MAT1320-Linear Algebra
Lecture Notes

Vectors

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- Many physical quantities, such as temperature and speed, possess only magnitude. These quantities can be represented by real numbers and are called scalars.
- On the other hand, there are also quantities, such as force and velocity, that possess both magnitude and direction.
- These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point A, are called vectors.

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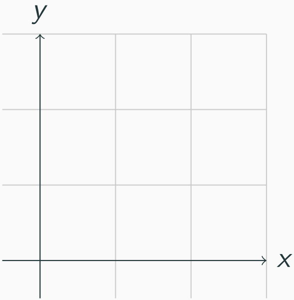

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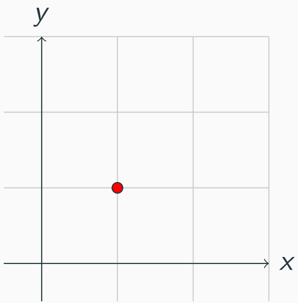

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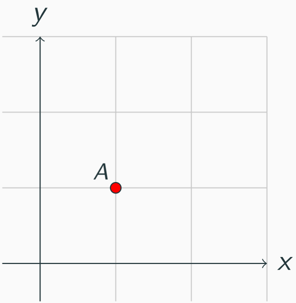

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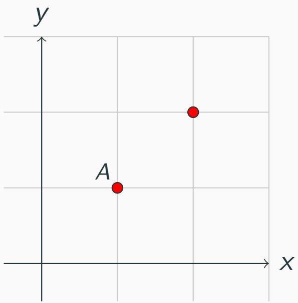

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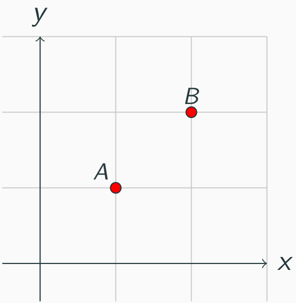

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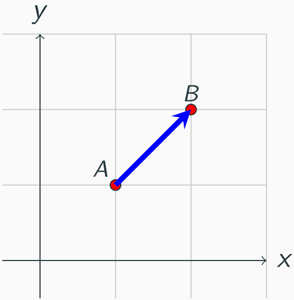

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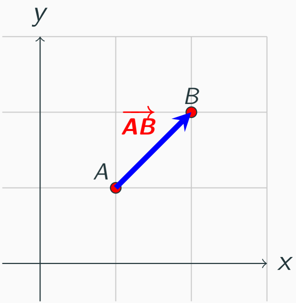

Norm (Length) of a Vector

- Let $P_0\left(x_0, y_0, z_0\right)$ and $P_1\left(x_1, y_1, z_1\right)$ be two points in $\mathbb{R}^3$. Then the vector with start point P_0 and end point P_1 is denoted by $\overrightarrow{P_0 P_1}$ and defined as

$$\overrightarrow{\mathbf{v}} = \overrightarrow{P_0 P_1} = \left(x_1 - x_0, y_1 - y_0, z_1 - z_0\right).$$

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Note: A vector $\overrightarrow{\mathbf{u}}$ is called unit vector if $\left|\overrightarrow{\mathbf{u}}\right| = 1$.

Note: For any nonzero vector $\overrightarrow{\mathbf{v}}$, the vector $\frac{\overrightarrow{\mathbf{v}}}{\left|\overrightarrow{\mathbf{v}}\right|}$ is the unique unit vector in the same direction as $\overrightarrow{\mathbf{v}}$.

Vector Addition

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$$\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

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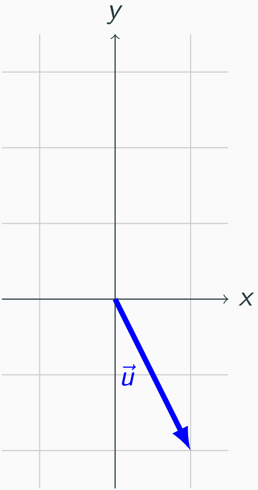

