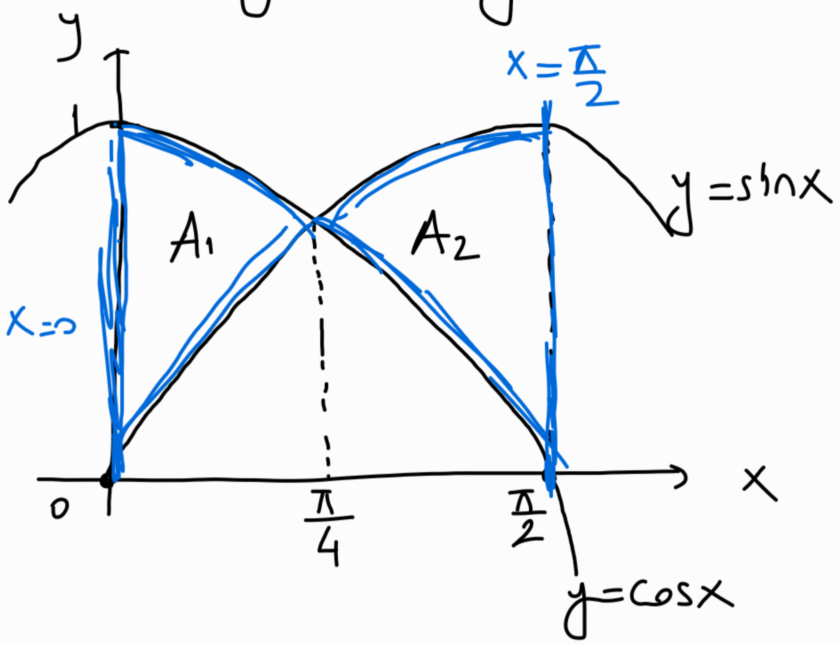
Ex.: Find the area of the region bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$, and $x=\frac{\pi}{2}$



$\cos x=\sin x \Rightarrow x=\frac{\pi}{4}$

$$A=A_1+A_2=\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)\,dx$$

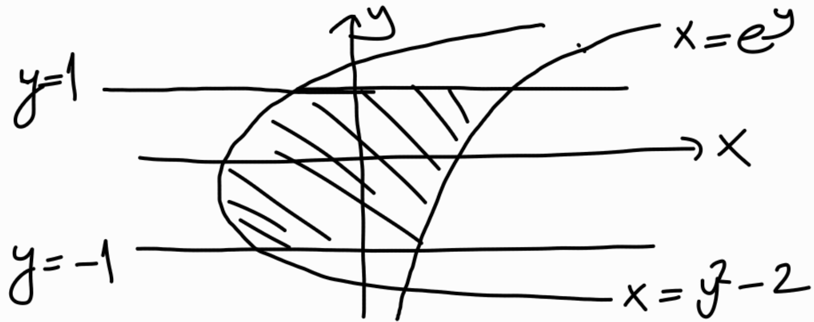
$$=\Big[\sin x+\cos x\Big]_0^{\pi/4}+\Big[-\cos x-\sin x\Big]_{\pi/4}^{\pi/2}$$

$$=\Big[\Big(\underbrace{\sin\tfrac{\pi}{4}}_{\frac{\sqrt2}{2}}+\underbrace{\cos\tfrac{\pi}{4}}_{\frac{\sqrt2}{2}}\Big)-\Big(\underbrace{\sin 0}_{1}+\underbrace{\cos 0}_{0}\Big)\Big]-\Big[\Big(\underbrace{\cos\tfrac{\pi}{2}}_{0}+\underbrace{\sin\tfrac{\pi}{2}}_{1}\Big)-\Big(\underbrace{\cos\tfrac{\pi}{4}}_{\frac{\sqrt2}{2}}+\underbrace{\sin\tfrac{\pi}{4}}_{\frac{\sqrt2}{2}}\Big)\Big]$$

