HW 4

1. Radiative Diffusion

a) $F = -\frac{c}{3\rho k_R} \frac{d}{dz}(aT^4)$

$d\tau = -\rho k_R \, dz$

$F = \frac{c}{3} \frac{d}{d\tau}(aT^4)$

$$\int_{\tau_0}^{\tau} \frac{3F}{c} \, d\tau = \int_{aT(\tau_0)^4}^{aT(\tau)^4} d(aT^4)$$

$$\frac{3F}{c}(\tau - \tau_0) = a\left(T(\tau)^4 - T(\tau_0)^4\right)$$

$$\boxed{T(\tau)^4 = \frac{3F}{ac}(\tau - \tau_0) + T(\tau_0)^4}$$

b) $\tau_0 = 2/3$

$$T(\tau)^4 = \frac{3F}{ac}(\tau - 2/3) + T_{eff}^4$$

$F = \sigma T_{eff}^4$

$$T(\tau)^4 = T_{eff}^4\left(\frac{3\sigma}{ac}(\tau - 2/3) + 1\right)$$

$$\frac{ac}{\sigma} = 4$$

$$T(\tau)^4 = T_{eff}^4\left(\frac{3}{4}\tau - \frac{1}{2} + 1\right)$$

$$= T_{eff}^4\left(\frac{3}{4}\tau + \frac{1}{2}\right)$$

$$\boxed{T(T_{eff},\tau)^4 = \frac{3}{4}T_{eff}^4\left(\tau + \frac{2}{3}\right)}$$

2. $$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$f(x,t) = A(t)e^{-x^2/[2w(t)^2]} \qquad w^2 = 2Dt$$

a) $$\int_{-\infty}^{\infty} A(t)e^{-x^2/[2w(t)^2]}\,dx = 1$$

$$A(t)\int_{-\infty}^{\infty} e^{\frac{-x^2}{2w(t)^2}}\,dx = 1$$

$$A(t)\sqrt{\frac{\pi}{1/(2w(t)^2)}} = 1 \qquad \boxed{A(t) = \frac{1}{\sqrt{2\pi w(t)^2}}}$$

b) $f = Ae^{u}$

$$A(t) = \frac{1}{\sqrt{4\pi D t}}$$

$$u(x,t) = \frac{-x^2}{4Dt}$$

$$\frac{dA}{dt} = \frac{-4\pi D}{2(4\pi D t)^{3/2}} = -\frac{4\pi D A^3}{2}$$

$$= -\frac{1}{2}\frac{1}{A^2 t} A^3$$

$$\boxed{\frac{dA}{dt} = -\frac{A}{2t}}$$

$$\frac{\partial u}{\partial t} = \frac{x^2}{4Dt^2} \boxed{= -\frac{u}{t}}$$

$$\frac{\partial u}{\partial x} = \frac{-2x}{4Dt} = \frac{-2x\,u}{-x^2} \boxed{= \frac{2u}{x}}$$

$$\frac{\partial^2 u}{\partial x^2} = \cancel{\frac{\partial}{\partial x}\left(\frac{2u}{x}\right)} \frac{\partial}{\partial x}\left(\frac{-2x}{4Dt}\right) = \frac{-2}{4Dt}$$

$$= \frac{-2u}{-x^2}$$

$$\boxed{= \frac{2u}{x^2}}$$

c) $\frac{\partial f}{\partial t} = e^{U}\frac{\partial A}{\partial t} + A e^{U}\frac{\partial U}{\partial t}$

$$\frac{\partial^2 f}{\partial x^2} = A\frac{\partial}{\partial x}\left(e^{U}\frac{\partial U}{\partial x}\right)$$

$$= A\left(e^{U}\frac{\partial^2 U}{\partial x^2} + e^{U}\left(\frac{\partial U}{\partial x}\right)^2\right)$$

$$\frac{\partial f}{\partial t} = e^{U}\left(\frac{-A}{2t}\right) + Ae^{U}\left(\frac{-U}{t}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = A\left(e^{U}\frac{2U}{x^2} + e^{U}\frac{4U^2}{x^2}\right)$$

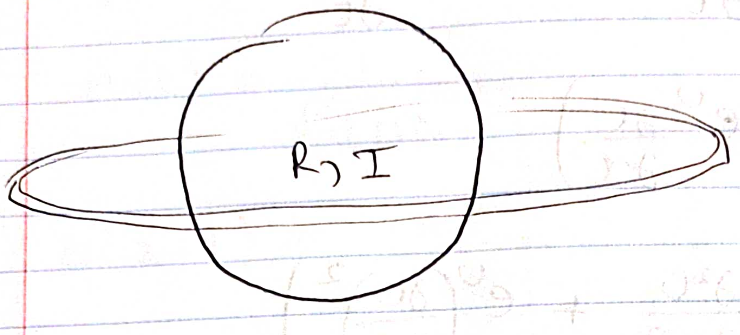
$$\frac{1}{f}\frac{\partial f}{\partial t} = \boxed{-\frac{1}{2t} - \frac{U}{t}}$$

$$\frac{D}{f}\frac{\partial^2 f}{\partial x^2} = \frac{D\,2U}{x^2} + \frac{D\,4U^2}{x^2}$$

$$D = -\frac{x^2}{4Ut} \qquad \frac{D}{f}\frac{\partial^2 f}{\partial x^2} = \boxed{-\frac{1}{2t} - \frac{U}{t}}$$

$$\therefore \frac{1}{\cancel{f}}\frac{\partial f}{\partial t} = \frac{D}{\cancel{f}}\frac{\partial^2 f}{\partial x^2} \qquad \boxed{\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}}$$

3.



