Introduction to the Linear Algebra

About Data

- What is your Weight?
  - Answer is a Real Number (Scalar)
- What is the Date?
  - Represented by Three Numbers [*Date* , *Month* , *Year*]
  - Can be called n-Tuple (n=3)
  - The Order Matters!
- Temporally Ordered Audio Signals
- Data in Tables or Spreadsheets
- Monochrome or Color (Digital) Images
- Digital Video Data

Data Types & Indices

x, y, z : Scalar Data

$\{x_1, \ldots x_i, \ldots x_{\mathrm{n}}\}$: Indexed Set of Numbers

$\{x_1, x_2, x_3\} \neq \{x_2, x_1, x_3\}$: Order Matters

$(1,3,2018) \neq (3,1,2018)$: Date

Data Types & Indices

$\{x_t; t = 1, \ldots\}$
Temporally Indexed
Scalar Data

Image as Spatially
Indexed (x, y) array
of Pixels

Video or Frame Sequence as
Spatio-Temporally Indexed
(x, y, t) Data

Vector Space

A Real/Complex Vector Space is a Set $\boldsymbol{V}$ with a Special Element $\mathbf{0}$

- Addition: IF $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, THEN $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{V}$
- Inverse: IF $\boldsymbol{x} \in \boldsymbol{V}$, THEN $\exists \boldsymbol{y} \in \boldsymbol{V}$ SUCH THAT $\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$
- Scalar Multiplication: IF $\boldsymbol{x} \in \boldsymbol{V}$, THEN $\mathrm{c}\boldsymbol{x} \in \boldsymbol{V}$ where $\boldsymbol{c}$ is Scalar

Additive Axioms

For every $x, y, z \in V$

- $x + y = y + x$ (Commutative Property)
- $(x + y) + z = x + (y + z)$ (Associative Property)
- $0 + x = x + 0 = x$ (Identity)
- $(-x) + x = x + (-x) = 0$ (Inverse)

Multiplicative Axioms

For every $\boldsymbol{x} \in \boldsymbol{V}$ and Real/Complex Number $\boldsymbol{c}, \boldsymbol{d}$

- $\mathbf{0} \cdot \boldsymbol{x} = \mathbf{0}$
- $\mathbf{1} \cdot \boldsymbol{x} = \boldsymbol{x}$
- $\boldsymbol{c}(\boldsymbol{d}\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{d}(\boldsymbol{c}\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{d}\boldsymbol{x}$

Distributive Axioms

For every $\boldsymbol{x} \in \boldsymbol{V}$ and Real/Complex Number $\boldsymbol{c}, \boldsymbol{d}$

