Case III: When $f(a)=0$ then

$$f(D)=(D-a)^r\psi(D) \quad \text{such that } \psi(a)\neq 0$$

$\therefore$ P.I.: $y_p = \frac{1}{f(D)} e^{ax}$

$$= \frac{1}{(D-a)^r\psi(D)} e^{ax}$$

$$= \frac{1}{\psi(a)} \frac{1}{(D-a)^r} e^{ax}$$

$$\boxed{y_p = \frac{1}{\psi(a)} \frac{x^r}{r!} e^{ax}}$$

Ex. $(D^2+2D+1)y = 2e^{3x}$

The A.E. is $m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1$

So, the C.F. is $y_h(x) = c_1e^{-x}+c_2xe^{-x}$

The P.I. is $y_p(x) = \frac{1}{D^2+2D+1} 2e^{3x}$

$$= \frac{2}{3^2+2\cdot 3+1} e^{3x}$$

$$= \frac{1}{8} e^{3x}$$

$\therefore$ The G.S. is $y = y_h(x)+y_p(x)$

$$= (c_1+c_2x)e^{-x}+\frac{1}{8}e^{3x}$$

Ex. $(D^3-2D^2-5D+6)y = 2e^{x}+4e^{3x}$

The A.E. is $m^3-2m^2-5m+6=0$

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$$(m-1)(m^2-m-6)=0$$
$$(m-1)(m-3)(m+2)=0$$
$$m=1,3,-2$$

So, the C.F. is $y_h(x) = c_1e^{x}+c_2e^{3x}+c_3e^{-2x}$

P.I. $$y_p(x) = \frac{1}{(D^3-2D^2-5D+6)}(2e^{x}+4e^{3x})$$

$$= \frac{1}{(D-1)(D-3)(D+2)}(2e^{x}+4e^{3x})$$

$$= \frac{2}{(-2)(3)}\frac{1}{(D-1)}e^{x} + \frac{4}{(2)(5)}\frac{1}{(D-3)}e^{3x}$$

$$= -\frac{1}{3}xe^{x}+\frac{2}{5}xe^{3x}$$

$\therefore$ The G.S. is $y = y_h(x)+y_p(x)$

$$= c_1e^{x}+c_2e^{3x}+c_3e^{-2x}-\frac{1}{3}xe^{x}+\frac{2}{5}xe^{3x}$$

Ex. $(D^4-2D^3+2D-1)y = 2e^{x}$

The A.E. is $m^4-2m^3+2m-1=0$

$$(m+1)(m-1)^3=0$$
$$\Rightarrow m=-1,1,1,1$$

So, the C.F. is $y_h(x) = c_1e^{-x}+c_2e^{x}+c_3xe^{x}+c_4x^2e^{x}$

P.I. $$y_p(x) = \frac{1}{f(D)}2e^{x} = \frac{1}{(D+1)(D-1)^3}2e^{x} = \frac{2}{2}\cdot\frac{1}{(D-1)^3}e^{x}$$

$$= \frac{x^3}{3!}e^{x} = \frac{1}{6}x^3e^{x}.$$