ASSIGNMENT 19

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Discrete Mathematics

Q1(a): Solve the following recurrence relation by substitution.

$$a_n = a_{n/2} + 2n - 1 \text{ where } n = 2^k \text{ and } a_1 = 1$$

Solution: We have:

$$\begin{aligned}
a_n &= a_{n/2} + 2n - 1 \\
a_{n/2} &= a_{n/4} + 2(n/2) - 1 \\
a_{n/4} &= a_{n/8} + 2(n/4) - 1 \\
&. \\
&. \\
a_{n/2^{k-1}} &= a_{n/2^k} + 2(n/2^{k-1}) - 1 \\
a_{n/2^k} &= a_1 = 1 \ (\because n = 2^k)
\end{aligned}$$

Rearranging above equations we get,

$$\begin{aligned}
a_n - a_{n/2} &= 2n - 1 \\
a_{n/2} - a_{n/4} &= 2(n/2) - 1 \\
a_{n/4} - a_{n/8} &= 2(n/4) - 1 \\
&. \\
&. \\
&. \\
a_{n/2^{k-1}} - a_{n/2^k} &= 2(n/2^{k-1}) - 1 \\
a_{n/2^k} &= 1
\end{aligned}$$

Summing all the equations we have,

$$\begin{aligned}
a_n &= 2n - 1 + 2(n/2) - 1 + 2(n/4) - 1 + ... + 2(n/2^{k-1}) - 1 + 1 \\
&= 2n(1 + 1/2 + 1/4 + ... + 1/2^{k-1}) - k + 1 \\
&= 2n(\frac{1-(1/2)^k}{1-(1/2)}) - k + 1 \\
&= 2n2(\frac{2^k - 1}{2^k}) - k + 1 \\
&= 4n(\frac{n-1}{n}) - log_2 n + 1 \ (\because n = 2^k \ \ and \ \ k = log_2 n) \\
&= 4n - log_2 n - 3
\end{aligned}$$

$$\therefore a_n = 4n - log_2 n - 3$$

Solution (b): Solve the following recurrence relation by substitution:

$$a_n - 7a_{n/3} = 2n \ \ where \ \ n = 3^k \ \ for \ \ k \geq 1 \ \ and \ \ a_1 = \frac{5}{2}$$

$$\begin{aligned}
a_n &= 7a_{n/3} + 2n \\
7a_{n/3} &= 7^2 a_{n/3^2} + (7)2(n/3) \\
7^2 a_{n/3^2} &= 7^3 a_{n/3^3} + (7^2)2(n/3^2) \\
&. \\
&. \\
&. \\
7^{k-1} a_{n/3^{k-1}} &= 7^k a_{n/3^k} + (7^{k-1})2(n/3^{k-1}) \\
7^k a_{n/3^k} &= a_1 = 7^k(\frac{5}{2}) \ (\because n = 3^k)
\end{aligned}$$

Rearranging above equations we get,

$$\begin{aligned}
a_n - 7a_{n/3} &= 2n \\
7a_{n/3} - 7^2 a_{n/3^2} &= (7)2(n/3) \\
7^2 a_{n/3^2} - 7^3 a_{n/3^3} &= (7^2)2(n/3^2) \\
&. \\
&. \\
&. \\
7^{k-1} a_{n/3^{k-1}} - 7^k a_{n/3^k} &= (7^{k-1})2(n/3^{k-1}) \\
7^k a_{n/3^k} &= 7^k(\frac{5}{2})
\end{aligned}$$

Summing all the equations we have,

$$\begin{aligned}
a_n &= 2n + (7)2(n/3) + (7^2)2(n/3^2) + ... + (7^{k-1})2(n/3^{k-1}) + 7^k(\frac{5}{2}) \\
&= 2n(1 + 7/3 + 7^2/3^2 + ... + 7^{k-1}/3^{k-1}) + 7^k(\frac{5}{2}) \\
&= 2n(\frac{1-(7/3)^k}{1-(7/3)}) + 7^k(\frac{5}{2}) \\
&= 2n\frac{3}{-4}(\frac{3^k - 7^k}{3^k}) + 7^k(\frac{5}{2}) \\
&= \frac{-3}{2}n(\frac{n - 7^{log_3 n}}{n}) + 7^{log_3 n}(\frac{5}{2}) \ \ (\because n = 3^k \ \ and \ \ k = log_3 n) \\
&= \frac{-3n + (8)7^{log_3 n}}{2} \\
&= \frac{-3n}{2} + (4)7^{log_3 n}
\end{aligned}$$

