Unsolved Exercises (3rd. week) [Continuity]

1) Let

$$f(x) = \begin{cases} x & \text{if } x \leq -2 \\ \dfrac{1}{x+2} & \text{if } -2 < x \leq 1 \\ \dfrac{\sin(1-\sqrt{x})}{x-1} & \text{if } x > 1 \end{cases}.$$

Find the points where $f(x)$ is discontinuous and classify their discontinuity types.

Solution: First, we need to check the domain of the function. If a point isn't in the domain, then it will cause discontinuity.

i) If $x \leq -2$, the function is $f(x) = x$ and it is defined on $(-\infty, -2]$. No problem.

ii) If $-2 < x \leq 1$, the function is $f(x) = \dfrac{1}{x+2}$ and it is defined on $\mathbb{R} \setminus \{-2\} \supset (-2, 1]$. No problem.

iii) If $x > 1$, the function is $f(x) = \dfrac{\sin(1-\sqrt{x})}{x-1}$ and it is defined on $[0, \infty) \setminus \{1\} \supset (1, \infty)$. No problem.

Now, we need to check for the transition points. ($x = -2$, $x = 1$) For these points, right-hand side and left-hand side limits must be equal to each other and the functions value.

i) $\lim_{x \to -2} f(x)$ needs to be $f(-2) = -2$.

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} x = -2$ ✓ (Finite and equal to $f(-2)$.)

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{1}{x+2} = +\infty \Rightarrow$ Limit doesn't exist.

For $x = -2$, the function has infinite discontinuity

ii) $\lim_{x \to 1} f(x)$ needs to be $f(1) = \frac{1}{1+2} = \frac{1}{3}$.

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x+2} = \frac{1}{3}$ ✓ (Finite and equal to $f(1)$)

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{\sin(1-\sqrt{x})}{x-1} \underset{(*)}{=} \lim_{x \to 1^+} \overbrace{\frac{\sin(1-\sqrt{x})}{1-\sqrt{x}}}^{1} \cdot \frac{1}{-(\sqrt{x}+1)}$

$= -\frac{1}{2}$ ✗ (Finite but not equal to $\frac{1}{3}$)

$$\boxed{(*) \quad x-1 = (\sqrt{x}-1).(\sqrt{x}+1) = -(1-\sqrt{x}).(\sqrt{x}+1)}$$

For $x = 1$, the function has jump discontinuity.

2) Let a, b be two positive real numbers and consider the functions

$$f(x) = e^{\left(\frac{ax^2+bx+19}{x^2+52}\right)} \quad \text{and} \quad g(x) = \left(1+\frac{b}{ax}\right)^x.$$

Determine, if possible, the relation between a and b for which $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$.

<u>Solution:</u>

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\left(\frac{ax^2+bx+19}{x^2+52}\right)} \underset{(*)}{=} e^{\lim_{x\to\infty}\left(\frac{ax^2+bx+19}{1x^2+52}\right)} \underset{(**)}{=} e^{a}$$

(*) exponential function is a continuous function. Thus, limit and e may change places. (We can apply the limit to the exponent of e).

(**) For $\lim_{x\to\infty}$ division of two polynomials, if the highest degrees (powers) of x's are equal, then limit is their coefficients, $\frac{a}{1}$ for this case.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(1+\frac{b}{ax}\right)^x \underset{(*)}{=} \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right]^{\frac{b}{a}} \underset{(**)}{=} e^{\frac{b}{a}}$$

(*) Let $t = \frac{ax}{b} \Rightarrow x = \frac{bt}{a}$ and as $x\to\infty$, $t\to\infty$

(**) $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$ is a property that will be proven in the future.

$$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) \Rightarrow e^{a} = e^{\frac{b}{a}} \Rightarrow a = \frac{b}{a} \Rightarrow \boxed{b = a^2}$$

3) For what values of a and b,

$$f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x^2} & \text{if } x<0 \\ a - \arcsin\left(\frac{x+1}{2}\right) & \text{if } 0 \le x < 1 \\ \frac{a}{b} + \arctan(\sqrt{3}x) & \text{if } x \ge 1 \end{cases}$$

is continuous for all $x \in \mathbb{R}$?

