Committee Meeting: Forces on an atom in a cavity, and the Abraham-Minkowski paradox

June 21, 2012

Nick Miladinovic

Supervisor: Duncan O'Dell

1 Research

1.1 Introduction to the paradox

The Abraham-Minkowski dilemma ammounts to assigning a photon traveling through a dielectric material the momentum $p_{Abr} = p_0/n$ or $p_{Min} = p_0 n$, where n is refractive index of the material. It's hard to believe that such a seemingly trivial problem has not been sorted out yet, but after 100 years, the correct form remains enigmatic. The difficulty lies in the fact that experiments have been done which apparently support both forms. Consider the following two gedanken experiments. In the first experiment, a photon travels through a slab of transparent material [1]. The total energy of the system before the photon enters the median is $E_{total} = \hbar\omega + Mc^2$, where M is the mass of the slab. While traveling through the block, it's speed slows down to c/n and therefore would take a time $t = Ln/c$ to transverse the slab, where L is the length of the slab. Upon exiting the slab, the photon would have traveled a shorter distance than it would have had it traveled in free space. This difference in distance is $L\,(n-1)$. By uniformity of center of mass-energy, we expect the block to be displaced by an amount Δz in the direction of propagation of the photon. The uniform motion of the center of mass-energy leads to

$$L\,(n-1)\,\hbar\omega = \Delta z M c^2$$

If we then assume that the momentum acquired from the slab came from the momentum lost by the photon, we obtain

$$p_{slab} = M\frac{\Delta z}{Ln/c}$$

This leads to the momentum transfer

$$p_{slab} = \left(1 - \frac{1}{n}\right)\frac{\hbar\omega}{c}$$

By conservation of momentum, we know the total momentum must be $p_{total} = \frac{\hbar\omega}{c}$, and therefore we obtain $p_{photon} = \frac{\hbar\omega}{cn} = p_{Abr}$.

For the second gedanken experiment [1], consider an atom of mass m with a transition frequency of ω traveling through a median with an index of refraction n, at velocity v. Let as also assume that the atom is moving away from a light source emitting at an angular frequency ω_0. The atom can absorb a photon if the doppler shifted frequency matches the transition frequency of the atom. In this case we require

$$\omega = \omega_0\left(1 - \frac{vn}{c}\right)$$

By conservation of energy and momentum we would then have

$$\frac{1}{2}mv_{final}^2 + \hbar\omega_0 = \frac{1}{2}mv_{initial}^2 + \hbar\omega$$

$$mv_{final} = mv_{initial} + p_{photon}$$

Combining these we obtain

$$p_{photon} = \frac{\hbar\omega n}{c}\frac{2v_{initial}}{v_{initial} + v_{final}} \approx \frac{\hbar\omega n}{c}$$

1.2 Delta function model

Our idea is to examine the effects that an atom has on an electromagnetic field. The atom interacts with light to change the wavemodes and therefore acts like a median with some effective index of refraction. Obviously this effect will be small for a single atom, however, if the atom is placed in a cavity, this effect is multiplied many thousands of times due to light bouncing back and forth. During each pass, the atom interacts with the light, slightly changing the electromagnetic field, and also the state of the atom. This coupled problem can be easily solved using a model which I studied during my masters research.

The setup consists of a perfect one-dimensional cavity, with an atom placed inside. The atom can be though of as a dielectric mirror with some index of refraction, which we must find. The result is a symmetric coupled cavity system which we allow to asymmetrize by perturbing the atom from it's central position. For the purpose of finding an analytic solution to the problem, we treat the atom as a point delta potential. The wave equation of an electromagnetic field in matter is given by

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \mu\varepsilon\frac{\partial^2 E(x,t)}{\partial t^2} = 0$$