- $\boldsymbol{c(x+y) = cx + cy}$
- $\boldsymbol{(c+d)x = cx + dx}$

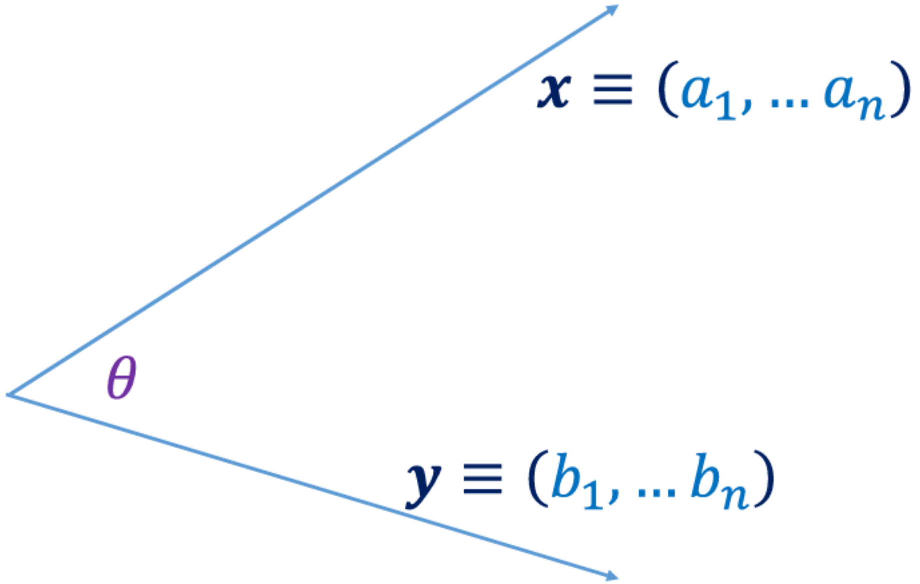
Norm of a Vector

Alternatively Length or Magnitude of a Vector

l^p Norm $\quad \|\boldsymbol{x}\|_p = \left(\sum_{r=1}^{n} |x[r]|^p\right)^{\frac{1}{p}}$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \Rightarrow \|\boldsymbol{x}\|_2 = \sqrt{|1|^2 + |-2|^2 + |4|^2} = 4.583$$

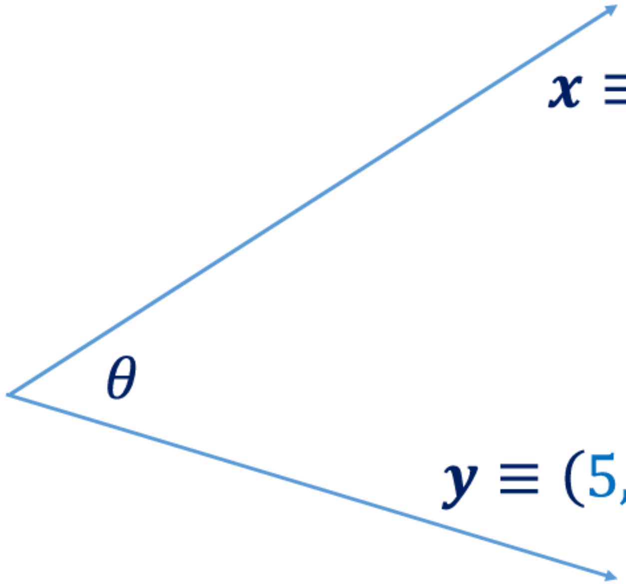
Inner Product



$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos \boldsymbol{\theta}$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} a_i b_i$$

Inner Product



$\boldsymbol{x} \equiv (1, -1, 2)$

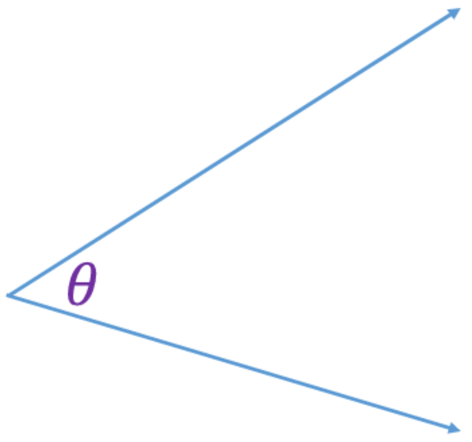
$\boldsymbol{y} \equiv (5, 2, 7)$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = ?$$

$$\theta = ?$$

Inner Product

$$\boldsymbol{x} \equiv (1, -1, 2)$$



$$\boldsymbol{y} \equiv (5, 2, 7)$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i = 17$$