$$=\underbrace{(\sqrt2-1)}-\underbrace{(1-\sqrt2)}=2\sqrt2-2$$

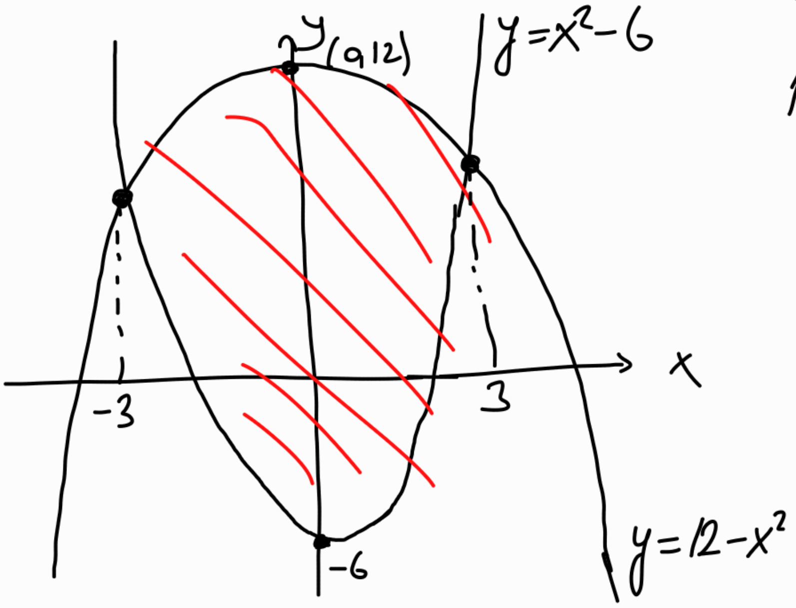
Actually, $A_1=A_2$. $\underbrace{A=2A_1=2\int_0^{\pi/4}(\cos x-\sin x)\,dx}$

Ex.: Find the area of the shaded region



$$A=\int_{-1}^{1}\left(e^{y}-y^{2}+2\right)dy=\left[e^{y}-\frac{y^{3}}{3}+2y\right]_{y=-1}^{y=1}=e-\frac{1}{e}+\frac{10}{3}$$

Ex.: Find the area of the region $y=12-x^2$, $y=x^2-6$

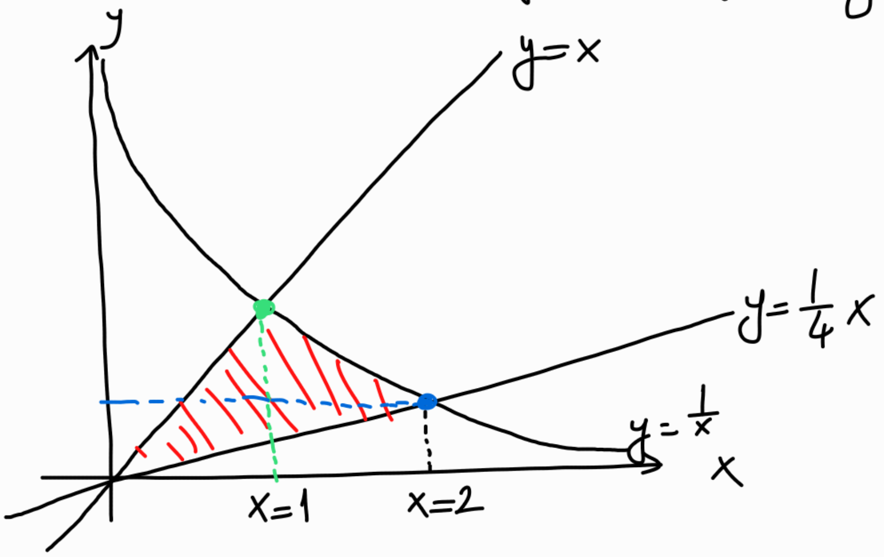


$$12-x^2=x^2-6$$
$$2x^2=18 \Rightarrow x^2=9 \Rightarrow x=\mp 3$$

$$A=\int_{-3}^{3}\left[(12-x^2)-(x^2-6)\right]dx \underset{\text{by symmetry}}{=} 2\cdot\int_{0}^{3}(18-2x^2)\,dx=2\left[18x-\frac{2}{3}x^3\right]_{0}^{3}$$

