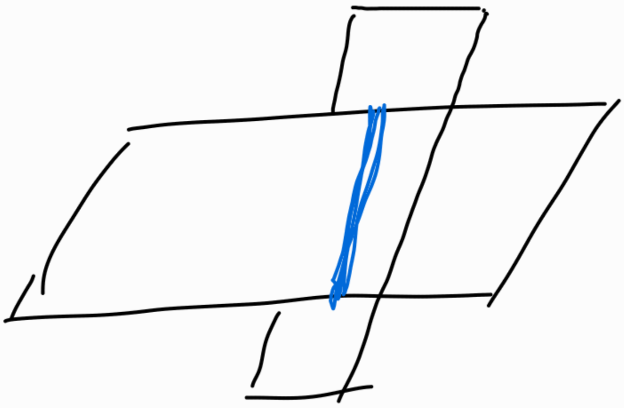
Lines of Intersection



Example: Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ and $2x+y-2z=5$. Then find parametric equations for the line of intersection.

$$\left.\begin{array}{l}\vec{n}_1=\langle 3,-6,-2\rangle \\ \vec{n}_2=\langle 2,1,-2\rangle\end{array}\right\} \quad \vec{n}_1 \neq k\cdot\vec{n}_2 \Rightarrow \text{not parallel}$$
$$\Rightarrow \text{They intersect}$$

Line of intersection lies on both planes.
Direction vector of the intersection line is perpendicular to the normal vectors.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 14, 2, 15\rangle = \vec{v}$$

vector parallel to the line of intersection (direction vector)

A point on the line has to satisfy both plane equations.

$$\begin{aligned} 3x-6y-2z&=15 \\ 2x+y-2z&=5 \end{aligned} \qquad \text{Take } z=0 \qquad \left.\begin{aligned} 3x-6y&=15 \\ 2x+y&=5 \end{aligned}\right\} \begin{aligned} x&=3 \\ y&=-1 \end{aligned}$$

$x = 3 + 14t$, $y = -1 + 2t$, $z = 15t \Rightarrow$ Intersection line.

Example: Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

$$3 \cdot \left(\frac{8}{3} + 2t\right) + 2 \cdot (-2t) + 6 \cdot (1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8 \Rightarrow t = -1 \qquad x = \frac{8}{3} - 2 = \frac{2}{3}, \quad y = 2, \quad z = 0$$

$P\left(\frac{2}{3}, 2, 0\right)$.

The Distance from a Point to a Plane

Definition: If P is a point on a plane with a normal $\vec{n}$, then the distance from a point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\vec{n}$, as given in the following formula:

$$d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

Example: Find the distance from $S(1,1,3)$ to the plane $3x + 2y + 6z = 6$.

$\vec{n} = \langle 3, 2, 6 \rangle$ Take $x = z = 0 \Rightarrow y = 3 \Rightarrow P(0, 3, 0)$

$\overrightarrow{PS} = \langle 1, -2, 3 \rangle$

$|\vec{n}| = \sqrt{3^2+2^2+6^2} = \sqrt{49} = 7$

$d = \left| \dfrac{3-4+18}{7} \right| = \dfrac{17}{7}$.

<u>Ex.</u>: Find a vector equation and parametric equations for the line through the point $(1,0,6)$ and perpendicular to the plane $x+3y+z=5$.

$\vec{n} = \langle 1,3,1 \rangle \Rightarrow$ direction vector

normal vector of the plane.

$\vec{r}_0 = \langle 1,0,6 \rangle$

$\vec{r} = \vec{r}_0 + t\cdot\vec{v} = (1+t)\vec{i} + (3t)\vec{j} + (6+t)\vec{k}$

direction vector

$x = 1+t, \quad y = 3t, \quad z = 6+t$

<u>Ex.</u>: Find parametric eqs. for the line through the points $(0, \frac{1}{2}, 1)$ and $(2,1,-3)$

$\vec{v} = \langle 2, +\frac{1}{2}, -4 \rangle$

$P_0 = (2,1,-3)$

$\vec{v} = \langle 4, 1, -8 \rangle$

$x = 2+2t$

$y = 1 + \frac{1}{2}t$

$z = -3-4t$

$\underline{Ex.}$: Is the line through $(-4,-6,1)$ and $(-2,0,-3)$ <u>parallel</u> to the line through $(10,18,4)$ and $(5,3,14)$?

