# Electromagnetic Momenta in a Double Cavity System

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We examine the electromagnetic momentum inside a double cavity system in which we allow the difference in cavity lengths to vary. We show that the dichotomy between the Abraham momentum $p_A = p_0 n_r$ and the Minkowski momentum $p_M = \frac{p_0}{n_r}$ is due to an ambiguity in interpreting the electromagnetic energy considered. Here $p_0$ is the free space momentum of the photon in the cavity, and $n_r$ is the refractive index due to the presence of the dielectric slab. We extend the model to a single atom in a cavity and show how experimental agreement with both representations may arise.

Figure 1. Double cavity setup consisting of two perfectly reflecting mirrors, along with a partially transmissive common central mirror. $\Delta L \equiv L_1 - L_2$ is the difference in length between the two cavities.

## I. INTRODUCTION

There has been a recent resurgence of interest in understanding the electromagnetic momentum density in a dielectric medium [1–7]. Two different forms of the momentum density were given by Minkowski and Abraham. Minkowski argued that the density be of the form $P_M = D \times B$, while Abraham held that $P_A = \frac{1}{c^2} E \times H$. For single photons these momenta are given by $p_M = p_0 n_r$ and $p_A = \frac{p_0}{n_r}$ respectively, where $p_0 = \hbar k$ is the momentum in free space . In this paper we consider an electromagnetic field in a cavity interacting with a dielectric slab. The model is solved analytically by making use of a simple $\delta$-function approximation for the central dielectric slab. This paper is organized as follows. In Section II we introduce a simple model for the spatial dependence of the dielectric permittivity function inside a double cavity. This model treats the central mirror as a Dirac $\delta$-function which facilitates analytic calculations. In Section III we find the global static solutions (normal modes) of Maxwell's wave equation subject to this dielectric function. From here we are able to extract the refractive index of the cavity-slab system. In Section IV we determine the optical force on the dielectric slab and wrestle with an ambiguity in system energy which crops up. In Section V we use the work-energy theorem to determine the form of the energy-momentum density. Conclusions are drawn in Section VII. Appendix A draws connections between the forces obtained using a classical beam splitter in a cavity, to the standard dipole force [11] justifying the model used.

## II. $\delta$-FUNCTION DIELECTRIC MODEL

Consider a double cavity formed from two end mirrors plus a common mirror located between them, as shown Figure 1.

A simple theoretical model describing a double cavity has been given in a classic paper by Lang, Scully and Lamb [8]. For the purposes of solving Maxwell's wave equation in the double cavity, they treated the end mirrors as perfect reflectors and the central mirror as a thin slab of dielectric material which is modelled by a Dirac $\delta$-function spatial profile. The double cavity model is thereby encoded in a dielectric permittivity function of the form

$$\varepsilon(x) = \begin{cases} \varepsilon_0(1 + \frac{a}{\varepsilon_0}\delta(x)) & -L_1 < x < L_2 \\ \infty & \text{elsewhere} \end{cases} \tag{1}$$

where $x = -L_1$, and $x = L_2$ are the positions of the end mirrors. $a$ is a parameter which determines the reflectivity of the common mirror. We have purposely written it in this suggestive manner in anticipation of the findings in Appendix A. The total length of the double cavity is $L \equiv L_1 + L_2$, and we also define the difference between the lengths of the two cavities to be $\Delta L \equiv L_1 - L_2$, which is also twice the displacement of the common mirror from the center of the whole cavity.

Maxwell's wave equation for the electric field $\mathcal{E}(x,t)$ in the double cavity is

$$\frac{\partial^2 \mathcal{E}(x,t)}{\partial x^2} - \mu_0 \varepsilon_0 (1 + \frac{a}{\varepsilon_0}\delta(x)) \frac{\partial^2 \mathcal{E}(x,t)}{\partial t^2} = 0 \ . \tag{2}$$

We use this $\delta$-mirror model because its simplicity facilitates analytic results. However, in Appendix A we compare the force calulations of the $\delta$-mirror model to the standard dipole force equation to obtain a relationship between $a$ and the polarizibility of an atom.

