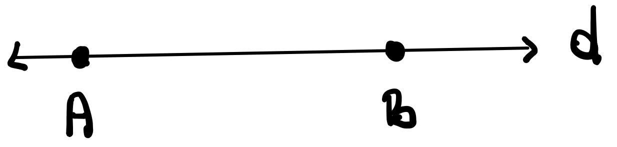
2) What is the point of intersection of the line passing through the points $A(-1,2,1)$, $B(2,3,-1)$ and the plane $x-3y-2z-3=0$?

A) $(1,4,-7)$ B) $(-11,-2,-4)$ C) $(5,4,-3)$ D) $(0,1,-3)$ E) $(8,5,-5)$



$\to B-A$

• Equation of the line d: $\vec{r} = \vec{OA} + \vec{AB}.t \Rightarrow$

$$\vec{r} = (-1,2,1) + (3,1,-2)t \Rightarrow$$

$$x = -1 + 3t$$
$$y = 2 + t$$
$$z = 1 - 2t$$

• Intersection: $x - 3y - 2z - 3 = 0 \Rightarrow$

$$(-1+3t) - 3(2+t) - 2(1-2t) - 3 = 0 \Rightarrow$$
$$-1 + 3t - 6 - 3t - 2 + 4t - 3 = 0 \Rightarrow$$
$$4t - 12 = 0 \Rightarrow t = 3$$

• Intersection Point:

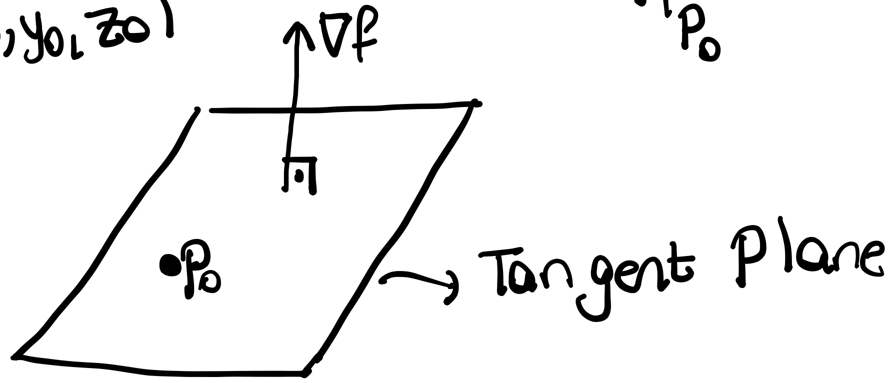
$$\left.\begin{array}{l} x = -1 + 3\cdot 3 = 8 \\ y = 2 + 3 = 5 \\ z = 1 - 2\cdot 3 = -5 \end{array}\right\} \Rightarrow (8,5,-5)$$

13) If the tangent plane equation of the $ae^{xz}+x^2y+yz=1$ surface at the point $P(0,b,1)$ is $2x+y+cz=d$, then what is $a+d$?

A) -3 B) -2 C) -1 D) 0 E) 1

• Equation of the tangent plane of a surface :

$P_0(x_0, y_0, z_0)$ $\qquad$ $\vec{\nabla} f\big|_{P_0} = \langle A, B, C \rangle$



$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

• $\vec{\nabla} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$

$= (aze^{xz}+2xy)\vec{i} + (x^2+z)\vec{j} + (axe^{xz}+y)\vec{k}$

$\vec{\nabla} f\big|_{P(0,b,1)} = a\vec{i} + \vec{j} + b\vec{k} = \langle \underset{A}{a}, \underset{B}{1}, \underset{C}{b} \rangle$

• $P_0 = P(\underset{x_0}{0}, \underset{y_0}{b}, \underset{z_0}{1})$

$a(x-0)+1\cdot(y-b)+b\cdot(z-1)=0 \Rightarrow$

$ax+y+bz-2b=0 \Rightarrow ax+y+bz=2b$

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad 2x+y+cz=d$ $\Bigg\} \Rightarrow$

$a=2$, $b=c$ and $d=2b$

• $P(0,b,1)$ and $ae^{xz}+x^2y+yz=1 \Rightarrow$

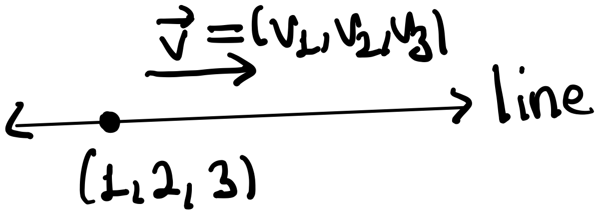
$\underset{\downarrow \atop 2}{a} + b = 1 \Rightarrow b = -1 \Rightarrow d = 2(-1) = -2$

so $a+d = 2+(-2) = 0$

7) Which of the following is a parametric equation of the line parallel to the normal line of the surface $x^2+2y^2+z^2=4$ at the point $(1,1,1)$ and passing through the point $(1,2,3)$?

