MAT1320-Linear Algebra
Lecture Notes

Determinants and Properties

Mehmet E. KÖROĞLU

Fall 2024

Yildiz Technical University, Department of Mathematics
mkoroglu@yildiz.edu.tr

Table of contents

Determinants

Determinants

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be a square matrix of order n. Then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$ and defined as follows:

Determinants

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be a square matrix of order n. Then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$ and defined as follows:

- For $n = 1$, $A = [a]_{1 \times 1}$ and $\det(A) = a$.

Determinants

Let $\mathbf{A} = [a_{ij}]_{n\times n}$ be a square matrix of order n. Then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$ and defined as follows:

- For $n = 1$, $A = [a]_{1\times 1}$ and $\det(A) = a$.
- For $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$ and

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

- For $n = 3$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times 3}$ and

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants

- For $n = 3$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times 3}$ and

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{aligned} = \quad & a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

- For $n = 3$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times 3}$ and

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{aligned} = \quad & a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) \\ & + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \end{aligned}$$

In order to generalize this concept for $n > 3$, we need to give definition of the minors and cofactors.

Minors and Cofactors

Definition

Given an $n \times n$ matrix $\mathbf{A}$, the (i,j)-th minor, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting the i-th row and the j-th column.

Minors and Cofactors

Definition
Given an $n \times n$ matrix $\mathbf{A}$, the (i,j)-th minor, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting the i-th row and the j-th column. Similarly, the (i,j)-th cofactor of $\mathbf{A}$ is denoted by A_{ij} is defined in terms of the minor by $A_{ij} = (-1)^{i+j} M_{ij}$.

Definition

Given an $n \times n$ matrix $\mathbf{A}$, the (i,j)-th minor, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting the i-th row and the j-th column. Similarly, the (i,j)-th cofactor of $\mathbf{A}$ is denoted by A_{ij} is defined in terms of the minor by $A_{ij} = (-1)^{i+j} M_{ij}$.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

we have

$$M_{23} = det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 8 - 14 = -6$$

We also find $A_{23} = (-1)^{2+3}(-6) = 6$

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

1. Choose any row or column of $\mathbf{A}$.

Laplace Expansion

Algorithm (Laplace Expansion): To compute the determinant of a square **A** matrix, do the following.

1. Choose any row or column of **A**.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.

Laplace Expansion

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

1. Choose any row or column of $\mathbf{A}$.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.
3. Multiply each cofactor by the associated matrix entry a_{ij}.

Laplace Expansion

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

1. Choose any row or column of $\mathbf{A}$.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.
3. Multiply each cofactor by the associated matrix entry a_{ij}.
4. The sum of these products is $det\left(\mathbf{A}\right)$.

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

1. Choose any row or column of $\mathbf{A}$.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.
3. Multiply each cofactor by the associated matrix entry a_{ij}.
4. The sum of these products is $det\left(\mathbf{A}\right)$.

Let $\mathbf{A} = \left[a_{ij}\right]_{n\times n}$ be a square matrix of order n.

Algorithm (Laplace Expansion): To compute the determinant of a square **A** matrix, do the following.

1. Choose any row or column of **A**.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.
3. Multiply each cofactor by the associated matrix entry a_{ij}.
4. The sum of these products is $det\left(\mathbf{A}\right)$.

Let $\mathbf{A} = \left[a_{ij}\right]_{n\times n}$ be a square matrix of order n. Then

$$det\left(\mathbf{A}\right) = \sum_{k=1}^{n} a_{ik}A_{ik} \text{ (LE along } i\text{-th row)}$$

Algorithm (Laplace Expansion): To compute the determinant of a square $\mathbf{A}$ matrix, do the following.

1. Choose any row or column of $\mathbf{A}$.
2. For each element a_{ij} of this row or column, compute the associated cofactor A_{ij}.
3. Multiply each cofactor by the associated matrix entry a_{ij}.
4. The sum of these products is $det\left(\mathbf{A}\right)$.