$\vec{v}_1 = \langle 2, 6, -4 \rangle$, $\vec{v}_2 = \langle -5, -15, 10 \rangle = \langle 1, 3, -2 \rangle$

Parallel $\Longleftrightarrow$ $\underline{\vec{v}_1 = k \cdot \vec{v}_2}$, $k \in \mathbb{R}$ Take $\underline{\underline{k=2}}$

$\underline{Ex.}$: Find parametric equations for the line through the point $(0,1,2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line $x=1+t$, $y=1-t$, $z=2t$.

$P_0 = (0,1,2) \rightarrow$ point on the line.

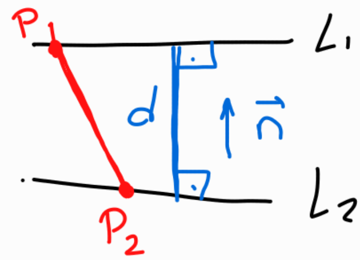
$\vec{n} = \langle 1,1,1 \rangle \Rightarrow$ perpendicular to the desired line.

$\vec{v}_1 = \langle 1,-1,2 \rangle \Rightarrow$ perpendicular " " " "

$$\vec{v} = \vec{n} \times \vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2 \rangle$$

$x = 3t$, $y = 1-t$, $z = 2-2t$

Ex.: Let L_1 be the line through the origin and the point $(2,0,-1)$. Let L_2 be the line through the points $(1,-1,1)$ and $(4,1,3)$. Find the distance between L_1 and L_2.



$L_1 : \vec{v}_1 = \langle 2,0,-1 \rangle$

$L_2 : \vec{v}_2 = \langle 3,2,2 \rangle$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 3 & 2 & 2 \end{vmatrix} = \langle 2,-7,4 \rangle \Rightarrow \text{perpendicular to both lines.}$$

$L_1 : P_1\,(0,0,0)$ $\qquad \overrightarrow{P_1P_2} = \langle 1,-1,1 \rangle$

$L_2 : P_2\,(1,-1,1)$

$$d = \left| \overrightarrow{P_1P_2} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| \frac{2+7+4}{\sqrt{4+49+16}} \right| = \frac{13}{\sqrt{69}}$$

Ex.: Find an equation of the plane through the line of intersection of the planes $x-z=1$ and $y+2z=3$ and perpendicular to the plane $x+y-2z=1$.

$\vec{n}_1 = \langle 1,0,-1 \rangle$

$\vec{n}_2 = \langle 0,1,2 \rangle$

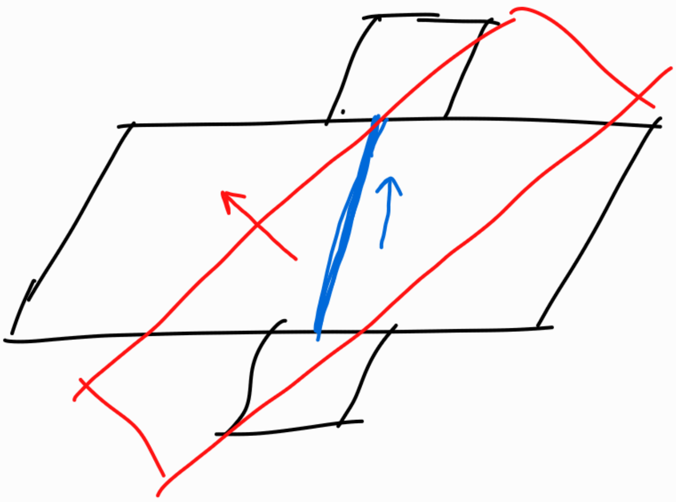
$$\vec{v}_1 = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \langle 1,-2,1 \rangle$$

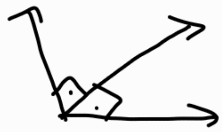
$\vec{n}_3 = \langle 1,1,-2 \rangle \Rightarrow$ lies on the desired plane

$$\vec{n} = \vec{v}_1 \times \vec{n}_3 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \langle 3, 3, 3 \rangle \Rightarrow \vec{n} = \langle 1, 1, 1 \rangle$$

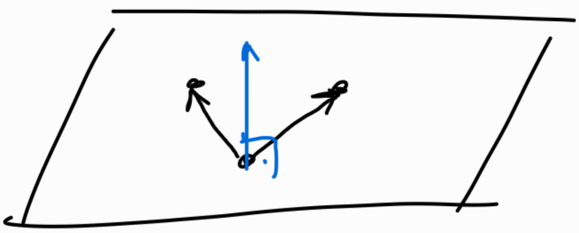
$$\begin{aligned} x - z &= 1 \\ y + 2z &= 3 \end{aligned} \qquad \text{Set } z = 0 \qquad \begin{aligned} x &= 1 \\ y &= 3 \end{aligned} \qquad P(1, 3, 0)$$

$$1(x-1) + 1 \cdot (y-3) + 1 \cdot z = 0 \Rightarrow x + y + z = 4 \quad //$$





Ex.: Find an equation of the plane through the points $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.