Here we may assume that $\mu = \mu_0$. For our model [5], we are assuming that the atom is equavalent to a delta-slab of dielectric material. Therefore we have that

$$\varepsilon = \varepsilon_0(1 + \alpha\delta(x))$$

where α is a parameter which determines the reflectivity of the delta-slab. We choose to write $E_n(x,t) = U_n(x)e^{-i\omega_n t}$ as separate variables, where $\omega_n = \frac{k_n}{\sqrt{\varepsilon_0\mu_0}}$. Inserting this into Maxwell's equation, we find:

$$\frac{d^2U_n(x)}{dx^2} + k_n^2(1 + \alpha\delta(x))U_n(x) = 0$$

Imposing the boundary condition $U_n(-L_1) = U_n(L_2) = 0$, the solution is found to be:

$$U_n(x) = \begin{cases} A_n \sin(k_n(x + L_1)) & -L_1 < x \leq 0 \\ B_n \sin(k_n(x - L_2)) & 0 < x < L_2 \end{cases}$$

We may calculate the reflection/transmission coefficients of light in the delta model

$$r = -\frac{1}{1 - \frac{2}{ik\alpha}}$$

$$t = \frac{1}{1 - \frac{ik\alpha}{2}}$$

and therefore obtain the probability amplitudes

$$R = \frac{1}{1 + \frac{4}{k^2\alpha^2}}$$

$$T = \frac{1}{1 + \frac{k^2\alpha^2}{4}}$$

This allows us to relate these results to the reflectivity and transmission amplitudes for a single atom, allowing us calculate α for a single atom. The dipole force on an atom is given by [2]

$$F_{dipole} = -\frac{\hbar\delta}{4}\frac{\vec{\nabla}\Omega^2}{\frac{\Gamma^2}{4} + \delta^2 + \frac{\Omega^2}{2}} = -\hbar u_{ss}\vec{\nabla}\Omega$$

The the change in momentum is given by [4](we assume dt=dz/c which is true for a slow moving atom)

$$\triangle P = u_{ss}\Omega\hbar/(2c)$$

Note that If we assume that the dipole force is a result of the reflection of photons from one mode to the other, we obtain

$$R = \frac{u_{ss}\Omega\hbar/(2c)}{2\hbar k} = \frac{u_{ss}\Omega\hbar}{4\hbar ck}$$

If we now equate this with the reflection probability for the delta-slab model, we arrive at

$$\alpha = \sqrt{\frac{u_{ss}\Omega}{ck^3 - \frac{u_{ss}\Omega k^2}{4}}}$$

It was also possible to find an analytic solution for the wavenumber k_n. By applying boundary conditions, we arrive at a transcendental equation for the wavenumber k

$$\cos(k\triangle L) - \cos(kL) = \frac{2L}{\alpha}\frac{\sin kL}{kL}$$

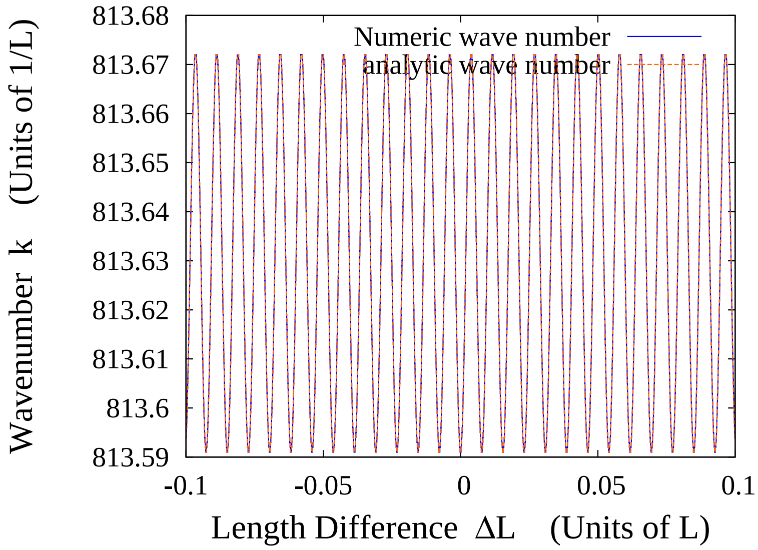
Here $\triangle L$ is central atom displacement. When α is very small, as we have found it indeed is, the right side must be of unitary order, thereforesin kL must be very close to zero. We therefore expand kL about $n\pi$ to first order. Note that $\pm$ corresponds to odd and even n respectively.

