10)

(x,y)	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$ A	$f_{yy}(x,y)$ C	$f_{xy}(x,y)$ B	B^2-AC
Saddle ← $(0,0)$	0	0	-6	6	6	72 → $A<0$, $B^2-AC>0$
Local minimum ← $(2,-2)$	0	0	18	6	6	-72 → $A>0$, $B^2-AC<0$

Let $f(x,y)$ be a partially differentiable function of all orders. Some of the values of the partial derivatives of $f(x,y)$ at the points $(0,0)$ and $(2,-2)$ is given in the table above. Then, which of the following is true about these points?

A) $(0,0)$ is a saddle point; $(2,-2)$ is a local maximum point

B) $(0,0)$ is a saddle point; $(2,-2)$ is a local minimum point

C) $(0,0)$ is a local minimum point; $(2,-2)$ is a local maximum point

D) $(0,0)$ is a local maximum point; $(2,-2)$ is a local minimum point

E) $(0,0)$ is a local maximum point; $(2,-2)$ is a saddle point

- Let $f_{xx}=A$, $f_{xy}=B$ and $f_{yy}=C$.

1. If $A<0$ and $B^2-AC<0$, then f has a local maximum at the point (a,b).
2. If $A>0$ and $B^2-AC<0$, then f has a local minumum at the point (a,b).
3. If $B^2-AC>0$, then f has a saddle point.
4. If $B^2-AC=0$, then the test is inconclusive at (a,b). f may have a local maximum, a local minumum, or a saddle point at (a,b).

18) How many critical points does the function $f(x,y) = (y-2)x^2 - y^2$ have?

A) 6 B) 5 C) 3 D) 4 E) 2

$f(x,y) = x^2y - 2x^2 - y^2$

$f_x = 2xy - 4x = 2x(y-2) = 0 \Rightarrow$

$x = 0$ or $y = 2$

$f_y = x^2 - 2y = 0 \Rightarrow 2y = x^2$

- If $x = 0 \Rightarrow 2y = 0 \Rightarrow y = 0 \Rightarrow (0,0)$
- If $y = 2 \Rightarrow 2y = x^2 \Rightarrow x^2 = 4 \Rightarrow x = \mp 2$

$\Rightarrow (-2,2)$ and $(2,2)$

- Critical Points:

$(0,0)$

$(-2,2)$

$(2,2)$

Example: Find and classify the critical points of the function $f(x,y)=2x^3-6xy+3y^2$.

$f_x = 6x^2 - 6y = 0 \Rightarrow y = x^2$

$f_y = -6x+6y \Rightarrow x = y$

$\Rightarrow x^2 = x \Rightarrow x(x-1)=0 \Rightarrow$

$x=0$ or $x=1$

$x=0 \Downarrow y=x=0$ $(0,0)$

$x=1 \Downarrow y=x=1$ $(1,1)$

• Critical Points:

$(0,0)$ and $(1,1)$

$A=f_{xx}=12x$, $B=f_{xy}=-6$, $C=f_{yy}=6$

	$A=12x$	$B=-6$	$C=6$	B^2-AC
$(0,0)$	0	-6	6	36
$(1,1)$	12	-6	6	-36

• For $(0,0)$, $B^2-AC=36>0$ so $(0,0)$ is saddle point

• For $(1,1)$, $A=12>0$ and $B^2-AC=-36<0$ so $(1,1)$ is local minumum point

Example: Find and classify the critical points of the function $f(x,y)=x^3-3x^2+3xy^2-3y^2$.

$$f_x=3x^2-6x+3y^2$$

$$f_y=6xy-6y=0\Rightarrow 6y(x-1)=0$$

$y=0$ or $x=1$

- If $y=0$, then $f_x=3x^2-6x$. If $f_x=3x(x-2)=0$, then $x=0$ or $x=2\Rightarrow$ $(0,0)$ and $(2,0)$.
- If $x=1$, then $f_x=-3+3y^2$. If $f_x=0\Rightarrow$ $3y^2=3\Rightarrow y^2=1\Rightarrow y=\mp 1\Rightarrow(1,-1)$ and $(1,1)$.
- $A=f_{xx}=6x-6$, $B=f_{xy}=6y$, $C=f_{yy}=6x-6$

critical points	$A=6x-6$	$B=6y$	$C=6x-6$	B^2-AC	
$(0,0)$	-6	0	-6	-36	Local max
$(2,0)$	6	0	6	-36	Local min
$(1,-1)$	0	-6	0	36	Saddle
$(1,1)$	0	6	0	36	Saddle

- For $(0,0)$, $A=-6<0$ and $B^2-AC=-36<0$
so $(0,0)$ is local maximum point.

- For $(2,0)$, $A=6>0$ and $B^2-AC=-36<0$
so $(2,0)$ is local minimum point.

