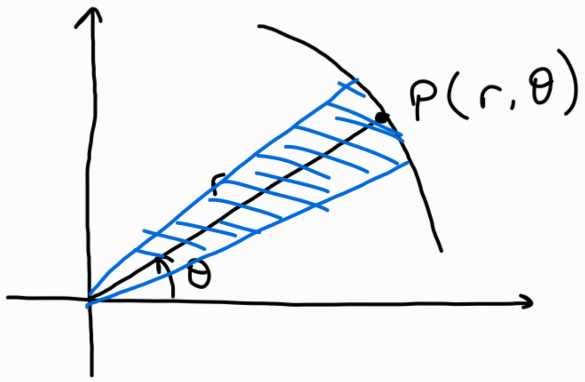
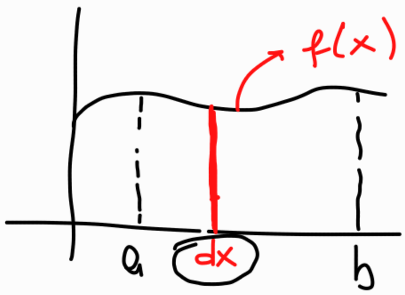
# Area in the Plane

Area of the Fan-Shaped region between the origin and the curve $r=f(\theta)$ when $\alpha \leq \theta \leq \beta$, and $\beta-\alpha \leq 2\pi$

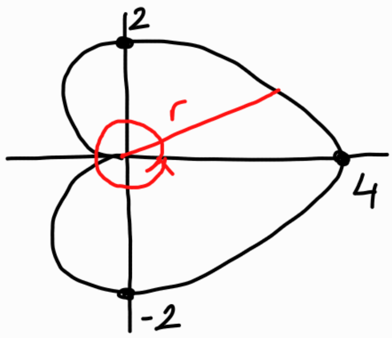
$$A=\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$




**Example:** Find the area of the region in the xy-plane enclosed by the cardioid $r=2(1+\cos\theta)$

$$A=\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{\cancel{2}} \cancel{4}^{\,2} (1+2\cos\theta+\cos^2\theta)\, d\theta$$

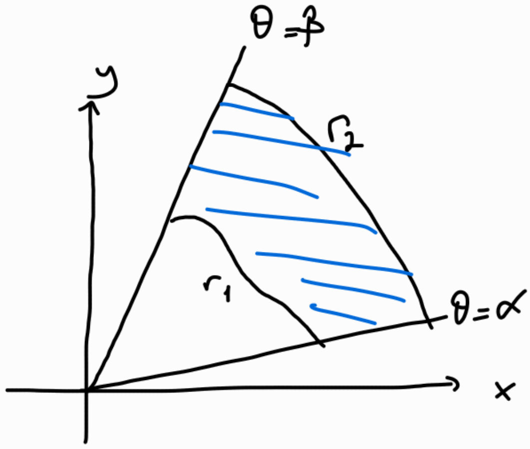



$$=\int_0^{2\pi} 2\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$=\int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)\, d\theta = 3\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \Big|_0^{2\pi}$$

$$=6\pi //$$

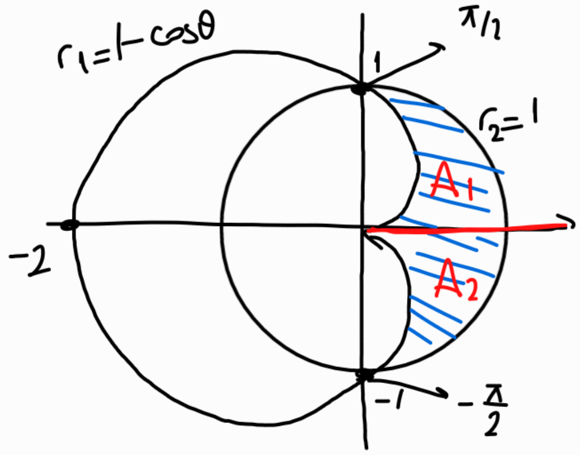
Area of the region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$, and $\beta - \alpha \leq 2\pi$