$$\cos(k\triangle L) \pm 1 = \mp\frac{2L}{\alpha}\left(\frac{1}{n\pi}(x - n\pi)\right)$$

Now $\cos(k\Delta L)$ has k in the argument, however as this doesn't deviate from $n\pi$ much, it is reasonable for small values of χ_E to replace it with $n\pi$ as the cosine function is insensitive to such small perturbations

$$k_n = \pm\frac{\alpha n\pi}{2L^2}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) + \frac{n\pi}{L}$$

Here is a plot for $k \approx 8 \times 10^7$ and $\alpha = 10^{-9}$ as a function of $\triangle L$. Both numeric and analytic solutions are plotted.



We now ask ourselves what the phase shift of the light is. For this we use the formula for the wavenumber k above.

$$k_n = \pm\frac{\alpha n\pi}{2L^2}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) + \frac{n\pi}{L}$$

We rewrite this in a form that allows us to extract the index of refraction

$$k_n = \frac{n\pi}{L}\left[\pm\frac{\alpha}{2L}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right) + 1\right]$$

and we can therefore read off the index of refraction of the atom-cavity system as

$$n_{atom-cavity} = 1 \pm \frac{1}{2L}\sqrt{\frac{u_{ss}\Omega}{ck^3 - \frac{u_{ss}\Omega k^2}{4}}}\left(\cos(n\pi\frac{\triangle L}{L}) \mp 1\right)$$

We can now calculate the force on the atom by noting that the force F is given by the momentum flux integrated about a gaussian pillbox containing the atom [3]. If we write the modes to the left and to the right of the gas as

$$E_L = E_1 e^{ikx} + E_2 e^{-ikx}$$

$$E_R = E_3 e^{ikx} + E_4 e^{-ikx}$$

Then we arrive at the force F

$$F = \frac{\varepsilon_0}{2}\left(|E_1|^2 + |E_2|^2 - |E_3|^2 - |E_4|^2\right)$$

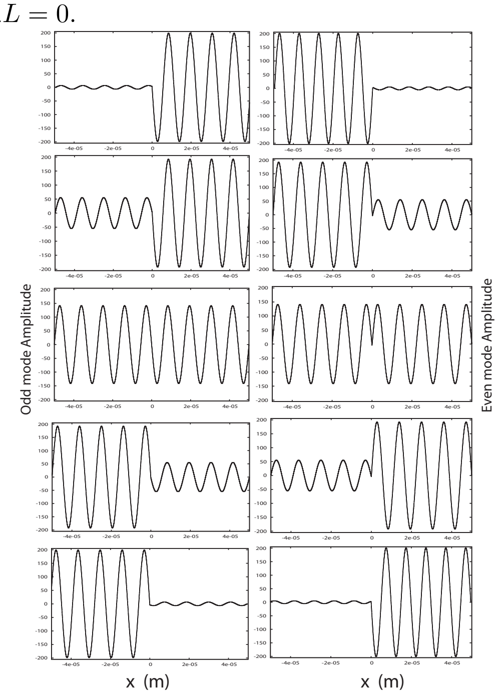
To procede we require the amplitudes E_1, E_2, E_3, E_4. What we have instead is an analytic solution to the amplitudes of the two standing waves to the left and to the right of the atom.

$$\frac{A_n}{B_n} = -\frac{\sin(k_n L_2)}{\sin(k_n L_1)}$$

If we rewrite the standing wave as being composed of two opposing travelling waves, then we require $A_n = 2E_1 = 2E_2$, and $B_n = 2E_3 = 2E_4$. This yields a force (force divided by transverse area) of

$$F = \frac{\varepsilon_0}{4}\left(|A|^2 - |B|^2\right) = \frac{1}{2L}\left(\frac{\varepsilon_0}{2}|B|^2 L\right)\left(\left|\frac{\sin(k_n L_2)}{\sin(k_n L_1)}\right|^2 - 1\right)$$