A) $x=2+t \quad y=2+2t \quad z=2+3t$ B) $x=1+2t \quad y=2+2t \quad z=3+2t$ C) $x=2+t \quad y=4+2t \quad z=2+3t$

D) $x=1+2t \quad y=2+4t \quad z=3+2t$ E) $x=1+t \quad y=2+t \quad z=3+t$



• Equation of the line : $x = 1+v_1t = 1+2t$

$y = 2+v_2t = 2+4t$

$z = 3+v_3t = 3+2t$

• $\vec{v} = \vec{\nabla} f \big|_{(1,1,1)}$

$$\left.\begin{aligned}\vec{\nabla} f &= f_x\vec{i} + f_y\vec{j} + f_z\vec{k}\\ &= (2x)\vec{i} + (4y)\vec{j} + (2z)\vec{k}\end{aligned}\right\} \Rightarrow$$

$\vec{\nabla} f \big|_{(1,1,1)} = 2\vec{i}+4\vec{j}+2\vec{k} = \langle \underset{v_1}{2}, \underset{v_2}{4}, \underset{v_3}{2} \rangle$

8) Which of the following is the linearization $L(x,y)$ of the function $f(x,y)=e^{2x-y}+\ln(1+x^2+3y^2)$ at the point $P(0,0)$?

$x_0 \quad y_0$

A) $1+2x-y$ B) $x+y$ C) $x+1$ D) $2+x+y$ E) $1+xy$

$$L(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(0,0) = e^0 + \ln 1 = 1 + 0 = 1$$

$$f_x = 2e^{2x-y} + \frac{2x}{1+x^2+3y^2} \Rightarrow$$

$$f_x(0,0) = 2e^0 + \frac{0}{1} = 2+0 = 2$$

$$f_y = -e^{2x-y} + \frac{6y}{1+x^2+3y^2} \Rightarrow$$

$$f_y(0,0) = -e^0 + \frac{0}{1} = (-1)+0 = -1$$

$$L(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \Rightarrow$$

$$L(x,y) \approx \underline{1+2x-y}$$

5) What is the value of the differential df of the function $f(x,y)=\sqrt{x^2+y^2}+\sin(xy)$ at the point $\underset{x_0 \quad y_0}{(0,1)}$ with $dx = 0{,}3$, $dy = 0{,}1$?

A) 0,5 B) 0,4 C) 0,2 D) 0,1 E) 0,3

$$df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

$$f_x = \cancel{2}x \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} + y\cos(xy) \Rightarrow$$

$$f_x(0,1) = \frac{0}{1} + 1 \cdot \underbrace{\cos 0}_{=1} = 0+1 = 1$$

$$f_y = \cancel{2}y \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} + x\cos(xy) \Rightarrow$$

