Ex.: Find the tangent and normal line equations of the following implicit function at the point $(\pi, \pi/2)$

$$x^2\cdot\cos y=\sin\left(\frac{2xy}{\pi}\right).$$

$$2x\cdot\cos y - x^2\cdot\sin y\cdot y' = \cos\left(\frac{2xy}{\pi}\right)\cdot\frac{2}{\pi}\cdot(y+x\cdot y')$$

$$2x\cdot\cos y - x^2\cdot\sin y\cdot y' = \frac{2}{\pi}\cos\left(\frac{2xy}{\pi}\right)\cdot y + \frac{2x}{\pi}\cos\left(\frac{2xy}{\pi}\right)\cdot y'$$

$$\left[\frac{2x}{\pi}\cos\left(\frac{2xy}{\pi}\right)+x^2\cdot\sin y\right]\cdot y' = 2x\cos y - \frac{2y}{\pi}\cos\left(\frac{2xy}{\pi}\right) \qquad P=\left(\pi, \frac{\pi}{2}\right)$$

$$\Bigg[\underbrace{\frac{2\cancel{\pi}}{\cancel{\pi}}\cos\left(\frac{\cancel{2}\pi\cdot\cancel{\pi}}{\cancel{2}\cdot\cancel{\pi}}\right)}_{-2}+\underbrace{\pi^2\cdot\sin\left(\frac{\pi}{2}\right)}_{+\pi^2}\Bigg]\cdot y'\Big|_P = \underbrace{2\cdot\pi\cdot\cos\left(\frac{\pi}{2}\right)}_{0} - \underbrace{\frac{\cancel{2}\cdot\pi}{2\cdot\cancel{\pi}}\cdot\cos\left(\frac{\cancel{2\pi}\cdot\pi}{\cancel{2\cdot\pi}}\right)}_{-1}$$

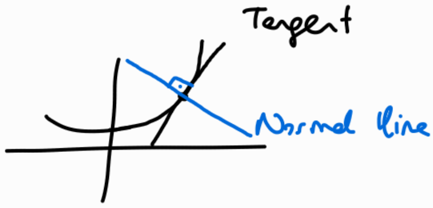
$$(-2+\pi^2)\cdot y'|_P = 1 \Rightarrow y'|_P = \frac{1}{\pi^2-2} = m_T$$

Tangent line: $y-\frac{\pi}{2} = \frac{1}{\pi^2-2}\cdot(x-\pi)$

$$m_T\cdot m_N = -1$$

$$\frac{1}{\pi^2-2}\cdot m_N = -1 \Rightarrow m_N = 2-\pi^2$$

Normal line: $y-\frac{\pi}{2} = (2-\pi^2)\cdot(x-\pi)$



Ex.: Let $y=y(x)$ is given implicitly by

$$2^{xy}+\ln\left[e+\arcsin\left(\frac{y}{x}\right)\right]=1+x.$$

Find the eq. of the tangent line to $y(x)$ at the point $P(1,0)$

$$2^{x\cdot y}\cdot\ln 2\cdot(y+x\cdot y') + \frac{\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\cdot\frac{y'\cdot x-y}{x^2}}{e+\arcsin\left(\frac{y}{x}\right)} = 1$$

$$\underbrace{\underbrace{2^{1\cdot 0}}_{1}\cdot\ln 2\left(0+1\cdot y'|_P\right)}_{\ln 2\cdot y'|_P} + \underbrace{\frac{\frac{1}{\sqrt{1-0}}\cdot\frac{y'|_P-0}{1}}{e+\arcsin\left(\frac{0}{1}\right)}}_{\frac{y'|_P}{e}} = 1$$

$$\ln 2 \cdot y'|_P + \frac{y'|_P}{e} = 1 \Rightarrow y'|_P = \frac{1}{\left(\ln 2 + \frac{1}{e}\right)} = m_T$$