$\vec{v}_1 = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$

$\vec{v}_2 = \langle 1-0, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$

lie on the plane.

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle \qquad \begin{aligned} \vec{n} \cdot \vec{v}_1 &= 0 \\ \vec{n} \cdot \vec{v}_2 &= 0 \end{aligned}$$

By using $(0,1,1) \Rightarrow 1\cdot(x-0)+1\cdot(y-1)+1\cdot(z-1)=0$

$$x+y+z=2.$$

Ex.: Find an eq. of the plane that passes through the point $(3,5,-1)$ and contains the line $x=4-t$, $y=2t-1$, $z=-3t$.

$\vec{v}=\langle -1,2,-3\rangle \Rightarrow$ parallel to the plane.

Any point on the line is also on the plane.

$t=0 \Rightarrow P(4,-1,0)$

$(3,5,-1), P \Rightarrow \vec{a}=\langle 3-4, 5-(-1), -1-0\rangle$

$$=\langle -1,6,-1\rangle$$

or $\vec{a}=\langle 1,-6,1\rangle \Rightarrow$ parallel to the plane.

$$\vec{n}=\vec{v}\times\vec{a}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 1 & -6 & 1 \end{vmatrix}=\langle -16,-2,4\rangle$$

$\vec{n}=\langle 8,1,-2\rangle \quad P_0=(3,5,-1)$

$$8(x-3)+1\cdot(y-5)-2\cdot(z+1)=0$$

$$8x+y-2z=31.$$

Ex.: Find an eq. of the plane that passes through the point $(3,1,4)$ and contains the line of intersection of the planes $x+2y+3z=1$ and $2x-y+z=-3$.

$\vec{n}_1 = \langle 1,2,3 \rangle$

$\vec{n}_2 = \langle 2,-1,1 \rangle$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \langle 5,5,-5 \rangle$$

$\vec{v} = \langle 1,1,-1 \rangle \Rightarrow$ direction vector for the line of intersection

normal vector is perpendicular to $\vec{v}$.

Set $z=0 \Rightarrow$

$$x+2y=1$$
$$2x-y=-3 \rightarrow 4x-2y=-6$$
$$5x=-5 \Rightarrow x=-1 \quad y=1$$

$P(-1,1,0) \Rightarrow$ on the intersection line.

$\vec{a} = \langle 3-(-1), 1-1, 4-0 \rangle = \langle 4,0,4 \rangle \Rightarrow \vec{a} = \langle 1,0,1 \rangle$

perpendicular to normal vector

$$\vec{n} = \vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \langle 1,-2,-1 \rangle$$

$$1\cdot(x-3) - 2(y-1) - 1\cdot(z-4) = 0$$

$$x-2y-z=-3.$$

Ex.: Find an eq. for the plane consisting of all points that are equidistant from the points $(1,0,-2)$ and $(3,4,0)$.

Take a point on the plane: (x,y,z)

$$\left.\begin{aligned} d_1 &= \sqrt{(x-1)^2+y^2+(z+2)^2} \\ d_2 &= \sqrt{(x-3)^2+(y-4)^2+z^2} \end{aligned}\right\} \Rightarrow d_1 = d_2 \Rightarrow d_1^2 = d_2^2$$

$$x^2-2x+1+y^2+z^2+4z+4 = x^2-6x+9+y^2-8y+16+z^2$$

$$-2x+4z+5 = -6x-8y+25 \Rightarrow 4x+8y+4z=20$$

$$\underline{x+2y+z=5}$$

Ex.: Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

$\vec{n}_1 = \langle 1,1,1\rangle$

$\vec{n}_2 = \langle 1,-2,3\rangle$

Angle between Planes $=$ Angle between normal vectors

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| \cdot |\vec{n}_2| \cdot \cos\theta \Rightarrow \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$\cos\theta = \frac{1\cdot 1+1\cdot(-2)+1\cdot 3}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+(-2)^2+3^2}} = \frac{2}{\sqrt{3}\cdot\sqrt{14}} \Rightarrow \theta = \arccos\left(\frac{2}{\sqrt{42}}\right)$$

$$\approx 72^\circ$$