Question 1: Which of the following is the domain of the function $f(x,y)=\arcsin(x^2+y^2-1)+\sqrt{x^2+y^2-1}$?

A) $D(x,y)=\{(x,y): 1<x^2+y^2<2\}$
B) $D(x,y)=\{(x,y): 0\le x^2+y^2<1\}$
C) $D(x,y)=\{(x,y): 1\le x^2+y^2\le 2\}$
D) $D(x,y)=\{(x,y): 1<x^2+y^2\le 2\}$
E) $D(x,y)=\{(x,y): 0<x^2+y^2\le 1\}$

Question 2: $\lim\limits_{(x,y)\to(1,0)} \dfrac{\tan(xy)}{y+2xy}=?$

A) 0 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{4}$ E) 1

Question 3: $\lim\limits_{(x,y)\to(0,1)} \dfrac{\tan(xy)}{x^2y+x}=?$

A) 0 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{4}$ E) 1

Question 4: $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin(x^2y)}{x^3y+2x^2}=?$

A) 0 B) 1 C) $\frac{1}{3}$ D) $\frac{1}{4}$ E) $\frac{2}{3}$

Question 5: Find the domain of the function $f(x,y,z)=\frac{1}{\sqrt{1-z-x^2-y^2}}$ and draw its graph.

Question 6: Let $f(x,y)=\begin{cases}\frac{3xy}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Examine whether the function f is continuous at the point $(0,0)$.

Question 7: $\lim_{(x,y)\to(0,0)}(x^2+y^2).\cos\left(\frac{1}{x^2+y^2}\right)=?$

Question 8: Find the domain of the function $z=\arcsin\left(\frac{y-1}{x}\right)$ and draw its graph.

Question 9: Let $f(x,y)=\begin{cases}\frac{xy-y^2}{\sqrt{x}+\sqrt{y}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Examine whether the function f is continuous at the point $(0,0)$.

Question 10: $\displaystyle\lim_{(x,y)\to(0^+,2^-)} \frac{x+y-2}{\sqrt{x}+\sqrt{2-y}} = ?$

A) -1 B) 0 C) 1 D) -2 E) 2

SOLUTIONS

Q1: ① If $\arcsin(x^2+y^2-1) = z$, then

$$x^2+y^2-1 = \sin z.$$

This implies that $-1 \leq x^2+y^2-1 \leq 1$

since $-1 \leq \sin z \leq 1$. Hence,

$$0 \leq x^2+y^2 \leq 2.$$

② $x^2+y^2-1 \geq 0$ for $\sqrt{x^2+y^2-1}$.

Then $1 \leq x^2+y^2$

① + ② $\rightarrow$ $1 \leq x^2+y^2 \leq 2$

so $D(f) = \{(x,y) : 1 \leq x^2+y^2 \leq 2\}$

Q2: $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\tan(xy)}{y+2xy} \cdot \frac{xy}{xy} =$

$$\lim_{(x,y)\to(1,0)} \frac{\tan(xy)}{xy} \cdot \lim_{(x,y)\to(0,0)} \frac{x\cancel{y}}{\cancel{y}(1+2x)} =$$

$$\underbrace{\lim_{(x,y)\to(1,0)} \frac{\sin(xy)}{xy}}_{=1} \cdot \underbrace{\lim_{(x,y)\to(1,0)} \frac{1}{\cos(xy)}}_{=\frac{1}{\cos 0}=\frac{1}{1}=1} \cdot \underbrace{\lim_{(x,y)\to(1,0)} \frac{x}{1+2x}}_{=\frac{1}{1+2}=\frac{1}{3}}$$

$$\left\{\begin{array}{l} \theta = xy \\ \left.\begin{array}{l} x\to 1 \\ y\to 0 \end{array}\right\} \to \theta = 1\cdot 0 \\ \qquad\qquad \theta\to 0 \end{array} \to \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \right\}$$

$$= 1 \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}$$

Q3: $\lim_{(x,y)\to(0,1)} \frac{\tan(xy)}{x^2y+x} \cdot \frac{xy}{xy} =$

$$\underbrace{\lim_{(x,y)\to(0,1)} \frac{\tan(xy)}{xy}}_{=1} \cdot \underbrace{\lim_{(x,y)\to(0,1)} \frac{\cancel{x}y}{\cancel{x}(xy+1)}}_{=\frac{1}{0+1}=1}$$

$$= 1 \cdot 1 = 1$$

Q4: $\lim_{(x,y)\to(0,2)} \frac{\sin(x^2y)}{x^3y+2x^2} \cdot \frac{x^2y}{x^2y} =$

$$\underbrace{\lim_{(x,y)\to(0,2)} \frac{\sin(x^2y)}{x^2y}}_{} \cdot \underbrace{\lim_{(x,y)\to(0,2)} \frac{\cancel{x^2}y}{\cancel{x^2}(xy+2)}}_{=\frac{2}{0+2}=\frac{2}{2}=1}$$

$$\begin{array}{l} \theta = x^2y \\ \left.\begin{array}{l} x\to 0 \\ y\to 2 \end{array}\right\} \to \theta\to 0 \end{array} \quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow 1 \cdot 1 = 1$$

