PARAMETRIC EQUATIONS

(1) Find the tangent line of the curve given by
$x=t^2$, $y=t^3-3t$ at the point $(3,0)$

$x=t^2 \Longrightarrow t^2=3 \Rightarrow t=\mp\sqrt{3}$

$y=t^3-3t \Longrightarrow t(t^2-3)=0 \Rightarrow t=0,\ t=\mp\sqrt{3}$

$\frac{dy}{dt}=3t^2-3$

$\frac{dx}{dt}=2t$

$\frac{dy}{dx}=\frac{3t^2-3}{2t}$ $\to$ $t_1=\sqrt{3} \Rightarrow m_1=\frac{6}{2\sqrt{3}}=\sqrt{3}$; $t_2=-\sqrt{3} \Rightarrow m_2=\frac{-6}{2\sqrt{3}}=-\sqrt{3}$

For $t_1=\sqrt{3}$

$y=\sqrt{3}(x-3)$

$y=\sqrt{3}x-3\sqrt{3}$

For $t_2=-\sqrt{3}$

$y=-\sqrt{3}(x-3)$

$y=-\sqrt{3}x+3\sqrt{3}$

(2) Find the equation of the tangent line for the curve
$x=t^4+2\sqrt{t}$, $y=\sin(\pi t)$ at $t=1$.

$$t=1 \Rightarrow \begin{cases} x(1)=1^4+2\sqrt{1}=3=x_0 \\ y(1)=\sin(\pi 1)=0=y_0 \end{cases}$$

$$\left.\begin{array}{l} \frac{dy}{dt}=\pi\cdot\cos(\pi t) \Longrightarrow \left.\frac{dy}{dt}\right|_{t=1}=\pi\cdot\cos\pi=-\pi \\ \frac{dx}{dt}=4t^3+\frac{1}{\sqrt{t}} \Longrightarrow \left.\frac{dx}{dt}\right|_{t=1}=4+1=5 \end{array}\right\} m_T=\left.\frac{dy}{dx}\right|_{t=1}=\frac{-\pi}{5}$$

Tangent line: $y=-\frac{\pi}{5}(x-3)$

(3) Find the equation of the tangent line for the curve
$x=e^{\sqrt{t}}$, $y=t-\ln t^2$ $(t>0)$ at $t=1$.

$$t=1 \Rightarrow \begin{cases} x(1)=e^1=e=x_0 \\ y(1)=1-2\ln 1=1=y_0 \end{cases}$$

$\frac{dy}{dt}=1-\frac{2}{t}$ $\qquad$ $\frac{dx}{dt}=\frac{1}{2\sqrt{t}}\cdot e^{\sqrt{t}}$

$\left.\frac{dy}{dt}\right|_{t=1}=1-2=-1$ $\qquad$ $\left.\frac{dx}{dt}\right|_{t=1}=\frac{e}{2}$ $\Rightarrow m_T=\frac{-2}{e}$

Tangent Line

$y-1=-\frac{2}{e}(x-e)$

$y=\frac{-2x}{e}+3$

4) At which point(s) slope of the tangent line for the curve $x=2t^3$, $y=1+4t-t^2$ is 1?

$$\left.\begin{aligned}\frac{dy}{dt}&=4-2t\\ \frac{dx}{dt}&=6t^2\end{aligned}\right\}\quad \frac{dy}{dx}=\frac{4-2t}{6t^2}=1 \Rightarrow 6t^2=4-2t \Rightarrow 3t^2=2-t$$

$$\Rightarrow 3t^2+t-2=0$$

$$\begin{matrix}3t & -2\\ t & 1\end{matrix}$$

$$(3t-2)(t+1)=0 \Rightarrow t_1=\frac{2}{3},\ t_2=-1$$

5) Find the length of the curve given by $x=2\cos t+3$, $y=2\sin t+4$ for $0\leqslant t\leqslant 2\pi$.

$$\left.\begin{aligned}x'(t)&=-2\sin t\\ y'(t)&=2\cos t\end{aligned}\right\}\quad (x')^2+(y')^2=4\sin^2 t+4\cos^2 t=4$$

$$L=\int_0^{2\pi}\sqrt{4}\,dt=2t\Big|_0^{2\pi}=4\pi$$

6) Evaluate the length of the curve given by the parametric equations $x=\theta-\sin\theta$, $y=1-\cos\theta$ for the interval $[0,2\pi]$.

$$\begin{aligned}x'(\theta)&=1-\cos\theta\\ y'(\theta)&=\sin\theta\end{aligned}\qquad (x')^2+(y')^2=1-2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}=2(1-\cos\theta)$$

$$L=\int_0^{2\pi}\sqrt{\underbrace{2(1-\cos\theta)}_{4\sin^2\frac{\theta}{2}}}\,d\theta=\int_0^{2\pi}2\left|\sin\frac{\theta}{2}\right|d\theta$$

$$0\leqslant\theta\leqslant 2\pi$$
$$0\leqslant\frac{\theta}{2}\leqslant\pi \Rightarrow \sin\frac{\theta}{2}\text{ is }(+)$$

$$=2\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta=-4\cos\frac{\theta}{2}\Big|_0^{2\pi}=-4(-1-1)=8$$

7) Write the integral that gives the area bounded by $x=1+e^t$, $y=t-t^2$, $0\leqslant t\leqslant 1$, and x-axis.

$$A=\int_a^b y(t).x'(t)dt=\int_0^1(t-t^2).e^t dt$$

8) Evaluate the area under the parametric curve $x=\theta-\sin\theta$, $y=1-\cos\theta$ for the interval $0\leq\theta\leq 2\pi$.

$$A=\int_0^{2\pi} y(t)\cdot x'(t)\,dt=\int_0^{2\pi}(1-\cos\theta)^2\,dt=\int_0^{2\pi}\Big(1-2\cos\theta+\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\Big)\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta=\frac{3}{2}\theta-2\sin\theta+\frac{\sin 2\theta}{4}\Big|_0^{2\pi}=\frac{3}{\cancel{2}}\cdot\cancel{2}\pi=3\pi.$$

9) Evaluate the area of the region bounded by the parametric curve $x=t^2-2t$, $y=\sqrt{t}$, and y-axis for the interval $0\leq t\leq 2$.

Remark: Since the area is bounded by a curve and y-axis the area formula will be $A=\int x(t)\cdot y'(t)\,dt$.

$$A=\int_0^2\left|(t^2-2t)\cdot\frac{1}{2\sqrt{t}}\right|dt$$

If checked, it can be seen that $\frac{t^2-2t}{2\sqrt{t}}$ is negative for the given interval.

$$A=-\int_0^2\frac{t^2-2t}{2\sqrt{t}}\,dt=\int_0^2\left[\frac{t}{\sqrt{t}}-\frac{t^2}{2\sqrt{t}}\right]dt=\int_0^2\left[t^{1/2}-\frac{t^{3/2}}{2}\right]dt$$

$$=\frac{2}{3}t^{3/2}-\frac{1}{5}t^{5/2}\Big|_0^2=\frac{2}{3}\cdot 2\sqrt{2}-\frac{1}{5}\cdot 4\sqrt{2}=\frac{8\sqrt{2}}{15}.$$