$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

**Example:** Find the area of the region that lies inside the circle $r=1$ and outside the cardioid $r = 1-\cos\theta$




$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 - \left(1 - 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_{0}^{\pi/2} \cancel{2 \cdot \frac{1}{2}} \left(2\cos\theta - \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4} \quad //$$

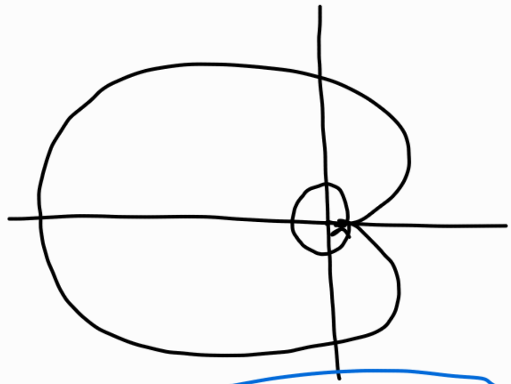
## Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$, and if the point $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as $\theta$ runs from $\alpha$ to $\beta$, then the length of the curve is

$$L=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

<u>Example</u>: Find the length of the cardioid $r=1-\cos\theta$.

$r=1-\cos\theta \qquad \frac{dr}{d\theta}=\sin\theta$



$$r^2+(r')^2=1-2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}$$

$$=2-2\cos\theta=2(1-\cos\theta)$$

$2\sin^2\theta=1-\cos 2\theta$

$$L=\int_0^{2\pi}\sqrt{2\cdot(1-\cos\theta)}\,d\theta=\int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta$$

$$=\int_0^{2\pi}2\left|\sin\frac{\theta}{2}\right|d\theta=\int_0^{2\pi}2\cdot\sin\frac{\theta}{2}\,d\theta$$

$0\leq\frac{\theta}{2}\leq 2\pi$

$0\leq\theta\leq\pi$ $\sin>0$

$$=2\cdot\left(\frac{-\cos\frac{\theta}{2}}{\frac{1}{2}}\right)\Bigg|_0^{2\pi}=-4(-1-1)=8$$

<u>Example</u>: Describe the graph of the polar equation $r=\tan\theta\sec\theta$.

$$r=\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}=\frac{\sin\theta}{\cos^2\theta}\Rightarrow r\cos^2\theta=\sin\theta$$

$$r^2\cos^2\theta=r\sin\theta$$

$$x^2=y\Rightarrow \text{Parabola.}$$

$x=r\cos\theta$

$y=r\sin\theta$

**Example**: Describe the graph of the polar equation

$$r = \frac{6}{\sqrt{9-5\sin^2\theta}}.$$

$$r^2 = \frac{36}{9-5\sin^2\theta} \Rightarrow 9r^2 - 5r^2\sin^2\theta = 36$$

$$9(x^2+y^2) - 5y^2 = 36$$

$$9x^2+4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \text{Ellipse}$$

**Example**: Describe the graph of the polar equation

$$r = 6(\sin\theta + \cos\theta)$$

$$r^2 = 6r\sin\theta + 6r\cos\theta \Rightarrow x^2+y^2 = 6x+6y$$

$$\Rightarrow x^2 - 6x \underset{+9}{\downarrow} + y^2 - 6y \underset{+9}{\downarrow} = 0$$

$$\Rightarrow (x-3)^2 + (y-3)^2 = 18 \Rightarrow \text{Circle, center } (3,3),\ r: 3\sqrt{2}$$

**Example**: $r^2\cos 2\theta = 1$

$$r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow x^2 - y^2 = 1$$

hyperbola.

