INTEGRATION TECHNIQUES 4

1) $\displaystyle\int \frac{x^3+x}{x-1}\,dx = ?$

$$
\begin{array}{l|l}
x^3+x & x-1 \\ \hline
-\;x^3-x^2 & x^2+x+2 \\ \hline
x^2+x & \\
-\;x^2-x & \\ \hline
2x & \\
-\;2x-2 & \\ \hline
2 &
\end{array}
$$

$$I=\int\left[x^2+x+2+\frac{2}{x-1}\right]dx$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+c.$$

2) $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = ?$

$$
\begin{array}{l|l}
x^4-2x^2+4x+1 & x^3-x^2-x+1 \\ \hline
-\;x^4-x^3-x^2+x & x+1 \\ \hline
x^3-x^2+3x+1 & \\
-\;x^3-x^2-x+1 & \\ \hline
4x &
\end{array}
$$

$$I=\int\left[x+1+\frac{4x}{x^3-x^2-x+1}\right]dx$$

$$x^3-x^2-x+1=x^2(x-1)-(x-1)=(x-1).\underbrace{(x^2-1)}_{(x-1)(x+1)}$$

$$\frac{4x}{x^3-x^2-x+1}=\frac{A}{\underset{(x^2-1)}{x-1}}+\frac{B}{\underset{(x+1)}{(x-1)^2}}+\frac{C}{\underset{((x-1)^2)}{x+1}}$$

$$Ax^2-A+Bx+B+Cx^2-2Cx+C=4x$$

$$\left.\begin{array}{l} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{array}\right\}\quad \begin{array}{l} A=1 \\ B=2 \\ C=-1 \end{array}$$

$$I=\int\left[x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right]dx=\frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+c$$

$$=\frac{x^2}{2}+x-\frac{2}{x-1}+\ln\left|\frac{x-1}{x+1}\right|+c.$$

③ $\displaystyle\int \frac{4x^2-3x+1}{4x^2-4x+3}\,dx = ?$

$$\frac{4x^2-3x+2+(-x+2)-(-x+2)}{4x^2-4x+3} = \frac{4x^2-4x+3+x-2}{4x^2-4x+3} = 1 + \frac{x-2}{4x^2-4x+3}$$

$$I = \int \left[1 + \frac{x-2}{\underbrace{4x^2-4x+3}_{\Delta<0}} \right] dx = x + \int \left(\frac{x-\frac{1}{2}}{\underbrace{4x^2-4x+3}_{\frac{1}{8}\ln|4x^2-4x+3|}} - \frac{-\frac{3}{2}}{(2x-1)^2+2} \right) dx$$

$2x-1 = \sqrt{2}\tan t$

$2\,dx = \sqrt{2}\sec^2 t\,dt$

$$I = x + \frac{1}{8}\ln|4x^2-4x+3| + \underbrace{\int \frac{\frac{3}{2}\cdot\frac{\sqrt{2}}{2}\sec^2 t\,dt}{2\tan^2 t+2}}_{J}$$

$$J = \frac{3\sqrt{2}}{8}\int \frac{\cancel{\sec^2 t}\,dt}{\cancel{\sec^2 t}} = \frac{3\sqrt{2}}{8}t + c = \frac{3\sqrt{2}}{8}\arctan\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

$$\Rightarrow I = x + \frac{1}{8}\ln|4x^2-4x+3| + \frac{3\sqrt{2}}{8}\arctan\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

④ $\displaystyle\int \frac{24\,dx}{(x^2-9)(x+1)} = ?$

$$\frac{24}{(x^2-9)(x+1)} = \frac{24}{(x-3)(x+3)(x+1)} = \frac{A}{\underset{(x+3)(x+1)}{x-3}} + \frac{B}{\underset{(x-3)(x+1)}{x+3}} + \frac{C}{\underset{(x^2-9)}{x+1}}$$

$$A(x+3)(x+1) + B(x-3)(x+1) + C(x^2-9) = 1$$

$x=3 \Rightarrow 24A = 24 \Rightarrow A = 1$

$x=-3 \Rightarrow -12B = 24 \Rightarrow B = -2$

$x=-1 \Rightarrow -8C = 24 \Rightarrow C = -3$

$$I = \int \left[\frac{1}{x-3} - \frac{2}{x+3} - \frac{3}{x+1} \right] dx = \ln|x-3| - 2\ln|x+3| - 3\ln|x+1| + c$$