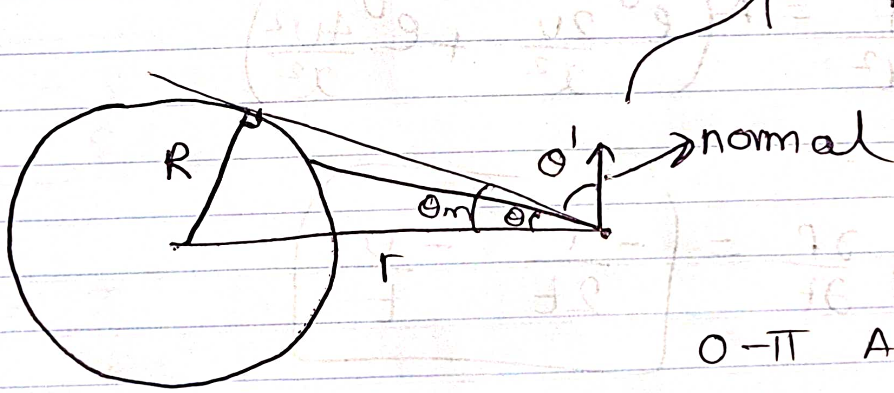
a) $F(r=R) = \pi I$

$L = 4\pi R^2 F$

$\boxed{L = 4\pi^2 R^2 I}$

$F = I\cos\theta'$

b)



0 – π Azimuthal

as only top surface of disk is hit by upper half of sphere

$\theta' = 90 - \theta$

$F = \iint I\cos\theta' \, d\Omega$

$\int_{90-\theta_m}^{90} \int_0^{\pi} I\cos\theta' \sin\theta' \, d\theta' \, d\phi$

$$F = \pi \int_{90-\theta_m}^{90} I \cos\theta' \sin\theta' \, d\theta'$$

$$\cos\theta' = \mu$$

$$F = \pi \int_{0}^{\sin\theta_m} I \mu \, d\mu$$

$$\boxed{F = \frac{\pi I \sin^2\theta_m}{2}} \qquad \sin\theta_m = \frac{R}{r}$$

$$F = \frac{\pi I}{2}\left(\frac{R^2}{r^2}\right)$$

c) ~~F~~ $\qquad \theta_m \ll 1$

$$F = \frac{\pi I}{2}\left(\frac{1-\cos 2\theta_m}{2}\right)$$

$$F = \frac{\pi I}{2}\left(\frac{1-\left(1-\frac{(2\theta_m)^2}{2!}+\dots\right)}{2}\right)$$

$$F = \frac{\pi I}{2}\left(\frac{\cancel{4}\theta_m^2}{\cancel{4}}\right) = \boxed{\frac{\pi I \theta_m^2}{2}}$$

$$\theta_m \ll 1$$
$$\sin\theta_m \sim \theta_m$$
$$\hookrightarrow \frac{R}{r}$$

$$F = \frac{\pi I}{2} \frac{R^2}{r^2}$$

$$L = 4\pi R^2 F(R=r)$$

$$L = 4\pi R^2 \pi I$$

$$L = 4\pi^2 R^2 I$$

$$L = 4\pi \pi I R^2$$

$$F_{disk} = \frac{L/4\pi}{2r^2} \boxed{= \frac{L}{8\pi r^2}}$$

$$F_{disk} = \sigma T^4$$

$$\frac{L_{sun}}{8\pi r^2} = \sigma T^4$$

$$r = 1AU$$

$$L_{sun} = 3.846 \times 10^{26}\ W$$

$$1AU \approx 1.496 \times 10^{11}$$

$$\sigma = 5.67e^{-8}\ W/m^2K^4$$

$$T^4 = \frac{3.846 \times 10^{26}}{(8\pi \times (1.496)^2 \times 10^{22} \times 5.67 \times 10^{-8})}$$

$$T^4 = \frac{3.846 \times 10^{26}}{(8\pi \times 5.67 \times 10^{14} \times (1.496)^2)}$$

$$T^4 = \frac{3.846 \times 10^{12}}{8\pi \times 5.67 \times (1.496)^2}$$

$$\cancel{\sim 0.0493e12}$$

$$\cancel{T^4 \sim 119.3 \times 10^8}$$

$$\cancel{T \approx 3.45 \times 10^2}$$

$$\boxed{T \sim 337.15K}$$

4.

