Eigenvalues and Eigenvectors:

Suppose $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$Av = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4v \quad ; \quad Au = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$\Rightarrow \boxed{Au \neq \lambda u}$ any scalar

$\Rightarrow \boxed{Av = 4v}$

eigenvector corresponding to Eigenvalue 4 of A

Def:- Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector v such that $Av = \lambda v$. Such a vector v is called an eigenvector of A corresponding to λ.

Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction that is stretched by the transformation and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.

In essence, an eigenvector v of a linear transformation T (A) is a non-zero vector that, when T is applied to it, does not change direction.

Ex. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ geometrically.

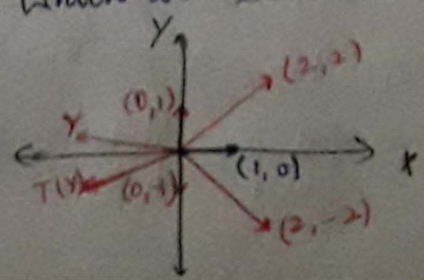
A is the matrix of a reflection transformation (L.T.) T in the x-axis.

The only vectors that don't change its direction (except reverse direction) under T(= A) are vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are reversed, so eigenvalue is -1.

And another vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which are sent to themselves, so eigenvalue is 1.

$\therefore \lambda_1 = 1, \; v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1, \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\therefore\ Av = \lambda v \Leftrightarrow Av = \lambda I v \Leftrightarrow (A-\lambda I)v = 0$ — This is a Homogeneous system of Linear equation

If $v \neq 0 \Rightarrow |A - \lambda I| = 0$ $\left(\det(A-\lambda I) = 0\right)$

$\Rightarrow$ This will give a polynomial of degree n, if A has order n. (Characteristic polynomial of A)

$\Rightarrow$ This will give at most n roots (real or complex).

$\Rightarrow$ These roots are called eigenvalues of A.

Def: The <u>characteristic polynomial</u> of A (square matrix of order n) is the polynomial defined by $\det(A-\lambda I)$ or $\det(\lambda I - A)$ where I denotes the $n\times n$ identity matrix.

$\longrightarrow$ Degree of characteristic polynomial = order of square matrix

<u>Example</u> Find all eigenvalues and eigenvectors corresponding to them of $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$.

Soln:- First find characteristic polynomial as:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 2 & -6-\lambda \end{vmatrix} = (-6-\lambda)(3-\lambda) + 8 = \lambda^2 + 3\lambda - 10$$

$\therefore\ \lambda^2 + 3\lambda - 10 = 0 \Rightarrow (\lambda - 2)(\lambda + 5) = 0 \Rightarrow \lambda = 2, -5$

So, eigenvalues of A are 2 and -5.

Now, find eigenvector corresponding to $\lambda = 2$ as:

$$(A - 2I)v = 0 \Rightarrow \begin{bmatrix} 3-2 & -4 \\ 2 & -6-2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} +1 & -4 \\ 2 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 - 4v_2 = 0$ & $2v_1 - 8v_2 = 0 \Rightarrow v_1 = 4v_2$.

$\therefore\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus, $(4, 1)$ is an eigenvector belonging to $\lambda = 2$.