HW 5

1. $P = 10^4 W$

a) 93.1 KXCI (as the name suggests) broadcasts at 93.1 MHz

$$\lambda = \frac{3\times10^8}{93.1\times10^6} = \frac{300}{93.1} \sim 3.22m$$

common freq of cell communication

$\sim$ 1000 MHz (800-1900 MHz) typically

$$\lambda = \frac{3\times10^8}{10^3\times10^6} = 0.3m$$

Dipole moment can roughly be given by $d = qL$

b) ~~$d = q\,dL$ (Dipole moment)~~ $q = \int I\,dt$

current in antenna

$$I = I_0 \sin(\omega t)$$

$$q = \frac{-I_0 \cos(\omega t)}{\omega}$$

$$\boxed{d = \frac{-I_0 \cos(\omega t)}{\omega} L}$$

$$\ddot{d} = \frac{I_0 \omega^2 \cos(\omega t) L}{\omega} = \boxed{I_0 \omega \cos(\omega t) L}$$

c) $E_{rad} = \frac{q}{Rc^2} n \times (n \times \dot{v})$

→ Elecnic field that radiates

$|E_{rad}| = \frac{q\dot{v}}{Rc^2} \sin\theta$

S (Poynting flux) $= \frac{c}{4\pi} E_{rad}^2 = \frac{c}{4\pi} \frac{q^2 \dot{v}^2}{R^2 c^4} \sin^2\theta$

$\frac{dW}{dt\, d\Omega} = S R^2 d\Omega$

$= \frac{q^2 \dot{v}^2}{4\pi c^3} \sin^2\theta$

$P = \frac{dW}{dt} = \int \frac{q^2 \dot{v}^2}{4\pi c^3} \sin^2\theta \, d\Omega$

$= \frac{q^2 \dot{v}^2}{2c^3} \int_{-1}^{1} (1-\mu^2)\, d\mu$

Larmor formula for radiated power by dipole $P = \frac{2 q^2 \dot{v}^2}{3c^3}$

~~$q\dot{v} = \ddot{d}$~~
~~$d = qx$~~
~~$\dot{d} = qx$~~

$P = \frac{2(\ddot{d})^2}{3c^3}$

$q\dot{v} = q\frac{d^2x}{dt^2} = \ddot{d}$

ignoring 2/3

$P \sim \frac{(\ddot{d})^2}{c^3} \sim \frac{I_0^2 \omega^2 \cos^2(\omega t) L^2}{c^3}$

$\lambda = \frac{c 2\pi}{\omega}$

$\langle P \rangle \sim \frac{I_0^2 \omega^2 L^2}{2c^3} \sim \frac{I_0^2}{c} \left(\frac{L}{\lambda}\right)^2 4\pi^2$

as $\langle \cos^2 \omega t \rangle = 1/2$

→ ignoring

→ not ignoring

$4\pi^2 \sim 40$ is big so I didnt remore, but if I ignore it
$I_0 \sim \sqrt{\langle P\rangle c} \sim 1.732e6$ A

$\pi^2 \sim 10$

$\frac{L}{\lambda} \simeq 1 \Rightarrow \langle P\rangle \sim 40 I_0^2/c$

(given)

