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On the electromagnetic force on a dielectric medium

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Abstract
The study of the mechanical effects of light on a dielectric medium has led to two quite distinct forms for the force density: one based on the microscopic distribution of charges and the other on the distribution of atomic dipoles. Both approaches are based directly on the Lorentz force, but it has been suggested that they lead, in a number of cases, to significantly different predictions. In this paper we address this paradoxical situation and show that in the majority of problems the force densities lead to identical results. Where the theories do differ we attempt to determine which of the two descriptions is the more reliable.

This paper is respectfully dedicated to the memory of Edwin Power

1. Introduction: the two force densities

The problem of calculating the force induced on a dielectric material by an electromagnetic field has a surprisingly long and complicated history. It is intimately tied up with such thorny issues as the correct form of the momentum of the electromagnetic field inside a medium [1]. Recent approaches have focused directly on the Lorentz force as the means by which to calculate the force on the medium. The problem, however, is that there are two competing forms of the Lorentz force density: one, based on treating the medium as formed from individual charges, has the form

$$\mathbf{f}^c = -(\nabla \cdot \mathbf{P})\mathbf{E} + \dot{\mathbf{P}} \times \mathbf{B}, \tag{1}$$

while the other, based on treating the medium as being formed from individual dipoles, has the form

$$\mathbf{f}^d = (\mathbf{P} \cdot \nabla)\mathbf{E} + \dot{\mathbf{P}} \times \mathbf{B}. \tag{2}$$

The first of these follows directly from the application of the Lorentz force law to the bound charges comprising the dielectric and has been applied to study the forces and torques on dielectric media in a wide variety of situations [2, 3]. The second follows from the force on a medium comprising point dipoles [4]; equivalent forms have also been given by Penfield and Haus [5] and by Landau, Lifshitz and Pitaevskii [6]. We have used it to study in some detail the forces acting on dielectric media [7–9]. It has been suggested [10] that the different force densities make distinct predictions concerning the interaction of light with the dielectric and it is important, therefore, to explore thoroughly the fundamental bases for these expressions and to investigate in detail the differences between them. This is the aim of the present paper. We shall work within the framework of classical electromagnetism, but will discuss briefly the implications for quantum phenomena at the end of the paper.

A good and safe place to start is with the microscopic Maxwell equations which, in SI units, take the familiar form:

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\dot{\mathbf{E}} + \mu_0 \mathbf{J}. \tag{3}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the full microscopic electric and magnetic fields, and ϱ and $\mathbf{J}$ are the true electric charge and current densities with contributions for every charged particle present, including all of those in the atoms forming any host medium. The description is completed by the form of the force exerted on the charges due to any electric and magnetic fields. We are interested in the total or nett force exerted on all of the charges in a given volume and, for this reason, it is simplest to work with a force density, or force per unit volume, in the form

$$\mathbf{f}^L = \varrho \mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{4}$$

One of our tasks will be to work from this force to obtain an appropriate form for the force on a dielectric. It is interesting to note that H A Lorentz himself wrote this force density [11], albeit in a more old-fashioned notation.

We can understand the origin of the two rival force densities (1) and (2) by looking at the dielectric medium on the microscopic scale. We begin by introducing the displacement and polarization fields, $\mathbf{D}$ and $\mathbf{P}$, which are related by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{5}$$

We note that the dielectric medium is electrically neutral and that all of the charges are bound and so set $\nabla \cdot \mathbf{D} = 0$. This, together with the first Maxwell equation, leads us to

$$\nabla \cdot \mathbf{P} = -\varrho. \tag{6}$$

Taking the divergence of the final Maxwell equation then leads us to

$$\nabla \cdot \dot{\mathbf{P}} = \nabla \cdot \mathbf{J}. \tag{7}$$

The relation (6) constrains only the longitudinal part of $\mathbf{P}$ and hence we can, without loss of generality, use (7) to define $\mathbf{P}$ so that it obeys

$$\dot{\mathbf{P}} = \mathbf{J}. \tag{8}$$

Substituting these relationships into the Lorentz force density gives us, directly, the charge-based force density $\mathbf{f}^c$ given in (1).

An alternative approach treats not the individual charges, but the individual atomic electric dipoles. The force on a single point dipole $\mathbf{d}$ is

$$\mathbf{F} = (\mathbf{d} \cdot \nabla)\mathbf{E} + \dot{\mathbf{d}} \times \mathbf{B}, \tag{9}$$

so that the force density is

$$\mathbf{f}^{\mathrm{spd}}(\mathbf{r}) = ((\mathbf{d} \cdot \nabla)\mathbf{E}(\mathbf{r}) + \dot{\mathbf{d}} \times \mathbf{B}(\mathbf{r}))\delta(\mathbf{r} - \mathbf{R}), \tag{10}$$

where $\mathbf{R}$ is the position of the dipole. We note that the polarization $\mathbf{P}$ is also the dipole density

$$\mathbf{P}(\mathbf{r}) = \sum_i \mathbf{d}_i \delta(\mathbf{r} - \mathbf{R}_i), \tag{11}$$

where the sum runs over all the dipoles. It follows immediately that the total force density is $\mathbf{f}^d$ given in (2). It seems that both of the candidate force densities are sensible and that any difference between them is likely to be subtle. This does indeed turn out to be the case. We will find that in most situations the two force densities lead to the same predictions. As a clue to where they do not, we note that the polarization $\mathbf{P}$, as it has been introduced here, is the full microscopic polarization and not the more familiar macroscopic polarization, related to the electric field by an effective susceptibility. We will conclude that it is in the replacement of this microscopic polarization by a macroscopic polarization that the source of the difficulties lies. This conclusion has been reached before, of course, and Sommerfeld even went as far as stating that 'ponderable bodies with their continuous material constants ε, μ are simply convenient abstractions and are not physical realities' [12]. The macroscopic dielectric constant ε can be used but care needs to be taken.

