Antiderivatives

Let we have $f(x) = F'(x)$. Find $F(x)$.

Definition: A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example: $f(x) = 2x$, $F'(x) = 2x \Rightarrow F(x) = x^2$

$g(x) = \sin x \Rightarrow G(x) = -\cos x$, $H(x) = -\cos x + 7$

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + \underline{\underline{C}}$$

where C is an arbitrary constant.

Example: Find an antiderivative F of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

$$f(x) = 3x^2 \Rightarrow F(x) = x^3 + C$$

$$F(1) = -1 \Rightarrow 1 + C = -1 \Rightarrow C = -2 \Rightarrow F(x) = x^3 - 2$$

Function ($k \in \mathbb{R} \setminus \{0\}$)	General Antiderivative
1) x^n	$\frac{1}{n+1} \cdot x^{n+1} + C$, $n \neq -1$
2) $\sin kx$	$-\frac{1}{k} \cos kx + C$
3) $\cos kx$	$\frac{1}{k} \sin kx + C$
4) $\sec^2 kx$	$\frac{1}{k} \tan kx + C$
5) $\csc^2 kx$	$-\frac{1}{k} \cot kx + C$

6) $\sec kx \cdot \tan kx$ $\qquad$ $\frac{1}{k}\sec kx + C$

7) $\csc kx \cdot \cot kx$ $\qquad$ $-\frac{1}{k}\csc kx + C$

8) e^{kx} $\qquad$ $\frac{1}{k}\cdot e^{kx} + C$

9) $\frac{1}{x}$ $\qquad$ $\ln|x| + C, \quad x \neq 0$

10) $\frac{1}{\sqrt{1-k^2x^2}}$ $\qquad$ $\frac{1}{k}\arcsin kx + C$

11) $\frac{1}{1+k^2x^2}$ $\qquad$ $\frac{1}{k}\arctan kx + C$

12) $\frac{1}{x\sqrt{k^2x^2-1}}$ $\qquad$ $\operatorname{arcsec} kx + C, \quad kx > 1$

13) a^{kx} $\qquad$ $\frac{1}{k \cdot \ln a}\cdot a^{kx} + C, \quad a > 0, a \neq 1$

<u>Example</u>: Find the general antiderivative of each of the following functions.

a) $f(x) = x^5 \Rightarrow F(x) = \frac{1}{6}\cdot x^6 + C$

b) $g(x) = \frac{1}{\sqrt{x}} \Rightarrow g(x) = x^{-\frac{1}{2}} \Rightarrow G(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C$

c) $h(x) = \sin(2x) \Rightarrow H(x) = \frac{-\cos(2x)}{2} + C$

d) $i(x) = \cos\frac{x}{2} \Rightarrow I(x) = \frac{\sin\left(\frac{x}{2}\right)}{\frac{1}{2}} + C = 2\sin\left(\frac{x}{2}\right) + C$

e) $j(x) = e^{-3x} \Rightarrow J(x) = \frac{e^{-3x}}{-3} + C$

f) $k(x)=2^x \Rightarrow K(x)=\dfrac{2^x}{\ln 2}+C$

Antiderivative Linearity Rules

1) Constant Multiple: $k f(x) \Rightarrow k.F(x)+C$, k is constant
2) Sum / Difference: $f(x) \mp g(x) \Rightarrow F(x) \mp G(x)+C$

Example: Find the general antiderivative of

$$f(x)=\frac{3}{\sqrt{x}}+\sin(2x).$$

$$F(x)=3.G(x)+H(x)+C$$
$$=\underbrace{3.2}_{6}\sqrt{x}-\frac{\cos(2x)}{2}+C$$

Indefinite Integrals

Definition: The collection of all antiderivatives of f is called the indefinite integral of f with respect to x; it is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral, and x is the variable of integration.