$\frac{c\,10^4}{40} \sim I_0^2$

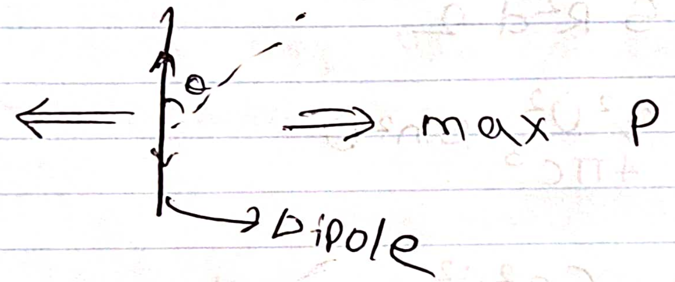
$\boxed{\cancel{I_0 \sim 100A}}$

$I_0^2 \sim 7.5\times10^{10}$

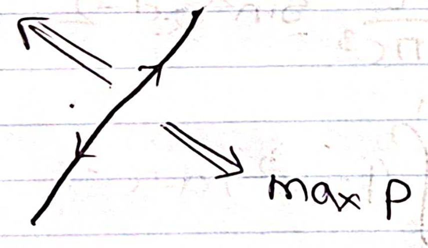
$\boxed{I_0 \sim 2.74\times10^{5}A}$

$P \propto \sin^2\theta$

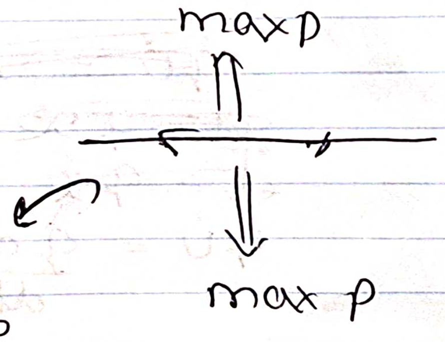
Best orientation depends on where do we want to transmit most energ



Better for ground.



Best position to recieve max P is probably somewhere in b/w vertical & horizontal



Better for space transmission?

To transmit max P from antenna to Phone

2. a) $\sigma_e = 0.665 \times 10^{-24}\ cm^2$

$X_H = X$ $\quad$ $X_{He} = Y = 1 - X$ $\quad$ $\boxed{m_p = m_n}$

mass fractions $\quad$ $m_e = 0$

absorption coeff $\alpha = n_e \sigma$ → cross section

$= \rho K$ $\quad$ Assuming fully ionized

$\Downarrow$

$K = \frac{n_e \sigma}{\rho}$ $\quad$ $K = \left(\frac{Y}{2m_p} + \frac{X}{m_p}\right)\sigma = \left(\frac{1-X}{2} + X\right)\frac{\sigma}{m_p}$

opacity

H → $1e^-$ $1p$

He^- → $2p, 2n$ $\quad 2e^-$

$\rho_{He} = Y\rho$

$\rho_H = X\rho$

$\cancel{n_e} = \frac{N}{V}$

mass density for He

$n_{He} = \frac{Y\rho}{4m_p}$ → (2p + 2n per He nucleus)

$n_e = 2\left(\frac{Y\rho}{4m_p}\right) + 1\left(\frac{X\rho}{m_p}\right)$ $\quad$ $n_H = \frac{X\rho}{m_p}$ → 1 p per H | particle

free $2e^-$ per He nucleus $\quad$ $1e^-$ per free H nucleus

$$K \sim \frac{(1+X)}{2}\frac{\sigma}{m_p}$$

$$= (1+X)\,\frac{0.665\times10^{-24}\ cm^2}{2\times1.67\times10^{-24}\ g}$$

$$\sim 0.199\,(1+X)\,\frac{cm^2}{g}$$

$$\boxed{K \sim 0.2\,(1+X)\,\frac{cm^2}{g}}$$

as H mass fraction ↓ opacity ↑

b) $d\tau = \alpha\, ds$ (optical depth formula)

$$\int d\tau = \int_0^R \rho K\, ds$$

(Assuming average value for ρ, K)

$$\tau \sim \langle \rho K \rangle R$$

$$\tau \sim \frac{\overset{\text{mass}}{M}}{\underbrace{\frac{4\pi R^3}{3}}_{\text{volume}}}(0.2)(1+X)\,\underset{\text{radius}}{R}$$

$$= \frac{3}{4\pi}\frac{M}{R^2}(0.2)(1+X)$$

$M = M_{sun}$ $X \sim 0.7$ (Assuming for Sun)