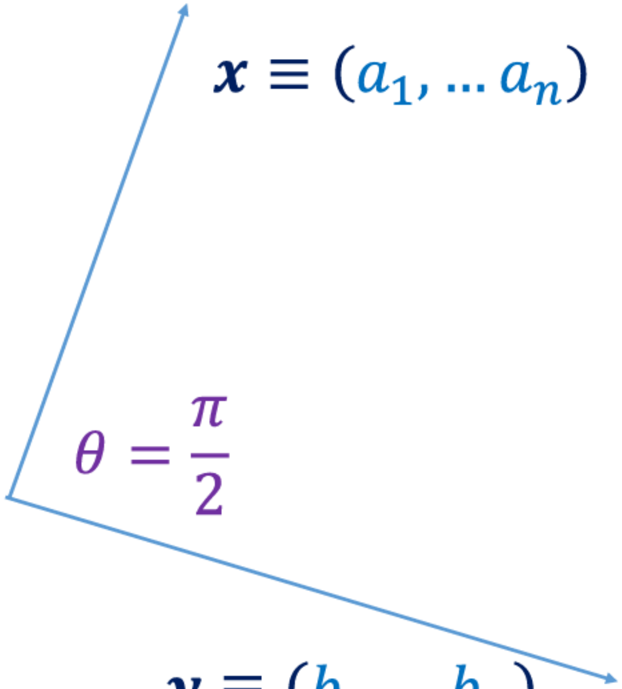
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos \boldsymbol{\theta}$$

$$\cos \boldsymbol{\theta} = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2}$$

$$\theta = cos^{-1}\left(\frac{17}{2.45 \times 8.83}\right)$$

$$\theta = 38^o$$

Orthogonal Vectors



$$\boldsymbol{x} \equiv (a_1, \ldots a_n)$$

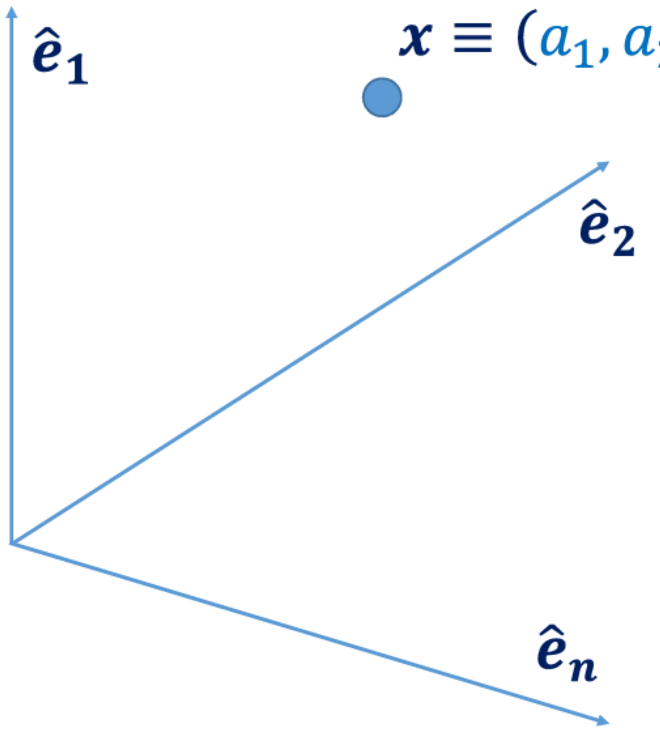
$$\theta = \frac{\pi}{2}$$

$$\boldsymbol{y} \equiv (b_1, \ldots b_n)$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} a_i b_i = 0$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos \frac{\pi}{2} = 0$$

