Introduction to Graph Theory

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Outline

Introduction

Definition

Graph Theory is a field of mathematics which studies mathematical structures called *graphs*.

What is a Graph?

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A graph consists of two sets, V and E. V is a set of *vertices*. E is a set of unordered pairs of vertices, called *edges*.

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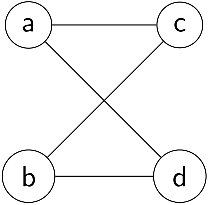
Example



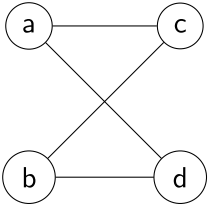
Graph G has a vertex set V and edge set E.

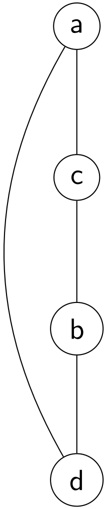
$$V = \{a, b, c, d\}$$

$$E = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$$

$$E = \{ac, ad, bc, bd\}$$

Graph Representations





Simple, Multi, and Pseudographs

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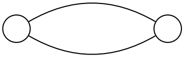

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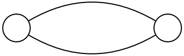

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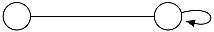
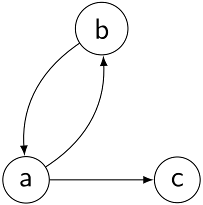


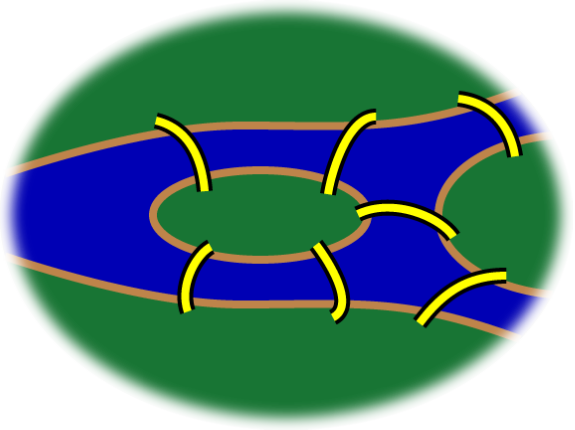
Directed Graphs

If we make the edges of the graph a set of ordered pairs, we get a directed graph, or digraph.



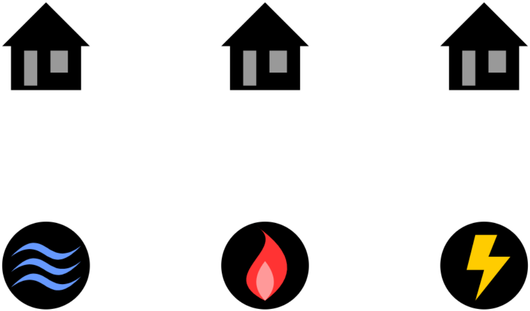
$$E = \{(a, b)\,,\ (b, a)\,,\ (a, c)\}$$

Seven Bridges of Königsberg



Find a walk through the city that crosses each bridge exactly once.

Three Utilities Problem

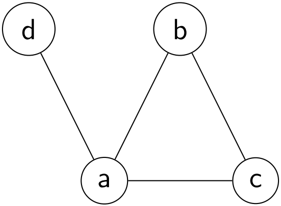


Connect every house to water, gas, and electric without crossing any lines.

Escaping a Maze

Adjacency, Incidence, and Degree

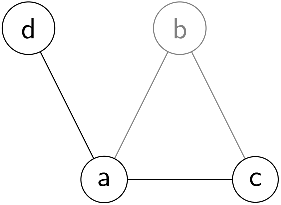
- Two vertices u and v are said to be adjacent if there is an edge uv connecting them. If there is no edge uv, then they are nonadjacent.
- If an edge has a vertex v as an end vertex, we say that edge is incident to v.
- The degree of a vertex v is the number of edges incident to v.



$$\deg(a) = 3$$

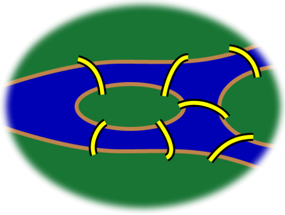
Walks, Paths, and Trails

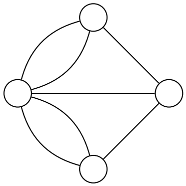
- A walk in a graph is a sequence of vertices such that each consecutive pair of vertices in the sequence are adjacent. The vertices need not be distinct.
- A walk with distinct vertices is called a path.
- A walk with distinct edges is called a trail.



