[Question Bank 5]

Q1 $A = 1, \; B = 0, \; C = 3.$

Here, $A = 1, \; B = 0, \; C = 3$ is given.

$$B^2 - 4AC = (0)^2 - 4(1)(3)$$
$$= 0 - 12$$
$$= -12.$$

i.e $-12 < 0$

So, $B^2 - 4AC < 0$, $\therefore$ the given equation is Elliptic.

Q2 $A = -2, \; B = 3, \; C = -4.$

Here, $A = -2, \; B = 3$ and $C = -4$ is given.

$$B^2 - 4AC = (3)^2 - 4(-2)(-4)$$
$$= 9 - 32$$
$$= -23$$

i.e $-23 < 0$

So, $B^2 - 4AC < 0$. $\therefore$ it is Elliptic.

Q3 $A = -4, \; B = -2, \; C = -1$

Here, $A = -4, \; B = -2, \; C = -1$ is given.

$$B^2 - 4AC = (-2)^2 - 4(-4)(-1)$$
$$= 4 - 16$$
$$= -12$$

i.e $-12 < 0$

So, $B^2 - 4AC < 0$, $\therefore$ it is Elliptic.

04 $A = 2, \quad B = 2, \quad C = 2$

Here, $A = 2$, $B = 2$ and $C = 2$ is given

$$
\begin{aligned}
B^2 - 4AC &= (2)^2 - 4(2)(2) \\
&= 4 - 16 \\
&= -12
\end{aligned}
$$

i.e $-12 < 0$

So, $B^2 - 4AC < 0$, $\therefore$ it is Elliptic.

05 $A = -5, \quad B = -5, \quad C = -5$

Here, $A = -5$, $B = -5$ and $C = -5$ is given

$$
\begin{aligned}
B^2 - 4AC &= (-5)^2 - 4(-5)(-5) \\
&= 25 - 100 \\
&= -75
\end{aligned}
$$

i.e $-75 = 0$

So, $B^2 - 4AC < 0$, $\therefore$ it is Elliptic.

06 $A = 100, \quad B = 10, \quad C = 10$

Here, $A = 100$, $B = 10$ and $C = 10$ is given.

$$
\begin{aligned}
B^2 - 4AC &= (10)^2 - 4(100)(10) \\
&= 100 - 4000 \\
&= -3900
\end{aligned}
$$

i.e $-3900 < 0$

So, $B^2 - 4AC < 0$, $\therefore$ it is Elliptic.

$\cancel{x^4(y^2-4x)}$

Q7 $A = x^2, \; B = x^2y, \; C = x^3$

Here, $A = x^2$, $B = x^2y$ and $C = x^3$ is given.

$$B^2 - 4AC = (x^2y)^2 - 4(x^2)(x^3)$$
$$= x^4y^2 - 4x^5$$

Case 1: if $x = 0$ ~~$or\ y = 0$~~

$\Rightarrow \quad B^2 - 4AC = x^4y^2 - 4x^5 = 0.$

$\Rightarrow \quad B^2 - 4AC = 0.$

it will be then ~~Elliptic~~ parabolic.

Case 2: if $y = 0$

$B^2 - 4AC = -4x^5$

then there are 3 possible values of x.

(i) $x = 0$

then it will be parabolic

(ii) $x > 0$

it will be elliptic (as $B^2 - 4AC < 0$).

(iii) $x < 0$

it will be hyperbolic (as $B^2 - 4AC > 0$).

Case 3: if $x > 0$, $y > 0$

$\Rightarrow \quad B^2 - 4AC = x^4y^2 - 4x^5$

i.e $x^4y^2 - 4x^5 < 0$

it will be then elliptic.

Case 4: if $x > 0$, $y < 0$

$\Rightarrow \quad B^2 - 4AC = x^4(y^2 - 4x)$

i.e $x^4y^2 - 4x^5 < 0$

it will be then elliptic.