$$= \ln\left| \frac{x-3}{(x+3)^2(x+1)^3} \right| + c.$$

(5) $\int \frac{x^4}{x^4-1}\,dx = ?$

$$\frac{x^4}{x^4-1} = \frac{x^4-1}{x^4-1} + \frac{1}{x^4-1} = 1 + \frac{1}{x^4-1} \qquad x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\frac{1}{x^4-1} = \underset{(x+1)(x^2+1)}{\frac{A}{x-1}} + \underset{(x-1)(x^2+1)}{\frac{B}{x+1}} + \underset{(x^2-1)}{\frac{Cx+D}{x^2+1}} \qquad x^3(A+B+C)+x^2(A-B+D) + x(A+B-C)+(A-B-D) = 1$$

$$\Rightarrow A = \frac{1}{4}, \quad B = \frac{-1}{4}, \quad C = 0, \quad D = \frac{-1}{2}$$

$$I = \int\left[1 + \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{2}\,\frac{1}{x^2+1}\right]dx = x + \frac{1}{4}\ln|x-1| - \frac{1}{4}|x+1| - \frac{1}{2}\arctan x + c$$

$$= x + \frac{1}{4}\cdot\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + c.$$

(6) $\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = ?$

$$\frac{x-1}{(x+2)(x+1)} = \frac{A}{\underset{x=-2}{x+2}} + \frac{B}{\underset{x=-1}{x+1}} \qquad A = \frac{x-1}{x+1}\bigg|_{x=-2} = 3 \qquad B = \frac{x-1}{x+2}\bigg|_{x=-1} = -2$$

$$I = \int_0^1\left[\frac{3}{x+2} - \frac{2}{x+1}\right]dx = 3\ln|x+2| - 2\ln|x+1|\Big|_0^1 = \ln\left|\frac{(x+2)^3}{(x+1)^2}\right|\Bigg|_0^1 = \ln\frac{27}{4} - \ln 8 = \ln\frac{27}{32}.$$

(7) $\int_1^2 \frac{4x^2-7x-12}{x(x+2)(x-3)}\,dx = ?$

$$\frac{4x^2-7x-12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$A = \frac{4x^2-7x-12}{(x+2)(x-3)}\bigg|_{x=0} = 2, \quad B = \frac{4x^2-7x-12}{x(x-3)}\bigg|_{x=-2} = \frac{9}{5}, \quad C = \frac{4x^2-7x-12}{x(x+2)}\bigg|_{x=3} = \frac{1}{5}$$

$$I = \int_1^2\left[\frac{2}{x} + \frac{9}{5}\cdot\frac{1}{x+2} + \frac{1}{5}\cdot\frac{1}{x-3}\right]dx = 2\ln|x| + \frac{9}{5}\cdot\ln|x+2| + \frac{1}{5}\ln|x-3|\,\Big|_1^2$$

$$= \left(2\ln 2 + \frac{9}{5}\ln 4 + \cancel{\frac{1}{5}\ln 1}\right) - \left(\cancel{2\ln 1} + \frac{9}{5}\ln 3 + \frac{1}{5}\ln 2\right) = \frac{27}{5}\ln 2 - \frac{9}{5}\ln 3.$$

(8) $\displaystyle\int_0^1 \frac{x^3}{x^3+1}\,dx = ?$

$$\frac{x^3}{x^3+1} = \frac{x^3+1}{x^3+1} - \frac{1}{x^3+1} = 1 - \frac{1}{x^3+1}$$

$x=-1$ is a root of x^3+1
$\Rightarrow x-(-1)=x+1$ divides x^3+1

$$\begin{array}{r|l} x^3+1 & x+1 \\ -\,x^3+x^2 & \overline{x^2-x+1} \\ \hline -x^2+1 & \\ -\,-x^2-x & \\ \hline x+1 & \end{array}$$

$$x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\Delta<0} \qquad \frac{1}{x^3+1} = \frac{A}{\underset{(x^2-x+1)}{x+1}} + \frac{Bx+C}{\underset{(x+1)}{x^2-x+1}}$$

$$x^2(A+B)+x(-A+B+C)+(A+C)=1 \Rightarrow A=\frac{1}{3},\ B=\frac{-1}{3},\ C=\frac{2}{3}$$

$$I = \int_0^1 \Big[\underbrace{1-\frac{1}{3(x+1)}}_{J} - \underbrace{\frac{-x+2}{3(x^2-x+1)}}_{K}\Big]dx \qquad J = x - \frac{1}{3}\ln|x+1|\Big|_0^1 = 1-\frac{1}{3}\ln 2$$

$$K = \frac{1}{3}\int_0^1 \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx \qquad x-\frac{1}{2} = \frac{\sqrt3}{2}\tan t \qquad x=0 \Rightarrow t=-\frac{\pi}{3}$$