$\int 2x\,dx = x^2+C$, $\int \cos x\,dx=\sin x+C$, $\int\left(\frac{1}{x}+2e^{2x}\right)dx$
$=\ln|x|+e^{2x}+C.$

Integration Formulas

1) $\int x^n = \frac{1}{n+1} \cdot x^{n+1} + C, \quad n \neq -1$

2) $\int \sin kx = -\frac{1}{k} \cos kx + C$

3) $\int \cos kx = \frac{1}{k} \sin kx + C$

4) $\int \sec^2 kx = \frac{1}{k} \tan kx + C$

5) $\int \csc^2 kx = -\frac{1}{k} \cot kx + C$

6) $\int \sec kx \cdot \tan kx = \frac{1}{k} \sec kx + C$

7) $\int \csc kx \cdot \cot kx = -\frac{1}{k} \csc kx + C$

8) $\int e^{kx} = \frac{1}{k} \cdot e^{kx} + C$

9) $\int \frac{1}{x} = \ln|x| + C, \quad x \neq 0$

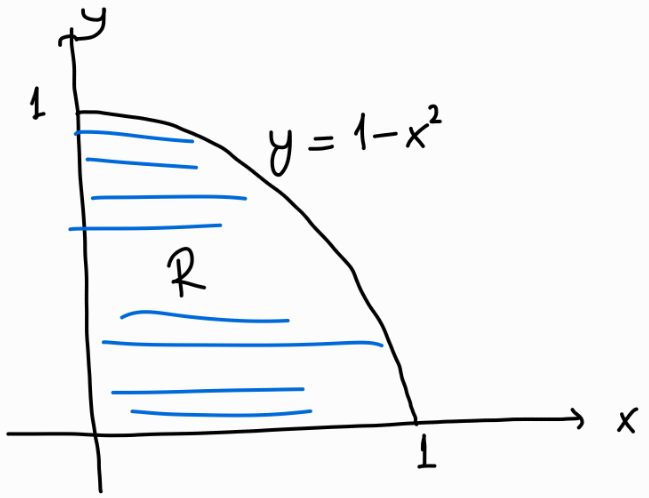
10) $\int \frac{1}{\sqrt{1-k^2x^2}} = \frac{1}{k} \arcsin kx + C$

11) $\int \frac{1}{1+k^2x^2} = \frac{1}{k} \arctan kx + C$

12) $\int \frac{1}{x\sqrt{k^2x^2-1}} = \operatorname{arcsec} kx + C, \quad kx > 1$

13) $\int a^{kx} = \frac{1}{k \cdot \ln a} \cdot a^{kx} + C, \quad a > 0, a \neq 1$