$R = R_{sun}$

$$\boxed{\tau \sim 3.35\times10^{10}}$$

8) c) $\lambda(\text{MFP}) = \frac{1}{\alpha} = \frac{1}{\rho K}$
(mean free path)

$$\lambda \sim \frac{R}{\rho K R} \sim \frac{R}{\tau} \sim 2\,\text{cm}$$

scattering of photons is a random walk

$$N \sim \frac{R^2}{\lambda^2} \sim \tau^2$$ (for optically thick)

$$N \sim (3.34)^2 \times 10^{22}$$

$$\sim \cancel{1.115 \times 10^{22}}$$

$$\boxed{N \sim 1.11 \times 10^{23}}$$ scatterings

$$t_{esc} \sim N \times \frac{\lambda}{c} \sim \cancel{1.12 \times 10^{23}} \times \frac{R}{\tau}$$

$\frac{\lambda}{c}$ → time b/w each scattering

$$\sim \tau^2 \times \frac{R}{\tau c} \sim \frac{R\tau}{c}$$

$$\boxed{t_{esc} \sim \cancel{7.74 \times 10^{11}}\ 2452\ \text{yr}}$$

so → long time stars are pretty opaque ↗ long stability

this is relevant for how much time it takes to transport energy in sun from center to surface

$\cancel{10^{23}}$ $\cancel{2\,\text{cm}}$ 3×10^{8} m

However we are assuming that ~~only~~ sun is fully ionized, there are no other sources of opacity & ignoring the interior structure of star by assuming average values. None of the above are true in practice. Only the core can be assumed to be nearly fully ionized. We are also ignoring other energy transport like convection

d) $$\frac{\partial T}{\partial r} = \frac{-3}{16\pi a c}\frac{K}{r^2}\frac{\rho L}{T^3}$$

local Luminosity $L = 4\pi r^2 F \rightarrow$ local

$$\frac{\partial T}{\partial r} = \frac{-3}{4\cancel{16\pi} ac}\frac{K}{\cancel{r^2}}\frac{\rho}{T^3}\cancel{4\pi r^2}F$$

$$\frac{\partial T}{\partial r} = \frac{-3}{4ac}\frac{K\rho}{T^3}F$$

Equation for energy transport by radiation

$F \Rightarrow$ constant ~~surface~~

$$\frac{\partial T}{\partial r} = -\frac{A}{T^3} \qquad A = \frac{3}{4ac}K\rho F$$

$$\left.\frac{T^4}{4}\right|_{T_s}^{T_c} = \int_R^0 -A\, dr$$

a: radiation constant

$$\frac{T_c^4}{4} = AR$$

$$\frac{ac}{\sigma} = 4$$

$$T_c^4 = \cancel{4}\frac{3}{\cancel{4}ac}K\rho F R$$

$$\sim \frac{3}{4}\frac{K\rho F R}{\sigma} \longrightarrow \text{substituting average values for sun}$$

$$\boxed{T_c \sim 2.3 \times 10^6 K}$$

$K = 0.2(1+0.7)$

3 dof

$$\frac{3}{2} kT \sim \frac{GMm_p}{R}$$

$$T \sim \frac{2}{3} \frac{GMm_p}{kR}$$

Substituting values for sun, we get

$$T \sim 1.54 \times 10^{7} K$$

(Alternate derivation in the end)

3. For rad reaction to be treated as perturbation,

$T \Rightarrow$ time to change particle KE significantly (mv^2)

$$T \sim \frac{mv^2}{P_{rad}} \sim \frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2 \qquad a = \dot{v}$$

time for which $P_{rad} \times T$ be ones similar to KE $\sim mv^2$

$$\frac{v}{a} \sim t_p \quad \text{(typical orbit time for particle)}$$

For $\frac{T}{t_p} \gg 1$, $\quad \frac{3mc^3}{2e^2} \frac{t_p^2}{t_p} \gg 1$

$$t_p \gg \frac{2e^2}{3mc^3} \equiv \tau$$

$\sim 10^{-23}$ s

If timescale $\gg \tau$, rad reaction is a perturbation

$\tau \sim$ approx. time for radiation to cross electron radius r_0 $\quad \tau \sim \frac{2r_0}{3c} \quad r_0 = \frac{e^2}{mc^2}$