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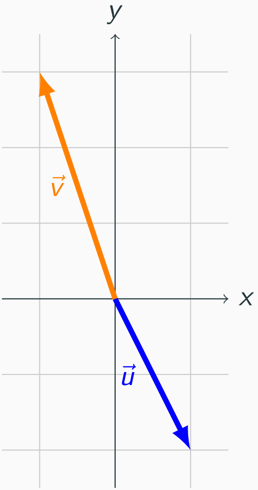

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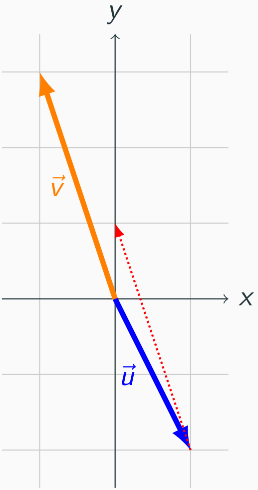

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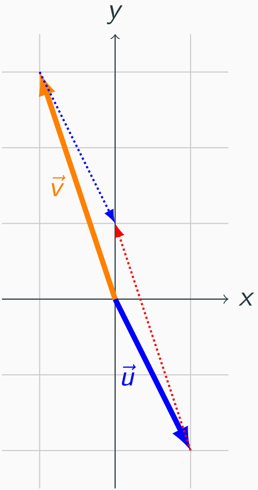

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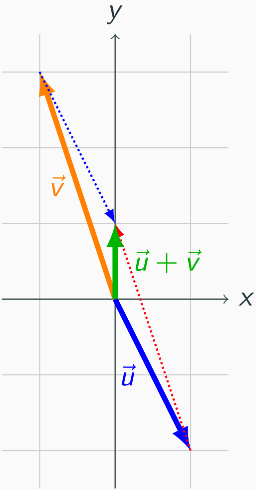

Scalar Multiplication

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The scaler product of vector $\overrightarrow{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$ by a real number λ, written $\lambda\overrightarrow{\mathbf{u}}$, is the vector obtained by multiplying each component of $\overrightarrow{\mathbf{u}}$ by λ. That is, $\lambda\overrightarrow{\mathbf{u}} = (\lambda u_1, \lambda u_2, \ldots, \lambda u_n)$.

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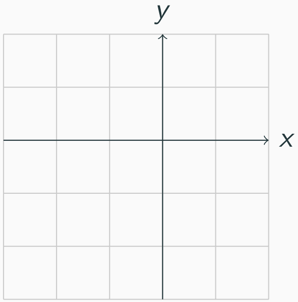

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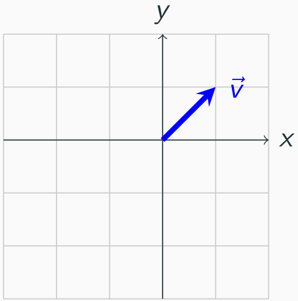

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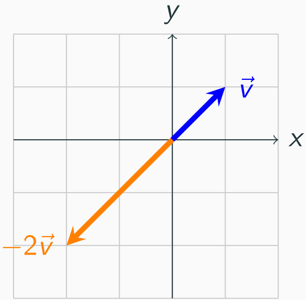

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The dot product or inner product of vectors $\overrightarrow{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$ and $\overrightarrow{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ is denoted by $\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}}$ or $\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \rangle$

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\begin{aligned}
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = \sum_{i=1}^{4} u_i v_i \\
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Note: The dot product of two vectors is a real number.

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Dot (Inner) Product: Properties

Let the vectors $\overrightarrow{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$, $\overrightarrow{\mathbf{w}} = (w_1, w_2, \ldots, w_n)$ the scalers α, $\beta \in \mathbb{R}$ be given. The inner product satisfies the following properties:

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The Angle Between Two Nonzero Vectors

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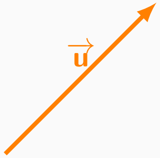

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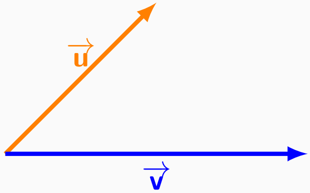

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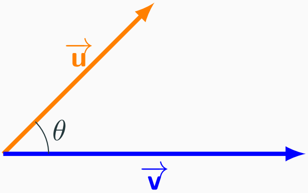

Example

Find the angle between the vectors $\overrightarrow{\mathbf{u}} = (2, 2, 0, 1)$ and $\overrightarrow{\mathbf{v}} = (-1, 1, 0, 2)$.