**Example**: Find the distance between the points with polar coordinates $(4, 4\pi/3)$ and $(6, 5\pi/3)$

$x = r\cos\theta$

$y = r\sin\theta$

$$(4, 4\pi/3) \Rightarrow \left(4\cos\tfrac{4\pi}{3}, 4\sin\tfrac{4\pi}{3}\right) = (-2, -2\sqrt{3})$$

$$(6, 5\pi/3) \Rightarrow \left(6\cos\tfrac{5\pi}{3}, 6\sin\tfrac{5\pi}{3}\right) = (3, -3\sqrt{3})$$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(3-(-2))^2 + (-3\sqrt{3}+2\sqrt{3})^2} = \sqrt{28} = 2\sqrt{7}$$

**Example:** Find the largest value of $y$ on the cardioid $r = 2(1+\cos\theta)$

$$\underbrace{r\sin\theta}_{y} = 2\sin\theta(1+\cos\theta)$$

$$y = 2\sin\theta + \underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}$$

$$\frac{dy}{d\theta} = 2\cos\theta + 2\cos 2\theta = 0$$

$$\cos\theta = -\cos 2\theta - 1 + 1$$

$$\cos\theta = 1 - 2\cos^2\theta$$

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1).(\cos\theta + 1) = 0$$

$$2\cos\theta - 1 = 0 \qquad \cos\theta + 1 = 0$$

$$\cos\theta = \frac{1}{2} \qquad \cos\theta = -1$$

$$\Downarrow \qquad \Downarrow$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \theta = \pi$$

$$\text{I} \quad \text{IV}$$

$$y = 2\sin\theta + \sin 2\theta$$

$$y(\pi) = 0$$

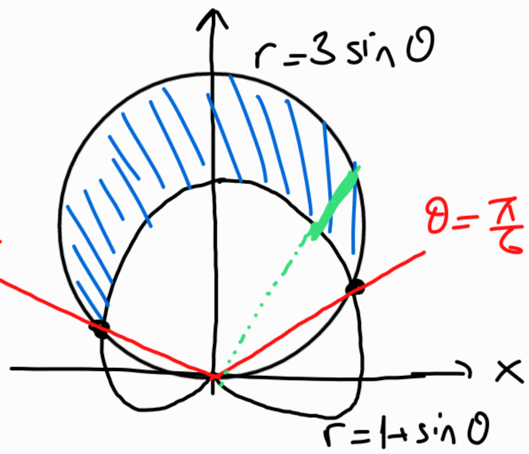
$$y\left(\frac{5\pi}{3}\right) < 0$$

$$y\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

**Example:** Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$

$$3\sin\theta = 1 + \sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}$$

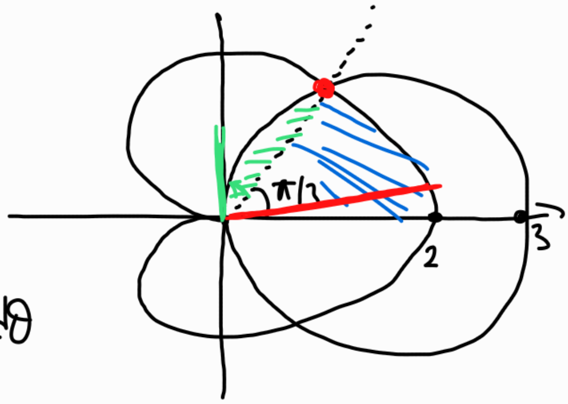



$$A = \int_{\pi/6}^{\pi/2} 2 \cdot \frac{1}{2}\left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right)d\theta$$

$$= \pi$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Ex.: Find the area common to the circle $r = 3\cos\theta$ and the cardioid $r = 1+\cos\theta$



