THE PHYSICS OF ASTROPHYSICS

Volume I

RADIATION

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To Helen

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Preface

Modern astronomers need to know a lot of physics. Unfortunately, the typical curriculum in most graduate astronomy departments leaves little time for the student to learn this material through formal course work. Ideally, while in graduate school, a prospective astronomer with a thorough preparation in undergraduate physics and mathematics would take a two-semester graduate sequence in quantum mechanics (leading up to Dirac's equation), a two-semester sequence in classical electrodynamics (including multipole radiation and the theory of special relativity), a semester of statistical mechanics (including Gibbs's ensembles and some applications in quantum statistical mechanics), a semester or more of gas dynamics and plasma physics (including shockwave theory and magnetohydrodynamics), and a year or more of advanced mathematical methods (including the theory of complex variables, ordinary and partial differential equations, numerical analysis, Fourier analysis, and various asymptotic approximation techniques). Unfortunately, such a program would take the budding professional astronomer through the first two years of graduate school without time for a single course in astronomy!

To cope with this dilemma, teachers of graduate courses in astronomy have traditionally included the relevant physics and mathematics as part of the background lecture material. This compromise, however, is very inefficient (e.g., there is a lot of repetition of subject matter such as radiative transfer or atomic and molecular physics) during an era when a veritable explosion of phenomenological knowledge increasingly demands more lecture time for exposure to the frontiers of the field.

At many universities this challenge has given birth to one or more basic courses in the physics prerequisite to many different subject areas in astronomy. I have taught such courses at the State University of New York at Stony Brook (1968–1973) and the University of California at Berkeley (1973–present). Innovations urged by J. Arons have caused the format at Berkeley to evolve from a single-semester course on "astrophyical processes" to the present format of a year-long sequence—the first semester of which deals with "radiation processes" and the second with "gas dynamics and magnetohydrodynamics."

This two-volume text on *The Physics of Astrophysics* grew from my lecture notes for the reorganized two-semester sequence. It is aimed at first-year graduate students and well-prepared seniors in astronomy and physics. The first volume deals with *Radiation*, the second with *Gas Dynamics*. The excellent text by G. Rybicki and A. Lightman on *Radiative Processes in Astrophysics* (Wiley) was my model for the first half of the sequence, but my presentation of the basic material pays more attention to low-energy phenomena (e.g., radio astronomy) and statistical astrophysics (e.g., rate equations).

Although these two volumes were written to form a coherent whole, they can be decoupled for use in separate courses. In particular, when I teach the two-course sequence, the first is not a prerequisite for the second. In writing both volumes, I have been guided by the following pedagogical assumptions and philosophy.

A. Although my discussion emphasizes processes rather than objects, so that the topics are arranged in a sequence formed more by the tradition of physics than by that of astronomy, I usually try to motivate the development by using concrete examples from astrophysics. When faced with a choice between abstract principles or practical applications, I always opt for the practical approach.

B. I have tried to make explanations detailed enough to indicate the important points in the reasoning, but I refrain from displaying every step of a derivation, in order not to divert the attention of the reader from more serious matters. The student may wish to read straight through each chapter to get the central thrust of the physical ideas, and then—as an aid to long-term memory and detailed understanding—return with paper and pencil to work out the missing steps in the formal mathematics.

C. For beginning students, who may not know in which subfields they wish to specialize, I believe it better to cover a lot of ground coherently than to delve deeply into any particular subject. Astronomers of the future will need tools that allow them to explore in many different directions.

D. Along the same lines, I prefer to assign long problems that require a sustained attack on a practical astronomical situation than to contrive short problems that require only a few simple steps to demonstrate a limited objective. I hope I have supplied enough hints along the way so that the student does not become frustrated by getting stuck in the midst of an extended calculation.

This first volume nominally has as its subject matter the emission, absorption, and scattering of radiation by matter, but it includes, in fact, many related topics as well: radiative transfer, statistical physics, classical electrodynamics, atomic and molecular structure. To use the book effectively, the student should have had: some atomic physics and kinetic theory, with exposure to the concepts of the mean-free path and the equilibrium thermodynamic distributions of matter and radiation; a course in quantum mechanics in which the Schrödinger equation is solved for the structure of the hydrogen atom, but not necessarily including the effects of electron spin; electrodynamics with Maxwell's equations expressed in differential form; special relativity with some exposure to the notion of four-vectors and Lorentz transformations; and classical mechanics at a level that uses the Lagrangian and Hamiltonian formulations of the subject. Gaps in a few of these subject areas will not prove fatal, but the student should then be prepared to use heavily the reference material listed at the beginning of each chapter.

This book adopts boldface symbols, e.g., $\mathbf{A}$, for vectors in ordinary three-space; in addition, we use the notation P and Q to represent pressure and quadrupole tensors (of dimension 3×3). We denote quantities having four-components by arrows, e.g., $\vec{u}$ for the four-velocity of special relativity. Double-arrowed characters mean 4×4 tensors or matrices, e.g., $\overleftrightarrow{\mathsf{R}}$ for the scattering matrix involving the radiative transfer of the Stokes parameters. To avoid overcomplicating the print face, we forego the arrow convention for variables that refer to dimensions in an internal space, e.g., Dirac matrices and spinors.

Vector calculus in three spatial dimensions constitutes a mathematical tool used freely throughout the book. Apart from the standard theorems of Green, Stokes, etc., the student should be conversant with the identity involving the triple vector product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, and the mnemonic that the minus sign accompanies the dot product that involves the vector in the middle. I also assume that he or she knows the Cartesian-tensor analogs for this expression: $\epsilon_{ikm}\epsilon_{mj\ell} = \delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{kj}$, but I do not use the latter formula very much. The reader should also know that the dot and the cross can be interchanged in the triple scalar product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$. Finally, I assume that the student knows how to modify these expressions if one or more of the quantities $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ represents the del operator $\boldsymbol{\nabla}$, and that the direct product of two vectors, as expressed by the dyadic $\mathbf{AB}$, yields a second-rank tensor.

References to journal articles and review papers occur in the text where they appear (usually for the first and last time). Citations of books, to which the reader may want to refer in depth, are given in full in the bibliography.

PART I

RADIATIVE TRANSFER AND STATISTICAL MECHANICS

1

Specific Intensity and the Equation of Radiative Transfer

Reference: Chandrasekhar, *Radiative Transfer*, Chapter 1.

Almost everything we know about the astronomical universe derives from the laborious gathering of light from faint celestial sources. The information encoded in electromagnetic radiation from the cosmos cannot be deciphered unless we know how to read the message. This book contains the fundamentals of the physics needed to analyze electromagnetic radiation from astronomical sources. As outlined in the Table of Contents, this volume separates into three related parts. Our discussion in Part I begins with the different concepts involved in a statistical treatment of the transport of radiation: *rays* when freely propagating and *waves* or *photons* when interacting with matter. We assume that the reader has some familiarity with the validity of the classical description of radiation as electromagnetic waves if many photons are involved (Part II). We also assume that he or she knows that a quantum treatment is warranted if the interaction with matter takes place one photon at a time (Part III).

DEFINITION OF SPECIFIC INTENSITY

Our definition of specific intensity follows from consideration of Figure 1.1. Let dE be the amount of radiant energy which crosses in time dt the area dA with unit normal $\hat{\mathbf{n}}$ in a direction within solid angle $d\Omega$ centered about $\hat{\mathbf{k}}$ and with photon frequency between ν and $\nu + d\nu$. The monochromatic specific intensity I_ν is then defined by the equation:

$$\text{energy}: \qquad dE = I_\nu(\hat{\mathbf{k}}, \mathbf{x}, t)\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, dA\, d\Omega\, d\nu\, dt. \tag{1.1}$$

The quantity I_ν has the cgs units, $[I_\nu] = \text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{steradian}^{-1}$. The rationale for this definition comes from the conservation of I_ν in the absence of interactions with matter.

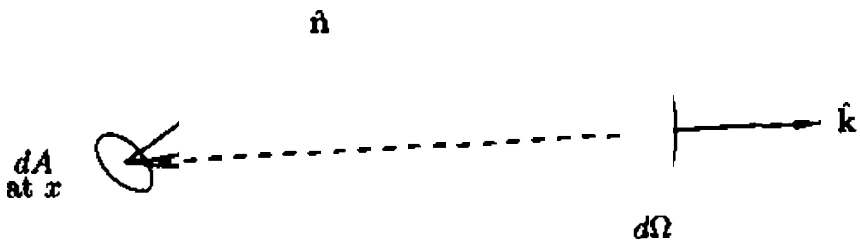

FIGURE 1.1
Definition of specific intensity $I_\nu(\hat{\mathbf{k}}, \mathbf{x}, t)$.

Proof: Consider the passage of radiant energy from element of area dA to dA' as depicted in Figure 1.2.

$$dE' = I_\nu(\hat{\mathbf{k}}, \mathbf{x}', t')\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}'\, dA'\, d\Omega'\, d\nu\, dt, \tag{1.2}$$

where

$$d\Omega = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}'\, dA'/s^2, \tag{1.3}$$

$$d\Omega' = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, dA/s^2, \tag{1.4}$$

with s being the distance along the ray path between the two elements of area,

$$\mathbf{x}' = \mathbf{x} + s\hat{\mathbf{k}}, \tag{1.5}$$

and t' is the delayed time allowing for a finite speed of propagation c,

$$t' \equiv t + s/c. \tag{1.6}$$

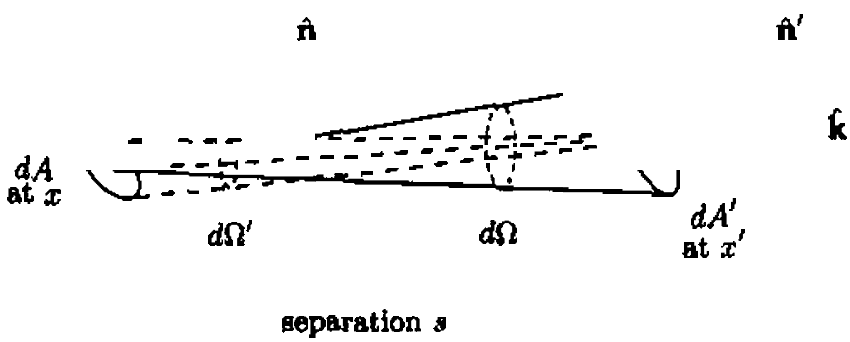

FIGURE 1.2
Proof of constancy of I_ν in a vacuum.

