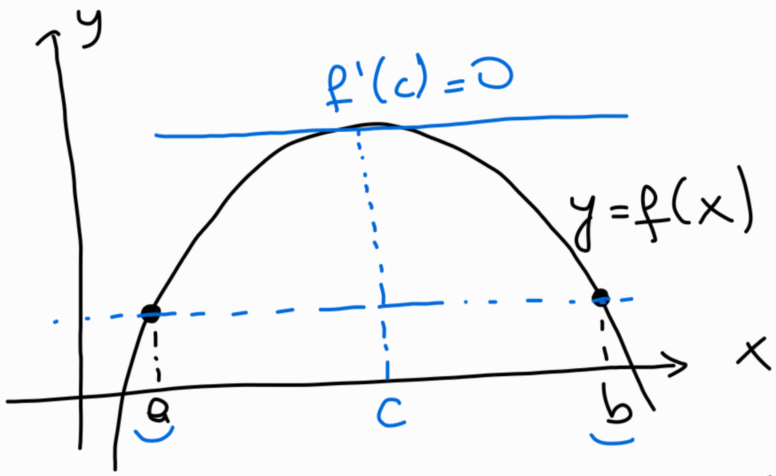
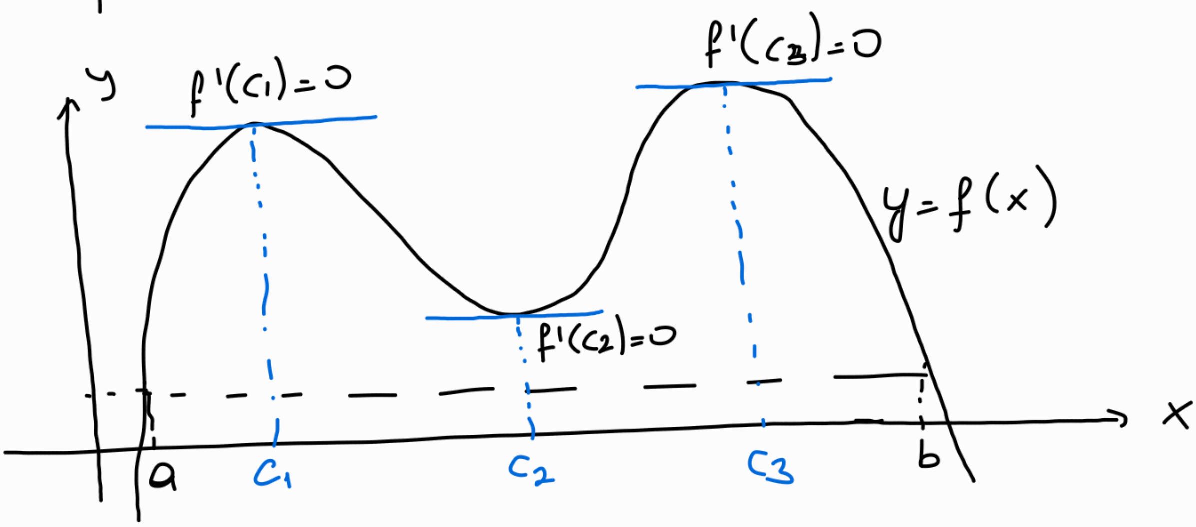
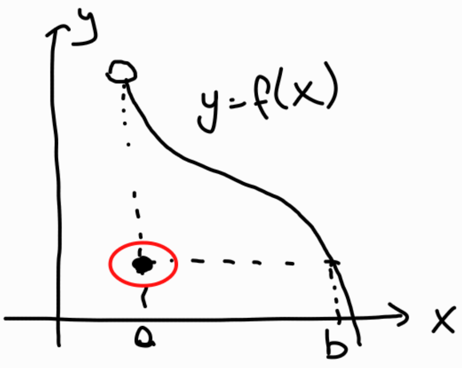
Rolle's Theorem

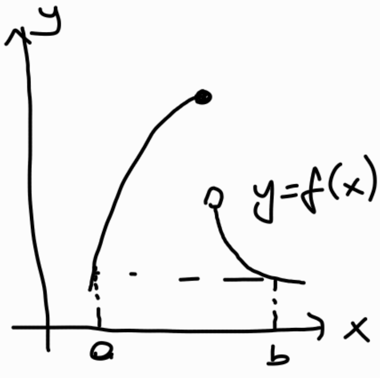
Suppose that $y=f(x)$ is continuous over the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If $f(a)=f(b)$, then there is at least one c in (a,b) at which $f'(c)=0$.

