

# Vector-Valued Functions and Motion in Space

## Definition

A vector-valued function is a function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \quad (3.1)$$

where the component functions  $f$ ,  $g$ , and  $h$ , are real-valued functions of the parameter  $t$ . Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle. \quad (3.2)$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

**EXAMPLE 1** If

$$\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$$

then the component functions are

$$f(t) = t^3 \quad g(t) = \ln(3 - t) \quad h(t) = \sqrt{t}$$

By our usual convention, the domain of  $\mathbf{r}$  consists of all values of  $t$  for which the expression for  $\mathbf{r}(t)$  is defined. The expressions  $t^3$ ,  $\ln(3 - t)$ , and  $\sqrt{t}$  are all defined when  $3 - t > 0$  and  $t \geq 0$ . Therefore the domain of  $\mathbf{r}$  is the interval  $[0, 3)$ .

To calculate the limit of a vector function, we find the limit of each component scalar function.

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \left( \lim_{t \rightarrow t_0} f(t) \right) \mathbf{i} + \left( \lim_{t \rightarrow t_0} g(t) \right) \mathbf{j} + \left( \lim_{t \rightarrow t_0} h(t) \right) \mathbf{k} \quad (3)$$

provides a practical way to calculate limits of vector functions.

**EXAMPLE 2** If  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ , then

$$\begin{aligned} \lim_{t \rightarrow \pi/4} \mathbf{r}(t) &= \left( \lim_{t \rightarrow \pi/4} \cos t \right) \mathbf{i} + \left( \lim_{t \rightarrow \pi/4} \sin t \right) \mathbf{j} + \left( \lim_{t \rightarrow \pi/4} t \right) \mathbf{k} \\ &= \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} + \frac{\pi}{4} \mathbf{k}. \end{aligned}$$

We define continuity for vector functions the same way we define continuity for scalar functions defined over an interval.

**EXAMPLE 2** Find  $\lim_{t \rightarrow 0} \mathbf{r}(t)$ , where  $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$ .

**SOLUTION** According to Definition 1, the limit of  $\mathbf{r}$  is the vector whose components are the limits of the component functions of  $\mathbf{r}$ :

$$\begin{aligned}\lim_{t \rightarrow 0} \mathbf{r}(t) &= \left[ \lim_{t \rightarrow 0} (1 + t^3) \right] \mathbf{i} + \left[ \lim_{t \rightarrow 0} te^{-t} \right] \mathbf{j} + \left[ \lim_{t \rightarrow 0} \frac{\sin t}{t} \right] \mathbf{k} \\ &= \mathbf{i} + \mathbf{k} \quad (\text{by Equation 3.3.2})\end{aligned}$$

**DEFINITION** A vector function  $\mathbf{r}(t)$  is **continuous at a point**  $t = t_0$  in its domain if  $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$ . The function is **continuous** if it is continuous over its interval domain.

### Example 5 – Continuity of Vector-Valued Functions

Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k} \quad a \text{ is a constant.}$$

at  $t = 0$ .

**Solution:**

As  $t$  approaches 0, the limit is

$$\begin{aligned}\lim_{t \rightarrow 0} \mathbf{r}(t) &= \left[ \lim_{t \rightarrow 0} t \right] \mathbf{i} + \left[ \lim_{t \rightarrow 0} a \right] \mathbf{j} + \left[ \lim_{t \rightarrow 0} (a^2 - t^2) \right] \mathbf{k} \\ &= 0\mathbf{i} + a\mathbf{j} + a^2\mathbf{k} \\ &= a\mathbf{j} + a^2\mathbf{k}.\end{aligned}$$

Because

$$\begin{aligned}\mathbf{r}(0) &= (0)\mathbf{i} + (a)\mathbf{j} + (a^2)\mathbf{k} \\ &= a\mathbf{j} + a^2\mathbf{k}\end{aligned}$$

you can conclude that  $\mathbf{r}$  is continuous at  $t = 0$ .