Basis Vectors



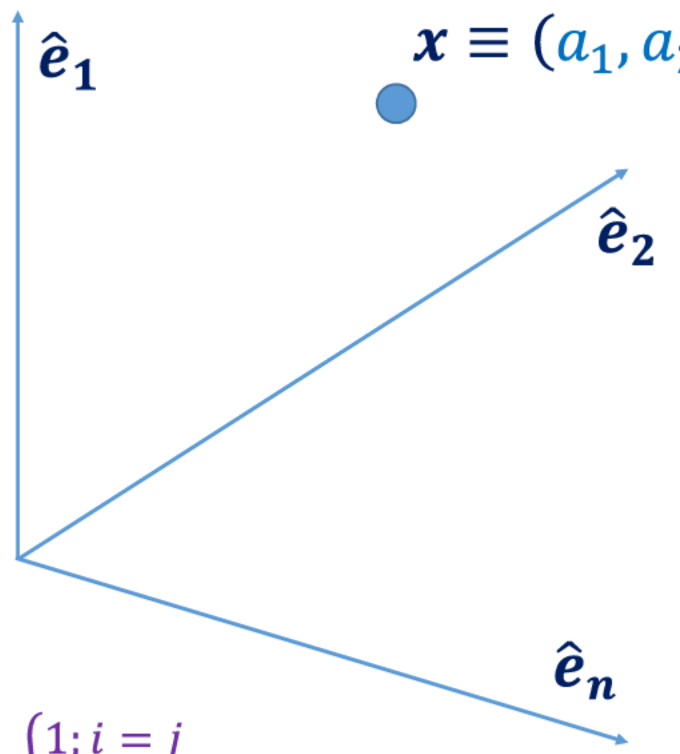
$$\boldsymbol{x} \equiv (a_1, a_2, \ldots a_n)$$

$$\|\hat{\boldsymbol{e}}_{\boldsymbol{i}}\|_2 = 1$$

$$\boldsymbol{x} = a_1\hat{\boldsymbol{e}}_{\mathbf{1}} + a_2\hat{\boldsymbol{e}}_{\mathbf{2}} + \cdots + a_n\hat{\boldsymbol{e}}_{\boldsymbol{n}}$$

Orthogonal Basis Vectors

$$\boldsymbol{x} \equiv (a_1, a_2, \ldots a_n)$$



$$\|\hat{\boldsymbol{e}}_i\|_2 = 1$$

$$\langle \hat{\boldsymbol{e}}_i, \hat{\boldsymbol{e}}_j \rangle = \delta[i-j]$$

$$a_i = \langle \boldsymbol{x}, \hat{\boldsymbol{e}}_i \rangle$$

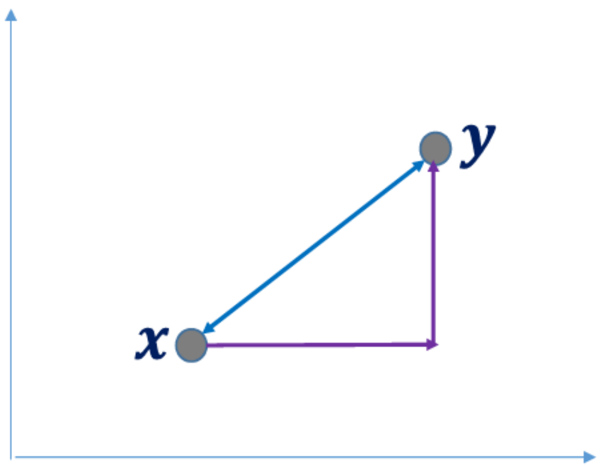
$$\boldsymbol{x} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + \cdots + a_n\hat{\boldsymbol{e}}_n$$

$$\delta[i-j] = \begin{cases} 1; i = j \\ 0; i \neq j \end{cases}$$

Distance between Two Vectors

- Required for Classification Problems
- Metric on $\boldsymbol{V}$ is a Function $\boldsymbol{d}: \boldsymbol{V} \times \boldsymbol{V} \rightarrow [\mathbf{0}, \infty)$
- Function $d(\boldsymbol{x}, \boldsymbol{y})$ satisfies the following for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$
  - $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \geq 0$
  - $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0} \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$
  - $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x})$
  - $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{z}) + \boldsymbol{d}(\boldsymbol{y}, \boldsymbol{z})$

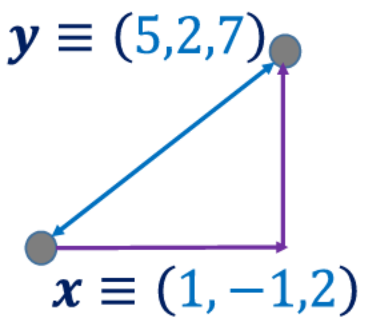
Euclidean and Manhattan Distance



$$d_E^2(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n} (\boldsymbol{x}_i - \boldsymbol{y}_i)^2$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n} |\boldsymbol{x}_i - \boldsymbol{y}_i|$$