2. Is there really a problem?

It is quite possible, of course, that there is no problem. This would be the case if, despite appearances, all possible predictions derived on the basis of the force densities (1) and (2) were identical. For most problems this does indeed turn out to be the case and we need to search very carefully to find any situations in which there are different predictions.

2.1. The microscopic scale

In the introduction we derived the two candidate force densities on the basis of rival, but good, theoretical models based on either the microscopic distribution of charges or on the microscopic distribution of point dipoles. For this reason we would expect the two force densities to give the same predictions at least at the microscopic level in which we consider a single point dipole at position $\mathbf{R}$. The polarization field associated with this dipole is

$$\mathbf{P}(\mathbf{r}) = \mathbf{d}\delta(\mathbf{r}-\mathbf{R}), \tag{12}$$

and so the two force densities are

$$\begin{aligned} \mathbf{f}^c &= -(\mathbf{d}\cdot\nabla\delta(\mathbf{r}-\mathbf{R}))\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B}\delta(\mathbf{r}-\mathbf{R}) \\ \mathbf{f}^d &= ((\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B})\delta(\mathbf{r}-\mathbf{R}). \end{aligned} \tag{13}$$

Despite the differences of form, the total *force* exerted on the dipole is the same for the two force densities. To see this we integrate the two force densities over a volume containing the dipole:

$$\begin{aligned} \mathbf{F}^d &= \int \mathbf{f}^d\,\mathrm{d}V = (\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B} \\ \mathbf{F}^c &= \int \mathbf{f}^c\,\mathrm{d}V = -\int \mathbf{E}\,(\mathbf{d}\cdot\nabla\delta(\mathbf{r}-\mathbf{R}))\,\mathrm{d}V + \dot{\mathbf{d}}\times\mathbf{B} \\ &= (\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B} \\ &= \mathbf{F}^d, \end{aligned} \tag{14}$$

where we have used integration by parts in order to treat the derivatives of the delta function. It is clear that, at least at the microscopic level, the two force densities give rise to the same force and should, therefore, be regarded as equivalent.

2.2. The macroscopic scale

At the opposite extreme to the single point dipole is the total force exerted on a macroscopic dielectric medium. We consider a dielectric surrounded by vacuum and calculate the total force associated with each of the two densities by integrating over the volume of the dielectric. In doing so, we need to be careful about defining the integration volume. We select a volume V, bounded by the surface S_j, which contains the dielectric and a thin volume of the surrounding vacuum, so that dielectric–vacuum interface is fully contained within the integration volume. There is, of course, no force exerted in the vacuum region and the polarization there is identically zero. It is convenient to work with Cartesian components of the force, labelled by an index i, which takes the three values x, y and z. We will also employ the familiar convention in which a summation is implied over the three values, corresponding to the Cartesian coordinates, of any index appearing twice [13]. The difference between the two total forces is then

$$\begin{aligned} F_i^d - F_i^c &= \int_V (P_j \nabla_j + (\nabla_j P_j)) E_i \, \mathrm{d}V \\ &= \int_V \nabla_j (P_j E_i) \, \mathrm{d}V. \end{aligned} \tag{15}$$

We should note that the fields, $\mathbf{E}$ and $\mathbf{P}$, are the microscopic fields; we shall introduce the familiar averaged macroscopic fields in the following subsection. Gauss's theorem allows us to write this difference of forces as an integral over the surface of the integration volume:

$$\begin{aligned} F_i^d - F_i^c &= \int P_j E_i \, \mathrm{d}S_j \\ &= 0, \end{aligned} \tag{16}$$

as the polarization is zero on the integration surface.

As with the point dipole, the difference in the total force on the dielectric given by the two force densities is *zero*. This is reasonable as $\mathbf{F}^d$ is based on the force exerted on the (centre of mass of) each dipole, while $\mathbf{F}^c$ is based on the force acting on each charge. The nett effect in each case is the same total force. It has been suggested that the total forces calculated on the basis of $\mathbf{f}^c$ or $\mathbf{f}^d$ are sometimes different [10]. This is at odds, however, with the simple proof presented here that they should be the same. We shall explain this contradiction in section 2.4.

2.3. The mesoscopic scale

The results of the preceding two sections suggest that there might not be a significant difference between the two force densities. This happy situation does not survive, however, when we examine the force acting on a small volume v of dielectric, enclosed by the surface s_j. The volume is supposed to be very large compared to the spacing of the individual dipoles, but small compared with the scales on which any externally applied electromagnetic field varies. The use of such volumes is familiar from elementary classes on electromagnetism where it is used to derive macroscopic fields by averaging over the volume [14].

The difference between the forces acting on our small volume is simply

$$F_i^d - F_i^c = \int P_j E_i \, \mathrm{d}s_j. \tag{17}$$

If we keep P_j as the microscopic polarization, then we can make the integral zero by carefully threading our surface between the dipoles. This is impractical for real calculations, however, and if we wish to employ a macroscopic polarization ($\bar{\mathbf{P}}$), as we must, then the difference persists and needs to be resolved. Might it be that this difference can be removed by introducing a distribution of charges onto the surface of our volume? The answer would appear to be no. To see this we suppose that our volume is within a larger region in which the material susceptibility is a constant. It is then clear that $\bar{\mathbf{P}}$ will be continuous, $\nabla \cdot \bar{\mathbf{P}}$ will be zero, and that there will be no buildup of charge on the surface of our volume v. It follows that there will be no surface force that we can easily add to restore the agreement between the charge and dipole forms of the force.