$$=2\,(\underbrace{54-18}_{36})=72$$

Ex.: Find the area of the region; $y=\frac{1}{x}$, $y=x$, $y=\frac{1}{4}x$, $x>0$.



$$x=\frac{1}{x} \Rightarrow x=1$$

$$\frac{x}{4}=\frac{1}{x} \Rightarrow x^2=4 \Rightarrow x=2$$

$$A = \int_0^1 \Big(\underbrace{x - \frac{1}{4}x}_{\frac{3x}{4}}\Big)\,dx + \int_1^2 \Big(\frac{1}{x} - \frac{1}{4}x\Big)\,dx$$

$$= \frac{3}{8}x^2 \Big|_0^1 + \Big[\ln x - \frac{x^2}{8}\Big]_1^2 = \cancel{\frac{3}{8}} + \Big[\Big(\ln 2 - \cancel{\frac{1}{2}}\Big) - \Big(0 \cancel{-\frac{1}{8}}\Big)\Big]$$

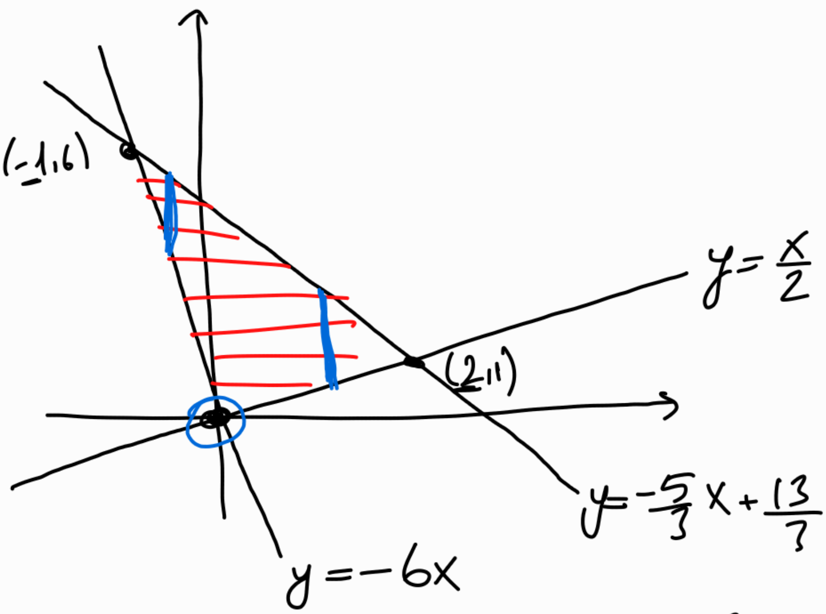
$$= \underline{\underline{\ln 2}}$$

<u>Ex.:</u> Use calculus to find the area of the triangle with the given vertices: $(0,0)$, $(2,1)$, $(-1,6)$.

