MAT1320-Linear Algebra Lecture Notes

Special Types of Square Matrices

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Periodic Matrix

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$$A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \Rightarrow A^2 = I_3$$

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Observe that patterns of 0's in the third matrix have been omitted.

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Note: A lower triangular matrix is a square matrix whose entries above the diagonal are all zero.

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- Note that a matrix A must be square if $A^T = A$ or $A^T = -A$.

Example

Let $A = \begin{pmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix}$,

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i. By inspection, the symmetric elements in A are equal, or $A^T = A$. Thus, A is symmetric.

ii. The diagonal elements of B are 0 and symmetric elements are negatives of each other, or $B^T = -B$. Thus, B is skew-symmetric.

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iii. Because C is not square, C is neither symmetric nor skew-symmetric.

Orthogonal Matrices

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Then

$$A^H = \begin{pmatrix} 2-8i & -6i \\ 5+3i & 1+4i \\ 4+7i & 3-2i \end{pmatrix}.$$

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A complex matrix A is unitary if $A^H A = AA^H = I$; that is, if $A^H = A^{-1}$.

Example

Consider the following complex matrices:

$$A = \begin{pmatrix} 3 & 1-2i & 4+7i \\ 1+2i & -4 & -2i \\ 4-7i & 2i & 5 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{pmatrix}$$

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ii. Multiplying B by B^H yields I; that is, $BB^H = I$. This implies $B^H B = I$, as well. Thus, $B^H = B^{-1}$ which means B is unitary.

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