HW 4: Diffusion, Einstein Coefficients

Due: Sept. 24, 9:30AM on D2L

ASTR 589, Prof. Youdin

1. **Radiative Diffusion**

   (a) Starting from the diffusive radiative flux in the vertical direction

   $$F = -\frac{c}{3\rho\kappa_R}\frac{d}{dz}(aT^4)$$

   solve for $T(\tau)^4$, in a constant F atmosphere, using τ as the vertical optical depth (for the opacity κ_R). Apply the boundary conditions at a reference optical depth τ_0, where the temperature is $T(\tau_0)$.

   (b) "Derive" the Eddington approximation for $T(T_{\rm eff}, \tau)$ by setting $\tau_0 = 2/3$ and $T(2/3) = T_{\rm eff}$, giving the usual definition for this effective temperature. *Note that this is not a real derivation for various reasons: Diffusion only holds for $\tau \gg 1$, not in the atmosphere. Also the $\tau = 2/3$ value is assumed, not derived. Thus a convenient consistency is actually what is demonstrated here.*

2. **Diffusion as a Random Walk** This problem demonstrates that the solutions to a 1D diffusion equation

   $$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

   obey a random walk. Do this by showing that a Gaussian distribution function

   $$f(x,t) = A(t)e^{-x^2/[2W(t)^2]}$$

   satisfies the diffusion equation for $W^2 = 2Dt$ and constant D. The growth of width as $W \propto \sqrt{t}$ demonstrates a continuous random walk.

   As a fairly lengthy derivation, it helps to be organized. One path to solution is spelled out below. Alternate methods are possible, and can be submitted, but should be clear and complete.

   (a) Work out the normalization $A(t)$ so that $\int_{-\infty}^{\infty} f dx = 1$

   (b) Define $u(x,t)$ so that $f = Ae^u$. Then give $dA/dt, \partial u/\partial t, \partial u/\partial x$ and $\partial^2 u/\partial x^2$, ideally in simple terms with just u, A, x, t and numbers.

   (c) Express $\partial f/\partial t$ and $\partial^2 f/\partial x^2$ in terms of A, u and derivatives of A, u. Then eliminate the A, u derivatives with the results from (b).

   (d) Finally show that $(1/f)\partial f/\partial t$ and $(D/f)\partial^2 f/\partial x^2$ are equal, eliminating u. (Division by f is optional but simplifies things, e.g. by removing A.)

3. **Irradiating a Flat Disk** Consider a flat, thin disk surrounding a spherical star of radius R and uniform intensity I. This problem will work out the equilibrium temperature of the disk. (*Note that real disks aren't flat or perfectly thin, and have other heating sources, but this problem is a start.*)

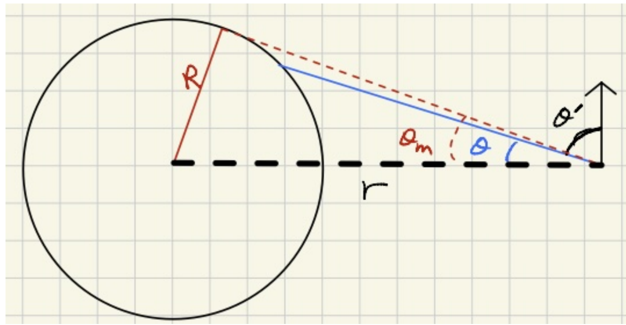

Figure 1: Geometry for the irradiation of a flat disk. The black arrow is perpendicular to the disk surface

(a) What is the luminosity, L of the star in terms of I (review)?

(b) Calculate the flux into the top disk surface (unit normal drawn). Give your result in terms of I and θ_m, the maximum value of θ (see figure) that grazes the stellar surface. Explain the integral you set up, with the aid of a labelled figure (hand-drawn is OK).

(c) Working in the small angle approximation, $\theta_m \ll 1$, series expand F to lowest (non-zero) order. Then express this flux in terms of distance from the star $r \ll R$, R and L.

(d) Give the effective temperature of the disk at distance $r \gg R$, in equilibrium. What is this temperature at 1AU from a Sun-like star? *Note that since each surface absorbs and emits equal amounts, it's OK to just consider one surface.*

4. **The hydrogen 21 cm line** arises due to the energy difference between electron and proton spins being aligned (the lower energy state with degeneracy $g_\ell = 1$) vs. anti-aligned (the upper energy state with degeneracy $g_u = 3$). The spontaneous emission coefficient is $A_{u\ell} = 2.8843 \times 10^{-15}\mathrm{s}^{-1}$, or 1/(11.0 Myr). More precisely, $\lambda_{u\ell} = c/\nu_{u\ell} =$ 21.106 cm, but 21 cm is sufficient for the numerical evaluations here.

   (a) What is the energy difference $E_{u\ell}$ between the two states, in eV and what is $E_{u\ell}/k$ in K? Assuming $kT \gg E_{u\ell}$, what is the density ratio n_u/n_l in LTE? And what are the ratios n_u/n_H and n_ℓ/n_H relative to the total hydrogen density n_H?

   (b) What is the spontaneous emission coefficient j_ν? Give all quantities at line center, except for the line shape $\phi(\nu)$. Use $A_{u\ell}$ and n_H in your answer (symbolically) but apply the integer values of g_u, g_ℓ. Use $\lambda_{u\ell}, \nu_{u\ell}$ or $E_{u\ell}$ as you prefer (but pick one).

   (c) What is the absorption opacity κ_ν? Include the correction for stimulated emission (as is usual). Express the answer as above (with $A_{u\ell}$) but additionally, remove the exponential by Taylor expansion and assume the gas is composed purely of hydrogen atoms with mass m_p (neglecting Helium for simplicity).

   (d) Evaluate κ_ν numerically (in cm^2/g) for $T = 100$ K and $\phi(\nu) = \lambda_{u\ell}/[\sqrt{2\pi}v]$ for $v = 1$ km/s, a relevant value at line center for molecular gas. Finally what is the optical depth through a hydrogen column density of $N_H = n_H L = 10^{21}$ atoms/cm^2.