Tangent line: $y = \dfrac{1}{\ln 2 + \frac{1}{e}} \cdot (x-1)$

Ex.: Find the eq. of the tangent line at the point $P\left(0, \frac{1}{\pi}\right)$ of the curve given by

$$\frac{1}{y} = y^{-1}$$

$$y \cdot \sin\left(\frac{1}{y}\right) + x \cdot \cos\left(\frac{1}{y}\right) = -2x$$

$$\underline{y' \cdot \sin\left(\frac{1}{y}\right)} + \underline{y \cdot \cos\left(\frac{1}{y}\right) \cdot (-1) \cdot y^{-2}} \cdot y' + \underline{\cos\left(\frac{1}{y}\right)} + \underline{x \cdot \sin\left(\frac{1}{y}\right) \cdot y^{-2}} \cdot y' = -2$$

$$\left[\sin\left(\frac{1}{y}\right) - \frac{1}{y} \cdot \cos\left(\frac{1}{y}\right) + \frac{x}{y^2} \cdot \sin\left(\frac{1}{y}\right)\right] \cdot y' = -2 - \cos\left(\frac{1}{y}\right)$$

$$y' = \frac{-2 - \cos\left(\frac{1}{y}\right)}{\sin\left(\frac{1}{y}\right) \cdot \left(1 - \frac{x}{y^2}\right) - \frac{1}{y} \cdot \cos\left(\frac{1}{y}\right)}$$

$$F(x,y) = \underline{\underline{0}} \qquad F(x,y) = \underline{y} \cdot \underline{\sin\left(\frac{1}{y}\right)} + x \cdot \cos\left(\frac{1}{y}\right) + 2x = 0$$

$$F'(x,y) = -\frac{F_x}{F_y} = -\frac{0 + \cos\left(\frac{1}{y}\right) + 2}{\sin\left(\frac{1}{y}\right) + \cancel{y} \cdot \cos\left(\frac{1}{y}\right) \cdot \left(\frac{-1}{y^2}\right) + x \cdot \sin\left(\frac{1}{y}\right) \cdot \left(\frac{1}{y^2}\right)} = y'$$

$$y'|_{\underset{\left(0, \frac{1}{\pi}\right)}{P}} = \frac{-2 - \cos(\pi)}{\underbrace{\sin(\pi) \cdot (1-0)}_{0} - \pi \cdot \cos(\pi)} = \frac{-1}{\pi} = m_T$$

Tangent line: $y - \frac{1}{\pi} = -\frac{1}{\pi} x \Rightarrow y = \frac{1-x}{\pi}$

Ex.: Let $g(t) = t^3 + 7t + 21$ be an invertible function and $g(-2) = -1$. Find the linearization to the inverse function $y = g^{-1}(t)$ at $t = -1$.

$$L(x) = f(a) + f'(a) \cdot (x-a) \Rightarrow L(t) = g^{-1}(\underset{-1}{t}) + \left(g^{-1}(\underset{-1}{t})\right)' \cdot (t+1)$$

$$g(-2) = -1 \Rightarrow \underline{-2} = g^{-1}(-1) \qquad \left(g^{-1}(\underset{-1}{t})\right)' = \frac{1}{g'(\underbrace{g^{-1}(\underset{-1}{t})}_{-2})} = \frac{1}{19}$$

$$g'(t) = 3t^2 + 7 \Rightarrow g'(-2) = 19$$

$$L(t) = -2 + \frac{1}{19} \cdot (t+1)$$

Ex.: $\lim\limits_{x\to\infty} \frac{\ln(1+e^x)}{1+x} \overset{L}{=} \lim\limits_{x\to\infty} \frac{\frac{e^x}{1+e^x}}{1} \overset{L}{=} \lim\limits_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}} = 1$

Ex.: $\lim\limits_{x\to 0} \frac{3^x-2^x}{x} \overset{L}{=} \lim\limits_{x\to 0} \frac{3^x.\ln 3 - 2^x.\ln 2}{1} = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right)$

Ex.: $\lim\limits_{x\to 0^+} \underbrace{x^{\sin x}}_{y} = ?$

$\ln y = \sin x.\ln x = (0.\infty)$

$\lim\limits_{x\to 0^+}(\ln y) = \lim\limits_{x\to 0^+} \frac{\ln x}{\csc x} = \left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x.\cot x}$