Area and Estimating with Finite Sums

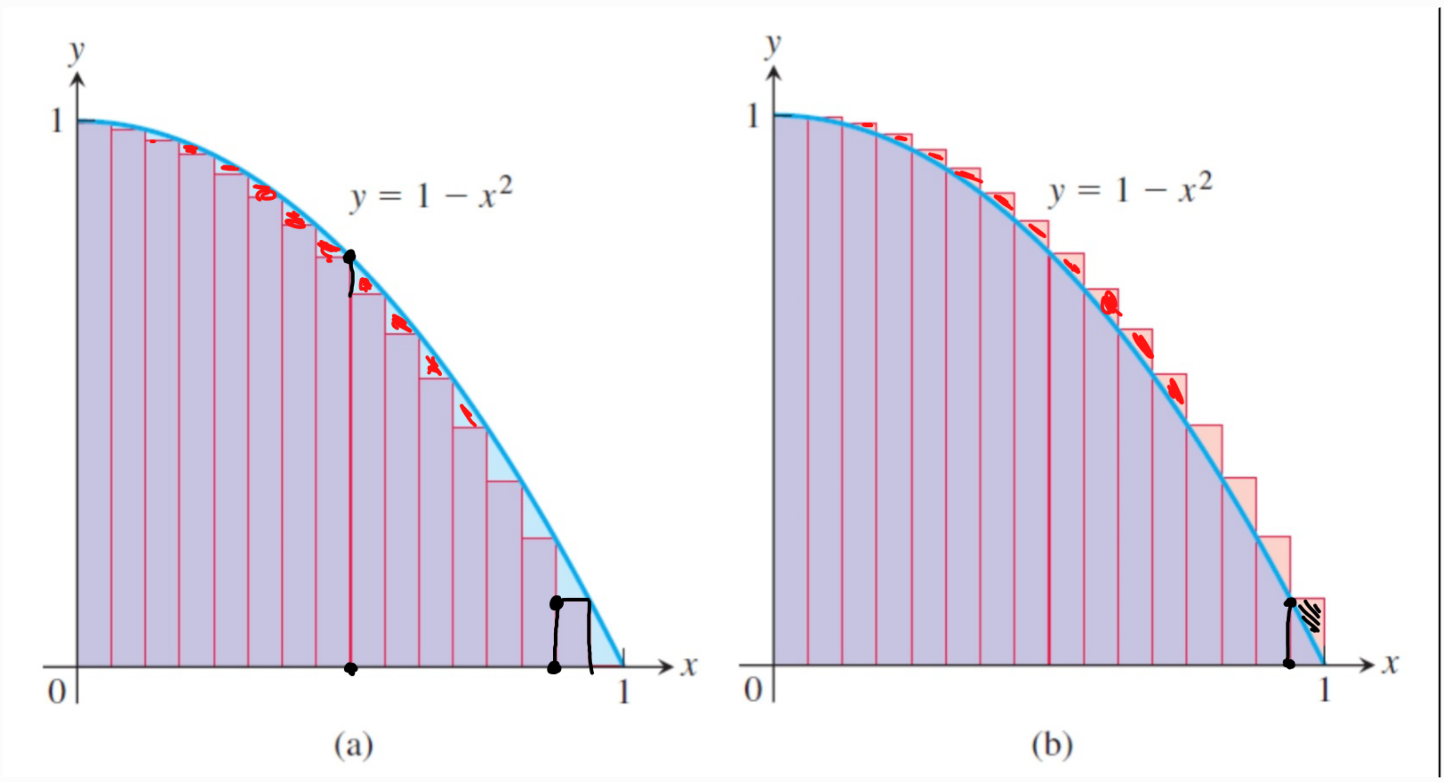


How can we compute the shaded area R?

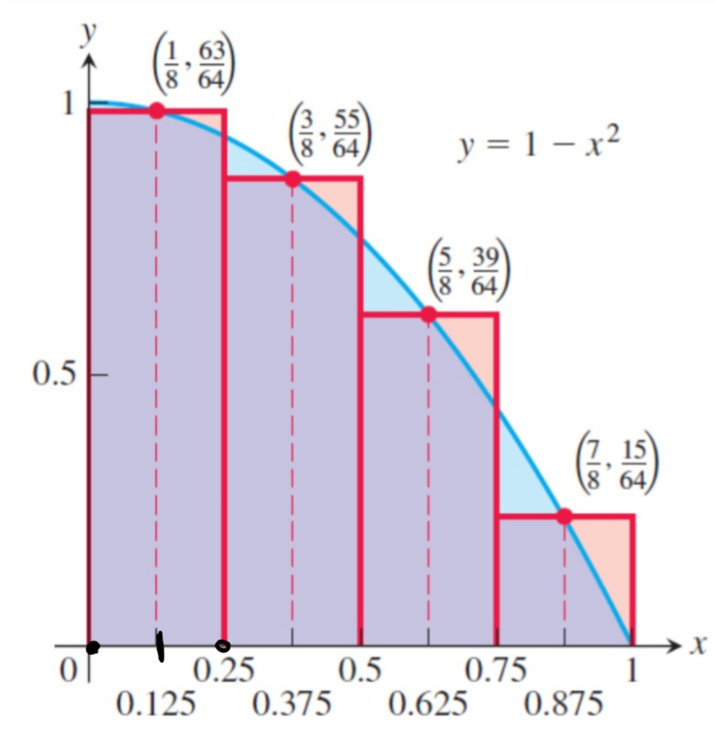
1) Subdivide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.

2) Choose point c_k in the k-th subinterval.

3) Construct rectangles:
   a) midpoint rule: Choose c_k in the middle of k-th subinterval.
   b) upper sum: Choose c_k such that $f(c_k)$ is maximal.
   c) lower sum: Choose c_k such that $f(c_k)$ is minimal.



lower sum   upper sum



(b)

midpoint rule

4) Form the sum $f(c_1)\Delta x + f(c_2)\cdot\Delta x + \dots + f(c_n)\Delta x$.