We write the solutions to the Maxwell wave equation as $\mathcal{E}_m(x,t) = U_m(x)\exp(-i\omega_m t)$, where $\omega_m = k_m/\sqrt{\varepsilon_0 \mu_0}$ is the angular frequency and $m = 1, 2, 3\ldots$ is an integer labeling the modes. The dimensionless mode functions $U_m(x)$ can be chosen to be orthogonal in the Sturm-Liouville sense by ensuring that they obey

$$\frac{1}{\varepsilon_0}\int_{-L_1}^{L_2} \varepsilon(x) U_l(x) U_m(x) dx = 0 \quad l \neq m \tag{3}$$

Inserting the above form for $\mathcal{E}(x,t)$ into Eq. (2) gives

$$\frac{\mathrm{d}^2 U_m(x)}{\mathrm{d}x^2} + k_m^2 (1 + \frac{a}{\varepsilon_0}\delta(x)) U_m(x) = 0 \ . \tag{4}$$

Figure 2. The relative amplitude ratio Eq. (7) is plotted (red) along side numeric solutions solved using Maxwell's equations in an open cavity system (blue). In the open system, the outer mirrors were set to be 10 times more reflective than the central mirror.

Solutions satisfying the boundary conditions $U_m(-L_1) = U_m(L_2) = 0$ are given by

$$U_m(x) = \begin{cases} \mathcal{A}_{Lm} \sin\left[k_m(x + L_1)\right] & -L_1 \leq x \leq 0 \\ \mathcal{A}_{Rm} \sin\left[k_m(x - L_2)\right] & 0 \leq x \leq L_2 \ . \end{cases} \tag{5}$$

Assuming the electric field is continuous across the $\delta$-mirror, so that $U_m(0^+) = U_m(0^-)$, we can integrate Eq. (4) over a vanishingly small interval containing the mirror and thereby find the last boundary condition $U'_m(0^+) - U'_m(0^-) = -\frac{a}{\varepsilon_0} k_m^2 U_m(0)$.

Combining all the boundary conditions one is led to the following equation for the wave numbers $k_m$ of the allowed modes [8]

$$\cos(k_m \Delta L) - \cos(k_m L) = 2\varepsilon_0 \ \frac{\sin k_m L}{a k_m} \ . \tag{6}$$

This transcendental equation can in general only be solved numerically. However, when $ak$ becomes large the sinc function on the right hand side becomes small. The left hand side may then be expanded around its roots and this permits approximate analytic solutions which will be supplied in Section III. We refer to the solutions for the wave number in the case of an empty cavity system (i.e when there is no central mirror) as $k_0$.

## III. ANALYTIC RESULTS

The mode amplitudes $\mathcal{A}_{Lm}$ and $\mathcal{A}_{Rm}$ on the two sides of the common mirror are calculated. From the continuity condition for the field across the mirror we find that

$$\frac{\mathcal{A}_{Lm}}{\mathcal{A}_{Rm}} = -\frac{\sin(k_m L_2)}{\sin(k_m L_1)} = -\frac{\sin[k_m(L - \Delta L)/2]}{\sin[k_m(L + \Delta L)/2]} \tag{7}$$

Throughout this paper we make use of results obtained by considering a closed cavity system. Although not physical, the results approximate an open cavity system in which the end mirrors are much more reflective than the central mirror. In Figure 2 we compare the amplitude ratio found in Eq. (7) with numerical solutions obtained for an open cavity system. The end mirrors are assumed to be 10 times more reflective than the central mirror. We see that the field distribution coincides very well with the closed cavity.

We now turn back to the transcendental equation Eq. (6) . It is possible to find an analytic solution for the wave number $k_m$ when $a$ is very small as it is for a low density atomic cloud. When $a$ is very small, the right side of Eq. (6) must still be of order one, therefore $\sin kL$ must be very close to zero. We therefore expand $kL$ about $m\pi$ to first order

$$\cos(k\triangle L) \pm 1 = \mp \frac{2\varepsilon_0 L}{a} \left( \frac{1}{m\pi} (x - m\pi) \right) \tag{8}$$

Now $\cos(k\Delta L)$ has a $k$ in the argument, however as this doesn't deviate from $m\pi$ much, it is reasonable for small values of $a$ to replace it with $m\pi$ as the cosine function is insensitive to such small perturbations

$$k_m = \pm \frac{am\pi}{2\varepsilon_0 L^2} \left( \cos(m\pi \frac{\triangle L}{L}) \mp 1 \right) + \frac{m\pi}{L} \tag{9}$$

Note that the upper signs correspond to odd $m$ and lower signs to even $m$ respectively. In Figure 3 we plot Eq. (9) against the numeric solution for the wave number.