By similar reasoning, you can conclude that the vector-valued function  $\mathbf{r}$  is continuous at all real-number values of  $t$ .

## Derivatives

The **derivative**  $\mathbf{r}'$  of a vector function  $\mathbf{r}$  is defined in much the same way as for real-valued functions:

1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$

2 **Theorem** If  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f, g$ , and  $h$  are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

### V EXAMPLE 1

(a) Find the derivative of  $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$ .

#### SOLUTION

(a) According to Theorem 2, we differentiate each component of  $\mathbf{r}$ :

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos 2t\mathbf{k}$$

### Differentiation Rules for Vector Functions

Let  $\mathbf{u}$  and  $\mathbf{v}$  be differentiable vector functions of  $t$ ,  $\mathbf{C}$  a constant vector,  $c$  any scalar, and  $f$  any differentiable scalar function.

1. *Constant Function Rule:*  $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:*  $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$   
 $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:*  $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:*  $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:*  $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:*  $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:*  $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

## Example 4 – Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \text{ and } \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k},$$

find

a.  $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

and

b.  $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)].$

### Example 4(a) – Solution

Because  $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$  and  $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ , you have

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \frac{1}{t} \\ &= 3 + \frac{1}{t}. \end{aligned}$$

### Example 4(b) – Solution

cont'd

Because  $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$  and  $\mathbf{u}''(t) = 2\mathbf{i}$ , you have

$$\begin{aligned} \frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}. \end{aligned}$$

## Example 1 – Differentiation of Vector-Valued Functions

For the vector-valued function given by  $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$ , find  $\mathbf{r}'(t)$ . Then sketch the plane curve represented by  $\mathbf{r}(t)$ , and the graphs of  $\mathbf{r}(1)$  and  $\mathbf{r}'(1)$ .

**Solution:**

Differentiate on a component-by-component basis to obtain

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \quad \text{Derivative}$$

From the position vector  $\mathbf{r}(t)$ , you can write the parametric equations  $x = t$  and  $y = t^2 + 2$ .

The corresponding rectangular equation is  $y = x^2 + 2$ .

When  $t = 1$ ,  $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$  and  $\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}$ .

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In Figure 12.9,  $\mathbf{r}(1)$  is drawn starting at the origin, and  $\mathbf{r}'(1)$  is drawn starting at the terminal point of  $\mathbf{r}(1)$ .

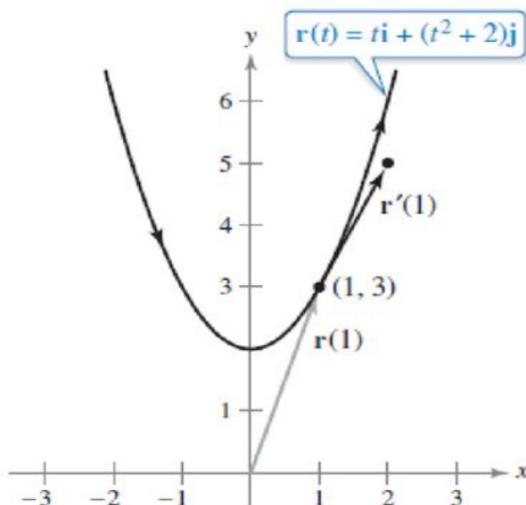


Figure 12.9

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**DEFINITIONS** If  $\mathbf{r}$  is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time  $t$ , the direction of  $\mathbf{v}$  is the **direction of motion**, the magnitude of  $\mathbf{v}$  is the particle's **speed**, and the derivative  $\mathbf{a} = d\mathbf{v}/dt$ , when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position:  $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ .
2. Speed is the magnitude of velocity: Speed =  $|\mathbf{v}|$ .
3. Acceleration is the derivative of velocity:  $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$ .
4. The unit vector  $\mathbf{v}/|\mathbf{v}|$  is the direction of motion at time  $t$ . ( $T = \frac{\mathbf{v}}{|\mathbf{v}|}$  → **unit tangent vector**)

**EXAMPLE 4** Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector  $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 5 \cos^2 t \mathbf{k}$ . Sketch the velocity vector  $\mathbf{v}(7\pi/4)$ .

