INTEGRATION TECHNIQUES 2

1) $\int \sqrt{x} \cdot \ln x \, dx = ?$

$\ln x = u \qquad \sqrt{x}\, dx = dv$

$\frac{dx}{x} = du \qquad \frac{2}{3} x^{3/2} = v$

$I = \ln x \cdot \frac{2}{3} x^{3/2} - \int \frac{2}{3} \cancel{x}\sqrt{x} \frac{dx}{\cancel{x}} = \frac{2}{3} x\sqrt{x} \ln x - \frac{4}{9} x\sqrt{x} + c$

2) $\int \frac{\ln x}{\sqrt{x}} dx = ?$

$\ln x = u \qquad \frac{dx}{\sqrt{x}} = x^{-1/2} dx = dv$

$\frac{dx}{x} = du \qquad 2x^{1/2} = 2\sqrt{x} = v$

$\frac{2}{\sqrt{x}} = 2x^{-1/2}$

$I = \ln x \cdot 2\sqrt{x} - \int 2\sqrt{x} \frac{dx}{x} = 2\sqrt{x} \ln x - 4\sqrt{x} + c$

3) $\int x^3 \cdot e^x dx = ?$

$x^3 = u \qquad e^x dx = dv$

$3x^2 dx = du \qquad e^x = v$

$I = x^3 . e^x - \int e^x . 3x^2 dx \qquad 3x^2 = u \quad e^x dx = dv \quad 6x dx = du \quad e^x = v$

$I = x^3 . e^x - \left[3x^2 . e^x - \int e^x 6x dx\right] \qquad 6x = u \quad e^x dx = dv \quad 6dx = du \quad e^x = v$

$I = x^3 . e^x - 3x^2 e^x + 6x . e^x - \int e^x 6 dx = e^x \left[x^3 - 3x^2 + 6x - 6\right] + c$

4) $\int x . \arctan x dx = ?$

$\arctan x = u \qquad x dx = dv$

$\frac{dx}{1+x^2} = du \qquad \frac{x^2}{2} = v$

$\frac{x^2}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}$

$I = \frac{x^2}{2} . \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{\arctan x}{2} + c .$

(5) $\int (x^2-3x+4).\ln x\,dx = ?$

$$\ln x = u \qquad (x^2-3x+4)dx = dv$$
$$\frac{dx}{x} = du \qquad \frac{x^3}{3}-\frac{3x^2}{2}+4x = v$$

$$I = \left(\frac{x^3}{3}-\frac{3x^2}{2}+4x\right).\ln x - \int\left(\frac{x^{\cancel{3}\,2}}{3}-\frac{3x^{\cancel{2}}}{2}+4\cancel{x}\right).\frac{dx}{\cancel{x}}$$
$$= \left(\frac{x^3}{3}-\frac{3x^2}{2}+4x\right).\ln x - \frac{x^3}{9}+\frac{3x^2}{4}-4x+c$$

(6) $\int \cos(\ln x)\,dx = ?$

$$\cos(\ln x) = u \qquad dx = dv$$
$$-\frac{\sin(\ln x)}{x}dx = du \qquad x = v$$
$$\Bigg\}\ I = x.\cos(\ln x) + \int \cancel{x}\,\frac{\sin(\ln x)}{\cancel{x}}dx$$

$$\sin(\ln x) = u \qquad dx = dv$$
$$\frac{\cos(\ln x)}{x}dx = du \qquad x = v$$
$$\Bigg\}\ I = x.\cos(\ln x) + x.\sin(\ln x) - \underbrace{\int \cancel{x}.\frac{\cos(\ln x)}{\cancel{x}}dx}_{I}$$

$$\Rightarrow 2I = x\left[\cos(\ln x)+\sin(\ln x)\right]+c \Rightarrow I = \frac{x}{2}\left[\cos(\ln x)+\sin(\ln x)\right]+c$$

(7) $\int x.\tan^2 x\,dx = ?$

$$x = u \qquad \tan^2 x\,dx = (\tan^2 x+1-1)dx = dv$$
$$dx = du \qquad \tan x - x = v$$

$$I = x.(\tan x - x) - \int(\tan x - x)dx = x.\tan x - \cancel{x}^2 + \ln|\cos x| + \frac{x^2}{2}+c$$

(8) $\int \arctan\left(\frac{1}{x}\right)dx = ?$

$$\arctan(x^{-1}) = u \Rightarrow \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}dx = \frac{-dx}{x^2+1} = du \qquad dx = dv \quad x = v$$

$$I = x.\arctan\left(\frac{1}{x}\right) + \int x.\frac{dx}{x^2+1} = x.\arctan\left(\frac{1}{x}\right)+\frac{1}{2}\ln(x^2+1)+c.$$