2 d) contd

U = radiation energy density

$= a T^4$

$\langle u \rangle = a T_{vir}^4$ (on average for order of magnitude estimate)

$$U_{rad} = \langle u \rangle V$$

$$= \langle u \rangle \frac{4}{3} \pi R^3$$

$$= a T_{vir}^4 \cdot \frac{4 \pi R^3}{3}$$

$$= \frac{4\sigma}{c} T_{vir}^4 \cdot \frac{4}{3} \pi R^3$$

substituting values for sun

$$U_{rad} \sim 2 \times 10^{40} \text{ J}$$

Rest mass energy $= M_{sun} c^2 \sim 1.79e47$ J

$$\boxed{\frac{U_{rad}}{M_{sun} c^2} \sim \cancel{1.12e^3}\ 1.12e-7}$$

b) $\ddot{x} = -\omega_0^2 x + \tau \dddot{x} + \frac{eE_0}{m} e^{i\omega t}$

To solve this we try a solution of the form

$x = x_0 e^{i\omega t}$

Simplifying, we get —

$$x_0\left(-\omega^2 + \omega_0^2 + \tau i\omega^3\right) = \frac{eE_0}{m}$$

$$x_0 = -\frac{eE_0}{m}\,\frac{1}{(\omega^2-\omega_0^2-i\omega^3\tau)}$$

we can also express x_0 as

~~x as x = x~~ $x_0 = |x_0|e^{i\delta}$ → Any complex

$$|x_0|e^{i\delta} = -\frac{eE_0}{m}\,\frac{1}{(\omega^2-\omega_0^2-i\omega^3\tau)}$$

$$= -\frac{eE_0}{m}\left(\omega^2-\omega_0^2-i\omega^3\tau\right)^{-1}$$

Raising to power $-1 \rightarrow$

$$|x_0|^{-1} e^{-i\delta} = \left(\frac{-eE_0}{m}\right)^{-1} (\omega^2 - \omega_0^2 - i\omega^3\tau)^{+1}$$

Taking absolute values on both sides -

$$\cancel{|x_0|^{-1} \cdot 1 = \left|\left(\frac{-eE_0}{m}\right)^{-1}\right|}$$

$$\frac{1}{|x_0|} \cdot 1 = \left|\frac{m}{-eE_0}\right| \left((\omega^2 - \omega_0^2)^2 + \omega^6\tau^2\right)^{1/2}$$

$$\boxed{|x_0| = \frac{eE_0}{m}\left[(\omega^2 - \omega_0^2)^2 + \omega^6\tau^2\right]^{-1/2}}$$

taking argument of both sides

$$-\tan\delta = \frac{-\omega^3\tau}{\omega^2 - \omega_0^2}$$

✓

$$e^{-i\delta} = \cos\delta - i\sin\delta$$

$$\rightarrow \boxed{\tan\delta = \frac{\omega^3\tau}{\omega^2 - \omega_0^2}}$$

$$-\frac{\sin\delta}{\cos\delta} = -\tan\delta$$

solution = $|x_0| e^{i\delta} e^{i\omega t}$

$$x = |x_0| e^{i(\omega t + \delta)}$$

~~the δ is the phase shift~~

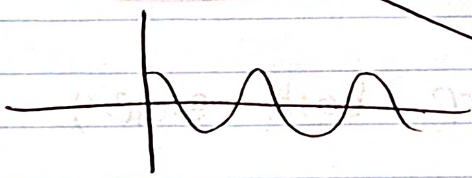
there is a phase shift in the particle response to the driving force

$\omega > \omega_0 \Rightarrow \delta > 0 \to$ particle leads

$\omega < \omega_0 \Rightarrow \delta < 0 \to$ particle lags

the actual soln to the differential eqn is the real part of x

$$= |x_0| \cos(\omega t + \delta)$$



amplitude $|x_0|$

frequency ω

Power (from dipole formula)