We now ask ourselves what the effective refractive index is for the system. For this we use Eq. (9) for the wave number $k_m$ and rewrite it in a form that allows us to extract the index of refraction $n_r$

$$k_m = \frac{m\pi}{L} \left[ \pm \frac{a}{2\varepsilon_0 L} \left( \cos(m\pi \frac{\triangle L}{L}) \mp 1 \right) + 1 \right] = k_0 n_r \tag{10}$$

and we see that

$$n_r = \left[ \pm \frac{a}{2\varepsilon_0 L} \left( \cos(m\pi \frac{\triangle L}{L}) \mp 1 \right) + 1 \right] \tag{11}$$

where the upper signs correspond to odd $m$ while lower signs give the result for even $m$.

Figure 3. Wavenumber is plotted as a function of central mirror position. The analytic result Eq. (9) is in blue, and the numeric solution is plotted in red. In the plot the value of $a$, which controls the strength of the $\delta$-potential, is set at $a = 10^{-5}$.

Figure 4. The force is found by integrating the Maxwell stress tensor around a small pillbox containing the central mirror

## IV. THE FORCE ON AN ATOM IN A CAVITY

We turn to the problem of calculating the electromagnetic force on the atom, which will allow for a simple calculation of the momentum transfer. We begin by noting that the force $F$ is given by the momentum flux integrated about a gaussian pillbox containing the atom [9]. See Figure 1. This optical force is the rate at which momentum is being extracted from the electromagnetic field due to the presence of the mirror [10].

$$F = \oint_S \overleftrightarrow{T} \cdot da - \varepsilon_0 \mu_0 \int_V S d\tau \tag{12}$$

where $T$ is the Maxwell stress tensor defined as

$$T_{xx} = \frac{\varepsilon_0}{2}\left(\mathcal{E}_x^2 - \mathcal{E}_y^2 - \mathcal{E}_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right) \quad (13)$$

$$T_{xy} = \varepsilon_0\left(\mathcal{E}_x\mathcal{E}_y\right) + \frac{1}{\mu_0}\left(B_xB_y\right) \quad (14)$$

$S$ in Eq. (12) is the poynting vector, which is zero for a stationary system. As we are only considering the stationary modes of the system this term drops out. If we write the modes to the left and to the right of the dielectric slab respectively as

$$\mathcal{E}_L = \mathcal{E}_1 e^{ikx} + \mathcal{E}_2 e^{-ikx} \quad (15)$$

$$\mathcal{E}_R = \mathcal{E}_3 e^{ikx} + \mathcal{E}_4 e^{-ikx} \quad (16)$$

We then arrive at the force per unit area $F$

$$F = \frac{\varepsilon_0}{2}\left(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 - |\mathcal{E}_3|^2 - |\mathcal{E}_4|^2\right) \quad (17)$$

To proceed we require the amplitudes $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$. What we have instead is an analytic solution for the amplitudes of the two standing waves to the left and to the right of the atom as is given by Eq. (7). If we rewrite the standing wave as the composite of two traveling waves moving in opposite direction, then we require $\mathcal{E}_{Lm} = 2\mathcal{E}_1 = 2\mathcal{E}_2$, and $\mathcal{E}_{Rm} = 2\mathcal{E}_3 = 2\mathcal{E}_4$. Plugging this into Eq. (17) we obtain

$$\begin{aligned} F &= \frac{\varepsilon_0}{4}\left(|\mathcal{E}_{Lm}|^2 - |\mathcal{E}_{Rm}|^2\right) \\ &= \left(\frac{\varepsilon_0}{4}|\mathcal{E}_{Rm}|^2\right)\left(\left|\frac{\sin(k_mL_2)}{\sin(k_mL_1)}\right|^2 - 1\right) \end{aligned} \quad (18)$$