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The projection of the vector $\overrightarrow{\mathbf{u}} = \overrightarrow{AB}$ onto the nonzero vector $\overrightarrow{\mathbf{v}} = \overrightarrow{AD}$ is denoted by $proj_{\overrightarrow{\mathbf{v}}}\left(\overrightarrow{\mathbf{u}}\right)$ and is defined as the vector $\overrightarrow{AC}$.

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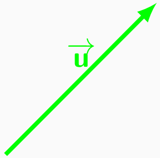

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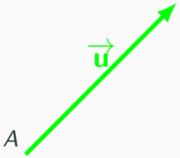

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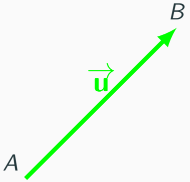

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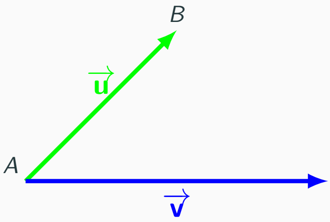

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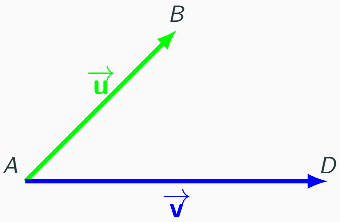

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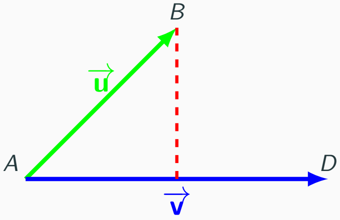

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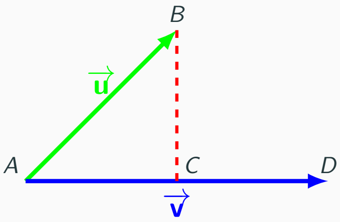

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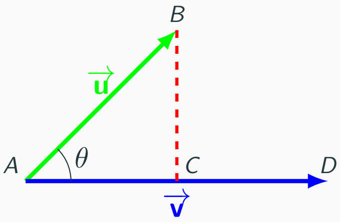

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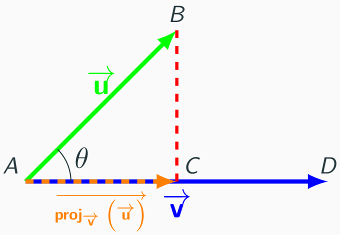

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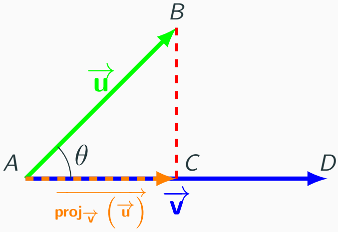

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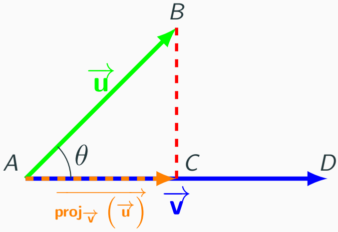

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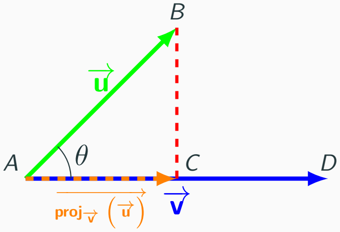
The projection of the vector $\overrightarrow{\mathbf{u}} = \overrightarrow{AB}$ onto the nonzero vector $\overrightarrow{\mathbf{v}} = \overrightarrow{AD}$ is denoted by $proj_{\overrightarrow{\mathbf{v}}}\left(\overrightarrow{\mathbf{u}}\right)$ and is defined as the vector $\overrightarrow{AC}$.