$$\therefore a_n = \frac{-3n}{2} + (4)7^{log_3 n}$$

Solution 2(a): Solve the following recurrence relation by generating functions:

$$a_n - 6a_{n-1} + 8a_{n-2} = n4^n \ \ where \ \ a_0 = 8 \ \ and \ \ a_1 = 22$$

Let $A(X) = \sum_{n=0}^{\infty} a_n X_n$ be the generating function for sequence $\{a_n\}_{n=0}^{\infty}$. Then note that

$$\begin{aligned}
A(X) &= a_0 + a_1 X + a_2 X^2 + a_3 X^3 + ... + a_n X^n + ... \\
6XA(X) &= 6a_0 X + 6a_1 X^2 + 6a_2 X^3 + ... + 6a_n X^{n+1} + ... \\
8X^2 A(X) &= 8a_0 X^2 + 8a_1 X^3 + 8a_2 X^4 + ... + 8a_n X^{n+2} + ...
\end{aligned}$$

Thus,

$$
\begin{aligned}
(1-6X+8X^2)A(X) &= a_0 + (a_1 - 6a_0)X + (a_2 - 6a_1 + 8_0)X^2 \\
&+ (a_3 - 6a_2 + 8_a 1)X^3 + ... + (a_n - 6_{n-1} + 8_{n-2})X^n + ... \\
&= a_0 + (a_1 - 6a_0)X + \sum_{n=2}^{\infty} n4^n X^n \\
&= a_0 + (a_1 - 6a_0)X + \sum_{n=2}^{\infty} n(4X)^n \\
&= a_0 + (a_1 - 6a_0)X + 4X\sum_{n=2}^{\infty} n(4X)^{n-1} \\
&= a_0 + (a_1 - 6a_0)X + 4X\sum_{n=2}^{\infty} (n(4X)^{n-1} + 1 - 1) \\
&= a_0 + (a_1 - 6a_0)X - 4X + 4X\sum_{n=2}^{\infty} (n(4X)^{n-1} + 1) \\
&= a_0 + (a_1 - 6a_0)X - 4X + \frac{4X}{(1-4X)^2} \\
\implies A(X) &= \frac{a_0 + (a_1 - 6a_0)X - 4X}{(1-6X+8X^2)} + \frac{4X}{(1-4X)^2(1-6X+8X^2)} \\
&= \frac{a_0 + (a_1 - 6a_0)X - 4X}{(1-2x)(1-4X)} + \frac{4X}{(1-4X)^3(1-2X)} \\
&= \frac{8 + (22 - 6(8))X - 4X}{(1-2x)(1-4X)} + \frac{4X}{(1-4X)^3(1-2X)} \quad (\because a_0 = 8 \ \ and \ \ a_1 = 22) \\
&= \frac{8 - 30X}{(1-2x)(1-4X)} + \frac{4X}{(1-4X)^3(1-2X)}
\end{aligned}
$$

Applying partial fraction decomposition on both the terms separately, we have

$$
\frac{8-30X}{(1-2x)(1-4X)} = \frac{7}{1-2X} + \frac{1}{1-4X}
$$

$$
\frac{4X}{(1-4X)^3(1-2X)} = \frac{-2}{1-2X} + \frac{4}{1-4X} + \frac{-4}{(1-4X)^2} + \frac{2}{(1-4X)^3}
$$

Thus,

$$
\begin{aligned}
A(X) &= \frac{7}{1-2X} + \frac{1}{1-4X} + \frac{-2}{1-2X} + \frac{4}{1-4X} + \frac{-4}{(1-4X)^2} + \frac{2}{(1-4X)^3} \\
&= \frac{5}{1-2X} + \frac{5}{1-4X} + \frac{-4}{(1-4X)^2} + \frac{2}{(1-4X)^3} \\
&= 5\sum_{n=0}^{\infty}\binom{1-1+n}{n}2^n X^n + 5\sum_{n=0}^{\infty}\binom{1-1+n}{n}4^n X^n + (-4)\sum_{n=0}^{\infty}\binom{2-1+n}{n}4^n X^n + 2\sum_{n=0}^{\infty}\binom{3-1+n}{n}4^n X^n \\
&= 5\sum_{n=0}^{\infty} 2^n X^n + 5\sum_{n=0}^{\infty} 4^n X^n + (-4)\sum_{n=0}^{\infty}(n+1)4^n X^n + 2\sum_{n=0}^{\infty}\frac{(2+n)(1+n)}{2}4^n X^n \\
&= \sum_{n=0}^{\infty}\left[2^n(5 + 2^n(n^2 - n + 3))\right]X^n \\
&= \sum_{n=0}^{\infty} a_n X^n
\end{aligned}
$$

