Functions of Several Variables

DEFINITION

A **function of two variables** $z = f(x, y)$ maps each ordered pair (x, y) in a subset $\mathbf{D}$ of the real plane $\mathbb{R}^2$ to a unique real number z. The set $\mathbf{D}$ is called the *domain* of the function. The *range* of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in \mathbf{D}$ such that $f(x, y) = z$ as shown in the following figure.

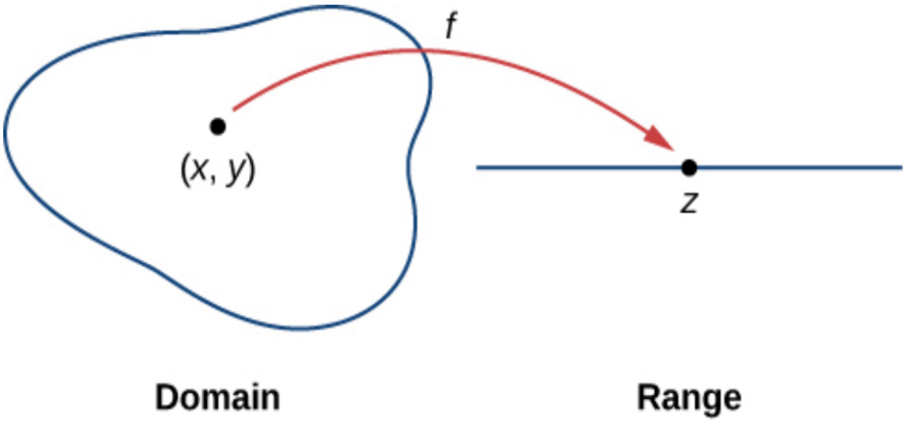


Figure 1. The domain of a function of two variables consists of ordered pairs (x, y).

EXAMPLE 1

- These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

Example 2: Find the domains of the functions given below and draw their graphs.

a) $z = \dfrac{1}{\sqrt{1-x^2-y^2}}$

b) $z = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$

c) $z = \dfrac{1}{\sqrt{9x^2 - y^2}}$

d) $z = f(x,y) = \ln(x \cdot \ln(y-x))$

Functions of Three Variables

- A **function of three variables**, x, y, and z, is a rule that assigns to each ordered triple, (x,y,z), **exactly one real number**.

$w = f(x, y, z)$

Example: These are functions of three variables with restrictions on some of their domains.

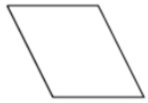
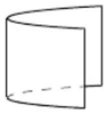
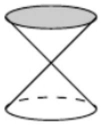
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

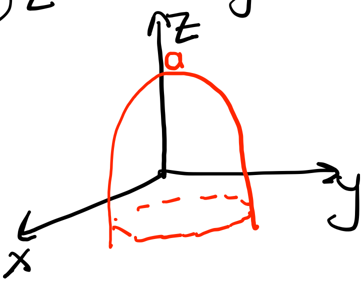
QUADRIC SURFACES

name	equation in standard form	x = const cross-section	y = const cross-section	z = const cross-section	sketch
plane	$ax + by + cz = d$	line	line	line	
elliptic cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	two lines	two lines	ellipse	
parabolic cylinder	$y = ax^2$	one line	two lines	parabola	
sphere	$x^2 + y^2 + z^2 = d^2$	circle	circle	circle	
ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	ellipse	ellipse	ellipse	
elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	parabola	parabola	ellipse	
elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	two lines if x = 0 hyperbola if x 6= 0	two lines if y = 0 hyperbola if y 6= 0	ellipse	
hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	hyperbola	hyperbola	ellipse	
hyperboloid of two sheets	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	hyperbola	hyperbola	ellipse	
hyperbolic paraboloid	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$	parabola	parabola	hyperbola	

