Committee Meeting: Forces on an atom in a cavity, and the Abraham-Minkowski paradox

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1 Progress

- Oct 13 2011 - First author publication in Physical Review A. "Adiabatic transfer of light in a double cavity and the optical Landau-Zener problem".
- Oct -Dec - Examined the differences between open cavity systems and closed systems. Understanding the parameter regime in which the two differ and overlap. Most of the time was spent writing programs to calculate variables of interest and plotting them for the two cases to better understand cavity structure.
- Dec-present - Understanding optical properties of atoms interacting with light and relating it to work done during my Masters. Working on understanding the forces a single atom feels inside a cavity in terms of atomic and cavity parameters. We are currently working on mapping our model onto the two level atom-photon model. We have two analytic expressions for the index of refraction of the atom-cavity system, and it is now a matter of unraveling them.

2 Current research

2.1 Introduction to the paradox

The Abraham-Minkowski dilemma ammounts to assigning a photon traveling through a dielectric material the momentum $p_{Abr} = p_0/n$ or $p_{Min} = p_0 n$, where n is refractive index of the material. It's hard to believe that such a seemingly trivial problem has not been sorted out yet, but after 100 years, the correct form remains enigmatic. The difficulty lies in the fact that experiments have been done which apparently support both forms. Consider the following two gedanken experiments. In the first experiment, a photon travels through a slab of transparent material [1]. While traveling through the block, it's speed slows down to c/n and therefore would take a time $t = Ln/c$ to transverse the slab, where L is the length of the slab. Upon exiting the slab, the photon would have traveled a shorter distance than it would have had it traveled in free space. This difference in distance is $L\,(n-1)$. By the uniform motion of mass-energy, we expect the block to be displaced by an amount Δz in the direction of propagation of the photon.

$$L\,(n-1)\,\hbar\omega = \Delta z M c^2 \tag{1}$$

It is important to note that this expression makes no assumption about the magnitude of the momentum in the medium. We then find that the momentum acquired from the slab is

$$p_{slab} = M\frac{\Delta z}{Ln/c} \tag{2}$$

This leads to the momentum transfer

$$p_{slab} = \left(1 - \frac{1}{n}\right)\frac{\hbar\omega}{c} \tag{3}$$

By conservation of momentum, we know the total momentum must be $p_{total} = \frac{\hbar\omega}{c}$, and therefore we obtain

$$p_{photon} = \frac{\hbar\omega}{cn} = p_{Abr} \tag{4}$$

This argument is in line with Einstein's box theory [7] on mass-energy equivalence.

For the second gedanken experiment [1], consider a single atom of mass m with a transition frequency of ω_0 traveling through a medium with an index of refraction n, at velocity v. Let as also assume that the atom is moving away from a light source emitting at an angular frequency ω. The atom can absorb a photon if the doppler shifted frequency matches the transition frequency of the atom. In this case we require

$$\omega_0 = \omega\left(1 - \frac{vn}{c}\right) \tag{5}$$

By conservation of energy and momentum we would then have

$$\frac{1}{2}mv_{final}^2 + \hbar\omega_0 = \frac{1}{2}mv_{initial}^2 + \hbar\omega \tag{6}$$

$$mv_{final} = mv_{initial} + p_{photon} \tag{7}$$

Combining these we obtain

$$p_{photon} = \frac{\hbar\omega n}{c}\frac{2v_{initial}}{v_{initial}+v_{final}} \approx \frac{\hbar\omega n}{c} \tag{8}$$

2.2 The δ-function model

Our idea is to examine the effects that a single atom has on an electromagnetic field. The atom interacts with light to change the wavemodes and therefore acts like a medium with some effective index of refraction. Obviously this effect will be small for a single atom, however, if the atom is placed in a cavity, this effect is multiplied many thousands (or millions) of times due to the light bouncing back and forth. During each pass, the atom interacts with the light, slightly changing the electromagnetic field, and also the state of the atom. This coupled problem can be easily solved using a model which I studied during my Masters research.

