Indefinite Integrals and the Substitution Method

Recall the chain rule for $F(g(x))$:

$$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$$

If F is an antiderivative of f, then

$$\frac{d}{dx} F(g(x)) = f(g(x)) \cdot g'(x)$$

Take integral for both sides

$$\int f(g(x)) \cdot g'(x) dx = \underbrace{\int}\left(\underbrace{\frac{d}{dx}} F(g(x))\right) dx$$

$$= F(g(x)) + c$$

$$u = g(x) \implies \quad = F(u) + c$$

$$= \int F'(u)\, du$$

$$= \int f(u)\, du$$

Theorem: If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u) du.$$

1) Substitute $u = g(x)$, $du = g'(x) dx$ to obtain $\int f(u) du$
2) Integrate w.r.t. u.
3) Replace $u = g(x)$.

Ex.: $\int x^4 \sqrt[3]{3-5x^5}\,dx$

$u = 3-5x^5 \Rightarrow du = -25x^4 dx \Rightarrow \underbrace{-\frac{1}{25}}\,du = x^4 dx$

$\int -\frac{1}{25}\cdot\sqrt[3]{u}\,du = -\frac{1}{25}\int u^{\frac{1}{3}}du = -\frac{1}{25}\,\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = -\frac{3}{100}u^{\frac{4}{3}} + C$

$= -\frac{3}{100}(3-5x^5)^{\frac{4}{3}} + C.$

Ex.: $\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

$\boxed{\sin^2 x = \frac{1-\cos(2x)}{2}}$, $\boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$

Ex.: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int -\frac{du}{u} = \underbrace{-\ln|u| + C}$

$\frac{f'(x)}{f(x)}$

$u = \cos x$

$du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C$

$|\cos x|^{-1}$

Ex.: $\int_{x=1}^{x=e} \frac{\ln x}{x}dx$

$\ln x = u$

$\frac{1}{x}dx = du$

$x = e \Rightarrow \ln e = u \Rightarrow u = 1$

$x = 1 \Rightarrow \ln 1 = u \Rightarrow u = 0$

$= \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

Ex.: $\int \cos^3 x\,dx = \int \underbrace{\cos^2 x}_{f(\sin x)} \cdot \underline{\cos x\,dx} = \int (1-\sin^2 x)\cdot \cos x\,dx$

$u = \sin x \Rightarrow du = \cos x\,dx$

$\int (1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$

Ex.: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \qquad u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

$\int e^u \cdot 2\,du = 2\cdot e^u + C = 2\cdot e^{\sqrt{x}} + C$

Ex.: $\int \tan^3 x \cdot \sec x\,dx = \int \tan^2 x \cdot \underline{\tan x \cdot \sec x\,dx}$

$u = \sec x \Rightarrow du = \tan x \cdot \sec x\,dx \qquad (\tan^2 x = \sec^2 x - 1)$

$\int (u^2-1)\cdot du = \frac{u^3}{3} - u + C = \frac{\sec^3 x}{3} - \sec x + C$

Ex.: $\int_{-2}^{5} |x-3|\,dx = \int_{-2}^{3} (3-x)\,dx + \int_{3}^{5} (x-3)\,dx = \left[3x - \frac{x^2}{2}\right]\Big|_{-2}^{3} + \left[\frac{x^2}{2} - 3x\right]\Big|_{3}^{5} = \frac{29}{2}$

$\Downarrow$ $x=3 \in [-2,5]$ $\qquad -(x-3)$

Ex.: $\int \frac{2x\,dx}{\sqrt[3]{x^2+1}}$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$\int \frac{du}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}}\,du = \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{3}{2}\cdot (x^2+1)^{\frac{2}{3}} + C$

$\underline{u^3 = x^2+1} \Rightarrow 3u^2\,du = 2x\,dx$

$\int \frac{3u^{\cancel{2}}\,du}{\cancel{u}} = 3\cdot \frac{u^2}{2} + C = \frac{3}{2}\cdot (x^2+1)^{\frac{2}{3}} + C$

Integration by Parts (Partial Integration)

$$d(u \cdot v) = u \cdot dv + v \cdot du$$

Integrate both sides

$$uv = \int d(uv) = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u(x) \cdot v'(x)\,dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x)\,dx$$

Ex.: $\int \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\Rightarrow \int \ln x\,dx = \ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx = x \cdot \ln x - x + C$$

Which to call u?