5) Refine your approximation by choosing more rectangles:

TABLE 5.1 Finite approximations for the area of R (Exact: $\frac{2}{3}$)

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.5313	0.6719	0.7813
16	0.6348	0.6670	0.6973
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Sigma Notation and Limits of Finite Sums

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \dots + a_n.$$

The summation symbol (Greek Letter sigma)

$$\sum_{k=1}^{n} a_k$$

n → The index k ends at $k=n$

a_k → a_k is a formula for the k-th term.

$k=1$ → The index k starts at $k=1$

Examples

1) $f(c_1)\cdot\Delta x + f(c_2)\cdot\Delta x + \dots + f(c_n)\,\Delta x = \sum\limits_{k=1}^{n} f(c_k)\,\Delta x$

2) $\sum\limits_{k=1}^{3} (-1)^k k = (-1)^1\cdot 1 + (-1)^2\cdot 2 + (-1)^3\cdot 3 = -1+2-3 = -2$

3) $1+3+5+7+9 = \sum\limits_{k=1}^{5}(2k-1)$

$(k=n+1) \qquad = \sum\limits_{n=0}^{4}(2n+1)$

$(n=x+3) \qquad = \sum\limits_{x=-3}^{1}(2x+7) = 25$

Algebra Rules for Finite Sums

1-2) Sum/Difference: $\sum\limits_{k=1}^{n}(a_k \mp b_k) = \sum\limits_{k=1}^{n} a_k \mp \sum\limits_{k=1}^{n} b_k$

3) Constant Multiple: $\sum\limits_{k=1}^{n} c\cdot a_k = c\cdot\sum\limits_{k=1}^{n} a_k$ (c: constant)

4) Constant Value: $\sum\limits_{k=1}^{n} c = c\sum\limits_{k=1}^{n} 1 = c\cdot\underbrace{1+1+\dots+1}_{n\text{-times}} = c\cdot n = nc$ (c: constant)

Ex.:

1) $$\sum_{k=1}^{n}(3k-k^2)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$$

2) $$\sum_{k=1}^{n}(-a_k)=\sum_{k=1}^{n}(-1).a_k=(-1).\sum_{k=1}^{n}a_k=-\sum_{k=1}^{n}a_k$$

3) $$\sum_{k=1}^{3}(k+4)=\sum_{k=1}^{3}k+\sum_{k=1}^{3}4=(1+2+3)+4.3=6+12=18$$

4) $$\sum_{k=1}^{n}\underbrace{\frac{1}{n}}_{\text{not depending on } k}=n.\frac{1}{n}=1 \quad \left(\frac{1}{n}:\text{constant}\right)$$

$$\sum_{k=1}^{n}k=\frac{n.(n+1)}{2}, \quad \sum_{k=1}^{n}k^2=\frac{n.(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}k^3=\left(\frac{n.(n+1)}{2}\right)^2$$

Limits of Finite Sums

Example: Compute the area R below the graph of $y=1-x^2$ and above the interval $[0,1]$.

1) Subdivide the interval into n subintervals of width $\Delta x=\frac{1}{n}$

$$\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\left[\frac{2}{n},\frac{3}{n}\right],\ldots,\left[\frac{n-1}{n},\frac{n}{n}\right]$$

$\frac{n}{n}=1$

upper

lower

1) Choose the lower sum: $c_k = \frac{k}{n}$, $k \in \mathbb{N}$ is the rightmost point.

$f(x) = 1 - x^2$

3) Summation:

$$f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + f\left(\frac{2}{n}\right)\cdot\frac{1}{n} + \dots + f\left(\frac{n}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$= \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right) = \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n}\cdot n - \frac{1}{n^3}\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

$$\boxed{* \text{ Lower sum: } R \geqslant \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}} \quad c_k = \frac{k}{n}$$