In the absence of interaction with matter, $dE' = dE$; substitution of equations (1.3) and (1.4) into equations (1.1) and (1.2) yields the desired result,

$$I_\nu(\hat{\mathbf{k}}, \mathbf{x}', t') = I_\nu(\hat{\mathbf{k}}, \mathbf{x}, t), \tag{1.7}$$

for propagation in a vacuum. When interactions do take place, we replace the above result with a transport equation of the form,

$$\frac{\partial I_\nu}{\partial t} + c\hat{\mathbf{k}} \cdot \nabla I_\nu = \text{sources} - \text{sinks}. \tag{1.8}$$

We shall shortly expand on what we mean by the right-hand side of equation (1.8), but in the interim, we find it useful to relate I_ν to some quantities that may be more familiar to physicists.

RELATIONSHIP TO PHOTON DISTRIBUTION FUNCTION AND OCCUPATION NUMBER

The photon distribution function $F_\alpha(\mathbf{x}, \mathbf{p}, t)$ is defined so that $F_\alpha(\mathbf{x}, \mathbf{p}, t)$ $d^3x\, d^3p$ = number of photons of spin state α at time t with position within d^3x centered on $\mathbf{x}$ and with momentum within d^3p centered on $\mathbf{p}$. The index α has only two values, 1 and 2, even though spin $s = 1$ has three possible projections along the direction of motion, $m_s = -1, 0, +1$, because the state $m_s = 0$ has no physical meaning for a particle traveling at the speed of light.

The vector momentum of a photon satisfies

$$\mathbf{p} = \hbar\mathbf{k} = (h\nu/c)\hat{\mathbf{k}}. \tag{1.9}$$

The radiant energy contained by all photons occupying the elemental phase-space volume $d^3x\, d^3p$ is

$$dE = \sum_{\alpha=1}^{2} h\nu F_\alpha(\mathbf{x}, \mathbf{p}, t)\, d^3x\, d^3p. \tag{1.10}$$

A beam of photons traveling for time dt in the direction $\hat{\mathbf{k}}$ through an element of area dA whose normal equals $\hat{\mathbf{n}}$ occupies a spatial volume

$$d^3x = (c\,dt)(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})\, dA.$$

If their momenta have a small spread of solid angle $d\Omega$ and magnitude dp, the corresponding element of momentum volume equals

$$d^3p = p^2\, d\Omega\, dp = \left(\frac{h^3\nu^2}{c^3}\right) d\Omega\, d\nu.$$

Comparison of equations (1.1) and (1.10) now yields the correspondence

$$I_\nu = \sum_{\alpha=1}^{2} \left(\frac{h^4 \nu^3}{c^2} \right) F_\alpha(\mathbf{x}, \mathbf{p}, t). \tag{1.11}$$

In quantum statistical mechanics, h^3 represents a fundamental unit of phase-space volume. We define the occupation number for each (manifested) photon spin state α as $\mathcal{N}_\alpha \equiv h^3 F_\alpha$. Equation (1.11) can now be written

$$I_\nu = \sum_{\alpha=1}^{2} \left(\frac{h \nu^3}{c^2} \right) \mathcal{N}_\alpha(\mathbf{x}, \mathbf{p}, t). \tag{1.12}$$

PROPERTIES UNDER THERMODYNAMIC EQUILIBRIUM

Toward the end of the nineteenth century Kirchhoff showed that (blackbody) radiation in thermodynamic equilibrium with matter has a distribution that depends on only the temperature T and on no other material properties of the enclosure.

Proof: Suppose I_ν differed for two enclosures at the same T over some frequency interval $\delta\nu$. Connect the enclosures with a filter that passes only radiation of that frequency interval. Radiant energy would now flow from one enclosure to another, making it possible to extract useful work from two reservoirs at the same temperature T, in violation of the second law of thermodynamics. Thus our assumption must be in error, and the monochromatic specific intensity of a radiation field in thermodynamic equilibrium with matter must be a universal function of T; in modern notation,

$$\text{Kirchhoff:} \qquad I_\nu = B_\nu(T). \tag{1.13}$$

Notice that blackbody radiation is isotropic and unpolarized.

Kirchhoff set a goal for future generations of finding the correct functional form of $B_\nu(T)$. Planck found the appropriate formula—first by interpolating (in entropy) between two forms (Rayleigh-Jeans and Wien laws) and next by coming up with a statistical argument using quantized oscillators to emit and absorb the radiation field. Both derivations gave

$$\text{Planck:} \qquad B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}. \tag{1.14}$$

Bose found a quantum-statistical method to treat an ideal gas of photons directly, and Einstein generalized Bose's treatment to indistinguishable material particles. Applied to a perfect gas of non-mutually interacting bosons (particles with integer spins), quantum statistical mechanics yields for the occupation number of each spin state at energy level ϵ:

$$\text{Bose-Einstein:} \qquad \mathcal{N}_\alpha = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}, \tag{1.15}$$

where μ is the chemical potential (Lagrange multiplier to assure number conservation). When photons are absorbed and emitted by matter (rather than simply scattered), as they must be to come into thermodynamic equilibrium with matter, number conservation becomes an irrelevant constraint. Thus the chemical potential μ of photons can be taken to be zero. Applied to photons, furthermore, $\epsilon = h\nu$; therefore the occupation number of blackbody photons of each spin state α at temperature T is given by

$$\text{Bose:} \qquad \mathcal{N}_\alpha = \frac{1}{e^{h\nu/kT} - 1}. \tag{1.16}$$

Substitution of the thermodynamic distribution (1.16) into equation (1.12), with $I_\nu = B_\nu(T)$, rederives equation (1.14).

Three aspects of the preceding discussion deserve comment. First, photons should behave like particles in the limit of high energies. Indeed, when $h\nu \gg kT$, equation (1.16) has the approximate form

$$\mathcal{N}_\alpha \approx e^{-h\nu/kT}$$

that Boltzmann found appropriate for material distributions.

Second, photons should behave like classical waves in the limit of large occupation numbers. From equation (1.16), we see that $\mathcal{N}_\alpha \gg 1$ when $h\nu \ll kT$, since in this limit, we may expand $e^{h\nu/kT} = 1 + h\nu/kT + \cdots$ to obtain

$$\mathcal{N}_\alpha \approx kT/h\nu \gg 1.$$

This case corresponds to the Rayleigh-Jeans limit, wherein the energy content, $h\nu\mathcal{N}_\alpha$, of photons of frequency ν within a phase-space volume h^3 is given by the equipartition theorem as kT ($kT/2$ each for vibrations of $\mathbf{E}$ and $\mathbf{B}$).

Third, equation (1.15) holds for material particles only if they are bosons. For fermions (particles of half-integer spin) satisfying the Pauli exclusion principle, quantum statistical mechanics yields

$$\text{Fermi-Dirac:} \qquad \mathcal{N}_\alpha = \frac{1}{e^{(\epsilon-\mu)/kT} + 1}. \tag{1.17}$$

Notice the $+1$ which replaces the -1 in the denominator. The $+1$ assures that no more than one fermion of a given spin state (or, more generally, of a given internal quantum state) can occupy a given translational state (of phase-space volume h^3). The difference between $+1$ for an ideal Fermi-Dirac gas and -1 for an ideal Bose-Einstein gas can be stated as follows:

fermions avoid being in a place where there are other fermions like themselves (with $\mathcal{N}_\alpha$ smaller than the classical value), whereas bosons like to be where there are other bosons like themselves (with $\mathcal{N}_\alpha$ larger than the classical value). As we shall see in a later chapter, this tendency is the basis for the phenomenon of the *stimulated emission* of photons (spin 1). In contrast, neutrinos (spin 1/2) are fermions—with, as far as we can tell, zero rest mass and zero chemical potential when they are copiously emitted and absorbed by hot dense matter. In thermodynamic equilibrium, neutrinos of energy $h\nu$ would have occupation number

$$\text{Neutrinos:} \qquad \mathcal{N}_\alpha = \frac{1}{e^{h\nu/kT}+1}, \tag{1.18}$$

and instead of suffering stimulated emission, they would exhibit the phenomenon of *suppressed emission*.

EQUATION OF RADIATIVE TRANSFER

Astronomers like to write the transport equation (1.8) per unit length rather than per unit time; so the equation of radiative transfer usually reads

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{k}}\cdot\nabla I_\nu = \frac{1}{4\pi}\rho j_\nu - \rho\kappa_\nu^{\text{abs}} I_\nu - \rho\kappa_\nu^{\text{sca}} I_\nu + \rho\kappa_\nu^{\text{sca}} \oint \phi_\nu(\hat{\mathbf{k}},\hat{\mathbf{k}}') I_\nu(\hat{\mathbf{k}}')\,d\Omega', \tag{1.19}$$

where ρ is the mass density per unit volume of the gas, j_ν is its emissivity per unit mass, κ_ν^{abs} is its total absorption opacity (absorption cross-section per unit mass), κ_ν^{sca} is its total scattering opacity, and $\phi_\nu(\hat{\mathbf{k}},\hat{\mathbf{k}}')$ is the scattering probability density (from $\hat{\mathbf{k}}'$ to $\hat{\mathbf{k}}$). The last satisfies the normalization and reversibility constraints:

$$\oint \phi_\nu(\hat{\mathbf{k}},\hat{\mathbf{k}}')\,d\Omega' = 1 = \oint \phi_\nu(\hat{\mathbf{k}},\hat{\mathbf{k}}')\,d\Omega. \tag{1.20}$$

On the right-hand side of equation (1.19), ρj_ν represents the source for radiation that comes from true emission (the term is reduced by a factor of 4π to make it per steradian); $-\rho\kappa_\nu^{\text{abs}} I_\nu$ represents the amount of light removed from the beam, per unit length of photon travel, by the effects of true absorption; $-\rho\kappa_\nu^{\text{sca}} I_\nu$ represents the analogous amount scattered out of the relevant beam; and the last term represents the integral contribution scattered into the beam from any other line of sight.

Later in this text we will practice calculating j_ν, κ_ν^{abs}, etc., for specific microscopic radiation processes. For now, we will assume that such calculations exist, and we regard equation (1.19) as one to solve for I_ν when j_ν, κ_ν^{abs}, etc. are given. In point of fact, not only do interactions of photons with matter change the radiation field, but, in general, they will also affect the physical state of the matter. That is, real interactions are often severely and nonlinearly coupled, and this fact constitutes the principal difficulty in computing their values. In any case, before we can proceed with the simpler problem of solving formally for I_ν given j_ν, κ_ν^{abs}, etc., we need to carefully decompose the qualitatively differing microscopic contributions to j_ν and κ_ν^{abs}.