Here we have made use of Eq. (7. The second factor in Eq. (18) tells us that the force is proportional to the amplitude ratio between the modes on the left and right of the atom as is expected with radiative pressure. The first factor is more interesting, and tells us that it is also proportional to the energy of the field. The energy per unit area $E$ for the cavity system without the central mirror is

$$E = \int_0^L \frac{\varepsilon_0|\mathcal{E}|^2}{2}dl = \frac{\varepsilon_0|\mathcal{A}_{Rm}|^2 L}{4} \quad (19)$$

Here we used the fact that without a central mirror $\mathcal{A}_{Lm} = \mathcal{A}_{Rm} = \mathcal{E}$. It is crucial to note that in using Eq. (19) we have decided to ignore energy stored in the medium itself. Later in this section we will revisit this choice and discuss its consequences. Substituting this back into Eq. (18) gives

$$F = \frac{E}{L}\left(\left|\frac{\sin(k_mL_2)}{\sin(k_mL_1)}\right|^2 - 1\right) \quad (20)$$

As we are interested in determining the average force per photon, we divide Eq. (20) by the number of photons $n_{photon}$. It is assumed that the total number of photons in the cavity is conserved during motion of the mirror.

$$n_{\mathrm{photon}} = \frac{E}{\hbar c k_0} \quad (21)$$

This gives the average force per photon.

$$F = \frac{\hbar c k_0}{L}\left(\left|\frac{\sin(k_mL_2)}{\sin(k_mL_1)}\right|^2 - 1\right) \quad (22)$$

We gain better intuition by noting that for very small $a$ we can approximate the amplitude ratio factor in Eq. (22) to first order

$$\left|\frac{\sin(k_mL_2)}{\sin(k_mL_1)}\right|^2 \approx \frac{\pm\frac{\varepsilon_0 L}{am\pi} - \frac{1}{2}\sin(m\pi\frac{\triangle L}{L})}{\pm\frac{\varepsilon_0 L}{am\pi} + \frac{1}{2}\sin(m\pi\frac{\triangle L}{L})} \quad (23)$$

for odd and even $m$ respectively. Substituting this into Eq. (22) and simplifying yields

$$\begin{aligned} F &= \frac{\hbar c k_0}{L}\frac{\pm\frac{\varepsilon_0 L}{am\pi} - \frac{1}{2}\sin(m\pi\frac{\triangle L}{L}) - \left(\pm\frac{\varepsilon_0 L}{am\pi} + \frac{1}{2}\sin(m\pi\frac{\triangle L}{L})\right)}{\pm\frac{\varepsilon_0 L}{am\pi} + \frac{1}{2}\sin(m\pi\frac{\triangle L}{L})} \\ &\approx \mp\frac{\hbar c k_0}{L}\frac{\sin(m\pi\frac{\triangle L}{L})}{\frac{\varepsilon_0 L}{am\pi}} \end{aligned} \quad (24)$$

We therefore may write the optical areal force per photon as

$$F_{\mathrm{Min}} = \mp\hbar c\frac{am^2\pi^2}{\varepsilon_0 L^3}\sin(m\pi\frac{\triangle L}{L}) \quad (25)$$

for odd/even $m$ respectively. We have written the force here with a subscript, foreshadowing results from the next segment.

Let us now retrace our steps in the derivation of the force. We began with the Maxwell stress tensor ( see Eq. (13)) and proceeded to rewrite the force in terms of the amplitude ratio (Eq. (18)). We then innocently used Eq. (19) to eliminate the amplitude $\mathcal{E}_{Rm}$ in Eq. (18) in favor of writing the force in terms of the energy density $E$. As was previously mentioned, in doing so we have chosen to ignore the electromagnetic energy stored in the polarization of the medium [10]. Let us now go back and take into account the energy stored in polarization. We use the electric displacement $D$, and rewrite the energy density Eq. (19) as [10]

$$E_{\text{total}} = \int_0^L \frac{D \cdot \mathcal{E}}{2} dl = \frac{n_r^2 \varepsilon_0 \left|\mathcal{E}_{Rm}\right|^2 L}{4} \tag{26}$$

where $D = \varepsilon\mathcal{E}$ and $n_r$ is the refractive index of the system as was found in Eq. (11). Using this energy in Eq. (18) we obtain

$$F_{\text{Abr}} = \frac{E}{n_r^2 L}\left(\left|\frac{\sin(k_m L_2)}{\sin(k_m L_1)}\right|^2 - 1\right) \tag{27}$$

We have now labeled this force with a different subscript to distinguish it from Eq. (25). Following the same logic used to obtain Eq. (25) we arrive at

$$F_{\text{Abr}} = \mp\hbar c \frac{am^2\pi^2}{\varepsilon_0 n_r^2 L^3} \sin(m\pi\frac{\triangle L}{L}) \tag{28}$$

Eq. (28) gives the force on the central mirror as a function of the total electromagnetic field energy density. This is in contrast to Eq. (25), in which we only selected for the force due to free electromagnetic fields. As we shall see in Section V this variance leads one to either realizing the Minkowski, or Abraham energy-momentum.