$$P = \frac{2e^2 \dot{u}^2}{3c^3} = \frac{2e^2 \ddot{x}^2}{3c^3}$$

$$P = \frac{2e^2}{3c^3} |x_0|^2 \omega^4 \cos^2(\omega t + \delta)$$

$\langle P \rangle$ using $\langle \cos^2(\omega t + \delta) \rangle = 1/2$

$$\langle P \rangle = \frac{2e^2}{3c^3} \frac{|x_0^2| \omega^4}{2}$$

$$= \frac{e^2}{3c^3} \omega^4 \left(\frac{eE_0}{m}\right)^2 \left[(\omega^2 - \omega_0^2)^2 + \omega^6 \tau^2\right]^{-1}$$

$$\boxed{\langle p \rangle = \frac{e^4 E_0^2}{3m^2c^3} \frac{\omega^4}{\left[(\omega^2 - \omega_0^2)^2 + (\omega^3 \tau)^2\right]}}$$

The ~~energy~~ average poynting flux due to the incident beam (forcing agent) is

$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

$\therefore$ cross section is the ratio of emitted power to incident i.e

$$\sigma(\omega) = \frac{\langle P \rangle}{\langle S \rangle} = \frac{e^4 8\pi}{3m^2c^4} \frac{\omega^4}{\left[(\omega^2 - \omega_0^2)^2 + (\omega^3 \tau)^2\right]}$$

Note that $\frac{8\pi}{3} \frac{e^4}{(mc^2)^2} = \frac{8\pi}{3} r_0^2$

where $r_0 = \frac{e^2}{mc^2}$ & $\sigma_T = \frac{8\pi}{3} r_0^2$

Thomson cross section

$$\therefore \boxed{\sigma(\omega) = \sigma_T \frac{\omega^4}{\left[(\omega^2 - \omega_0^2)^2 + (\omega^3 \tau)^2\right]}}$$

[I did not make the approxim

c) $\sigma(\omega) = \sigma_T \dfrac{\omega^4}{[(\omega^2-\omega_0^2)^2+(\omega^3\tau)^2]}$

$\omega \ll \omega_0$ → low frequency incident radiation compared to oscillator strength

$\omega^2 - \omega_0^2 \sim -\omega_0^2$

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{[\omega_0^4+(\omega^3\tau)^2]}$$

since radiation reaction is only valid for $\omega_0\tau \ll 1$,

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{\omega_0^4\left[1+\left(\frac{\omega^3\tau}{\omega_0^2}\right)^2\right]}$$

$\because$ $\omega_0\tau \ll 1$

$\omega_0^3 \dfrac{\tau}{\omega_0^2} \ll 1$

since $\omega \ll \omega_0 \Rightarrow \dfrac{\omega^3\tau}{\omega_0^2} \ll \dfrac{\omega_0^3\tau}{\omega_0^2} \ll 1$

$\therefore$ $\boxed{\sigma(\omega) = \sigma_T \dfrac{\omega^4}{\omega_0^4}}$ → Intuitive makes sense from rayleigh scattering

- $\omega \gg \omega_0$ we have (incident beam energy high compared to oscillator strength)

$\cancel{\sigma(\omega)}$ $(\omega^2 - \omega_0^2)^2 \sim (\omega^2)^2$

$$\therefore \sigma(\omega) = \sigma_T \frac{\omega^4}{\omega^4 + (\omega^3 \tau)^2}$$

$$= \frac{\sigma_T}{1 + (\omega\tau)^2}$$

$\omega_0 \tau \ll 1$ & $\omega\tau \ll 1$ for

- radiation reaction formula to be valid

$\therefore \sigma(\omega) = \sigma_T$ thomson scattering independent of frequency as at high energies, all e^- are unbinded and scattering of free e^- is independent of frequency

$\omega \simeq \omega_0$ (incident energy comparable to oscillator strength)

$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \sim 2\omega_0(\omega - \omega_0)$
as $\omega \sim \omega_0$ (replace $\omega \to \omega_0$)