$$f_y(0,1) = 1 + 0 \cdot \underbrace{\cos 0}_{=1} = 1+0 = 1$$

$$df = f_x(0,1) \cdot dx + f_y(0,1)\,dy \Rightarrow$$

$$df = 1 \cdot (0{,}3) + 1 \cdot (0{,}1) = \underline{0{,}4}$$

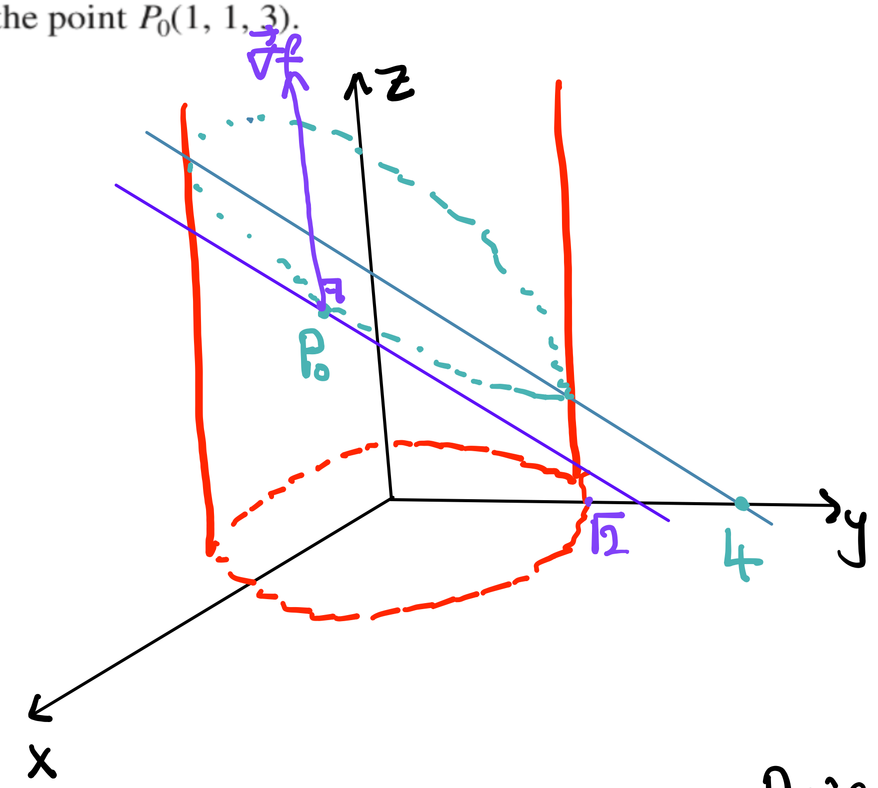
EXAMPLE 3 The surfaces

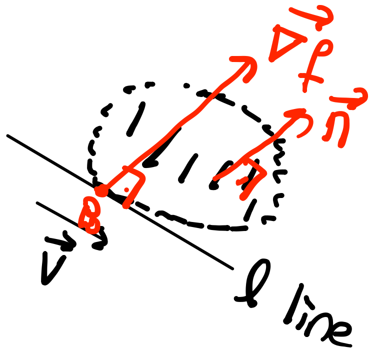
$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.





• Equation of line ℓ: • Point $P_0(1,1,3)$

• $\vec{v}$

ℓ is tangent to cylinder

$\Downarrow$

$\vec{v} \perp \vec{\nabla f}$

ℓ is tangent to plane

$\Downarrow$

$\vec{v} \perp \vec{n}$

$$\vec{v} = \vec{\nabla f} \times \vec{n}$$

- $\left.\begin{aligned} \vec{\nabla f} &= f_x \vec{i} + f_y \vec{j} + f_z \vec{k} \\ &= (2x)\vec{i} + 2y\vec{j} + 0 \cdot \vec{k} \end{aligned}\right\} \Rightarrow$

$$\vec{\nabla f}\,\big|_{P_0} = 2\vec{i} + 2\vec{j} = \langle 2, 2, 0 \rangle$$

- plane : $1 \cdot x + 0 \cdot y + 1 \cdot z = 4 \Rightarrow$

$$\vec{n} = \langle 1, 0, 1 \rangle$$

- $\vec{v} = \vec{\nabla f} \times \vec{n} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$$\begin{array}{ccc} i & j & k \\ 2 & 2 & 0 \end{array}$$

$$= (2\vec{i} + 0 + 0) - (2\vec{k} + 0 + 2\vec{j})$$

$$= 2\vec{i} - 2\vec{j} - 2\vec{k} \Rightarrow$$

$$\vec{v} = \langle \underset{\downarrow\, v_1}{2}, \underset{\downarrow\, v_2}{-2}, \underset{\downarrow\, v_3}{-2} \rangle$$

- Equation of the tangent line :

$$x = x_0 + v_1 t = 1 + 2t$$

$$y = y_0 + v_2 t = 1 - 2t$$

$$z = z_0 + v_3 t = 3 - 2t$$

EXAMPLE 6 Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Volume $\rightarrow V = \pi r^2 h \Rightarrow V_r = 2\pi r h$ and $V_h = \pi r^2$

$dV = V_r\, dr + V_h\, dh$

$dr = 0{,}03$ and $dh = -0{,}1$

$r_0 = 1$ and $h_0 = 5 \Rightarrow$

$V_r\big|_{(r_0, h_0)} = 2.\pi.1.5 = 10\pi$ and

$V_h(r_0, h_0) = \pi.1^2 = \pi$

$dV = 10\pi.(0{,}03) + \pi(-0{,}1)$

$= 0{,}3\pi - 0{,}1\pi$

$= 0{,}2\pi$

Example: Find an approximate value for the number $(1,1)^2+(2,5)^3$.

- If we say $f(x,y)=x^2+y^3$ and we take $P_0(1,2)$, then

$f(x_0,y_0)=f(1,2)=1^2+2^3=1+8=9$

- $L(x,y)\approx f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)$

$f_x=2x \Rightarrow f_x(1,2)=2$

$f_y=3y^2 \Rightarrow f_y(1,2)=12$

- In here, $x=1,1$ and $y=2,5$

$L(x,y)\approx 9+2(1,1-1)+12(2,5-2)=$

$\approx f(1,1\ ,2,5)$

$9+2(0,1)+12(0,5)=$

$9+(0,2)+6=$

$15,2$