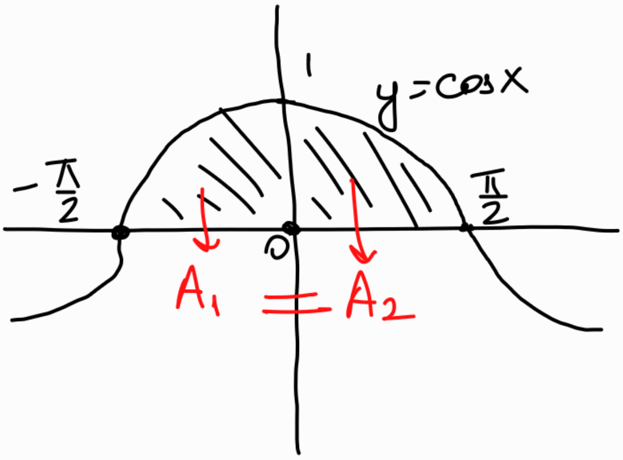
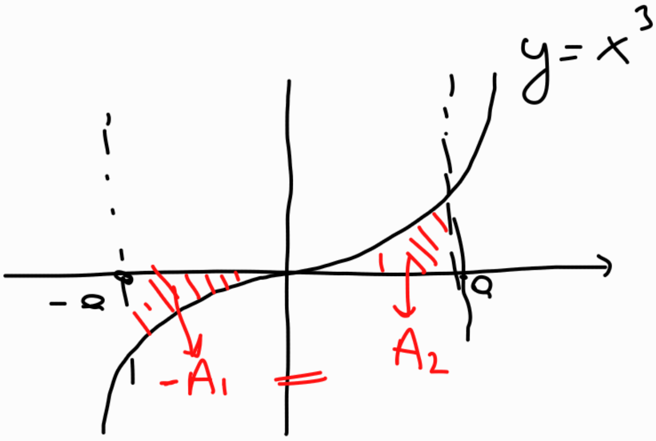
Theorem: Let f be continuous on the symmetric interval $[-a,a]$.

a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\cdot\int_{0}^{a} f(x)dx$.

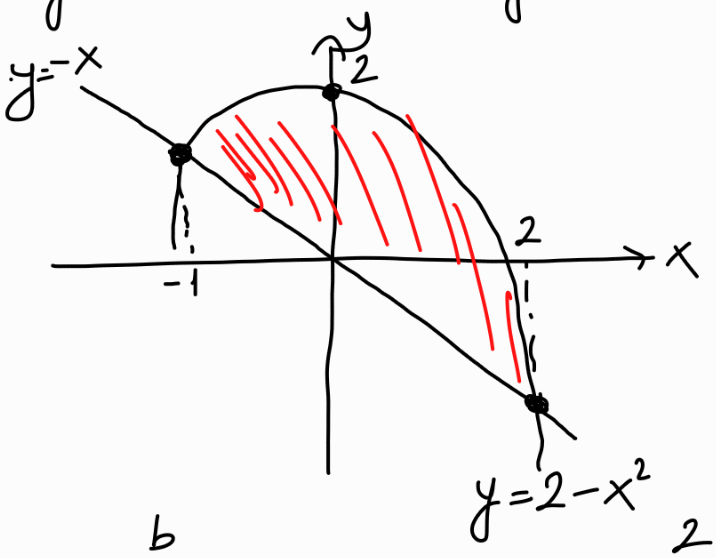
b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.





Area Between Curves

Example: Find the area of the region enclosed by the parabola $y=2-x^2$ and the line $y=-x$.