The setup consists of a perfect one-dimensional cavity, with an atom placed inside. The atom can be thought of as a dielectric mirror with some index of refraction, which we must find. The result is a coupled atom-cavity system. For the purposes of finding an analytic solution to the problem, we treat the atom as a point δ-dielectric. The wave equation of an electromagnetic field in matter is given by

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \mu\varepsilon\frac{\partial^2 E(x,t)}{\partial t^2} = 0 \tag{9}$$

Here we may assume that $\mu = \mu_0$. For our model [5], we are assuming that the atom is equavalent to a delta-slab of dielectric material. Therefore we have that

$$\varepsilon(x) = \varepsilon_0(1+\alpha\delta(x)) \tag{10}$$

where α is a parameter which determines the reflectivity of the delta-slab. We write the solutions to the Maxwell wave equation as $E_n(x,t) = U_n(x)e^{-i\omega_n t}$, where $\omega_n = \frac{k_n}{\sqrt{\varepsilon_0\mu_0}}$. Here $n = 1, 2, 3...$ is an integer labeling the normal modes. Inserting this into the Maxwell wave equation, we find:

$$\frac{d^2 U_n(x)}{dx^2} + k_n^2(1+\alpha\delta(x))U_n(x) = 0 \tag{11}$$

Imposing the boundary condition $U_n(-L_1) = U_n(L_2) = 0$, where $x = -L_1$ and $x = L_2$ are the positions of the cavity mirrors, we find:

$$U_n(x) = \begin{cases} A_n \sin(k_n(x+L_1)) & -L_1 < x \leq 0 \\ B_n \sin(k_n(x-L_2)) & 0 < x < L_2 \end{cases} \tag{12}$$

We may calculate the reflection/transmission coefficients of light in the delta model

$$r = -\frac{1}{1-\frac{2}{ik\alpha}} \tag{13}$$

$$t = \frac{1}{1 - \frac{ik\alpha}{2}} \tag{14}$$

and therefore obtain the probability amplitudes

$$R = \frac{1}{1 + \frac{4}{k^2\alpha^2}} \tag{15}$$

$$T = \frac{1}{1 + \frac{k^2\alpha^2}{4}} \tag{16}$$

This allows us to relate these results to the reflectivity and transmission amplitudes for a single atom, allowing us calculate α for a single atom. We now step back from the δ-function model and consider the dipole force on an atom. The expression for the coherent force on a two level atom is given by [2]

$$F_{dipole} = -\frac{\hbar\delta}{4} \frac{\overrightarrow{\nabla}\Omega^2}{\frac{\Gamma^2}{4} + \delta^2 + \frac{\Omega^2}{2}} = -\hbar u \overrightarrow{\nabla}\Omega \tag{17}$$

Here δ is the detuning, Ω is the Rabi frequency, Γ is the spontaneous decay rate, and U is the steady state Bloch vector. The change in momentum is given by [4] (we assume dt=dz/c which is true for a slow moving atom)

$$\triangle P = u\Omega\hbar/(2c) \tag{18}$$

Note that if we assume that the dipole force is a result of the reflection of photons from one mode to the other, we obtain

$$R = \frac{u\Omega\hbar/(2c)}{2\hbar k} = \frac{u\Omega}{4ck} \tag{19}$$

If we now equate this with the reflection probability for the delta-slab model eq.(15), we arrive at

$$\alpha = \sqrt{\frac{u\Omega}{ck^3 - \frac{u\Omega k^2}{4}}} \approx \sqrt{\frac{u\Omega}{ck^3}} \tag{20}$$

It was also possible to find an analytic solution for the wavenumber k_n when α is very small as it is for an atom. By applying boundary conditions, we arrive at a transcendental equation for the allowed wavenumbers k_n in the cavity-atom system

$$\cos(k_n \triangle L) - \cos(k_n L) = \frac{2L}{\alpha} \frac{\sin k_n L}{k_n L} \tag{21}$$

