LIMIT (BASIC)

1) Evaluate the limit $\lim\limits_{x\to 5}\left(1+(x-5)^2\cdot\sin\left(\csc(\pi x)\right)\right)$ by using the Sandwich Theorem.

$$-1 \leq \sin x \leq 1 \quad (\forall x\in\mathbb{R})$$ (take $\csc(\pi x)$ instead of x)

$$-1 \leq \sin\left(\csc(\pi x)\right) \leq 1$$ (multiplicate all by $(x-5)^2>0$)

$$-(x-5)^2 \leq (x-5)^2\cdot\sin\left(\csc(\pi x)\right) \leq (x-5)^2$$ (add 1 to all)

$$1-(x-5)^2 \leq 1+(x-5)^2\cdot\sin\left(\csc(\pi x)\right) \leq 1+(x-5)^2$$ (take limit)

$$\underbrace{\lim_{x\to 5}\left(1-(x-5)^2\right)}_{1} \leq L \leq \underbrace{\lim_{x\to 5}\left(1+(x-5)^2\right)}_{1}$$

$\Rightarrow$ From Sandwich Theorem, $L=1$.

2) Let $f(x)=\cos(1+x)+\sqrt[3]{x}\cdot\cos\left(\frac{1}{x}\right)$ where $x\neq 0$.
Evaluate the limit $\lim\limits_{x\to 0} f(x)$ by using Pinching Theorem.

$$-1 \leq \cos x \leq 1 \quad (\forall x\in\mathbb{R})$$ (take $\frac{1}{x}$ instead of x)

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$ (multiplicate all by $\sqrt[3]{x}$)

$$-\sqrt[3]{x} \leq \sqrt[3]{x}\cdot\cos\left(\frac{1}{x}\right) \leq \sqrt[3]{x}$$ (add $\cos(1+x)$ to all)

$$\cos(1+x)-\sqrt[3]{x} \leq \cos(1+x)+\sqrt[3]{x}\cdot\cos\left(\frac{1}{x}\right) \leq \cos(1+x)\cdot\sqrt[3]{x}$$ (take limit)

$$\underbrace{\lim_{x\to 0}\left[\cos(1+x)-\sqrt[3]{x}\right]}_{\cos 1} \leq \lim_{x\to 0} f(x) \leq \underbrace{\lim_{x\to 0}\left[\cos(1+x)\cdot\sqrt[3]{x}\right]}_{\cos 1}$$

$\Rightarrow$ From Pinching Theorem, $\lim\limits_{x\to 0} f(x)=\cos 1$.

③ Let $f(x)=\begin{cases}\dfrac{x^2-4}{x+2}\cdot\sin(\pi x), & \text{if } x\leq -2\\ \sqrt{x+2}, & \text{if } x>-2\end{cases}$. $\lim\limits_{x\to -2} f(x)=?$

i) $\lim\limits_{x\to -2^+} f(x)=\lim\limits_{x\to -2^+}\sqrt{x+2}=0$

ii) $\lim\limits_{x\to -2^-} f(x)=\lim\limits_{x\to -2^-}\dfrac{x^2-4}{x+2}\cdot\sin(\pi x)=\lim\limits_{x\to -2^-}\dfrac{(x-2)\cdot\cancel{(x+2)}\cdot\sin(\pi x)}{\cancel{x+2}}=0$

$\Rightarrow \lim\limits_{x\to -2^+} f(x)=\lim\limits_{x\to -2^-} f(x)=\lim\limits_{x\to -2} f(x)=0$

④ $\lim\limits_{x\to 0}\dfrac{|x^3|}{(x^2-1)\cdot x}=?$

i) $\lim\limits_{x\to 0^+}\dfrac{|x^3|}{(x^2-1)\cdot x}=\lim\limits_{x\to 0^+}\dfrac{x^{\cancel{3}\,2}}{(x^2-1)\cdot\cancel{x}}=0$

ii) $\lim\limits_{x\to 0^-}\dfrac{|x^3|}{(x^2-1)\cdot x}=\lim\limits_{x\to 0^-}\dfrac{-x^{\cancel{3}\,2}}{(x^2-1)\cdot\cancel{x}}=0$