$(0,0)$ and $(2,1) \Rightarrow y = \frac{1}{2}x$

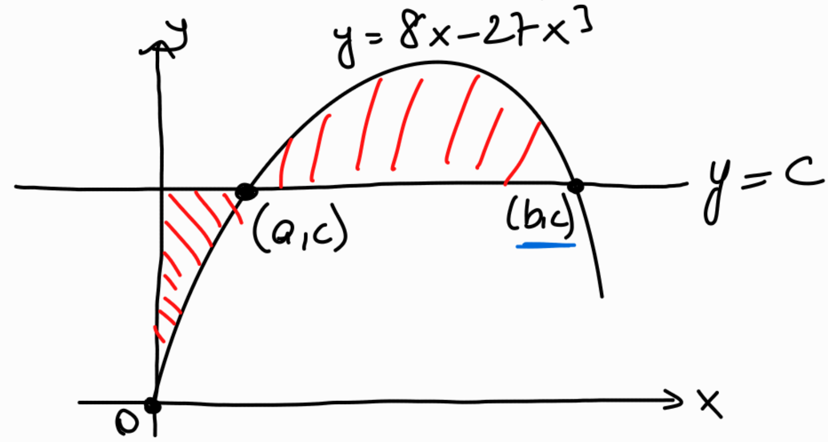
$(0,0)$ and $(-1,6) \Rightarrow y = -6x$

$(2,1)$ and $(-1,6) \Rightarrow y = -\frac{5}{3}x + \frac{13}{3}$



$$A = \int_{-1}^{0} \Big(\frac{-5x+13}{3} - (-6x)\Big)\,dx + \int_0^2 \Big(\frac{-5x+13}{3} - \frac{x}{2}\Big)\,dx = \frac{13}{2}$$

Ex.: The figure shows a horizontal line $y=c$ intersecting the curve $y=8x-27x^3$. Find the number c s.t. the areas of the shaded regions are equal



$$\int_0^a \left[c-(8x-27x^3)\right]dx = \int_a^b \left[(8x-27x^3)-c\right]dx$$

$$\left[cx-4x^2+\frac{27}{4}x^4\right]_0^a = \left[4x^2-\frac{27}{4}x^4-cx\right]_a^b$$

$$\cancel{ac}-\cancel{4a^2}+\cancel{\frac{27}{4}}a^4 = \left(4b^2-\frac{27}{4}b^4-bc\right)-\left(\cancel{4a^2}-\cancel{\frac{27}{4}}a^4-\cancel{ac}\right)$$

$$0=4b^2-\frac{27}{4}b^4-bc = 4b^2-\frac{27}{4}b^4-b\left(\underbrace{8b-27b^3}_{\text{polynomial}}\right)$$

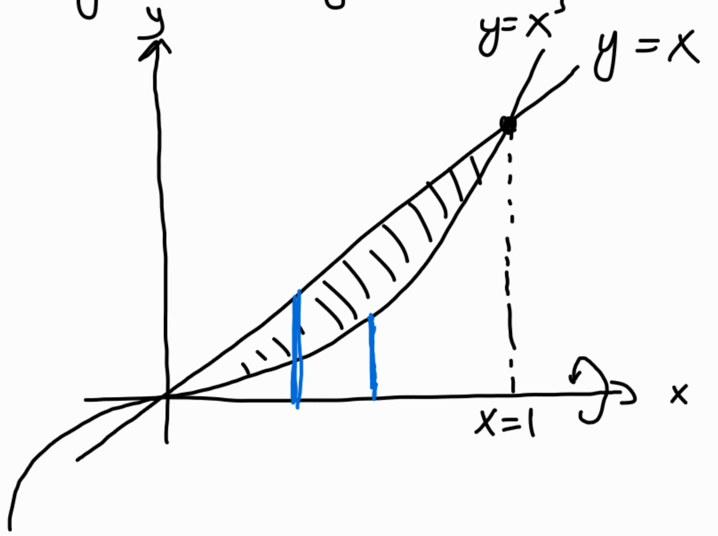
$$0=\frac{81}{4}b^4-4b^2 = \underbrace{b^2}_{\substack{\neq 0\\(b>0)}}\underbrace{\left(\frac{81}{4}b^2-4\right)}_{0}$$

$$\frac{81}{4}b^2-4=0 \Rightarrow \frac{81b^2}{4}=4 \Rightarrow b^2=\frac{16}{81} \Rightarrow b=\frac{4}{9}$$

$$\Rightarrow c = 8\cdot\frac{4}{9}-27\cdot\left(\frac{4}{9}\right)^3=\frac{32}{27}$$