- $$\sigma(\omega) = \sigma_T \frac{\omega_0^4}{\left[4\omega_0^2(\omega - \omega_0)^2 + (\omega_0^3 \tau)^2\right]}$$

$$\sigma(\omega) = \sigma_T \frac{\omega_0^{\cancel{4}\,2}}{\cancel{\omega_0^2}\left[4(\omega-\omega_0)^2 + (\omega_0^2\tau)^2\right]}$$

$$\omega_0^2 \tau \equiv \Gamma \text{ (line width)}$$

$$\sigma(\omega) = \frac{\sigma_T}{\tau} \frac{\omega_0^2 \tau}{\left[4(\omega-\omega_0)^2 + \Gamma^2\right]}$$

$$= \frac{\sigma_T\,\Gamma}{\tau\ 4} \frac{1}{\left[(\omega-\omega_0)^2 + (\Gamma/2)^2\right]} \rightarrow D$$

$(\times \div \text{ by } \pi)$

$$= \frac{\pi \sigma_T}{2\tau} \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

$$\sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2c^4} \qquad \tau = \frac{2r_0}{3c} = \frac{2}{3c}\frac{e^2}{mc^2}$$

$$= \frac{2e^2}{3mc^3}$$

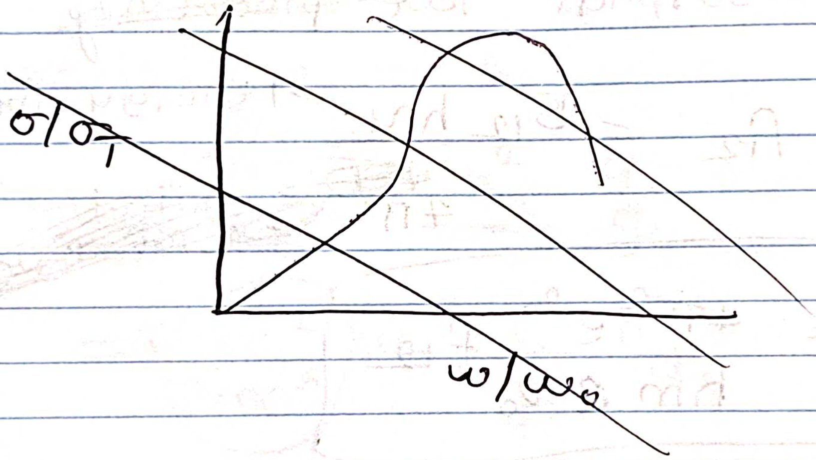
$$\sigma(\omega) = \frac{\frac{\cancel{8}\pi}{\cancel{3}} r_0^2 \Gamma}{\frac{\cancel{2} r_0}{\cancel{3}c}\ \cancel{4}} \frac{1}{D}$$

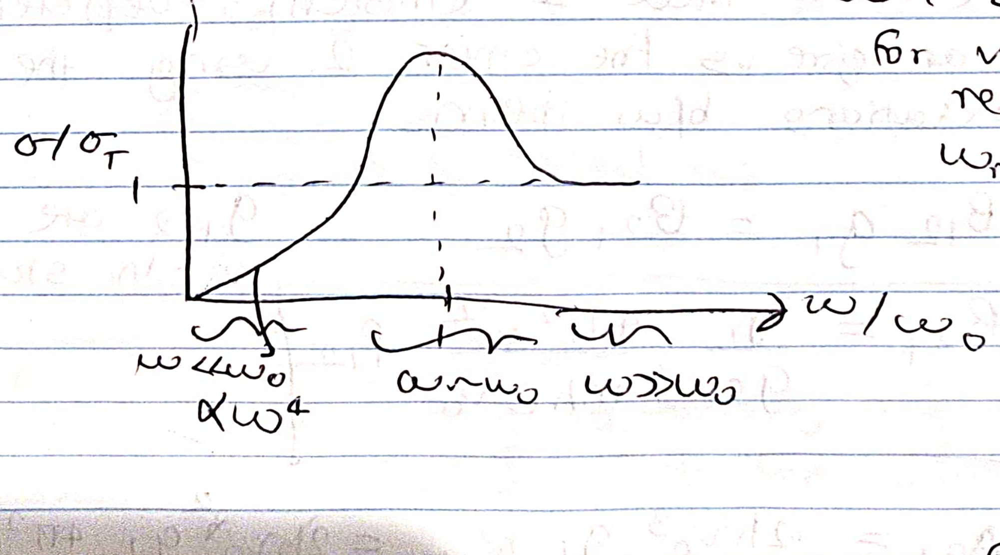
$$= \cancel{c}\,\pi \frac{e^2 \Gamma}{mc^{\cancel{2}}} \frac{1}{D}$$