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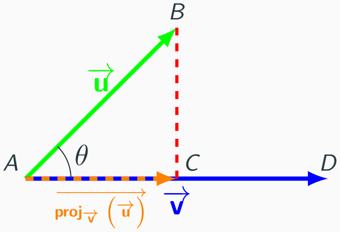

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Projection of A Vector onto Another

Example

Let $\overrightarrow{\mathbf{u}} = (2, 2, 0, 1)$ and $\overrightarrow{\mathbf{v}} = (-1, 1, 0, 2)$ be two nonzero vectors. Find $proj_{\overrightarrow{\mathbf{v}}}\left(\overrightarrow{\mathbf{u}}\right)$.

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$$proj_{\overrightarrow{\mathbf{v}}}\left(\overrightarrow{\mathbf{u}}\right) = \frac{2}{6}(-1, 1, 0, 2) = \left(\frac{-1}{3}, \frac{1}{3}, 0, \frac{2}{3}\right)$$

Cross Product

Definition

Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$ and $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The cross product of the vectors $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ is denoted by $\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}$ or $\overrightarrow{\mathbf{u}} \wedge \overrightarrow{\mathbf{v}}$ and defined as follows:

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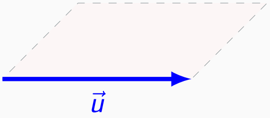

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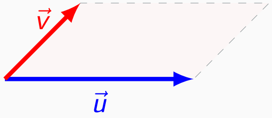

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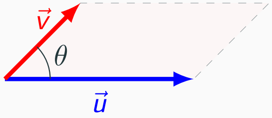

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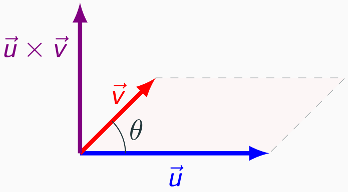

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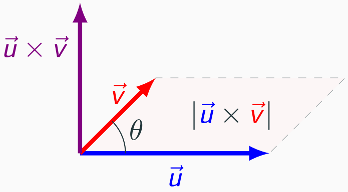

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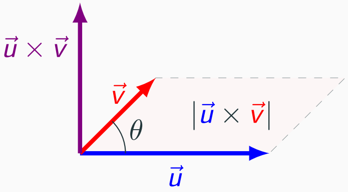

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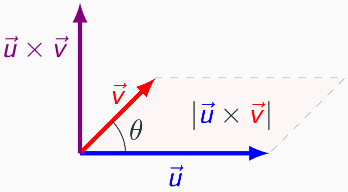

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Cross Product

$$\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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Note: The cross product is only defined over $\mathbb{R}^3$.

Example

Let $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (-2, 3, 4) \in \mathbb{R}^3$. Find the cross product of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$.

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Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$ and $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

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4. $(\overrightarrow{\mathbf{u}} \mp \overrightarrow{\mathbf{v}}) \times \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{w}} \mp \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}$
5. $\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = (\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}) \cdot \overrightarrow{\mathbf{w}}$
6. $\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}})\, \overrightarrow{\mathbf{v}} - (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}})\, \overrightarrow{\mathbf{w}}$
7. $\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = (|\overrightarrow{\mathbf{u}}|\,|\overrightarrow{\mathbf{v}}| \sin\theta)\, \overrightarrow{\mathbf{n}}$
   $(\overrightarrow{\mathbf{u}} \perp \overrightarrow{\mathbf{n}},\ \overrightarrow{\mathbf{v}} \perp \overrightarrow{\mathbf{n}},\ \text{ve } |\overrightarrow{\mathbf{n}}| = 1)$

Cross Product: Properties

Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$ and $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

1. $\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = -\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{u}}$
2. $(c\overrightarrow{\mathbf{u}}) \times \overrightarrow{\mathbf{v}} = c(\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}) = \overrightarrow{\mathbf{u}} \times (c\overrightarrow{\mathbf{v}})$
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8. $\theta = 0$ or $\theta = \pi \Rightarrow \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}$

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8. $\theta = 0$ or $\theta = \pi \Rightarrow \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}$
9. $\overrightarrow{\mathbf{u}} \perp (\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}})$ and $\overrightarrow{\mathbf{u}} \perp (\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}})$

Mixed Product

Definition
Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$ and $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3)$ be three vectors in $\mathbb{R}^3$. Then the mixed product of the vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$ is denoted by $\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$ or $(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}})$

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$$\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

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Note: Mixed product is only defined over $\mathbb{R}^3$.