We note that direct application of Maxwell's equations to (2) leads to the equivalent form [4]

$$f_i^d = P_j \nabla_i E_j + \frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B})_i. \tag{18}$$

This form is useful in electrostatic problems, where the time derivative vanishes, and on introducing the macroscopic fields and writing $\bar{\mathbf{P}} = \varepsilon_0(\varepsilon - 1)\bar{\mathbf{E}}$ we reach the simple form

$$\mathbf{f}^d = \tfrac{1}{2}\varepsilon_0(\varepsilon - 1)\nabla \bar{E}^2. \tag{19}$$

For a beam of (quasi) monochromatic radiation, it is convenient to introduce the complex macroscopic electric field $\bar{\boldsymbol{\mathcal{E}}}$, the real part of which is the electric field. The corresponding macroscopic polarization is then $\bar{\boldsymbol{\mathcal{P}}} = \varepsilon_0(\varepsilon - 1)\bar{\boldsymbol{\mathcal{E}}}$ and leads to distinct forms for the force density:

$$\begin{aligned} f_i^c &= \tfrac{1}{2}\varepsilon_0(\varepsilon - 1)\,\mathrm{Re}(\bar{\mathcal{E}}_j^* \nabla_i \bar{\mathcal{E}}_j - \bar{\mathcal{E}}_j^* \nabla_j \bar{\mathcal{E}}_i) \\ f_i^d &= \tfrac{1}{4}\varepsilon_0(\varepsilon - 1)\nabla_i |\bar{\mathcal{E}}|^2. \end{aligned} \tag{20}$$

Consider a linearly polarized beam propagating in the z-direction in which the y-component of the electric field is zero. If the beam is not too tightly focused then there will be a component of the electric field pointing in the x-direction and a very much smaller component along the z-direction, associated with the transverse confinement of the beam, which can be neglected for the purposes of this exercise. Both of the force densities (20) predict a force tending to constrict the dielectric within the beam. The difference, however, is that $f_x^c = 0$, while $f_x^d = \frac{1}{4}\varepsilon_0(\varepsilon - 1)\partial|\bar{\boldsymbol{\mathcal{E}}}|^2/\partial x$ [2].

2.4. Dielectric interfaces

It is clear from the foregoing analysis that the introduction of the macroscopic electric and polarization fields introduces a significant difference between the two force densities and that this is manifest at the mesoscopic scale. It is important to check that the use of macroscopic fields does not invalidate the conclusion, derived in section 2.2, that the nett forces are the same. This leads us to investigate the forces acting at dielectric interfaces.

The problem is most simply addressed by reference to a classic and exactly solvable problem in electrostatics: the attraction by a point charge on a large piece of the dielectric material. We consider a semi-infinite dielectric medium, with the dielectric constant ε, filling the half-space $z < 0$. We shall work in cylindrical polar coordinates (ρ, ϕ, z) and consider a point charge q placed at $(0, 0, d)$ (see figure 1). The macroscopic electric field is readily calculated using the method images and leads to [15]

$$\bar{\mathbf{E}}^{\mathrm{out}} = \frac{1}{4\pi\varepsilon_0}\left[\hat{\rho}\left(\frac{q\rho}{R_1^3} + \frac{q'\rho}{R_2^3}\right) + \hat{\mathbf{z}}\left(\frac{q(z-d)}{R_1^3} + \frac{q'(z+d)}{R_2^3}\right)\right] \tag{21}$$

outside the dielectric ($z > 0$) and

$$\bar{\mathbf{E}}^{\mathrm{in}} = \frac{q''}{4\pi\varepsilon_0\varepsilon}\left(\hat{\rho}\frac{\rho}{R_1^3} + \hat{\mathbf{z}}\frac{(z-d)}{R_2^3}\right) \tag{22}$$

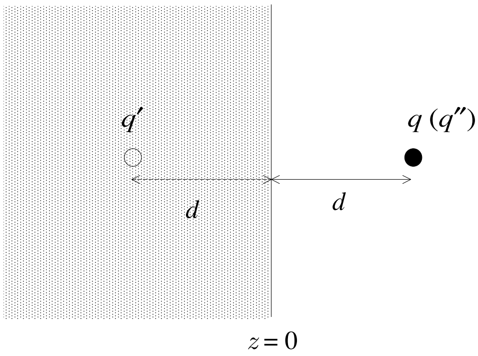

Figure 1. Configuration for calculating the force exerted by a charge q on a dielectric medium. The electric field can be calculated by the method of images: the field outside the dielectric is that due to the charge q and its image q' and the field inside the dielectric is equivalent to that associated with a charge q'' at the position of q.

inside the dielectric ($z < 0$). Here $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$ are unit vectors in the radial and z-directions and the two distances are $R_1 = [\rho^2 + (z-d)^2]^{1/2}$ and $R_2 = [\rho^2 + (z+d)^2]^{1/2}$. The charges q' and q'' are related to q by

$$q' = -\left(\frac{\varepsilon - 1}{\varepsilon + 1}\right) q \qquad q'' = \left(\frac{2\varepsilon}{\varepsilon + 1}\right) q. \tag{23}$$

Our task is to calculate the force exerted on the dielectric and no less than five possible methods come to mind. These are (i) to extend the image concept by calculating the force exerted by the charge q on its image q', (ii) to calculate the force exerted by q on the surface charge density induced by q, (iii) to use the force density $-(\nabla \cdot \bar{\mathbf{P}})\bar{\mathbf{E}}$, (iv) to use the force density $(\bar{\mathbf{P}} \cdot \nabla)\bar{\mathbf{E}}$ and (v) to use the force density $\frac{1}{2}\varepsilon_0(\varepsilon - 1)\nabla \bar{E}^2$. We shall examine each of these in turn.

