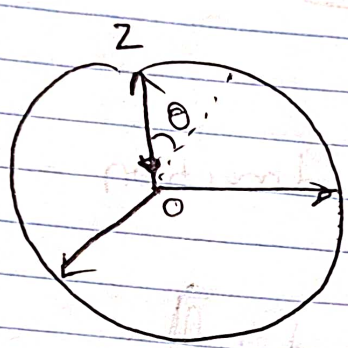
HW 2 Moments & Transfer equation Neev

1. $I(\mu, \phi) = \begin{cases} I_+ & \mu > 0 \\ I_- & \mu < 0 \end{cases}$



Assuming monochromatic (ν)

$$E = U_\nu = \int U_\nu(\Omega)\, d\Omega$$

$$U_\nu(\Omega) = \frac{I_\nu(\Omega)}{c}$$

$I_\nu \equiv I$

$$U_\nu = \int \frac{I_\nu(\Omega)}{c}\, d\Omega = \int \frac{I(\Omega)}{c} \sin\theta\, d\theta\, d\phi$$

$$= -\int \frac{I(\theta,\phi)}{c}\, d(\cos\theta)\, d\phi$$

$$= -\frac{2\pi}{c}\int_{1}^{-1} I(\mu)\, d\mu \quad \cancel{= \frac{2\pi}{c}\left(I_+ - I_-\right)}$$

$$= \frac{2\pi}{c}\left[\int_0^1 I(\mu)\,d\mu + \int_{-1}^{0} I(\mu)\,d\mu\right]$$

$$E = \boxed{\frac{2\pi}{c}\left[I_+ + I_-\right]}$$

flux $F \equiv F_{\nu}$ in $\hat{z}$ direction

$$F = \int I(\theta,\phi)\,\cos\theta\, d\Omega$$

$$= -\int I(\theta)\cos\theta\, d(\cos\theta)\, d\phi$$

$$= -2\pi \int_1^{-1} I(\mu)\,\mu\, d\mu$$

$$= 2\pi\left[\int_0^1 I_+\,\mu\, d\mu + \int_{-1}^{0} I_-\,\mu\, d\mu\right]$$

$$= 2\pi\left[\frac{I_+}{2} - \frac{I_-}{2}\right] = \boxed{\pi\left(I_+ - I_-\right)}$$

$$P_{zz} \equiv P_{zz,\nu}$$

$$P_{zz} = \frac{1}{c}\int I \cos^2\theta \, d\Omega$$

$$= \frac{2\pi}{c}\left[\int_0^1 I_+ \mu^2 d\mu + \int_{-1}^0 I_- \mu^2 d\mu\right]$$

$$= \frac{2\pi}{c}\left[\frac{I_+}{3} + \frac{I_-}{3}\right]$$

$$= \boxed{\frac{2\pi}{3c}\left[I_+ + I_-\right]}$$

$$\frac{P}{E} = \frac{\frac{2\pi}{3c}\cancel{\left[I_+ + I_-\right]}}{\frac{2\pi}{c}\cancel{\left[I_+ + I_-\right]}} = \frac{1}{3}$$

$$\boxed{\frac{P}{E} = \frac{1}{3}}$$

2. $$\frac{dI}{d\tau} = I - S$$

$\tau \downarrow$ along ray

$$S(\tau) = B[T(\tau)]$$

a) $$\frac{dI}{d\tau} - I = e^{\tau}\frac{d}{d\tau}(Ie^{-\tau})$$

$$e^{\tau}\frac{d}{d\tau}(Ie^{-\tau}) = -S$$

$$d(Ie^{-\tau}) = -Se^{-\tau}\,d\tau$$

$$\int_{\tau=\tau_0}^{\tau=0} d(Ie^{-\tau}) = \int_{\tau_0}^{0} -B(T(\tau))\,e^{-\tau}\,d\tau$$

$$I(0) - I(\tau_0)e^{-\tau_0} = -\int_{\tau_0}^{0} B(T(\tau))e^{-\tau}\,d\tau$$

$$\boxed{I(0) = I(\tau_0)e^{-\tau_0} - \int_{\tau_0}^{0} B(T(\tau))\,e^{-\tau}\,d\tau}$$

