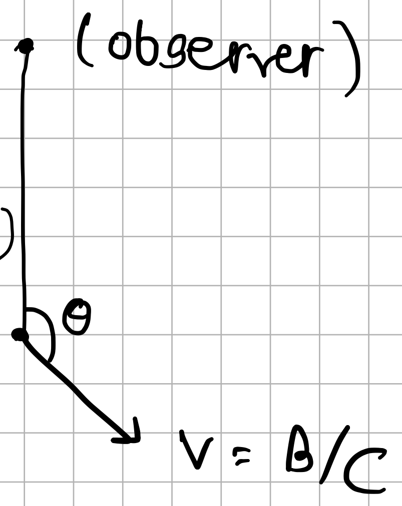
HW 6 ASTR 589

1. $$\nu = \frac{\nu_0}{\gamma(1-\beta\cos\theta)}$$

a) ν_0 is the frequency of the emitter moving with speed β away from the observer at an angle θ as shown below

Convention
$v > 0$ (away)
$v < 0$ (towards)



v : speed of source relative to observer

For cosmological redshift, the universe is isotropically expanding away from any observer, so the correct choice of $\underline{\underline{\theta = \pi}}$

$$\nu = \frac{\nu_0}{\gamma(1-\beta(-1))} = \frac{\nu_0}{\gamma(1+\beta)}$$

Now $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$\therefore$ ν (observed freq) $= \dfrac{\nu_0 \text{ (source freq)}}{\frac{1}{\sqrt{(1-\beta)(1+\beta)}} \cdot (1+\beta)}$

$$\nu = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}}$$

As $|\beta| < 1$ we can clearly see that $\nu < \nu_0$, i.e things are getting redshifted (moving to lower freq/ higher wavelength)

Now. $\nu = c/\lambda$

$$\therefore \quad \frac{\nu}{\nu_0} = \frac{\lambda_0}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}$$

we define z (redshift) as $z = \dfrac{\lambda}{\lambda_0} - 1$

$$\therefore \quad z = \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

$$\Rightarrow \quad \boxed{1 + z = \sqrt{\frac{1+\beta}{1-\beta}}}$$ where $\beta = v/c$

b) At a redshift $z = 1$, we can find β by using the above formula, assuming we are only measuring cosmological expansion velocities and local peculiar velocities are negligible

$$1 + 1 = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\Rightarrow \quad 4(1-\beta) = 1 + \beta$$

$$4 - 4\beta = 1 + \beta$$

$$\boxed{\beta = 3/5} \quad \Rightarrow \quad v = \frac{3}{5} c$$

for JADES-GSZ14-0 at $z = 14.32$ (farthest galaxy known)

$$1 + 14.32 = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$(15.32)^2 \, 1 - (15.32)^2 \beta = 1 + \beta$$

$$(15.32)^2 - 1 = \beta\left((15.32)^2 + 1\right)$$

$$233.7024 = \beta \cdot 235.7024$$

$$\boxed{\beta = \frac{233.7024}{235.7024} \sim 0.9915} \quad \Rightarrow v \approx 0.995 \, c$$

c) If we recive a collimated beam from a laser from the 214 galaxy with source power $P_0 = \frac{h\nu_0}{t_0}$

Note that $\frac{1}{t_0}$ transforms in the same way as ν_0

So we want to find $\frac{P}{P_0}$ where

$P = \frac{h\nu}{t}$ with $\frac{\nu}{\nu_0} = \sqrt{\frac{1-\beta}{1+\beta}}$

and $\frac{1/t}{1/t_0} = \sqrt{\frac{1-\beta}{1+\beta}}$ since they transform in same way

$$\therefore \frac{P}{P_0} = \frac{\frac{h\nu}{t}}{\frac{h\nu_0}{t_0}} = \frac{\nu}{\nu_0} \cdot \frac{t_0}{t}$$

$$= \sqrt{\frac{1-\beta}{1+\beta}} \cdot \sqrt{\frac{1-\beta}{1+\beta}} = \frac{1-\beta}{1+\beta}$$

$$\boxed{\frac{P}{P_0} = \frac{1-\beta}{1+\beta}}$$ $\Rightarrow$ lower power observed.