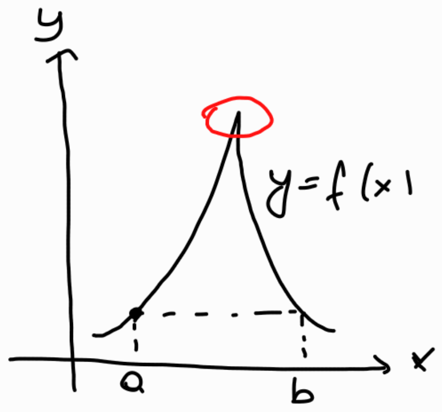




The hypotheses of Rolle's Theorem are essential. If they fail at even one point, the graph may not have a horizontal tangent.







Discont. at an endpoint

Discont. at an interior point

Continuous on [a,b] but not diff.ble at (a,b)

Example: Verify that the function $f(x) = \frac{x^3}{3} - 3x$ satisfies the three hypotheses of Rolle's Theorem on the interval $[-3,3]$.

i) Continuity on $[-3,3]$ (Yes, polynomials are cont.)

ii) Differentiability on $(-3,3)$ (Yes, polynomials are diff.ble)

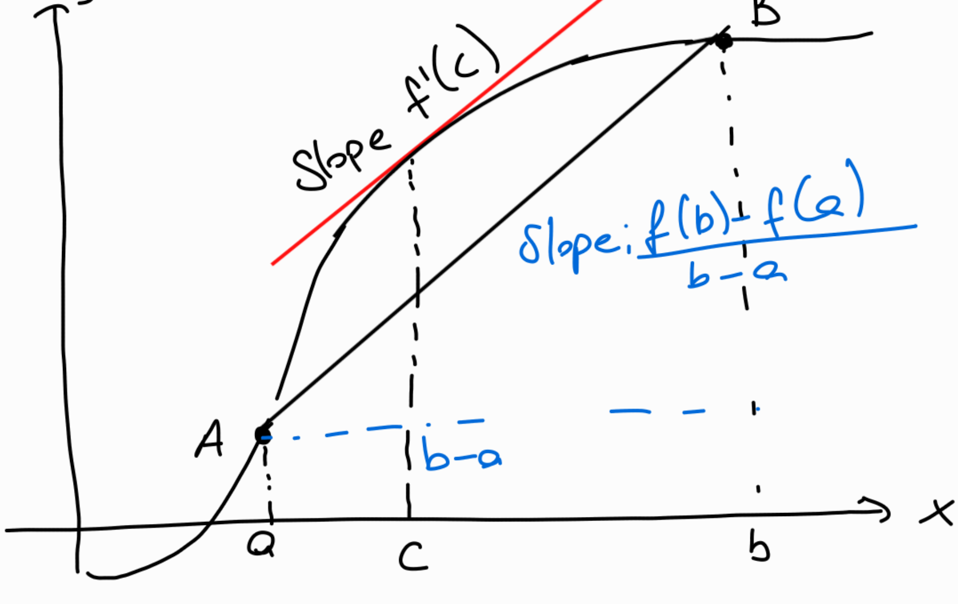
iii) $f(-3) = f(3) = 0$

$\Rightarrow$ We can find at least one $c \in (a,b)$ s.t. $f'(c) = 0$.

$$f'(x) = x^2 - 3 = 0 \Rightarrow x = \mp\sqrt{3}, \quad c_1 = -\sqrt{3}, \quad c_2 = \sqrt{3}$$

The Mean Value Theorem (M.V.T)

Suppose $y = f(x)$ is continuous over a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one point c in (a,b) at which $\frac{f(b)-f(a)}{b-a} = f'(c)$.



Geometrically the M.V.T. says that somewhere between a and b the curve has at least one tangent line parallel to the secant line that joins A and B.

Example: Verify that the function $f(x)=x^3-x$ satisfies the hypotheses of the M.V.T. on the interval $[0,2]$.

i) function is cont. on $[0,2]$

ii) function is diff.ble on $(0,2)$

$$\underline{f'(c)}=\frac{f(2)-f(0)}{2-0}=\frac{6-0}{2-0}=3$$

$$3c^2-1=3$$

$$3c^2=4 \Rightarrow c^2=\frac{4}{3} \Rightarrow c=\mp\frac{2}{\sqrt{3}} \qquad c_1=\frac{2}{\sqrt{3}}\in(0,2)$$

$$c_2=-\frac{2}{\sqrt{3}} \times \notin (0,2)$$

Corollary: If $f'(x)=0$ at each point x of an open interval (a,b), then $f(x)=C$ for all $x\in(a,b)$, where C is a constant.

Example: Prove the identity $\arctan x+\operatorname{arccot} x=\frac{\pi}{2}$.

$$f(x)=\arctan x+\operatorname{arccot} x=\frac{\pi}{2}$$

$$f'(x)=\frac{1}{1+x^2}-\frac{1}{1+x^2}=0, \quad \forall x.$$

From the corollary, $f(x)$ is a constant.