Logarithmic

Inverse trigonometric

Algebraic

Trigonometric

Exponential

Ex.: $\int \arctan x\,dx$

$u = \arctan x \qquad dx = dv$

$du = \frac{1}{1+x^2}dx \qquad x = v$

$$\int \arctan x\,dx = x.\arctan x - \frac{1}{2}\underbrace{\int \frac{2x}{1+x^2}\,dx}$$

$\frac{f'(x)}{f(x)}$ $\quad t = 1+x^2$, $\quad dt = 2x\,dx$

$$= x.\arctan x - \frac{1}{2}\ln(1+x^2) + c$$

Ex.: $I = \int \sec^3 x\,dx = \int \underbrace{\sec x}_{u} . \underbrace{\sec^2 x\,dx}_{dv}$

$\sec x = u$ $\qquad \sec^2 x\,dx = dv$

$\sec x \tan x\,dx = du$ $\qquad \tan x = v$

$$I = \sec x.\tan x - \int \underbrace{\tan x.\sec x.\tan x}_{\sec x.\tan^2 x}\,dx \qquad (\tan^2 x = \underline{\sec^2 x - 1})$$

$\sec x = t \Rightarrow \sec x.\tan x\,dx = dt$ X

$\int \tan x\,dt$

$$I = \sec x.\tan x - \underbrace{\int \sec^3 x\,dx}_{I} + \int \sec x\,dx$$

$\frac{f'(x)}{f(x)}$

$$2I = \sec x.\tan x + \int \sec x\,dx$$

$$\int \sec x\,dx = \int \sec x.\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\,dx$$

$\tan x + \sec x = t \Rightarrow (\sec^2 x + \sec x \tan x)\,dx = dt$

$$\int \sec x\,dx = \int \frac{dt}{t} = \ln|t| + c = \ln|\sec x + \tan x| + c$$

$$I = \frac{\sec x.\tan x + \ln|\sec x + \tan x|}{2} + c$$

Ex.: $\int_0^{1/2} \frac{x\cdot e^{2x}}{(1+2x)^2}\,dx$ (Homework, result: $\frac{1}{8}e - \frac{1}{4}$)

$x\cdot e^{2x} = u \qquad \frac{dx}{(1+2x)^2} = dv \qquad (1+2x)^{-2}$

Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ $(\cos \geq 0)$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\cdot\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $(\cos > 0)$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\cdot\sec\theta,\ 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Ex.: $\int \frac{dx}{\sqrt{4+x^2}}$

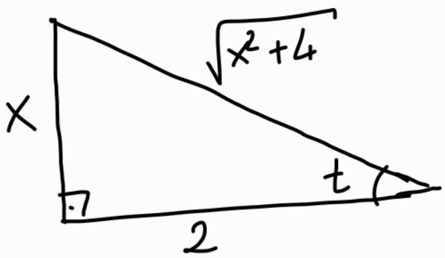
$x = 2\tan t \Rightarrow dx = 2\cdot\sec^2 t\,dt \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$\int \frac{\cancel{2}\sec^2 t\,dt}{\underbrace{\sqrt{4+4\tan^2 t}}_{\cancel{\sqrt{4}}\underbrace{\sqrt{(1+\tan^2 t)}}_{\sqrt{\sec^2 t}}}} = \int \frac{\sec^2 t\,dt}{|\sec t|} = \int \frac{\sec^{\cancel{2}} t}{\cancel{\sec t}}\,dt = \ln|\sec t + \tan t| + c$$

$\int \sec t\,dt = \ln|\sec t + \tan t|$

$x = 2\tan t$

$\tan t = \frac{x}{2}$



$\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c$

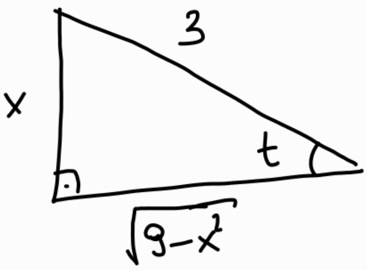
Ex.: $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

$x = 3\cdot\sin t \Rightarrow dx = 3\cdot\cos t\,dt \qquad \underline{(\cos > 0)}$

$9 - x^2 \Rightarrow 9 - 9\sin^2 t \Rightarrow 3^2\cdot(\underbrace{1-\sin^2 t}_{\cos^2 t})$

$$\int \frac{9\cdot\sin^2 t\cdot\cancel{3}\cdot\cancel{\cos} t\,dt}{\cancel{3}\cdot\cancel{\cos} t} = 9\cdot\int \frac{1-\cos(2t)}{2}\,dt = \frac{9}{2}\left(t - \frac{\sin(2t)}{2}\right) + c$$

$\sin t = \frac{x}{3} \Rightarrow \underbrace{\arcsin(\sin t)}_{t} = \arcsin\left(\frac{x}{3}\right)$



$\frac{\sin(2t)}{2} = \sin t\cdot\cos t$

$$I = \frac{9}{2}\left(\arcsin\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c$$

Ex.: $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x^3\sqrt{x^2-1}}$ $\qquad x = \sec t \Rightarrow dx = \sec t\cdot\tan t\,dt$