Case 5: if $x < 0$, $y > 0$

$\Rightarrow B^2 - 4AC = x^4y^2 - 4x^5$

i.e $x^4y^2 - 4x^5 > 0$

it will be then hyperbolic.

Case 6:

if $x < 0$, $y < 0$

$\Rightarrow B^2 - 4AC = x^4y^2 - 4x^5$

i.e $x^4y^2 - 4x^5 > 0$

it will be then hyperbolic.

Q8 $A = x^2y$, $B = y^2$, $C = x^5$

Here, $A = x^2y$, $B = y^2$ and $C = x^5$ is given

$$B^2 - 4AC = (y^2)^2 - 4(x^2y)(x^5)$$

$$= y^4 - 4x^7y$$

$$= y(y^3 - 4x^7)$$

Cases →.

Q9 $A = y^3$, $B = x^2$, $C = y^2$

$$B^2 - 4AC = (x^2)^2 - 4(y^3)(y^2)$$

$$= x^4 - 4y^5$$

cases →

Q10 $A = x^{-3}$, $B = (xy)^{-3/2}$, $C = y^{-3}$

$$B^2 - 4AC = \left[(xy)^{-3/2}\right]^2 - 4(x^{-3})(y^{-3})$$

$$= (xy)^{-3} - 4x^{-3}y^{-3} \Rightarrow = (xy)^{-3} - 4(xy)^{-3}$$

$$= (xy)^{-3}[1 - 4]$$

$$= 3(xy)^{-3}$$

Cases $\rightarrow$.

Q11 $A = x^3, \quad B = x^2, \quad C = y^2$

$$B^2 - 4AC = (x^2)^2 - 4(x^3)(y^2)$$

$$= x^4 - 4x^3y^2$$

Cases $\rightarrow$

Q12 $A = x^2, \quad B = x^2, \quad C = x^2.$

$$B^2 - 4AC = (x^2)^2 - 4(x^2)(x^2)$$

$$= x^4 - 4x^4$$

$$= x^4(1 - 4)$$

$$= 3x^4$$

Cases $\rightarrow$.

Question Bank - 5

Q17 PDE: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\rightarrow$ (1)

Let the complete solution of (1) be.

$$u(x,t) = X(x).T(t) \rightarrow (2)$$

Now, $\frac{\partial^2 u}{\partial t^2} = XT''$ and $\frac{\partial^2 u}{\partial x^2} = X''T$

put them in (1)

$$XT'' = c^2 X''T$$

$$\frac{T''}{T} = \frac{c^2 X''}{X} \rightarrow (3)$$

Since in (3) RHS is a function of x and LHS is a function of t. only. So, we have-

$$\frac{T''}{c^2 T} = \frac{X''}{X} = K \text{ (constant)}.$$

Now,

$$\frac{T''}{c^2 T} = K \quad \text{and} \quad \frac{X''}{X} = K$$

$$T'' - Kc^2 T = 0 \rightarrow (4) \quad \text{and} \quad X'' - KX = 0 \rightarrow (5)$$

Here, there are 3 cases arises for value of K.

Case I when $K = 0$, then we have

in (4)

$$T'' = 0$$

Integrating w.r.t t.

$$T' = c_1 \Rightarrow \int T' = \int c_1 dt + c_2$$

$$\Rightarrow \boxed{T = c_1 t + c_2}$$

in (5)

$$X'' = 0$$

Integrating w.r.t x.

