# ASSIGNMENT 17

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Discrete Mathematics

**Q1(a):** Prove using $PIE$, if $A, B$ and $C$ are finite sets, then

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

**Solution:**

$$\begin{gathered}
= |A_1 \cup A_2 \cup A_3| \\
= |(A_1 \cup A_2) \cup A_3| \\
= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3| \\
= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)| \\
= |A_1| + |A_2| - |A_1 \cap A_2| + A_3 - [|A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|] \\
\implies |A_1 \ + |A_2| + A_3| - |A_1 \cap A2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\end{gathered}$$

**Q1(b):** Prove using $PIE$, if $A_1, A_2 \dots$ An are finite sets, then,

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \ \cdots\cdots \ + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

**Solution:** We can take,

$$P(n) : |\bigcup_{i=0}^{n} A_i| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \ \cdots\cdots \ + (-1)^{n-1} |\bigcap_{i=0}^{n} A_i|$$

**Base Case:** For $n = 2$, we have, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, by PIE which implies$P(2)$.

**Taking the inductive step here:** We can assume that $P(r)$:

$$|\bigcup_{i=0}^{r} A_i| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \ \cdots\cdots \ + (-1)^{r-1} |\bigcap_{i=0}^{r} A_i|$$

As $P(2)$, we have:

$$\begin{aligned}
|\bigcup_{i=0}^{r+1} A_i| &= |(\bigcup_{i=0}^{r} A_i) \bigcup A_{r+1}| \\
&= |\bigcup_{i=0}^{r} A_i| + |A_{r+1}| - |(\bigcup_{i=0}^{r} A_i) \bigcap A_{r+1}|
\end{aligned}$$

Consider $|(\bigcup_{i=0}^{r} A_i) \bigcap A_{r+1}|$

This can also be written as:
$|\bigcup_{i=0}^{r} (A_i \cap A_{r+1})|$

using inductive method:

$$|\bigcup_{i=0}^{r} (A_i \cap A_{r+1})| = \sum_{i=1}^{r} (A_i \cap A_{r+1}) \qquad - \sum_{1 \le i < j \le k} (A_i \cap A_j \cap A_{r+1}) + \ \cdots + (-1)^{r-1} (\bigcap_{i=1}^{r} A_i \cap A_{r+1})$$

Therefore, we have:

$$\begin{aligned}
\left|\bigcup_{i=0}^{r+1} A_i\right| &= |A_{r+1}| + \left[\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,m} |A_i \cap A_j \cap A_m| + \ \ . \ . \ . \ + (-1)^{r-1} \left|\bigcap_{i=0}^{r} A_i\right|\right] \\
&- \left[\sum_{i=1}^{r} (A_i \cap A_{r+1}) - \sum_{1 \le i < j \le k} (A_i \cap A_j \cap A_{r+1}) + ... + (-1)^{r-1} (\bigcap_{i=1}^{r} A_i \cap A_{r+1})\right] \\
&= \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \ \ . \ . \ . \ + (-1)^r |\bigcap_{i=0}^{r+1} A_i| \implies P(r+1)
\end{aligned}$$

**Q2(a):** Among the permutations of $\{1, 2, \cdots, n\}$, there are some called de-arangements, in which none of the $n$ integers appears in its natural place. Thus, $(i_1, i_2, ..., i_n)$ is a de-rangement if $i_1 \neq 1$, $i_2 \neq 2$,..., and $i_n \neq n$.Let $D_n$ be the number of de-rangements of $\{1, 2, ..., n\}$.Prove that

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + \frac{(-1)^n}{n!}\right]$$

**Solution:** Total number of permutations of $n$ objects is $n!$.
Let us assume, there are $N$ number of ways of arranging the objects in such a way that atleast one object goes to its right.
So, number of de-arrangerment will be:

$$D_n = n! - N$$

Let $A_r$ be the set of permutations in which the $r^{th}$ object goes into its right position. Then,

$$N = \left|\bigcup_{r=1}^{n} A_r\right| \qquad = \left[\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \quad \cdots\cdots \quad + (-1)^n \left|\bigcap_{i=0}^{n} A_i\right|\ \right]$$

Observe that $|A_i| = n - 1!$, $|A_i \cap A_j| = n - 2!$, and so on.
When we are ficing $k$ elemets, it requires permuting of the remaining terms. So,

$$\sum_{i_1,\ldots,i_k} |(A_{i_1} \cap A_{i_2} \cap .\ .\ . \cap A_{i_k})| = \binom{n}{k}(n-k)!$$

So the expression for $D_n$ reduces to,

$$D_n = n! - \left[\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \ .\ .\ .\ + (-1)^n\binom{n}{n}(n-n)!\right] \implies D_n \ = \sum_{i=0}^{n}\binom{n}{i}(n-i)!$$

$$\implies D_n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\ (n-i)! \ = \ n!\sum_{i=0}^{n}\frac{1}{i!}$$

$$\implies D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ .\ .\ .\ + \frac{(-1)^n}{n!}\right]$$