MAT 1320 LINEAR ALGEBRA
FURTHER EXERCISES

Note: Rank A : the number of nonzero rows in an echelon form of A.

1. If $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ -1 & 1 & -2 \end{pmatrix}$, then find $rank\,(A)$.

Answer: rank A = 2

2. If $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{pmatrix}$, then find $rank\,(A)$.

$$\xrightarrow[\substack{r_4 \to r_4 - 5r_1 \\ r_5 \to r_5 - r_1}]{\substack{r_2 \to r_2 - 3r_1 \\ r_3 \to r_3 - 2r_1}} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & -2 & -3 \end{pmatrix} \xrightarrow[r_3 \cdot \frac{-1}{5}]{r_4 \to r_4 - r_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \end{pmatrix} \xrightarrow[r_4 \leftrightarrow r_5]{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_4 \to r_4 - r_2]{r_3 \to r_3 + 4r_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_4 \to r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_3 \cdot \frac{-1}{2}]{} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank A = 3.

Note: $[A : I_n] \longrightarrow \cdots \longrightarrow [I_n : A^{-1}]$ (If A^{-1} exists)

(via elementary row operations)

3. Find the inverse of $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ (if exists).

$$\left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right).$$

$$\xrightarrow{r_1 \to r_1 - 3r_2 - 3r_3} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} 4 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

I_3 $\qquad$ A^{-1}

4. Calculate $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$.

$$\underset{r_3 \to r_3 - r_1}{\overset{r_2 \to r_2 - r_1}{=}} \begin{vmatrix} 1 & x & y+z \\ 0 & y-x & x-y \\ 0 & z-x & x-z \end{vmatrix} \overset{\text{Laplace for first column}}{=} 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} y-x & x-y \\ z-x & x-z \end{vmatrix}$$

$$= (y-x)(x-z) - (x-y)(z-x)$$
$$= (y-x)(x-z) + (y-x)(z-x) = 0.$$

5. Calculate $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

$$\underset{r_3 \to r_3 - r_1}{\overset{r_2 \to r_2 - r_1}{=}} \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} b-a & b^3-a^3 \\ c-a & c^3-a^3 \end{vmatrix} = (b-a)(c-a) \cdot \begin{vmatrix} 1 & b^2+ab+a^2 \\ 1 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a) \cdot \left(c^2+ca+\cancel{a^2} - b^2 - ab - \cancel{a^2} \right)$$
$$= (b-a)(c-a)\left(c^2 - b^2 + a(c-b) \right)$$

$$= (b-a)(c-a)(c-b)(a+b+c).$$

6. Calculate $\begin{vmatrix} a & a^3 & bc \\ b & b^3 & ca \\ c & c^3 & ab \end{vmatrix}$.

$$= \frac{1}{abc} \cdot \begin{vmatrix} a^2 & a^4 & abc \\ b^2 & b^4 & abc \\ c^2 & c^4 & abc \end{vmatrix} = \frac{abc}{abc} \begin{vmatrix} a^2 & a^4 & 1 \\ b^2 & b^4 & 1 \\ c^2 & c^4 & 1 \end{vmatrix}$$

$$\underset{r_3 \to r_3 - r_1}{\overset{r_2 \to r_2 - r_1}{=}} \begin{vmatrix} a^2 & a^4 & 1 \\ b^2-a^2 & b^4-a^4 & 0 \\ c^2-a^2 & c^4-a^4 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+3} \begin{vmatrix} b^2-a^2 & b^4-a^4 \\ c^2-a^2 & c^4-a^4 \end{vmatrix} = (b^2-a^2)(c^2-a^2) \begin{vmatrix} 1 & b^2+a^2 \\ 1 & c^2+a^2 \end{vmatrix}$$