Ex.: Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. (Disk or Washer)

a) $y=x^3$, $y=x$, $x\geq 0$; about the x-axis

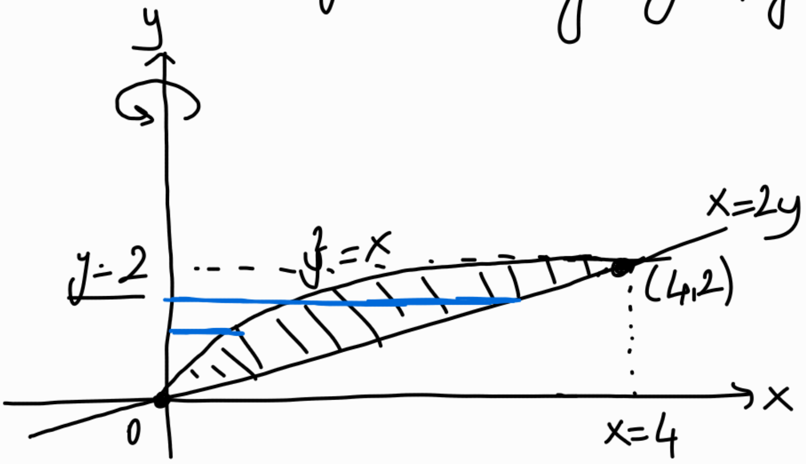


$$V=\int_0^1 \pi\left(x^2-(x^3)^2\right)dx=\frac{4}{21}\pi$$

(Washer)