Mixed Product: Geometric Interpretation

Geometrically, mixed product of the vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$ is the volume of the parallelepiped having edges as $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$.

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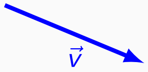

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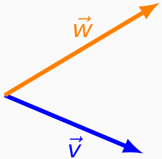

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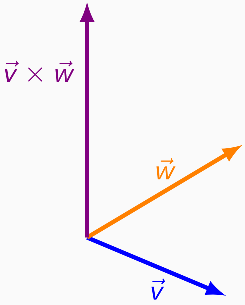

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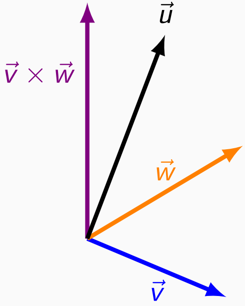

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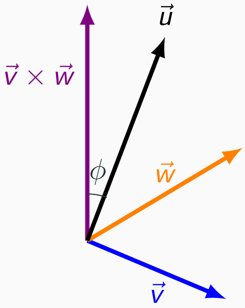

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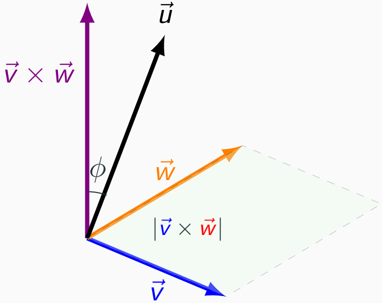

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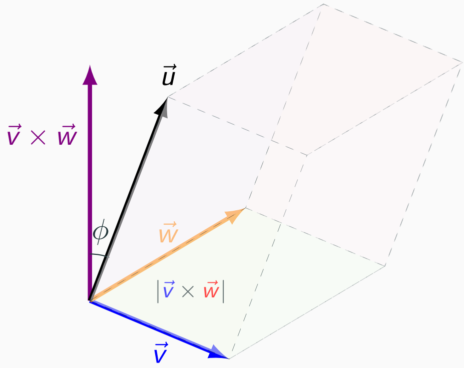

Example
The mixed product $\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$, of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$, $\overrightarrow{\mathbf{v}} = (-2, 3, 4)$ and $\overrightarrow{\mathbf{w}} = (2, 1, 0) \in \mathbb{R}^3$ is

$$\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix}$$

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$$\begin{aligned} \overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) &= \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} \\ &= 1 \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} -2 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} \end{aligned}$$

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&= 1 \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} -2 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} \\
&= -4 + 16 + 8 = 20.
\end{aligned}$$

Mixed Product: Properties

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Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$, $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3)$ and $\overrightarrow{\mathbf{r}} = (r_1, r_2, r_3) \in \mathbb{R}^3$ $c \in \mathbb{R}$.

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1. $\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = \overrightarrow{\mathbf{w}} \cdot (\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}) = \overrightarrow{\mathbf{v}} \cdot (\overrightarrow{\mathbf{w}} \times \overrightarrow{\mathbf{u}})$

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2. $-\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{w}} \times \overrightarrow{\mathbf{v}}) = -\overrightarrow{\mathbf{v}} \cdot (\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{w}}) = -\overrightarrow{\mathbf{w}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{u}})$

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3. $(c\overrightarrow{\mathbf{u}}) \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = \overrightarrow{\mathbf{u}} \cdot ((c\overrightarrow{\mathbf{v}}) \times \overrightarrow{\mathbf{w}}) = \overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times (c\overrightarrow{\mathbf{w}}))$

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Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$, $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3)$ and $\overrightarrow{\mathbf{r}} = (r_1, r_2, r_3) \in \mathbb{R}^3$ $c \in \mathbb{R}$.