$$dx = \frac{\sqrt3}{2}\sec^2 t\,dt \qquad x=1 \Rightarrow t=\frac{\pi}{3}$$

$$K = \frac{1}{3}\int_{-\pi/3}^{\pi/3} \frac{\frac{\sqrt3}{2}\tan t - \frac{3}{2}}{\underbrace{\frac{3}{4}\tan^2 t+\frac{3}{4}}_{\frac{3}{4}\cancel{\sec^2} t}} \cdot \frac{\sqrt3}{2}\cancel{\sec^2} t\,dt = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{\sqrt3}{2}\cdot\frac{1}{2}\int_{-\pi/3}^{\pi/3}(\sqrt3\underbrace{\tan t}_{\frac{\sin t}{\cos t}}-3)\,dt$$

$$= \frac{\sqrt3}{9}\Big[-\sqrt3\ln|\cos t| - 3t\Big]_{-\pi/3}^{\pi/3} = \frac{\sqrt3}{9}\Big[\Big(\cancel{-\sqrt3\ln\big(\tfrac{1}{2}\big)}-\pi\Big)-\Big(\cancel{-\sqrt3\ln\big(\tfrac{1}{2}\big)}+\pi\Big)\Big] = \frac{-2\sqrt3\,\pi}{9}$$

$$I = J+K = 1-\ln\sqrt[3]{2} - \frac{2\sqrt3\,\pi}{9}.$$

(9) $\displaystyle\int_1^2 \frac{x^2+1}{3x-x^2}\,dx = ?$

$$\begin{array}{r|l} x^2+1 & 3x-x^2 \\ -\,x^2-3x & \overline{-1} \\ \hline 3x+1 & \end{array} \qquad \frac{3x+1}{3x-x^2} = \frac{A}{x} + \frac{B}{3-x} \qquad A = \frac{3x+1}{3-x}\Big|_{x=0} = \frac{1}{3},\ B = \frac{3x+1}{x}\Big|_{x=3} = \frac{10}{3}.$$

$$I = \int_1^2 \Big[-1+\frac{1}{3x}+\frac{10}{3(3-x)}\Big]dx = \Big[-x+\frac{1}{3}\ln|x| - \frac{10}{3}\ln|3-x|\Big]_1^2$$

$$= \Big(-2+\frac{1}{3}\ln 2 - \cancel{\frac{10}{3}\ln 1}\Big) - \Big(-1+\cancel{\frac{1}{3}\ln 1} - \frac{10}{3}\ln 2\Big) = -1+\frac{11}{3}\ln 2.$$

10 $\displaystyle\int \frac{x^2+4}{x^2(x-4)}\,dx = ?$

$$\frac{x^2+4}{x^2(x-4)} = \underset{(x^2-4x)}{\frac{A}{x}} + \underset{(x-4)}{\frac{B}{x^2}} + \underset{(x^2)}{\frac{C}{x-4}}$$

$$\underbrace{(A+C)}_{=1}x^2 + \underbrace{(-4A+B)}_{=0}x + \underbrace{(-4B)}_{=4} = x^2+4$$

$$A = -\frac{1}{4}, \quad B = -1, \quad C = \frac{5}{4}$$

$$I = \int\left[-\frac{1}{4x} - \frac{1}{x^2} + \frac{5}{4(x-4)}\right]dx = -\frac{1}{4}\ln|x| + \frac{1}{x} + \frac{5}{4}\ln|x-4| + c.$$

11 $\displaystyle\int \frac{x^2}{(x+3)^3}\,dx = ?$

$$\frac{x^2}{(x+3)^3} = \underset{(x^2+6x+9)}{\frac{A}{x+3}} + \underset{(x+3)}{\frac{B}{(x+3)^2}} + \frac{C}{(x+3)^3}$$

$$\underbrace{A}_{A=1}x^2 + \underbrace{(6A+B)}_{B=-6}x + \underbrace{(9A+3B+C)}_{C=9} = x^2$$

$$I = \int\left[\frac{1}{x+3} - \frac{6}{(x+3)^2} + \frac{9}{(x+3)^3}\right]dx = \ln|x+3| + \frac{6}{x+3} - \frac{9}{2(x+3)^2} + c$$