Let $\mathbf{A} = \left[a_{ij}\right]_{n\times n}$ be a square matrix of order n. Then

$$\begin{aligned} det\left(\mathbf{A}\right) &= \sum_{k=1}^{n} a_{ik}A_{ik} \text{ (LE along } i\text{-th row)} \\ det\left(\mathbf{A}\right) &= \sum_{k=1}^{n} a_{kj}A_{kj} \text{ (LE along } j\text{-th column)} \end{aligned}$$

Note: For $i \neq j$ we have

$$\sum_{k=1}^{n} a_{ik} A_{jk} = 0$$

Laplace Expansion

Note: For $i \neq j$ we have

$$\sum_{k=1}^{n} a_{ik} A_{jk} = 0 \text{ and } \sum_{k=1}^{n} a_{ki} A_{kj} = 0$$

Note: For $i \neq j$ we have

$$\sum_{k=1}^{n} a_{ik} A_{jk} = 0 \text{ and } \sum_{k=1}^{n} a_{ki} A_{kj} = 0$$

Example
We find the determinant of

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}.$$

Example (1)

We make the arbitrary choice to expand along the first row. We compute the cofactors as

Laplace Expansion

Example (1)

We make the arbitrary choice to expand along the first row. We compute the cofactors as

$$A_{11} \quad = \quad (-1)^{1+1}\, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4$$

Example (1)
We make the arbitrary choice to expand along the first row. We compute the cofactors as

$$\begin{aligned} A_{11} &= (-1)^{1+1}\,det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\ A_{12} &= (-1)^{1+2}\,det\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \end{aligned}$$

Laplace Expansion

Example (1)

We make the arbitrary choice to expand along the first row. We compute the cofactors as

$$
\begin{aligned}
A_{11} &= (-1)^{1+1} \, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\
A_{12} &= (-1)^{1+2} \, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\
A_{13} &= (-1)^{1+3} \, det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2
\end{aligned}
$$

Example (1)

We make the arbitrary choice to expand along the first row. We compute the cofactors as

$$\begin{aligned} A_{11} &= (-1)^{1+1} \, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\ A_{12} &= (-1)^{1+2} \, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{13} &= (-1)^{1+3} \, det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2 \end{aligned}$$

Thus,

$$det\,(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2(4) + (1) + 3(2) = 15$$

Example (1-cont.)

$$A_{11} = (-1)^{1+1}\,det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4$$

Laplace Expansion

Example (1-cont.)

$$
\begin{aligned}
A_{11} &= (-1)^{1+1}\, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\
A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1
\end{aligned}
$$

Example (1-cont.)

$$\begin{aligned} A_{11} &= (-1)^{1+1}\, det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\ A_{12} &= (-1)^{1+2}\, det\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{13} &= (-1)^{1+3}\, det\begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2 \end{aligned}$$

Example (1-cont.)

$$\begin{aligned}
A_{11} &= (-1)^{1+1}\, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\
A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\
A_{13} &= (-1)^{1+3}\, det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2
\end{aligned}$$

Notice that

$$a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = (-1)\,(4) + 2(1) + 1(2) = 0$$

Example (1-cont.)

$$\begin{aligned} A_{11} &= (-1)^{1+1}\, det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\ A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{13} &= (-1)^{1+3}\, det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2 \end{aligned}$$

Notice that

$$\begin{aligned} a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} &= (-1)(4) + 2(1) + 1(2) = 0 \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} &= (-2)(4) + 2(1) + 3(2) = 0 \end{aligned}$$

Example (2)

We make the arbitrary choice to expand along the second column.

We compute the cofactors as:

Example (2)

We make the arbitrary choice to expand along the second column.

We compute the cofactors as:

$$A_{12} \;=\; (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1$$

Example (2)

We make the arbitrary choice to expand along the second column.