Lorentz profile

$$\boxed{\sigma(\omega) = \frac{2\pi^2 e^2}{mc}\left(\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}\right)}$$

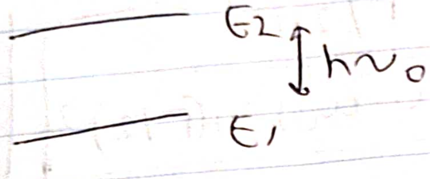
∴ Near the resonance of harmonic oscillator, the spectrum has the same shape as that of ~~undamp~~ undriven oscillar (free e^-) emission from





$\omega\tau << 1$
for valid reaction
$\omega_{max} << \frac{1}{\tau}$
upper limit for valid formula

4.



$$\int \sigma_\nu \, d\nu = \frac{\pi e^2}{mc} f_{12}$$

$B_{12} \rightarrow$ absorption ~~rate probability~~ Einstein coefficient

$$\frac{\pi e^2}{mc} f_{12} = B_{12} \frac{h\nu_0}{4\pi}$$ → energy for transition

$$\boxed{B_{12} = \frac{4\pi^2 e^2}{h m c \nu_0} f_{12}}$$

We know that 1 Einstein coefficient can give us the other 2 using the derived relations b/w them

$$B_{12} g_1 = B_{21} g_2$$ $g_{1,2}$ are degeneracies in state 1,2

$$\boxed{B_{21} = \frac{g_1}{g_2} \frac{4\pi^2 e^2}{m h c \nu_0} f_{12}}$$

$$A_{21} = \frac{2h\nu_0^3}{c^2} \frac{g_1}{g_2} B_{12} = \frac{2\cancel{h}\nu_0^{\cancel{3}\,2}}{c^2} \frac{g_1}{g_2} \frac{4\pi^2 e^2}{\cancel{h} m c \cancel{\nu_0}} f_{12}$$

$$\boxed{A_{21} = \frac{8\pi^2 e^2 \nu_0^2}{mc^3} \frac{g_1}{g_2} f_{12}}$$

b) $\omega_0 = 2\pi\nu_0$ $\qquad \tau = \frac{2r_0}{3c} = \frac{2e^2}{3c\,mc^2} = \frac{2e^2}{3mc^3}$

radiation reaction timescale

substituting in A_{21}

$$A_{21} = \frac{8\pi^2 e^2}{mc^3} \frac{\omega_0^2}{4\pi^2} \frac{g_1}{g_2} f_{12}$$

$$= 3\left(\frac{2e^2}{3mc^3}\right)\omega_0^2 \frac{g_1}{g_2} f_{12}$$

$$A_{21} = 3\tau\omega_0^2 \frac{g_1}{g_2} f_{12}$$

classical line width is defined as

$$\Gamma = \omega_0^2 \tau$$

$$\therefore \boxed{A_{21} = 3\Gamma \frac{g_1}{g_2} f_{12}}$$

A_{21} directly proportional to line width

2 c) alternate

$$T^4 = \frac{3}{4} T_{eff}^4 (\tau + 2/3)$$ (Eddington approximation)

$$T^4 = \frac{3}{4} \frac{F}{\sigma} (\tau + 2/3)$$

τ from surface to center
$= 3.34e10$

(Assuming this formula holds throughout sun)

$$T^4 = \frac{3}{4} \frac{F}{\sigma} \times \tau$$

substituting F_{sun} & $\tau = 3.34e10$ from 2b)

we get $\boxed{T_c \sim 2.3e6\ K}$ same as the prev method, which makes sense because the Eddington approxn was derived using the radiation ~~trans~~ transport equation assuming constant flux