MAT1320-Linear Algebra
Lecture Notes

Cramer's Rule, Inverse Matrix Method

Mehmet E. KÖROĞLU

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Yildiz Technical University, Department of Mathematics

mkoroglu@yildiz.edu.tr

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Cramer's Rule

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Let

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{1}$$

be a linear system of n equations with n unknowns.

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be a linear system of n equations with n unknowns. Let denote the coefficient matrix of the system by $A = \left[a_{ij}\right]$.

Cramer's Rule

Then for the vector of unknowns $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and the vector of constants $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, the system given in (1) can be written as

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where A_i $(i = 1, 2, \ldots, n)$ is the matrix obtained from A by replacing i-th column with the vector of constants $\mathbf{b}$.

Example (Cramer's Rule)
By using Carmer's Rule solve the system

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = -3 \\ -3x_1 + 4x_2 - x_3 = 4 \end{cases}$$

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Example

Let the determinant of the matrix $\mathbf{F} = \begin{bmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{bmatrix}$ be $|\mathbf{F}|$

such that $|\mathbf{F}| \neq 0$.

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1. By using properties of determinants, find $|\mathbf{F}|$.
2. For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ solve $\mathbf{Fx} = \mathbf{b}$ by Cramer's Rule.

Solution (1)

$$abcd\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \begin{matrix} H_{21}(-1) \\ H_{31}(-1) \\ H_{41}(-1) \\ = \end{matrix}$$

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Solution (1 cont.)

$$= \underbrace{abcd\,(b-a)\,(c-a)\,(d-a)}_{r} \begin{vmatrix} 1 & b+a & b^2+ab+a^2 \\ 1 & c+a & c^2+ac+a^2 \\ 1 & d+a & d^2+ad+a^2 \end{vmatrix} \begin{matrix} \\ H_{21}(-1) \\ H_{31}(-1) \end{matrix}$$

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&= \underbrace{r(c-b)\,(d-b)}_{s} \begin{vmatrix} 1 & b+a & b^2+ab+a^2 \\ 0 & 1 & c+b+a \\ 0 & 1 & d+b+a \end{vmatrix}
\end{aligned}
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$$= \quad rs\begin{vmatrix} 1 & c+b+a \\ 1 & d+b+a \end{vmatrix} H_{21}(-1) = rs\begin{vmatrix} 1 & c+b+a \\ 0 & d-c \end{vmatrix}$$

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Solution (2)
We know that $\triangle = |\mathbf{F}| \neq 0$ *and* $\triangle_i$ $(i = 1, 2, 3, 4)$ *is the determinant of the matrix obtained from* $\mathbf{F}$ *by replacing i-th column with* $\mathbf{b}$.

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We know that $\triangle = |\mathbf{F}| \neq 0$ *and* $\triangle_i$ $(i = 1, 2, 3, 4)$ *is the determinant of the matrix obtained from* $\mathbf{F}$ *by replacing i-th column with* $\mathbf{b}$. *Then* $\triangle_1 = |\mathbf{F}|$ *and* $\triangle_2 = \triangle_3 = \triangle_4 = 0$. *So, we have* $x_1 = 1$, $x_2 = x_3 = x_4 = 0$.

?