2.4.1. Image method. The method of images replaces the dielectric by a point charge q' positioned at $(0, 0-d)$ (see figure 1). The force exerted on the dielectric should be equivalent to that exerted by the charge q on q'. This gives

$$\mathbf{F} = \frac{q'q}{4\pi\varepsilon_0(2d)^2}(-\hat{\mathbf{z}}) = \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}}. \tag{24}$$

2.4.2. Force exerted on the surface charges. The point charge induces a charge distribution on the dielectric surface. This polarization–charge density is [15]

$$\sigma_{\mathrm{pol}} = -\frac{q}{2\pi}\left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\frac{d}{(\rho^2 + d^2)^{3/2}}. \tag{25}$$

The force exerted by the point charge q on the surface charges is

$$\mathbf{F} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \rho\,\mathrm{d}\rho\,\sigma_{\mathrm{pol}}\frac{qd}{4\pi\varepsilon_0(\rho^2 + d^2)^{3/2}}(-\hat{\mathbf{z}}), \tag{26}$$

where we have used the cylindrical symmetry to see that the force points along the z-axis. Substituting for σ_{pol} gives

$$\mathbf{F} = \frac{q^2 d^2}{4\pi\varepsilon_0}\left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\int_0^{\infty} \rho\,\mathrm{d}\rho\,\frac{1}{(\rho^2 + d^2)^3}\hat{\mathbf{z}} = \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}}, \tag{27}$$

which agrees with (24).

2.4.3. Force on the charges, $-(\nabla\cdot\bar{\mathbf{P}})\bar{\mathbf{E}}$. The term $\nabla\cdot\bar{\mathbf{P}}$ is intimately related to the charge density (6). For our problem, it is simply related to σ_{pol}:

$$-\nabla\cdot\bar{\mathbf{P}} = \sigma_{\mathrm{pol}}\delta(z). \tag{28}$$

Hence the force density is

$$\mathbf{f}^{c} = \sigma_{\mathrm{pol}}\delta(z)\bar{\mathbf{E}} \tag{29}$$

and the total force on the dielectric is

$$\mathbf{F} = \int \mathbf{f}^{c}\,\mathrm{d}V = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\rho\,\mathrm{d}\rho\,\sigma_{\mathrm{pol}}\bar{\mathbf{E}}(z=0). \tag{30}$$

The field $\bar{\mathbf{E}}$ is the macroscopic field obtained by a local volume average. It is clear, therefore, that $\bar{\mathbf{E}}(\rho,\phi,0)$ should be equal to its local average, that is

$$\bar{\mathbf{E}}(\rho,\phi,0) = \tfrac{1}{2}[\bar{\mathbf{E}}(\rho,\phi,0^{+}) + \bar{\mathbf{E}}(\rho,\phi,0^{-})]. \tag{31}$$

Again we can invoke the cylindrical symmetry of the problem to write

$$\begin{aligned}\mathbf{F} &= \int_0^{2\pi}\int_0^{\infty}\rho\,\mathrm{d}\rho\,\sigma_{\mathrm{pol}}\frac{1}{2}[\bar{E}_z(\rho,\phi,0^{+}) + \bar{E}_z(\rho,\phi,0^{-})]\hat{\mathbf{z}}\\ &= 2\pi\int_0^{\infty}\rho\,\mathrm{d}\rho\,\sigma_{\mathrm{pol}}\frac{1}{8\pi\varepsilon_0}\frac{(-q+q'-q'')d}{(\rho^2+d^2)^{3/2}}\hat{\mathbf{z}}\\ &= \left(\frac{\varepsilon-1}{\varepsilon+1}\right)\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}},\end{aligned} \tag{32}$$

which is in exact agreement with (24) and (27).

2.4.4. Force on the dipoles, $(\bar{\mathbf{P}}\cdot\nabla)\bar{\mathbf{E}}$. In contrast with the preceding calculations, the force density $(\bar{\mathbf{P}}\cdot\nabla)\bar{\mathbf{E}}$ leads to two contributions to the total force: one due to the bulk dielectric and a second surface contribution due to the discontinuity of $\bar{E}_z$ at the dielectric interface.

We begin by examining the bulk contribution. Inside the dielectric we can write $\bar{\mathbf{P}} = \varepsilon_0(\varepsilon-1)\bar{\mathbf{E}}$ and hence integrating $(\bar{\mathbf{P}}\cdot\nabla)\bar{\mathbf{E}}$ over the dielectric volume is equivalent to integrating $\frac{1}{2}\varepsilon_0(\varepsilon-1)\nabla\bar{E}^2$. Hence the bulk contribution to the force is

$$\begin{aligned}\mathbf{F}^{\mathrm{bulk}} &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\rho\,\mathrm{d}\rho\int_{-\infty}^{0}\mathrm{d}z\frac{1}{2}\varepsilon_0(\varepsilon-1)\frac{\partial}{\partial z}\bar{E}^2\hat{\mathbf{z}}\\ &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\rho\,\mathrm{d}\rho\frac{1}{2}\varepsilon_0(\varepsilon-1)\bar{E}^2(z=0^{-})\hat{\mathbf{z}}\\ &= \frac{q^2(\varepsilon-1)}{8\pi\varepsilon_0 d^2(\varepsilon+1)^2}\hat{\mathbf{z}}.\end{aligned} \tag{33}$$

The surface contribution arises from the derivative of $\bar{\mathbf{E}}$ across the interface leading to the force density

$$\bar{P}_z\delta(z)(\bar{E}_z(z=0^{+}) - \bar{E}_z(0^{-}))\hat{\mathbf{z}} \tag{34}$$

corresponding to the surface force

$$\begin{aligned}\mathbf{F}^{\mathrm{surface}} &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\rho\,\mathrm{d}\rho\int_{-\infty}^{\infty}\mathrm{d}z\,\bar{P}_z\delta(z)(\bar{E}_z(z=0^{+}) - \bar{E}_z(0^{-}))\hat{\mathbf{z}}\\ &= 2\pi\int_0^{\infty}\rho\,\mathrm{d}\rho\,\bar{P}_z(\bar{E}_z(z=0^{+}) - \bar{E}_z(0^{-}))\hat{\mathbf{z}}.\end{aligned} \tag{35}$$