$2-x^2=-x$

$x^2-x-2=0 \Rightarrow (x-2)\cdot(x+1)=0$

$x \quad -2$

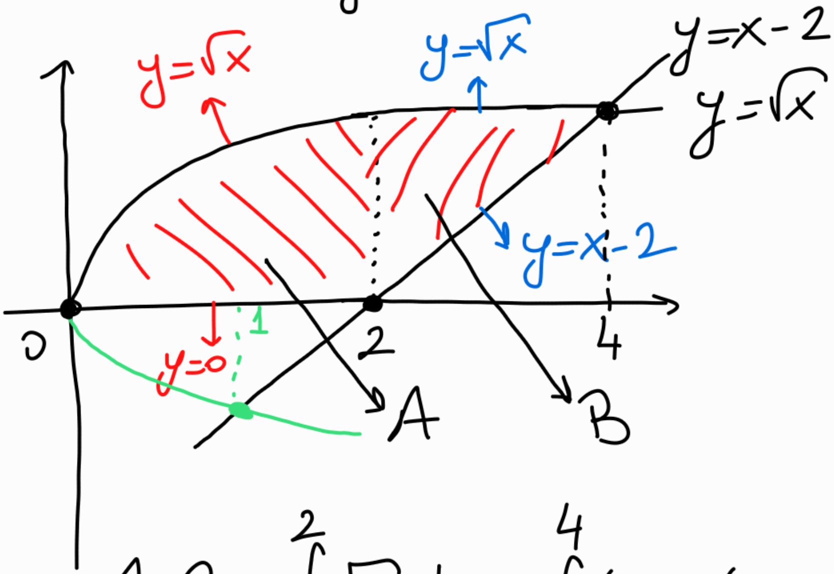
$x \quad +1$

$x=2 \qquad x=-1$

$$A=\int_a^b [\underbrace{f(x)}_{\text{above}}-\underbrace{g(x)}_{\text{below}}]dx = \int_{-1}^{2}[(2-x^2)-(-x)]dx = \left(-\frac{x^3}{3}+\frac{x^2}{2}+2x\right)\Bigg|_{-1}^{2}$$

$$=\frac{9}{2}$$

Example: Find the area of the region in the first quadrant that is bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$.



$$\sqrt{x} = x-2$$
$$x = x^2-4x+4$$
$$x^2-5x+4=0$$
$$x \quad -4$$
$$x \quad -1$$
$$x=1, \quad x=4$$

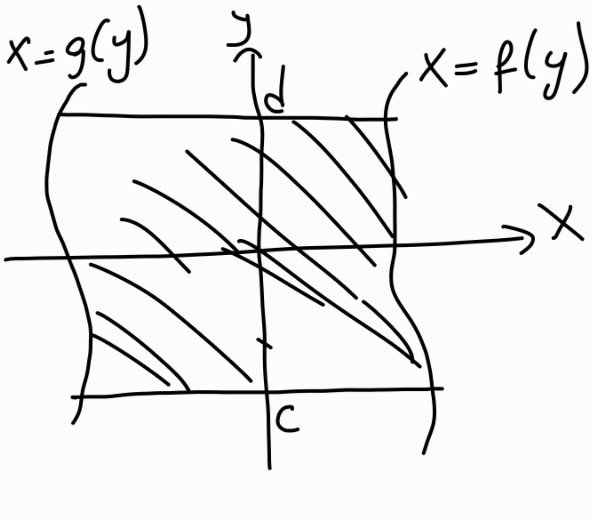
$$A+B = \underbrace{\int_0^2 \sqrt{x}\,dx}_{A} + \underbrace{\int_2^4 \left(\sqrt{x}-(x-2)\right)dx}_{B}$$

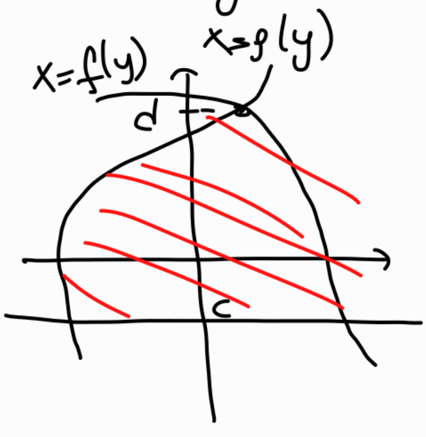
$$= \frac{2}{3}\cdot x^{3/2}\Big|_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

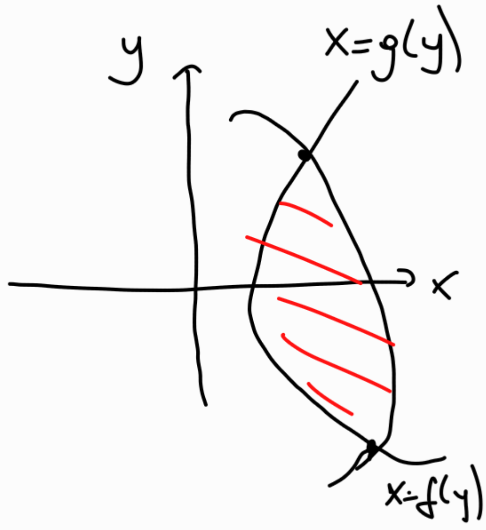
$$= \frac{2}{3}\cdot 2\sqrt{2} + \left(\frac{2}{3}\cdot 8 \ldots\right) = \frac{10}{3}$$