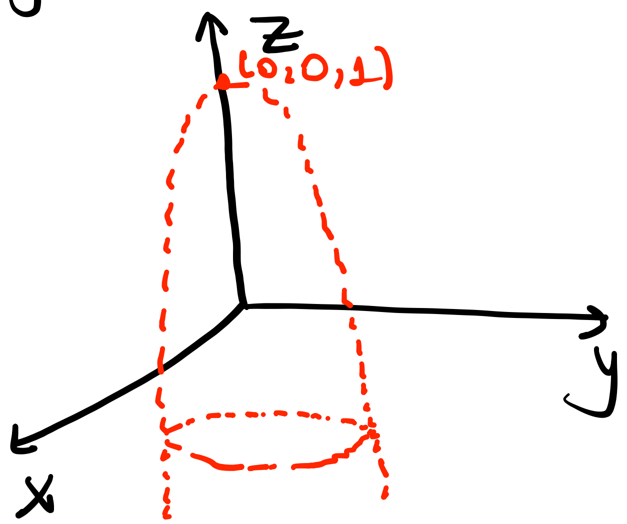
Q5: ① $\sqrt{1-z-x^2-y^2} \neq 0$
② $1-z-x^2-y^2 \geq 0$ $\Big\} \Rightarrow$

① + ② $\rightarrow 1-z-x^2-y^2 > 0$.

This implies that $z < 1-x^2-y^2$

$\circledast$ $z = 1-x^2-y^2$ is a paraboloid and

if $x=y=0$, then $z=1$.



Q6: Let we take $y=kx$ for $k \in \mathbb{R}-\{0\}$. Then,

$$\lim_{x\to 0} \frac{3kx^2}{x^4+k^2x^2} = \lim_{x\to 0} \frac{3k\cancel{x^2}}{\cancel{x^2}(x^2+k^2)} = \frac{3k}{k^2} = \frac{3}{k}$$

In here, limit is depending on k so there is no limit. Hence, f is not continuous.

Q7: For $(x,y)\to(0^+,0^+)$ and $(x,y)\to(0^-,0^-)$ $\Big\} \to (x,y)\neq(0,0)$,

$$-1 \leq \cos\left(\frac{1}{x^2+y^2}\right) \leq 1 \text{. Then,}$$

$$-(x^2+y^2) \leq (x^2+y^2)\cdot\cos\left(\frac{1}{x^2+y^2}\right) \leq x^2+y^2 \text{.}$$

$$\lim_{(x,y)\to(0,0)} -(x^2+y^2) = \lim_{(x,y)\to(0,0)} x^2+y^2 = 0 \text{ so}$$

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cdot\cos\left(\frac{1}{x^2+y^2}\right) = 0$$ by the Sandwich theorem.

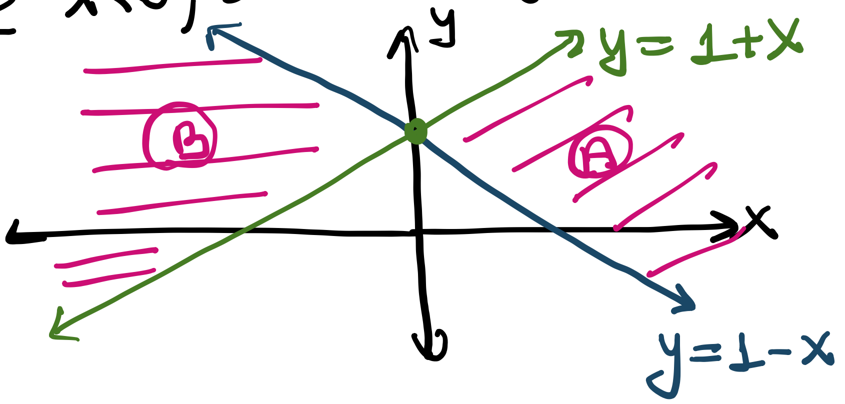
Q8: ① $x\neq 0$ for $\frac{y-1}{x}$

② $z=\arcsin\left(\frac{y-1}{x}\right) \to \sin z = \frac{y-1}{x}$.

Then, $-1 \leq \frac{y-1}{x} \leq 1$ since $-1\leq \sin z \leq 1$.

Ⓐ If $x>0$, then $-x \leq y-1 \leq x \Rightarrow 1-x \leq y \leq x+1$

Ⓑ If $x<0$, then $x \leq y-1 \leq -x \Rightarrow 1+x \leq y \leq 1-x$



Q9 : ① $f(0,0)=0$ so f is defined at the point $(0,0)$.

② $$\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{y(\underbrace{x-y}_{=(\sqrt{x})^2-(\sqrt{y})^2})}{\sqrt{x}+\sqrt{y}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{y(\sqrt{x}-\sqrt{y})\cancel{(\sqrt{x}+\sqrt{y})}}{\cancel{(\sqrt{x}+\sqrt{y})}} =$$

$$\lim_{(x,y)\to(0,0)} y.(\sqrt{x}-\sqrt{y}) = 0.0 = 0$$

③ $$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

f is continuous at the point $(0,0)$.

Q10 : $$\lim_{(x,y)\to(0^+,2^-)} \frac{x+y-2}{\sqrt{x}+\sqrt{2-y}} \cdot \frac{\sqrt{x}-\sqrt{2-y}}{\sqrt{x}-\sqrt{2-y}} =$$

$$\lim_{(x,y)\to(0^+,2^-)} \frac{(x+y-2).(\sqrt{x}-\sqrt{2-y})}{x-(2-y)} =$$

$$\lim_{(x,y)\to(0^+,2^-)} \frac{\cancel{(x+y-2)}.(\sqrt{x}-\sqrt{2-y})}{\cancel{(x+y-2)}} = 0$$