Inner product Space

An inner product is a generalization of the dot product. As dot product tells about length related to two vectors of $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$ (Euclidean space) in some way. Similarly inner product will tell you about length of two vectors of any vector space like vector space of polynomials, matrices and etc. So, you might be thinking that what I am taking about, length of polynomials, matrices!
But this is what that mathematicians always do. They take a thing (dot product here) that is very much geometrical or physical then generalize that to any extent which is sometimes unbelievable. A mathematician proved that counting of natural numbers and integers are equal !!!! . . .

Definition: An inner product on a vector space V is a function that takes each ~~each~~ pair of vectors u and v in V a real (or complex) number $\langle u, v\rangle$ such that the following properties hold for all vectors u, v and w in V and all scalars c:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ [conjugate symmetry]
2. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ [additive in first slot)
3. $\langle cu, v\rangle = c\langle u, v\rangle$ [homogeneity in Ist Slot)
4. $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$ [Positive definiteness]

A vector space with an inner product is called an inner product space.

Properties 2 & 3 can be written combinly as

$$\langle c_1u + c_2v, w\rangle = c_1\langle u, w\rangle + c_2\langle v, w\rangle \quad \text{[linear property]}$$

When we take about real inner product space then conjugate symmetry becomes symmetry $\langle u, v\rangle = \langle v, u\rangle$

Ex: $\mathbb{R}^2$ is an inner product space with $\langle u, v\rangle = u\cdot v = u^T v$.

Let $u = (a_1, a_2)$, $v = (b_1, b_2) \in \mathbb{R}^2$

1. $\langle u,v\rangle = (a_1,a_2)\cdot(b_1,b_2) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [a_1 \ a_2]\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1b_1 + a_2b_2$

$\langle v,u\rangle = (b_1,b_2)\cdot(a_1,a_2) = [b_1 \ b_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = b_1a_1 + b_2a_2$

$= a_1b_1 + a_2b_2$

(Real no. are commutative)

$\therefore \langle u,v\rangle = \langle v,u\rangle$

2. $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$; $w = (c_1, c_2) \in \mathbb{R}^2$

$\langle u+v, w\rangle = \langle (a_1+b_1, a_2+b_2), (c_1, c_2)\rangle$

$= (a_1+b_1, a_2+b_2)\cdot(c_1, c_2)$

$= a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2$

$\langle u,w\rangle + \langle v,w\rangle = u\cdot w + v\cdot w = a_1c_1 + a_2c_2 + b_1c_1 + b_2c_2$

$\therefore$ Addition is linear in Ist slot.

3. $\langle cu, v\rangle = c\langle u,v\rangle$

$\langle cu, v\rangle = cu\cdot v = (ca_1, ca_2)\cdot(b_1, b_2) = ca_1b_1 + ca_2b_2$

$= c(a_1b_1 + a_2b_2) = c\langle u,v\rangle$

4. $\langle u,u\rangle = u\cdot u = (a_1,a_2)\cdot(a_1,a_2) = a_1^2 + a_2^2$

$\therefore$ Both are positive and so their sum

$\therefore \langle u,u\rangle > 0$ But if $\langle u,u\rangle = 0 \Rightarrow a_1^2 + a_2^2 = 0$

$\Rightarrow a_1 = 0$ & $a_2 = 0 \Rightarrow u = (0,0)$.

Therefore, all conditions are satisfied, $\mathbb{R}^2$ is an inner product space with dot product.

Ex2. $\mathbb{R}^n$ is an inner product space (IPS) with usual dot product
$\langle u,v\rangle = u\cdot v = a_1b_1 + a_2b_2 + \dots + a_nb_n$ where
$$u = (a_1, a_2, \dots, a_n) \ \& \ v = (b_1, b_2, \dots, b_n)$$
We can show all properties as done in previous example.

[Note:- The dot product is not the only inner product that can be defined on $\mathbb{R}^n$.]

Ex3. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Then
$\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$ defines an inner product.

Ex4. In P_2, let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. Then
$\langle p(x), q(x)\rangle = a_0b_0 + a_1b_1 + a_2b_2$ defines inner product on P_2.

Ex5. Let f and g be in $C[a,b]$, the vector space of all continuous functions on the closed interval $[a,b]$. Then
$\langle f, g\rangle = \int_a^b f(x)\,g(x)\,dx$ defines inner product on $C[a,b]$.

Ex.6 Let $M = M_{m,n}$, the vector space of all real $m\times n$ matrices.
An inner product is defined on M by
$$\langle A, B\rangle = tr(B^TA)$$

Ex7. Let $V = \mathbb{C}^n$ and let $u = (z_i)$ and $v = (w_i)$ be vectors in $\mathbb{C}^n$.
Then $\langle u, v\rangle = u^T\bar{v} = z_1\bar{w}_1 + z_2\bar{w}_2 + \dots + z_n\bar{w}_n$ is an inner product on $V = \mathbb{C}^n$.