## V. ENERGY AND MOMENTUM

We want to determine how much energy is required to realize a given mirror configuration. We will use this as a check to confirm that the inclusion of polarization energy leads to Abraham momentum, while neglecting it will give Minkowski. It is assumed that the polarizbility factor $\alpha$ is very small, and we make use of the analytic results obtained in Sections III and IV. The work-energy theorem tells us how much energy must be used in moving the central mirror to some position $\triangle L$.

$$\text{Work} = -\int F dx \tag{29}$$

We first tackle the energy required to assemble the system by assuming the force is due to the free fields $F_{Min}$. We begin by considering the case in which $m$ is odd.

$$\begin{aligned}\text{Work} &= -\int F_{Min}\frac{d\left(\triangle L\right)}{2} \\ &= \int_0^{\triangle L} \hbar c\frac{\alpha m^2\pi^2}{2L^3}\sin(m\pi\frac{\triangle L'}{L})d\left(\triangle L'\right) \\ &= -\hbar\frac{m\pi}{L}\left[\frac{\alpha}{2L}\left(\cos(m\pi\frac{\triangle L}{L}) - 1\right)\right]\end{aligned} \tag{30}$$

What Eq. (30) gives us is the energy per photon required to move the mirror from a central position $\triangle L = 0$ , to some other position $\triangle L$. By subtracting out this extracted energy from the initial configuration energy, we are able to arrive at an expression for the energy per photon remaining in the system. When the mirror is in the central position, the odd mode does not "see" the mirror. The starting energy of the configuration is nothing but the initial photon energy $\hbar\omega_0 = \hbar c\frac{n\pi}{L}$. By subtracting Eq. (30) from the initial energy, we find the change in energy for a given mirror configuration to be

$$E_{\text{photon}} = \hbar c\frac{n\pi}{L} + \hbar\frac{n\pi}{L}\left[\frac{\alpha}{2L}\left(\cos(n\pi\frac{\triangle L}{L}) - 1\right)\right] \tag{31}$$

Using Eq. (11)

$$E_{\text{photon}} = \hbar c\frac{n\pi}{L}\left[+\frac{\alpha}{2L}\left(\cos(n\pi\frac{\triangle L}{L}) - 1\right) + 1\right] = \hbar c k_0 n_r \tag{32}$$

To obtain the momentum we note that the electromagnetic fields are propagating in a vacuum, and hence we divide Eq. (32) by $c$ to obtain the Minkowski momentum for a photon. The importance of this derivation lies in the realization that we had the option to either consider only the field energy, or to also include the bound electromagnetic energy of the medium. When only the energy of free fields was considered, we ended up with the Minkowski momentum. We now show that if one includes the polarization energy of the dielectric, the Abraham result follows.

The procedure here is exactly the same as what we have done above, but instead of using $F_{\text{Min}}$, we use $F_{\text{Abr}}$.

$$\begin{aligned}\text{Work} &= -\int F_{\text{Abr}}\frac{d\left(\triangle L\right)}{2} \\ &= \int_0^{\triangle L} \frac{\hbar c\frac{\alpha m^2\pi^2}{2L^3}\sin(m\pi\frac{\triangle L'}{L})}{\left[+\frac{\alpha}{2L}\left(\cos(m\pi\frac{\triangle L}{L}) - 1\right) + 1\right]^2} d\left(\triangle L'\right) \\ &= -\hbar c\frac{m\pi}{L}\left[1 + \frac{\alpha}{2L}\left(\cos(m\pi\frac{\triangle L}{L}) - 1\right)\right]^{-1} \\ &+ \hbar c\frac{m\pi}{L}\end{aligned} \tag{33}$$

The remaining energy is therefore

$$\begin{aligned}E_{\text{photon}} &= \hbar c\frac{m\pi}{L} - \hbar c\frac{m\pi}{L} \\ &+ \hbar c\frac{m\pi}{2L}\left[1 + \frac{\alpha}{2L}\left(\cos(m\pi\frac{\triangle L}{L}) - 1\right)\right]^{-1} \\ &= \frac{\hbar c k_0}{n_r}\end{aligned} \tag{34}$$

Dividing Eq. (34) by $c$ we find that we have ended up with the Abraham momentum.

