HW 3: Radiation Moments

Due: Sept. 17, 9:30AM on D2L

ASTR 589, Prof. Youdin

1. **Eddington Approximation** This problem derives the Eddington approximation

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3}\right)$$

where τ is the *vertical* optical depth. Use the approximation from HW 2 of half isotropic radiation. That problem gives the definition of I_+, I_- and the resulting energy density, vertical flux and pressure as E, F, P.

This problem assumes a constant value of flux $F = \sigma T_{\text{eff}}^4$ and no incoming radiation at $\tau = 0$, i.e. $I_-(0) = 0$. We use the moment equations (frequency averaged and assuming grey opacities)

$$\frac{dF}{d\tau^{(abs)}} = \frac{\kappa}{\kappa^{(abs)}} \frac{dF}{d\tau} = cE - 4\pi B = 0 \tag{1}$$

$$\frac{dP}{d\tau} = \frac{F}{c} \tag{2}$$

Note that scattering, which affects ratio of $\kappa/\kappa^{(abs)}$, is irrelevant since we apply $F =$ constant.

   (a) Give the surface ($\tau = 0$) values for E, P in terms of F (using $I_-(0) = 0$).

   (b) Solve for $P(\tau)$ by integrating a moment equation and applying the above boundary condition. From this result, give $E(\tau)$

   (c) From radiative equilibrium (i.e. $F =$ constant) derive $B(\tau)$, the (frequency integrated) Planck function. Express in the form of the Eddington approximation.

2. **Radiation Pressure** In this problem, we first review the Eddington luminosity, then show that radiation pressure forces are generally important whenever the thermal radiation pressure $P_{\text{rad}} = aT^4/3$ is $\gtrsim P_{\text{gas}}$, the (ideal) gas pressure.

   (a) Derive the Eddington luminosity, L_{edd}, balancing the radiation pressure force on matter with spherical gravity. Express your result in terms of mass M and opacity κ (which you can assume is grey or appropriately frequency averaged).

   (b) Use hydrostatic balance of gas, $dP_{\text{gas}}/dz = -\rho g$ to estimate P_{gas} at the $\tau = 2/3$ photosphere, changing variables to the vertical optical depth as $d\tau = -\rho\kappa dz$. Use $P_{\text{gas}}(\tau = 0) = 0$ and constant g, κ to simplify the integral estimate.

   (c) Express P_{rad} at the photosphere in terms of the luminosity L. The photosphere is at radius, R. Assume thermal radiation, and then use the Stephan-Boltzmann law (and/or the Eddington Approximation).

   (d) Show that $P_{\text{rad}} \gtrsim P_{\text{gas}}$ roughly (not exactly) reproduces the Eddington luminosity condition for mass loss (by using the above results).

3. **SED of grey atmosphere** We will compare the SED of a grey atmosphere (frequency independent opacities) to that of a black body, using the Eddington approximation $T = T_{\rm eff} q(\tau)$ for $q(\tau) = [(3/4)(\tau + 2/3)]^{1/4}$. (Here τ is the vertical optical depth in the atmosphere.)

   For frequency and flux, use the scaled variables $x = h\nu/(kT_{\rm eff})$ and $\mathcal{F} = \nu F_\nu/(\sigma T_{\rm eff}^4)$.

   (a) For a perfect blackbody with $I_\nu = B_\nu(T = T_{\rm eff})$ on the surface, calculate the flux in scaled coordinates, $\mathcal{F}_B(x)$ (labelled B for blackbody). Make a log-log plot of this curve. (Your answer should include the derivation of the scaled result and the plot. You need to use $\sigma(k, c, h)$ for all physical constants to scale out.) Check that your plot peaks at $x = 3.921$, following Wein's Law (a faint vertical line should suffice).

   (b) Calculate the emergent spectrum from an Eddington atmosphere, using the result of Shu (Vol. 1, Eq. 4.20) that

   $$F_\nu(0) = 2\pi \int_0^\infty E_2(\tau) B_\nu(T(\tau)) d\tau$$

   where the exponential integral $E_2(\tau)$ is dimensionless and available as a special function in e.g. `scipy.special.expn` as `expn(2, tau)`. First derive the scaled version of this integral for $\mathcal{F}_E(x)$ (labelled E for Eddington). Then compute $\mathcal{F}_E(x)$ numerically for a range of x values, and overlay it with the above plot.

   (c) Interpret this result. At what frequencies does the grey atmosphere emit more or less flux than the corresponding blackbody atmosphere. Include a plot of the fractional deviation $\mathcal{F}_E(x)/\mathcal{F}_B(x) - 1$ to see the deviation more clearly in regions where it is more subtle (smaller x). Attempt to explain this effect, by the fact that the emission is just a superposition of Planck functions.