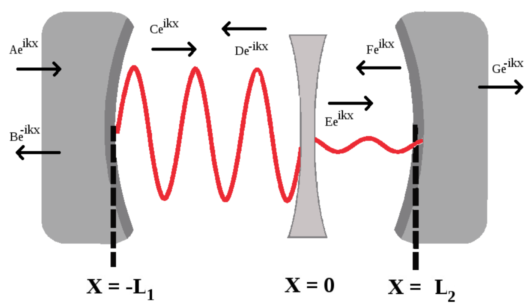
The factor $\frac{\varepsilon_0}{2}|B|^2 L$ is the initial cross-sectional energy density of the field in the cavity. We may set this to 1 for simplicity. Below is a series of graphs for the mode structure as the atom is moved from slightly left displaced from center, to slightly right displaced. In the third row we see the odd and even modes when $\triangle L = 0$.



1.3 Infinite reflectivity vs. losses

The last consideration was to examine the effects that having an open system as opposed to the closed system we were working with had on the mode structure.

Consider a double cavity system in which the two end mirrors are not perfectly reflective. This scenario is fundamentally very different from the case in which we have perfectly reflecting end mirrors. In the latter case, the mirror position determines the allowed wavenumbers of the system. In the former case, the positioning does not change the wavenumber of the pumped laser, it only changes the amplitude. We wish to find the force on the central mirror as a function of the mirror reflectivity, wave number, and mirror position. To accomplish this, we solve Maxwell's equations in the four zones. We treat the mirrors as delta potentials similar to how we treated the atom. Using the ansatz that in each region we have a plane wave propagating to the right and a plane wave propagating to the left, we may solve for the amplitude of the waves by matching boundary values.



Continuity of the electric field at the mirrors give the following relations:

$$Ae^{ikx_1} + Be^{-ikx_1} = Ce^{ikx_1} + De^{-ikx_1}$$

$$C + D = E + F$$

$$Ee^{ikx_3} + Fe^{-ikx_3} = Ge^{ikx_3}$$

Integrating over an infinitesimal region about each delta mirror gives the final 3 relations:

$$\frac{i}{k}\left(-Ae^{ikx_1} + Be^{-ikx_1} + Ce^{ikx_1} - De^{-ikx_1}\right) = -\alpha_{left}\left(Ae^{ikx_1} + Be^{-ikx_1}\right)$$

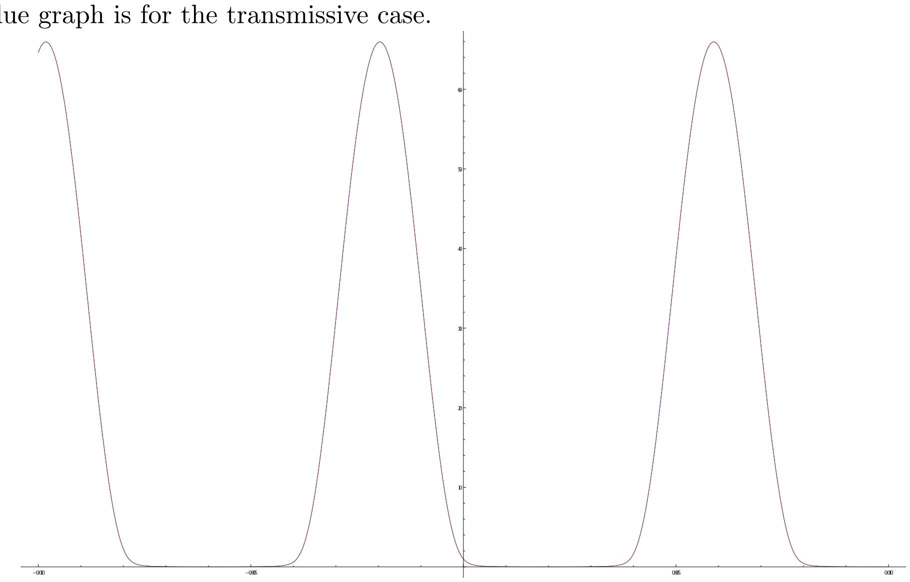
$$\frac{i}{k}\left(-C + D + E - F\right) = -\alpha_{central}\left(C + D\right)$$

$$\frac{i}{k}\left(-Ee^{ikx_3} + Fe^{-ikx_3} + Ge^{ikx_3}\right) = -\alpha_{right}\left(Ee^{ikx_3} + Fe^{-ikx_3}\right)$$