(9) $\int (3x^2-x+2)\cdot e^{-x}\,dx = ?$

$$\left.\begin{array}{ll} 3x^2-x+2=u & e^{-x}dx=dv \\ (6x-1)dx=du & -e^{-x}=v \end{array}\right\} I=-e^{-x}(3x^2-x+2)+\int e^{-x}(6x-1)dx$$

$$\left.\begin{array}{ll} 6x-1=u & e^{-x}dx=dv \\ 6dx=du & -e^{-x}=v \end{array}\right\} I=-e^{-x}(3x^2-x+2)-e^{x}\cdot(6x-1)+\underbrace{\int e^{-x}\cdot 6dx}_{-6e^{-x}+C}$$

$$\Rightarrow I=e^{-x}(-3x^2-5x-7)+c$$

(10) $\int (x^2+x+1)\cdot \sin x\,dx = ?$

$$\left.\begin{array}{ll} x^2+x+1=u & \sin x\,dx=dv \\ (2x+1)dx=du & -\cos x=v \end{array}\right\} I=-(x^2+x+1)\cdot\cos x+\int \cos x\cdot(2x+1)dx$$

$$\left.\begin{array}{ll} 2x+1=u & \cos x\,dx=dv \\ 2dx=du & \sin x=v \end{array}\right\} I=-(x^2+x+1)\cos x+(2x+1)\cdot\sin x-\underbrace{\int 2\sin x\,dx}_{-2\cos x+C}$$

$$\Rightarrow I=(-x^2-x+1)\cdot\cos x+(2x+1)\cdot\sin x+c.$$

(11) $\int \sin x\cdot\cos x\,dx = ?$ (Can be solved by substitution or directly.)

<u>1st way</u>: $\left.\begin{array}{ll} \sin x=u & \cos x\,dx=dv \\ \cos x\,dx=du & \sin x=v \end{array}\right\} I=\sin^2 x-\underbrace{\int \sin x\cdot\cos x\,dx}_{I} \Rightarrow I=\frac{\sin^2 x}{2}+c$

<u>2nd way</u>: $\left.\begin{array}{ll} \cos x=u & \sin x\,dx=dv \\ -\sin x\,dx=du & -\cos x=v \end{array}\right\} I=-\cos^2 x-\underbrace{\int \sin x\cos x\,dx}_{I} \Rightarrow I=\frac{-\cos^2 x}{2}+c$

(12) $\int e^x\cdot\cos(3x)\,dx = ?$

<u>1st way</u>: $\left.\begin{array}{ll} \cos(3x)=u & e^x dx=dv \\ -3\sin(3x)dx=du & e^x=v \end{array}\right\} I=e^x\cdot\cos(3x)+\int e^x\cdot 3\sin(3x)dx$

$$\left.\begin{array}{ll} \sin(3x)=u & e^x dx=dv \\ 3\cos(3x)=du & e^x=v \end{array}\right\} I=e^x\cdot\cos(3x)+3e^x\cdot\sin(3x)-3\underbrace{\int e^x\cdot 3\cdot\cos(3x)dx}_{3I}$$

$$\Rightarrow I=\frac{1}{10}\cdot e^x\left[\cos(3x)+3\sin(3x)\right]+c.$$

2nd way: $e^x = u$ $\quad \cos(3x)dx = dv$
$e^x dx = du$ $\quad \frac{1}{3}\cdot\sin(3x) = v$

$$I = e^x\cdot\frac{\sin(3x)}{3} - \int\frac{\sin(3x)}{3}\cdot e^x dx$$

$e^x = u$ $\quad \sin(3x)dx = dv$
$e^x dx = du$ $\quad -\frac{1}{3}\cos(3x) = v$

$$I = \frac{e^x.\sin(3x)}{3} + \frac{1}{9}\cdot e^x.\cos(3x) - \underbrace{\int\frac{e^x.\cos(3x)}{9}dx}_{I/9}$$

$$\frac{10}{\cancel{9}}I = \frac{1}{\cancel{9}}e^x\left[3\sin(3x) + \cos(3x)\right] + C \Rightarrow I = \frac{e^x}{10}\left[3\sin(3x) + \cos(3x)\right] + C$$

13 $\int 9x^2\ln x\,dx = ?$

$\ln x = u$ $\quad 9x^2dx = dv$
$\frac{dx}{x} = du$ $\quad 3x^3 = v$

$$I = 3x^3\ln x - \int 3x^{\cancel{3}\,2}\cdot\frac{dx}{\cancel{x}} = 3x^3\ln x - x^3 + C.$$

