10.13 Radius and Interval of Convergence of Power Series

Solutions **Practice**

Calculus

Find the interval of convergence for each power series.

1. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$

$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(x-1)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-1)}{4}\right| < 1$

$-1 < \frac{x-1}{4} < 1$

$-4 < x-1 < 4 \Rightarrow |x-1| < 4 \Rightarrow \boxed{R=4}$

$-3 < x < 5$

Check endpoints!

$x=-3$: $\sum_{n=0}^{\infty}\frac{(-4)^n}{4^n}$ diverges

$x=5$: $\sum_{n=0}^{\infty}\frac{4^n}{4^n}=1$ diverges

$\boxed{-3 < x < 5}$

$(-3,5)$

2. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$

$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x+2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x+2)}{3}\right| < 1$

$-1 < \frac{x+2}{3} < 1$

$-3 < x+2 < 3 \Rightarrow |x-2| < 3$

$-5 < x < 1 \Rightarrow \boxed{R=3}$

Check endpoints!

$x=-5$: $\sum_{n=0}^{\infty}\frac{(-3)^n}{3^n}$ diverges

$x=1$: $\sum_{n=0}^{\infty}\frac{3^n}{3^n}=1$ diverges

$\boxed{-5 < x < 1}$

$(-5,1)$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n\cdot 2^n}{(x-2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-2)}{2}\cdot\underbrace{\frac{n}{n+1}}_{\text{approaches } 1}\right| < 1$

$-1 < \frac{x-2}{2} < 1$

$-2 < x-2 < 2 \Rightarrow |x-2| < 2$

$0 < x < 4 \Rightarrow \boxed{R=2}$

Check endpoints

$x=0$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n}$ Diverges

$x=4$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cancel{(2)^n}}{n\cancel{2^n}}$ converges!

$\boxed{0 < x \le 4}$

$(0,4]$

4. $\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(2n)!}{3^n}\cdot(x-0)^n$

$\lim_{n\to\infty}\left|(2n+2)!\left(\frac{x}{3}\right)^{n+1}\cdot\frac{1}{(2n)!}\left(\frac{3}{x}\right)^n\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}}\cdot\left(\frac{x}{3}\right)\right| = \infty$

∞

$\boxed{\text{Converges only at center } x=0.}$

$\Downarrow$

$\boxed{R=0}$

Find the radius of convergence for each series.

5. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

$\lim_{n\to\infty} \left| \frac{(4x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x)^n} \right|$

$\lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot (4x) \right| < 1$

approaches 1

$-1 < 4x < 1$

$-\frac{1}{4} < x < \frac{1}{4}$

$\boxed{\text{radius} = \frac{1}{4}}$

6. $\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{2 \cdot 3^{n+1}}$

$\lim_{n\to\infty} \left| \frac{(x-4)^{n+2}}{2 \cdot 3^{n+2}} \cdot \frac{2 \cdot 3^{n+1}}{(x-4)^{n+1}} \right|$

$\lim_{n\to\infty} \left| (x-4) \cdot \frac{1}{3} \right| < 1$

$-1 < \frac{x-4}{3} < 1$

$-3 < x-4 < 3$

$1 < x < 7$

$\boxed{\text{radius} = 3}$

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$

$\lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \cdot \frac{x^2}{1} \right| = 0$

converges for all values of x.

$\boxed{\text{Radius} = \infty}$

8. $\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{n!}$

$\lim_{n\to\infty} \left| \frac{(2n+2)!\, x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)!\, x^{2n}} \right|$

$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{n+1} \cdot \frac{x^2}{1} \right| = \infty$

approaches ∞

converges to center.

$\boxed{\text{radius} = 0}$

What are all values of x for which each series converges?

9. $\sum_{n=1}^{\infty} \left(\frac{4}{x^2+1}\right)^n$

$\lim_{n\to\infty} \left| \left(\frac{4}{x^2+1}\right)^{n+1} \cdot \left(\frac{x^2+1}{4}\right)^n \right|$

$\lim_{n\to\infty} \left| \frac{4}{x^2+1} \right| < 1$

Only true if $x^2+1 > 4$

$x^2 > 3$

$x > \sqrt{3}$ or $x < -\sqrt{3}$

Both points lead to divergent series.

$\boxed{x > \sqrt{3} \text{ or } x < -\sqrt{3}}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(x + \frac{3}{2}\right)^n$

$\lim_{n\to\infty} \left| \frac{1}{n+1} \left(x+\frac{3}{2}\right)^{n+1} \cdot \frac{n}{(x+\frac{3}{2})^n} \right|$

$\lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \left(x+\frac{3}{2}\right) \right| < 1$

$-1 < x + \frac{3}{2} < 1$

$-\frac{5}{2} < x < -\frac{1}{2}$

$x = -\frac{5}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^n$

$\frac{1}{n}$

diverges!