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4. $(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{r}}) \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = \overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{r}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$

Mixed Product: Properties

Let $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$, $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3)$ and $\overrightarrow{\mathbf{r}} = (r_1, r_2, r_3) \in \mathbb{R}^3$ $c \in \mathbb{R}$.

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5. $\overrightarrow{\mathbf{u}} \cdot (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = 0 \Leftrightarrow \overrightarrow{\mathbf{u}} \parallel \overrightarrow{\mathbf{v}}$ or $\overrightarrow{\mathbf{u}} \parallel \overrightarrow{\mathbf{w}}$ or $\overrightarrow{\mathbf{v}} \parallel \overrightarrow{\mathbf{w}}$ or $\overrightarrow{\mathbf{u}} \parallel \overrightarrow{\mathbf{v}} \parallel \overrightarrow{\mathbf{w}}$

Two Fold Cross Product

Two Fold Cross Product

Definition

The two fold cross product of the vectors $\overrightarrow{\mathbf{u}} = (u_1, u_2, u_3)$, $\overrightarrow{\mathbf{v}} = (v_1, v_2, v_3)$ and $\overrightarrow{\mathbf{w}} = (w_1, w_2, w_3) \in \mathbb{R}^3$ is defined by

$$\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) = (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}})\, \overrightarrow{\mathbf{v}} - (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}})\, \overrightarrow{\mathbf{w}}.$$

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Note: The result of two fold cross product is a vector over $\mathbb{R}^3$.

Example
Let $\overrightarrow{\mathbf{u}} = (1, 2, -1)$, $\overrightarrow{\mathbf{v}} = (-2, 3, 4)$ and $\overrightarrow{\mathbf{w}} = (2, 1, 0) \in \mathbb{R}^3$ given. Find $\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$.

$$\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}} \quad = \quad 1.2 + 2.1 + (-1).0 = 4$$

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$$\begin{aligned} \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}} &= 1.2 + 2.1 + (-1).0 = 4 \\ \overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} &= 1.(-2) + 2.3 + (-1).4 = 0 \end{aligned}$$

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$$\begin{aligned}
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}} &= 1.2 + 2.1 + (-1).0 = 4 \\
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} &= 1.(-2) + 2.3 + (-1).4 = 0 \\
\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) &= (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}})\, \overrightarrow{\mathbf{v}} - (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}})\, \overrightarrow{\mathbf{w}}
\end{aligned}$$

Example

Let $\overrightarrow{\mathbf{u}} = (1, 2, -1)$, $\overrightarrow{\mathbf{v}} = (-2, 3, 4)$ and $\overrightarrow{\mathbf{w}} = (2, 1, 0) \in \mathbb{R}^3$ given. Find $\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$.

$$\begin{aligned}
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}} &= 1.2 + 2.1 + (-1)\,.0 = 4 \\
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} &= 1.(-2) + 2.3 + (-1)\,.4 = 0 \\
\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) &= (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}})\,\overrightarrow{\mathbf{v}} - (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}})\,\overrightarrow{\mathbf{w}} \\
&= 4\overrightarrow{\mathbf{v}} - 0\overrightarrow{\mathbf{w}} = 4\,(-2, 3, 4)
\end{aligned}$$

Two Fold Cross Product

Example

Let $\overrightarrow{\mathbf{u}} = (1, 2, -1)$, $\overrightarrow{\mathbf{v}} = (-2, 3, 4)$ and $\overrightarrow{\mathbf{w}} = (2, 1, 0) \in \mathbb{R}^3$ given. Find $\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}})$.

$$
\begin{aligned}
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}} &= 1.2 + 2.1 + (-1).0 = 4 \\
\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} &= 1.(-2) + 2.3 + (-1).4 = 0 \\
\overrightarrow{\mathbf{u}} \times (\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}) &= (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{w}})\,\overrightarrow{\mathbf{v}} - (\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}})\,\overrightarrow{\mathbf{w}} \\
&= 4\overrightarrow{\mathbf{v}} - 0\overrightarrow{\mathbf{w}} = 4(-2, 3, 4) \\
&= (-8, 12, 16)
\end{aligned}
$$

?