^2\frac{dy}{dx} + 1 = 0$$

or, $du = \underline{(2xy+1)}dx + x^2dy = 0$. —— (A)

~~Here~~ Now, find derivative of $u(x, y)$ w.r.t. x & y,

$$\frac{\partial u}{\partial x} = 2xy + 1 \quad \& \quad \frac{\partial u}{\partial y} = x^2.$$

which are multipliers of dx and dy in eqn (1).

This shows that $\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$

A first-Order ODE $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ or

(1) —— $M(x, y)dx + N(x, y)dy = 0$ is called an exact differential equation if the DE is of the form $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ of some function $u(x, y)$. Then (1) can be written as $du = 0$. By Integrating, the general solution is $\underline{u(x, y) = c}$.

Let $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$ be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-interaction. Then by partial differentiation

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

By the assumption of continuity the two second partial derivatives are equal. Thus

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

This is necessary and sufficient condition for exact differential equation.

If ① is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way,

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int M dx + k(y) \, ;$$

treat y as a constant

To find $k(y)$, derive $\frac{\partial u}{\partial y}$ from above u, and equate $\frac{\partial u}{\partial y} = N$ and then integrate $\frac{dk}{dy}$ to get $k(y)$.
The solution can also be found by $\frac{\partial u}{\partial y} = N$ then
$u = \int N dy + m(x)$ and so on.

Ex: Solve $\cos(x+y)\,dx + (3y^2+2y+\cos(x+y))\,dy = 0$

Test for exactness:- $M = \cos(x+y)$ & $N = 3y^2+2y + \cos(x+y)$

So, $\frac{\partial M}{\partial y} = -\sin(x+y)$ & $\frac{\partial N}{\partial x} = -\sin(x+y)$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ DE is exact DE.

General solution :- $u = \int M\,dx + k(y) = \int \cos(x+y)\,dx + k(y)$

$= \sin(x+y) + k(y)$

To find $k(y)$, differentiate $u = \sin(x+y) + k(y)$ w.r.t. y,

$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = 3y^2+2y+\cos(x+y)$

$\Rightarrow \frac{dk}{dy} = 3y^2+2y$

$\Rightarrow k = y^3+y^2+c'$

So, General soln is $\boxed{u = \sin(x+y) + y^3 + y^2 = c}$

Q. $e^y\,dx + (xe^y + 2y)\,dy = 0 \Rightarrow xe^y + y^2 = c$

Q. $(3x^2y + \frac{y}{x})\,dx + (x^3 + \ln x)\,dy = 0 \Rightarrow x^3y + y\ln x = c$

Q. $(\cos x - x\cos y)\,dy - (\sin y + y\sin x)\,dx = 0 \Rightarrow y\cos x - x\sin y = c$