b) $S(\tau) = S$

$$I(0) = I(\tau_0)e^{-\tau_0} - S\int_{\tau_0}^{0} e^{-\tau}d\tau$$

$$= I(\tau_0)e^{-\tau_0} - S\left(-e^{-\tau}\right)_{\tau_0}^{0}$$

$$\boxed{I(0) = I(\tau_0)e^{-\tau_0} + S\left(1 - e^{-\tau_0}\right)}$$

$$\boxed{I(0) = S + e^{-\tau_0}\left(I(\tau_0) - S\right)}$$

$\tau_0 >> 1 \Rightarrow e^{-\tau_0} \Rightarrow 0$

$\boxed{I(0) = S}$ (optically thick)

$\tau_0 << 1 \Rightarrow e^{-\tau_0} \sim 1 - \tau_0$

$$I(0) = \cancel{S} + I(\tau_0) - \cancel{S} - \tau_0\left(I(\tau_0) - S\right)$$

$$\boxed{I(0) = I(\tau_0)(1 - \tau_0) + \tau_0(S)}$$

c) $I(\tau_0) = 0$

$$\Rightarrow \frac{I(0)}{S} = 1 - e^{-\tau_0}$$

$\tau_0 = 0.1 \Rightarrow 0.09516$

$= 2/3 \Rightarrow 0.48658$

$= 1 \Rightarrow 0.63212$

$= 3 \Rightarrow 0.95021$

$= 10 \Rightarrow 0.99995$

d) $I(0) = I(\tau_0)(1 - \tau_0) + \tau_0 S \quad (\tau_0 \ll 1)$

$I_\nu(\tau_0) \equiv B_\nu(T_b)$

$S = B_\nu(T_S)$

If no background source

$I(0) = \tau_0 B_\nu(T_S)$

For ν in vicinity of line frequency ν_0, τ_0 is higher because k is higher as compared

to the rest of the continuum. (remember
Hence we see emission lines $d\tau = \rho \kappa ds$)
at the line frequencies ~~for~~ from
a optically thin medium w/o background source
as $I(0) \propto \tau_0$

Lets say there is backgroud $I = B_\nu(T_b)$
and low temp optically thin medium in
between $S = B_\nu(T_S)$

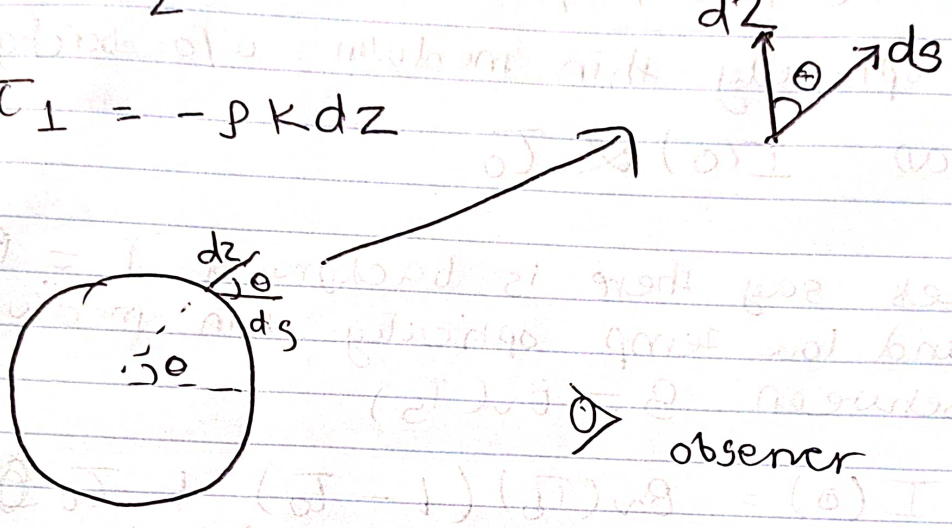
$$I(0) = B_\nu(T_b)(1-\tau_0) + \tau_0 B_\nu(T_S)$$

$$= B_\nu(T_b) - \tau_0 \underbrace{\left(B_\nu(T_b) - B_\nu(T_S)\right)}_{>0}$$

Now at line freq ν_0, τ_0 is higher in vicinity
as compared to the rest of the continuum
so $I(0)$ is suppressed in this vicinity compared
to the rest due to the $-\tau_0(B_\nu(T_b) - B_\nu(T_S))$
term and we see an absorption line at
those line freq.