$\lim\limits_{x\to 0}\dfrac{|x^3|}{(x^2-1)\cdot x}=0$

⑤ $\lim\limits_{x\to 3}\dfrac{|x^2-x-6|}{x-3}=?$ $\quad\lim\limits_{x\to 3}\dfrac{|(x-3)\cdot(x+2)|}{x-3}=\lim\limits_{x\to 3}\dfrac{|x-3|\cdot(x+2)}{(x-3)}$

i) $\lim\limits_{x\to 3^+}\dfrac{|x-3|\cdot(x+2)}{(x-3)}=\lim\limits_{x\to 3^+}\dfrac{\cancel{(x-3)}\cdot(x+2)}{\cancel{(x-3)}}=5$

ii) $\lim\limits_{x\to 3^-}\dfrac{|x-3|\cdot(x+2)}{(x-3)}=\lim\limits_{x\to 3^-}\dfrac{-\cancel{(x-3)}\cdot(x+2)}{\cancel{(x-3)}}=-5$

$\lim\limits_{x\to 3}\dfrac{|x^2-x-6|}{x-3}$ does not exist.

⑥ $\lim\limits_{x\to 0}\dfrac{x}{|x|}=?$

i) $\lim\limits_{x\to 0^+}\dfrac{x}{|x|}=\lim\limits_{x\to 0^+}\dfrac{\cancel{x}}{\cancel{x}}=1$

ii) $\lim\limits_{x\to 0^-}\dfrac{x}{|x|}=\lim\limits_{x\to 0^-}\dfrac{\cancel{x}}{-\cancel{x}}=-1$

Limit does not exist.

⑦ $\lim\limits_{x\to 0} \dfrac{\sin|x|}{x} = ?$

i) $\lim\limits_{x\to 0^+} \dfrac{\sin|x|}{x} = \lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1$ (Property $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$)

ii) $\lim\limits_{x\to 0^-} \dfrac{\sin|x|}{x} = \lim\limits_{x\to 0^-} \dfrac{\sin(-x)}{x} = -1$

Limit doesn't exist.

⑧ $\lim\limits_{x\to 0} \dfrac{(x^2+2)\cdot|x|}{x^3-x} = ?$

i) $\lim\limits_{x\to 0^+} \dfrac{(x^2+2)\cdot|x|}{x^3-x} = \lim\limits_{x\to 0^+} \dfrac{(x^2+2)\cdot\cancel{x}}{\cancel{x}(x^2-1)} = -2$

ii) $\lim\limits_{x\to 0^-} \dfrac{(x^2+x)\cdot|x|}{x^3-x} = \lim\limits_{x\to 0^-} \dfrac{(x^2+2)\cdot(-\cancel{x})}{\cancel{x}(x^2-1)} = 2$

Limit doesn't exist.

⑨ $\lim\limits_{x\to 3} \dfrac{1}{x^2-7x+12} = ?$

i) $\lim\limits_{x\to 3^+} \dfrac{1}{\underbrace{(x-4)}_{-}\cdot\underbrace{(x-3)}_{+}} = -\infty \Rightarrow$ No limit

ii) $\lim\limits_{x\to 3^-} \dfrac{1}{\underbrace{(x-4)}_{-}\cdot\underbrace{(x-3)}_{-}} = \infty \Rightarrow$ No limit

⑩ $\lim\limits_{x\to 0} \dfrac{x}{\cos x-1} = ?$

$(\cos x+1)$

$\lim\limits_{x\to 0} \dfrac{x\cdot(\cos x+1)^{2}}{\cos^2 x-1} = \lim\limits_{x\to 0} \dfrac{2x}{-\sin^2 x} = \underbrace{\lim\limits_{x\to 0} \dfrac{x}{\sin x}}_{1} \cdot \underbrace{\lim\limits_{x\to 0} \dfrac{-2}{\sin x}}_{-\infty} = -\infty$