Here $\triangle L$ is the atom displacement from the centre of the cavity. When α is very small, the right side must still be of order unit, thereforesin kL must be very close to zero. We therefore expand kL about $n\pi$ to first order, (note that $\pm$ corresponds to odd and even n respectively)

$$\cos(k\triangle L) \pm 1 = \mp \frac{2L}{\alpha} \left(\frac{1}{n\pi} (x - n\pi) \right) \tag{22}$$

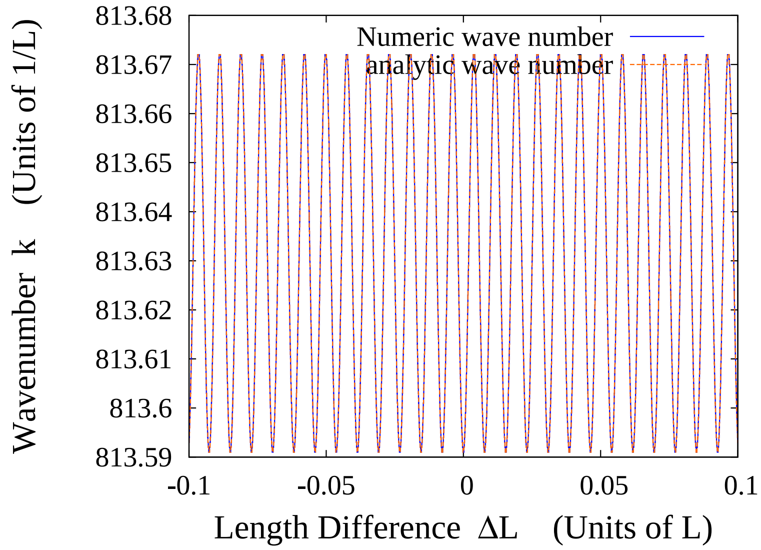

Figure 1: This plot above shows wavenumbers for the numeric vs. analytic results. Here is a plot for $k \approx 800$ in units of $1/L$ and $\alpha = 10^{-5}$ in units of L as a function of $\triangle L$.

Now $\cos(k\Delta L)$ has k in the argument, however as this doesn't deviate from $n\pi$ much, it is reasonable for small values of α to replace it with $n\pi$ as the cosine function is insensitive to such small perturbations

$$k_n = \pm\frac{\alpha n\pi}{2L^2}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) + \frac{n\pi}{L} \tag{23}$$

Here is a plot for $k \approx 800$ in units of $1/L$ and $\alpha = 10^{-5}$ in units of L as a function of $\triangle L$. Both numeric and analytic solutions are plotted.

This is an opportune time to state that most models of cavity QED (atoms and photons in a cavity) assume that the atom does not affect the structure of the light mode [8]. We have shown in a recently published paper [5] that these affects can be quite substantial in the strong coupling regime. We now ask ourselves what the phase shift of the light is. For this we use eq.(23)for the wavenumber k_n and rewrite it in a form that allows us to extract the index of refraction

$$k_n = \frac{n\pi}{L}\left[\pm\frac{\alpha}{2L}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) + 1\right] \tag{24}$$

and we can therefore read off the index of refraction of the atom-cavity system as

$$n_{atom-cavity} = 1 \pm \frac{1}{2L}\sqrt{\frac{u\Omega}{ck^3 - \frac{u\Omega k^2}{4}}}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) \tag{25}$$

In order to better understand the form of momentum used here, we wish to recalculate the refractive index of the atom-photon system using a different method. We can calculate the force on the atom by noting that the force F is given by the momentum flux integrated about a gaussian pillbox containing the atom [3]. If we write the modes to the left and to the right of the dielectric slab as

$$E_L = E_1 e^{ikx} + E_2 e^{-ikx} \tag{26}$$

$$E_R = E_3 e^{ikx} + E_4 e^{-ikx} \tag{27}$$

Then we arrive at the force F

$$F = \frac{\varepsilon_0}{2}\left(|E_1|^2 + |E_2|^2 - |E_3|^2 - |E_4|^2\right) \tag{28}$$