$$X' = c_3 \Rightarrow \int X' = \int c_3 dx + c_4$$

$$\Rightarrow \boxed{X = c_3 x + c_4}$$

$\therefore$ Complete Sol^n is

$$u(x,t) = (c_1 t + c_2)(c_3 x + c_4)$$

Case II when K is positive i.e. $K = p^2$ (where p is any no.) then we have,

in (4)

$$T'' - p^2c^2T = 0 \Rightarrow (D^2 - p^2c^2)T = 0 \rightarrow (6)$$

in (5)

$$X'' - p^2X = 0 \Rightarrow (D'^2 - p^2)X = 0 \rightarrow (7)$$

where, $D = \frac{d}{dt}$ and $D' = \frac{d}{dx}$

Auxiliary Eq^n for (6)

$$m^2 - p^2c^2 = 0$$
$$m = \pm pc$$
$$\boxed{T = c_1 e^{pct} + c_2 e^{-pct}}$$

Auxiliary Eq^n for (7)

$$m^2 - p^2 = 0$$
$$m = \pm p$$
$$\boxed{X = C_3 e^{px} + C_4 e^{-px}}$$

$\therefore$ Complete Sol^n is -

$$u(x,t) = \left(C_1 e^{pct} + C_2 e^{-pct}\right)\left(C_3 e^{px} + C_4 e^{-px}\right).$$

Case III when K is $-$ve i.e. $K = -p^2$ (where p is any no.) then we have,

in (4)

$$T'' + p^2c^2 T = 0 \Rightarrow (D^2 + p^2c^2)T = 0 \rightarrow (8)$$

in (5)

$$X'' + p^2 X = 0 \Rightarrow (D'^2 + p^2)X = 0 \rightarrow (9)$$

A.E for (8)

$$m^2 + p^2c^2 = 0$$
$$m = \pm ipc$$
$$\boxed{T = c_1 \cos pct + c_2 \sin pct}$$

A.E for (9)

$$m^2 + p^2 = 0$$
$$m = \pm ip$$
$$\boxed{X = C_3 \cos px + C_4 \sin px}$$

$\therefore$ Complete Sol^n is.

$$u(x,t) = (c_1 \cos pct + c_2 \sin pct)(c_3 \cos px + C_4 \sin px)$$

Since periodic function are present in Case III solution and it shows the physical nature of wave, So, it is only solution which become the complete solution of (1).

Q16 $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

Let the complete of (1) be

$$u(x,t) = X(x) \cdot T(t) \rightarrow (2)$$

Now, $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$

put them in (1)

$$XT' = c^2 X''T$$

$$\frac{T'}{c^2 T} = \frac{X''}{X}$$

Since, RHS is a function of x and LHS is a function of t only. So, we have,

$$\frac{T'}{c^2 T} = \frac{X''}{X} = k \text{ (constant)}$$

$T' - kc^2T = 0 \rightarrow (3)$ and $X'' - kX = 0 \rightarrow (4)$

Here, there are 3 cases arises for value of k.

Case I when $K = 0$, then we have from (3) and (4)

$T' = 0 \Rightarrow \boxed{T = c_1}$ (Integrating w.r.t t)

$X'' = 0 \Rightarrow X' = c_2 \Rightarrow \int X' = \int c_2 dx + c_3$

$\Rightarrow \boxed{X = c_2 x + c_3}$ (integration w.r.t x)

$\therefore$ Complete Solution is $\rightarrow u(x,t) = (c_1)(c_2 x + c_3)$

Case II when K is +ve, i.e $K = p^2$ (where p is any no.) then we have from (3) and (4)

$T' - p^2c^2T = 0 \Rightarrow \frac{T'}{T} = p^2c^2$

Integrating w.r.t t

$\int \frac{T'}{T} = p^2 \int c^2 dt$

$\Rightarrow \log T = p^2c^2t + c_1$

$\Rightarrow T = e^{p^2c^2t + c_1}$

$T = e^{p^2c^2t} \cdot e^{c_1}$

$T = Ae^{p^2c^2t}$ (where $A = e^{c_1}$)