CORRECTION FOR STIMULATED EMISSION

According to Einstein, atoms and molecules (which emit and absorb photons one at a time) have two components to their emissivity j_ν:

$$j_\nu = j_\nu^{\text{spontaneous}} + j_\nu^{\text{induced}}, \tag{1.21}$$

where

$$j_\nu^{\text{induced}} \propto I_\nu. \tag{1.22}$$

Notice that j_ν^{induced} is not isotropic (it will depend on the propagation direction $\hat{\mathbf{k}}$ if I_ν does), so that stimulated emission occurs coherently and in the same direction as the carrier photons. In contrast, $j_\nu^{\text{spontaneous}}$ can usually (but not always) be taken as isotropic in the local frame of rest of the gas. Only for a static medium (bulk or fluid velocity = 0, although individual particles can have random thermal motions) will the local rest frame of the matter correspond to the laboratory frame. When fluid motions exist (especially when relativistic motions are present), a proper treatment of radiative transfer becomes much more complicated than it is for the static case; hence, for most of this text, we will assume that the effects of fluid motion are negligible, except when we explicitly state otherwise.

In any case, because of the proportionality in equation (1.22), astronomers conventionally incorporate $\rho j_\nu^{\text{induced}}/4\pi$ as part of the term $-\rho\kappa_\nu^{\text{abs}} I_\nu$, and they speak of the resulting κ_ν^{abs} as "true absorption corrected for stimulated emission." We will illustrate how to do this in detail for some important microscopic examples in future chapters. For now, we will use j_ν though we really mean $j_\nu^{\text{spontaneous}}$. The importance of carrying through the preceding decomposition is that we can then assume that j_ν and κ_ν^{abs} are independent of $\hat{\mathbf{k}}$ (for a static isotropic medium).

What about "stimulated scattering"? Is there such a thing? Yes, although most textbooks on radiative transfer ignore any discussion of the effect, because for *coherent scattering* (propagation redirection but no change of photon frequency—i.e., momentum exchange but no energy exchange), the effects of stimulated scattering exactly cancel out in the source and sink terms. (For nearly coherent scattering, they would nearly cancel out.) We shall return to this topic in a later chapter, but for now we will adopt the usual treatment and leave equation (1.19) as it stands. Moreover, for many applications, we may make a further simplification, and assume

$$\oint \phi_\nu(\hat{\mathbf{k}}, \hat{\mathbf{k}}')(\hat{\mathbf{k}}\, d\Omega \text{ or } \hat{\mathbf{k}}' d\Omega') = 0, \tag{1.23}$$

since electrons, atoms, molecules, and even grains scatter equally in the forward and backward directions when $\lambda = c/\nu$ is greater than the "size" of the scattering particle. Scattering problems where the phase function ϕ_ν does not possess forward-backward symmetry—or, even worse, where the scattering does not occur coherently—add computational complications but no conceptual difficulties.

2

Moment Equations and Conduction Approximation

Reference: Schwarzschild, *Structure and Evolution of the Stars*, Chapter 2.

Solution of the transfer equation (1.19), which represents a complicated integro-partial-differential equation in six phase-space variables (three components of $\mathbf{x}$, two angles $\hat{\mathbf{k}}$, and ν) plus time, would give us a lot of information that we often do not need. For many purposes, including a first step in numerical iterative methods to solve equation (1.19), we may be satisfied with a contracted description which integrates out the dependence on photon propagation direction $\hat{\mathbf{k}}$, but retains the information on the distribution with space $\mathbf{x}$ (and time t) and photon frequency ν. In preparation for such a contracted description, we first define some angular moments of I_ν:

$$\begin{pmatrix} cE_\nu \\ \mathbf{F}_\nu \\ c\mathbf{P}_\nu \end{pmatrix} = \oint \begin{pmatrix} 1 \\ \hat{\mathbf{k}} \\ \hat{\mathbf{k}}\hat{\mathbf{k}} \end{pmatrix} I_\nu \, d\Omega. \tag{2.1}$$

Some texts (e.g., Mihalas 1978) denote successive angular moments, when divided by 4π, as J_ν, $\mathbf{H}_\nu$, $\mathbf{K}_\nu$, etc., but we prefer the notation of equation (2.1) to remind ourselves that E_ν is the monochromatic energy density in the radiation field, $\mathbf{F}_\nu$ is the monochromatic energy flux, and $\mathbf{P}_\nu$ is the monochromatic pressure tensor. The integrands for E_ν and $\mathbf{F}_\nu$ in (2.1) are obvious from the fundamental definition of I_ν, equation (1.1), but what about $\mathbf{P}_\nu$?

We recall from electrodynamics or fluid mechanics that the different components of a stress tensor refer to the rate of transfer of momentum across surfaces with certain orientations. The rate of momentum transfer per photon of momentum $\mathbf{p}$ across a surface of unit normal $\hat{\mathbf{k}}$ equals $c\hat{\mathbf{k}}\mathbf{p}$; therefore the total rate of momentum transfer, the radiation stress tensor $\mathbf{P}_{\text{rad}}$, by all photons in the distribution, is locally given by

$$\mathbf{P}_{\text{rad}} = \sum_{\alpha=1}^{2} \int c\hat{\mathbf{k}}\mathbf{p} F_\alpha \, d^3p = \sum_{\alpha=1}^{2} \int \hat{\mathbf{k}}\hat{\mathbf{k}} F_\alpha \frac{h^4\nu^3}{c^3} \, d\Omega \, d\nu, \tag{2.2}$$

where we have used $d^3p = (h^3\nu^2/c^3)d\Omega d\nu$. Invoking equation (1.11), we obtain the identification

$$\mathbf{P}_{\rm rad} = \frac{1}{c}\int \hat{\mathbf{k}}\hat{\mathbf{k}} I_\nu \, d\Omega d\nu, \tag{2.3}$$

which corresponds to the third row of equation (2.1) if we equate $\mathbf{P}_{\rm rad}$ with the integral of $\mathbf{P}_\nu$ over all frequencies ν. More generally, we denote the radiation energy density, flux, and pressure tensor by

$$\begin{pmatrix} E_{\rm rad} \\ \mathbf{F}_{\rm rad} \\ \mathbf{P}_{\rm rad} \end{pmatrix} = \int_0^\infty \begin{pmatrix} E_\nu \\ \mathbf{F}_\nu \\ \mathbf{P}_\nu \end{pmatrix} d\nu. \tag{2.4}$$

CONSERVATION RELATIONS FOR THE RADIATION FIELD

Given the preceding, we now begin our derivation of the moment equations. Begin by noting that $\hat{\mathbf{k}}$ commutes with ∇ and $\partial/\partial t$. If we multiply equation (1.19) by 1 and integrate over all solid angles of photon propagation, we obtain

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot \mathbf{F}_\nu = \rho(j_\nu - c\kappa_\nu^{\rm abs} E_\nu), \tag{2.5}$$

where we have assumed that j_ν and $\kappa_\nu^{\rm abs}$ have no dependence on $\hat{\mathbf{k}}$, and where we have used equation (1.20) to cancel the contributions from the scattering terms. Equation (2.5) has the generic form of a conservation equation,

$$\frac{\partial}{\partial t}(\text{density of quantity}) + \nabla \cdot (\textbf{flux} \text{ of quantity}) = \text{sources} - \text{sinks}. \tag{2.6}$$

Here the quantity is the energy of radiation of frequency ν, and the zeroth moment of the radiative transfer equation therefore states simply that the time-rate of change of photon energy equals the difference between what is emitted and what is absorbed by matter; i.e., energy is conserved in the matter-radiation system. Coherent scattering does not contribute to this budget, because no energy is exchanged between matter and radiation, photon by photon, if we make the approximation that all scatterings preserve $h\nu$.

Notice that equation (2.5) plays the natural role as a scalar partial differential equation (PDE) in $\mathbf{x}$ and t to solve for E_ν. Unfortunately, even if j_ν and $\kappa_\nu^{\rm abs}$ were known, equation (2.5) contains three components of another variable, $\mathbf{F}_\nu$, that we have no way of obtaining without solving the original equation (1.19), and then integrating $\hat{\mathbf{k}}I_\nu$—see equation (2.1). Alternatively, we could try to obtain an equation for $\mathbf{F}_\nu$ by multiplying equation (1.19) by $\hat{\mathbf{k}}$ and integrating over all solid angles. By inspection again, this produces the result

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t} + c\nabla \cdot \mathbf{P}_\nu = -\rho(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}})\mathbf{F}_\nu. \tag{2.7}$$

To derive equation (2.7) we have used the forward-backward symmetry assumption, equation (1.23), to eliminate the contribution of the photons scattered into our beam. Physically, if such photons are as likely to be scattered forward as backward, then they can contribute nothing to the momentum exchange. In contrast, the photons scattered out of a beam into all other directions, the term represented by $-\rho\kappa_\nu^{\mathrm{sca}}\mathbf{F}_\nu$, can yield a net momentum transfer that is opposite to the net flow of radiative energy $\mathbf{F}_\nu$. The same goes for the absorption of photons, which involves momentum exchange, followed by *isotropic* remission, which does not. The quantity

$$\kappa_\nu \equiv \kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}} \tag{2.8}$$

appearing in equation (2.7) is called the total opacity (corrected for stimulated emission but not for stimulated scattering), or simply, the opacity. In any case, if we now recall that the net momentum flow of photons is $1/c$ times the net energy flow, we see that the first moment equation (2.7) represents the law of conservation of momentum, just as the zeroth moment equation (2.5) represents the law of conservation of energy.

Notice also the following general rule: what appears as a source for the radiation field must reappear as a sink for the matter, and vice versa. In other words, when we add the energy or momentum equations for matter and radiation, the internal exchanges between them must not survive as net terms on the right-hand sides of the resulting sum. In particular, if momentum is removed at a certain rate from the radiation field, the back reaction must appear as a force for the matter. Thus the radiative force per unit area per unit length, i.e., the force per unit volume, acting on the gas must equal

$$\mathbf{f}_{\mathrm{rad}} = \frac{\rho}{c}\int_0^\infty \kappa_\nu \mathbf{F}_\nu \, d\nu. \tag{2.9}$$

THE NEED FOR A CLOSURE CONDITION

Having dispensed with these digressions, let us return to the original issue. Equation (2.7) gives, as expected, an equation for $\mathbf{F}_\nu$; unfortunately, the set composed of the scalar equation (2.5) for E_ν and the vector equation (2.7) for $\mathbf{F}_\nu$ still does not form a closed set, because we have gained *five* new variables in the independent components of the symmetric tensor $\mathbf{P}_\nu$. The number 5 enters because the third row in equation (2.1) shows explicitly that $\mathbf{P}_\nu$ is a symmetric tensor; so three off-diagonal elements are redundant. The last fact explains why we do not need to specify which index is being summed in the "dot" product $\nabla \cdot \mathbf{P}_\nu$. Moreover, since $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$ for a unit vector, the trace of $\mathbf{P}_\nu$ equals E_ν and cannot vary independently of the latter variable; this removes one more degree of freedom from the nine entries of the tensor represented as a 3×3 matrix. Thus the appearance of $\mathbf{P}_\nu$ introduces $9 - 3 - 1 = 5$ new independent quantities.