- For $(1,-1)$ and $(1,1)$, $B^2-AC=36>0$
so $(1,-1)$ and $(1,1)$ are saddle points.

Example: Find and classify the critical points of the function $f(x,y) = 8x^3 + y^3 - 12xy + 8$.

$f_x = 24x^2 - 12y = 0 \Rightarrow y = 2x^2$

$f_y = 3y^2 - 12x = 0 \Rightarrow y^2 = 4x$

$(2x^2)^2 = 4x \Rightarrow$

$4x^4 = 4x \Rightarrow$

$4x^4 - 4x = 0 \Rightarrow$

$4x(x^3 - 1) = 0 \Rightarrow$

$x = 0$ or $x = 1$

$\Downarrow$ $\quad\quad$ $\Downarrow$

$y = 2 \cdot 0^2 = 0$ $\quad\quad$ $y = 2 \cdot 1^2 = 2$

$(0,0)$ $\quad\quad$ $(1,2)$

$A = f_{xx} = 48x$, $B = f_{xy} = -12$ and

$C = f_{yy} = 6y$

Critical points	$A = 48x$	$B = -12$	$C = 6y$	$B^2 - AC$	
$(0,0)$	0	-12	0	$144 > 0$	$\Rightarrow$ Saddle Point
$(1,2)$	48	-12	12	-432	$\Rightarrow$ $A = 48 > 0$ and $B^2 - AC = -432 < 0$ so $(1,2)$ is local minimum point

Example: Find and classify the critical points of the function $f(x,y) = 2y^3 + 3x^2 - 3y^2 - 12xy$.

$f_x = 6x - 12y = 0 \Rightarrow x = 2y$

$f_y = 6y^2 - 6y - 12x = 6y^2 - 6y - 12(2y)$
$= 6y^2 - 6y - 24y$
$= 6y^2 - 30y$

If $f_y = 6y^2 - 30y = 6y(y-5) = 0 \Rightarrow y = 0$ or $y = 5$

- If $y = 0$, then $x = 2 \cdot 0 = 0$, $(0,0)$
- If $y = 5$, then $x = 2 \cdot 5 = 10$, $(10,5)$

$A = f_{xx} = 6$, $B = f_{xy} = -12$, $C = f_{yy} = 12y - 6$

Critical Points	$A=6$	$B=-12$	$C=12y-6$	B^2-AC
$(0,0)$	6	-12	-6	$180 > 0 \Rightarrow$ Saddle Point
$(10,5)$	6	-12	54	$-180 \Rightarrow A = 6 > 0$ and $B^2 - AC = -180 < 0$ so $(10,5)$ is local minumum point.

Example: Find and classifying the critical points of the function $f(x,y)=xy+\frac{1}{x}+\frac{8}{y}$.

$$f_x = y-\frac{1}{x^2}=0 \Rightarrow y=\frac{1}{x^2}$$

$$f_y = x-\frac{8}{y^2}=0 \Rightarrow x=\frac{8}{y^2}=\frac{8}{\left(\frac{1}{x^2}\right)^2} \Rightarrow$$

$$x=\frac{8}{\frac{1}{x^4}} \Rightarrow x=8x^4 \Rightarrow$$

$$8x^4-x=0 \Rightarrow$$

$$x(8x^3-1)=0 \Rightarrow$$

$x=0$ or $x=\frac{1}{2}$

$x=0 \notin D(f)$

$$\Downarrow$$

$$y=\frac{1}{\left(\frac{1}{2}\right)^2}=4$$

• **Critical Point:**

$\left(\frac{1}{2},4\right)$

$$A=f_{xx}=\frac{2}{x^3}$$

$$B=f_{xy}=1$$

$$C=f_{yy}=\frac{16}{y^3}$$

• For $x=1/2$, $A=\frac{2}{\left(\frac{1}{2}\right)^3}=16>0$, $B=1$ and

$C=\frac{16}{4^3}=1/4$ so

$$B^2-AC=1-16\cdot\frac{1}{4}=1-4=-3<0.$$

$A=16>0$ and $B^2-AC=-3<0$ so

$\left(\frac{1}{2},4\right)$ is local minimum point.

Example : Let $f(x,y) = y^2 \cdot \sqrt{x}$. Find all critical points and local extrema values of the function f.

- $f_x = \dfrac{y^2}{2\sqrt{x}} = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$, $\forall x > 0$
- $f_y = 2y\sqrt{x} = 0 \Rightarrow y = 0$ or $x = 0$

- If $x = a > 0$, $a \in \mathbb{R}$, then critical point is $(a, 0)$.

$f(a,0) = 0^2 \cdot \sqrt{a} = 0$

$f(x,y) = \underbrace{y^2}_{\geq 0} \underbrace{\sqrt{x}}_{\geq 0} \geq 0 = f(a,0)$

So the point $(a,0)$ is a local minimum point.