For the JADES galaxy we got

$$\beta = \frac{233.7024}{235.7024}$$

$$\therefore \boxed{\frac{P}{P_0} \sim 0.00426}$$

by plugging β in $\frac{P}{P_0} = \frac{1-\beta}{1+\beta}$

d) we know that

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\therefore 1-\beta^2 = \frac{1}{\gamma^2}$$

$$(1-\beta)(1+\beta) = \frac{1}{\gamma^2}$$

$$1-\beta = \frac{1}{\gamma^2(1+\beta)}$$

For $\beta \to 1$ $\frac{1}{1+\beta} \sim \frac{1}{2}$ ✱

$$\therefore \boxed{1-\beta = \frac{1}{2\gamma^2} + \text{higher order terms}}$$

*

Proof $\frac{1}{1+\beta} = \frac{1}{1+\beta+1-1} = \frac{1}{2+\beta-1}$

$$= \frac{1}{2\left(1+\frac{\beta-1}{2}\right)} = \frac{1}{2}\Big(1+\underbrace{\frac{\beta-1}{2}}_{\text{small}}\Big)^{-1}$$

$$= \frac{1}{2}\left(1-\left(\frac{\beta-1}{2}\right)\right)$$

$$= \frac{1}{2}\left(1-\frac{\beta}{2}+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{3-\beta}{2}\right)$$

$\beta \to 1 \quad \therefore \quad = \frac{1}{2}\left(\frac{2}{2}\right) = \frac{1}{2}$

Proved

$$\boxed{a = \frac{1}{2} \quad b = -2}$$

Q2.

a) collision rate

$$\boxed{\nu_c = n\sigma_c \overline{v_{th}}}$$

This makes sense because,
if there are more atoms per unit volume,

i.e $n \uparrow$, the collision rate would certainly increase as things are more densly packed.

Similarly if σ_c is higher, there is more chance of a colision as the interaction area σ_c becomes larger.

Lastly we know that $\bar{v}_{th} \propto \sqrt{T}$ and at higher T there is more random motion among atoms, so they are more likely to collide.

Now we know the frequency of collisions $\nu_c = n \sigma_c \bar{v}_{th}$. So the approx time b/w each colission is $1/\nu_c = \frac{1}{n\sigma_c \bar{v}_{th}}$. Now the mean free path is defined to be the distance traveled b/w each colission. Since an atom on average has speed $\bar{v}_{th}$, the MFP is $l_c = \frac{1}{\nu_c} \cdot \bar{v}_{th} = \frac{1}{n\sigma_c \bar{v}_{th}} \cdot \bar{v}_{th}$

$$\boxed{l_c = \frac{1}{n\sigma_c}}$$

→ This dimensionally makes sense and also we exceped $l_c \downarrow$ as $n \uparrow$ or $\sigma_c \uparrow$ as there is more chance of collission, so atom travels less b/w each collission!

b) we are given line profile width due to colissional broadening $\Gamma = 2\nu_c$

and doppler width $\Delta\nu_D \simeq \nu_0 \frac{v_{th,a}}{c}$

where $v_{th,a} = \sqrt{kT/m_a}$ → thermal speed of emitting atom of mass $m_a = A m_p$