3. $\tau_\perp = \int_z^\infty \rho \kappa \, dz'$

$d\tau_\perp = -\rho \kappa \, dz$



$$\frac{dI}{d\tau} = I - S \qquad d\tau = -\rho \kappa \, ds$$

$ds \cos\theta = dz$

$$d\tau = \frac{-\rho \kappa \, dz}{\cos\theta}$$

$$d\tau = \frac{d\tau_\perp}{\cos\theta}$$

$$\frac{dI}{d\tau_\perp} \frac{d\tau_\perp}{d\tau} = I - S$$

$$\frac{dI}{d\tau_\perp}\cos\theta = I - S$$

$$\mu = \cos\theta$$

$$\boxed{\frac{dI}{d\tau_\perp} = \frac{I}{\mu} - \frac{S}{\mu}}$$

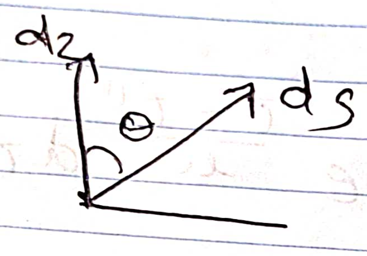
b) $$\mu \frac{dI}{d\tau_\perp} = I - S$$

$$\cancel{\frac{d}{d\tau_\perp}\left(e^{-\mu\tau_\perp} I\right) = e^{-\mu\tau_\perp}}$$

$$\frac{d\left(e^{-\frac{\tau_\perp}{\mu}} I\right)}{d\tau_\perp} = e^{-\frac{\tau_\perp}{\mu}}\frac{dI}{d\tau_\perp} - \frac{1}{\mu} e^{-\frac{\tau_\perp}{\mu}} I$$

$$e^{\frac{\tau_\perp}{\mu}} \frac{d}{d\tau_\perp}\left(e^{-\frac{\tau_\perp}{\mu}} I\right) = -\frac{S}{\mu}$$

$$\int_{\tau_{\nu,0\perp}}^{\tau_{\nu_\perp}} d\left(I_\nu e^{-\tau'_{\nu_\perp}/\mu}\right) = \int_{\tau_{\nu_{0\perp}}}^{\tau_{\nu_\perp}} -\frac{S}{\mu} e^{-\tau''_{\nu_\perp}/\mu} \, d\tau''_{\nu_\perp}$$



$\mu = \cos\theta$

$1 > \mu > 0 \Rightarrow$ Ray outward
$-1 < \mu < 0 \Rightarrow$ Ray inward

$\mu > 0$: $\tau_{\nu,0} = \infty$ (~~deep~~ inside star $\tau \to \infty$)
~~&~~ outward

$$I_\nu(\tau_{\nu_\perp}, \mu) \, e^{-\frac{\tau_{\nu_\perp}}{\mu}}$$

$$= \int_{\tau_{\nu_\perp}}^{\infty} \frac{S}{\mu} e^{-\tau''_{\nu_\perp}/\mu} \, d\tau''_{\nu_\perp}$$ cont'd

$\mu < 0$: $\tau_{\nu,0} = 0$ (surface where inward $I_\nu = 0$)

$$I_\nu(\tau_{\nu_\perp}, \mu) e^{-\frac{\tau_{\nu_\perp}}{\mu}} = \int -\frac{S}{\mu} e^{-\tau''_{\nu_\perp}/\mu} \, d\tau''_{\nu_\perp}$$