(11) $\lim\limits_{x\to 0} \dfrac{x.\cos x+\sin x}{x+\tan x} = ?$

$$\lim_{x\to 0} \frac{\cancel{x}\left(\cos x+\underbrace{\frac{\sin x}{x}}_{1}\right)}{\cancel{x}\left(1+\underbrace{\frac{\tan x}{x}}_{1}\right)} = \lim_{x\to 0}\frac{\cos x+1}{2} = 1$$

(12) $\lim\limits_{x\to 1} \dfrac{x^{1/4}-1}{x^{1/3}-1} = ?$ Let $x=t^{12}$ $\quad x\to 1 \Rightarrow t\to 1$ $\quad x^{1/4}=t^3$ $\quad x^{1/3}=t^4$

$$\lim_{x\to 1}\frac{x^{1/4}-1}{x^{1/3}-1} = \lim_{t\to 1}\frac{t^3-1}{t^4-1} = \lim_{t\to 1}\frac{\cancel{(t-1)}.(t^2+t+1)}{\cancel{(t-1)}.(t+1).(t^2+1)} = \frac{3}{4}$$

(13) $\lim\limits_{x\to \pi} \dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = ?$ Let $x-\pi=t$ $\quad (x=\pi+t)$ $\quad x\to\pi \Rightarrow t\to 0$

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{t\to 0}\frac{\sqrt{2+\cos(\pi+t)}-1}{(-t)^2} = \lim_{t\to 0}\frac{\sqrt{2-\cos t}-1}{t^2}\ \left(\sqrt{2-\cos t}+1\right)$$

$$= \lim_{t\to 0}\frac{2-\cos t-1}{t^2\underbrace{\left(\sqrt{2-\cos t}+1\right)}_{2}} = \lim_{t\to 0}\frac{1-\cos t}{2t^2}\ (1+\cos t) = \lim_{t\to 0}\frac{1-\cos^2 t}{2t^2\underbrace{(1+\cos t)}_{2}}$$

$$= \lim_{t\to 0}\underbrace{\frac{\sin^2 t}{4t^2}}_{1} = \frac{1}{4}$$

(14) $\lim\limits_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right) = ?$ $\quad \left(\sqrt{x+1}+\sqrt{x}\right)$

$$\lim_{x\to\infty}\frac{\cancel{x}+1-\cancel{x}}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} = 0$$

(15) $\lim_{x\to 0}\dfrac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}=?$

$(\sqrt{2+x^2}+\sqrt{2-x^2})$

$$\lim_{x\to 0}\frac{\cancel{2}+x^2-\cancel{2}+x^2}{x^2(\underbrace{\sqrt{2+x^2}+\sqrt{2-x^2}}_{2\sqrt{2}})}=\lim_{x\to 0}\frac{\cancel{2x^2}}{\cancel{2}\sqrt{2}\cancel{x^2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

(16) $\lim_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos(2x)}}{\sin^2 x}=?$

$\boxed{\sin^2 x=\dfrac{1-\cos(2x)}{2}}$

$(\sqrt{2}+\sqrt{1+\cos(2x)})$

$$\lim_{x\to 0}\frac{2-1-\cos(2x)}{\sin^2 x\cdot\underbrace{(\sqrt{2}+\sqrt{1+\cos(2x)})}_{2\sqrt{2}}}=\lim_{x\to 0}\frac{1-\cos(2x)}{2\sqrt{2}\cdot\sin^2 x}=\lim_{x\to 0}\frac{\cancel{2\sin^2 x}}{\cancel{2}\sqrt{2}\cancel{\sin^2 x}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

(17) $\lim_{x\to\infty}\sin\left(\dfrac{\pi x^2+4}{6x^2+9}\right)=?$

sin is continuous

$$\lim_{x\to\infty}\sin\left(\frac{\pi x^2+4}{6x^2+9}\right)=\sin\left(\lim_{x\to\infty}\frac{\pi x^2+4}{6x^2+9}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