$\frac{\Gamma}{\Delta\nu_D}$, the ratio of the 2 line widths

is $$\frac{\Gamma}{\Delta\nu_D} = \frac{2\nu_c}{\nu_0 \frac{v_{th,a}}{c}}$$

Plugging the ν_c from 2a)

and we know $\nu_0 \lambda_0 = c$

$$\therefore \frac{\Gamma}{\Delta\nu_D} = \frac{2 n \sigma_c \overline{v_{th}}}{\frac{1}{\lambda_0} v_{th,a}}$$

we know $l_c = \frac{1}{n\sigma_c}$

$$\therefore \frac{\Gamma}{\Delta\nu_D} = \frac{2\lambda_0}{l_c} \cdot \frac{\overline{v_{th}}}{v_{th,a}}$$

$$\overline{v_{th}} \propto \sqrt{\frac{kT}{1 m_p}}$$ (we mostly have H atoms)

$$v_{th,a} \propto \sqrt{\frac{kT}{A m_p}}$$ (emitting atom)

Since we are ignoring order unity constants,

$$\boxed{\frac{\Gamma}{\Delta\nu_D} \approx \frac{\lambda_0}{l_c}\sqrt{A}}$$

c) For sodium doublet line, $\lambda_0 = 589\,nm$

$\sigma_c \sim 4\pi a_0^2$ → Bohr radius

$$\frac{\Gamma}{\Delta\nu_D} \sim \lambda_0 n \sigma \sqrt{23}$$ $A_{Na} = 23$

$P = n k_B T$ (ideal gas law)

$$n = \frac{P}{k_B T}$$

$$\therefore \frac{\Gamma}{\Delta\nu_D} \sim \lambda_0 \frac{P}{k_B T} \cdot 4\pi a_0^2 \sqrt{23}$$

From HW 3, we know

$$P_{gas}(\tau = 0) = \frac{2g}{3K} \quad \text{where } g = \frac{GM}{R^2}$$

and K = mean opacity
$= 0.3 \, cm^2/g$

$$\therefore \frac{\Gamma}{\Delta\nu_D} \sim \frac{\lambda_0}{k_B T} \frac{2}{3K_R} \cdot \frac{GM}{R^2} \, 4\pi a_0^2 \sqrt{23}$$

Plugging in appropriate values for Sun, a_0, $T = 5800K$ we get

$$\boxed{\frac{\Gamma}{\Delta\nu_D} \sim 0.0076}$$

i.e $\Gamma << \Delta\nu_D$ which means that at line center doppler broadening is much more important than colissional broadening

Q3 Doppler : $\Delta\nu_D$

collisional : Γ

$\Delta\nu = \nu - \nu_0$ relative to line center

$\nu_0 = c/\lambda_0$

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} H(u,a)$$

$$H(u,a) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a e^{-y^2}}{(u-y)^2 + a^2}\, dy$$

$$u = \frac{\Delta\nu}{\Delta\nu_D} \qquad a = \frac{\Gamma}{4\pi\Delta\nu_D}$$

a) As $a \to 0$, voigt reduces to doppler

Given : $\lim_{a \to 0} \left(\frac{a}{x^2 + a^2} \right) \Rightarrow \pi\,\delta(x)$

Using this in $H(u,a)$

$$\lim_{a\to 0} H(v,a) = \lim_{a\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a e^{-y^2}}{(v-y)^2+a^2}\, dy$$

No we have $\lim_{a\to 0} \frac{a}{(v-y)^2+a^2}$

$$= \pi\,\delta(v-y)$$

Plugging it in,

$$\lim_{a\to 0} H(v,a) = \frac{1}{\pi}\int_{-\infty}^{\infty} \pi\,\delta(v-y)\,e^{-y^2}\,dy$$

Using $\int_{-\infty}^{\infty} \delta(v-y)\,e^{-y^2}\,dy = e^{-v^2}$

we get

$$\lim_{a\to 0} H(v,a) = \frac{1}{\pi}\cdot \pi\, e^{-v^2} = e^{-v^2}$$

Plugging this in $\phi(\Delta\nu)$ we get,

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D}\, e^{-v^2}$$

Using $v = \dfrac{\Delta\nu}{\Delta\nu_D}$ we get

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D}\, e^{-\frac{(\Delta\nu)^2}{(\Delta\nu_D)^2}}$$

$$\boxed{\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D}\, e^{-\frac{(\nu-\nu_0)^2}{\Delta\nu_D^2}}}$$

This is the doppler profile hence proved that as $a \to 0$, Voigt reduces to doppler while at line wings, collisional will be more important as the doppler broadening shape is a gaussian (exponential dropoff) and for collisional it is Lorentzian, and away from their centers, Lorentzian has slower dropoff than a gaussian.