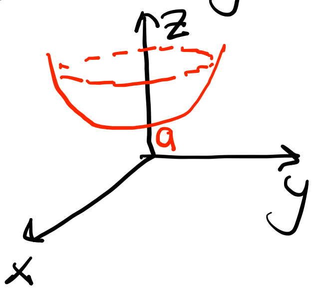
Paraboloid

① $z = ax^2 + by^2$ $(a, b > 0)$

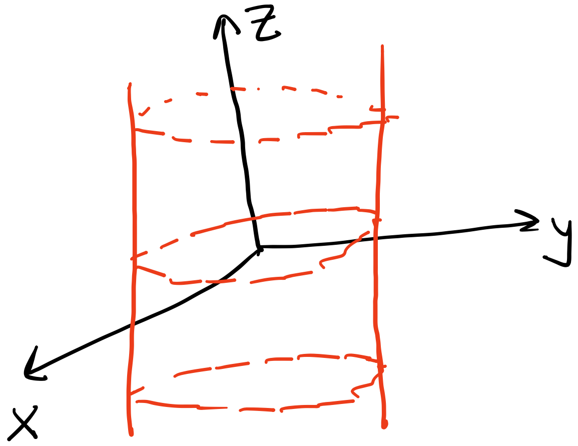
② $z = a - x^2 - y^2$



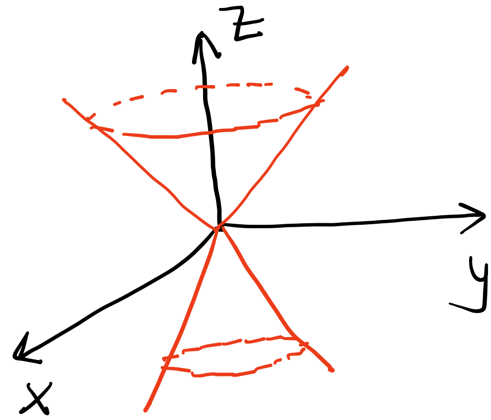
③ $z = a + x^2 + y^2$



Cylinder: $x^2+y^2=r^2$



Circular Cone: $x^2+y^2=z^2$



Example: Find the domain of the function
$w=\sqrt{1-x^2-y^2-z^2}$.

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $\boldsymbol{z = f(x, y)}$.

EXAMPLE [redacted] Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure 14.5.

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.5) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$

$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

Note: If there is a limit, it is unique.

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x- and y-values of the point being approached into the functional expression to find the limiting value.

(a) $$\lim_{(x,\, y) \to (0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,\, y) \to (3,\, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

EXAMPLE 2 Find $\displaystyle\lim_{(x,\, y) \to (0,\, 0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

[redacted]

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\lim_{(x,\, y) \to (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,\, y) \to (0,0)} \frac{\left(x^2 - xy\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$ Multiply by a form equal to 1.

$$= \lim_{(x,\, y) \to (0,0)} \frac{x\left(x - y\right)\left(\sqrt{x} + \sqrt{y}\right)}{x - y}$$ Algebra

$$= \lim_{(x,\, y) \to (0,0)} x\left(\sqrt{x} + \sqrt{y}\right)$$ Cancel the nonzero factor $(x - y)$.

$$= 0\left(\sqrt{0} + \sqrt{0}\right) = 0$$ Known limit values

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$f(x, y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$
2. *Difference Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$
3. *Constant Multiple Rule:* $$\lim_{(x, y)\to(x_0, y_0)} kf(x, y) = kL \qquad \text{(any number } k\text{)}$$
4. *Product Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$
5. *Quotient Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \qquad M \neq 0$$
6. *Power Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \left[f(x, y) \right]^n = L^n, \ n \text{ a positive integer}$$
7. *Root Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$
   n a positive integer, and if n is even, we assume that $L > 0$.

(The Squeeze Theorem)

The Sandwich Theorem for functions of two variables states that if $g(x,y) \le f(x,y) \le h(x,y)$ for all $(x,y) \ne (x_0,y_0)$ in a disk centered at (x_0,y_0) and if g and h have the same finite limit L as $(x,y) \to (x_0,y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Example: Let $f(x,y) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{y}\right), & y \ne 0 \\ 0, & y = 0 \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

DOUBLE LIMIT

Let $\lim\limits_{x\to a}\left[\lim\limits_{y\to b} f(x,y)\right] = L_1$ and $\lim\limits_{y\to b}\left[\lim\limits_{x\to a} f(x,y)\right] = L_2$.

(a) If $L_1 = L_2$, then there exists the double limit of the function $f(x,y)$ at the point (a,b).

Note: The existence of the double limit does not guarantee that the function f(x,y) has a limit at the point (a,b).

(b) If $L_1 \neq L_2$, then there is no the double limit of the function $f(x,y)$ at the point (a,b). Hence, there is no limit of the function $f(x,y)$ at the point (a,b).

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$. We examine the values of f along parabolic curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$. ■

Example: Find $\lim_{(x,y)\to(0,0)} \dfrac{3x^2-y^2}{3y^2+x^2}$ if it exists.

1. way: Let $y=x$. Then,

$$\lim_{x\to 0} \frac{3x^2-x^2}{3x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{4x^2} = \frac{1}{2}$$

Let $y=x^2$. Then,

$$\lim_{x\to 0} \frac{3x^2-x^4}{3x^4+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}(3-\overset{0}{\overbrace{x^2}})}{\cancel{x^2}(\underset{0}{\underbrace{3x^2}}+1)} = 3$$

$\neq$ so there is no limit!

2. way: $$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{3x^2-y^2}{3y^2+x^2}\right] = \lim_{x\to 0}\left(\frac{3x^2}{x^2}\right) = 3$$

$\neq$

$$\lim_{y\to 0}\left[\lim_{x\to 0} \frac{3x^2-y^2}{3y^2+x^2}\right] = \lim_{y\to 0}\left(-\frac{y^2}{3y^2}\right) = -\frac{1}{3}$$

So there is no double limit. Hence, there is no limit.

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist (see Example 1). □

$$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2}\right] = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

$\neq$

$$\lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2}\right] = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

So there is no double limit. Hence, there is no limit. Thus, the function is not continuous.

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos \frac{xy}{x^2 + 1}, \qquad \ln \left(1 + x^2y^2\right)$$

are continuous at every point (x, y).

EXAMPLE 2 If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y)$ exist?

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y)$ exist?