14 $\int\left(\frac{\ln x}{x}\right)^2 dx = ?$

$\ln^2 x = u$ $\quad \frac{dx}{x^2} = dv$
$\frac{2\ln x}{x}dx = du$ $\quad -\frac{1}{x} = v$

$$I = -\frac{\ln^2 x}{x} + \int\frac{2\ln x}{x^2}dx$$

$\ln x = u$ $\quad \frac{2}{x^2}dx = dv$
$\frac{dx}{x} = du$ $\quad -\frac{2}{x} = v$

$$I = -\frac{\ln^2 x}{x} - \frac{2\ln x}{x} + \int\frac{2}{x^2}dx = -\frac{\ln^2 x}{x} - \frac{\ln x^2}{x} - \frac{2}{x} + C.$$

15 $\int x\sec x\tan x\,dx$

$x = u$ $\quad \sec x\tan x\,dx = dv$
$dx = du$ $\quad \sec x = v$

$$I = x\sec x - \int\sec x\,dx = x.\sec x - \ln|\sec x + \tan x| + C$$

16 $\int e^{\sqrt{x}}dx = ?$

$x = t^2$
$dx = 2t\,dt$

$$I = \int e^t.2t\,dt$$

$2t = u$ $\quad e^t dt = dv$
$2dt = du$ $\quad e^t = v$

$$I = 2.t.e^t - \int 2e^t dt = 2te^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

17) $\displaystyle\int_0^1 \ln(1+x^2)\,dx = ?$

$$\left.\begin{array}{ll} \ln(1+x^2)=u & dx=dv \\ \dfrac{2x}{1+x^2}dx=du & x=v \end{array}\right\} \quad I = x\cdot\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{\overset{2x^2+2-2}{2x^2}}{1+x^2}dx$$

$$I = \ln 2 - \int_0^1\left[2-\frac{2}{1+x^2}\right]dx = \ln 2 - \Big[2x - 2\arctan x\Big]_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

18) $\displaystyle\int_0^1 x\ln(1+x)\,dx = ?$

$$\left.\begin{array}{ll} \ln(1+x)=u & x\,dx=dv \\ \dfrac{dx}{1+x}=du & \dfrac{x^2}{2}=v \end{array}\right\} \quad I = \frac{x^2}{2}\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{dx}{1+x}$$

$$\begin{array}{r|l} x^2 & x+1 \\ -\ x^2+x & x-1 \\ \hline -x & \\ -\ -x-1 & \\ \hline 1 & \end{array}$$

$$I = \frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1\left[x-1+\frac{1}{x+1}\right]dx = \frac{\ln 2}{2} - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln|x+1|\right]_0^1$$

$$\Rightarrow I = \cancel{\frac{\ln 2}{2}} - \frac{1}{2}\left[\left(\frac{1}{2} - 1 + \cancel{\ln 2}\right) - 0\right] = \frac{1}{4}.$$

19) $\displaystyle\int_{-1}^0 9x^2\ln(2+x)\,dx = ?$

$$\left.\begin{array}{ll} \ln(2+x)=u & 9x^2dx=dv \\ \dfrac{dx}{2+x}=du & 3x^3=v \end{array}\right\} \quad I = \underbrace{3x^3\ln(2+x)\Big|_{-1}^0}_{0} - \int_{-1}^0 3x^3\frac{dx}{2+x}$$

$$\begin{array}{r|l} x^3 & x+2 \\ -\ x^3+2x^2 & x^2-2x+4 \\ \hline -2x^2 & \\ -\ -2x^2-4x & \\ \hline 4x & \\ -\ 4x+8 & \\ \hline -8 & \end{array}$$

$$I = -3\int_{-1}^0\left[x^2-2x+4-\frac{8}{x+2}\right]dx = -x^3+3x^2-12x+24\ln|x+2|\,\Big|_{-1}^0$$

$$\Rightarrow I = 24\ln 2 - (1+3+12) = 24\ln 2 - 16.$$

(20) $\int \cos x . \ln(\cos x)dx = ?$

$\ln(\cos x) = u$ $\quad$ $\cos x dx = dv$

$-\tan x dx = du$ $\quad$ $\sin x = v$

$$I = \sin x . \ln(\cos x) + \underbrace{\int \frac{\sin^2 x}{\cos x} dx}_{J}$$

$$J = \int \frac{(1-\cos^2 x)}{\cos x} dx = \int (\sec x - \cos x) dx = \ln|\sec x + \tan x| - \sin x + c$$

$$\Rightarrow I = \sin x . \ln(\cos x) + \ln|\sec x + \tan x| - \sin x + C$$

(21) $\int \sec^2\sqrt{x}\, dx = ?$

$\sqrt{x} = t$

$\frac{dx}{2\underbrace{\sqrt{x}}_{t}} = dt$

$$I = \int 2t . \sec^2 t\, dt$$

$t = u$ $\quad$ $\sec^2 t\, dt = dv$

$dt = du$ $\quad$ $\tan t = v$

$$\Rightarrow I = 2 . \left[t . \tan t - \int \tan t\, dt \right] = 2t . \tan t + 2 \ln|\cos t| + c$$