Euclidean and Manhattan Distance



$$d_E^2(\boldsymbol{x},\boldsymbol{y}) = (1-5)^2 + (-1-2)^2 + (2-7)^2$$

$$d_E^2(\boldsymbol{x},\boldsymbol{y}) = 16 + 9 + 25 = 50$$

$$d_E(\boldsymbol{x},\boldsymbol{y}) = 7.07$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = |1-5| + |-1-2| + |2-7|$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = 4 + 3 + 5$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = 12$$

Weighted Euclidean Distance

In Practical Applications, we want to Scale the Dimensions

$$d^2(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} (\boldsymbol{x}_i - \boldsymbol{y}_i)^2$$

$$d^2(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} s_i (\boldsymbol{x}_i - \boldsymbol{y}_i)^2$$

Matrix

A Matrix is a Two Dimensional Arrangement of Numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Square Matrix

$$m = n$$

Row Vector

$$A(i,:) = [a_{i1} \quad \cdots \quad a_{in}]$$

Column Vector

$$A(:,j) = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Transpose of Matrix A

Symmetric Matrix: $A = A^T$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 11 & 12 & 13 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 11 \\ 2 & 12 \\ 3 & 13 \end{bmatrix}$

Matrix Transpose

$$A = \begin{bmatrix} 4 & -8 \\ -3 & 0 \\ 11 & -5 \end{bmatrix} \qquad A^T = ?$$

Matrix Transpose

$$A = \begin{bmatrix} 4 & -8 \\ -3 & 0 \\ 11 & -5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & -3 & 11 \\ -8 & 0 & -5 \end{bmatrix}$$

Matrix Transpose

How will you Generate a Random Symmetric Matrix?

Symmetric Matrix: $A = A^T$

Matrix Transpose

Generate Random Matrix: A

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 5 & -7 \\ -9 & 10 & 4 \end{bmatrix}$$

Evaluate: A^T

$$A^T = \begin{bmatrix} -1 & 2 & -9 \\ 3 & 5 & 10 \\ 0 & -7 & 4 \end{bmatrix}$$

Compute: $\mathrm{B} = \mathrm{A} + A^T$

$$B = \begin{bmatrix} -2 & 5 & -9 \\ 5 & 10 & 3 \\ -9 & 3 & 8 \end{bmatrix}$$

Check: $\mathrm{B} = B^T$

Matrix Operations: Scalar Multiplication

Multiplication of Matrix A and a Scalar $c \in \mathbb{R}^1$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$c \cdot A = A \cdot c = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Matrix Operations: Addition

Matrices A and B are of the same size $m \times n$, and $c, d \in \mathbb{R}^1$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$c \cdot A + d \cdot B = \begin{bmatrix} ca_{11} + db_{11} & ca_{12} + db_{12} & \cdots & ca_{1n} + db_{1n} \\ ca_{21} + db_{21} & ca_{22} + db_{22} & \cdots & ca_{2n} + db_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} + db_{m1} & ca_{m2} + db_{m2} & \cdots & ca_{mn} + db_{mn} \end{bmatrix}$$

Matrix Operations: Addition

$$A = \begin{bmatrix} 1 & 5 & 2 \\ -2 & 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 & 7 \\ -1 & -2 & 6 \end{bmatrix}$$

Evaluate $\mathrm{C} = 2A - 3B$

Matrix Operations: Addition

$$C = 2 * A + (-3) * B$$

$$C = (2) * \begin{bmatrix} 1 & 5 & 2 \\ -2 & 3 & 7 \end{bmatrix} + (-3) * \begin{bmatrix} 2 & -3 & 7 \\ -1 & -2 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 10 & 4 \\ -4 & 6 & 14 \end{bmatrix} + \begin{bmatrix} -6 & 9 & -21 \\ 3 & 6 & -18 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 19 & -17 \\ -1 & 12 & -4 \end{bmatrix}$$