The derivation for the case in which $m$ is even is similar to the odd case, with the added twist that $\triangle L = 0$ no longer corresponds to a situation in which the fields don't "see" the mirror. We must therefore adjust the initial setup such that

$$\cos(m\pi \frac{\triangle L}{L}) + 1 = (q + \frac{1}{2})\pi \tag{35}$$

where $q$ is an integer. By doing so we can follow the same strategy used for the odd case and arrive at the same conclusion. We omit the derivation for sake of repetitiveness.

Let us now instead begin with the Lorentz force law for a dipole $d$ [**?** ]

$$F = d \cdot \frac{\partial \mathcal{E}(x,t)}{\partial x} + \frac{\partial}{\partial t}(d \times B) \tag{36}$$

This may be rewritten as a force density $f$ using $P = D - \varepsilon_0 \mathcal{E}$

$$f = D \cdot \frac{\partial \mathcal{E}}{\partial x} - \varepsilon_0 \mathcal{E} \cdot \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial}{\partial t}(D \times B - \varepsilon_0 \mathcal{E} \times B) \tag{37}$$

This leads to 4 different terms. The last two terms are identified with the Minkowski and Abraham momenta $P_M = D \times B$ and $P_A = \frac{1}{c^2} E \times H$ respectively. We can rewrite equation Eq. (37)

$$f = D \cdot \frac{\partial \mathcal{E}}{\partial x} - \varepsilon_0 \mathcal{E} \cdot \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial}{\partial t}(P_M - P_A) \tag{38}$$

The first term originating from the dipole force $F = d \cdot \frac{\partial \mathcal{E}}{\partial x}$ can be written in terms of the energy

## VI. EXPERIMENTAL AMBIGUITY

What does the above analysis tell us about why the Abraham and Minkowski momentum both are present in experimental findings? Section IV tells us that the disagreement arises due to an ambiguity in attributing the total force owing to one subsystem or another.

## VII. CONCLUSION

We'll finish this later... The Minkowski-Abraham paradox is shown to originate from interpreting the momentum transfer as arising from different components of the system energy. By including or neglecting the energy bound in the polarization one either obtains the Abraham or the Minkowski field momentum.

Here we calculate the amplification effect that the cavity has on the refractive index of an atom. From [11] the average refractive index $n_{free}$ of a an atom in free space is given by

$$n_{\text{free}} \approx 1 + \frac{\alpha_p N}{2\varepsilon_0} \tag{39}$$

where $N$ is the density. For comparison sake, we take $N = 1/L$. Comparing this to Eq. (11) we see that the refractive index $n_r$ of the cavity-atom system is at most double the value of the free refractive index.

## VIII. A CONNECTING THE MAXWELL STRESS TENSOR AND THE DIPOLE FORCE

In this appendix, we connect the microscopic description of optical forces on atoms [11] with the classical derivation obtained in Section IV. This relationship will link the $\delta$-function factor $a$ in Eq. (1) with the polarizability of an atom. We begin by examining the force derived using the Maxwell stress tensor. Suppose we have a dielectric slab, which we approximate with a $\delta$-function, interacting with two opposing plane waves. From Eq. (17) we can write the force as

$$F = \frac{\varepsilon_0}{2}\left(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 - |\mathcal{E}_3|^2 - |\mathcal{E}_4|^2\right) \tag{40}$$

Let us rewrite the outgoing fields $\mathcal{E}_1$ and $\mathcal{E}_4$ as a linear combination of the incoming fields $\mathcal{E}_2 = \mathcal{E}_{\text{left}} e^{ikx+i\phi}$ and $\mathcal{E}_3 = \mathcal{E}_{\text{right}} e^{-ikx}$ (see Figure 4).