12 $\displaystyle\int \frac{\cot x}{3+2\sin x}\,dx = ?$

$3+2\sin x = t$

$2\cos x\,dx = dt$

$$I = \int \frac{\cos x\,dx}{\sin x\cdot(3+2\sin x)} = \int \frac{\frac{dt}{\cancel{2}}}{\left(\frac{t-3}{\cancel{2}}\right)\cdot t} = \int \frac{dt}{t(t-3)}$$

$$\frac{1}{t(t-3)} = \frac{A}{t} + \frac{B}{t-3} \qquad A = \frac{1}{t-3}\Big|_{t=0} = \frac{-1}{3}, \quad B = \frac{1}{t}\Big|_{t=3} = \frac{1}{3}$$

$$I = \int\left[\frac{-1}{3t} + \frac{1}{3(t-1)}\right]dt = -\frac{1}{3}\ln|t| + \frac{1}{3}\ln|t-1| + c = \frac{1}{3}\ln\left|\frac{t-1}{t}\right| + c$$

$$\Rightarrow I = \frac{1}{3}\ln\left|\frac{2+2\sin x}{3+2\sin x}\right| + c.$$

13 $\displaystyle\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = ?$

$$\begin{array}{r|l} 6t^5 & t^3-t^2 \\ \hline -\;6t^5-6t^4 & 6t^2+6t+6 \\ \hline 6t^4 & \\ -\;6t^4-6t^3 & \\ \hline 6t^3 & \\ -\;6t^3-6t^2 & \\ \hline 6t^2 & \end{array}$$

$x = t^6$

$dx = 6t^5 dt$

$$I = \int \frac{6t^5 dt}{t^3-t^2}$$

$$I = \int \left[6t^2+6t+6+\underbrace{\frac{6t^2}{t^3-t^2}}_{\frac{6}{t-1}} \right] dt$$

$$I = 2t^3+3t^2+6t+6\ln|t-1|+c = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+c.$$

14 $\displaystyle\int \frac{\sec^2 x\, dx}{\tan x\,(\underbrace{\sec^2 x-2}_{\underbrace{\sec^2 x-1}_{\tan^2 x}-1})} = ?$

$\tan x = t$

$\sec^2 x\,dx = dt$

$$I = \int \frac{dt}{t(t^2-1)} \Rightarrow \frac{1}{t(t^2-1)} = \frac{A}{t}+\frac{B}{t-1}+\frac{C}{t+1}$$

$$A = \frac{1}{t^2-1}\bigg|_{t=0} = -1, \quad B = \frac{1}{t^2+t}\bigg|_{t=1} = \frac{1}{2}, \quad C = \frac{1}{t^2-t}\bigg|_{t=-1} = \frac{1}{2}$$

$$I = \int \left[\frac{-1}{t}+\frac{1}{2}\cdot\frac{1}{t-1}+\frac{1}{2}\cdot\frac{1}{t+1} \right] dt = -\ln|t|+\frac{1}{2}\ln|t-1|+\frac{1}{2}\ln|t+1|+c$$

$$= \ln\left(\frac{\sqrt{(t-1)(t+1)}}{t}\right)+c = \ln\left(\frac{\sqrt{t^2-1}}{t}\right)+c = \ln\left(\frac{\sqrt{\tan^2 x-1}}{\tan x}\right)+c.$$

15 $\displaystyle\int \frac{x^2+5}{x(x^2+2x+5)}\,dx = ?$

$$\frac{x^2+5}{x(x^2+2x+5)} = \frac{A}{\underset{(x^2+2x+5)}{x}}+\frac{Bx+C}{\underset{(x)}{x^2+2x+5}} \Rightarrow (A+B)x^2+(2A+C)x+5A = x^2+5$$

$A=1,\quad B=0,\quad C=-2$

$$I = \int \left[\frac{1}{x}+\frac{-2}{\underbrace{x^2+2x+5}_{(x+1)^2+4}} \right] dx = \ln|x| - \cancel{2}\cdot\frac{1}{\cancel{2}}\arctan\left(\frac{x+1}{2}\right)+c$$

$$⑯\ \int \frac{4dx}{x(4+x^{2024})} = ?\quad (x>0)$$

$$\left.\begin{aligned} 4+x^{2024} &= t \\ 2024\,x^{2023}dx &= dt \end{aligned}\right\} \quad I = \int \frac{dt}{506\,t\,(t-4)} = \frac{1}{506}\int \frac{dt}{t(t-4)}$$

$$\frac{1}{t(t-4)} = \frac{A}{t} + \frac{B}{t-4} \qquad A = \frac{1}{t-4}\Big|_{t=0} = -\frac{1}{4}\ ,\quad B = \frac{1}{t}\Big|_{t=4} = \frac{1}{4}$$

$$I = \frac{1}{2024}\int\left[-\frac{1}{t} + \frac{1}{t-4}\right]dt = \frac{1}{2024}\left[-\ln|t| + \ln|t-4|\right] + c$$