Say $x=1 \Rightarrow f(1)=\arctan 1+\operatorname{arccot} 1=\frac{\pi}{4}+\frac{\pi}{4}=\underline{\underline{\frac{\pi}{2}}}$.

Thus, $\arctan x+\operatorname{arccot} x=\frac{\pi}{2}$.

Monotonic Functions and the First Derivative Test

Definition: Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1) If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$ then f is said to be increasing on I.
2) If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then f is said to be decreasing on I.

Corollary: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).

1) If $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$.
2) If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.

Example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and those on which f is decreasing.

$$f'(x) = 0 \Rightarrow f'(x) = 3x^2 - 12 = 0$$

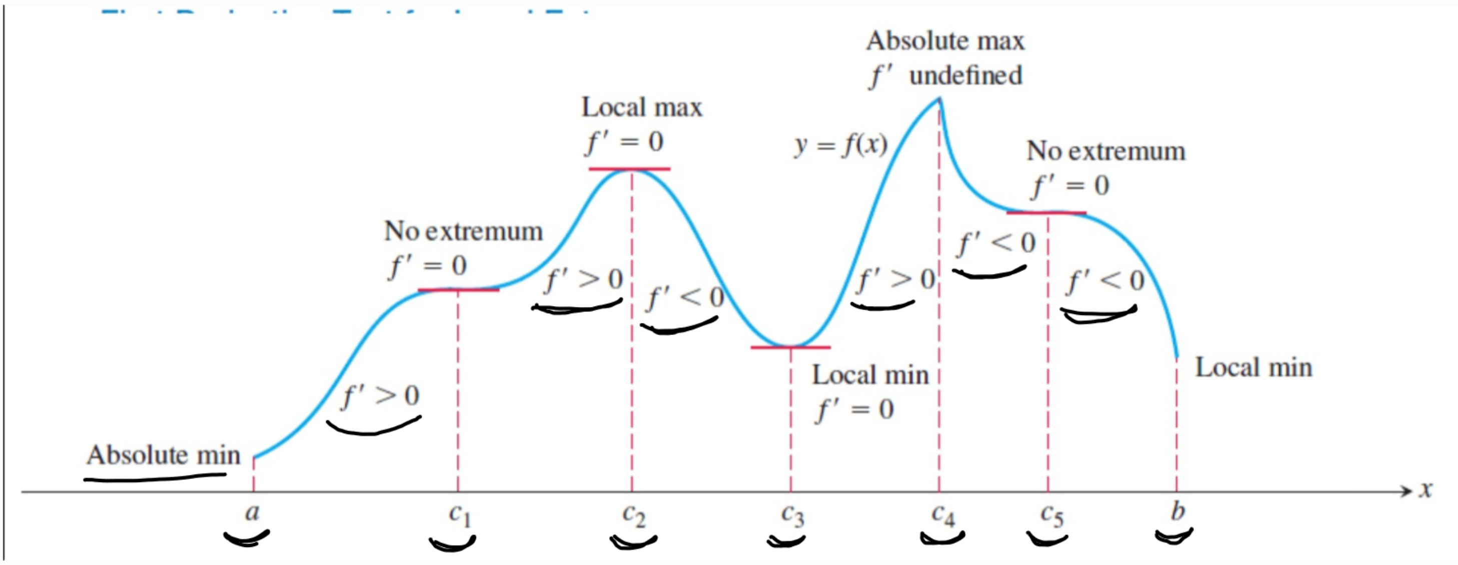
$$3(x^2-4) = 0 \Rightarrow 3(x+2)\cdot(x-2) = 0$$

$$\underset{x=-2}{\downarrow} \qquad \underset{x=2}{\downarrow}$$

		-2		2	3
f'	+	0	−	0	+
	↗		↘		↗

Increasing: $(-\infty,-2) \cup (2,\infty)$

Decreasing: $(-2,2)$



First Derivative Test for Local Extrema

Suppose that c is a critical point of a continous function f, and that is f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1) if f′ changes from negative to positive at c, then f has a local minimum at c;
2) if f′ changes from positive to negative at c, then f has a local maximum at c;
3) if f′ doesn't change sign at c (that is, f′ is positive or negative on both sides), then f has no local extremum at c.

Exemple: Find the critical points of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

$$f'(x) = \frac{4}{3} \cdot x^{1/3} - \frac{4}{3} \cdot x^{-2/3} = \frac{4}{3} x^{-2/3} \cdot (x-1) = \frac{4(x-1)}{3 \cdot x^{2/3}}.$$

$f'(x) = 0 \Rightarrow x = 1$ and $f'(x)$ is undefined at $x = 0$.

Endpoints: There are none.