We compute the cofactors as:

$$\begin{aligned} A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \end{aligned}$$

Example (2)
We make the arbitrary choice to expand along the second column.

We compute the cofactors as:

$$
\begin{aligned}
A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\
A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\
A_{32} &= (-1)^{3+2}\, det \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = -5
\end{aligned}
$$

Example (2)
We make the arbitrary choice to expand along the second column.

We compute the cofactors as:

$$\begin{aligned} A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\ A_{32} &= (-1)^{3+2}\, det \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = -5 \end{aligned}$$

Thus,

$$det\,(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 1(1) + 2(12) + 2(-5) = 15$$

Laplace Expansion

Example (2-cont.)

$$A_{12} \;=\; (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1$$

Example (2-cont.)

$$A_{12} = (-1)^{1+2} \, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1$$

$$A_{22} = (-1)^{2+2} \, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12$$

Example (2-cont.)

$$\begin{aligned} A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\ A_{32} &= (-1)^{3+2}\, det \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = -5 \end{aligned}$$

Example (2-cont.)

$$
\begin{aligned}
A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\
A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\
A_{32} &= (-1)^{3+2}\, det \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = -5
\end{aligned}
$$

Note that

$$
a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 2(1) + (-1)(12) + (-2)(-5) = 0
$$

Example (2-cont.)

$$\begin{aligned}
A_{12} &= (-1)^{1+2}\, det\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\
A_{22} &= (-1)^{2+2}\, det\begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\
A_{32} &= (-1)^{3+2}\, det\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} = -5
\end{aligned}$$

Note that

$$\begin{aligned}
a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} &= 2(1) + (-1)(12) + (-2)(-5) = 0 \\
a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} &= 3(1) + 1(12) + 3(-5) = 0
\end{aligned}$$

Properties of Determinants

Properties of Determinants

- $\det I = 1$

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

  Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

Properties of Determinants

- det $I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

  Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- If we multiply one row of a matrix by t, the determinant is multiplied by t.

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

  Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- If we multiply one row of a matrix by t, the determinant is multiplied by t. For example; $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

  Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- If we multiply one row of a matrix by t, the determinant is multiplied by t. For example; $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
- The determinant behaves like a linear function on the rows of the matrix:

Properties of Determinants

- $\det I = 1$
- If you exchange two rows or columns of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive. Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

  Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- If we multiply one row of a matrix by t, the determinant is multiplied by t. For example; $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
- The determinant behaves like a linear function on the rows of the matrix:

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

- If two rows or columns of a matrix are equal, its determinant is zero.

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant.

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant. In two dimensions, this argument looks like:

$$\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix}$$

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant. In two dimensions, this argument looks like:

$$\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant. In two dimensions, this argument looks like:

$$\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant. In two dimensions, this argument looks like:

$$\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- If A has a row or column that is all zeros, then $det\,(A) = 0$.

Properties of Determinants

- If two rows or columns of a matrix are equal, its determinant is zero.
- If $i \neq j$, subtracting t times row i from row j doesn't change the determinant. In two dimensions, this argument looks like:

$$\begin{vmatrix} a & b \\ c - ta & d - tb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- If A has a row or column that is all zeros, then $det\left(A\right) = 0$.
- The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_{11}, d_{22}, \ldots, d_{nn}$.

Properties of Determinants

- $det\left(AB\right) = det\left(A\right)det\left(B\right).$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right) det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}.$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right) det\left(B\right)$.
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$.

Properties of Determinants

- $det\left(AB\right) = det\left(A\right)det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.

Properties of Determinants

- $det\left(AB\right) = det\left(A\right)det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^{2}\right) = \left(det\left(A\right)\right)^{2}$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right) det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^{2}\right) = \left(det\left(A\right)\right)^{2}$ and $det\left(2A\right) = 2^{n} det\left(A\right).$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right)det\left(B\right)$.
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^{2}\right) = \left(det\left(A\right)\right)^{2}$ and $det\left(2A\right) = 2^{n}det\left(A\right)$. We assume A is an $n \times n$ square matrix.