To procede we require the amplitudes E_1, E_2, E_3, E_4. What we have instead is an analytic solution to the amplitudes of the two standing waves to the left and to the right of the atom. From 12 we find that

$$\frac{A_n}{B_n} = -\frac{\sin(k_n L_2)}{\sin(k_n L_1)} \tag{29}$$

If we rewrite the standing wave as being composed of two opposing travelling waves, then we require $A_n = 2E_1 = 2E_2$, and $B_n = 2E_3 = 2E_4$. This yields a pressure (force divided by transverse area)

$$F = \frac{\varepsilon_0}{4}\left(|A|^2 - |B|^2\right) = \frac{1}{2L}\left(\frac{\varepsilon_0}{2}|B|^2 L\right)\left(\left|\frac{\sin(k_n L_2)}{\sin(k_n L_1)}\right|^2 - 1\right) \tag{30}$$

The factor $\frac{\varepsilon_0}{2}|B|^2 L$ is the cross-sectional energy density of the field in the cavity. We may set this to 1 for simplicity. The next step is to approximate the above expression using eq.(24) and compare it with the form of the refractive index found in eq.(25). To help better understand how the intensity ratio changes with $\triangle L$, below is a series of graphs for the mode structure as the atom is displaced.

2.3 Slightly transmitting end mirrors

The last consideration was to examine the effects that having an open system as opposed to a closed system (perfectly reflective end mirrors) has on the mode structure. Consider a double cavity system in which the two end mirrors are not