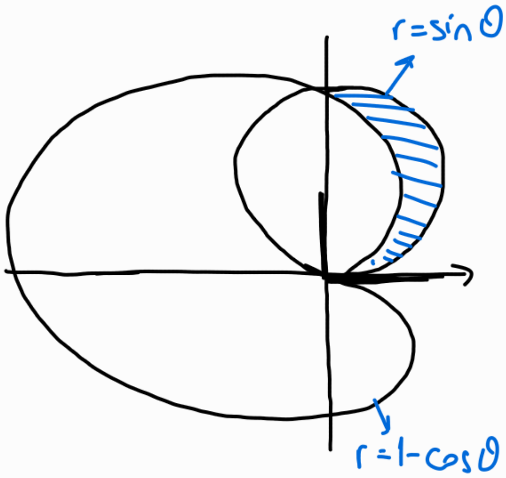

$$1+\cos\theta = 3\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$A = \int_0^{\pi/3} 2\cdot\frac{1}{2}(1+\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} 2\cdot\frac{1}{2}\cdot 9\cos^2\theta\,d\theta$$

$$= \frac{5\pi}{4}$$

Ex.: Find the area inside the circle $r = \sin\theta$ and outside the cardioid $r = 1-\cos\theta$




$$A = \int_0^{\pi/2} \frac{1}{2}\left(\sin^2\theta - \left(1-2\cos\theta+\cos^2\theta\right)\right)d\theta$$

$$= 1 - \frac{\pi}{4}$$

Ex.: Find the area of the region that is bounded by the given curve and lies inside the specified sector.

a) $r = e^{-\theta/4}$, $\frac{\pi}{2} \le \theta \le \pi$

$$A = \int_{\pi/2}^{\pi} \frac{1}{2} e^{-\theta/2}\,d\theta = \cancel{\frac{1}{2}}\cdot \frac{e^{-\theta/2}}{-\cancel{\frac{1}{2}}}\Bigg|_{\pi/2}^{\pi} = -\left(e^{-\pi/2} - e^{-\pi/4}\right) = e^{-\pi/4} - e^{-\pi/2}$$

b) $r = \sin\theta + \cos\theta, \quad 0 \leq \theta \leq \pi$

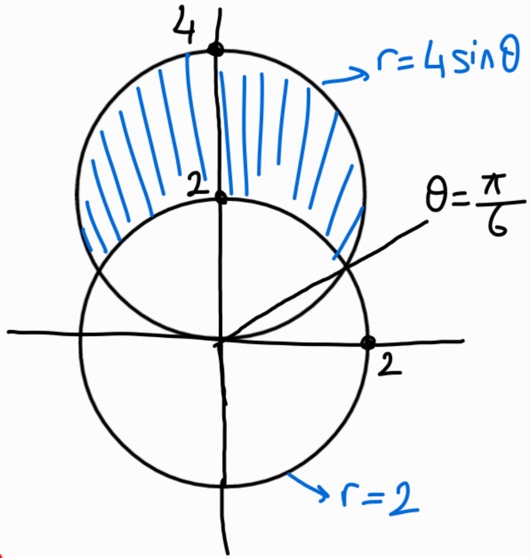
$$A = \int_0^{\pi} \frac{1}{2}\Big(\underbrace{\sin^2\theta + \cos^2\theta}_{1} + \underbrace{2.\sin\theta\cos\theta}_{\sin 2\theta}\Big)\, d\theta = \frac{1}{2}\left(\theta - \frac{\cos 2\theta}{2}\right)\Bigg|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Ex.: Find the area of the region that lies inside $r = 4\sin\theta$ and outside $r = 2$.

