LINEARIZATION AND DIFFERENTIALS

(1) Evaluate the approximate value of $(1.003)^4 + 4.(1.003)^{3/2} - 5$ by using differentials.

Let $f(x) = x^4 + 4x^{3/2} - 5. \Rightarrow f'(x) = 4x^3 + 6x^{1/2}$

$x_0 = 1$, $\Delta x = 0.03$, $f(1) = 1+4-5=0$, $f'(1) = 4+6=10.$

$$\left.\begin{array}{l} dy = f'(x_0)\,dx \\ \Delta y = f(x_0+\Delta x) - f(x_0) \\ dy \approx \Delta y \end{array}\right\} \begin{array}{l} f'(1).(0.003) \approx f(1.003) - f(1) \\ 10.(0.003) \approx f(1.003) - 0 \\ \Rightarrow f(1.003) \approx 0.03 \mathbin{/\!/} \end{array}$$

(2) Find an approximate value for $\sqrt[5]{33^3}$ by using linearization.

$f(x) = x^{3/5} \Rightarrow f'(x) = \frac{3}{5}x^{-2/5}$, $a = 32$

$f(32) = 8$, $f'(32) = \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20}$

$L(x) = f(32) + f'(32).(x-32) = 8 + \frac{3}{20}(x-32)$

$f(33) \approx L(33) = 8 + \frac{3}{20} \cdot (33-32) = \frac{163}{20} = 8.15. \Rightarrow \sqrt[5]{33^3} \approx 8.15$

(3) Find an approximate value for $\sqrt[3]{81}$ by using differential.

$\sqrt[3]{81} = \sqrt[3]{9^2} = 9^{2/3} \Rightarrow f(x) = x^{2/3}$

$x_0 = 8$, $\Delta x = 1 = dx$

$x_0 + \Delta x = 9$

we need something close to a number that we know its 1/3 rd power $(8+1=9)$

$\Downarrow$

$f'(x) = \frac{2}{3}x^{-1/3}$

$f(8)=4, \quad f'(8)=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}$

$$\left.\begin{array}{l} dy=f'(x_0)\,dx \\ \Delta y=f(x_0+\Delta x)-f(x_0) \\ dy\approx\Delta y \end{array}\right\} \begin{array}{l} f'(8).1\approx f(9)-f(8) \\ \frac{1}{3}\approx f(9)-4 \Rightarrow f(9)=9^{2/3}\approx 4.33 \end{array}$$

4) Find an approximate value for $(0.99)^5+3.(0.99)^{5/3}-2$ by using linearization.

$f(x)=x^5+3.x^{5/3}-2 \Rightarrow f'(x)=5x^4+5x^{2/3}$

$a=1, \quad f(1)=1+3-2=2, \quad f'(1)=5.1+5.1=10$

$L(x)=f(1)+f'(1).(x-1)=2+10.(x-1)$

$f(0.99)\approx L(0.99)=2+10(\underbrace{0.99-1}_{-0.01})=2-0.1=1.9\approx f(0.99).$

5) Find the linearization of the function $f(x)=\ln\left(\frac{x^3+1}{x^2+1}\right)+\arctan(x^2-1)+2$ at $x=2$ and evaluate the approximate value of $f(1.1)$.

$f(x)=\ln(x^3+1)-\ln(x^2+1)+\arctan(x^2-1)+2 \Rightarrow f(1)=2$

$f'(x)=\frac{3x^2}{x^3+1}-\frac{2x}{x^2+1}+\frac{2x}{1+(x^2-1)^2} \Rightarrow f'(1)=\frac{3}{2}-\frac{2}{2}+\frac{2}{1}=\frac{5}{2}$

$L(x)=f(1)+f'(1).(x-1)=2+\frac{5}{2}.(x-1)$

$f(1.1)\approx L(1.1)=2+\frac{5}{2}(1.1-1)=2.25 \Rightarrow f(1.1)\approx 2.25.$

6) Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(-1)=3$, $g'(-1)=-3$ and $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x)=[g(x)]^2 \cdot (x^3+2)$. Find an approximate value for $f(-0.9)$.

$a=-1=x_0, \quad \Delta x=0.1, \quad x_0+\Delta x=-0.9, \quad f(-1)=9$

$f'(x)=2 \cdot g(x) \cdot g'(x) \cdot (x^3+2)+[g(x)]^2 \cdot 3x^2$

$f'(-1)=2 \cdot \underbrace{g(-1)}_{3} \cdot \underbrace{g'(-1)}_{-3} \cdot \underbrace{(-1+2)}_{1}+\underbrace{[\underbrace{g(-1)}_{3}]^2}_{9} \cdot 3 \cdot 1=-18+27=9$

i) Linearization

$L(x)=f(-1)+f'(-1) \cdot (x-(-1))=9+9 \cdot (x+1)$

$f(-0.9) \approx L(-0.9)=9+9 \cdot (-0.9+1)=9+9 \cdot (0.1)=9.9$

ii) Differential

$$\left.\begin{array}{l} dy=f'(-1)dx \\ \Delta y=f(-0.9)-f(-1) \\ dy \approx \Delta y \end{array}\right\} \begin{array}{l} f'(-1)dx \approx f(-0.9)-f(-1) \\ 9 \cdot (0.1) \approx f(-0.9)-9 \\ f(-0.9) \approx 9.9 \end{array}$$