The field $\bar{\mathbf{P}}$ is the macroscopic field obtained by a local volume average (in much the same way as for $\bar{\mathbf{E}}$ above). It is clear, therefore that $\bar{P}_z(\rho, \phi, 0)$ should be equal to

$$\begin{aligned}\bar{P}_z(\rho, \phi, 0) &= \tfrac{1}{2}[\bar{P}_z(\rho, \phi, 0^+) + \bar{P}_z(\rho, \phi, 0^-)] \\ &= \tfrac{1}{2}\bar{P}_z(\rho, \phi, 0^-).\end{aligned} \tag{36}$$

Substituting this into (35) gives

$$\begin{aligned}\mathbf{F}^{\text{surface}} &= \pi \int_0^\infty \rho\, \mathrm{d}\rho \frac{(\varepsilon - 1)q''(-d)}{32\pi^2\varepsilon_0\varepsilon(\rho^2 + d^2)^3}\left(-qd + q'd + \frac{q''d}{\varepsilon}\right)\hat{\mathbf{z}} \\ &= \frac{q^2(\varepsilon - 1)^2}{16\pi\varepsilon_0(\varepsilon + 1)^2 d^2}\hat{\mathbf{z}}.\end{aligned} \tag{37}$$

Adding the bulk and surface contributions gives the total force:

$$\begin{aligned}\mathbf{F} &= \mathbf{F}^{\text{bulk}} + \mathbf{F}^{\text{surface}} \\ &= \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}},\end{aligned} \tag{38}$$

which agrees exactly with (24), (27) and (32).

We have arrived at precisely the same result in four very different ways, each suggestive of a different physical interpretation. These are (i) the force on an effective point charge, (ii) the force on a surface density of bound charges due to the Coulomb interaction with our point charge (iii) the force due to the (average) electric field at the boundary acting on a surface density of charges and (iv) the Coulomb force on the elementary dipoles forming the dielectric. The last of these comprises both surface and bulk contributions. Distinguishing between $\mathbf{f}^c$ and $\mathbf{f}^d$ on the basis of the total force is, therefore, impossible.

2.4.5. Force based on permittivity, $\frac{1}{2}\varepsilon_0(\varepsilon - 1)\nabla\bar{E}^2$. It is tempting, even if only for computational convenience, to work with the form (19) of the force density. There is, however, a cautionary note that should be mentioned and this concerns its application at boundaries. To illustrate the point we proceed to calculate, one final time, the force exerted by our charge q on its neighbouring dielectric.

As in section 2.4.4, we find that the force comprises a bulk and a surface contribution. The bulk contribution is clearly the same as obtained using $(\bar{\mathbf{P}} \cdot \nabla)\bar{\mathbf{E}}$:

$$\mathbf{F}^{\text{bulk}} = \frac{q^2(\varepsilon - 1)}{8\pi\varepsilon_0 d^2(\varepsilon + 1)^2}\hat{\mathbf{z}}. \tag{39}$$

A surface contribution arises from the derivative of the discontinuous $\bar{E}^2$ at the interface. It follows that the surface contribution to the force density is

$$\tfrac{1}{2}\varepsilon_0(\varepsilon - 1)\delta(z)(\bar{E}^2(\rho, \phi, 0^+) - \bar{E}^2(\rho, \phi, 0^-))\hat{\mathbf{z}} \tag{40}$$

corresponding to the surface force

$$\begin{aligned}\mathbf{F}^{\text{surface}} &= \int \mathrm{d}V \frac{1}{2}\varepsilon_0(\varepsilon(\mathbf{r}) - 1)\delta(z)(\bar{E}^2(\rho, \phi, 0^+) - \bar{E}^2(\rho, \phi, 0^-))\hat{\mathbf{z}} \\ &= 2\pi \int_0^\infty \rho\, \mathrm{d}\rho \frac{1}{2}\varepsilon_0(\varepsilon(z = 0) - 1)(\bar{E}^2(z = 0^+) - \bar{E}^2(z = 0^-))\hat{\mathbf{z}}.\end{aligned} \tag{41}$$

At this stage we are stuck! The reason is that we have no physical principle by which to determine the appropriate form of $\varepsilon(z = 0)$. *If* we simply let

$$\varepsilon(z = 0) = \tfrac{1}{2}[\varepsilon(z = 0^+) + \varepsilon(z = 0^-)] = \tfrac{1}{2}(\varepsilon + 1) \tag{42}$$

then we get into difficulties. This substitution leads to a surface force

$$\mathbf{F}'^{\text{surface}} = 2\pi \int_0^\infty \rho\, \mathrm{d}\rho \frac{1}{4} \frac{(\varepsilon - 1)}{16\pi^2\varepsilon_0(\rho^2 + d^2)^3} \frac{q''^2(\varepsilon^2 - 1)}{\varepsilon^2}\hat{\mathbf{z}}$$
$$= \frac{(\varepsilon - 1)^2 q^2}{32\pi\varepsilon_0(\varepsilon + 1)d^2}\hat{\mathbf{z}}, \tag{43}$$

and addition to the bulk contribution gives the total force

$$\mathbf{F}' = \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right) \frac{q^2}{16\pi\varepsilon_0 d^2} \left(\frac{\varepsilon^2 + 3}{2(\varepsilon + 1)}\right)\hat{\mathbf{z}}. \tag{44}$$

This differs from the expressions calculated above and it is certainly incorrect.