$g_u = 3$ $\quad A_{UL} = 2.8843e{-15}\ s^{-1}$

$g_l = 1$ $\quad$ or $1/11.0$ Myr

$$\lambda_{UL} = \frac{c}{\nu_{UL}} = 21.106\ cm$$

a) $E = \frac{hc}{\lambda}$

$$= \frac{6.626 \times 10^{-34} \times 3\times10^{8}}{0.21106 \times 1.6\times10^{-19}}\ eV$$

$$= \frac{6.626 \times 3}{0.21106 \times 1.6} \times 10^{-7} = 58.88e{-7}\ eV$$

$$= 5.888\ \mu eV$$

$$\boxed{E \sim 5.89\ \mu eV}$$

$k_B = 8.617e{-5}\ eV/K$

$E = k_B T$

$$T = \frac{5.89 \times 10^{-6}\ eV}{8.617 \times 10^{-5}\ eV/K}$$

$$= \frac{5.89}{8.617} \times 10^{-1}\ K$$

$$\sim 0.682 \times 10^{-1}\ K \sim \boxed{68.2\ mK}$$

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\frac{E_{ul}}{k_B T}}$$

$$kT \gg E_{ul}$$

$$e^{-\frac{E_{ul}}{k_B T}} \sim 1$$

$$\boxed{\frac{n_u}{n_l} = \frac{g_u}{g_l} = 3}$$

$$n_u + n_l = n_H$$

$$4n_l = n_H$$

$$\boxed{\frac{n_l}{n_H} = \frac{1}{4} \qquad \frac{n_u}{n_H} = \frac{3}{4}}$$

b) $$j_\nu = \frac{h\nu_{ul}}{4\pi} n_u A_{ul} \phi(\nu)$$

$$\boxed{j_\nu = \frac{E_{ul}}{4\pi} \frac{3}{4} n_H A_{ul} \phi(\nu)}$$

c) $$\rho_\nu \kappa_\nu = \alpha_\nu$$

(in LTE)

$$\alpha_\nu = \frac{h\nu_{UL}}{4\pi} n_L B_{LU} \left(1 - e^{-\frac{h\nu_{UL}}{kT}}\right) \phi(\nu)$$

$$g_L B_{LU} = g_U B_{UL}$$

$$A_{UL} = \frac{2h\nu^3}{c^2} B_{UL}$$

$$A_{UL} = \frac{2h\nu^3}{c^2} \frac{g_L}{g_U} B_{LU}$$

$$\alpha_\nu = \frac{\cancel{h\nu_{UL}}}{4\pi} n_L \frac{g_U}{g_L} \frac{c^2}{2\cancel{h}\nu_{UL}^{\cancel{3}\,2}} A_{UL} \left(1 - e^{-\frac{h\nu_{UL}}{kT}}\right) \phi(\nu)$$

$$= \frac{1}{4\pi} n_L \frac{g_U}{g_L} \frac{c^2}{2\nu_{UL}^2} A_{UL} \left(1 - e^{-\frac{h\nu_{UL}}{kT}}\right) \phi(\nu)$$

again assuming $kT \gg h\nu_{UL}$ $\quad e^{-x} \sim 1 - x$

$1-(1-x)$

$$= \frac{1}{4\pi} n_U \frac{c^2}{2\nu_{UL}^2} A_{UL} \left(\frac{h\nu_{UL}}{kT}\right) \phi(\nu) \qquad \sim x$$

$$\boxed{K_\nu = \frac{1}{8\pi\rho} \frac{3}{4} n_H \left(\frac{c}{\nu_{UL}}\right)^2 A_{UL} \left(\frac{h\nu_{UL}}{kT}\right) \phi(\nu)}$$

$\rho = m_p n_H$

$$\boxed{K_\nu = \frac{3}{32\pi}\frac{1}{m_p}\frac{hc^2}{\nu_{UL}}\frac{A_{UL}}{kT}\phi(\nu)}$$

d) $T = 100K$

$$\phi(\nu) = \frac{\lambda_{UL}}{v\sqrt{2\pi}} \qquad v = 1 km/s$$

$$K_\nu = \frac{3}{32\pi}\frac{1}{m_p}\frac{hc^2}{\nu_{UL}}\frac{A_{UL}}{kT}\frac{\lambda_{UL}}{v\sqrt{2\pi}}$$

$$= \frac{3}{32\pi}\frac{1}{m_p}\frac{hc^3}{\nu_{UL}^2}\frac{A_{UL}}{kT}\frac{1}{v\sqrt{2\pi}}$$

$$= \frac{3}{32\pi}\frac{1}{m_p} hc\lambda_{UL}^2 \frac{A_{UL}}{kT}\frac{1}{v\sqrt{2\pi}}$$

$$\boxed{K_\nu \approx 1317.3 \frac{cm^2}{g}}$$

$$\tau_\nu = \rho K_\nu L$$

$N_H = 10^{21}$ atoms / cm^2

$$\tau_\nu = m_p K_\nu n_H L$$

$$= m_p K_\nu N_H$$

$\tau_\nu \approx 2.2033$