		0		1	
f'	$-$		$-$	0	$+$
f	$\searrow$		$\searrow$	local min (global) (absolute)	$\nearrow$

$$f'(-1) = \frac{4 \cdot (-2)}{3 \cdot (+1)} < 0$$

$$f(1) = 1 \cdot (-3) = \underline{\underline{-3}}$$

Exomple: Find the critical points of

$$f(x) = (x^2 - 3) \cdot e^x.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

$$f'(x) = 2x \cdot e^x + (x^2-3) \cdot e^x = \underbrace{(x^2+2x-3)}_{=0} \cdot \underbrace{e^x}_{\neq 0} = 0$$

$$x^2 + 2x - 3 = 0 \Rightarrow (x+3) \cdot (x-1) = 0 \Rightarrow x = -3, \quad x = 1.$$

$x \quad +3$

$x \quad -1$

		-3		1	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	local (absolute) max	$\searrow$	local (absolute) min	$\nearrow$

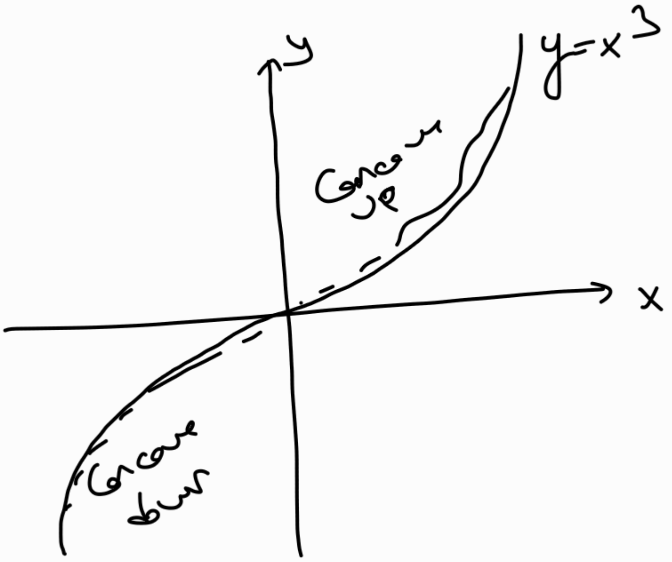
Increasing: $(-\infty,-3)\cup(1,\infty)$
Decreasing: $(-3,1)$
$f(-3)=0$ (max)
$f(1)=-2.e$ (min)

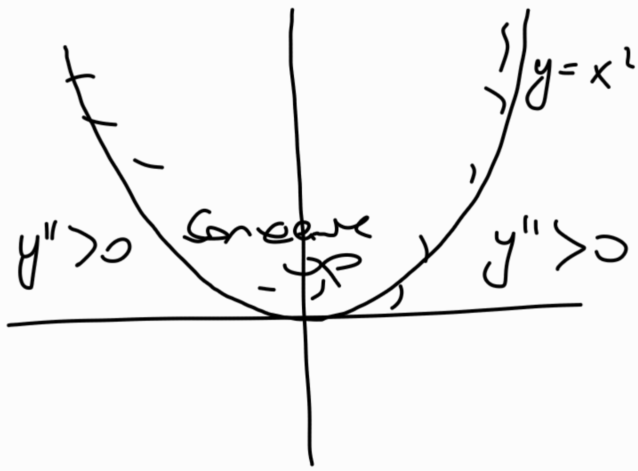
Concavity and Curve Sketching

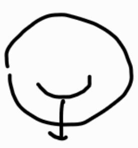
<u>Definition</u>: The graph of a differentiable function $y=f(x)$ is

(a) concave up on an open interval I if f' is increasing on I;

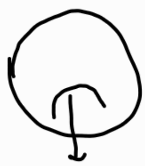
(b) concave down on an open interval I if f' is decreasing on I.







concave up



concave down

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on I.

1) If $f'' > 0$ on I, the graph of f over I is concave up.
2) If $f'' < 0$ on I, the graph of f over I is concave down.

Example:

(a) The curve $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow y'' = 6x$

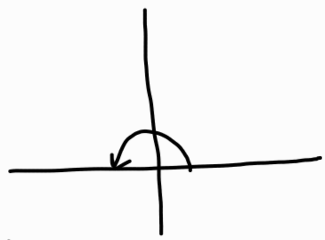
Concave down: $(-\infty, 0)$

Concave up: $(0, \infty)$

(b) The curve $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2 > 0$

Concave up: $(-\infty, \infty)$

Example | Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.



$y' = \cos x \Rightarrow y'' = -\sin x$

$(0, \pi)$, $y'' = -\sin x < 0 \Rightarrow$ concave down

$(\pi, 2\pi)$, $y'' = -\sin x > 0 \Rightarrow$ concave up.

Points of Inflection

Definition: A point $(c, f(c))$ where graph of a function has a tangent line and where the concavity changes is a point of inflection.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Example: Determine the concavity and find the inflection points of the function $f(x) = x^3 - 3x^2 + 2$.