We see, therefore, the emergence of a general pattern. Without making some physical assumption, we find that no finite number of the moment equations will ever form a closed set; new variables always appear, making the number of variables larger than the set of equations. Taking the $(\ell - 1)$-th moment of the transfer equation generally introduces $2\ell + 1$ new quantities associated with the ℓ-th moment variable. Only the *infinite* set of all moment equations (multiplication by 1, $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}\hat{\mathbf{k}}$, ... and integrating over $d\Omega$) would contain the same amount of information as the original kinetic equation (1.19). An analogy holds with taking the Fourier-series expansion of a function. No finite number of Fourier coefficients contains the same amount of information as the original function; only the infinite set of Fourier coefficients can reproduce all of the bumps and wiggles. Since we need here to specify *two* angles of photon propagation (ϑ, φ), the actual decomposition would correspond to one in *spherical harmonics*, $Y_{\ell m}(\vartheta, \varphi)$. Thus the scalar E_ν relates to the single coefficient of the Y_{00} term; the vector $\mathbf{F}_\nu$ relates to the three coefficients of the Y_{1m} term, with $m = -1, 0, +1$; the symmetric tensor $\mathbf{P}_\nu$, apart from the trace, $\mathrm{tr}[\mathbf{P}_\nu] = E_\nu$, relates to the five coefficients of the Y_{2m} term, with $m = -2, -1, 0, +1, +2$; etc.

This analogy provides us, however, with some hope. If small bumps and wiggles do not interest us, we don't need all the Fourier (or spherical harmonic) coefficients; the lowest-order coefficients contain the gross behavior of the function. In other words, if we don't need to know all the details of a distribution function, the lowest-order moments specify its important properties (average value, degree of anisotropy, spread about mean, etc.). Thus, for many purposes, if we could use only equations (2.5) and (2.7) to compute E_ν, $\mathbf{F}_\nu$, and $\mathbf{P}_\nu$, even in some approximate manner, we might be quite content. To carry out such a program, we need to find or postulate a supplementary (tensor) equation that relates these three variables, i.e., that gives the high-order moments in terms of the lower-order ones. This constitutes the problem of finding a *closure relation*. In principle, we could search for a closure condition at a higher-order set of equations than just equations (2.5) and (2.7); in practice, people have generally limited searches to this level, for good reasons. Physically, equations (2.5) and (2.7) correspond to *conservation laws*, suggesting that they are special members of a much larger and less-interesting set of possible moment equations. Being related to familiar physical quantities, E_ν, $\mathbf{F}_\nu$, and $\mathbf{P}_\nu$ allow us to use physical intuition to find a useful approximate relation among them much more easily than if we thought in the abstract about some other set of higher-order moments. For many people, as it is, tensors that exert stress already border on the unfamiliar and bring on feelings of strain. Finally, when approximate methods work well, they usually do so after just a few terms, and if they don't work well, taking more terms usually doesn't improve matters very rapidly!

Under what conditions can we find good closure conditions? At the extremes. If a region is optically thin (photon mean-free path long compared to the macroscopic distances of interest), we can consider photons to fly in straight lines, basically unimpeded by the presence of matter. We shall consider in Chapter 5 the corresponding closure relation that might work for such situations. On the other hand, if a region is optically thick (photon mean-free path short compared to the macroscopic scales of interest), repeated scatterings, emissions, and absorptions must produce a nearly isotropic radiation field. We consider this situation in its simplest context in the next section.

THE RADIATION CONDUCTION APPROXIMATION

Consider the situation in the interior of a star like the Sun. The mean-free path $\ell_\nu \equiv 1/\rho\kappa_\nu$ for a typical photon in the Sun's interior might amount to 0.5 cm, with roughly equal contributions from absorption and scattering. This is roughly 10^{11} times smaller than the radius of the Sun, $R_\odot = 7\times 10^{10}$ cm; i.e., the "optical depth" from the surface of the Sun to a typical point in its interior is $\sim 10^{11}$, making it the closest thing to a perfect thermodynamic enclosure known in the solar system. For the interior of the Sun, not only must the radiation field I_ν be nearly completely isotropic, but it must also be nearly Planckian,

$$I_\nu \approx B_\nu(T), \tag{2.10}$$

where T is the local matter temperature. If we use this approximation to calculate E_ν and $\mathbf{P}_\nu$ from equation (2.1), we obtain

$$E_\nu \approx 4\pi B_\nu(T)/c, \qquad \mathbf{P}_\nu \approx [4\pi B_\nu(T)/3c]\mathbf{I}, \tag{2.11}$$

where $\mathbf{I}$ is the unit tensor that has the Kronecker delta δ_{ij} as its components in Cartesian form. In other words, $\mathbf{P}_\nu = P_\nu\mathbf{I}$, where P_ν is a monochromatic scalar pressure that satisfies the familiar relationship $P_\nu = E_\nu/3$ valid for a relativistic isotropic gas. Usually this relationship is stated in the frequency-integrated form: $P_{\rm rad} = E_{\rm rad}/3$, where $P_{\rm rad}$ and $E_{\rm rad}$ actually have the dimensions of pressure and energy density, i.e., erg cm^{-3}.

Notice that we do not use equation (2.10) to evaluate $\mathbf{F}_\nu$ because only the small *nonisotropic* part of I_ν can lead to a nonvanishing energy flux.

In other words, photons going in opposite directions cancel in their contribution to the integral of $\hat{\mathbf{k}} I_\nu$ if we approximate $I_\nu(\hat{\mathbf{k}}, \mathbf{x})$ to be independent of $\hat{\mathbf{k}}$.

To see how accurate are the approximations given in equation (2.11), we can arrive at them by a different line of reasoning. Under conditions of close thermodynamic coupling between matter and radiation, we expect the emissivity j_ν to be given by Kirchhoff's law:

$$j_\nu = 4\pi\kappa_\nu^{\text{abs}} B_\nu(T). \tag{2.12}$$

For a blackbody, which absorbs everything that falls on it (κ_ν having the same uniformly high value for all ν), the specific emissivity would be exactly proportional to $B_\nu(T)$; for more realistic materials in thermodynamic equilibrium with matter at temperature T, $B_\nu(T)$ is weighted by the factor κ_ν^{abs} that measures the ability of the material to *absorb* (*and therefore to emit*) at frequency ν. As we shall prove below, this makes the rate of local radiative emission j_ν a large term in equation (2.5), which can be balanced only by an almost exactly equal amount of local radiative absorption, represented by the term $c\kappa_\nu^{\text{abs}} E_\nu$, thereby recovering the blackbody relation $E_\nu = 4\pi B_\nu(T)/c$.

To begin, notice that, in order of magnitude, $\partial E_\nu/\partial t \sim E_\nu/t_\odot$, where $t_\odot$ is the evolutionary timescale of the interior of the Sun (several billion years). Thus its ratio to $c\rho\kappa_\nu^{\text{abs}} E_\nu$ must be as the mean-free path $\ell_\nu = 1/\rho\kappa_\nu$, typically 0.5 cm, is to $ct_\odot$, several billion light-years. This ratio, $\sim 10^{-28}$, is negligibly small by anyone's standards; so we may safely ignore the term $\partial E_\nu/\partial t$ in equation (2.5) when dealing with the Sun's interior. In a similar manner, we can argue that the term $(1/c)\partial \mathbf{F}_\nu/\partial t$ in equation (2.7) is smaller than the term $\rho\kappa_\nu \mathbf{F}_\nu$ by the same factor. Thus $\rho\kappa_\nu \mathbf{F}_\nu$ can be balanced only by the remaining term $c\nabla \cdot \mathbf{P}_\nu$, with a magnitude of $\sim cE_\nu/R_\odot$, since $\mathbf{P}_\nu$ and E_ν have comparable orders of magnitude. This demonstrates that $|\mathbf{F}_\nu|$ is $\sim (\ell_\nu/R_\odot)$ times smaller than cE_ν, which equals the monochromatic energy transport that photons could carry if they could only fly instead of having to random walk.

What about the term $\nabla \cdot \mathbf{F}_\nu$ in equation (2.5)? It must be of order $|\mathbf{F}_\nu|/R_\odot \sim (\ell_\nu/R_\odot^2)cE_\nu$, which is smaller than the term $c\rho\kappa_\nu^{\text{abs}} E_\nu$ in equation (2.5) by the factor $(\ell_\nu/R_\odot)^2 \sim 10^{-22}$. We have now demonstrated the desired result: that the only way to satisfy equation (2.5) results in a very near equality between E_ν and $j_\nu/c\kappa_\nu^{\text{abs}} = 4\pi B_\nu(T)/c$.

En route to this conclusion, we have also shown that equation (2.7) is well approximated by [see equation (2.11)]

$$\mathbf{F}_\nu = -\frac{c}{\rho\kappa_\nu}\nabla \cdot \mathbf{P}_\nu = -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\nabla T. \tag{2.13}$$

Integrating equation (2.13) over all ν yields

$$\mathbf{F}_{\text{rad}} = -\frac{4\pi}{3\rho\kappa_{\text{R}}}(\nabla T)\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu, \tag{2.14}$$

where we have defined the *Rosseland mean opacity* κ_{R} by the weighted transmission average:

$$\frac{1}{\kappa_{\text{R}}} \equiv \frac{\int_0^\infty (1/\kappa_\nu)(\partial B_\nu/\partial T)\,d\nu}{\int_0^\infty (\partial B_\nu/\partial T)\,d\nu} \tag{2.15}$$

Since

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu = \frac{d}{dT}\int_0^\infty B_\nu(T)\,d\nu = \frac{d}{dT}\left(\frac{caT^4}{4\pi}\right), \tag{2.16}$$

we may now write equation (2.14) as

$$\mathbf{F}_{\text{rad}} = -\frac{c}{3\rho\kappa_{\text{R}}}\nabla(aT^4). \tag{2.17}$$

Equation (2.17) is often called the radiation conduction equation, because it implies that the radiative flux is proportional to minus the gradient of the temperature T ("Fourier's law"). Indeed, recognizing aT^4 as the energy density of the blackbody radiation, we see that equation (2.17) has the general form for diffusive fluxes ("Fick's law"):

$$\text{diffusive } \textbf{flux} = -\mathcal{D}\nabla(\text{density of quantity being diffused}),$$

where $\mathcal{D}$ is the diffusivity. Indeed, this comparison allows us to identify the radiative diffusivity as having the characteristic formula,

$$\mathcal{D}_{\text{rad}} = \frac{1}{3}c\ell,$$

where $\ell \equiv 1/\rho\kappa_{\text{R}}$ is the (Rosseland) mean-free path of the diffusing particles (photons). A "random walk" slows down the free-flight speed c by a typical factor of $\ell/R_\odot$, so that the time $R_\odot^2/\mathcal{D}_{\text{rad}}$ for photons to diffuse to the surface of the Sun is roughly $3R_\odot/\ell$ times longer than the free-flight time $R_\odot/c$ of about 2 s. This process prevents the Sun from releasing its considerable internal reservoir of photons in one powerful blast, but instead regulates it to the stately observed luminosity of $L_\odot = 3.86 \times 10^{33}$ erg s^{-1}. In any case, apart from being useful for rough order-of-magnitude arguments, the accurate equation (2.17) constitutes one of the fundamental equations underlying the whole theory of stellar structure and evolution.

