HW2: Moments and Transfer Equation

Due: Sept. 10, 9:30 AM on D2L
ASTR 589, Prof. Youdin

1. **Hemispherically isotropic radiation** Consider an idealized radiation field with

$$I(\mu, \phi) = \begin{cases} I_+ & \text{if } \mu > 0, \\ I_- & \text{if } \mu < 0. \end{cases} \tag{1}$$

where I_+, I_- are hemispherically constant for outgoing and incoming radiation respectively. Take $\hat{z}$ aligned with $\theta = 0$, i.e. $\mu = \cos(\theta) = 1$.

Calculate the radiation energy density E, flux $\boldsymbol{F}$ and pressure $P = P_{zz}$. (The pressure tensor is isotropic with $P_{ij} = P\delta_{ij}$, but you don't need to show this.) Finally, give P/E.

2. **Solving I along rays** Consider the transfer equation along a ray

$$\frac{dI}{d\tau} = I - S$$

with τ decreasing along the ray, with $S(\tau)$ known, e.g. as $B[T(\tau)]$. We ignore frequency dependence initially, but consider its effects in (d).

   (a) Derive the formal solution for the emergent intensity $I(0)$ by integrating from a total optical depth τ_0, where the background intensity is $I(\tau_0)$. Use the integrating factor trick that

$$\exp(\tau)\frac{d}{d\tau}\left[I \exp(-\tau)\right] = \frac{dI}{d\tau} - I$$

   (b) Evaluate this integral for constant $S(\tau) = S$. Give the limiting results for $\tau_0 \gg 1$ and $\tau_0 \ll 1$. For $\tau_0 \ll 1$, keep terms linear in τ_0.

   (c) With the background $I(\tau_0) = 0$ calculate the ratio $I(0)/S$ for $\tau_0 = 0.1$, $2/3$, 1, 3, 10.

   (d) How can the above result for $\tau_0 \ll 1$ explain both absorption and emission lines in the frequency dependent I_ν? Assume that both the background, $I_\nu(\tau_0) = B_\nu(T_b)$ and the source function $S = B_\nu(T_s)$ are Planck functions, characterized by T_b and T_s. For a line, κ_ν is significantly larger in the vicinity of the line frequency ν_0.

3. **Limb Darkening** Consider a horizontally uniform atmosphere, where $\tau_\perp = \int_z^\infty \rho\kappa dz'$ is the vertical optical depth, from a height z to the top of the atmosphere (taken as ∞). Note that $d\tau_\perp = -\rho\kappa dz$.

   (a) Derive the transfer equation for $dI/d\tau_\perp$, valid for all μ, starting from $dI/d\tau = I - S$ with $d\tau = -\rho\kappa ds$. Start by relating ds, the path length along a ray in direction $\mu = \cos(\theta)$, to dz the vertical component of that path.

   (b) Give the solution for the emergent $I(\mu)$, at $\tau_\perp = 0$ in terms of an integral over $\tau_\perp$. You can translate the above result for τ instead of re-deriving, and should take the total optical depth, $\tau_0 \to \infty$.

(c) For a source function of the form $S = a + b\tau_\perp$, what is the emergent $I(\mu)$ (at $\tau_\perp = 0$)? For the Eddington approximation

$$S(\tau_\perp) = B[T(\tau_\perp)] = \frac{\sigma T^4}{\pi} = \frac{3\sigma}{4\pi} T_{\rm eff}^4 \left(\tau_\perp + \frac{2}{3}\right)$$

what is the limb-to-center contrast, $I(0)/I(1)$?

(d) Now consider frequency dependent $S_\nu = B_\nu[T(\tau_\perp)]$, with $T(\tau_\perp)$ again from the Eddington approximation. Perform numerical integrations (e.g. with `scipy`) to solve for $I_\nu(\mu)$ for three different frequencies, at $h\nu/(kT_{\rm eff}) = x_w/3, x_W, 3x_W$ for $x_W = 3.921$ the logarithmic peak of the Planck function. Make a plot with $I_\nu(\mu)/I_\nu(1)$ vs. $\theta = \cos^{-1}(\mu)$ for these three cases, and add the bolometric result for $I(\mu)/I(1)$ from above for comparison.

For numerical integrations you should: (i) Take the upper limit of integrals to be $\sim$ 10–100. Exponential damping will make negligible contributions beyond this point, and can cause underflow error messages, as you can check. (ii) Avoid dividing by $\mu = 0$, instead choosing a small, non-zero values of μ to see the limb behavior.