Integration with Respect to y

To find the areas of regions like these:



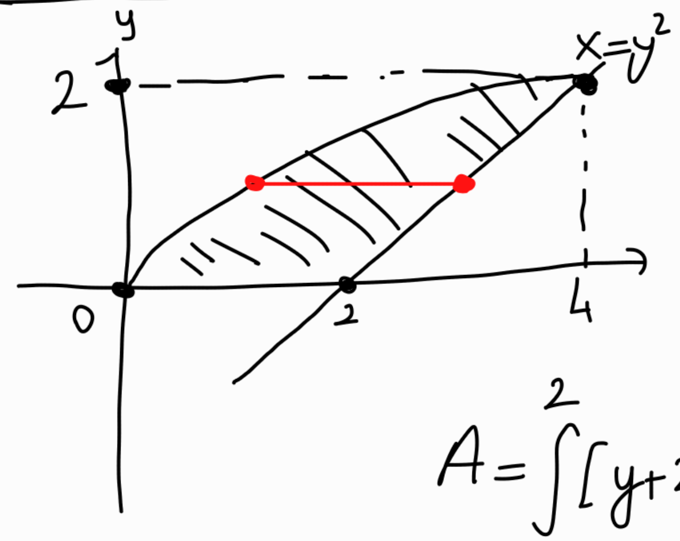




$$A = \int_c^d [f(y) - g(y)]\,dy$$

$f(y)$: right, $g(y)$: left

Example: Previous one.



$y = \sqrt{x} \Rightarrow x = y^2$

$y = x - 2 \Rightarrow x = y + 2$

$4 = y^2 \Rightarrow y = \mp 2$

$4 = y + 2 \Rightarrow y = 2$

$$A = \int_0^2 [y + 2 - y^2]\,dy = \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$= 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{10}{3}$$

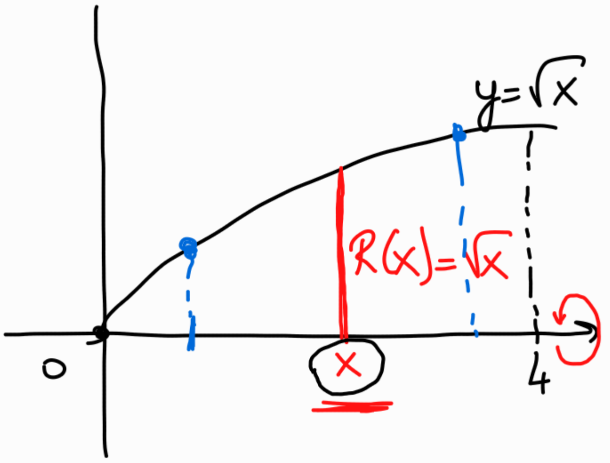
Solids of Revolution: The Disk Method

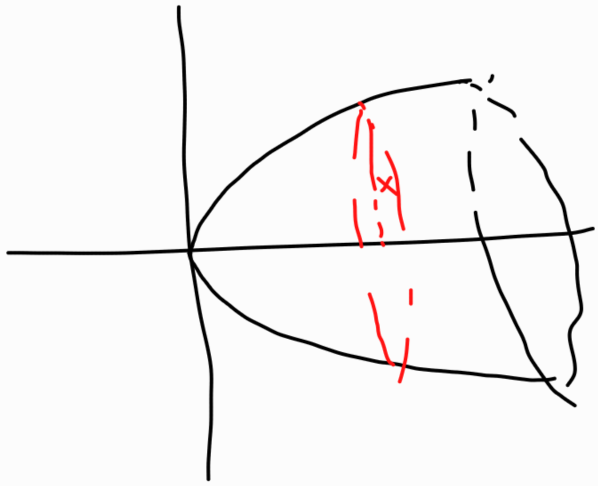
Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x) = \int_a^b \pi [R(x)]^2\,dx$$