To complete this section, we notice that the radiative force per unit volume acting on the matter can be obtained by substituting equation (2.13) into equation (2.9):

$$\mathbf{f}_{\text{rad}} = -(\nabla T)\frac{4\pi}{3c}\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu = -\frac{1}{3}\nabla(aT^4), \tag{2.18}$$

where we use equation (2.16). We recognize $aT^4/3$ as the radiation pressure $P_{\rm rad}$ of blackbody radiation of temperature T; so equation (2.9) reduces, in the limit of large optical thickness, to the familiar expression,

$$\mathbf{f}_{\rm rad} = -\nabla P_{\rm rad}. \tag{2.19}$$

Finally, note that none of the formulae derived in this section assumed spherical symmetry. They apply to stars severely distorted by rotation or tidal forces (in a close binary) as well as to rounder examples, like the Sun.

3

Solution of the Problem of Radiative Transfer

Reference: Mihalas, *Stellar Atmospheres*, Chapter 3.

In this chapter, we consider the solution of problems of radiative transfer where too little optical depth exists to make radiation conduction a good approximation. We begin by giving a general discussion that permits arbitrary spatial variations in all three dimensions.

FORMAL SOLUTION OF THE EQUATION OF TRANSFER

If light travels across an object in an interval short compared with its evolutionary time (see Chapter 2), we may ignore time-dependence in the corresponding problem of radiation transport. Equation (1.19) then can be written

$$\hat{\mathbf{k}} \cdot \nabla I_\nu + \rho\kappa_\nu I_\nu = \rho\left(\frac{j_\nu}{4\pi} + \kappa_\nu^{\mathrm{sca}}\Phi_\nu\right), \tag{3.1}$$

where $\kappa_\nu = \kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}}$ is the total opacity and Φ_ν is the intensity weighted by the angular phase function for scattering:

$$\Phi_\nu(\hat{\mathbf{k}}, \mathbf{x}) \equiv \oint \phi_\nu(\hat{\mathbf{k}}, \hat{\mathbf{k}}')I_\nu(\hat{\mathbf{k}}', \mathbf{x})\, d\Omega'. \tag{3.2}$$

For isotropic scattering, $\phi_\nu = 1/4\pi$, and Φ_ν becomes the mean intensity J_ν.

By introducing the *source function*,

$$S_\nu(\hat{\mathbf{k}}, \mathbf{x}) \equiv \frac{1}{\kappa_\nu}\left(\frac{j_\nu}{4\pi} + \kappa_\nu^{\mathrm{sca}}\Phi_\nu\right), \tag{3.3}$$

and by defining the ray-path derivative $\hat{\mathbf{k}} \cdot \nabla$ to be d/ds, we may write equation (3.1) as

$$\frac{dI_\nu}{ds} + \rho\kappa_\nu I_\nu = \rho\kappa_\nu S_\nu. \tag{3.4}$$

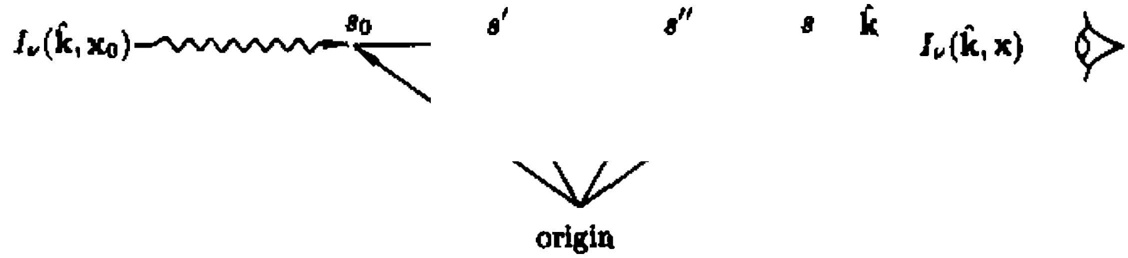

FIGURE 3.1
Geometry for ray-path integration for I_ν.

Introducing an optical depth variable

$$\tau_\nu \equiv \int_{s_0}^{s} \rho\kappa_\nu ds, \tag{3.5}$$

where s_0 represents the back boundary of our object, we see that the left-hand side of equation (3.4) possesses an integrating factor

$$e^{-\tau_\nu} \frac{d}{ds}\left(e^{\tau_\nu} I_\nu\right) = \rho\kappa_\nu S_\nu. \tag{3.6}$$

Recognizing $\rho\kappa_\nu ds$ as $d\tau_\nu$ and integrating in s from s_0 to s (in τ_ν from 0 to τ_ν), we obtain

$$I_\nu(\hat{\mathbf{k}}, \tau_\nu)e^{\tau_\nu} - I_\nu(\hat{\mathbf{k}}, 0) = \int_0^{\tau_\nu} S_\nu e^{t_\nu}\, dt_\nu.$$

Introducing a new dummy variable $\tau'_\nu = \tau_\nu - t_\nu$, we obtain, upon transposing terms,

$$I_\nu(\hat{\mathbf{k}}, \tau_\nu) = I_\nu(\hat{\mathbf{k}}, 0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\hat{\mathbf{k}}, \tau'_\nu)e^{-\tau'_\nu}\, d\tau'_\nu. \tag{3.7}$$

Equation (3.7) looks deceptively simple. The source function (3.3) depends, in reality, not on the monochromatic optical depth τ_ν, but on the spatial position $\mathbf{x}$. The trouble with using τ'_ν as an integration variable in place of source position $\mathbf{x}'$ is that the former has a different value at different frequencies ν and different propagation directions $\hat{\mathbf{k}}$ for each field (observer) position $\mathbf{x}$. Except for special circumstances, if we are interested in more than one frequency and one propagation direction per field point, we would be well advised to perform the integration in physical space and with a more explicit notation (see Figure 3.1)

$$I_\nu(\hat{\mathbf{k}},\mathbf{x}) = I_\nu(\hat{\mathbf{k}},\mathbf{x}_0)\exp[-\tau_\nu(\hat{\mathbf{k}},\mathbf{x},\mathbf{x}_0)] + \int_{s_0}^{s} S_\nu(\hat{\mathbf{k}},\mathbf{x}')\exp[-\tau_\nu(\hat{\mathbf{k}},\mathbf{x},\mathbf{x}')]\rho(\mathbf{x}')\kappa_\nu(\mathbf{x}')\,ds', \quad (3.8)$$

where

$$\tau_\nu(\hat{\mathbf{k}},\mathbf{x},\mathbf{x}') \equiv \int_{s'}^{s} \rho(\mathbf{x}'')\kappa_\nu(\mathbf{x}'')\,ds'' \quad (3.9)$$

represents the optical depth along a ray path from $\mathbf{x}'$ to $\mathbf{x}$, with

$$\mathbf{x}' = \mathbf{x} - (s-s')\hat{\mathbf{k}}, \quad \mathbf{x}'' = \mathbf{x} - (s-s'')\hat{\mathbf{k}}, \quad \mathbf{x}_0 = \mathbf{x} - (s-s_0)\hat{\mathbf{k}} \quad (3.10)$$

denoting backward positions along the ray path at distances $(s-s')$, $(s-s'')$, and $(s-s_0)$ from $\mathbf{x}$. In this notation, $\mathbf{x}_0$ represents the incident point at which a ray from a background source first enters the region of interest (see Figure 3.1).

Equation (3.8) states that, apart from an exponential attenuation of an initial value, I_ν results from the superposition of all sources (emitted and scattered light) along the line of sight, weighted by the extinction factor $e^{-\tau_\nu'}$ from the source position $\mathbf{x}'$ to the field (observation) point $\mathbf{x}$. To verify that equation (3.8) satisfies equation (3.1) in general, notice that $\hat{\mathbf{k}}\cdot\nabla$ operating on the attenuated initial term $[\propto I_\nu(\hat{\mathbf{k}},\mathbf{x}_0)]$ exactly cancels $-\rho\kappa_\nu$ times the same term; so this term can be ignored in what follows. When $\hat{\mathbf{k}}\cdot\nabla$ operates on the upper limit s of the integral, we get $\hat{\mathbf{k}}\cdot\nabla s$ times the integrand evaluated at $s' = s$, i.e., $\hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$ times $\rho(j_\nu/4\pi + \kappa_\nu^{\text{sca}}\Phi_\nu)$ evaluated at $\mathbf{x}' = \mathbf{x}$. On the other hand, when $\hat{\mathbf{k}}\cdot\nabla$ operates on the integrand, we get $-\hat{\mathbf{k}}\cdot\nabla\tau_\nu$ times the original integral, i.e., $-\rho\kappa_\nu$ times the original integral. Thus the solution (3.8) satisfies equation (3.1) (Q.E.D.).

