ASSIGNMENT 1

Neeraj Pandey

Discrete Mathematics

Q1. Show that $2^n > n^5 + 1$ for all $n \geq 23$

Solution: Let's take the base case as $n = 23$

Base Case: $2^{23} - (23^5 + 1) = 1952264 > 0 \implies P(23)$
Inductive Step: Assume $P(n = k)$. So,

$$2^k > k^5 + 1, \qquad \forall\ k \geq 23 \tag{1}$$

Let's take the inequality to check if it's true or not,

$$2k^5 > (k+1)^5$$
$$\implies 2 > (1 + \frac{1}{k})^5$$

Put $k = 23$ in the above inequality we have,

$$(1 + \frac{1}{23})^5 \approx 1.2371 < 2$$
$$\implies (1 + \frac{1}{k})^5 < 2$$
$$\implies (k+1)^5 < 2k^5, \quad \forall\ k \geq 23$$

$$\tag{2}$$

Now take 2^{k+1},

$$\begin{aligned} 2^{k+1} = 2(2^k) &> 2(k^5 + 1), from \quad (1) \\ &> 2(k^5) + 1 \\ &> (k+1)^5 + 1, from \quad (2) \\ \implies & P(n = k+1) \end{aligned}$$

Therefore, by *Induction* method $P(n)$ is true for all $n \geq 23$

Q2. A Fibonacci sequence F_n satisfies the recurrence relation $F_n = F_n - 1 + F_n - 2$ for $n \geq 3$, where $F_1 = 1$ and $F_2 = 1$. Prove by mathematical induction that for all $n \geq 1$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

.

Solution:

Base Case:

$$F_1 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right\}$$

$$= \left(\frac{2\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} = 1 \implies P(1)$$

Inductive Step:

Assume $P(n = 1), P(n = 2).........P(n = k - 1), P(n = k)$

Now,

$$F_{k+1} = F_k + Fk - 1$$

or,

$$F_{k+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{1 + \frac{1+\sqrt{5}}{2}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{1 + \frac{1-\sqrt{5}}{2}\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1}{2}\left(\frac{6+2\sqrt{5}}{2}\right)\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1}{2}\left(\frac{6-2\sqrt{5}}{2}\right)\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2\right) - \left(\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right)\right\}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

$$\implies P(K = n + 1)$$

$$Hence, proved.$$