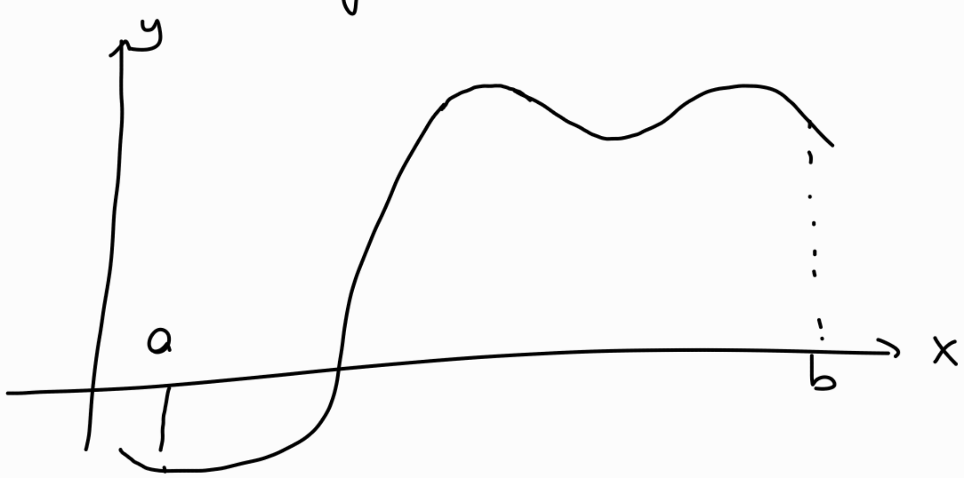
$$* \text{ Upper sum: } R \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2} \text{ (Exercise)} \quad c_k = \frac{k-1}{n}$$

As $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore $R = \frac{2}{3}$

$$\lim_{n\to\infty}\left[\frac{2}{3} - \cancel{\frac{1}{2n}}_{0} - \cancel{\frac{1}{6n^2}}_{0}\right] = \frac{2}{3}.$$