b) $y^2=x$, $x=2y$; about the y-axis

$y=\frac{x}{2}$ $\qquad$ $y^2=2y \Rightarrow y=0,\ y=2$

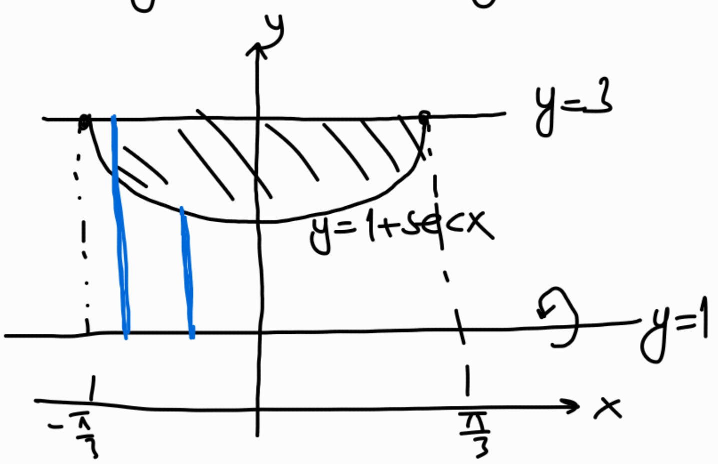


$$V=\int_0^2 \pi\left(4y^2-y^4\right)dy$$

$$=\frac{64}{15}\pi$$

c) $y=1+\sec x$, $y=3$; about $y=1$

$1+\sec x=3$
$\sec x=2$
$\frac{1}{\cos x}=2 \Rightarrow \cos x=\frac{1}{2}$

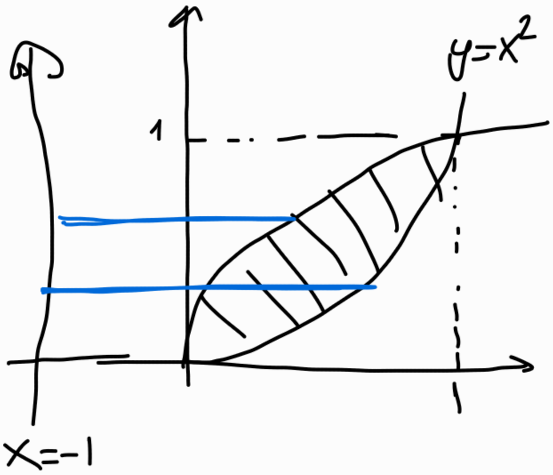


$R(x)=(3-1)^2=2^2$ $\qquad$ $r(x)=(1+\sec x-1)^2$
$=\sec^2 x$

$$V=2\int_0^{\pi/3} \pi\left(4-\sec^2 x\right)dx$$

$$=2\pi\left(\frac{4\pi}{3}-\sqrt{3}\right)$$

d) $y=x^2$, $x=y^2$; about $x=-1$

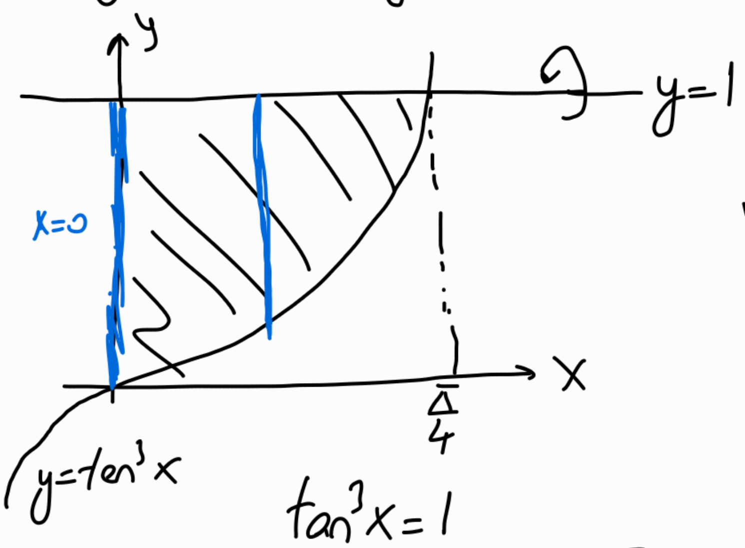


$$R(y)=\left(\sqrt{y}-(-1)\right)^2=\left(\sqrt{y}+1\right)^2$$

$$r(y)=\left(y^2-(-1)\right)^2=\left(y^2+1\right)^2$$

$$V=\int_0^1 \pi\left[\left(\sqrt{y}+1\right)^2-\left(y^2+1\right)^2\right]dy=\frac{29}{30}\pi$$