$x = 2 \Rightarrow t = \frac{\pi}{3} \qquad / \; x = \sqrt{2} \Rightarrow t = \frac{\pi}{4}$

$$\int_{\pi/4}^{\pi/3} \frac{\cancel{\sec} t\cdot\cancel{\tan t}\,dt}{\sec^{\cancel{3}\,2} t\cdot\cancel{\tan t}} = \int_{\pi/4}^{\pi/3} \cos^2 t\,dt = \int_{\pi/4}^{\pi/3} \frac{1+\cos(2t)}{2}\,dt = \frac{t}{2} + \frac{\sin(2t)}{4}\Bigg|_{\pi/4}^{\pi/3}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{8} - \left(\frac{\pi}{8} + \frac{1}{4}\right) = \underline{\frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}}$$

Integration of Rational Functions by Partial Functions

Ex.: $$I = \int \frac{x^2}{x-4}\,dx \underset{\substack{\downarrow \\ \text{polynomial} \\ \text{division}}}{=} \int \left(x+4+\frac{16}{x-4} \right) dx = \frac{x^2}{2} + 4x + 16\ln|x-4| + C$$

Case I: The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1 x + b_1).(a_2 x + b_2) \ldots (a_k x + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

Ex.: $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

$$2x^3+3x^2-2x = x\,(\underset{\substack{2x \\ x}}{2x^2} + 3x - \underset{\substack{-1 \\ 2}}{2}) = x.(2x-1).(x+2)$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \underset{(2x^2+3x-2)}{\frac{A}{x}} + \underset{(x^2+2x)}{\frac{B}{2x-1}} + \underset{(2x^2-x)}{\frac{C}{x+2}}$$

$$x^2+2x-1 = (2A+B+2C)x^2 + (3A+2B-C)x - 2A$$

$$\left.\begin{aligned} 2A+B+2C &= 1 \\ 3A+2B-C &= 2 \\ -2A &= -1 \end{aligned}\right\} \quad A=\frac{1}{2},\ B=\frac{1}{5},\ C=-\frac{1}{10}$$

$$\int\left(\frac{1}{2}\cdot\frac{1}{x}+\frac{1}{5}\cdot\frac{1}{2x-1}-\frac{1}{10}\cdot\frac{1}{x+2}\right)dx$$

$$\left.\begin{aligned} &= \underset{(5)}{\frac{1}{2}}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\cdot\ln|x+2| + C \\ &= \frac{1}{10}\left(\ln|x|^5 + \ln|2x-1| - \ln|x+2|\right) + C \\ &= \frac{1}{10}\ln\left|\frac{x^5\cdot(2x-1)}{x+2}\right| + C \end{aligned}\right\} =$$

<u>Case II</u>: Suppose the first linear factor (a_1x+b_1) is repeated r times; that is $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\frac{A_1}{a_1x+b_1}$ we would use

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

<u>Ex. 1</u> $$\frac{x^3-x+1}{x^2(x-1)^3} = \underbrace{\frac{A}{x}+\frac{B}{x^2}} + \frac{C}{(x-1)^1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Ex.: $\int \frac{6x+7}{(x+2)^2}\,dx$

$$\frac{6x+7}{(x+2)^2} = \underset{(x+2)}{\frac{A}{x+2}} + \frac{B}{(x+2)^2} \Rightarrow A(x+2)+B = 6x+7$$

$A=6$, $2A+B=7$

$B=-5$

$$\int\left(\frac{6}{x+2} - \underbrace{\frac{5}{(x+2)^2}}_{5\cdot(x+2)^{-2}}\right)dx = 6\cdot\ln|x+2| - 5\cdot\frac{(x+2)^{-1}}{-1} + c$$

$$= 6\ln|x+2| + \frac{5}{(x+2)} + c$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$ then the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Ex.: $\int \frac{2x^2-x+4}{x^3+4x}\,dx$ $\qquad x^3+4x = x\cdot(x^2+4)$

$$\frac{2x^2-x+4}{x^3+4x} = \underset{(x^2+4)}{\frac{A}{x}} + \underset{(x)}{\frac{Bx+C}{x^2+4}}$$

$A+B=2$ $\qquad 4A=4$

$C=-1$

$A=B=1$, $C=-1$

$$\int\left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)dx = \ln|x| + \frac{1}{2}\cdot\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$(x^2+4)' = 2x \qquad \frac{x-1}{x^2+4} = \frac{x}{x^2+4} - \frac{1}{x^2+4}$$

$a^2=4 \quad a=2$

<u>Case IV</u>: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

<u>Ex.</u>: $\dfrac{1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

$\quad (x^2+1)^2 \quad (x(x^2+1)) \quad (x)$

$A=1,\ B=-1,\ C=0,\ D=-1,\ E=0$

$$\int\frac{dx}{x(x^2+1)^2} = \int\left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx$$

$$= \underbrace{\ln|x| - \frac{1}{2}\ln|x^2+1|}_{\ln\left|\frac{x}{\sqrt{x^2+1}}\right|} + \frac{1}{2}\cdot\frac{1}{(x^2+1)} + C$$

$x^2+1=u$

$\cancel{2}x\,dx = \dfrac{du}{2}$

$$\int\frac{du}{2\cdot u^2} = \int\frac{1}{2}u^{-2}du$$