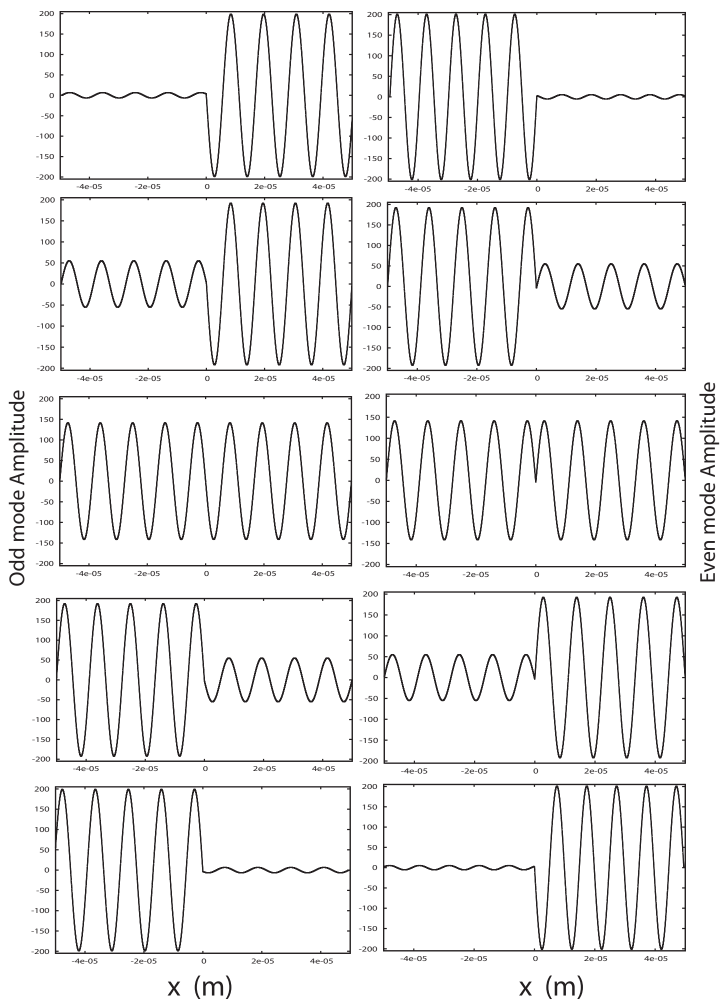

Figure 2: This series of plots shows the odd and even mode profiles as $\triangle L$ is changed. We have set $\alpha = 10^{-2}$ in units of L for asthetic purposes. The left column shows the odd mode, and the right column is the corresponding even mode. In the top row, we have assumed the central mirror is displaced to the left. By the third row we are in a situation in which the slab is at the central location. The last two rows shows the mode profiles as the central mirror is shifted to the right.

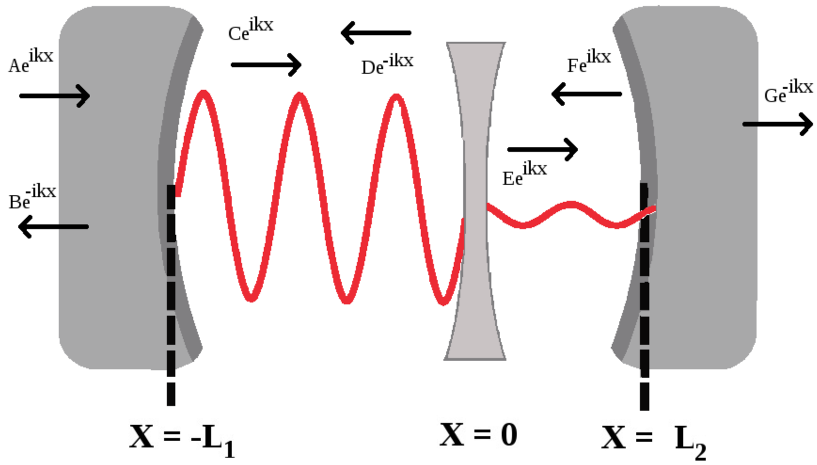

Figure 3: Double cavity setup consisting of 2 strongly reflecting mirrors, along with a common central mirror which is allowed to move.

perfectly reflective. This scenario is fundamentally very different from the case in which we have perfectly reflecting end mirrors. In the latter case, the mirror position determines the allowed wavenumbers of the system. In the former case, the positioning does not change the wavenumber of the light in the cavity, it only changes the amplitude. We wish to find the force on the central mirror as a function of the mirror reflectivity, wave number, and mirror position. To accomplish this, we solve Maxwell's equations in the four zones. We treat the mirrors as delta potentials similar to how we treated the atom. Using the ansatz that in each region we have a plane wave propagating to the right and a plane wave propagating to the left, we may solve for the amplitude of the waves by matching boundary values.

Continuity of the electric field at the mirrors give the following relations:

$$Ae^{ikx_1} + Be^{-ikx_1} = Ce^{ikx_1} + De^{-ikx_1} \tag{31}$$

$$C + D = E + F \tag{32}$$

$$Ee^{ikx_3} + Fe^{-ikx_3} = Ge^{ikx_3} \tag{33}$$

Integrating over an infinitesimal region about each delta mirror gives the final 3 relations:

$$\frac{i}{k}\left(-Ae^{ikx_1}+Be^{-ikx_1}+Ce^{ikx_1}-De^{-ikx_1}\right)=-\alpha_L\left(Ae^{ikx_1}+Be^{-ikx_1}\right) \tag{34}$$

$$\frac{i}{k}\left(-C+D+E-F\right)=-\alpha_C\left(C+D\right) \tag{35}$$

$$\frac{i}{k}\left(-Ee^{ikx_3}+Fe^{-ikx_3}+Ge^{ikx_3}\right)=-\alpha_R\left(Ee^{ikx_3}+Fe^{-ikx_3}\right) \tag{36}$$