e) $y=\tan^3 x$, $y=1$, $x=0$; about $y=1$

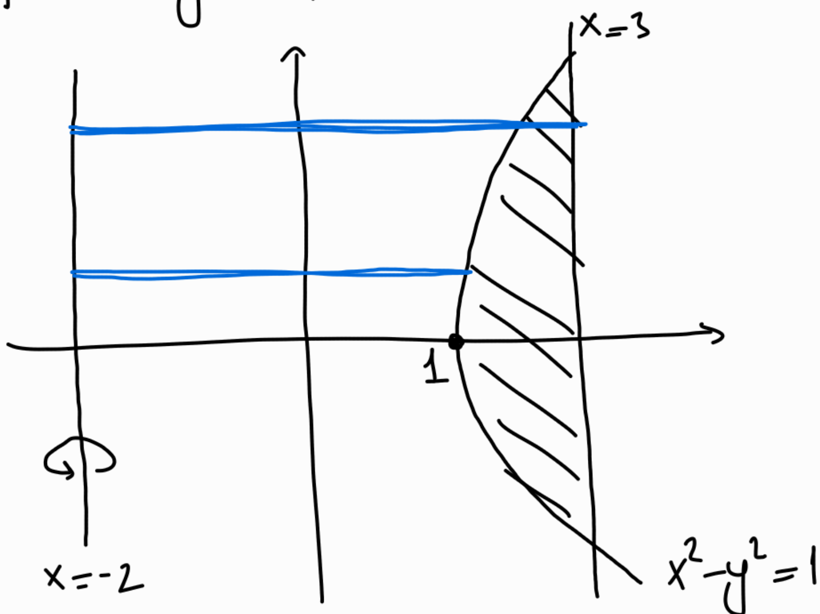


$$V=\int_0^{\pi/4} \pi\left(1-\tan^3 x\right)^2 dx$$

$\tan^3 x=1$

$\tan x=1 \Rightarrow x=\frac{\pi}{4}$

f) $x^2-y^2=1$, $x=3$; about $x=-2$



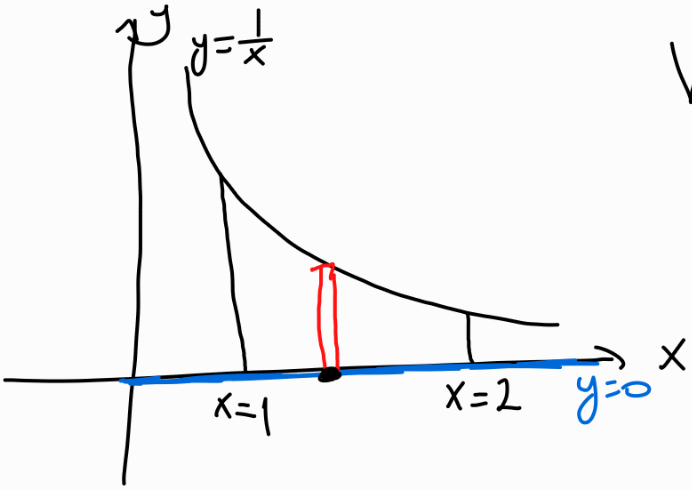
$x^2-y^2=1 \Rightarrow x=\sqrt{1+y^2}$

$x=3 \Rightarrow y=\mp 2\sqrt{2}$

$$V=2\int_0^{2\sqrt{2}} \pi\left[5^2-\left(\sqrt{1+y^2}+2\right)^2\right]dy$$

Ex.: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified line.

a) $y=\frac{1}{x}$, $y=0$, $x=1$, $x=2$; about y-axis



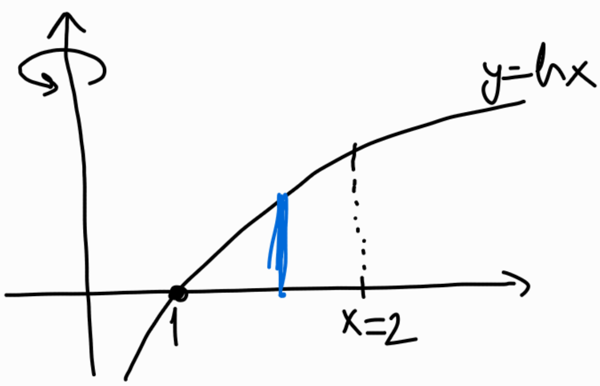
$$V=\int_1^2 2\pi x\cdot\frac{1}{x}\,dx=2\pi\int_1^2 dx=2\pi$$