<u>Solution</u>: Check the right and left limits for the transition points. ($x=0$ and $x=1$) Must be equal.

i) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \cdot \sin\frac{1}{x^2} = 0$ (as if 0 times constant)

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} a - \arcsin\left(\frac{x+1}{2}\right) = a - \arcsin\frac{1}{2} = a - \frac{\pi}{6}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \Rightarrow a - \frac{\pi}{6} = 0 \Rightarrow \boxed{a = \frac{\pi}{6}}$.

ii) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\underset{\downarrow \frac{\pi}{6}}{a} - \arcsin\left(\frac{x+1}{2}\right)\right) = \frac{\pi}{6} - \arcsin 1 = \frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{3}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(\frac{a}{b} + \arctan(\sqrt{3}x)\right) = \frac{\pi}{6b} + \arctan\sqrt{3} = \frac{\pi}{6b} + \frac{\pi}{3}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow \frac{\pi}{6b} + \frac{\pi}{3} = -\frac{\pi}{3} \Rightarrow \frac{\cancel{\pi}}{{}_2\cancel{6}b} = -\frac{2\cancel{\pi}}{\cancel{3}}$

$\Rightarrow \boxed{b = -\frac{1}{4}}$

4) Let $f(x) = \dfrac{x^2}{\cos 5x - \cos 2x}$ for $x \neq 0$. Find an extension $F(x)$ to make the function $f(x)$ continuous at $x = 0$.

Solution: We need a value for $f(0)$ such that this value is equal to $\lim\limits_{x \to 0} f(x)$.

$$\lim_{x\to 0} f(x) = \lim_{x \to 0} \frac{x^2}{\cos 5x - \cos 2x} \underset{(*)}{=} \lim_{x\to 0} \frac{x^2}{-2 \cdot \sin\left(\frac{5x+3x}{2}\right) \cdot \sin\left(\frac{5x-3x}{2}\right)}$$

$$= \lim_{x\to 0} \frac{-x^2}{2 \cdot \sin(4x) \cdot \sin x} = \lim_{x \to 0} \frac{-1}{2} \cdot \underbrace{\frac{x}{\sin(4x)}}_{\frac{1}{4}} \cdot \underbrace{\frac{x}{\sin x}}_{1} = -\frac{1}{8} = F(0)$$

$(*)$ $\cos A - \cos B = -2 \cdot \sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$

$$\Rightarrow F(x) = \begin{cases} \dfrac{x^2}{\cos 5x - \cos 2x} & \text{if } x \neq 0 \\ -\dfrac{1}{8} & \text{if } x = 0 \end{cases}$$

5) Let $f(x) = \begin{cases} 4\sin x & \text{if } x \leq -\frac{\pi}{2} \\ a\sin x + b & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 2\cos x & \text{if } x \geq \frac{\pi}{2} \end{cases}$. If this function is continuous everywhere, then find a and b.

Solution: sin and cos are continuous on $\mathbb{R}$. Hence we need to check the limits for $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

$\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^-} 4\sin x = -4$, $\lim\limits_{x\to -\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^+} a\sin x + b = -a + b$

$\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^+} f(x) \Rightarrow \boxed{-a+b=-4}$ (1)

$\lim\limits_{x\to \frac{\pi}{2}^+} f(x) = \lim\limits_{x\to \frac{\pi}{2}^+} 2\cos x = 0$, $\lim\limits_{x\to \frac{\pi}{2}^-} f(x) = \lim\limits_{x\to \frac{\pi}{2}^-} a\sin x + b = a + b$

$(1)+(2) \Rightarrow 2b = -4$

$\lim\limits_{x\to \frac{\pi}{2}^+} f(x) = \lim\limits_{x\to \frac{\pi}{2}^-} f(x) \Rightarrow \boxed{a+b=0}$ (2)

$\Rightarrow \boxed{b=-2 \quad a=2}$