$(\csc x)' = \left(\frac{1}{\sin x}\right)'$

$= \lim\limits_{x\to 0^+} \frac{-1}{x.\frac{1}{\sin x}.\frac{\cos x}{\sin x}} = \lim\limits_{x\to 0^+} -\frac{\sin x.\sin x}{x.\cos x} = 0$

1

$\lim\limits_{x\to 0^+}(\ln y) = 0 \Rightarrow \ln\left(\lim\limits_{x\to 0^+} y\right) = 0 \Rightarrow \lim\limits_{x\to 0^+} y = e^0 = 1$ //

Ex.: $\lim\limits_{x\to\infty}\left(\cos\frac{1}{x}\right)^x \Rightarrow \lim\limits_{x\to\infty}(\ln y) = \lim\limits_{x\to\infty} x.\ln\left(\cos\left(\frac{1}{x}\right)\right)$

$\lim\limits_{x\to\infty} \frac{\ln\left(\cos\left(\frac{1}{x}\right)\right)}{\frac{1}{x}} \overset{L}{=} \lim\limits_{x\to\infty} \frac{\frac{-\sin\left(\frac{1}{x}\right).\cancel{\left(-\frac{1}{x^2}\right)}}{\cos\left(\frac{1}{x}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim\limits_{x\to\infty} -\tan\left(\frac{1}{x}\right) = 0 \Rightarrow e^0 = 1$ //

Ex.: $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

$\lim\limits_{x\to\infty}(\ln y) = \lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \overset{L}{=} \lim\limits_{x\to\infty} \frac{\frac{\cancel{-\frac{a}{x^2}}}{\frac{a}{x}+1}}{\cancel{-\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \frac{a.x}{a+x} = a$

$\boxed{\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a}$

$\left(1-\frac{4}{x}\right)^x = e^{-4}$

Ex.: $\lim\limits_{x\to\infty}\left(1+\frac{4}{x}\right)^{2x} = \lim\limits_{x\to\infty}\left[\left(1+\frac{4}{x}\right)^x\right]^2 = (e^4)^2 = e^8$

Ex.: $\lim\limits_{x\to 0^+}(\sin x)^{\tan x} = (0^0) = e^0 = 1$

$\lim\limits_{x\to 0^+}(\tan x.\ln(\sin x)) = \lim\limits_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} \underset{\left(\frac{\infty}{\infty}\right)}{\overset{L}{=}} \lim\limits_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim\limits_{x\to 0^+} \frac{\cos x.\sin^{\cancel{2}} x}{-\cancel{\sin x}} = 0$

Ex.: $\lim\limits_{x\to 1} \csc(\pi x).\ln x = \lim\limits_{x\to 1} \frac{\ln x}{\sin(\pi x)} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim\limits_{x\to 1} \frac{\frac{1}{x}}{\pi.\cos(\pi x)} = \frac{1}{-\pi}$

Ex.: $\lim\limits_{x\to\infty} x.\sin(\pi/x) = ?$ $\underbrace{\lim\limits_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}}_{1.\pi} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim\limits_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right).\cancel{\left(\frac{-\pi}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \pi.\cos\left(\frac{\pi}{x}\right) = \pi$

Ex.: $\lim\limits_{x\to 1}\left(\underset{(\ln x)}{\frac{x}{x-1}} - \underset{(x-1)}{\frac{1}{\ln x}}\right) = (\infty - \infty)$

$$\lim_{x\to 1}\frac{x.\ln x - x + 1}{(x-1).\ln x} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to 1}\frac{\ln x + \overset{1}{\cancel{\frac{x}{x}}} - \cancel{1}}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1}\frac{x.\ln x}{x.\ln x + x - 1} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to 1}\frac{\ln x + 1}{\ln x + 1 + 1} = \frac{1}{2}$$

$$\frac{1}{x^2} = x^{-2} \Rightarrow -2.x^{-3}$$

Ex.: $\lim\limits_{x\to\infty}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = ?\ (1^\infty) = e^3$

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\frac{1}{x}} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to\infty}\frac{\frac{\cancel{-}3}{\cancel{x^2}} \cancel{-} \frac{10}{\cancel{x^3}}\big/\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = \frac{3}{1} = 3$$