$$= 2\sqrt{x} . \tan\sqrt{x} + \ln \cos^2\sqrt{x} + C .$$

(22) $\int \cos\sqrt[3]{x}\, dx = ?$

$x = t^3$

$dx = 3t^2 dt$

$$I = \int \cos t . 3t^2 dt$$

$3t^2 = u$ $\quad$ $\cos t\, dt = dv$

$6t dt = du$ $\quad$ $\sin t = v$

$$\Rightarrow I = 3t^2 . \sin t - \int 6t . \sin t\, dt$$

$6t = u$ $\quad$ $\sin t\, dt = dv$

$6 dt = du$ $\quad$ $-\cos t = v$

$$\Rightarrow I = 3t^2 . \sin t + 6t . \cos t - \underbrace{\int 6 \cos t\, dt}_{6 \sin t + C}$$

$$= 3x^{2/3} . \sin\sqrt[3]{x} + 6 . \sqrt[3]{x} . \cos\sqrt[3]{x} - 6 \sin\sqrt[3]{x} + C .$$

23) $\displaystyle\int_0^{1/2} \frac{x.e^{2x}}{(1+2x)^2}dx = ?$

$x.e^{2x} = u$

$(e^{2x}+2xe^{2x})dx = du$

$e^{2x}(1+2x)dx = du$

$\dfrac{dx}{(1+2x)^2} = (1+2x)^{-2}dx = dv$

$\dfrac{-(1+2x)^{-1}}{2} = \dfrac{-1}{2(1+2x)} = v$

$$I = \left[x.e^{2x}\cdot\frac{-1}{2(1+2x)} - \int\frac{-1}{2\cancel{(1+2x)}}e^{2x}\cancel{(1+2x)}dx\right]_0^{1/2} = \left[\frac{-x.e^{2x}}{2(1+2x)}+\frac{e^{2x}}{4}\right]_0^{1/2}$$

$$= \left[\frac{-\frac{1}{2}.e}{2(1+1)}+\frac{e}{4}\right]-\left[0+\frac{e^0}{4}\right] = -\frac{e}{8}+\frac{e}{4}-\frac{1}{4} = \frac{e}{8}-\frac{1}{4}$$

24) $\displaystyle\int_1^2 \frac{\arcsin(1-x)}{x^2}dx = ?$

$\arcsin(1-x) = u$

$\dfrac{-1}{\sqrt{1-(1-x)^2}}dx = du$

$\dfrac{dx}{x^2} = dv$

$-\dfrac{1}{x} = v$

$$I = -\frac{\arcsin(1-x)}{x}\Bigg|_1^2 - \int_1^2\left(-\frac{1}{x}\right)\cdot\left(\frac{-1}{\sqrt{1-(1-x)^2}}\right)dx$$

$\dfrac{1}{x} = t$ $\quad -\dfrac{1}{x^2}dx = dt$

$\Rightarrow dx = -x^2 dt$

$x = \dfrac{1}{t} \Rightarrow dx = -\dfrac{1}{t^2}dt$

$x = 1 \Rightarrow t = 1$

$x = 2 \Rightarrow t = \dfrac{1}{2}$

$$= \left(-\frac{\pi}{2}\right)\cdot\left(-\frac{1}{2}\right) - 0.(-1) - \int_1^2\frac{dx}{x\sqrt{1-(1-x)^2}}$$

$$= \frac{\pi}{4} - \int_1^{1/2}\frac{-\frac{1}{t^{\cancel{2}}}dt}{\frac{1}{\cancel{t}}\sqrt{1-\left(1-\frac{1}{t}\right)^2}} = \frac{\pi}{4} + \int_1^{1/2}\frac{dt}{t.\sqrt{-\frac{1}{t^2}+\frac{2}{t}}} = \frac{\pi}{4} + \int_1^{1/2}\frac{dt}{\cancel{t}.\frac{\sqrt{2t-1}}{\cancel{t}}}$$

$$= \frac{\pi}{4} + \int_1^{1/2}(2t-1)^{-1/2}dt = \frac{\pi}{4} + \cancel{2}.\frac{(2t-1)^{1/2}}{\cancel{2}}\Bigg|_1^{1/2} = \frac{\pi}{4} + \sqrt{\cancel{2}.\frac{1}{\cancel{2}}-1} - \sqrt{2.1-1}$$

$$= \frac{\pi}{4} - 1$$