MAT1320 LINEAR ALGEBRA EXERCISES III

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$[A : I_4] \longrightarrow [I_4 : A^{-1}]$

1. For the matrix $A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$, which of the followings is the inverse matrix of A (if exists). (Hint: You can make use of elementary row operations.)

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Then, } r_2 \leftrightarrow r_3]{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

a) $\begin{bmatrix} -4/5 & 3/2 & 1/2 & 4/5 \\ 3/5 & 0 & 0 & 2/5 \\ 1/5 & -1 & 0 & -1/5 \\ 1/5 & 0 & 0 & -1/5 \end{bmatrix}$

$$\xrightarrow[r_3 \cdot \frac{1}{2},\ r_2 \cdot -1]{r_4 \to r_4 - 2r_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_4 \to r_4 - r_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 8 & -5 & 0 & -2 & 1 & 1 \end{array}\right] \xrightarrow[r_4 \to r_4 - 8r_3]{r_2 \to r_2 + 3r_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 & -2 & 1 & 1 \end{array}\right]$$

b) $\begin{bmatrix} -4/5 & 3/5 & 1/5 & 1/5 \\ 3/2 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 4/5 & 2/5 & -1/5 & -1/5 \end{bmatrix}$

$$\xrightarrow{r_4 \cdot -\frac{1}{5}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4/5 & 2/5 & -1/5 & -1/5 \end{array}\right]$$

c) $\begin{bmatrix} -4/5 & 3/5 & 3/2 & 4/5 \\ 1/5 & 0 & 0 & 1/5 \\ 2/5 & -1 & 0 & -2/5 \\ 2/5 & 0 & 0 & -2/5 \end{bmatrix}$

$$\xrightarrow{r_1 \to r_1 - r_4} \left[\begin{array}{c|cccc} & -4/5 & 3/5 & 1/5 & 1/5 \\ I_4 & 3/2 & 0 & -1 & 0 \\ & 1/2 & 0 & 0 & 0 \\ & 4/5 & 2/5 & -1/5 & -1/5 \end{array}\right]$$

d) $\begin{bmatrix} -4 & 3 & 1 & 4 \\ 3/5 & 0 & 0 & 2/5 \\ 1 & -1 & 0 & -1 \\ 5 & 0 & 0 & -1 \end{bmatrix}$

e) A is not invertible.

2. If $\text{rank}(A) = 2$ for the matrix $A = \begin{bmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{bmatrix}$, then which of the followings is all the possible values of a?

a) $a = -1$ or $a = -2$
b) $a = -2$ or $a = 2$
c) $a = -1$ or $a = 1$
d) $a \neq -1$ or $a \neq -2$
e) $a = -1$ or $a = 2$

Try getting echelon form:

$$\begin{bmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1-a \\ a & 1 & 2 \end{bmatrix} \xrightarrow[r_3 \to r_3 - ar_1]{r_2 \to r_2 + r_1} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2-a \\ 0 & 1-a & 2-a \end{bmatrix}}_{\text{That is enough}}$$

If rank(A) = 2, then there exist a zero row.

Clearly, if $a = -1$, then we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{r_3 \to r_3 - r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \text{rank}(A) = 2$

If $a = 2$, then we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{r_3 \to r_3 + \frac{1}{2} r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \text{rank}(A) = 2$.

$\Rightarrow$ All possible values of a are -1 and 2.

3. (C points) Which of the following matrices are of reduced row echelon form?

$$\checkmark\ \mathcal{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \times\ \mathcal{B} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \end{bmatrix},$$

$$\times\ \mathcal{C} = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ \checkmark$$

a) $\mathcal{B}$ and $\mathcal{C}$ b) Only $\mathcal{D}$ c) $\mathcal{A}$ and $\mathcal{D}$ d) Only $\mathcal{A}$ e) Only $\mathcal{B}$

4. Which of the followings is the reduced row echelon form of the matrix $\begin{bmatrix} 2 & 3 & 3 & 5 \\ 6 & 7 & 8 & 9 \\ 1 & 0 & 0 & 4 \end{bmatrix}$?

a) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -9/2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -8 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 3 & 5 \\ 6 & 7 & 8 & 9 \\ 1 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 3 & 3 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

$$\xrightarrow[r_3 \to r_3 - 6r_1]{r_2 \to r_2 - 2r_1} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 3 & 3 & -3 \\ 0 & 7 & 8 & -15 \end{bmatrix} \xrightarrow{r_2 \cdot \frac{1}{3}} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 7 & 8 & -15 \end{bmatrix}$$

$$\xrightarrow{r_3 \to r_3 - 7r_2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 - r_3} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

5. Which of the followings is the rank of the matrix

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 2 \\ 2 & 2 & -4 & 0 & 1 \\ 3 & 3 & -6 & 0 & -7 \\ 4 & -2 & -8 & 0 & 17 \\ 5 & 4 & -10 & 0 & -4 \end{bmatrix}?$$

a) 3 b) 4 c) 5 d) 2 e) 1

Since rank is the number of nonzero rows in R.R.E.F or E.F, zero column doesn't change the rank. So, we don't have to write it.

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 2 & 2 & -4 & 1 \\ 3 & 3 & -6 & -7 \\ 4 & -2 & -8 & 17 \\ 5 & 4 & -10 & -4 \end{bmatrix}$$

Also, recall that $\text{rank}(A) = \text{rank}(A^T)$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 2 & 3 & -2 & 4 \\ -2 & -4 & -6 & -8 & -10 \\ 2 & 1 & -7 & 17 & -4 \end{bmatrix} \xrightarrow[\substack{r_3 \to r_3 + 2r_1 \\ r_4 \to r_4 - 2r_1}]{r_2 \to r_2 + r_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 6 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -13 & 9 & -14 \end{bmatrix} \xrightarrow{r_2 \to r_2 + r_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -7 & 11 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -13 & 9 & -14 \end{bmatrix}$$

$$\xrightarrow{r_4 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -7 & 11 & -5 \\ 0 & -3 & -13 & 9 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 3r_2]{r_1 \to r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 17 & -18 & 15 \\ 0 & 1 & -7 & 11 & -5 \\ 0 & 0 & -34 & 42 & -29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_3 \cdot \frac{-1}{34}} \underbrace{\begin{bmatrix} 1 & 0 & 17 & -18 & 15 \\ 0 & 1 & -7 & 11 & -5 \\ 0 & 0 & 1 & -42/34 & 29/34 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{In Echelon form.}}$$

$\Rightarrow$ The number of the nonzero rows is 3. $\Rightarrow \text{rank}(A) = 3$.

6. Let the matrix $B = [b_{ij}]$ be given as $1 \leq i, j \leq n$, $n \geq 3$ and $b_{ij} = i.j$. Then, which of the followings is the rank of the matrix B?

a) 0 b) 1 c) 2 d) 3 e) 4

Write some rows of B.

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 4 & 6 & 8 & \cdots & 2n \\ 3 & 6 & 9 & 12 & \cdots & 3n \\ & & \vdots & & & \end{bmatrix}$$

$$\xrightarrow[\substack{\vdots \\ r_n \to r_n - n.r_1}]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1}} \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\Rightarrow \text{rank}(B) = 1$.