RADIATIVE EQUILIBRIUM

In order to use equation (3.8), we must be able to specify j_ν and Φ_ν. The latter, unfortunately, depends on an integral [equation (3.2)] of I_ν. Even j_ν will usually turn out to depend on the solution I_ν. In particular, for *static* distributions of matter near a strong source of radiation (e.g., a star), the condition of steady state requires that radiation neither add nor subtract a net amount of energy from the gas:

$$\rho\int_0^\infty (j_\nu - c\kappa_\nu^{\text{abs}}E_\nu)\,d\nu = 0, \quad (3.11)$$

where E_ν is the monochromatic energy density of the ambient radiation field [see equation (2.1)] and numerically equals $4\pi J_\nu/c$. Equation (3.11) places an integral constraint at each point $\mathbf{x}$ in space on the possible variation of j_ν and I_ν. Equivalently, we may integrate equation (2.5) over all frequencies ν and apply equation (3.11) to obtain the constraint of radiative equilibrium as [see the frequency-integrated and time-independent version of equation (2.5)]:

$$\nabla \cdot \mathbf{F}_{\text{rad}} = 0. \tag{3.12}$$

LOCAL THERMODYNAMIC EQUILIBRIUM

To fix ideas let us consider a simple but important example. Suppose all the level populations of the internal and external degrees of freedom that can contribute to electromagnetic radiation are characterized by their thermodynamic values at a common temperature T. In such a situation, the matter is said to be in *local thermodynamic equilibrium* (LTE). The assumption of LTE differs from TE (complete thermodynamic equilibrium) in that the ambient radiation field need not be Planckian at the same T. Thus matter in LTE with $I_\nu \neq B_\nu(T)$ constitutes a first generalization beyond the "radiation conduction" approximation considered in the last chapter. The assumption of LTE constitutes a good approximation if collisional processes among particles either dominate competing photoprocesses or are in equilibrium with them at a common matter and radiation temperature. In all other situations they may fail badly, especially for the internal degrees of freedom. For rarefied matter in the spaces between stars and planets, non-LTE effects generally come into play, and we then need a full microscopic treatment of how atomic and molecular levels are populated and depopulated.

When LTE does apply, the emissivity j_ν has a thermal value given by Kirchhoff's law:

$$j_\nu = 4\pi\kappa_\nu^{\text{abs}} B_\nu(T). \tag{3.13}$$

Specifying j_ν now boils down to specifying $T(\mathbf{x})$. [We assume that $\kappa_\nu(\mathbf{x})$ is known if $\rho(\mathbf{x})$ and $T(\mathbf{x})$ are given.] For situations in which radiation provides the dominant means for heating and cooling the gas, we cannot really specify $T(\mathbf{x})$ in advance, but must find it as part of the overall problem.

If we substitute equation (3.13) into equation (3.11), we require that the distribution of matter temperature $T(\mathbf{x})$ satisfies the integral constraint of *radiative equilibrium* (first formulated for stellar photospheres by Karl Schwarzschild in 1906):

$$4\pi \int_0^\infty \kappa_\nu^{\text{abs}} B_\nu(T)\, d\nu = c \int_0^\infty \kappa_\nu^{\text{abs}} E_\nu\, d\nu. \tag{3.14}$$

The left-hand side is some function of T (and possibly $\mathbf{x}$ through κ_ν^{abs}); the right, some function of $\mathbf{x}$. Thus the two sides fix $T(\mathbf{x})$ if $E_\nu(\mathbf{x})$ is known, i.e., if we have the solution for I_ν. [Given E_ν, we can find the value for

T at each $\mathbf{x}$ from equation (3.14), for example, by Newton's method for extracting roots.]

But we don't have the solution (3.8) until we specify $T(\mathbf{x})$; thus, even without the coupling due to (coherent) scattering, equation (3.8) often amounts to a complicated nonlinear integral equation for I_ν. Exceptions occur if agents (e.g., cosmic rays in the interstellar medium) other than photons maintain the matter temperature T. Also, even if radiative processes dominate the heating and cooling, we might attack equation (3.8) as an integral equation for most of the photons, i.e., for the continuum, and then use the resulting self-consistent temperature distribution and continuum radiation field as a background for computing radiative transfer in the lines.

In this chapter, we will concern ourselves primarily with a broad overview. Various methods exist to solve the associated integral equation; the most direct involves an iterative technique, as follows. Make an initial guess for $J_\nu(\mathbf{x}) = cE_\nu/4\pi$; approximate Φ_ν by J_ν (okay, if the scattering is nearly isotropic); and solve equation (3.14) to obtain $T(\mathbf{x})$. With equation (3.13) now giving j_ν, integrate (3.8) to obtain $I_\nu(\hat{\mathbf{k}}, \mathbf{x})$. Substitute this value into the first row of equation (2.1) to compute a new E_ν and into equation (3.2) to obtain a new Φ_ν. Continue to iterate in this manner until satisfactory convergence is achieved.

In practice, straight iterative schemes have, at best, a 50–50 chance of converging (see Problem Set 1), and other techniques (e.g., "complete linearization"; see Mihalas 1978) are better for extracting precise numerical solutions, at least for problems involving a restricted number of spatial dimensions and angular propagation directions (e.g., one each). We leave further discussion of such methods to texts on stellar atmospheres theory, and focus here on intuitive discussions of the physical nature of the solution.

EMISSION AND ABSORPTION LINES, LIMB DARKENING

Suppose T and Φ_ν are given to us somehow (consistent with radiative equilibrium for a stellar atmosphere, more arbitrary for the interstellar medium or some other environment). What does the resulting radiation field look like in the continuum and in lines? Under LTE conditions, the source function (3.3) becomes

$$S_\nu = (1 - A_\nu)B_\nu(T) + A_\nu \Phi_\nu, \tag{3.15}$$

where

$$A_\nu \equiv \kappa_\nu^{\text{sca}}/\kappa_\nu \tag{3.16}$$

is the scattering albedo. Consider now the problem of the formation of LTE spectral lines under optically thick and thin conditions. For simplicity, let us ignore the effects of scattering at first, so that $S_\nu = B_\nu(T)$, a smooth function of ν.

When the medium is optically thin at all relevant frequencies (e.g., the upper chromosphere and corona of the Sun; emission nebulae in our own Galaxy and in external galaxies) and there is no background source $I_\nu(\hat{\mathbf{k}}, \mathbf{x}, \mathbf{x}_0) = 0$, we may approximate the exponential $\exp[-\tau_\nu(\hat{\mathbf{k}}, \mathbf{x}, \mathbf{x}')]$ in equation (3.8) by unity. Moreover, we expect κ_ν^{abs} to be larger at the frequency of a resonance line than at neighboring spectral intervals. The indicated integration in equation (3.8) therefore gives a larger I_ν at the line frequency than at neighboring frequencies for every line of sight, *independent of the details of the density and temperature distributions ρ and T*. In other words, both angularly resolved and angularly unresolved observations would yield emission brighter in lines than in the continuum. An *emission-line spectrum* appears in this situation, because gas has more emissive power where it has more absorptive power, and if the medium is optically thin, every emitted photon flying in the proper direction will reach the observer.

An upper limit exists, however, to how bright thermal emission lines can become with increasing optical depth. For simplicity, suppose the emitting medium has a uniform temperature T. If $\kappa_\nu^{\text{sca}} = 0$, the source function $S_\nu = B_\nu(T)$ is independent of position if T is. With no background source, equation (3.7) now has the integral,

$$I_\nu = B_\nu(T)(1 - e^{-\tau_\nu}), \tag{3.17}$$

where τ_ν is the total optical depth at frequency ν across the source along our particular line of sight. Since $(1 - e^{-\tau_\nu})$ cannot exceed 1, equation (3.17) states that a thermal body at a uniform temperature T cannot yield a specific intensity, in lines or a continuum, that is higher than a blackbody of the same temperature. (Non-LTE effects can provide counterexamples, the most celebrated being emission from masers.) For small optical depths, $(1 - e^{-\tau_\nu}) \approx \tau_\nu$, and the emission, normalized relative to a blackbody, is larger at frequencies where the optical depth is larger, i.e., in lines (see Figure 3.2).

The advantage that lines have over the continuum saturates when τ_ν begins to approach and exceed unity in both. Moreover, if the medium is (at least partially) optically thick in the continuum, we expect the temperature T to decrease outward from the source toward the observer. Recall that the discussion in Chapter 2 of radiation conduction shows the energy flux to be directed down a temperature gradient; conversely, a matter-temperature gradient will develop whenever the radiation flux in optically thick regions is nonzero. When we observe at the frequency of a line, the higher optical depth causes us to look less deeply into the object than at neighboring frequencies. Thus, at a resonant frequency, we see to less-hot layers than in the continuum, and equation (3.8) would then yield dark spectral lines

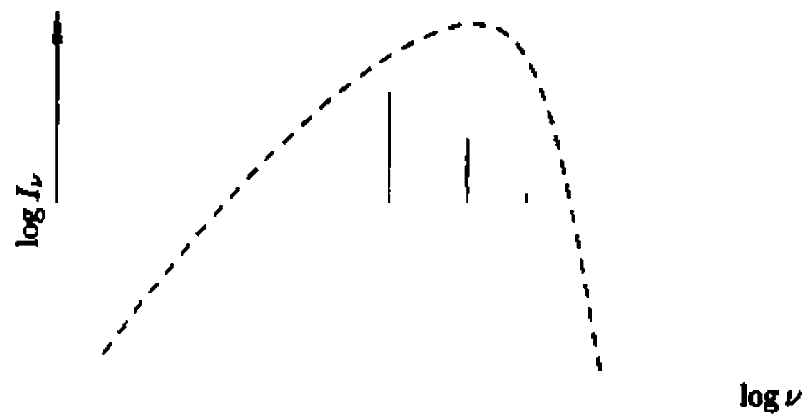

FIGURE 3.2
Limitation of emission lines and continuum to blackbody value in region of uniform temperature T and where LTE applies.

superimposed on a brighter continuum (Figure 3.3). In other words, this situation—e.g., the photosphere of the Sun and other stars—tends to produce an *absorption-line spectrum.*

The effect described in the preceding has an extreme example where there is a bright background source, of specific intensity $I_\nu(0)$, in addition to an emitting and absorbing medium in LTE at a uniform temperature T. If $\kappa_\nu^{sca} = 0$, equation (3.7) implies for this case

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)(1 - e^{-\tau_\nu}), \tag{3.18}$$

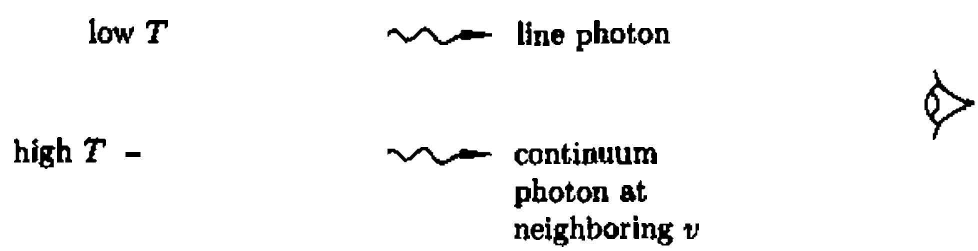

FIGURE 3.3
Radiation emergent from an atmosphere where the temperature increases inward tends to produce an absorption-line spectrum.