$4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\left(\theta = \frac{5\pi}{6}\right)$



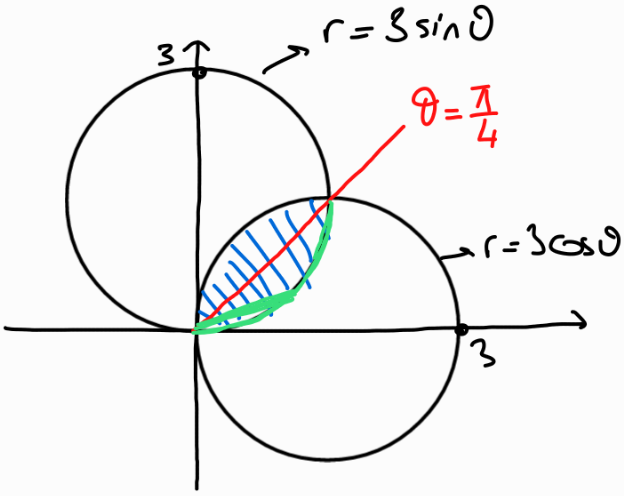

$$A = \int_{\pi/6}^{\pi/2} \cancel{2} \cdot \frac{1}{\cancel{2}}\left[16\sin^2\theta - 4\right] d\theta = \frac{4\pi}{3} + 2\sqrt{3}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Ex.: Find the area of the region that lies inside both $r = 3\sin\theta$, $r = 3\cos\theta$

$3\sin\theta = 3\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$




$$\int_0^{\pi/4} 2 \cdot \frac{1}{2} \cdot 9 \cdot \sin^2\theta \, d\theta = \frac{9\pi}{8} - \frac{9}{4}$$

$$\boxed{\int_0^{\pi/4} \frac{1}{2} 9\sin^2\theta \, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} 9\cos^2\theta \, d\theta} \rightarrow \text{no symmetry}$$

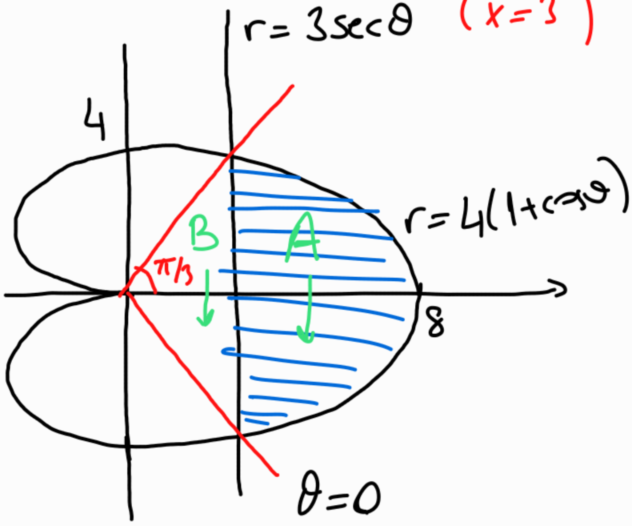
**H.W.:** Find the area of the region that lies inside both of the circles $r=2\sin\theta$ and $r=\sin\theta+\cos\theta$. $\left(\frac{1}{2}(\pi-1)\right)$

**H.W.:** Find the area of the region common to the two regions bounded by the curves $r=-6\cos\theta$, $r=2-2\cos\theta$

$2.(1-\cos\theta)$

$\left(\frac{5\pi}{2}\right)$

**Ex.:** Find the area of the region outside of $r=3\sec\theta$ and inside of $r=4+4\cos\theta$.




$r=4(1+\underbrace{\cos\theta}_{1})=8$

$3\sec\theta=4+4\cos\theta$

$\frac{3}{\cos\theta}=4+4\cos\theta$

$4\cos^2\theta+4\cos\theta-3=0$

$2\cos\theta \qquad -1$

$2\cos\theta \qquad 3$

$(2\cos\theta-1)(2\cos\theta+3)=0$

$\Downarrow \qquad \Downarrow$

$\cos\theta=\frac{1}{2} \qquad \cancel{\cos\theta=\frac{-3}{2}}$

$\theta=\frac{\pi}{3} \qquad -1\leq\cos\theta\leq 1$


$4(1+\cos\frac{\pi}{3})$

$4.\frac{3}{2}=6$

$A=(A+B)-B$

$\Downarrow$

$\int_0^{\pi/3} 2.\frac{1}{2}.16(1+\cos\theta)^2 d\theta-\frac{9}{2}\sqrt{3}$

$B=\frac{1}{2}.3.6.\sin\left(\frac{\pi}{3}\right)$

$=\frac{9}{2}\sqrt{3}$

$=8\pi+9\sqrt{3}$ //

Ex.: Find the length of the polar curve.