Q3 b) We know that the emergent intensity I_ν through a slab of source S_ν, optical depth τ_ν

is $$I_\nu = \underbrace{I_{\nu,b}}_{\text{Background}} e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)$$

Equivalent width EW

$$\equiv \int \left| 1 - \frac{I_\nu}{I_{\nu,b}} \right| d\lambda$$

Plugging in I_ν, we get

$$EW = \int \left| 1 - \frac{\left[I_{\nu,b} e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})\right]}{I_{\nu,b}} \right| d\lambda$$

$$EW = \int \left| \frac{I_{\nu,b}(1-e^{-\tau_\nu}) - S_\nu(1-e^{-\tau_\nu})}{I_{\nu,b}} \right| d\lambda$$

$$EW = \int \left| \frac{(I_{\nu,b} - S_\nu)(1-e^{-\tau_\nu})}{I_{\nu,b}} \right| d\lambda$$

$$EW = \int \left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| (1-e^{-\tau_\nu})\, d\lambda \qquad 1-e^{-\tau_\nu} > 0$$

always

Since the EW integral is over a range of freq $\ll \nu_0$, $\Rightarrow I_{\nu,b}$, S_ν are constant almost in the range

$$\therefore \quad EW = \left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| \int (1-e^{-\tau_\nu})\, d\lambda$$

range of freq $\gg \Delta\nu_D \Rightarrow u \to \infty$

$$\lambda = \frac{c}{\nu} \quad \text{(change of variable)}$$

$$\frac{d\lambda}{d\nu} = -\frac{c}{\nu^2} \qquad \frac{d\lambda}{d\nu} = -\frac{\lambda}{\nu}$$

in the range of freq we are interested

in $\dfrac{d\lambda}{d\nu} \sim \dfrac{-\lambda_0}{\nu_0}$ $\qquad \lambda_0, \nu_0 \rightarrow$ line center

$$\therefore EW = \left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| \int (1 - e^{-\tau_\nu})\, d\nu \left(\frac{-\lambda_0}{\nu_0} \right)$$

$$\therefore \boxed{\frac{EW}{\lambda_0} = -\left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| \int (1 - e^{-\tau_\nu}) \frac{d\nu}{\nu_0}}$$

multiplying by $\frac{\nu_0}{\Delta \nu_D}$ we get

$$\frac{EW}{\lambda_0} \frac{\nu_0}{\Delta \nu_D} = -\left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| \int (1 - e^{-\tau_\nu}) \frac{d\nu}{\cancel{\nu_0}} \frac{\cancel{\nu_0}}{\Delta \nu_D}$$

$$\frac{\Delta \lambda_D}{\Delta \nu_D} \sim -\frac{\lambda_0}{\nu_0} \quad \left(\begin{array}{c} \text{because} \\ \Delta \nu_D \ll \nu_0 \\ \Delta \lambda_D \ll \lambda_0 \end{array} \right)$$

$$\therefore \frac{EW}{\cancel{\lambda_0}} \cdot \frac{\cancel{\nu_0}}{-\Delta \lambda_D \frac{\cancel{\nu_0}}{\cancel{\lambda_0}}} = -\frac{EW}{\Delta \lambda_D}$$

$\therefore$ we get $\frac{EW}{\Delta \lambda_D} = \left| 1 - \frac{S_\nu}{I_{\nu,b}} \right| \int (1 - e^{-\tau_\nu}) \frac{d\nu}{\Delta \nu_D}$

we have defined $u = \frac{\Delta \nu}{\Delta \nu_D}$.