$A(x)$: Area of the disk

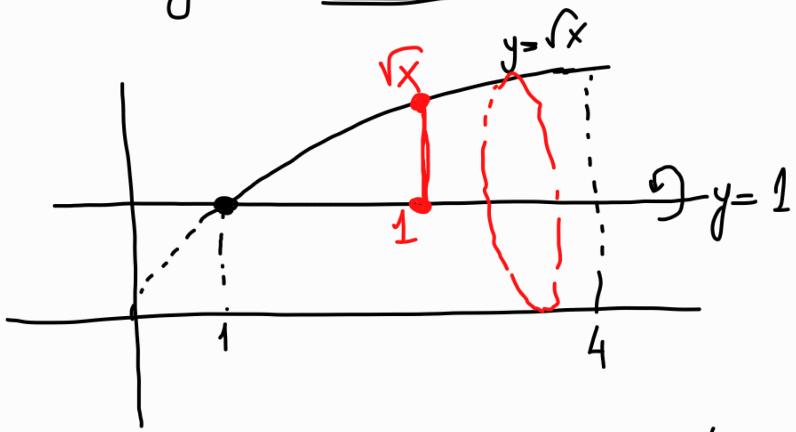
Example: The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.





$$V=\int_0^4 \pi \cdot [\sqrt{x}]^2 dx = \int_0^4 \pi x\, dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$$

<u>Example:</u> Find the volume of the solid generated by revolving region bounded by $y=\sqrt{x}$ and the lines $y=1$, $\underline{x=4}$ about the line $y=1$.



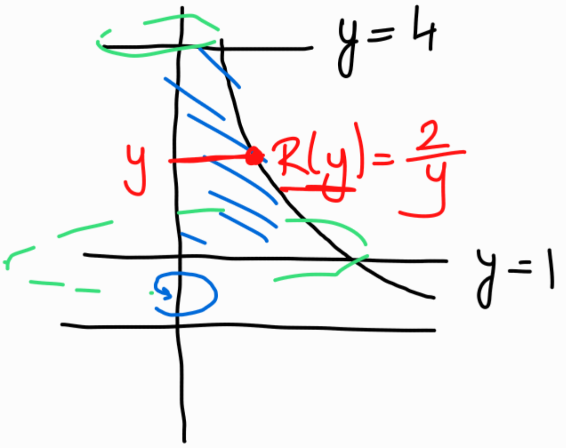
$$V=\int_1^4 \pi[\sqrt{x}-1]^2 dx = \pi \int_1^4 (x-2\sqrt{x}+1)\,dx = \pi\left[\frac{x^2}{2}-\frac{4}{3}x^{3/2}+x\right]_1^4$$

$$=\frac{7\pi}{6}$$

Volume by Disks for Rotation About the y-Axis

$$V=\int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2 dy$$

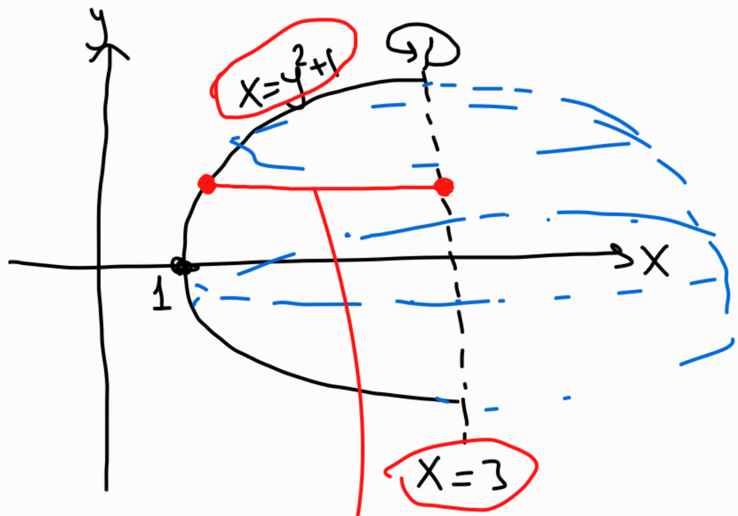
<u>Example</u>: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=\frac{2}{y}$, $1\le y\le 4$, about the <u>y-axis.</u>



$$V=\int_1^4 \pi\left(\frac{2}{y}\right)^2 dy=\int_1^4 \pi\,\frac{4}{y^2}\,dy$$

$$=4\pi\left[-\frac{1}{y}\right]_1^4=4\pi\underbrace{\left[-\frac{1}{4}+1\right]}_{\frac{3}{4}}=3\pi$$