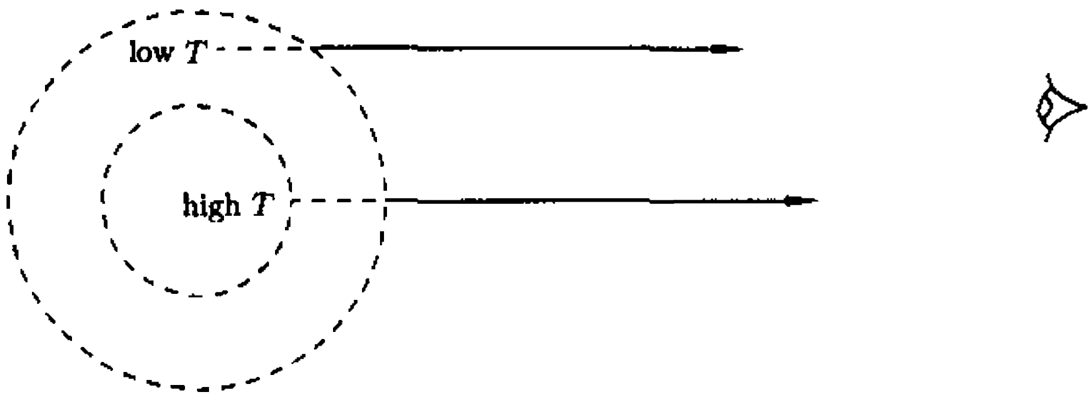

FIGURE 3.4
The physical situation behind limb darkening.

an equation much used in radio studies of the interstellar medium. If the background is brighter than the emitting and absorbing foreground, the positive combination $[I_\nu(0) - B_\nu(T)]e^{-\tau_\nu}$ will decrease with increasing τ_ν; i.e., lines will appear darker than the surrounding continuum. Historically, Arthur Schuster advanced in 1902–1905 the "reversing layer" type of analysis represented by equation (3.18) to explain the dark Fraunhofer lines of the solar spectrum. (He considered a foreground scattering layer rather than an absorbing one.) Modern stellar-atmosphere theory has generally abandoned the naive concept of two discrete layers at different temperatures in favor of more continuous variations, but models of comparable sophistication have yet to make their way into the literature of other fields (e.g., the interstellar medium, quasars, etc.).

Applied to the continuum, the same type of reasoning explains the phenomenon of *limb darkening* (see Figure 3.4). When we look at the limb (edge) of the Sun, we see—for a given optical depth—to less-hot layers than we do when we look toward the center. Thus photographs of the Sun in "white light" show its limb to be darker than its central face. In Chapter 4 we will give a quantitative estimate of the magnitude of this effect (first discussed by Karl Schwarzschild in 1906).

Since coherent scattering conserves the total number of photons at each frequency, we must reason more carefully when we come to "absorption" lines formed by the scattering of light by material distributed near a continuum source. Such scattering occurs more strongly in the line than in the neighboring continuum, so we naively expect the formation of a spectral feature. If, however, coherent scattering constitutes the only agent present, one observer would be able to see an "absorption" line only at the expense of other observers seeing an "emission" line. For a system in which the scattering medium is distributed spherically symmetrically with respect to the continuum source, all observers located on the celestial sphere are equivalent; so no net effect can occur! Nevertheless, spectral lines would remain absent only as long as the region remains unresolved. For a resolved source, e.g., the shell of a planetary nebula, the continuous background appears much brighter when an observer looks directly toward the central source than when he or she looks at the limb of nebula. Thus, as astronomers have recently demonstrated for actual observations of a planetary nebula, when scattering in a spectral line takes place, a line of sight toward the central star produces an absorption feature; a line of sight toward the limb of the nebula shows an emission feature. Averaged over the whole source, however, the spectral feature disappears!

How do dark lines form by scattering, then, in the atmosphere of an unresolved star? Line photons traveling out of the star in the direction of the observer can scatter out of the line of sight as well as scatter toward the deeper and denser layers of the atmosphere, where collisions among emitting and absorbing atoms can thermalize the radiation field (redistribute the frequencies into a more continuous spread). A compensating effect does not arise for inward-going photons scattering into our line of sight, since the presence of a boundary introduces a basic asymmetry to the problem (even if the atmosphere had an isothermal structure). Thus, in general, dark lines in stellar atmospheres will form by a combination of true absorption and true scattering.

4

Plane-Parallel Atmospheres

Reference. Kourganoff, *Basic Methods in Transfer Problems*, Chapters 1–4.

In this chapter, we consider approximate solutions for the problem of radiative transfer in the continuum, with a particular goal of extracting the implied *emergent spectral energy distribution* (distribution with ν) and *law of limb darkening* (variation with angle). We restrict ourselves to an idealized plane-parallel geometry and to the case where the assumptions of LTE and radiative equilibrium both hold. We shall not discuss methods for the exact solution of the problems so posed, but choose to emphasize approximate ones that lay bare the relevant physics and obtain 90 percent of the correct answer at 10 percent of the calculational effort. For more accurate (but also computationally more intensive) methods, consult Kourganoff (1963) or Chandrasekhar (1960).

PLANE-PARALLEL LTE ATMOSPHERE

If the transition between optically thick and optically thin regions occurs in a very narrow layer, calculations are simpler. For many stellar atmospheres, this transition layer (the photosphere) spans typically only $10^{-3}R_*$, where the stellar radius R_* is defined by the location where $F_{\rm rad} = \sigma T^4$ holds in an integration that proceeds outward from the deep interior. The resulting temperature, at which the star radiates *as if it were a blackbody* (in total flux), is called the *effective temperature* $T_{\rm eff}$. When variations of r can be ignored, radiative equilibrium [see equation (3.12)] requires $F_{\rm rad} = \text{constant} \equiv \sigma T_{\rm eff}^4$. Furthermore, for a slab geometry,

$$\hat{\mathbf{k}} \cdot \nabla = -\mu \frac{d}{dz}, \tag{4.1}$$

where μ is the cosine of the angle ϑ of a ray path with respect to the vertical, and z is a depth variable (increasing *inward*) equal to zero at the

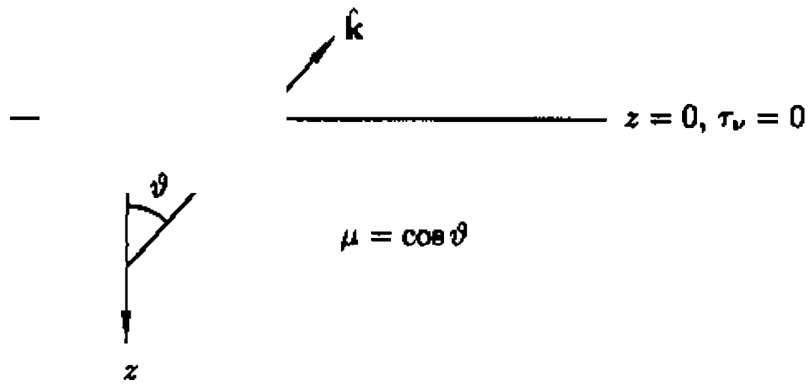

FIGURE 4.1
Geometry for plane-parallel atmosphere.

"top" of the atmosphere (where optical depth zero is reached). Figure 4.1 illustrates the geometry.

Such a configuration allows us to separate the angle dependence and the position dependence of the path interval ds in equation (3.1),

$$ds = -dz/\mu, \tag{4.2}$$

so that the angle variable μ enters effectively as a parameter only in the equation of transfer, rather than as something whose variations we need to consider when taking derivatives or performing integrals along ray paths. (If the radius of curvature r were important, a dispacement ds along a nonradial ray path would change the angle ϑ between $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$, with the complications that we will discuss in a later chapter.)

We assume that the atmosphere can be taken to possess infinite total optical depth. Recall that e^{-100} is already effectively zero, not to mention $\exp(-10^{11})$. If we further assume isotropic scattering and LTE (a reasonable approximation for transitions involving the continuum), the source function (3.15) becomes

$$S_\nu = (1 - A_\nu)B_\nu(T) + A_\nu J_\nu, \tag{4.3}$$

where A_ν is the scattering albedo $\kappa_\nu^{\text{sca}}/\kappa_\nu$ and J_ν is the mean specific intensity. The specific intensity emergent from the top of the atmosphere has the expression [see equation (3.7) with upper and lower limits of the integral reversed]

$$I_\nu(\mu, \tau_\nu = 0) = \int_0^\infty S_\nu e^{-\tau_\nu/\mu}\,\frac{d\tau_\nu}{\mu}, \tag{4.4}$$

where $\mu > 0$ for outgoing rays and τ_ν is now the optical depth measured in the *vertical* direction (rather than along a ray path),

$$\tau_\nu \equiv \int_0^z \rho\kappa_\nu \, dz. \tag{4.5}$$

Previously, e.g., in equations (3.8) and (3.7), we let τ_ν denote what we now call the *slant optical depth* τ_ν/μ. If we take care to distinguish between outgoing rays ($\mu > 0$) and ingoing rays ($\mu < 0$), we can find expressions similar to equation (4 4) for the specific intensity at interior points.

GREY OPACITIES

A particularly clean problem arises for the idealized case when the opacities are grey, i.e., when $\kappa_\nu^{\rm abs}$ and $\kappa_\nu^{\rm sca}$ are independent of ν (roughly true in the solar atmosphere). If the scattering albedo $A_\nu \equiv A$ does not depend on ν, the integration of the source function (4.3) over all frequencies yields the identification:

$$S = (1 - A)B(T) + AJ, \tag{4.6}$$

where we have denoted the integral of $B_\nu(T)$ over all ν as $B(T) = caT^4/4\pi$. On the other hand, with $\kappa_\nu \equiv \kappa$ independent of ν, radiative equilibrium, equation (3.14) with $E = 4\pi J/c$, requires $J = B$ for $A \neq 1$. The result that the frequency-integrated mean specific intensity equals the local blackbody value (for total emission to balance total absorption when the opacity is grey) does not imply, as we shall see later, that the *monochromatic* specific intensity I_ν has either the spectral composition or the angular distribution of a true Planckian $B_\nu(T)$. Moreover, for $A = 1$ (conservative isotropic scattering), radiative energy conservation enforces only $F_{\rm rad}$ = constant, without requiring $S = J$ to also equal B. For any other value of A, equation (4.6) does require

$$S = J = B. \tag{4.7}$$

Except for the issue whether the mean intensity J is related to the local matter temperature [as expressed through $B(T)$], we see from the preceding discussion that many of the distinctions between absorption and scattering vanish when the opacities are grey and the scattering is isotropic. Indeed, Chandrasekhar (1960, p. 293) explicitly shows that the grey problems of pure absorption ($A = 0$) and pure isotropic scattering ($A = 1$) have identical angular distributions and optical depth variations for the integrated light $I(\mu, \tau)$; consequently, we may take any linear combination of the two ($0 < A < 1$) and come to the same conclusion.