$$\mathcal{E}_1 = r\mathcal{E}_2 + t\mathcal{E}_3 \tag{41}$$

$$\mathcal{E}_4 = t\mathcal{E}_2 + r\mathcal{E}_3 \tag{42}$$

where the reflectivity $r$ and the transmission $t$ for the $\delta$-model are given by [12]

$$r = \frac{i\frac{ka}{2\varepsilon_0}}{1 - \frac{ika}{2\varepsilon_0}} \tag{43}$$

$$t = \frac{1}{1 - \frac{ika}{2\varepsilon_0}} \tag{44}$$

Substituting these equations into Eq. (17) yields

$$F=\frac{\frac{\varepsilon_0}{2}\left|\frac{ka}{2\varepsilon_0}\right|^2}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\left|\mathcal{E}_{\text{left}}\right|^2+\frac{\frac{\varepsilon_0}{2}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\left|\mathcal{E}_{\text{right}}\right|^2$$
$$+\frac{i\frac{\varepsilon_0 ka}{4\varepsilon_0}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\mathcal{E}_{\text{left}}\mathcal{E}_{\text{left}}e^{2ikx+i\phi}$$
$$-\frac{i\frac{ka^*}{4}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}e^{-2ikx-i\phi}$$
$$+\frac{\varepsilon_0}{2}\left|\mathcal{E}_{\text{left}}\right|^2-\frac{\varepsilon_0}{2}\left|\mathcal{E}_{\text{right}}\right|^2-\frac{\frac{\varepsilon_0}{2}\left|\frac{ka}{2\varepsilon_0}\right|^2}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\left|\mathcal{E}_{\text{right}}\right|^2$$
$$-\frac{\frac{\varepsilon_0}{2}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\left|\mathcal{E}_{\text{left}}\right|^2-\frac{i\frac{ka}{4}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}e^{-2ikx-i\phi}$$
$$+\frac{i\frac{ka^*}{4}}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}e^{2ikx+i\phi} \tag{45}$$

Here $a$ is a complex parameter which we break up in its real and imaginary components $a=a_1+ia_2$. Simplifying the expression above yields

$$F=-\frac{ka_1\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}\sin(2kx+\phi)}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}$$
$$+\frac{\frac{ka_2}{2}\left(\left|\mathcal{E}_{\text{left}}\right|^2-\left|\mathcal{E}_{\text{right}}\right|^2\right)}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2}$$
$$+\frac{\varepsilon_0\left|\frac{ka}{2\varepsilon_0}\right|^2\left(\left|\mathcal{E}_{\text{left}}\right|^2-\left|\mathcal{E}_{\text{right}}\right|^2\right)}{\left|1-\frac{ika}{2\varepsilon_0}\right|^2} \tag{46}$$

where $\phi$ is the phase difference between the two incoming waves at $x=0$. Let us examine Eq. (46) term by term to gain a better understanding of what each term represents. The first term, which we label as $F_1$, is the reactive part of the force more commonly known as the dipole force. To see this let us consider the standard reactive force on an atom as given by Cohen-Tannoudji [11] for a field of the form $\mathcal{E}(x)=\mathcal{E}_{\text{left}}e^{ikx+i\phi}+\mathcal{E}_{\text{right}}e^{-ikx}$

$$F_{\text{reactive}}=-\frac{\hbar\Delta}{4}\frac{\overrightarrow{\nabla}\Omega^2}{\frac{\Gamma^2}{4}+\Delta^2+\frac{\Omega^2}{2}}=\frac{1}{4}\alpha_1\nabla\mathcal{E}^2 \tag{47}$$

Here $\Omega$ is the atomic Rabi frequency, $\Gamma$ is the spontaneous decay rate, $d$ is the dipole coherence, and $\Delta$ is the detuning. We also introduce $\alpha_1$ as the real component of the atomic polarizibility given by

Figure 5. This plot compares the complete force obtained using the Maxwell stress tensor (red) against the reactive component - the first term $F_1$ of Eq. (46) (blue). Here $alpha=10^{-8}$ was used. It is seen that for small $alpha$, the other two components of Eq. (46) may be neglected.

$$\alpha=\frac{-\Delta\left|d\right|^2}{\hbar\left[\frac{\Gamma^2}{4}+\Delta^2+\frac{\Omega^2}{2}\right]}\approx\frac{-\left|d\right|^2}{\hbar\Delta} \tag{48}$$