Matrix as Linear Operator

$A \in \mathbb{R}^{m \times n}$

$y \in \mathbb{R}^m \longrightarrow$ $$y = Ax$$

$x \in \mathbb{R}^n$

$A: \mathbb{R}^n \to \mathbb{R}^m$

$$\boldsymbol{y} = A(\alpha \boldsymbol{x}_1 + \beta \boldsymbol{x}_2) = \alpha(A\boldsymbol{x}_1) + \beta(A\boldsymbol{x}_2)$$

Matrix Operations: Matrix-Vector Multiplication

Matrix A of size $m \times n$ and Column Vector $\boldsymbol{x}$ of size $n \times 1$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = A\boldsymbol{x}$$

$$y_i = \sum_{j=1}^{n} a_{ij} \times x_j$$

Matrix Operations: Matrix-Vector Multiplication

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 7 & -1 & 2 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{y} = A\boldsymbol{x} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) + (-3) \times 2 \\ 7 \times 1 + (-1) \times (-1) + 2 \times 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \end{bmatrix}$$

Matrix Operations: Multiplication

Matrices A and B are of respective sizes $m \times n$ and $n \times p$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

Matrix $\mathrm{C} = A \times B$ is of size $m \times p$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \times b_{kj}$$

Matrix Operations: Multiplication

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 7 & -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 5 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$C = A * B = ?$$

Matrix Operations: Multiplication

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 7 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 5 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$C = A * B$$

$$C = \begin{bmatrix} 1 \times (-2) + 2 \times 2 + (-3) \times 1 & 1 \times 5 + 2 \times (-1) + (-3) \times 3 \\ 7 \times (-2) + (-1) \times 2 + 2 \times 1 & 7 \times 5 + (-1) \times (-1) + 2 \times 3 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -6 \\ -14 & 42 \end{bmatrix}$$

Matrix Operations: Multiplication

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

$$\boldsymbol{x}^T \boldsymbol{y} = ? \qquad \boldsymbol{x}\boldsymbol{y}^T = ?$$

Matrix Operations: Multiplication

$$\boldsymbol{x}^T\boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} = 17$$

$$\boldsymbol{x}\boldsymbol{y}^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ -5 & -2 & -7 \\ 10 & 4 & 14 \end{bmatrix}$$

Matrix Operations: Multiplication

In General $\mathrm{AB} \neq BA$, might be possible in special cases

Associativity: $\mathrm{A(BC)} = (AB)C$

Distributivity: $\mathrm{A(B + C)} = AB + AC$ and $(\mathrm{A + B})C = AC + BC$

Transpose of Product: $(AB)^T = B^T A^T$ and $\left(\prod_{i=1}^{n} A_i\right)^T = \prod_{j=0}^{n-1} A_{n-j}^T$

Multiplication Rules Must be Strictly Followed

Matrix Operations: Determinant

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(B) = |B| = ad - bc$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Minors

$$\det(A) = a \times det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \times det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \times det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Minor Expansion Formula

Matrix Operations: Determinant

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \qquad |A| = ?$$

Matrix Operations: Determinant

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

$$|A| = 1 \times \begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix} - (-2) \times \begin{bmatrix} 7 & 2 \\ -1 & 0 \end{bmatrix} + 5 \times \begin{bmatrix} 7 & -3 \\ -1 & 2 \end{bmatrix}$$

$$|A| = 1 \times \{(-3 \times 0) - (2 \times 2)\} + 2 \times \{(7 \times 0) - (-1 \times 2)\} + 5 \times \{(7 \times 2) - (-1 \times -3)\}$$

$$|A| = -4 + 4 + 55 = 55$$

Matrix Operations: Inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Cofactor Matrix of A