In any case, with $\tau_\nu \equiv \tau$ independent of ν, equation (4.4) may now be integrated over all frequencies to yield

$$I(\mu,0) = \int_0^\infty S(\tau)e^{-\tau/\mu}\,\frac{d\tau}{\mu}. \tag{4.8}$$

With the same assumptions, the frequency-integrated counterpart of equation (2.7) becomes

$$c\frac{d}{d\tau}(P_{\rm rad}) = F_{\rm rad}, \tag{4.9}$$

where $d\tau \equiv \rho\kappa dz$ increases inward toward the center of the star. Equation (4.9) has the solution

$$cP_{\rm rad} = F_{\rm rad}(\tau + \tau_0), \tag{4.10}$$

where τ_0 is an integration constant.

EDDINGTON APPROXIMATION

So far our equations are exact, to the extent that the physical assumptions (slab geometry, isotropic scattering, grey opacities) are valid. To make further progress, we adopt Eddington's closure approximation:

$$P_{\rm rad} \approx \frac{1}{3}E_{\rm rad} = \frac{4\pi J}{3c}. \tag{4.11}$$

This relationship holds under more general conditions than completely isotropic radiation; for example, it remains valid if the radiation field is separately isotropic for the outward and inward directions. Combined with equation (4.10) and $S = J$, equation (4.11) leads to the identification that the source function is a linear function of optical depth,

$$S(\tau) = \frac{3}{4\pi}F_{\rm rad}(\tau + \tau_0), \tag{4.12}$$

for any value of A.

Substitution of equation (4.12) into equation (4.8) now yields (with the substitution of factorials for complete Γ functions):

$$I(\mu,0) = \frac{3}{4\pi}F_{\rm rad}(\mu + \tau_0). \tag{4.13}$$

To determine the integration constant τ_0, we require that the outwardly emergent intensity carry the flux $F_{\rm rad}$, i.e.,

$$F_{\rm rad} = 2\pi\int_0^1 \mu I(\mu,0)\,d\mu. \tag{4.14}$$

If we substitute equation (4.13) into the right-hand side of equation (4.14), a little manipulation yields the identification

$$\tau_0 = 2/3. \tag{4.15}$$

Equation (4.13) now gives the well-known limb-darkening result (for any combination of thermal emission or isotropic scattering) that the emergent specific intensity,

$$I(\mu, 0) = I(1, 0)\frac{3}{5}\left(\mu + \frac{2}{3}\right), \tag{4.16}$$

has 2/5 the value at the limb ($\mu = 0$) that it does at the center ($\mu = 1$). (Figure out for yourself the equivalence between rays traveling from different points in a thin spherical shell toward a given observer and rays traveling from the same point in a plane-parallel atmosphere toward different observers.) The exact solution by Chandrasekhar and by Hopf of this problem yields $I(0,0)/I(1,0) = 0.3439$, not bad agreement with the result obtained by adopting Eddington's approximation, when we consider the anisotropy that this ratio implies for the radiation field at the boundary of the atmosphere.

When $A \neq 1$ (so that some true absorption and emission occur to allow energy interchange between matter and radiation), equations (4.7), (4.10), and (4.11), with $F_{\text{rad}} = \sigma T_{\text{eff}}^4$ (and $\sigma = ca/4$), imply a temperature variation with depth given by the famous law

$$T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right). \tag{4.17}$$

Equation (4.17) represents a special property of the present problem that fails to be valid more generally, namely, the existence of a T-τ relationship that holds for the problem of radiative transfer *independently of the mechanical structure of the atmosphere* (the distribution of ρ and T as functions of z). In any case, equation (4.17) implies that the gas temperature equals the effective temperature at a continuum optical depth of 2/3; whereas the boundary temperature T_0 at $\tau = 0$ equals $2^{-1/4}T_{\text{eff}} = 0.8409T_{\text{eff}}$. The exact result is $T_0 = 0.8112T_{\text{eff}}$.

EMERGENT SPECTRAL ENERGY DISTRIBUTION

For an LTE grey stellar atmosphere in which we neglect the effects of scattering ($A = 0$), the emergent monochromatic intensity is given by equation (4.4) when τ_ν is set equal to τ and S_ν is set equal to $B_\nu(T)$:

$$I_\nu(\mu, 0) = \int_0^\infty B_\nu[T(\tau)]e^{-\tau/\mu}\,\frac{d\tau}{\mu}, \tag{4.18}$$

where $T(\tau)$ is given in the Eddington approximation by equation (4.17). For no star other than the Sun do we have access to (angularly resolved) measurements of $I_\nu(\mu,0)$. For other stars, we must be content with the *emergent spectral energy distribution*:

$$F_\nu(0) \equiv 2\pi \int_0^1 \mu I_\nu(\mu,0)\,d\mu. \tag{4.19}$$

Substituting equation (4.18) into equation (4.19), we reverse the order of integration and obtain

$$F_\nu(0) = 2\pi \int_0^\infty E_2(\tau) B_\nu[T(\tau)]\,d\tau, \tag{4.20}$$

where $E_n(\tau)$ is the n-th exponential integral,

$$E_n(\tau) \equiv \int_1^\infty x^{-n} e^{-\tau x}\,dx, \tag{4.21}$$

with convenient numerical approximations given in Abramowitz and Stegun 1965.

As an empirical fact (see Problem Set 1), the observed energy distributions for actual stars (e.g., the Sun) with a given $T_{\rm eff}$ agree qualitatively but not quantitatively with the theoretical calculations for an LTE grey atmosphere. Starting around the 1920s and continuing into the 1940s, astronomers (principally Milne, Stromgren, and Munch) began to use this discrepancy to try to infer the frequency variation of the unknown source for the continuum opacity κ_ν in the Sun (and other late-type stars). The theoretical tool they used assumed that if the model grey opacity κ were chosen equal to the Rosseland mean of the true opacity (assumed to be purely absorptive), then the T-τ relation given by equation (4.17) would continue to be a good approximation. Recall that in the underlying optically thick regions, the radiative flux can be expressed as

$$F_{\rm rad} = \frac{c}{3\rho\kappa_{\rm R}} \frac{d}{dz}(aT^4). \tag{4.22}$$

If we define $d\tau \equiv \rho\kappa_{\rm R}\,dz$ and set $F_{\rm rad} = caT_{\rm eff}^4/4$, we may integrate equation (4.22) to obtain

$$T^4 = \frac{3}{4} T_{\rm eff}^4 (\tau + \tau_0), \tag{4.23}$$

which agrees with equation (4.17) if we identify the integration constant τ_0 as 2/3.

The astronomers now assumed that the unknown κ_ν could be expressed as a function $\alpha(\nu)$ of ν alone (with no dependence on z) times $\kappa_{\rm R}$, i.e., that $\tau_\nu = \alpha(\nu)\tau$ in equation (4.4). By varying $\alpha(\nu)$, they could try to match the observed energy distributions of the actual stars. They discovered that the frequency dependence $\alpha(\nu)$ of the unknown continuum opacity in the Sun and other late-type stars had to increase from a minimum at about 16,000 Å, reach a maximum around 8,000 Å, and decline again toward shorter wavelengths. No then-known terrestrial material had these properties. There arose, then, a serious "opacity puzzle" in that, as late as 1940, no one knew what source of radiation determines the color of the Sun, and ultimately acts as the source of all hues on Earth!

Rupert Wildt made the suggestion that solved the puzzle. He proposed that the negative ion H^- (atomic hydrogen with a second electron loosely attached to it, with a binding energy of 0.754 eV) provides the continuum opacity of the Sun and most other late-type stars. Both *bound-free* transitions (with $h\nu > 0.754\,\mathrm{eV}$, i.e., $\lambda < 16{,}500\,\text{Å}$) and *free-free* transitions (with any $h\nu$) contribute to the process. The free-free process is somewhat unusual in comparison with the normal example that we will study in Chapter 15 in that the free (ionized) state of the H^- ion is a neutral H atom plus an electron. We would not expect the interaction of a neutral atom (even one with an asymmetric distribution of electronic charge) and an electron to be able to emit (or absorb) much radiation, but the large abundance of neutral H atoms in the solar atmosphere makes up for the deficiency. The free electrons, as well as those that attach themselves to the H^-, come mainly from easily ionized metallic species in the solar atmosphere, e.g., sodium.

Nevertheless, Wildt's suggestion overtaxed the ability of quantum physicists to make quantitative calculations, because none of the standard perturbation techniques for multi-electron atoms (see Chapter 27) yields very good results for the H^- ion. In 1945 the greatest astrophysicist of our time, Chandrasekhar (and a year later with Breen), undertook a delicate quantum-mechanical calculation of this unusual species with a single bound state, and found that its absorption cross section for bound-free and free-free transitions, combined with the likely abundance of the H^- ion in the solar atmosphere (see Chapter 7), satisfies all of the observational requirements (see Figure 4.2.) It was a glorious moment for radiative-transfer theory.

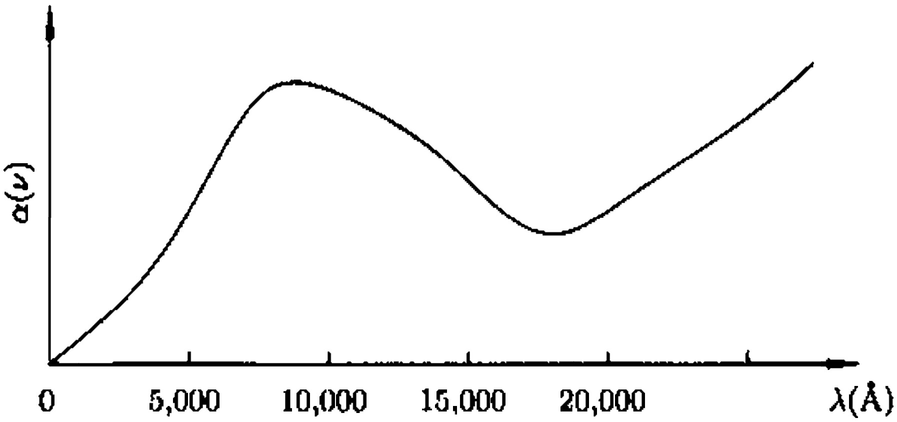

FIGURE 4 2
Frequency dependence of H^- bound-free and free-free opacity at conditions typical of the solar atmosphere.