$$= (b^2-a^2)(c^2-a^2)(c^2-b^2)$$

7. If $\begin{vmatrix} r & t & 1 \\ p & -1 & w \\ 2 & s & u \end{vmatrix} = 1$, then $\begin{vmatrix} r-2 & t-s & 1-u \\ -p+2u & 1+su & -w+u^2 \\ 4 & 2s & 2u \end{vmatrix} = ?$

$$= 2 \cdot \begin{vmatrix} r-2 & t-s & 1-u \\ -p+2u & 1+su & -w+u^2 \\ 2 & s & u \end{vmatrix} \underset{r_1 \to r_1 + r_3}{\overset{r_2 \to r_2 - u r_3}{=}} 2 \cdot \begin{vmatrix} r & t & 1 \\ -p & 1 & -w \\ 2 & s & u \end{vmatrix} = -2 \begin{vmatrix} r & t & 1 \\ p & -1 & -w \\ 2 & s & u \end{vmatrix}$$

$$= -2 \cdot 1 = -2.$$

8. Show that $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n+1 & n+2 & n+3 & \cdots & 2n-1 & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n-1 & 3n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2-1 & n^2 \end{vmatrix} = 0.$

$$\overset{\substack{r_2 - r_1 \\ r_3 - r_1 \\ \vdots \\ r_n - r_1}}{=} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \\ 2n & 2n & \cdots & \cdots & 2n & 2n \\ \vdots & & & & & \\ n^2-n & n^2-n & \cdots & \cdots & \cdots & n^2-n \end{vmatrix} \overset{r_3 \to r_3 - n r_2}{=} \begin{vmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n & \cdots & n & n \\ 0 & 0 & \cdots & 0 & 0 \\ & & & & \\ n^2-n & \cdots & \cdots & \cdots & n^2-n \end{vmatrix}$$

There is zero row. Then,

$= 0.$

9. Show that $\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}_{n\times n} = (-1)^{n-1}(n-1).$

See that the sum of entries in each row is $n-1$. Then,

$$\overset{C_1 \to C_1+C_2+\dots+C_n}{=} \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ n-1 & 0 & 1 & \cdots & 1 \\ n-1 & 1 & 0 & \cdots & 1 \\ \vdots & & & & \\ n-1 & 1 & 1 & \cdots & 0 \end{vmatrix} = (n-1).\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & & \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = (n-1).\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$= (n-1).1.(-1)^{1+1}.\begin{vmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{vmatrix}_{(n-1)\times n-1} = (n-1).(-1)^{n+1}$$

10. Calculate $\begin{vmatrix} x+y & x & \cdots & x \\ x & x+y & \cdots & x \\ \vdots & \vdots & \cdots & \vdots \\ x & x & \cdots & x+y \end{vmatrix}_{n\times n}.$

$$= \begin{vmatrix} nx+y & x & \cdots & x \\ nx+y & x+y & \cdots & x \\ \vdots & \vdots & & \\ nx+y & x & \cdots & x+y \end{vmatrix} = (nx+y).\begin{vmatrix} 1 & x & \cdots & x \\ 1 & x+y & \cdots & x \\ \vdots & \vdots & & \\ 1 & x & \cdots & x+y \end{vmatrix} = (nx+y).\begin{vmatrix} 1 & x & \cdots & x \\ 0 & y & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & y \end{vmatrix}$$

$$= (nx+y).1.(-1)^{1+1}.\begin{vmatrix} y & & 0 \\ & \ddots & \\ 0 & & y \end{vmatrix}_{(n-1)\times(n-1)} = (nx+y).y^{n-1}$$

11. Calculate $\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}.$

$$\overset{r_1+r_2}{=} \begin{vmatrix} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} = (a+b).\begin{vmatrix} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2b-2a-2c & c & a+c \\ a-b & a-b & b \end{vmatrix} = 1.(-1)^{1+3}.\begin{vmatrix} 2b-2a-2c & c \\ a-b & a-b \end{vmatrix} = (a-b).\begin{vmatrix} 2b-2a-2c & c \\ 1 & 1 \end{vmatrix}$$

$$= (a-b)(2b-2a-3c)$$

12. Calculate $\begin{vmatrix} 1+x & 1 & 1 & \vdots & 1 \\ 1 & 1+x & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \ddots & 1+x & 1 \\ 1 & \cdots & \cdots & 1 & 1+x \end{vmatrix}_{n\times n}$.

