Cantor's diagonalisation argument (cantor diagonal thm) [CDA]

CDA is an important method to prove that the set of real numbers is not countable.

→ Show that the set of real numbers is an uncountable set.

To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then $(0,1) \subset \mathbb{R}$ is countable. Under this assumption, the real numbers between 0 and 1 can be listed in a sequence, say, $r_1, r_2, r_3, \ldots$ Let the decimal representation of these real numbers be

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$$
$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$$
$$r_4 = 0.d_{41}d_{42}d_{43}d_{44}\ldots$$
$$\vdots$$

where $d_{ij} \in \{0,1,2,3,\ldots,9\}$.

Now, form a new real number with decimal expansion

$r = 0.d_1d_2d_3d_4\ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

Every real number has a unique decimal expansion (when the possibility that the expansion has a tail end that consists entirely of the digit 9 is excluded).

Therefore, r is not equal ~~a~~ to any of $r_1, r_2, \ldots$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Because there is a real number r between 0 and 1 that is not in the list, the assumption that all real numbers between 0 and 1 could be listed must be false. Thus, $(0,1)$ is uncountable so $\mathbb{R}$ is uncountable.

<u>Schröder-Bernstein Thm</u>: If A and B are sets with $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$. In other words, if there are one-to-one fns f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Ex Show that $|(0,1)| = |(0,1]|$

We will use above SB Thm. We will find a function $(0,1)$ to $(0,1]$ which is 1-1 and conversely.

Since $(0,1) \subset (0,1]$, $f(x) = x$ is a one-one function from $(0,1)$ to $(0,1]$. So, $|(0,1)| \le |(0,1]|$

The function $g(x) = \frac{x}{2}$ is also one-to-one function from $(0,1]$ to $(0,1)$. So, $|(0,1]| \le |(0,1)|$

$\therefore |(0,1)| = |(0,1]|$

<u>Cantor's theorem (Power set theorem)</u>: The cardinality of a set is always less than the cardinality of its power set.

Ex. Show that Power set theorem.

Define $f: X \to P(X)$ by $f(x) = \{x\}$. The function f is one-to-one and so $|X| \le |P(X)|$.

To prove that $|X| \ne |P(X)|$. We prove that there is no function from X onto $P(X)$.

On contrary, suppose there exists a function f from X onto $P(X)$. Define $Y = \{x \in X \mid x \notin f(x)\}$. We show that Y has no preimage under f by contradiction.

Suppose $Y = f(x)$ for some $x \in X$. As $f(x)$ is a subset of X, we will find out whether x is in $f(x)$ or not.

$x \in f(x)$ iff $x \notin Y$ (by definition of Y)

iff $x \notin f(x)$ ($\because Y = f(x)$) This is a contradic[tion]