<u>Example</u>: Find the volume of the solid generated by revolving the region between the parabola $x=y^2+1$ and the line $x=3$ about the line $x=3$



$$R(y)=3-(y^2+1)$$
$$=2-y^2$$

$$x=3\Rightarrow 3=y^2+1$$
$$y^2=2$$
$$y=\mp\sqrt{2}$$

$$V=\int_{-\sqrt{2}}^{\sqrt{2}}\pi\left(2-y^2\right)^2dy$$

$$=2\int_0^{\sqrt{2}}\pi\left(2-y^2\right)^2dy$$

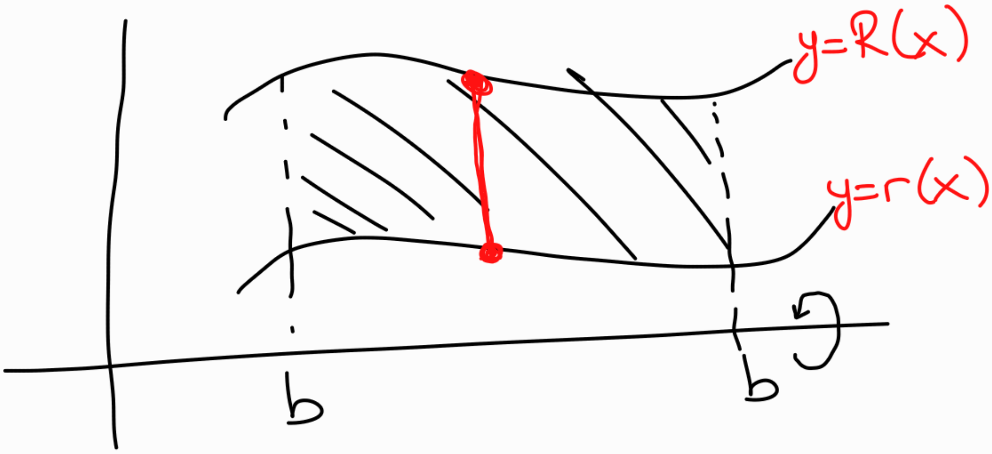
$$=2\pi\int_0^{\sqrt{2}}\left(4-4y^2+y^4\right)dy$$

$$=2\pi\left[4y-\frac{4}{3}y^3+\frac{y^5}{5}\right]_0^{\sqrt{2}}$$

$$=2\pi\left[4\sqrt{2}-\frac{8}{3}\sqrt{2}+\frac{4\sqrt{2}}{5}\right]$$

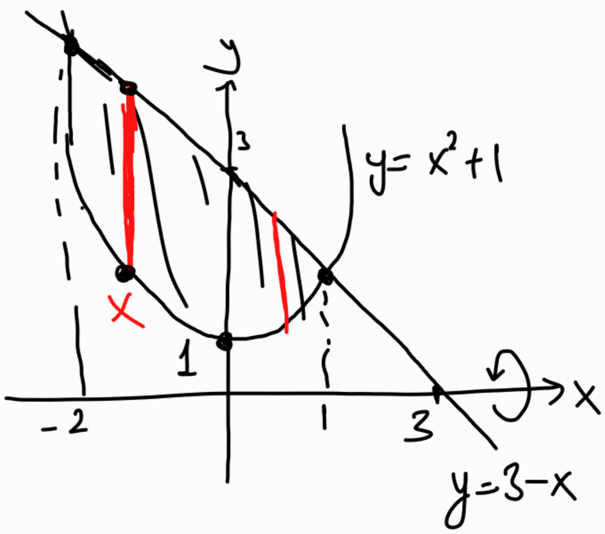
$$=2\sqrt{2}\pi\left[\frac{60-40+12}{15}\right]=\frac{64\sqrt{2}\,\pi}{15}$$