(Similar to exercise 10.)

(Replace x and y by 1 and x respectively.)

13. Calculate $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$.

14. Calculate $\begin{vmatrix} x^2 & x^2+2x+1 & x^2+4x+4 \\ x^2+2x+1 & x^2+4x+4 & x^2+6x+9 \\ x^2+4x+4 & x^2+6x+9 & x^2+8x+16 \end{vmatrix}$.

$$= \begin{vmatrix} x^2 & 2x+1 & 4x+4 \\ x^2+2x+1 & 2x+3 & 4x+8 \\ x^2+4x+4 & 2x+5 & 4x+12 \end{vmatrix} \underset{r_3-r_1}{\overset{r_2-r_1}{=}} \begin{vmatrix} x^2 & 2x+1 & 4x+4 \\ 2x+1 & 2 & 4 \\ 4x+4 & 4 & 8 \end{vmatrix} \overset{c_3-2c_2}{=} \begin{vmatrix} x^2 & 2x+1 & 2x+2 \\ 2x+1 & 2 & 0 \\ 4x+4 & 4 & 0 \end{vmatrix}$$

Laplace expansion for 3rd column.

$$= (2x+2)\cdot(-1)^{1+3}\cdot \begin{vmatrix} 2x+1 & 2 \\ 4x+4 & 4 \end{vmatrix} = -(2x+2)\cdot 4 = 8x+8.$$

15. Find the solution of the system of linear equations
$$\begin{aligned} 3x_1 - 2x_2 - x_4 &= 7 \\ 2x_2 + 2x_3 + x_4 &= 5 \\ x_1 - 2x_2 - 3x_3 - 2x_4 &= 1. \end{aligned}$$

16. For which values of a, the system of linear equations
$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + (a^2 - 1)\, x_3 &= a + 1 \\ 2x_1 + 3x_2 + 2x_3 &= 5 \end{aligned}$$

The number of the unknows $n = 3$.

a) is inconsistent? $\text{rank} A \neq \text{rank}(A:b)$

b) has unique solution? $\text{rank} A = \text{rank}(A:b) = n = 3$

c) has infinitely many solutions? $\text{rank} A = \text{rank}(A:b) < n = 3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & a^2-1 & a+1 \\ 2 & 3 & 2 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & a^2-3 & a-3 \\ 0 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & a^2-3 & a-3 \\ 0 & 0 & 3-a^2 & 4-a \end{array}\right]$$

c) In the case there are infinite solutions, the last row must be zero. This is not possible because there is no real number a such that $3-a^2=0$ and $4-a$ at the same time.

a) In the case there is no solution, $3-a^2$ must be 0 and $4-a$ must be nonzero. $\Rightarrow 3-a^2=0 \Rightarrow a = \mp\sqrt{3}$ (Then, clearly, $4-a = 4 \mp \sqrt{3} \neq 0$)

b) In the case there is unique solution, $3-a^2 \neq 0$. This guarantees that $\text{rank} A = \text{rank}(A:b) = 3 \Rightarrow 3 \neq a^2 \Rightarrow a \neq \mp\sqrt{3}$.

17. Determine the solution of the system of linear equations $\begin{matrix} x + 2y + \lambda z = 1 \\ 2x + \lambda y + 8z = 3 \end{matrix}$ with respect to λ.