One finds that for $\mathcal{E}(x)=\mathcal{E}_{\text{left}}e^{ikx+i\phi}+\mathcal{E}_{\text{right}}e^{-ikx}$

$$\nabla\mathcal{E}^2=-4k\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}\sin(2kx+\phi) \tag{49}$$

Substituting this back into Eq. (47) we get

$$F_{\text{reactive}}=-\alpha k\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}\sin(2kx+\phi) \tag{50}$$

We now compare Eq. (50) to the first term $F_1$ of Eq. (46). If we are considering a single atom in the dispersive regime, then $a$ may be assumed very small. We may therefore approximate $F_1$ to first order in $a$

$$F_1\approx-a_1\mathcal{E}_{\text{left}}\mathcal{E}_{\text{right}}\sin(2kx+\phi) \tag{51}$$

Comparing Eq. (50) with Eq. (51) we see that for an atom, $\alpha_1=a_1$, and that indeed $F_1$ is the reactive component of the optical force.
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Let us now return to Eq. (46) and consider the second term $F_2$ in the equation. This term can be shown to be nothing more than the dispersive force. Following a similar scheme to that used above we have

$$F_2=\frac{\frac{1}{2}ka_2\left(\left|\mathcal{E}_{\text{left}}\right|^2-\left|\mathcal{E}_{\text{right}}\right|^2\right)}{\left|1-\frac{ik\alpha}{2}\right|^2} \tag{52}$$

For small $a$ we approximate $F_2$ to first order

$$F_2\approx\frac{1}{2}ka_2\left(\left|\mathcal{E}_{\text{left}}\right|^2-\left|\mathcal{E}_{\text{right}}\right|^2\right) \tag{53}$$

We now wish to compare this to the dispersive force as given by Cohen-Tannoudji [11]. The dispersive force felt by an atom under the influence of a field of the form $\mathcal{E}(x)=\mathcal{E}_{\text{left}}e^{ikx+i\phi}+\mathcal{E}_{\text{right}}e^{-ikx}$ is given by

$$F_{\text{dispersivee}}=-\hbar\Gamma\left(\overrightarrow{\nabla}\phi(\overrightarrow{r})\right)\frac{\Omega^2}{\Gamma^2+4\Delta^2+2\Omega^2}$$
$$=\left(\left|\mathcal{E}_{\text{left}}\right|^2-\left|\mathcal{E}_{\text{right}}\right|^2\right)\frac{k\left|d\right|^2\Gamma}{4\hbar\Delta^2} \tag{54}$$

The imaginary component of the polarizibility of an atom is [11]

$$\alpha_2 = \frac{\frac{\Gamma}{2}\left|d\right|^2}{\hbar\left[\frac{\Gamma^2}{4} + \Delta^2 + \frac{\Omega^2}{2}\right]} \approx \frac{\Gamma d^2}{2\hbar\Delta^2} \tag{55}$$

Thus we can rewrite the dispersive force as

$$F_{\text{dispersive}} = \frac{\alpha_2 k}{2}\left(\left|\mathcal{E}_{\text{left}}\right|^2 - \left|\mathcal{E}_{\text{right}}\right|^2\right) \tag{56}$$

Comparing Eq. (56) with Eq. (53) and see that $\alpha_2 = a_2$, consistent with what we found for the reactive component of the force. Now the third component $F_3$ of Eq. (46) is interpreted as the radiation pressure due to incoherent scattering. To see this we note that

$$R = \left|r\right|^2 = \frac{\left|\frac{ka}{2\varepsilon}\right|^2}{\left|1 - \frac{ika}{2\varepsilon}\right|^2} \tag{57}$$

Comparing the coefficient in Eq. (46) with Eq. (57) we see that indeed

$$F_3 = 2R\left(\left|\mathcal{E}_{\text{left}}\right|^2 - \left|\mathcal{E}_{\text{right}}\right|^2\right) \tag{58}$$

This is a second order effect in $a$ which is why it is neglected in the conventional optical force on an atom. For higher densities however, it dominates $F_1$ and $F_2$ which explains why the radiative pressure equation of classical electrodynamics [10] agrees well for high density objects such as mirrors.


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