$$\therefore \boxed{\frac{EW}{\Delta\lambda_D} = \left|1 - \frac{S_\nu}{I_{\nu,0}}\right| \int \left(1 - e^{-\tau_\nu}\right) du}$$

c) we have

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\Delta\nu_D} H(u,a)$$

where $H(u,a) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a e^{-y^2}}{(u-y)^2 + a^2} dy$

Given $a = 0.1$

$$-4 < \frac{\Delta\nu}{\nu_D} = u < 4$$

$$\underbrace{\phi(\Delta\nu) \cdot \Delta\nu_D}_{y} = \underbrace{\frac{1}{\sqrt{\pi}} H(u,a)}_{f(u,\, a=0.1)}$$

Have to plot y vs $f(u)$
(log scale)

For pure doppler,

$$\phi(\Delta\nu)\,\Delta\nu_D = \frac{1}{\sqrt{\pi}}\, e^{-\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2}$$

$$= \frac{1}{\sqrt{\pi}}\, e^{-v^2}$$

For pure pressure,

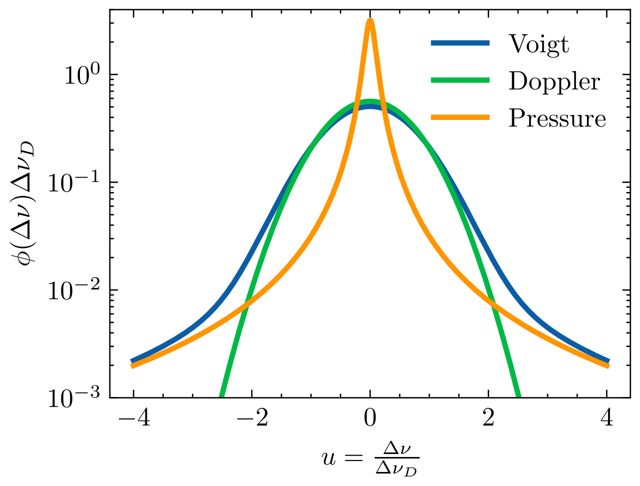
$$\phi(\Delta\nu) = \frac{\Gamma/4\pi^2}{\Delta\nu^2 + (\Gamma/4\pi)^2}$$

where $\Gamma = 4\pi\,\Delta\nu_D\, a$

$$\therefore\ \phi(\Delta\nu) = \frac{\frac{1}{\pi}\,\Delta\nu_D\, a}{\Delta\nu^2 + (\Delta\nu_D a)^2}$$

$$\phi(\Delta\nu)\,\Delta\nu_D = \frac{a/\pi}{\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2 + a^2}$$

$$= \frac{a/\pi}{v^2 + a^2}$$


Voigt
Doppler
Pressure
10^0
10^{-1}
10^{-2}
10^{-3}
$\phi(\Delta\nu)\Delta\nu_D$
−4
−2
0
2
4
$u = \frac{\Delta\nu}{\Delta\nu_D}$

we can clearly see that the tail of pressure matches voigt, and towards center, doppler matches voigt well

d) we want to plot

$$e^{-\tau_\nu} \quad \text{vs} \quad u = \frac{\Delta\nu}{\Delta\nu_D}$$

where $\tau_\nu = \tau_0 \frac{\phi(\Delta\nu)}{\phi(0)}$

$\tau_0 = 0.1, 1, 10$

Doppler & $a = 0.3$ Voigt profile

For doppler

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} e^{-\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2}$$

$$\phi(0) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D}$$

$$\tau_\nu = \tau_0 e^{-\left(\frac{\Delta\nu}{\Delta\nu_D}\right)^2} = \tau_0 e^{-u^2}$$