a) $r = e^{\theta}$, $\quad 0 \leq \theta \leq \ln 2$.

$$r^2 + (r')^2 = e^{2\theta} + e^{2\theta} = 2e^{2\theta}$$

$$L = \int_0^{\ln 2} \sqrt{2e^{2\theta}}\, d\theta = \int_0^{\ln 2} \sqrt{2}\cdot e^{\theta}\, d\theta = \sqrt{2}\, e^{\theta}\Big|_0^{\ln 2} = \sqrt{2}(2-1) = \sqrt{2}$$

b) $r = \frac{1}{\theta}$, $\quad 1 \leq \theta \leq \sqrt{3}$

$1 + \tan^2 t = \sec^2 t$
$1 + \cot^2 t = \csc^2 t$

$$r' = -\frac{1}{\theta^2} \qquad r^2 + (r')^2 = \frac{1}{\theta^2} + \frac{1}{\theta^4} = \frac{\theta^2+1}{\theta^4}$$

$$L = \int_1^{\sqrt{3}} \sqrt{\frac{\theta^2+1}{\theta^4}}\, d\theta = \int_1^{\sqrt{3}} \frac{\sqrt{\theta^2+1}}{\theta^2}\, d\theta$$

$\theta = \tan t \qquad \theta = 1 \Rightarrow t = \frac{\pi}{4}$

$d\theta = \sec^2 t\, dt \qquad \theta = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$

$$= \int_{\pi/4}^{\pi/3} \frac{\overbrace{\sqrt{\tan^2 t + 1}}^{\sec t}}{\tan^2 t} \cdot \sec^2 t\, dt = \int_{\pi/4}^{\pi/3} \frac{1}{\cos^3 t} \cdot \frac{\cancel{\cos^2 t}}{\sin^2 t}\, dt = \int_{\pi/4}^{\pi/3} \sec t \cdot \csc^2 u\, du$$

$$= \int_{\pi/4}^{\pi/3} (\sec t + \sec t \cdot \cot^2 t)\, dt = \ln|\sec t + \tan t| - \csc t \Big|_{\pi/4}^{\pi/3}$$

$$\Downarrow$$

$$\frac{1}{\cancel{\cos t}} \cdot \frac{\cos^2 t}{\sin^2 t} = \cot t \cdot \csc t$$

$$= \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right) + \sqrt{2} - \frac{2}{\sqrt{3}}.$$

c) $r = 2\cos\theta$, $\quad 0 \leq \theta \leq \pi \qquad$ (H.W.)

d) $r=\theta^2, \quad 0\le\theta\le 2\pi$

$$r' = 2\theta \Rightarrow r^2+(r')^2 = \theta^4+4\theta^2$$

$$L = \int_0^{2\pi}\sqrt{\underbrace{\theta^4+4\theta^2}_{\theta^2(\theta^2+1)}}\,d\theta = \int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$$

$$\theta^2+4=t^2$$
$$\cancel{2}\theta\, dt = \cancel{2}t\, dt$$
$$\theta=0 \Rightarrow t=2$$
$$\theta=2\pi \Rightarrow \sqrt{4\pi^2+4}$$

$$= \int_2^{\sqrt{4\pi^2+4}} \underbrace{\underset{\downarrow \atop t}{\sqrt{t^2}}\cdot t\,dt}_{t^2\,dt} = \frac{t^3}{3}\Bigg|_2^{\sqrt{4\pi^2+4}} = \frac{8}{3}\left[(\pi^2+1)^{3/2}-1\right]$$