Properties of Determinants

- $det\left(AB\right) = det\left(A\right) det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^2\right) = \left(det\left(A\right)\right)^2$ and $det\left(2A\right) = 2^n det\left(A\right)$. We assume A is an $n \times n$ square matrix.
- $det\left(A^T\right) = det\left(A\right).$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right)det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^2\right) = \left(det\left(A\right)\right)^2$ and $det\left(2A\right) = 2^n det\left(A\right)$. We assume A is an $n \times n$ square matrix.
- $det\left(A^T\right) = det\left(A\right).$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\Rightarrow \quad \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\Rightarrow \quad \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} = \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix}$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\Rightarrow \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} = \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix}$$

$$= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix}$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\Rightarrow \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} = \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix}$$

$$= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix} = r \begin{vmatrix} 1 & z & t \\ 0 & x+y-z & 0 \\ 0 & t-z & x+y-t \end{vmatrix}$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\Rightarrow \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} = \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix}$$

$$= \underbrace{(x+y+z+t)}_{r}\begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix} = r\begin{vmatrix} 1 & z & t \\ 0 & x+y-z & 0 \\ 0 & t-z & x+y-t \end{vmatrix}$$

$$= r\begin{vmatrix} x+y-z & 0 \\ t-z & x+y-t \end{vmatrix}$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

$$\begin{aligned}
\Rightarrow \quad \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} &= \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix} \\
&= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix} = r \begin{vmatrix} 1 & z & t \\ 0 & x+y-z & 0 \\ 0 & t-z & x+y-t \end{vmatrix} \\
&= r \begin{vmatrix} x+y-z & 0 \\ t-z & x+y-t \end{vmatrix} \\
&= (x+y+z+t)(x+y-z)(x+y-t).
\end{aligned}$$

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} r_1 \leftrightarrow r_2 \\ -r_1 \to r_1 \\ = \end{array}$$

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{matrix} r_1 \leftrightarrow r_2 \\ -r_1 \rightarrow r_1 \\ = \end{matrix} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 3 \end{vmatrix} \begin{matrix} -2r_1 + r_2 \rightarrow r_2 \\ 2r_1 + r_3 \rightarrow r_3 \\ = \end{matrix}$$

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} r_1 \leftrightarrow r_2 \\ -r_1 \rightarrow r_1 \\ = \end{array} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} -2r_1 + r_2 \rightarrow r_2 \\ 2r_1 + r_3 \rightarrow r_3 \\ = \end{array}$$

$$\begin{vmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & 1 \end{vmatrix} \begin{array}{c} \frac{1}{5}r_2 \rightarrow r_2 \\ = \end{array}$$

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{matrix} r_1 \leftrightarrow r_2 \\ -r_1 \to r_1 \\ = \end{matrix} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 3 \end{vmatrix} \begin{matrix} -2r_1 + r_2 \to r_2 \\ 2r_1 + r_3 \to r_3 \\ = \end{matrix}$$

$$\begin{vmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} \frac{1}{5}r_2 \to r_2 \\ = \end{matrix} 5\begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} 2r_2 + r_3 \to r_3 \\ = \end{matrix}$$

Example
By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} r_1 \leftrightarrow r_2 \\ -r_1 \to r_1 \\ = \end{array} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} -2r_1 + r_2 \to r_2 \\ 2r_1 + r_3 \to r_3 \\ = \end{array}$$

$$\begin{vmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & 1 \end{vmatrix} \begin{array}{c} \frac{1}{5} r_2 \to r_2 \\ = \end{array} 5 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} \begin{array}{c} 2r_2 + r_3 \to r_3 \\ = \end{array}$$

$$\Rightarrow \quad 5 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} \Rightarrow |A| = 5.(1.1.3) = 15.$$

?