**Solution** The velocity and acceleration vectors at time  $t$  are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} - 10 \cos t \sin t \mathbf{k} \\ &= -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} - 5 \sin 2t \mathbf{k}, \\ \mathbf{a}(t) &= \mathbf{r}''(t) = -2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} - 10 \cos 2t \mathbf{k},\end{aligned}$$

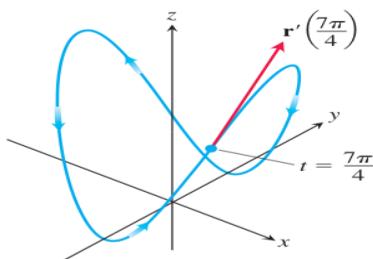
and the speed is

$$|\mathbf{v}(t)| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (-5 \sin 2t)^2} = \sqrt{4 + 25 \sin^2 2t}.$$

When  $t = 7\pi/4$ , we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 5\mathbf{k}, \quad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}, \quad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

A sketch of the curve of motion, and the velocity vector when  $t = 7\pi/4$ , can be seen in Figure 13.7. ■



**FIGURE 13.7** The curve and the velocity vector when  $t = 7\pi/4$  for the motion given in Example 4.

**EXAMPLE 3**

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (1 + 3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$$

**Solution** In that example, we found

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3 \sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3 \cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}. \quad \blacksquare$$

## Integrals of Vector Functions

**DEFINITION** The **indefinite integral** of  $\mathbf{r}$  with respect to  $t$  is the set of all antiderivatives of  $\mathbf{r}$ , denoted by  $\int \mathbf{r}(t) dt$ . If  $\mathbf{R}$  is any antiderivative of  $\mathbf{r}$ , then

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}.$$

**EXAMPLE 1**

To integrate a vector function, we integrate each of its components.

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}) dt = \left( \int \cos t dt \right) \mathbf{i} + \left( \int dt \right) \mathbf{j} - \left( \int 2t dt \right) \mathbf{k} \quad (1)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \quad (2)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \quad \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

As in the integration of scalar functions, we recommend that you skip the steps in Equations (1) and (2) and go directly to the final form. Find an antiderivative for each component and add a *constant vector* at the end. ■

**DEFINITION** If the components of  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$  are integrable over  $[a, b]$ , then so is  $\mathbf{r}$ , and the **definite integral** of  $\mathbf{r}$  from  $a$  to  $b$  is

$$\int_a^b \mathbf{r}(t) dt = \left( \int_a^b f(t) dt \right) \mathbf{i} + \left( \int_a^b g(t) dt \right) \mathbf{j} + \left( \int_a^b h(t) dt \right) \mathbf{k}.$$

**EXAMPLE 2** As in Example 1, we integrate each component.

$$\begin{aligned} \int_0^\pi ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}) dt &= \left( \int_0^\pi \cos t dt \right) \mathbf{i} + \left( \int_0^\pi dt \right) \mathbf{j} - \left( \int_0^\pi 2t dt \right) \mathbf{k} \\ &= [\sin t]_0^\pi \mathbf{i} + [t]_0^\pi \mathbf{j} - [t^2]_0^\pi \mathbf{k} \\ &= [0 - 0] \mathbf{i} + [\pi - 0] \mathbf{j} - [\pi^2 - 0^2] \mathbf{k} \\ &= \pi \mathbf{j} - \pi^2 \mathbf{k} \end{aligned}$$



## Arc Length in Space

1

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

2

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

3

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

**V EXAMPLE 1** Find the length of the arc of the circular helix with vector equation  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$  from the point  $(1, 0, 0)$  to the point  $(1, 0, 2\pi)$ .

**SOLUTION** Since  $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$ , we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from  $(1, 0, 0)$  to  $(1, 0, 2\pi)$  is described by the parameter interval  $0 \leq t \leq 2\pi$  and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