$n = 3$

$$\left(\begin{array}{ccc:c} 1 & 2 & \lambda & 1 \\ 2 & \lambda & 8 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} 1 & 2 & \lambda & 1 \\ 0 & \lambda-4 & 8-2\lambda & 1 \end{array}\right)$$

If $\lambda - 4 = 0$ and $8 - 2\lambda = 0$ at the same time, then rank $A = 2$ and rank $(A:b) = 3$ and so there is no solution. That is, if $\lambda = 4$, there is no solution.

If $\lambda \neq 4$, then rank $A = 2 =$ rank $(A:b)$ and there are infinite solutions depending on $n -$ rank $A = 3 - 2 = 1$ parameter.

18. Find the solution of $\begin{matrix} x + 3y - 2z + t = 4 \\ x + 2y - 4z - 3t = 6 \\ 2x + 5y - 5z - 2t = 10 \end{matrix}$?

19. For which values of a and b, the system of linear equations
$$\begin{aligned} y - 2z &= b \\ x - y + z &= 2 \\ x + ay &= 3 \end{aligned}$$
$\Rightarrow n = 3$.

- is inconsistent? $a = -\frac{1}{2},\ b \neq 2$
- has unique solution? $a \neq -1/2,\ b \in \mathbb{R}$
- has infinitely many solutions? $a = -\frac{1}{2},\ b = 2$

$$\left(\begin{array}{ccc:c} 0 & 1 & -2 & b \\ 1 & -1 & 1 & 2 \\ 1 & a & 0 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 1 & a & 0 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 0 & a+1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - (a+1) r_2} \left(\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & b \\ 0 & 0 & 2a+1 & -ab-b+1 \end{array}\right)$$

- If $2a+1 = 0$ and $-ab-b+1 \neq 0$, then there is no solution. $\Rightarrow a = -\frac{1}{2},\ b \neq 2$.
- If $2a+1 = 0$ and $-ab-b+1 = 0$, then there is infinite solutions. $\Rightarrow a = -\frac{1}{2},\ b = 2$
- If $2a+1 \neq 0$, then there is unique solution. $\Rightarrow a \neq -1/2,\ b \in \mathbb{R}$.

20. For which values of a and b, the system
$$\begin{aligned} 2x - y + 2az + t &= b \\ 2x - y + (2a+1)\,z + (a+1)\,t &= 0 \\ -2x + y + (1-2a)\,z - 2t &= -2b - 2 \end{aligned}$$
$\longrightarrow \underline{n = 4}$.

- is inconsistent? $-1 - a = 0 \Rightarrow a = -1,\ b \in \mathbb{R}$
- has unique solution? No possible because no possible for rank $A = 4$.
- has infinitely many solutions? $-1 - a \neq -1 \Rightarrow a \neq -1,\ b \in \mathbb{R}$.

$$\left(\begin{array}{cccc:c} 2 & -1 & 2a & 1 & b \\ 2 & -1 & 2a+1 & a+1 & 0 \\ -2 & 1 & 1-2a & -2 & -2b-2 \end{array}\right) \xrightarrow[r_3 \to r_3 + r_2]{r_1 \to r_2 - r_1} \left(\begin{array}{cccc:c} 0 & 0 & -1 & -a & b \\ 2 & -1 & 2a+1 & a+1 & 0 \\ 0 & 0 & 1 & -1 & -b-2 \end{array}\right)$$

Firstly $r_2 \leftrightarrow r_1$, and then $r_2 + r_3$

$$\longrightarrow \left(\begin{array}{cccc:c} 2 & -1 & 2a+1 & a+1 & 0 \\ 0 & 0 & -1 & -a & b \\ 0 & 0 & 0 & -1-a & -2 \end{array}\right)$$

Observe that rank$(A \,\vdots\, b) = 3$

- If $-1 - a = 0$, then rank $A = 2 \neq$ rank$(A \,\vdots\, b) = 3$ and so there is no solution
- If $-1 - a \neq 0$, then rank $A = 3 =$ rank$(A \,\vdots\, b) < n = 4$ and so infinite solutions with $4 - 3 = 1$ parameter.