$x = -\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1)^n$

converges

$\boxed{-\frac{5}{2} < x \leq -\frac{1}{2}}$

11. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$ $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\, 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{n}{(n+1)} \cdot \frac{x-2}{3} \right| < 1$

$-1 < \frac{x-2}{3} < 1$

$-3 < x-2 < 3$

$-1 < x < 5$

$\underline{x=-1}$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\, 3^n}$ Converges

$\underline{x=5}$ $\sum_{n=1}^{\infty} \frac{3^n}{n\, 3^n}$ diverges

$\boxed{-1 \le x < 5}$

12. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$ $\lim_{n\to\infty} \left| \frac{x^{5n+5}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right|$

$\lim_{n\to\infty} \left| \frac{1}{(n+1)} \cdot x^5 \right| = 0$

converges for all values of x.

$\boxed{(-\infty, \infty)}$

10.13 Radius and Interval of Convergence of Power Series

Test Prep

13. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

centered at $x=4$

$3 < x < 5$

$\boxed{3 < x < 5}$

Check endpoints

$\underline{x=3}$ $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \frac{1}{n}$ diverges

$\underline{x=5}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

14. If the power series $\sum_{n=0}^{\infty} a_n (x-5)^n$ converges at $x = 8$ and diverges at $x = 10$, which of the following must be true?

maybe I. The series converges at $x = 2$.

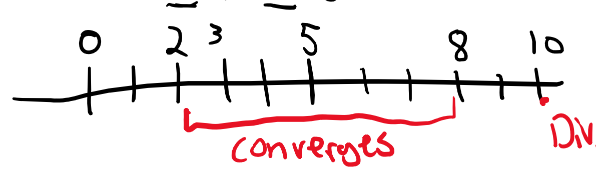
Yes II. The series converges at $x = 3$.

maybe III. The series diverges at $x = 0$.

center at $x=5$

radius is between 3 and 5

$3 \le r \le 5$



(A) I only (B) II only (C) I and II only (D) II and III only

15. The coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ satisfy $a_0 = 6$ and $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$ for all $n \geq 1$. What is the radius of convergence?

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = ?$$

$$b_n = a_n(x-3)^n$$

$$b_{n+1} = a_{n+1}(x-3)^{n+1}$$

$$a_{n+1} = \left(\frac{2(n+1)+1}{3(n+1)+1}\right)a_n$$

$$a_{n+1} = \frac{2n+3}{3n+4}a_n$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{3n+4}$$

$$\lim_{n\to\infty} \left|\frac{2n+3}{3n+4} \cdot \frac{(x-3)^{n+1}}{(x-3)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{2n+3}{3n+4} \cdot (x-3)\right| < 1$$

$$-1 < \tfrac{2}{3}(x-3) < 1$$

$$-\tfrac{3}{2} < x-3 < \tfrac{3}{2}$$

$$\tfrac{3}{2} < x < \tfrac{9}{2}$$

Interval $= 3$ $\boxed{\text{radius} = \tfrac{3}{2}}$

16. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ is 5, what is the interval of convergence?

centered at $x=5$
radius $=5$

$0 < x < 10$

Endpoints

$x=0$ diverges

$x=10$ converges

(A) $-5 < x < 5$ (B) $-5 < x \leq 5$ (C) $0 < x < 10$ (D) $0 < x \leq 10$

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17. Let $a_n = \frac{1}{n \ln n}$ for $n \geq 3$ and let f be the function given by $f(x) = \frac{1}{x \ln x}$.

a. The function f is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=3}^{\infty} a_n$.

$$\int_3^\infty \frac{1}{n \ln n}\,dn = \lim_{t\to\infty} \int_3^t \frac{1}{n \ln n}\,dn$$

let $u = \ln n$
$du = \frac{1}{n}dn$
$n\,du = dn$

$$\lim_{t\to\infty} \int_3^t \frac{1}{u}\,du$$

$$\lim_{t\to\infty} \ln|u| \Big|_3^t$$

$$\infty - 3 = \infty$$

Diverges by the Integral Test.

b. Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+2}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(x-2)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^n (x-2)^2}{(x-2)^n (x-2)^1} \cdot \frac{n \ln n}{(n+1) \ln(n+1)} \right|$$

1

$\frac{\infty}{\infty} \to$ use L'Hopital's Rule

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{1}{n} \cdot \frac{n+1}{1} = 1$$

$$\lim_{n\to\infty} |x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

✩ Check endpoints!

$\underline{x=1}$

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

converges

$\underline{x=3}$

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$

diverges

$$\boxed{1 \leq x < 3}$$