Minor of A_{ij}

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

$$adj(A) = \boldsymbol{C}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Matrix Operations: Inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A \times adj(A) = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = |A| \times I_2$$

$$A^{-1} \times A \times adj(A) = |A| \times I_2 \times A^{-1}$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

Matrix Operations: Inverse

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \qquad A^{-1} = ?$$

Matrix Operations: Inverse – Cofactor Matrices

$$M_{11} = \begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} 7 & 2 \\ -1 & 0 \end{bmatrix} \qquad M_{13} = \begin{bmatrix} 7 & -3 \\ -1 & 2 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} -2 & 5 \\ 2 & 0 \end{bmatrix} \qquad M_{22} = \begin{bmatrix} 1 & 5 \\ -1 & 0 \end{bmatrix} \qquad M_{23} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} -2 & 5 \\ -3 & 2 \end{bmatrix} \qquad M_{32} = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} \qquad M_{33} = \begin{bmatrix} 1 & -2 \\ 7 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

Matrix Operations: Inverse – Cofactor Matrices

$M_{11} = \begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix} = -4$ $M_{12} = \begin{bmatrix} 7 & 2 \\ -1 & 0 \end{bmatrix} = 2$ $M_{13} = \begin{bmatrix} 7 & -3 \\ -1 & 2 \end{bmatrix} = 11$

$M_{21} = \begin{bmatrix} -2 & 5 \\ 2 & 0 \end{bmatrix} = -10$ $M_{22} = \begin{bmatrix} 1 & 5 \\ -1 & 0 \end{bmatrix} = 5$ $M_{23} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = 0$

$M_{31} = \begin{bmatrix} -2 & 5 \\ -3 & 2 \end{bmatrix} = 11$ $M_{32} = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} = -33$ $M_{33} = \begin{bmatrix} 1 & -2 \\ 7 & -3 \end{bmatrix} = 11$

$A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ $C_{ij} = (-1)^{i+j}|M_{ij}|$ $C = \begin{bmatrix} -4 & -2 & 11 \\ 10 & 5 & 0 \\ 11 & 33 & 11 \end{bmatrix}$

Matrix Operations: Inverse – Adjoint Matrix

$$C = \begin{bmatrix} -4 & -2 & 11 \\ 10 & 5 & 0 \\ 11 & 33 & 11 \end{bmatrix}$$

$$adj(A) = \boldsymbol{C}^T$$

$$adj(A) = \begin{bmatrix} -4 & 10 & 11 \\ -2 & 5 & 33 \\ 11 & 0 & 11 \end{bmatrix}$$

Matrix Operations: Inverse

$$adj(A) = \begin{bmatrix} -4 & 10 & 11 \\ -2 & 5 & 33 \\ 11 & 0 & 11 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$A^{-1} = \frac{1}{55}\begin{bmatrix} -4 & 10 & 11 \\ -2 & 5 & 33 \\ 11 & 0 & 11 \end{bmatrix} = \begin{bmatrix} -0.073 & 0.182 & 0.2 \\ -0.036 & 0.091 & 0.6 \\ 0.2 & 0 & 0.2 \end{bmatrix}$$

Equation Solving

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$

n-Variables , n-Equations

$$a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n$$

$$Ax = b$$

$$x = A^{-1}b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Equation Solving

$$x - 2y + 5z = 2$$

$$7x - 3y + 2z = 5$$

$$2y - x = 3$$

$$x, y, z = ?$$

Equation Solving

$$x - 2y + 5z = 2$$
$$7x - 3y + 2z = 5$$
$$2y - x = 3$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 5 \\ 7 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}}_{B}$$

$$AP = B \Rightarrow P = A^{-1}B$$

$$P = \begin{bmatrix} -0.073 & 0.182 & 0.2 \\ -0.036 & 0.091 & 0.6 \\ 0.2 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.364 \\ 2.182 \\ 1.000 \end{bmatrix}$$