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6 = 0 \Rightarrow$ $\underline{x = 1 \text{ point of inflection.}}$

$(-\infty, 1)$: concave down

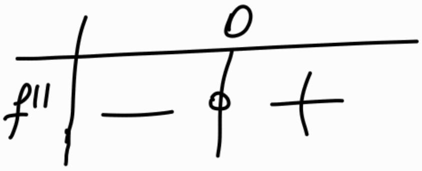
$(1, \infty)$: concave up.

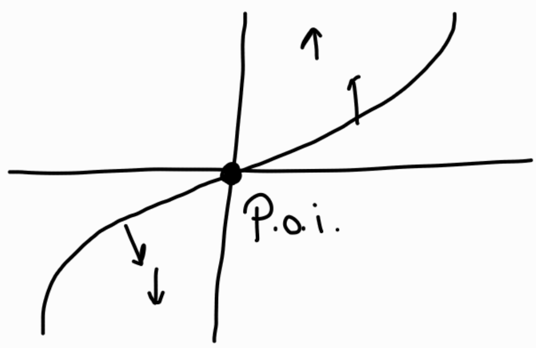
Example: $f(x) = x^{5/3}$

$f'(x) = \frac{5}{3} \cdot x^{2/3}$, $f''(x) = \frac{10}{9} \cdot x^{-1/3} = \frac{10}{9 \cdot x^{1/3}}$

at $x = 0$, f'' is undefined.

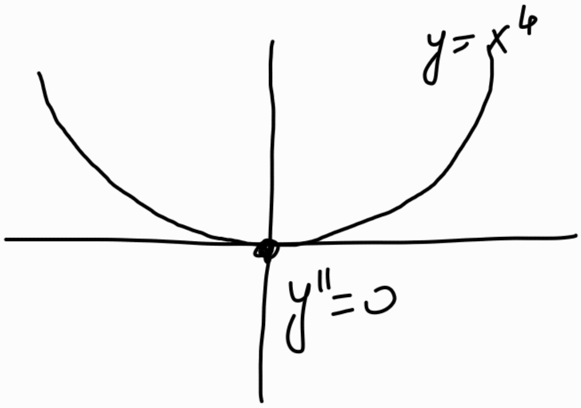
$x = 0$ is point of inflection





Example: $f(x) = x^4$

$f'(x) = 4x^3$. $f''(x) = 12x^2$ $x=0$ is a root (double)



$x=0$ is NOT p.o.i.

Theorem: (Second Derivative Test for Local Extrema)

Suppose f'' is continuous on an open interval that contains $x=c$.

1) If $f'(c)=0$ and $f''(c)<0$, then f has a local max. at $x=c$.

2) If $f'(c)=0$ and $f''(c)>0$, then f has a local min. at $x=c$.

3) If $f'(c)=0$ and $f''(c)=0$, then the test fails. The function f may have a local max, local min, or neither.

Horizontal Asymptotes

Definition: A line $y=b$ is a horizontal asymptote of the graph of a function $y=f(x)$ if either

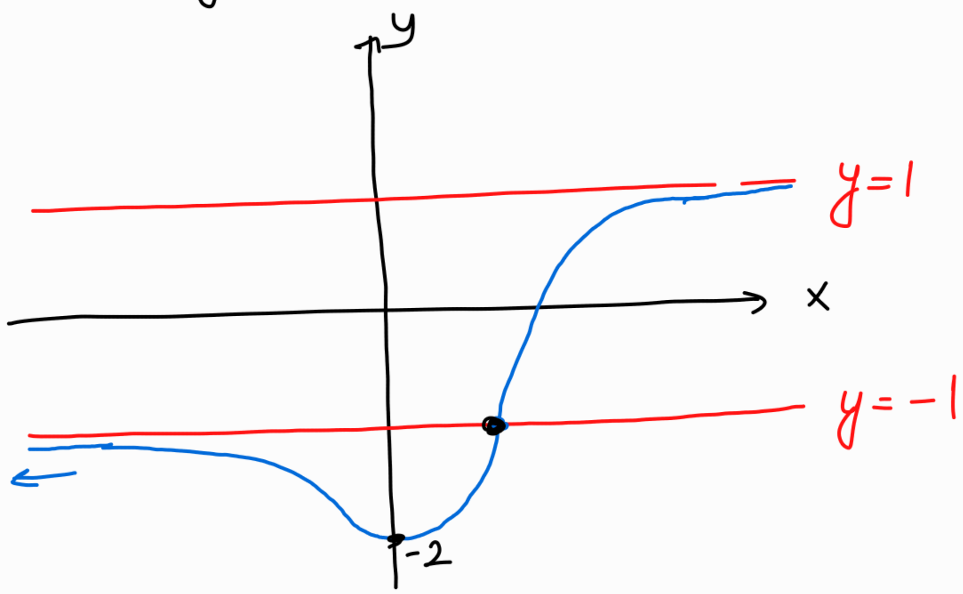
$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Example: Find the horizontal asymptotes of the graph of $f(x)=\dfrac{x^3-2}{|x|^3+1}$.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^3-2}{x^3+1}=1$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^3-2}{-x^3+1}=-1$$

$y=1$ and $y=-1$ are the horizontal asymptotes.