$$d, \ a, \ c$$

Seven Bridges of Königsberg Revisited





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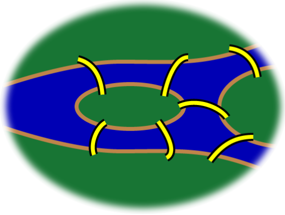

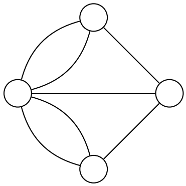

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- Known today as an Euler walk or Eulerian trail
- Leonhard Euler (1707–1783)

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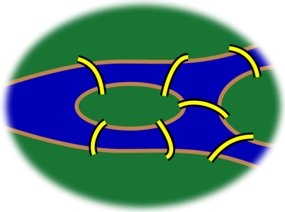
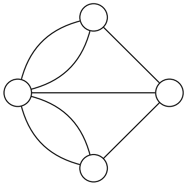
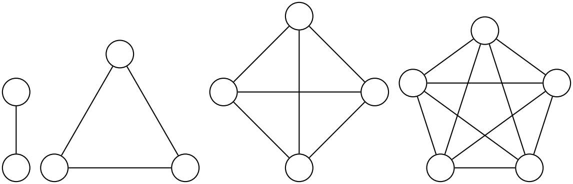




- Is there a trail which includes every edge?
- Known today as an Euler walk or Eulerian trail
- Leonhard Euler (1707–1783)
- An undirected graph has an Euler walk if and only if exactly zero or two vertices have odd degree.

Complete Graphs

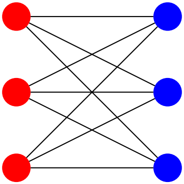
A graph is complete if every vertex is adjacent to every other vertex.



K_2, K_3, K_4, and K_5

Bipartite Graphs

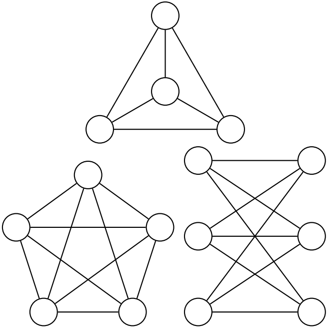
A graph is bipartite if all vertices can be colored red or blue such that every edge connects a red vertex to a blue vertex.



$K_{3,3}$

The complete bipartite graph of 3 red vertices and 3 blue vertices.

Graph Planarity



A graph is planar if it can be drawn on a plane in such a way that edges do not intersect, except at vertices.

Kuratowski's Theorem

A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Labyrinth of Daedalus

- Graph = maze
  Vertex = intersection
  Edge = passage
- How to search for the exit in a maze without getting lost and wandering in circles?
- Legend of Theseus and the Minotaur

Graph Search

Trémaux Exploration

1. *Follow any unmarked passage, unspooling string behind you*
2. *Mark all passages and intersections when first visiting them*
3. *Backtrack by rewinding the string when you approach a marked intersection*
4. *Backtrack when no unvisited options remain at an intersection*
5. *Repeat until the exit is found, or the entire maze is explored*

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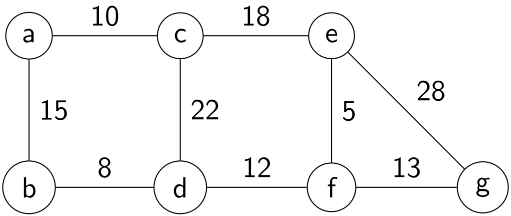
Depth-First Search

1. *Mark vertex as visited*
2. *Recursively visit all adjacent unmarked vertices*

Shortest Path

An edge-weighted graph is a graph where each edge has a weight or cost.



A *shortest path algorithm* finds a path between two vertices with the minimum cost.

Real-World Problems

- Maps
- Webpages
- Electric Circuits
- Social Networks
- Scheduling
- Commerce

  Arbitrage : shortest-path algorithm

References

- Harris, John M., Hirst, Jeffry L., and Michael J. Mossinghoff. 2008. *Combinatorics and Graph Theory*. New York: Springer.
- Sedgewick, Robert, and Kevin Wayne. 2011. *Algorithms*. Upper Saddle River: Pearson Education.