b) $y=e^{-x^2}$, $y=0$, $x=0$, $x=1$; about y-axis

$$V=2\pi\int_0^1 \underline{x}\cdot e^{-x^2}\,dx=\pi\left(1-\frac{1}{e}\right)$$

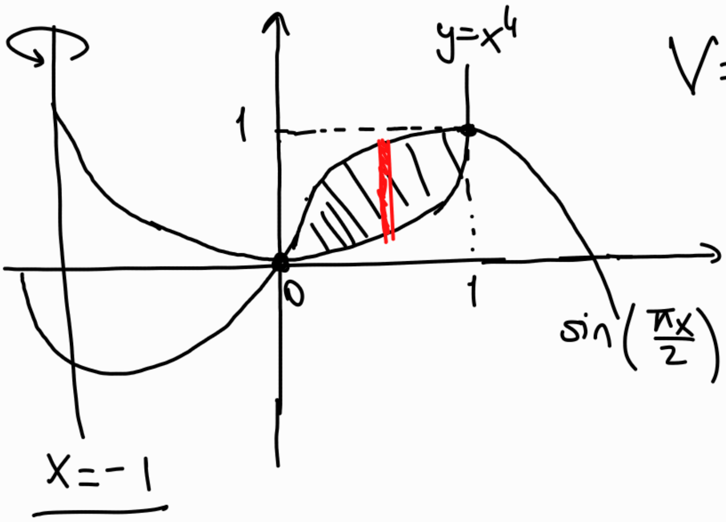
$x^2=t \Rightarrow 2x\,dx=dt$

c) $y=\ln x$, $y=0$, $x=2$; about y-axis



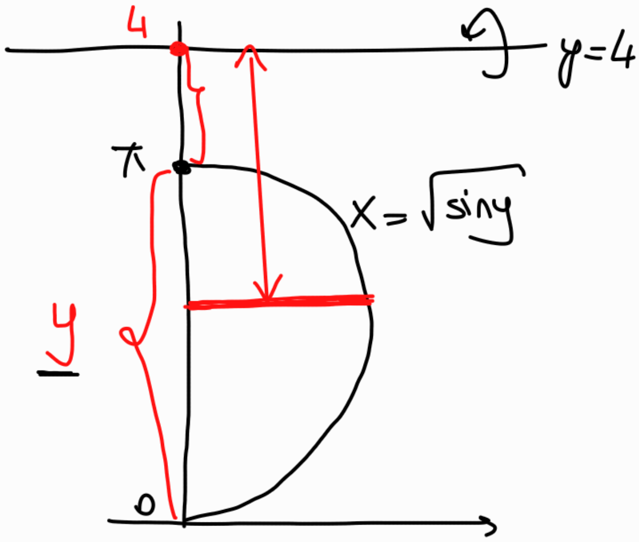
$$V=2\pi\int^2 x\cdot\ln x\,dx$$

d) $y=x^4$, $y=\sin\left(\frac{\pi x}{2}\right)$; about $x=-1$



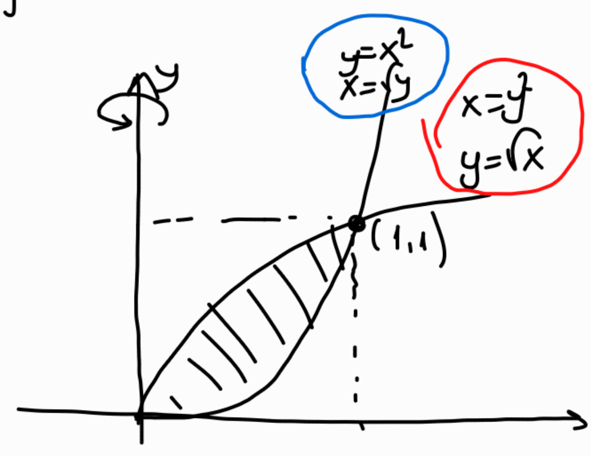
$$V=2\pi\int_0^1 [x-(-1)]\cdot\left[\sin\left(\frac{\pi x}{2}\right)-x^4\right]dx$$

e) $x=\sqrt{\sin y}$, $0\leq y\leq\pi$, $x=0$; about $y=4$



$$V=2\pi\int_0^{\pi}\underbrace{(4-y)}_{\substack{\text{radius}\\(>0)}}\cdot\sqrt{\sin y}\,dy$$

Ex.: Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y=\sqrt{x}$ and $y=x^2$. Find V both by slicing (washer) and by cylindrical shells.



Washer: $V=\int_0^1 \pi\left[(\sqrt{y})^2-(y^2)^2\right]dy$

Cylindrical Shells: $V=2\pi\int_0^1 x\cdot(\sqrt{x}-x^2)\,dx$

$=\frac{3\pi}{10}$