Eigenspace:

If u and v are eigenvectors of a matrix A corresponding to the same eigenvalue λ, so are $u+v$ (provided $u \neq -v$) and ku for any $k \neq 0$.

If A is an $n \times n$ square matrix and λ is an eigenvalue of A then the set of all eigenvectors corresponding to eigenvalues λ, together with 0 (zero vector), form a vector space, called the eigenspace (E_λ) of A corresponding to that λ.

Ex. $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Av = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $(1,1)$ is an eigenvector for $\lambda = 4$.

Now take any scalar multiplication of $(1,1)$, that becomes eigenvectors of A for $\lambda = 4$. But since all eigenvectors $(2,2)$, $(3,3)$, $(-1,-1)$, $(4,4)$, ... are obtained from $(1,1)$ ~~so~~ all these eigenvectors are L.D. And set of all these eigenvectors alongwith 0 is called eigenspace (E_λ) for $\lambda = 4$. This eigenspace has basis $= \{(1,1)\}$ and dim is 1.

$$E_4 = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Eigenbasis: If A is an $n \times n$ square matrix and corresponding each eigenvalues of A there exists n linearly independent eigenvectors then the set of all L.I. eigenvectors of A is called eigenbases for $\mathbb{R}^n$.

Ex. The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has 3 L.I. eigenvectors $(1,-1,0)$, $(1,1,-2)$ and $(1,1,1)$.

$\therefore$ $\{(1,-1,0), (1,1,-2), (1,1,1)\}$ is an eigenbases for $\mathbb{R}^3$.

Ex. The matrix $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & -5 & 4\end{bmatrix}$ has only two L.I. eigenvectors $(1,1,1)$ and $(1,2,4)$ but not 3. So, A has not eigenbases for $\mathbb{R}^3$.

Ex The matrix $A=\begin{bmatrix}-1 & 0 & 1\\ 3 & 0 & -3\\ 1 & 0 & -1\end{bmatrix}$ has 3 L.I. eigenvectors. So $\{(1,0,1),(0,1,0),(-1,3,1)\}$ is the eigenbases for $\mathbb{R}^3$.

Symmetric, Skew-symmetric, [redacted]:

Thm: The eigenvalues of a symmetric matrix A are real.

Pf:- Let λ be an eigenvalue and v be its eigenvector of A, i.e.

① — $Av=\lambda v \quad (v\neq 0)$

Take $\lambda \bar{v}^T v$; now $\lambda \bar{v}^T v = \bar{v}^T(\lambda v) = \bar{v}^T(Av)$ [By ①]

$\bar{v}^T(Av) = (A^T\bar{v})^T v = (A\bar{v})^T v$ [Since A is symmetric matrix]

$= (\bar{A}\bar{v})^T v$ [$\because$ A is real matrix so conjugate of A will be A]

$= (\bar{\lambda}\bar{v})^T v$ [By ①]

$= \bar{\lambda}\bar{v}^T v$

$\therefore \lambda \bar{v}^T v = \bar{\lambda}\bar{v}^T v \Rightarrow (\lambda-\bar{\lambda})\bar{v}^T v = 0 \Rightarrow \lambda=\bar{\lambda} \Rightarrow \lambda$ is real

Thm: The eigenvalues of a skew-symmetric matrix A are either zero or purely imaginary.

Pf:- Let λ be an eigenvalue and v be its eigenvector of A i.e. $Av=\lambda v$. By same way in previous thm;

$\lambda \bar{v}^T v = -\bar{\lambda}\bar{v}^T v \Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda$ is zero or purely imaginary.