Hence,

$$
a_n = 2^n(5 + 2^n(n^2 - n + 3))
$$

Solution 2(b): Solving the following recurrence relation by generating functions:

$$
a_n - 5a_{n-1} + 6a_{n-2} = n^2 4^n \ \ for \ \ n \geq 2
$$

Let $A(X) = \sum_{n=0}^{\infty} a_n X_n$ be the generating function for sequence $\{a_n\}_{n=0}^{\infty}$. Then note that

$$
\begin{aligned}
A(X) &= a_0 + a_1 X + a_2 X^2 + a_3 X^3 + ... + a_n X^n + ... \\
5XA(X) &= 5a_0 X + 5a_1 X^2 + 5a_2 X^3 + ... + 5a_n X^{n+1} + ... \\
6X^2 A(X) &= 6a_0 X^2 + 6a_1 X^3 + 6a_2 X^4 + ... + 6a_n X^{n+2} + ...
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(1-5X+6X^2)A(X) &= a_0 + (a_1 - 5a_0)X + (a_2 - 5a_1 + 6_0)X^2 \\
&+ (a_3 - 5a_2 + 6a_1)X^3 + ... + (a_n - 5_{n-1} + 6_{n-2})X^n + ... \\
&= a_0 + (a_1 - 5a_0)X + \sum_{n=2}^{\infty}(n^2 + n - n)4^n X^n \\
&= a_0 + (a_1 - 5a_0)X + \sum_{n=2}^{\infty}(n^2 - n)4^n X^n + \sum_{n=2}^{\infty} n4^n X^n \\
&= a_0 + (a_1 - 5a_0)X + \sum_{n=2}^{\infty}(n(n-1))4^n X^n - 4X + \frac{4X}{(1-4X)^2} \\
&= a_0 + (a_1 - 5a_0)X + \frac{32X^2}{(1-4X)^3} - 4X + \frac{4X}{(1-4X)^2} \\
\implies A(X) &= \frac{a_0 + (a_1 - 5a_0)X - 4X}{(1-5X+6X^2)} + \frac{32X^2}{(1-5X+6X^2)(1-4X)^3} + \frac{4X}{(1-5X+6X^2)(1-4X)^2} \\
&= \frac{a_0 + (a_1 - 5a_0)X - 4X}{(1-2X)(1-3X)} + \frac{32X^2}{(1-2X)(1-3X)(1-4X)^3} + \frac{4X}{(1-2X)(1-3X)(1-4X)^2} \\
\\
&\mathit{Applying\ partial\ fraction\ decomposition:} \\
&= \frac{C_1}{1-2X} + \frac{C_2}{1-3X} + \frac{C_3}{1-4X} + \frac{C_4}{(1-4X)^2} + \frac{C_5}{(1-4X)^3} \\
&= C_1 \sum_{n=0}^{\infty} 2^n X^n + C_2 \sum_{n=0}^{\infty} 3^n X^n + C_3 \sum_{n=0}^{\infty} 4^n X^n + C_4 \sum_{n=0}^{\infty} \binom{2-1+n}{n} 4^n X^n + C_5 \sum_{n=0}^{\infty} \binom{3-1+n}{n} 4^n X^n \\
&= \sum_{n=0}^{\infty} \left[C_1 2^n + C_2 3^n + \left(C_3 + C_4(n+1) + C_5 \frac{(2+n)(1+n)}{2} \right) 4^n \right] X^n \\
&= \sum_{n=0}^{\infty} \left[C_1 2^n + C_2 3^n + \left(C_3 + C_4 n + C_4 + C_5 \frac{n^2+3n+2}{2} \right) 4^n \right] X^n \\
&= \sum_{n=0}^{\infty} \left[C_1 2^n + C_2 3^n + \left(D_1 + D_2 n + D_3 n^2 \right) 4^n \right] X^n \\
&= \sum_{n=0}^{\infty} a_n X^n
\end{aligned}
$$

Hence,

$$
a_n = C_1 2^n + C_2 3^n + \left(D_1 + D_2 n + D_3 n^2 \right) 4^n
$$