Where α_L, α_C, andα_R are the reflectivity parameters for the left, central, and right mirrors respectively. I also set $A=1$. Now we may use the solution to these expressions to find the relative amplitudes of the modes 29. By squaring this quantity, we are able to create a plot for the relative intensities. The parameters used here are $k_n=800$ in units of $1/L$, and$\alpha=0.01$ in units of L. The plot below shows the scenario in which we set the two end mirrors to $\alpha_L=\alpha_R=10^3\alpha_C$. We have also plotted the perfectly reflective case on top to show how well they match up. It is important to note that in the open case, the wavenumber of the light being pumped in is set by the laser. Thus in this graph, and all that follow, we have also changed the wavenumber in the open case to match the resonance wavenumber that would have been found in the perfectly reflecting case. By this I mean that for each change in$\triangle L$, I recalculate the wave number k_n using the transcendental equation for the perfect cavity case and then I solve the system of equations using that value. The purple graph is the perfectly reflective case, while the blue graph is for the transmissive case.

This plot shows that for a high reflectivity ratio of the outer mirrors to that of the central mirror (atom), the open cavity modes tend towards the perfectly reflective closed cavity system. This justifies our use of the closed cavity model when dealing with this problem.

3 Progress and Degree Requirments

I have completed the necessary course requirements. Below is a list of completed courses:

Physics 750 (Statistical Mechanics I)
Physics 739 (Quantum Mechanics I)
Physics 6G03 (Computational Methods)
Math 744 (Asymptotic Methods)
Physics 2203 (Quantum Optics I took at the University of Toronto)
Physics 740 (Quantum Mechanics II)

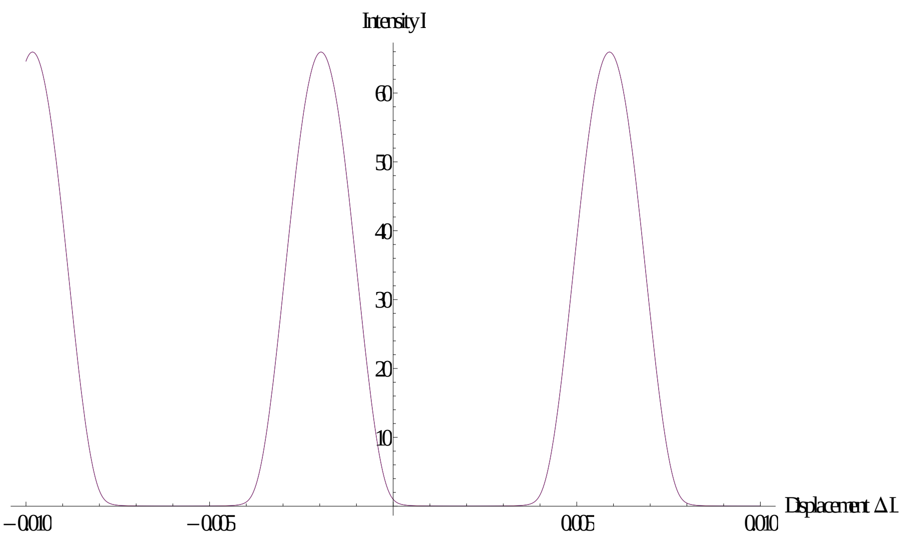

Figure 4: The plot above shows the intensity ratios of both the open case and the closed case. The purple graph represents the case in which the end mirrors are taken to be perfectly reflective, while in the blue graph we have set $\alpha_L = \alpha_R = 10^3\alpha_C$. We see that they coincide when the end mirrors are much more reflective than the central medium.

4 Short term research timeline:

•**June 25 - July 25:** Complete the analysis of cavity forces on an atom, and expand to a model in which the atom has a non-delta density profile. It would also be interesting to examine

the case in which an atom is dropped into a cavity and understanding the diffracting which would occur.

•**July 26 - Sept 1:** Write the first draft for the paper

•**Sept 2- Oct 1 :** Revisions and corrections

•**Oct 2 - :** Let's see how these results turn out.

References

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