HW 6: Doppler Effect, Line Shapes

Due: Oct. 8, 9:30AM on D2L

ASTR 589, Prof. Youdin

1. **Doppler shifts** are given by the relativistic formula

$$\nu = \frac{\nu_0}{\gamma(1 - \beta \cos\theta)}$$

   (a) Derive the formula for cosmological redshift in an expanding universe, $z = \lambda/\lambda_0 - 1$, in terms of $\beta = v/c$, explaining the choice of θ.

   (b) What is the recession speed, as β, for a $z = 1$ galaxy? For the current record, $z = 14.32$ (JWST, JADES-GS-z14-0)?

   (c) Assume (rather unrealistically) that we now see light from a perfectly collimated laser that was fired from JADES-GS-z14-0 with rest frame power $P_0 = h\nu_0/t_0$. The rate, $1/t_0$, transforms just like frequency, ν_0. What is P/P_0 the ratio of observed to emitted power, first for general β then numerically evaluated?

   (d) As $\beta \to 1$, express $\epsilon = 1 - \beta \ll 1$ as $\epsilon \approx a\gamma^b$ giving the constant a and exponent b. *Hint, if needed, not the only way: Taylor expand $\gamma(\epsilon)$ about $\epsilon = 0$.*

2. **Doppler vs. Pressure/Collisional Broadening** We will estimate the line profile width of due to collisional broadening, $\Gamma = 2\nu_c$, and compare to the Doppler width $\Delta\nu_D \simeq \nu_0 v_{\mathrm{th,a}}/c$, where $v_{\mathrm{th,a}} = \sqrt{kT/m_a}$ is the thermal speed of the emitting atom of mass $m_a = Am_p$.

   (a) Express the collision rate, ν_c, and mean free path, ℓ_c, in terms of the number density of all atoms, n, the collisional cross section σ_c and the average thermal speed $\overline{v}_{\mathrm{th}}$. Justify these expressions.

   (b) Express the ratio $\Gamma/\Delta\nu_D$ in terms of λ_0 (line wavelength), ℓ_c and A only. Assume that $\overline{v}_{\mathrm{th}}$ is given by atomic hydrogen (ignoring free electrons). Ignore order unity constants.

   (c) Now estimate $\Gamma/\Delta\nu_D$ for the Sodium doublet line with $\lambda_0 = 589$ nm in the Solar photosphere. For a rough estimate, use $\sigma_c \simeq 4\pi a_0^2$ (including this larger order unity factor for once), where a_0 is the Bohr radius. Apply the ideal gas law (with P, n, T) at the Solar photosphere, where T is known and the P can be estimated from the mean opacity $\kappa_R \simeq 0.3$ cm^2/g (see assignment 3). From your estimate, is thermal or collision broadening more important at line center? What about the line wings?

3. **Voigt profiles** Consider the Voigt profile for the line shape including Doppler broadening and collisional/pressure broadening, with widths $\Delta\nu_D$ and Γ, for $\Delta\nu = \nu - \nu_0$, the frequency relative to line center $\nu_0 = c/\lambda_0$.

$$\phi(\Delta\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} H(u, a)$$

$$H(u, a) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a \exp(-y^2)}{(u - y)^2 + a^2} dy$$

with $u = \Delta\nu/\Delta\nu_D$ and $a = \Gamma/(4\pi\Delta\nu_D)$ [matching Rybicki & Lightman].

(a) Show that the Voigt profile reduces to the Doppler profile as $a \to 0$ (i.e. no pressure broadening), using the identity for the delta function:

$$\lim_{a\to 0}\left(\frac{a}{x^2+a^2}\right) \to \pi\delta(x).$$

(b) Remind yourself of the solution for the emergent intensity I_ν when a background intensity $I_{\nu,b}$ passes though a slab of material with constant source function S_ν and with optical depth τ_ν. For this standard solution, show that the equivalent width

$$\mathrm{EW} \equiv \int \left|1 - \frac{I_\nu}{I_{\nu,b}}\right| d\lambda$$

can be written as

$$\frac{\mathrm{EW}}{\lambda_0} = \left|1 - \frac{S_\nu}{I_{\nu,b}}\right| \int (1 - e^{-\tau_\nu})\frac{d\nu}{\nu_0} \tag{1}$$

where the integral is over a range of frequencies $\ll \nu_0$ so that $I_{\nu,b}, S_\nu$ are constant, but $\gg \Delta\nu_D$, allowing the limits of $u \to \infty$. Finally, multiply both sides by $\nu_0/\Delta\nu_D$ to get a result for $\mathrm{EW}/\Delta\lambda_D$, changing the integration variable to u. (You can simply relate $\Delta\nu_D \ll \nu_0$ to $\Delta\lambda_D \ll \lambda_0$, the Doppler width in wavelength.)

(c) Plot the profile shape for $a = 0.1$ as dimensionless $\phi(\Delta\nu)\cdot\Delta\nu_D$ (log-scale) against $-4 < \Delta\nu/\Delta\nu_D < 4$. Overplot the individual contributions from a purely Doppler and purely collisional profile (widths $\Delta\nu_D$ and $\Gamma = 4\pi a\Delta\nu_D$. The tails of the Voigt distribution should match onto the collisional profile.

(d) Now plot line shapes, as $\exp(-\tau_\nu)$ vs. $u = \Delta\nu/\Delta\nu_D$. Use

$$\tau_\nu = \tau_0\phi(\Delta\nu)/\phi(0)$$

Include three choices of the optical depth at line center, $\tau_0 = 0.1, 1, 10$ and two cases: Doppler profiles and $a = 0.3$ Voigt proliles.

(e) Next plot equivalent widths vs. τ_0 for $0.01 < \tau_0 < 10^3$. Plot 5 cases, pure Doppler ($a = 0$), pure pressure ($a \to \infty$) and $a = 0.01, 0.1, 1$ Voigt profiles. To best compare results for different widths plot $\mathrm{EW}/\Delta\lambda_V$ where the with of the Voight profile is $\Delta\lambda_V = \Delta\lambda_D/H(0,a)$. Take $S_\nu = 0$ (pure absorption).

The $a \to \infty$ limit for pure pressure broadening (with width $\Delta\lambda_P$) is

$$\frac{\mathrm{EW}}{\Delta\lambda_P} = \int_{-\infty}^{\infty}\left[1 - \exp\left(-\frac{\tau_0}{1+y^2}\right)\right]\frac{dy}{\sqrt{\pi}} \tag{2}$$

Note: For symmetric integrals over line shapes (like Eqs. 1, 2) the replacement $\int_{-\infty}^{\infty} \to 2\int_0^{\infty}$ saves a factor of 2 in computation. However $H(u,a)$ is an asymmetric integral that can't use this trick. You can use $H(0,a) = \exp(a^2)\mathrm{erfc}(a)$, with the complementary error function, erfc.