The problem lies *not* with the force density $\frac{1}{2}\varepsilon_0(\varepsilon - 1)\nabla\bar{E}^2$ and certainly not with the form $(\bar{\mathbf{P}} \cdot \nabla)\bar{\mathbf{E}}$. Rather it is with the unjustified, if appealing, replacement (42). The forms of $\bar{\mathbf{E}}$ and $\bar{\mathbf{P}}$ follow from their definition as local volume averages, but there is no corresponding physical motivation for (42). We can demonstrate the error rather directly by considering the force density in the form (18):

$$f_i^d = \bar{P}_j \nabla_i \bar{E}_j. \tag{45}$$

For our problem of a charge acting on a dielectric, this leads to the surface force density

$$\bar{P}_j(\rho, \phi, z)(\bar{E}_j(\rho, \phi, 0^+) - \bar{E}_j(\rho, \phi, 0^-))\delta(z)\hat{\mathbf{z}}$$
$$= \tfrac{1}{2}\bar{P}_z(\rho, \phi, 0^-)(\bar{E}_z(\rho, \phi, 0^+) - \bar{E}_z(\rho, \phi, 0^-))\delta(z)\hat{\mathbf{z}}$$
$$= \tfrac{1}{2}\varepsilon_0(\varepsilon - 1)\bar{E}_z(\rho, \phi, 0^-)(\bar{E}_z(\rho, \phi, 0^+) - \bar{E}_z(\rho, \phi, 0^-))\delta(z)\hat{\mathbf{z}}, \tag{46}$$

where we have used the fact that only the z-component of $\bar{\mathbf{E}}$ is discontinuous at the interface. Using the form $\frac{1}{2}\varepsilon_0(\varepsilon - 1)\nabla\bar{E}^2$ together with the *incorrect* procedure of using the average value of ε, however, leads to the force density

$$\tfrac{1}{2}\varepsilon_0(\varepsilon(\rho, \phi, z) - 1)\left(\bar{E}_z^2(\rho, \phi, 0^+) - \bar{E}_z^2(\rho, \phi, 0^-)\right)\delta(z)\hat{\mathbf{z}}$$
$$= \tfrac{1}{2}\varepsilon_0(\varepsilon - 1)(\bar{E}_z(\rho, \phi, 0^+) + \bar{E}_z(\rho, \phi, 0^-))$$
$$\times(\bar{E}_z(\rho, \phi, 0^+) - \bar{E}_z(\rho, \phi, 0^-))\delta(z)\hat{\mathbf{z}}, \tag{47}$$

which contains an additional and incorrect contribution proportional to the z-component of the electric field *outside* the dielectric.

It has been stated that use of the dipole form of the force density produces inconsistencies when applied to problems such as the torque exerted on a dielectric slab and the force exerted on a dielectric wedge [10]. It seems to us that this conclusion arises from the application of the incorrect averaging procedure (42). We have demonstrated by explicit calculation that the correct boundary conditions together with the force density (2) produce forces that are both consistent and in full agreement with those calculated using (1). We will present these calculations elsewhere.

3. A resolution?

We have seen that use of either of our candidate force densities, (1) and (2) produces precisely the same total force, but that this can be formed from apparently different contributions. Only at the level of the force density itself does any significant difference remain. In particular, the force density $\mathbf{f}^c$ seems to predict polarization-dependent forces in a homogeneous medium while $\mathbf{f}^d$ does not. It may be argued that force density is not truly an observable and that only forces on determined volumes have physical significance. Alternatively, it is possible that a satisfactory resolution can only be obtained by appealing to experiment. In this section, however, we attempt to advise on which force density is likely to be more reliable.

It has been pointed out that 'one must avoid the use of ad hoc formulas and, instead, embrace the universal form of the Lorentz force $\varrho(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t)$, where ϱ is the local charge density' [2]. This seems an eminently sensible suggestion and we will use this fundamental approach to evaluate the force on a volume v of dielectric as introduced in section 2.3.

It is clear that the difference between the two candidate forces arises solely from the electric parts of the respective force densities. For this reason it suffices to consider only the electric part of the Lorentz force density:

$$\mathbf{f}_E^L = \varrho\mathbf{E}. \tag{48}$$

We stress that this force includes all charges comprising the medium and the full microscopic electric field. We can calculate the electric force acting on the matter within our volume v by integration:

$$\mathbf{F}_E^L = \int_v \varrho(\mathbf{r})\mathbf{E}(\mathbf{r})\,\mathrm{d}V. \tag{49}$$

The electric field $\mathbf{E}$ can be divided into two parts, one arising from sources within the volume v and the second arising from sources outside. The former, which includes the fields responsible for the local integrity of the medium, can be neglected by appealing to Newton's third law of motion: they cannot affect the centre of mass of the volume and so do not contribute to the nett force. In using (49) we need only include the electric field due to sources *outside* of the volume v. These fields will have a much slower variation than those due to sources inside the volume and, provided that the volume v is not too big, we can replace $\mathbf{E}$ by the local macroscopic electric field, $\bar{\mathbf{E}}$, due to the externally applied field and the averaged affect of the material surrounding our volume. The slow variation of this field means that we can replace $\mathbf{E}(\mathbf{r})$ by the Taylor expansion of $\bar{\mathbf{E}}(\mathbf{r})$. Let $\mathbf{r}_0$ be a point within the volume v. We can then write the electric field as

$$\mathbf{E}(\mathbf{r}) \approx \bar{\mathbf{E}}(\mathbf{r}_0) + ((\mathbf{r} - \mathbf{r}_0)\cdot\nabla)\,\bar{\mathbf{E}}(\mathbf{r}_0) + \cdots, \tag{50}$$

where $\nabla\bar{\mathbf{E}}(\mathbf{r}_0)$ denotes the derivatives of $\bar{\mathbf{E}}(\mathbf{r})$ evaluated at $\mathbf{r}_0$. Note that no such averaging or introduction of a Taylor expansion is possible for the local charge density ϱ as this quantity varies wildly on Ångstrom lengthscales. The electric field also varies on these scales but, as noted above, the rapidly varying part of the electric field does not contribute to the force.