(18) $\lim_{x\to\infty}[\ln x-\ln(x+1)]=?$

$$\lim_{x\to\infty}\ln\left(\frac{x}{x+1}\right)=\ln\left(\lim_{x\to\infty}\frac{x}{x+1}\right)=\ln 1=0$$

(19) $\lim_{x\to 0}\dfrac{\sin(2x)}{\sqrt{x+1}-1}=?$

$(\sqrt{x+1}+1)$

$$\lim_{x\to 0}\frac{\sin(2x)\cdot\overbrace{(\sqrt{x+1}+1)}^{2}}{x+\cancel{1}\cancel{-1}}=\lim_{x\to 0}2\overbrace{\frac{\sin(2x)}{x}}^{2}=4$$

(20) $\lim_{x\to\infty}\left(\sqrt{x+5}-\sqrt{x+1}\right)=?$

$\left(\sqrt{x+5}+\sqrt{x+1}\right)$

$$\lim_{x\to\infty}\frac{\cancel{x}+5-\cancel{x}-1}{\left(\sqrt{x+5}+\sqrt{x+1}\right)}=\lim_{x\to\infty}\frac{4}{\left(\sqrt{x+5}+\sqrt{x+1}\right)}=0$$

(21) $\lim_{x\to\pi^+}\frac{\sin(\pi\sin x).\sin\left(\frac{x}{2}\right)}{\sqrt{1+\cos x}}=?$ $\quad 1+\cos x=2\cos^2\left(\frac{x}{2}\right)$

$$\lim_{x\to\pi^+}\frac{\sin(\pi\sin x).\overset{1}{\sin\left(\frac{x}{2}\right)}}{\sqrt{2\cos^2\left(\frac{x}{2}\right)}}=\lim_{x\to\pi^+}\frac{\sin(\pi\sin x)}{\sqrt{2}.\left(-\cos\left(\frac{x}{2}\right)\right)}$$

$x-\pi=t$
$x\to\pi^+$
$t\to 0^+$

$$=\lim_{t\to0^+}\frac{\sin(\pi\sin(t+\pi))}{-\sqrt{2}\cos\left(\frac{t}{2}+\frac{\pi}{2}\right)}=\lim_{t\to0^+}\frac{\sin(\pi.(-\sin t))}{-\sqrt{2}.\left(-\sin\left(\frac{t}{2}\right)\right)}$$

$$=\lim_{t\to0^+}\frac{-\sin(\pi\sin t)}{\sqrt{2}.\sin\left(\frac{t}{2}\right)}=\lim_{t\to0^+}\frac{-\sin(\pi\sin t)}{\sqrt{2}.\underset{\frac{1}{\frac{1}{2}}}{\sin\left(\frac{t}{2}\right)}}.\frac{\sin t}{\underset{\pi}{\sin t}}=\frac{-2\pi}{\sqrt{2}}=-\sqrt{2}\pi$$

(22) $\lim_{x\to0^+}\frac{1-\sqrt{\cos x}}{1-\cos(\sqrt{x})}=?$

$\left(1+\cos(\sqrt{x})\right)$

$$\lim_{x\to0^+}\frac{\left(1-\sqrt{\cos x}\right)\overset{2}{\left(1+\cos(\sqrt{x})\right)}}{1-\cos^2(\sqrt{x})}=\lim_{x\to0^+}\frac{2\left(1-\sqrt{\cos x}\right)}{\sin^2(\sqrt{x})}.\overset{1}{\frac{(\sqrt{x})^2}{(\sqrt{x})^2}}$$

$\left(1+\sqrt{\cos x}\right)$

$$=\lim_{x\to0^+}\frac{\cancel{2}(1-\cos x)}{x.\left(1+\sqrt{\cos x}\right)_2}=\lim_{x\to0^+}\frac{1-\cos^2 x}{x.(1+\cos x)_2}=\lim_{x\to0^+}\overset{1}{\frac{\sin^2 x}{2x}}.\frac{x}{x}$$

$(1+\cos x)$

$$=\lim_{x\to0^+}\frac{x}{2}=0$$