Now,

$X'' - p^2X = 0$

$\Rightarrow (D^2 - p^2)X = 0$ $\left(D = \frac{d}{dx} \right)$

~~Integrating w.r.t x~~

A.E

$m^2 - p^2 = 0$

$m^2 = p^2$

$m \pm p$ $\Rightarrow \boxed{X = c_2e^{px} + c_3e^{-px}}$

$\therefore$ Complete Solution of (1)

$$u(x,t) = \left(Ae^{p^2c^2t}\right)\left(C_2e^{px} + C_3e^{-px}\right)$$

Case III when K is $-ve$ i.e $K = -p^2$ (p is any no.) then from (3) and (4) we have,

$$\frac{T'}{T} = -p^2c^2$$

integrate w.r.t t.

$$\int\frac{T'}{T} = -p^2\int c^2dt$$

$$\log T = -p^2c^2t + C_1$$

$$T = e^{-p^2c^2t + C_1}$$

$$T = e^{-p^2c^2t} \cdot e^{C_1}$$

$$T = Be^{-p^2c^2t}, \quad (\text{where } B = e^{C_1})$$

and

$$X'' + p^2X = 0 \Rightarrow D^2 + p^2 = 0$$

$$m^2 + p^2 = 0$$

$$m = \pm ip$$

$$X = C_2\cos px + C_3\sin px$$

$\therefore$ Complete Solution of (1)

$$u(x,t) = \left(Be^{-p^2c^2t}\right)\left(C_2\cos px + C_3\sin px\right)$$

Since, physical nature of any heat flow equation is shown by Case III solution So, it become complete solution of heat equation (1).

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Q29 with boundary condition.

$0 < x < l \,,\; t > 0$

$u(0,t) = u(l,t) = 0, \quad u(x,0) = x$

Applying condition on Q16

(i) put $x = 0$

$u(0,t) = Be^{-p^2c^2t} \cdot (c_2 \cdot 1 + 0)$

$\Rightarrow \boxed{c_3 = 0}$

$\therefore$ complete solution is

$u(x,t) = Be^{-p^2c^2t} \cdot c_2 \cos px \quad \rightarrow (a)$

(ii) put $x = l$. in (a)

$u(l,t) = Be^{-p^2c^2t} \cdot c_2 \cos pl$

$\cos pl = 0$

$pl = n\pi$

$p = \frac{n\pi}{l}$

$\therefore$ Complete solution is.

$u(x,t) = Be^{-p^2c^2t} c_2 \cos\left(\frac{n\pi x}{l}\right) \quad \rightarrow (b)$

$u(x,t) = z\, e^{-n^2\pi^2c^2t/l^2} \cos\left(\frac{n\pi x}{l}\right)$

where, $(z = B \cdot C_2)$.

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Now, the general sol^n of (1).

$$u(x,t) = \sum_{n=1}^{\infty} \cancel{\phi} Z_n e^{-n^2\pi^2c^2t/l^2} \cos\left(\frac{n\pi x}{l}\right) \rightarrow (c)$$

(iii) put $t=0$ in (c).

$$u(x,0) = \sum_{n=1}^{\infty} Z_n \, 1 . \cos\left(\frac{n\pi x}{l}\right) = f(x)$$

$$\text{or } f(x) = \sum_{n=1}^{\infty} Z_n \cos\left(\frac{n\pi x}{l}\right)$$

multiply both side by $\cos\left(\frac{m\pi x}{l}\right)$ and then integrating w.r.t x b/w the limits $x=0$ to $x=l$.

$$\int_0^l f(x) \cos\left(\frac{m\pi x}{l}\right) dx = \sum_{n=1}^{\infty} Z_n \int_0^l \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right)$$

$$\Rightarrow \int_0^l \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) = \begin{cases} l/2, & m=n \\ 0, & m \neq n \end{cases}$$

$$\Rightarrow \int_0^l f(x) \cos\left(\frac{m\pi x}{l}\right) dx = Z_m \frac{l}{2}$$

$$\Rightarrow Z_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Hence, the final general solution of ①

$$u(x,t) = \sum_{n=1}^{\infty} Z_n e^{-n^2\pi^2c^2t/l^2} \cos\left(\frac{n\pi x}{l}\right)$$

where, $Z_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$