Volume by Washers for Rotation About the x-Axis



$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

Example: The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



$$x^2+1 = -x+3$$

$$x^2+x-2=0$$

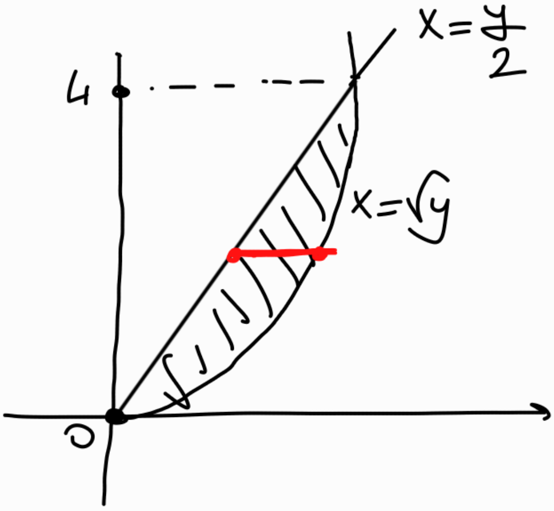
$$\begin{array}{ll} x & +2 \\ x & -1 \end{array} \Rightarrow x=+1 \quad x=-2$$

$$V = \pi \int_{-2}^{1} \left[(3-x)^2 - (x^2+1)^2\right] dx = \frac{117\pi}{5}$$

Example: The region bounded by the parabola $y=x^2$ and the line $y=2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

$y=x^2 \Rightarrow x=\sqrt{y}$

$y=2x \Rightarrow x=\frac{y}{2}$



$\sqrt{y}=\frac{y}{2}$

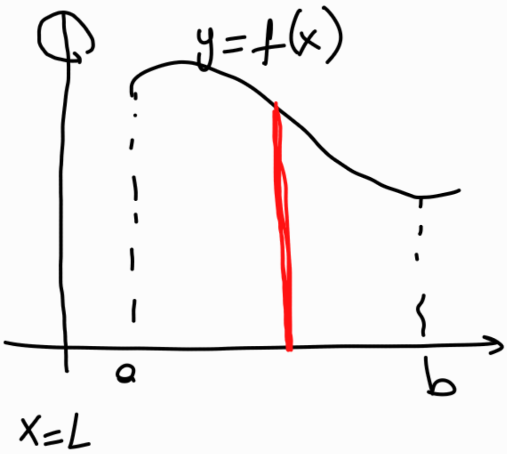
$y=\frac{y^2}{4}$

$y=0$
$y=4$

$$V=\int_0^4 \pi\left[(\sqrt{y})^2-\left(\frac{y}{2}\right)^2\right]dy$$

$$V=\pi\int_0^4\left[y-\frac{y^2}{4}\right]dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{12}\right]_0^4=\pi\left[8-\underbrace{\frac{64}{12}}_{\frac{16}{3}}\right]=\frac{8\pi}{3}$$

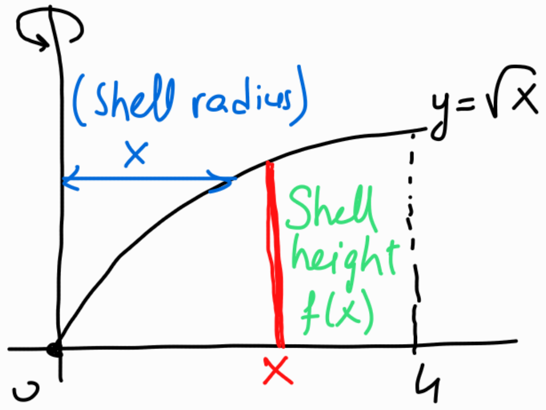
Volumes Using Cylindrical Shells



The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y=f(x)\geq 0$, $L\leq a\leq x\leq b$, about a vertical line $x=L$ is

$$V=\int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right)dx$$

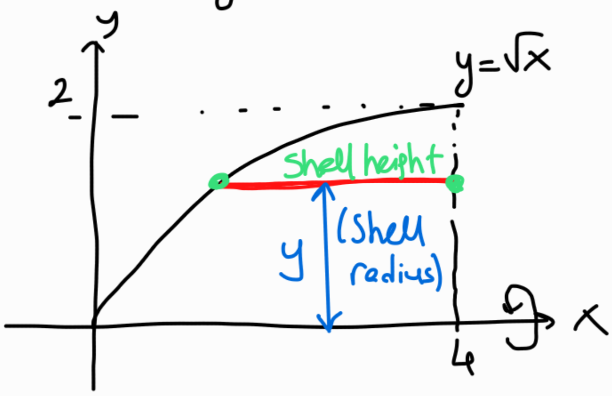
Example: The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$V = 2\pi \int_0^4 x \cdot \sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx$$

$$= 2\pi \cdot \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{4\pi}{5} \cdot 32 = \frac{128\pi}{5}$$