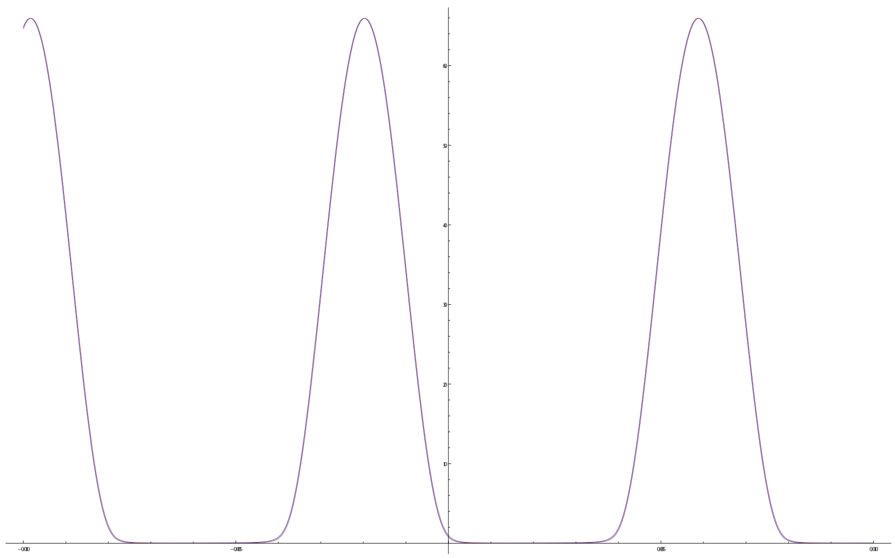
We assume here that $A = 1$. Now we may use the solution to these expressions to find the relative amplitude modes

$$\frac{A_n}{B_n} = -\frac{\sin(k_n L_2)}{\sin(k_n L_1)}$$

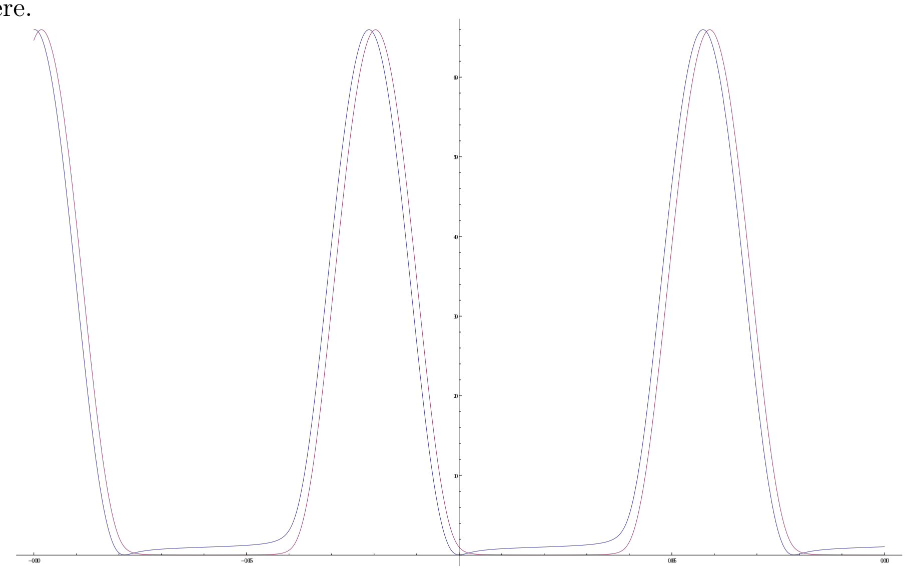
Here A_n and B_n are the field amplitudes in the first and second cavity respectively. By squaring this quantity, I was able to create a plot for the relative intensities. The parameters used here are $k = 800000$, $L = 0.001$,$\alpha = 0.00001$. The first plot shows the scenario in which we let the two end mirrors are set to $\alpha_{left} = \alpha_{right} = 10^3 \alpha_{central}$. I have also plotted the perfectly reflective case on top to show how well they match up. Note that in this graph, and all that follow, I have also changed the wavenumber to match the resonance wavenumber that would have been found in the perfectly reflecting case. By this I mean that for each change in$\triangle L$, I recalculate the wave number k using the transcendental equation for the perfect cavity case and then I solve the system of equations using that value. The purple graph is the perfectly reflective case, while the blue graph is for the transmissive case.



