

Partial Derivatives

DEFINITION The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$z = f(x, y)$$

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \frac{\partial z}{\partial x}$$

provided the limit exists.

DEFINITION The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$z = f(x, y)$$

$$f_y(x_0, y_0) = \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = \frac{d}{dy} f(x_0, y) \Big|_{y=y_0} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = \frac{\partial z}{\partial y}$$

provided the limit exists.

Example: Let $f(x, y) = x^2 y$. Find the f_y using by the definition of partial derivative.

$$\begin{aligned} f_y(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2(y+h) - x^2 y}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 y + x^2 h - x^2 y}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 h}{h} = x^2$$

NOTATIONS FOR PARTIAL DERIVATIVES If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

EXAMPLE 3 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example: Let $f(x,y) = e^{xy} \cdot \cos(x+y)$. $f_x(0, \pi) = ?$

FUNCTIONS OF MORE THAN TWO VARIABLES

$$\bullet f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$\bullet f_y(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

$$\bullet f_z(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

EXAMPLE 5 Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$.

In general, if u is a function of n variables, $u = f(x_1, x_2, \dots, x_n)$, its partial derivative with respect to the i th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0) , however, then f is continuous at (x_0, y_0) , as we see at the end of this section.

EXAMPLE 8

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

- (a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
- (b) Prove that f is not continuous at the origin.
- (c) Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

Solution

- (a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy -plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

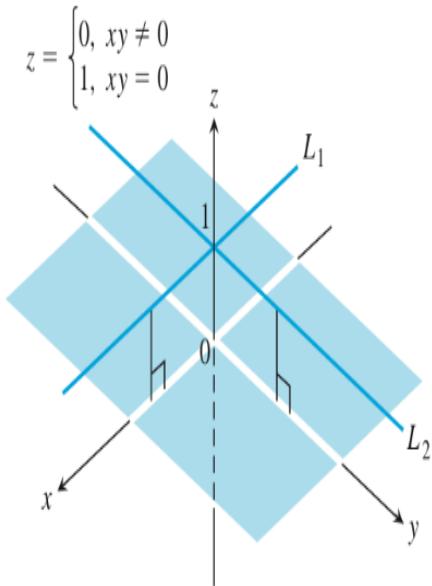


FIGURE 14.21 The graph of

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \Big|_{y=x} = \lim_{(x,y) \rightarrow (0,0)} 0 = 0.$$

- (b) Since $f(0, 0) = 1$, the limit in part (a) proves that f is not continuous at $(0, 0)$.
- (c) To find $\partial f / \partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x , and the graph of f is the line L_1 in Figure 14.21. The slope of this line at any x is $\partial f / \partial x = 0$. In particular, $\partial f / \partial x = 0$ at $(0, 0)$. Similarly, $\partial f / \partial y$ is the slope of line L_2 at any y , so $\partial f / \partial y = 0$ at $(0, 0)$. ■

Second-Order Partial Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f . If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x \cos y + ye^x) & \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x \cos y + ye^x) \\ &= \cos y + ye^x & &= -x \sin y + e^x \end{aligned}$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \cos y. \quad \blacksquare$$

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

EXAMPLE 10 Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\begin{aligned}\frac{\partial^3 f}{\partial x \partial y^2} &= f_{yyx}, \\ \frac{\partial^4 f}{\partial x^2 \partial y^2} &= f_{yyxx},\end{aligned}$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y , then x , then y again, and finally with respect to z :

$$\begin{aligned}f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4.\end{aligned}$$

2 THE CHAIN RULE (CASE 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Since we often write $\partial z / \partial x$ in place of $\partial f / \partial x$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

3 THE CHAIN RULE (CASE 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z / \partial s$ and $\partial z / \partial t$.

THEOREM 8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x . Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \quad (1)$$

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

• Let $z = f(g(x, y))$. Then,

$$\frac{\partial z}{\partial x} = f'(g(x, y)) \cdot g_x(x, y)$$

$$\frac{\partial z}{\partial y} = f'(g(x, y)) \cdot g_y(x, y)$$

Example: Let $z = f\left(\frac{x}{y}\right)$. Show that

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 0.$$

Question 1: Let the function $f(x,y) = \sin(xy - y^2x) + \ln\left(\frac{1}{x+y}\right)$ be given. Then, which of the following is the value of $\frac{f_y(1,1)}{f_{yy}(1,1)}$?

- A) $\frac{16}{23}$ B) $\frac{10}{23}$ C) $\frac{13}{12}$ D) $\frac{17}{12}$ E) 2

Question 2: Let $z = \sin(x^2y^2) + \tan(xy)$.

If $\frac{\partial^2 z}{\partial x^2} \Big|_{(0,0)} = A$ and $\frac{\partial^2 z}{\partial y^2} \Big|_{(0,0)} = B$, then

what is the value of $A+B$?

- A) 0 B) 1 C) 2 D) 3 E) 4

Question 3: Let $f(x,y) = (x^2+y^2) \cdot \tan(\ln(xy)) + e^{(x-y)}$. Then, what is the value of $2f_x(1,1) - f_y(1,1)$?

- A) 3 B) 4 C) 5 D) 6 E) 7

Question 4: Let $f(x,y) = \begin{cases} \frac{x^3 - xy^3}{2x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

What is the value of $\frac{\partial f}{\partial x} \Big|_{(0,0)}$?

- A) -1 B) $-\frac{1}{2}$ C) 0 D) $\frac{1}{2}$ E) 1

Question 5: Let $\omega = (x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$ and $z=\sin(r+s)$. What is the value of $\frac{\partial \omega}{\partial r} \Big|_{r=1, s=1}$?

- A) 6 B) 8 C) 10 D) 12 E) 14