2b) cont'd

$\tau_1 = 0$

$$\boxed{I(\mu) = \int_0^{\infty} \frac{S}{\mu} e^{-\tau_1''/\mu} \, d\tau_1'}$$

c) $S = a + b\tau_{\perp}$ Let $\tau_{\nu_\perp} \equiv \tau$

For outward $I_\nu \equiv I$

$I(\tau, \mu)$

$$= \int_{\tau_0}^{\infty} \frac{(a + b\tau)}{\mu} e^{\frac{\tau - \tau''}{\mu}} d\tau''$$

$$= \frac{1}{\mu}\left[(a+b\tau)(-\mu) e^{\frac{\tau-\tau''}{\mu}} \Big|_{\tau}^{\infty} - \int_{\tau}^{\infty} b(-\mu) e^{\frac{\tau-\tau''}{\mu}} \right]$$

$$= \frac{1}{\mu}\left[+\mu(a+b\tau) + \mu b(-\mu) e^{\frac{\tau-\tau''}{\mu}} \Big|_{\tau}^{\infty} \right]$$

$$= \frac{1}{\mu}\left[+\mu(a+b\tau) + \mu^2 b \right]$$

$$\boxed{I(\tau,\mu) = a + b\tau + \mu b}$$

At surface or outside for observer

At $\tau = 0$

$$I(\mu) = a + \mu b$$

$$\cancel{I(\theta) = a + b\cos\theta}$$

$$I(0) = a \quad : \text{limb}$$
$$I(1) = a+b \quad : \text{center}$$

$$\frac{I(0)}{I(1)} = \frac{a}{a+b}$$

$$a = \frac{2}{3}\left(\frac{3}{4}\frac{\sigma}{\pi}T_{eff}^4\right)$$

$$b = \frac{3}{4}\frac{\sigma}{\pi}T_{eff}^4$$

$$a = \frac{2}{3}b$$

$$a + b = \frac{5}{2}a$$

$$\boxed{\frac{I(0)}{I(1)} = \frac{2}{5} = 0.4}$$

d) $$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} = \frac{2\left(\frac{h^3\nu^3}{k^3T^3}\right)\frac{1}{h^2c^2}}{e^{\frac{h\nu}{kT}} - 1}$$

$$= \frac{2\left(\frac{h\nu}{kT}\right)^3 \frac{1}{h^2c^2}}{e^{\frac{h\nu}{kT}} - 1} = \frac{\left(\frac{2}{h^2c^2}\right)x^3}{e^x - 1} \qquad \frac{h\nu}{kT} = x$$

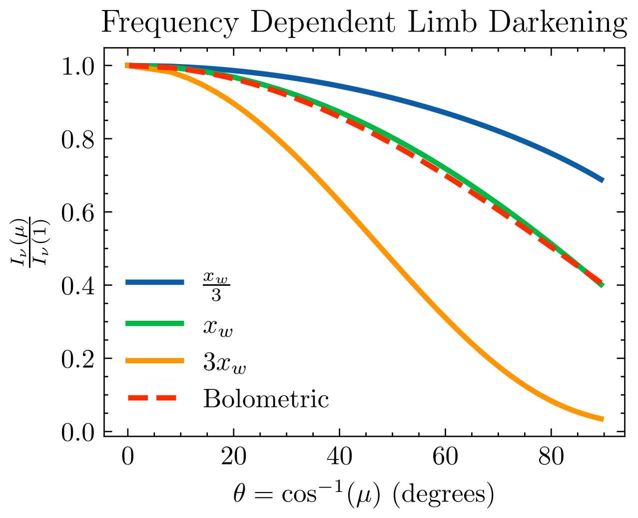
$$\cancel{\frac{\sigma T^4}{\pi}} = \frac{3\sigma T_{eff}^{\,4}}{4\cancel{\pi}}\left(\tau_1 + \frac{2}{3}\right)$$

$$T = T_{eff}\left[\frac{3}{4}\left(\tau_1 + \frac{2}{3}\right)\right]^{1/4}$$

$$I_\nu(\tau,\mu) = \int_{\tau}^{\infty} \frac{S_\nu}{\mu} e^{\frac{\tau - \tau''}{\mu}} d\tau''$$

$$S_\nu(T(\tau)) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

$$I_\nu(0,\mu) = \int_0^\infty \frac{S_\nu}{\mu} e^{-\frac{\tau''}{\mu}} d\tau''$$



Frequency Dependent Limb Darkening
$\frac{I_\nu(\mu)}{I_\nu(1)}$
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
$\theta = \cos^{-1}(\mu)$ (degrees)
$\frac{x_w}{3}$
x_w
$3x_w$
Bolometric