If we substitute the expanded electric field (50) into the electric Lorentz force acting on our volume of medium (49) then we obtain

$$\begin{aligned}\mathbf{F}_E^L &= \int_v \varrho(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r}_0)\,\mathrm{d}V + \int_v \varrho(\mathbf{r})\,((\mathbf{r} - \mathbf{r}_0)\cdot\nabla)\,\bar{\mathbf{E}}(\mathbf{r}_0)\,\mathrm{d}V \\ &= \left(\int_v \varrho(\mathbf{r})\,\mathrm{d}V\right)\bar{\mathbf{E}}(\mathbf{r}_0) + \left(\int_v \varrho(\mathbf{r})(\mathbf{r} - \mathbf{r}_0)\,\mathrm{d}V\right)\cdot\nabla_0\bar{\mathbf{E}}(\mathbf{r}_0),\end{aligned} \tag{51}$$

where ∇_0 denotes the 'del' operator for the coordinate $\mathbf{r}_0$. The first term is zero as the atomic dipoles comprising the medium are all neutrally charged and hence a volume containing just these will be electrically neutral so that[3]

$$\int_v \varrho(\mathbf{r})\,\mathrm{d}V = 0. \tag{52}$$

[3] Indeed were this not the case then the force on the volume would have to include the Coulomb interaction with the necessarily residually charged surrounding dielectric.

The integral in the second term does not vanish, but rather gives the macroscopic polarization as it is usually introduced

$$\bar{\mathbf{P}}(\mathbf{r}_0) = \frac{1}{v}\int_v \varrho(\mathbf{r})(\mathbf{r}-\mathbf{r}_0)\,\mathrm{d}V. \tag{53}$$

It is this quantity, and not the microscopic polarization $\mathbf{P}$, that we can write in terms of a dielectric constant:

$$\bar{\mathbf{P}}(\mathbf{r}) = \varepsilon_0(\varepsilon(\mathbf{r})-1)\bar{\mathbf{E}}(\mathbf{r}). \tag{54}$$

The electric force acting on our small volume of dielectric is therefore

$$\mathbf{F}_E^L(\mathbf{r}_0) = v(\bar{\mathbf{P}}(\mathbf{r}_0)\cdot\nabla_0)\bar{\mathbf{E}}(\mathbf{r}_0). \tag{55}$$

Dividing this by the volume v then gives the electric force density at a general point $\mathbf{r}$ as

$$\mathbf{f}_E^L(\mathbf{r}) = (\bar{\mathbf{P}}(\mathbf{r})\cdot\nabla)\bar{\mathbf{E}}(\mathbf{r}), \tag{56}$$

which is the same as the expression based on microscopic dipoles (2). It seems that starting with the fundamental Lorentz force density leads to the force density $\mathbf{f}^d$ and not to $\mathbf{f}^c$, as was suggested in [2]. This suggests that if we wish to work with macroscopic electric and polarization fields, related by a material susceptibility, then we probably should use $\mathbf{f}^d$ and not $\mathbf{f}^c$. Using $\mathbf{f}^d$, moreover, obviates the need to introduce any surface charges at dielectric interfaces.

The problem with $\mathbf{f}^c$ appears when we introduce the macroscopic polarization. Averaging $\mathbf{P}$ over a small volume to get $\bar{\mathbf{P}}$ is not a problem, but averaging the term $\nabla\cdot\mathbf{P} = -\varrho$ does *not* give $\nabla\cdot\bar{\mathbf{P}}$. The force density $\mathbf{f}^d$, by contrast, does not contain any spatial derivatives of the microscopic polarization and so the averaging procedure can be applied safely. An analysis based on the universal form of the Lorentz force suggests, somewhat surprisingly, that the force density is $\mathbf{f}^d$ rather than $\mathbf{f}^c$.

4. Electromagnetic torque

An electromagnetic field can also exert a torque on a dielectric and this has been used in demonstrations of both the orbital and spin angular momenta of light [16]. It has also been proposed that this mechanism might shed some light on the form of optical angular momentum within a dielectric [17].

We can use the preceding discussion to elucidate the form of the electromagnetic torque acting on a dielectric medium. We start by considering the single dipole introduced in section 2.1. The force density $\mathbf{f}^d$ gives the nett force on the *centre of mass* of our point dipole. The associated torque density, about the origin of coordinates, acting on this centre of mass is $\mathbf{r}\times\mathbf{f}^d$ and to this we need to add the internal torque $\mathbf{P}\times\mathbf{E}$, which acts to orient the dipole:

$$\begin{aligned}\mathbf{t}^d &= \mathbf{r}\times\mathbf{f}^d + \mathbf{P}\times\mathbf{E}\\ &= \mathbf{r}\times((\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B})\delta(\mathbf{r}-\mathbf{R}) + \mathbf{d}\times\mathbf{E}\delta(\mathbf{r}-\mathbf{R}).\end{aligned} \tag{57}$$

Integrating this over a volume containing the dipole gives the total torque

$$\mathbf{T}^d = \mathbf{R}\times((\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B}) + \mathbf{d}\times\mathbf{E}. \tag{58}$$

We might expect that the force, $\mathbf{f}^c$, being based on the microscopic distribution of charges should also give the correct torque. In this case, however, there is no separation of a centre of mass and so the full torque density is

$$\mathbf{t}^c = \mathbf{r}\times[-\mathbf{E}(\mathbf{d}\cdot\nabla\delta(\mathbf{r}-\mathbf{R})) + \dot{\mathbf{d}}\times\mathbf{B}\delta(\mathbf{r}-\mathbf{R})]. \tag{59}$$

The total torque is again obtained by integrating over a volume containing the dipole:

$$\begin{aligned}\mathbf{T}^{c} &= -\int \mathbf{r}\times\mathbf{E}(\mathbf{d}\cdot\nabla\delta(\mathbf{r}-\mathbf{R}))\,\mathrm{d}V + \int \mathbf{r}\times(\dot{\mathbf{d}}\times\mathbf{B})\delta(\mathbf{r}-\mathbf{R})\,\mathrm{d}V \\ &= \int \delta(\mathbf{r}-\mathbf{R})\mathbf{d}\cdot\nabla(\mathbf{r}\times\mathbf{E})\,\mathrm{d}V + \mathbf{R}\times(\dot{\mathbf{d}}\times\mathbf{B}) \\ &= \mathbf{T}^{d}. \end{aligned} \tag{60}$$

At the microscopic level, both force densities give the correct torque. The only distinction is that using $\mathbf{f}^{d}$ requires us to add the contribution associated with orientation of the dipole.