Example: The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.



$$V = \int_0^2 2\pi y (4-y^2)\, dy$$

$$= 2\pi \int_0^2 (4y - y^3)\, dy = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2$$

$$= 2\pi \left[8 - 4 \right] = 8\pi$$

Arc Length

Definition: If f' is continuous on $[a,b]$, then the length (arc length) of the curve $y=f(x)$ from the point $A=(a, f(a))$ to the point $B=(b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Find the length of the curve given by
$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1.$$

$$y' = 2\sqrt{2}\, x^{1/2} \Rightarrow (y')^2 = 8x$$

$$L = \int_0^1 \sqrt{1+8x}\, dx = \frac{2}{3}\cdot(1+8x)^{3/2}\cdot\frac{1}{8}\Big|_0^1 = \frac{2}{24}(27-1) = \frac{2\cdot 26}{24} = \frac{13}{6}$$

$$\sqrt{1+8x} \downarrow (1+8x)^{1/2}$$

Example: Find the length of the graph of
$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4.$$

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow [f'(x)]^2 = \frac{x^4}{16} - 2\cdot\frac{\cancel{x^2}}{4}\cdot\frac{1}{\cancel{x^2}} + \frac{1}{x^4} = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$1 + [f'(x)]^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\sqrt{1+[f'(x)]^2} = \sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2} = \left|\frac{x^2}{4}+\frac{1}{x^2}\right| = \frac{x^2}{4}+\frac{1}{x^2}$$

$$L = \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = 6$$

Formula for the Length of $x = g(y)$, $c \le y \le d$

If g' is continuous on $[c,d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Example: Find the length of the curve $y = (x/2)^{2/3}$

from $x=0$ to $x=2$.

$$\frac{dy}{dx} = \frac{2}{3}\cdot\left(\frac{x}{2}\right)^{-1/3}\cdot\frac{1}{2} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

$x \neq 0$ f' is not continuous

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y^{3/2} = \frac{x}{2} \Rightarrow x = 2\cdot y^{3/2}$$

$x=0 \Rightarrow y=0$ / $x=2 \Rightarrow y=1$

$$\frac{dx}{dy} = g'(y) = 3\cdot y^{1/2} \Rightarrow [g'(y)] = 9y$$

$$L = \int_0^1 \underbrace{\sqrt{1+9y}}_{(1+9y)^{1/2}}\,dy = (1+9y)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{9}\Big|_0^1 = \frac{2}{27}\left(10\sqrt{10}-1\right)$$

Areas of Surfaces of Revolution

Definition: If the function $f(x) \geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S = \int_a^b 2\pi \underbrace{y}_{f(x)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

Example: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis.

$$y' = \underset{\substack{\downarrow \\ \frac{1}{\sqrt{x}}}}{(x)^{-1/2}} \Rightarrow (y')^2 = \frac{1}{x} \Rightarrow 1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_1^2 2\cancel{\sqrt{x}} \cdot \frac{\sqrt{x+1}}{\cancel{\sqrt{x}}} = 4\pi (x+1)^{3/2} \cdot \frac{2}{3} \Big|_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

Definition: If the function $g(y) \geq 0$ is continuously differentiable on $[c,d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi \underset{\substack{\downarrow \\ g(y)}}{x} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi\, g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

Example: The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate a cone. Find its lateral surface area.

$$x' = -1 \Rightarrow (x')^2 = 1$$

$$S = 2\pi \int_0^1 (1-y)\sqrt{2}\, dy = 2\sqrt{2}\pi \cdot \left(y - \frac{y^2}{2}\right)\Big|_0^1 = 2\sqrt{2}\pi \cdot \frac{1}{2} = \sqrt{2}\pi$$