Example: 1) $f(x)=\arctan x$ 2) $g(x)=e^x$

1) $\lim\limits_{x\to\infty}\arctan x=\dfrac{\pi}{2}$ $\lim\limits_{x\to-\infty}\arctan x=-\dfrac{\pi}{2}$

2) $\lim\limits_{x\to\infty} e^x=\underline{\underline{\infty}}$ $\lim\limits_{x\to-\infty} e^x=0 \Rightarrow y=0$ is horizontal asymptote for e^x.

$\arctan\infty=a$

$\tan a=\infty \Rightarrow \dfrac{\sin a}{\cos a}=\infty$, $\cos a=0 \Rightarrow a=\dfrac{\pi}{2}$

$-\infty \Rightarrow a=-\dfrac{\pi}{2}$

Oblique Asymptotes

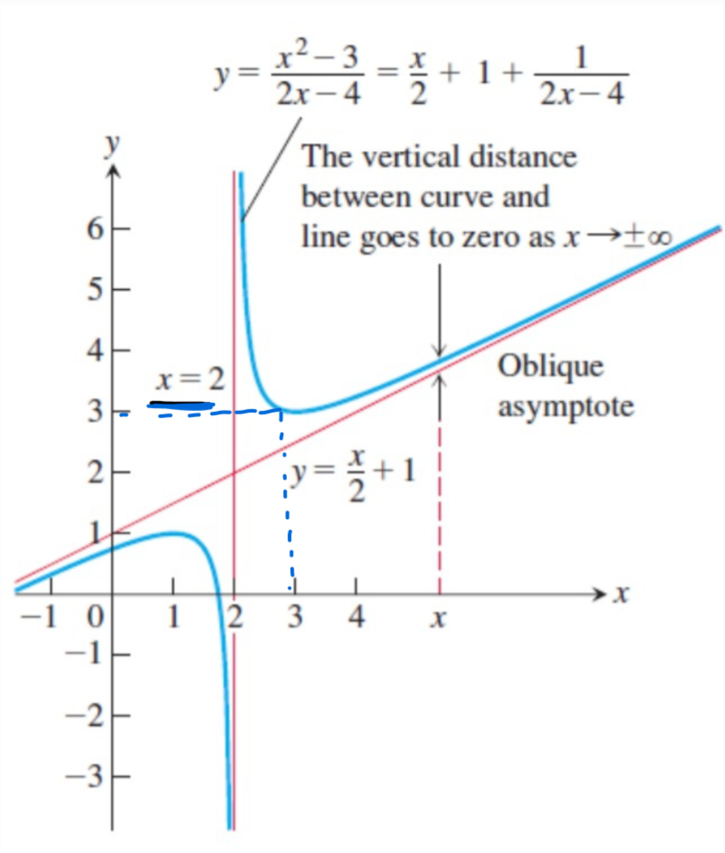
If the degree of the nominator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asympetete.

Example: Find the oblique asymptote of the graph of $f(x) = \dfrac{x^2-3}{2x-4}$

$$\begin{array}{l|l} x^2 \qquad -3 & 2x-4 \\ \hline \underline{-\;x^2-2x} & \underbrace{\tfrac{1}{2}x+1} \\ \quad 2x-3 & \\ \underline{-\;2x-4} & \\ \qquad 1 & \end{array}$$

$$f(x) = \underbrace{\left(\frac{x}{2}+1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x-4}\right)}_{\text{remainder}}$$

$g(x)$ is oblique asymptote.