At the other extreme we would like to calculate the nett torque on a dielectric medium by integrating the torque density (57) or (59) expressed in terms of the macroscopic fields:

$$\begin{aligned}\mathbf{t}^{d} &= \mathbf{r}\times((\bar{\mathbf{P}}\cdot\nabla)\bar{\mathbf{E}} + \dot{\bar{\mathbf{P}}}\times\bar{\mathbf{B}}) + \bar{\mathbf{P}}\times\bar{\mathbf{E}} \\ \mathbf{t}^{c} &= \mathbf{r}\times((-\nabla\cdot\bar{\mathbf{P}})\bar{\mathbf{E}} + \dot{\bar{\mathbf{P}}}\times\bar{\mathbf{B}}). \end{aligned} \tag{61}$$

This leads to the torque

$$\begin{aligned}\mathbf{T}^{d} &= \int \mathrm{d}V[\mathbf{r}\times((\bar{\mathbf{P}}\cdot\nabla)\bar{\mathbf{E}} + \dot{\bar{\mathbf{P}}}\times\bar{\mathbf{B}}) + \bar{\mathbf{P}}\times\bar{\mathbf{E}}] \\ \mathbf{T}^{c} &= \int \mathrm{d}V[\mathbf{r}\times((-\nabla\cdot\bar{\mathbf{P}})\bar{\mathbf{E}} + \dot{\bar{\mathbf{P}}}\times\bar{\mathbf{B}})] \\ &= -\int (\mathbf{r}\times\bar{\mathbf{E}})\bar{\mathbf{P}}\cdot\mathrm{d}\mathbf{S} + \int \mathrm{d}V((\bar{\mathbf{P}}\cdot\nabla)(\mathbf{r}\times\bar{\mathbf{E}}) + \mathbf{r}\times(\dot{\bar{\mathbf{P}}}\times\bar{\mathbf{B}})) \\ &= \mathbf{T}^{d} \end{aligned} \tag{62}$$

if we choose the volume to include all of the dielectric and its surfaces so that $\bar{\mathbf{P}} = 0$ on the surface of the integration volume. As with the total force, we find that the total torque is the same for either torque density provided we apply the correct boundary conditions at any interfaces.

5. Conclusion

We have considered the form of the electromagnetic force on a dielectric medium. Two rival forms, (1) and (2), have been suggested for the force density and it is highly desirable to determine the extent to which they are different and lead to distinct predictions. Contrary to recent claims we have shown that both force densities lead to the same total forces and torques, although the component parts of the forces have rather different forms. An experiment designed to measure a total force or torque cannot distinguish between the two rival force densities. The origin of the inconsistencies noted previously [10] appears to be the incorrect replacement of the dielectric constant at an interface by the average of its values on either side. We have shown, by reference to a classic problem in electrostatics, that this does indeed lead to an apparently erroneous distinction. We will report further examples from optics elsewhere.

Much of the interest in the force on a dielectric stems from an experiment by Ashkin and Dziedzic in which the propagation of laser pulses was reported to form bulges on the surface of water [18]. The estimated radiation pressure force at the *peak* laser power of 4 kW used in the experiments is 4×10^{-6}N. They interpreted the effect as caused by a longitudinal force associated with the change in photon momentum as the light passes from air to water. Gordon proposed an alternative explanation in which the bulge is caused by inward radial forces associated with the Gaussian intensity profile of the laser beam within the dielectric [4]. This is essentially the tube-of-toothpaste effect, where a transverse squeeze produces a longitudinal flow [8]. Gordon's analysis showed that the longitudinal force on the surface is negligible and that this transverse force produces an average effective outward force on the surface of magnitude

$$F = \frac{2}{c}\left(\frac{\eta - 1}{\eta + 1}\right) Q \approx 10^{-9}\mathrm{N}, \tag{63}$$

where Q is the average beam power (1 W) and $\eta = \sqrt{\varepsilon} = 1.33$ is the refractive index. Note that, although the physical origin of the radial force is quite distinct, its effect on the surface is the same as that of the longitudinal force used by Ashkin and Dziedzic in their numerical estimate quoted above.

Mansuripur has recently proposed a further explanation of the experiment [10]. He assumes a focused laser beam of elliptical intensity profile incident normally on the liquid surface. The force arises from an interaction of the normal component of the optical electric field with a surface charge density from the normal field discontinuity and it has a longitudinal character. Its magnitude depends on the optical polarization direction relative to the elliptical axes but its maximum value is approximately 4.4×10^{-12} N for the parameter values as used above. This is nearly three orders of magnitude smaller than the Gordon result and it does not provide a quantitative explanation of the observed bulges. Gordon's result is based on (2) and Mansuripur's on (1). Our work suggests that the nett forces should be the same in both cases and that one of the two mechanisms suggested is therefore incorrect. We will return to this point elsewhere.

Finally, we note that our analysis of the forces acting on a dielectric has assumed a linear isotropic medium described by the real dielectric constant ε. It is straightforward to quantize the electromagnetic field in the presence of such a medium and so we could use the quantized forms of the force densities to explore such effects as Casimir forces and the Casimir–Polder interaction [19, 20]. These forces are usually obtained by considering the modification of the vacuum field by the new boundary conditions. The quantized force densities, however, should provide a more mechanical method of calculation.

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