as $\quad v = \dfrac{\Delta\nu}{\Delta\nu_D}$

Plot $\quad e^{-\tau_\nu} \quad$ vs $\quad v$

For voigt $a = 0.3$

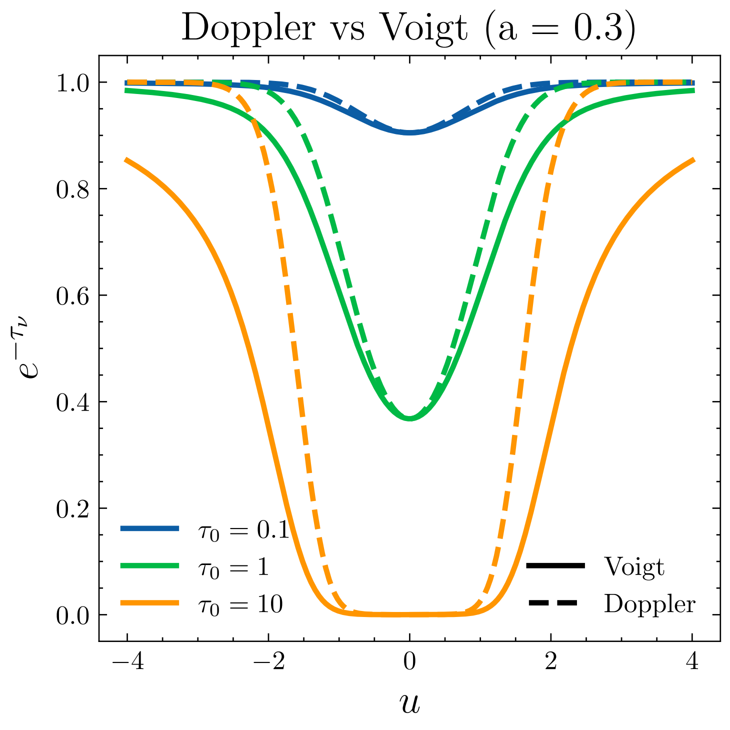
$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} H(v,a)$$

$$\phi(0) = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} H(0,a)$$

$$\therefore \tau_\nu = \tau_0 \frac{H(v,a)}{H(0,a)}$$

where $a = 0.3$

Plot $\quad e^{-\tau_\nu} \quad$ vs $\quad v$ [redacted]



Doppler vs Voigt (a = 0.3)
$e^{-\tau_\nu}$
u
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4
$\tau_0 = 0.1$
$\tau_0 = 1$
$\tau_0 = 10$
Voigt
Doppler

e) Plot EW vs τ_0 for $0.01 < \tau_0 < 1000$

$a = 0$ (Pure doppler)

$a \to \infty$ (pure pressure)

$a = 0.01, 0.1, 1$ Voigt

we are given $S_\nu = 0$ (only absorption & no emission)

$$\frac{EW}{\Delta\lambda} = \int_{-\infty}^{\infty} \left(1 - e^{-\tau_v}\right) dv$$

Note that for doppler,

$1 - e^{-\tau_v}$ is symmetric about v

$$\therefore \frac{EW}{\Delta\lambda_D} = 2\int_0^{\infty} \left(1 - e^{-\tau_v}\right) dv$$

For pressure, we are given

$$\frac{EW}{\Delta\lambda_p} = \int_{-\infty}^{\infty} \left[1 - e^{-\tau_0/(1+y^2)} \right] \frac{dy}{\sqrt{\pi}}$$

