MAT1320-Linear Algebra
Lecture Notes

Eigenvalues and Eigenvectors, Characteristic Polynomial, Diagonalization, Cayley-Hamilton Theorem

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Eigenvalues and Eigenvectors

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Note: Note that an eigenvector cannot be $\overrightarrow{\mathbf{0}}$, but an eigenvalue can be $0 \in \mathbb{R}$. If 0 is an eigenvalue of A, then there must be some nontrivial vector $\overrightarrow{\mathbf{x}}$ for which $A\overrightarrow{\mathbf{x}} = 0\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ which implies that A is not invertible.

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Note: The eigenspace of the $n \times n$ matrix A corresponding to the eigenvalue λ of A is the set of all eigenvectors of A corresponding to λ.

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Note: $P_A(\lambda) = \lambda^n + (-1)^{n-1}\, Tr(A)\, \lambda^{n-1} + \ldots + \det(A)$. For example, for a 2×2 square matrix A,
$P_A(\lambda) = \lambda^2 - Tr(A)\,\lambda + \det(A)$.

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Note 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Note 2: If $P_A(\lambda)$ has multiple roots, then there exists multiple eigenvalues.

Example

Let $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$. Find

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Solution (1)
$P_A(\lambda) = \det(\lambda I_3 - A)$

$$\lambda I_3 - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

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&\Rightarrow \quad P_A(\lambda) = \lambda(\lambda - 2)(\lambda - 3).
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Since, eigenvalues are the roots of $P_A(\lambda)$, *we have*

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Since, eigenvalues are the roots of $P_A(\lambda)$ *, we have*

$$\begin{aligned} P_A(\lambda) &= \lambda(\lambda-2)(\lambda-3)=0 \\ &\Rightarrow \lambda_1=0, \lambda_2=2, \lambda_3=3. \end{aligned}$$

Solution (3)
For $\lambda_1 = 0$,

$$(0I_3 - A) \quad = \quad -A = \begin{pmatrix} -3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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&\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.
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Finding Eigenvalues and Eigenvectors

Solution (3)
For $\lambda_2 = 2$,

$$(2I_3 - A) \quad = \quad \begin{pmatrix} -1 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

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Solution (3)
For $\lambda_3 = 3$,

$$(3I_3 - A) \quad = \quad \begin{pmatrix} 0 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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Note: $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$

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Theorem (The Diagonalization Theorem)

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2. *If $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$ are linearly independent eigenvectors of A and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are their corresponding eigenvalues, then $A = PDP^{-1}$, where*

$$P = \begin{pmatrix} \overrightarrow{\mathbf{v}}_1 & \ldots & \overrightarrow{\mathbf{v}}_n \end{pmatrix} \text{ and } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Example

We have found the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$ as $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3$ and $\overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \overrightarrow{\mathbf{v}}_2 = \begin{pmatrix} -10 \\ 3 \\ 1 \end{pmatrix}, \overrightarrow{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then

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1. show that $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \overrightarrow{\mathbf{v}}_3$ are linear independent,
2. show that there exists matrices P and D such that $A = PDP^{-1}$,
3. and find A^{125}.

Solution

1. $P = \begin{pmatrix} \overrightarrow{\mathbf{v}}_1 & \overrightarrow{\mathbf{v}}_2 & \overrightarrow{\mathbf{v}}_3 \end{pmatrix} = \begin{pmatrix} -2 & -10 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *and since* $\det(P) = 1 \neq 0$, $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \overrightarrow{\mathbf{v}}_3$ *are linearly independent.*

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2. $P = \begin{pmatrix} -2 & -10 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ *and*

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3. $A^{125} = PD^{125}P^{-1} = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^{125} & 0 \\ 0 & 0 & 3^{125} \end{pmatrix} P^{-1}$.

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$$\begin{aligned} P_A(A) &= A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n = 0 \\ &\Rightarrow A^n = -\left(a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n\right) \end{aligned}$$

If $a_0 \neq 0$, then

$$I_n = A\underbrace{\frac{-1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I_n\right)}_{A^{-1}}$$

Example

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find

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5. A^{-1} and A^5 (by using Cayley-Hamilton Theorem).

Solution (1)
$P_A(\lambda) = \det(\lambda I_n - A)$

$$\lambda I_2 - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

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$P_A(\lambda) = \det(\lambda I_n - A)$

$$\begin{aligned}\lambda I_2 - A &= \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\\ &= \begin{pmatrix}\lambda - 1 & -4\\ -2 & \lambda - 3\end{pmatrix}\end{aligned}$$

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&= \begin{pmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{pmatrix} \\
&\Rightarrow P_A(\lambda) = (\lambda - 1)(\lambda - 3) - 8 \\
&= \lambda^2 - 4\lambda - 5.
\end{aligned}$$

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The eigenvalues are the roots of $P_A(\lambda)$ *such that*

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P_A(\lambda) &= \lambda^2 - 4\lambda - 5 = 0 \\
&\Rightarrow \lambda_1 = -1,\ \lambda_2 = 5.
\end{aligned}
$$

Solution (3)
For $\lambda_1 = -1$,

$$(I_2 + A) = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

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&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\end{aligned}$$

Solution (3)
For $\lambda_2 = 5$,

$$(5I_2 - A) \quad = \quad \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

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&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Solution (4)
$P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ *and* $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow$
$A = PDP^{-1}$.

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$$P_A(A) = A^2 - 4A - 5I_2 = 0$$

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A^2 &= 4A + 5I_2 \\
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$$
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A^2 &= 4A + 5I_2 \\
A^4 &= (4A + 5I_2)(4A + 5I_2) \\
A^5 &= A(4A + 5I_2)(4A + 5I_2)
\end{aligned}
$$

?