$$= \frac{1}{2024}\left[-\ln\left(4+x^{2024}\right) + \ln x^{2024}\right] + c = \ln x - \frac{\ln\left(4+x^{2024}\right)}{2024} + c$$

$$⑰\ \int \frac{x^3 \cdot e^{x^2}}{(x^2+1)^2}\,dx = ?$$

$$\begin{aligned} x^2 &= t \\ 2x\,dx &= dt \end{aligned} \qquad I = \int \frac{t \cdot e^t}{(t+1)^2}\cdot\frac{dt}{2} \qquad \frac{t}{(t+1)^2} = \frac{A}{\underset{(t+1)}{t+1}} + \frac{B}{(t+1)^2}$$

$$At + A + B = t \Rightarrow A = 1\ ,\ B = -1$$

$$I = \frac{1}{2}\int\left[\frac{e^t}{t+1} - \frac{e^t}{(t+1)^2}\right]dt \qquad \begin{aligned} \frac{1}{t+1} &= u \\ -\frac{1}{(t+1)^2}dt &= du \end{aligned} \qquad \begin{aligned} e^t dt &= dv \\ e^t &= v \end{aligned}$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{e^t}{t+1} + \cancel{\int \frac{e^t dt}{(t+1)^2}} - \cancel{\int \frac{e^t dt}{(t+1)^2}}\right] = \frac{1}{2}\cdot\frac{e^t}{t+1} + c\,.$$

(18) $\displaystyle\int \frac{3x^2+5x+3}{x^3(x+1)}\,dx = ?$

$$\frac{3x^2+5x+3}{x^3(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$x^2(x+1)$ $x(x+1)$ $(x+1)$ (x^3)

$$(A+D)x^3 + (A+B)x^2 + (B+C)x + C = 3x^2+5x+3$$

$C=3$, $B+C=5 \Rightarrow B=2$, $A+B=3 \Rightarrow A=1$, $A+D=0 \Rightarrow D=-1$

$$\int\left[\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3} - \frac{1}{x+1}\right]dx = \ln|x| - \frac{2}{x} - \frac{3}{2x^2} - \ln|x+1| + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \frac{4x+3}{2x^2} + C.$$

(19) $\displaystyle\int \frac{7x^3-3x^2+9x-6}{(x^2+2)(x^2+1)}\,dx = ?$

$$\frac{7x^3-3x^2+9x-6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$

(x^2+1) (x^2+2)

$$(A+C)x^3 + (B+D)x^2 + (A+2C)x + (B+2D) = 7x^3-3x^2+9x-6$$

$A=5$ $B=0$ $C=2$ $D=-3$

$$I = \int\left[\frac{5x}{x^2+2} + \frac{2x-3}{x^2+1}\right]dx = \int\left[\frac{5x}{x^2+2} + \frac{2x}{x^2+1} - \frac{3}{x^2+1}\right]dx$$

$$= \frac{5}{2}\ln(x^2+2) + \ln(x^2+1) - 3\arctan x + C.$$

$$(20)\quad \int \frac{x^2+2x}{(x^2+1)^2}\,dx = ?$$

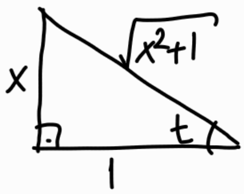
$$\frac{x^2+2x}{(x^2+1)^2} = \underset{(x^2+1)}{\frac{Ax+B}{x^2+1}} + \frac{Cx+D}{(x^2+1)^2} \qquad Ax^3+Bx^2+Cx+(B+D) = x^2+2x$$

$$A=0,\quad B=1,\quad C=2,\quad D=-1$$

$$I = \int\left[\frac{1}{x^2+1} + \frac{2x-1}{(x^2+1)^2}\right]dx = \int\left[\frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2} - \underbrace{\frac{1}{(x^2+1)^2}}_{J}\right]dx$$

For J: $x = \tan t$

$dx = \sec^2 t\,dt$



$$J = \int \frac{\sec^2 t\,dt}{\underbrace{(\tan^2 t+1)^2}_{\sec^2 t}} = \int \underbrace{\cos^2 t}_{\frac{1+\cos(2t)}{2}}\,dt = \frac{t}{2} + \underbrace{\frac{\sin 2t}{4}}_{\frac{\sin t\cdot\cos t}{2}}$$

$$I = \arctan x - \frac{1}{x^2+1} - \frac{\arctan x}{2} - \frac{x}{2(x^2+1)} + C = \frac{\arctan x}{2} - \frac{x+2}{2(x^2+1)} + C$$