Note that this is also symmetric by y

$$\therefore \int_{-\infty}^{\infty} \rightarrow 2\int_{0}^{\infty}$$

Now for voight, we are given

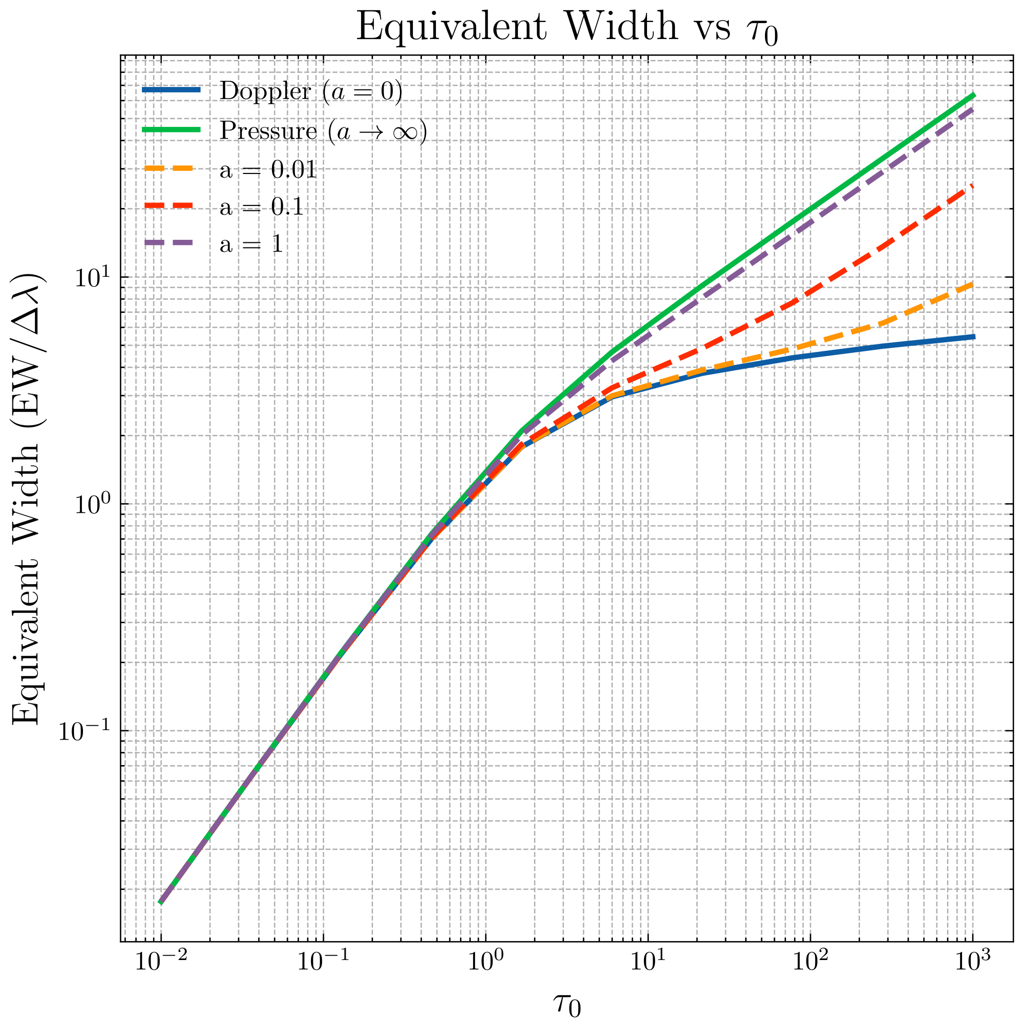
$$\Delta\lambda_V = \frac{\Delta\lambda_D}{H(0,a)}$$

$$\therefore \frac{EW}{\Delta\lambda_V} = H(0,a) \int (1 - e^{-\tau_v}) dv$$

where $\tau_v = \dfrac{H(v,a)}{H(0,a)} \tau_0$

we have to plot

$\dfrac{EW}{\Delta\lambda_V}$ vs τ_0 for $a = 0.01, 0.1, 1$



Equivalent Width vs τ_0
Doppler ($a = 0$)
Pressure ($a \to \infty$)
a = 0.01
a = 0.1
a = 1
Equivalent Width (EW/$\Delta\lambda$)
τ_0
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}