$$\frac{x^2-3}{2x-4} = 3$$

$$x^2-3 = 6x-12$$

$$x^2-6x+9=0$$

$$(x-3)^2=0$$

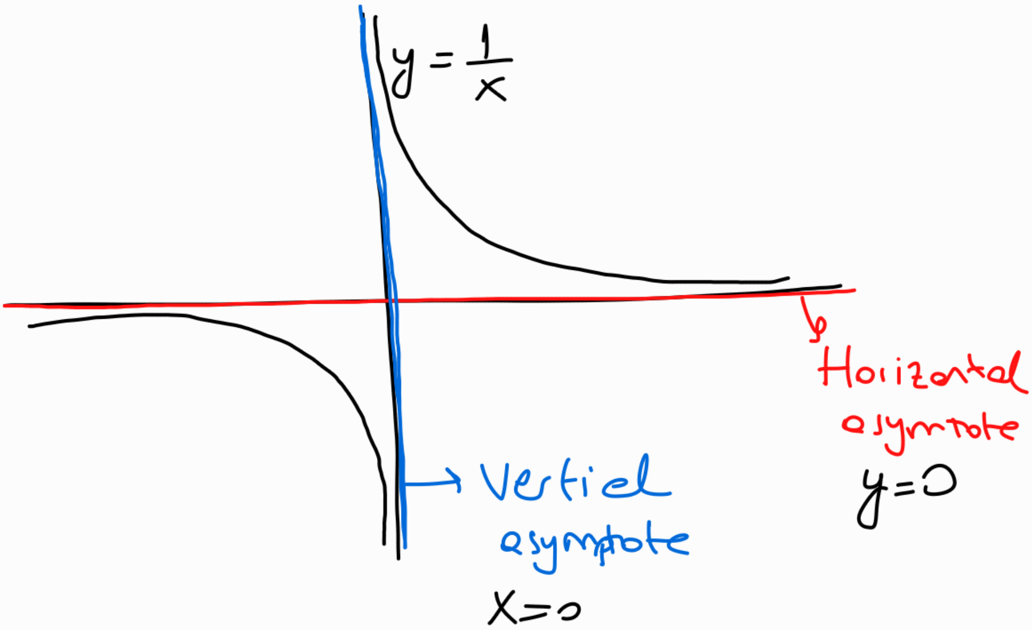
$$\Downarrow$$

$$3$$

Vertical Asymptotes

Definition: A line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either

$$\lim_{x\to a^+} f(x) = \mp\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \mp\infty$$

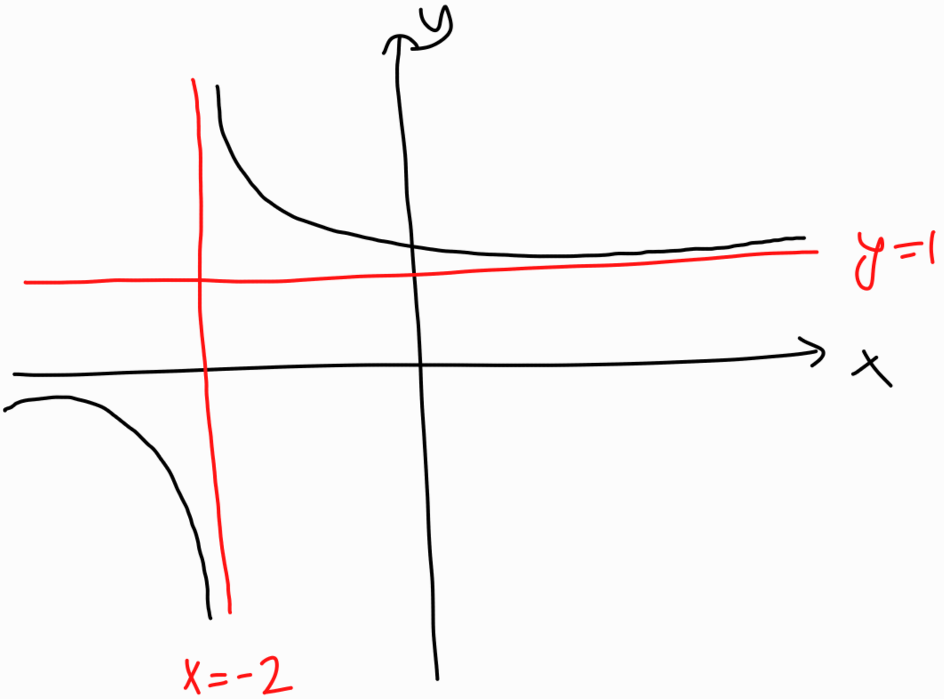


Ex.: Find the horizontal and vertical asymptotes of the curve $y=\dfrac{x+3}{x+2}$

Vertical $x=-2$

Horizontal: $\displaystyle\lim_{x\to\infty} \frac{x+3}{x+2} = 1 \qquad \lim_{x\to-\infty} \frac{x+3}{x+2} = 1$

$y=1$: H.A.



Procedure for Graphing $y=f(x)$

1) Identify the domain of f and any symmetries the curve may have.
2) Find y' and y''.
3) Find the critical points of f, if any, and identify the functions behavior at each one.
4) Find where the curve is increasing-decreasing
5) Find the points of inflection, determine the concavity of the curve.
6) Identify any asymptotes that may exist.
7) Plot.

Ex.: Sketch the grap of $f(x) = x^4 - 4x^3$.

1) Domain of f: $\mathbb{R} = (-\infty, \infty)$

2) $f'(x) = 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0$

$$\Downarrow \qquad \Downarrow$$
$$x = 0 \qquad x = 3$$

3) $f''(x) = 12x^2 - 24x = 0 \Rightarrow 12x(x-2) = 0$

$$\Downarrow \qquad \Downarrow$$
$$x = 0 \qquad x = 2$$

4) No slant (oblique) and vertical asymptotes.