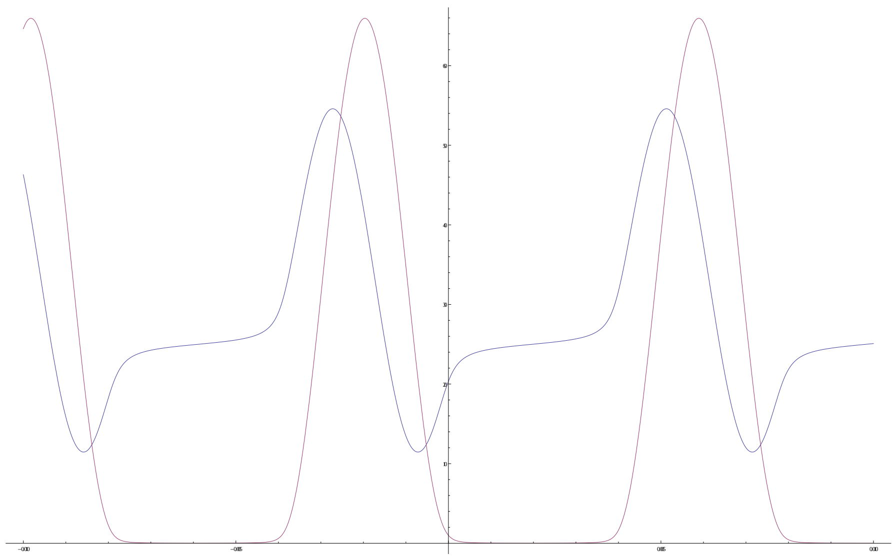
In the next graph, I have changed $\alpha_{left} = \alpha_{right} = 10\alpha_{central}$.



Here I set $\alpha_{left} = \alpha_{right} = \alpha_{central}$, which means all three mirrors are equal here.



Here I set $\alpha_{left} = \alpha_{right} = 0.1\alpha_{central}$



These graphs show that as the reflectivity ratio of the outer mirrors to the reflectivity of the central mirror (atom) increases, the open cavity modes tend towards the perfectly reflective closed cavity system. This justifies our use of the closed cavity model when dealing with this problem.

2 Progress and Degree Requirments

I have completed the necessary course requirements. Below is a list of completed courses:

Physics 750 (Statistical Mechanics I)
Physics 739 (Quantum Mechanics I)
Physics 6G03 (Computational Methods)
Math 744 (Asymptotic Methods)
Physics 2203 (Quantum Optics I took at the University of Toronto)
Physics 740 (Quantum Mechanics II)

3 Short term research timeline:

•**June 25 - July 25:** Complete the analysis of cavity forces on an atom, and expand to a model in which the atom has a non-delta density profile. It would also be interesting to examine

the case in which an atom is dropped into a cavity and understanding the diffracting which would occur.

•**July 26 - Sept 1:** Write the first draft for the paper

•**Sept 2- Oct 1 :** Revisions and corrections

•**Oct 2 - :** Let's see how these results turn out.

References

[1] Stephen M. Barnett*, Phys. Rev. Lett.* **104** 070401 (2010)

[2] J. Dalibard, and C. Cohen-Tannoudji, *Opt. Coc. Am. B*, **Vol 2** (1985)

[3] J.K. Asboth, H. Ritch, and P. Domokos, *Phys. Rev. A* **77** 063424 (2008)

[4] E.A Hinds, and M. Barnett, *Phys. Rev. Lett.* **102** 050403 (2009)

[5] N. Miladinovic et al., *Phys. Rev. A* **84** 043822 (2011)

[6] R. J. Cook, *Phys. Rev.* **Vol 20** (1979)