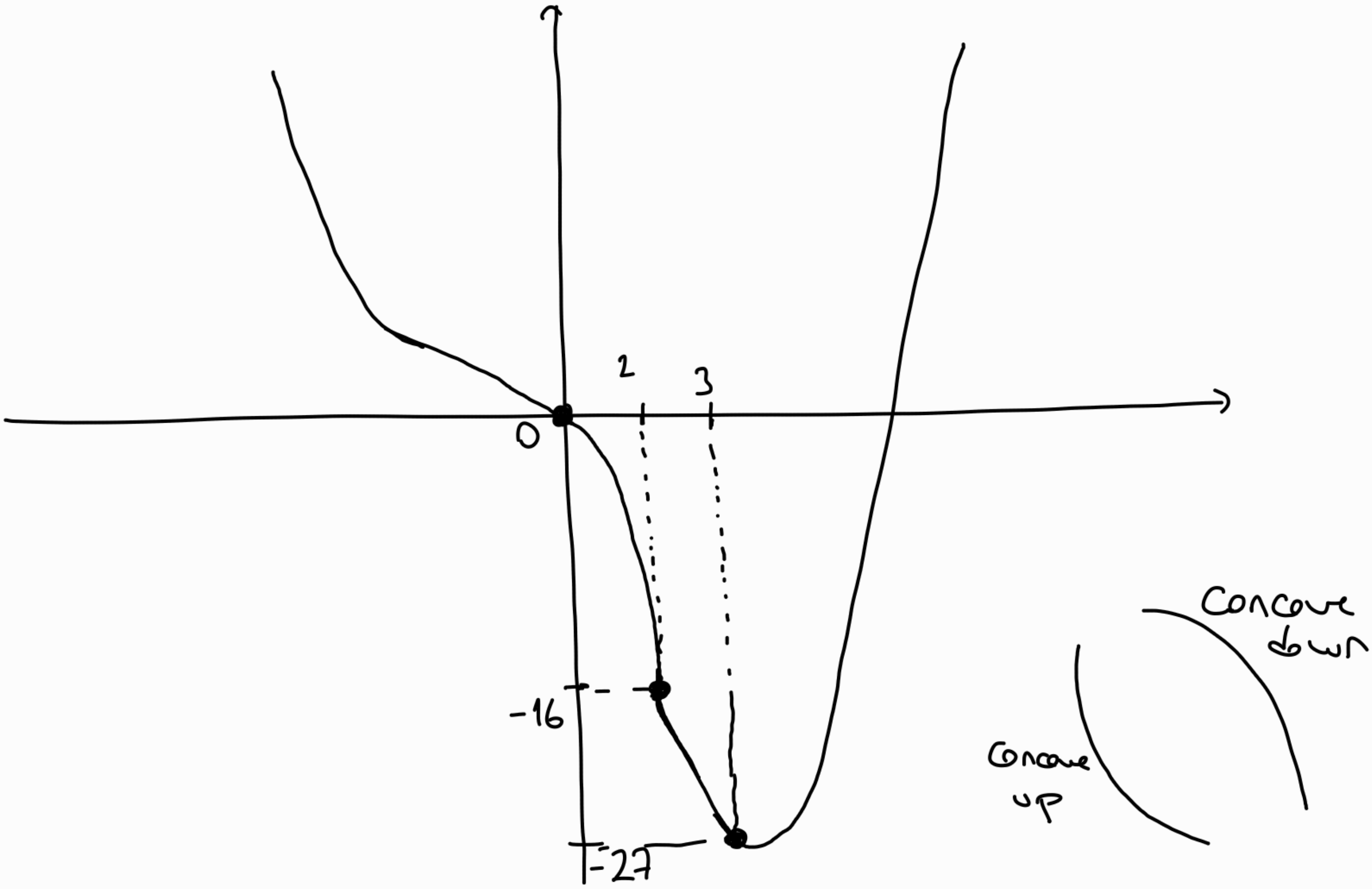
$\lim_{x \to \mp\infty} f(x) = \mp\infty$; No horizontal asymptotes.

5) $f(0) = 0$, $f(3) = -27$, $f(2) = -16$

$x = 0 \rightarrow y = 0$

	0	2	3	
f'	$-$	$-$	$-$	$+$
f''	$+$	$-$	$+$	$+$
f	$\searrow$ $\cup$	$\searrow$ $\cap$	$\searrow$ $\cup$	$\nearrow$ $\cup$

$x=0$ point of inf. $\qquad$ $x=2$ point of inf. $\qquad$ $x=3$ local min. $\qquad f(3) = -27$


2
3
0
-16
-27
Concave down
Concave up