Riemann Sums

Consider a typical continuous function over $[a,b]$

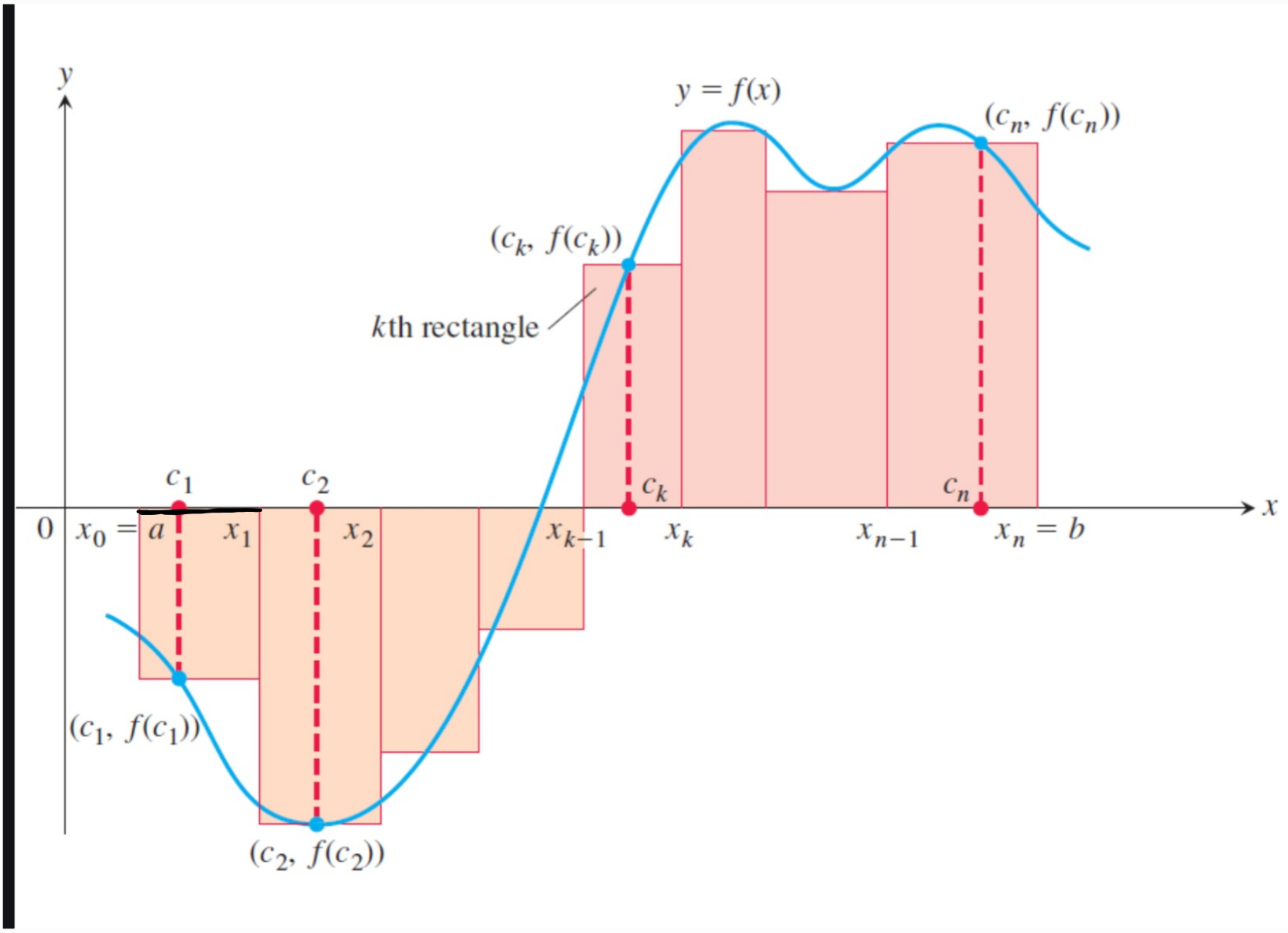


Partition: $[a,b]$ by choosing $n-1$ points between a and b.

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Note that $\Delta x_k = x_k - x_{k-1}$, the width of the subinterval $[x_{k-1}, x_k]$, may vary.

Choose $c_k \in [x_{k-1}, x_k]$ and construct rectangles:



The resulting sums $S_p = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ are called Riemann Sums for f on $[a,b]$.

Then choose finer and finer partitions by taking the limit such that the width of the largest subinterval goes to zero.

For a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ we write $\|P\|$ (called "norm") for the width of the largest subinterval.

The Definite Integral

$$I = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \int_a^b f(x)\,dx$$

Upper limit of integration

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

The function $f(x)$ is the integrand

x is the variable of integration

Theorem: (Continuous Functions are Integrable): If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Ex.: $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Upper sum is always 1; lower sum is always 0

$\Rightarrow \int_0^1 f(x)\,dx$ does not exist.

Properties of Definite Integrals

1) Order of integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2) Zero Width Interval: $\int_a^a f(x)dx = 0$

3) Constant Multiple: $\int_a^b k\cdot f(x)dx = k\int_a^b f(x)dx$ (k is constant)

4) Sum/Difference: $\int_a^b (f(x) \mp g(x))dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

5) Additivity: $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ $\quad [a,c]\cup[c,b]=[a,b]$

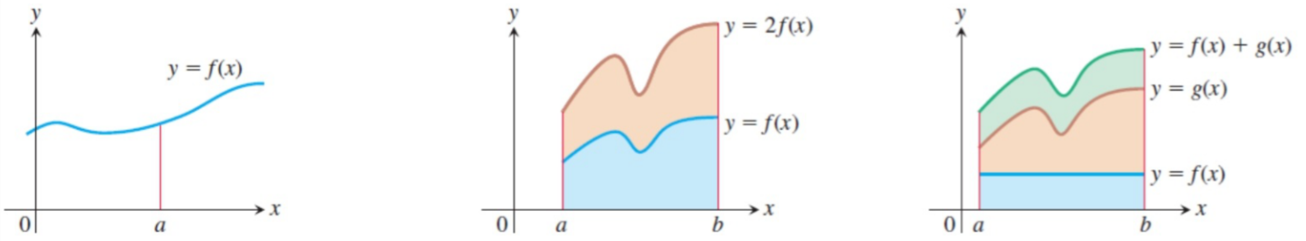
6) Max-Min Inequality: If f has max. value $\max f$ and min. value $\min f$ on $[a,b]$, then

$$\min f\cdot(b-a) \leq \int_a^b f(x)dx \leq \max f\cdot(b-a)$$

7) Domination: If $f(x) \geqslant g(x)$ on $[a,b]$, then

$$\int_a^b f(x)dx \geqslant \int_a^b g(x)dx.$$

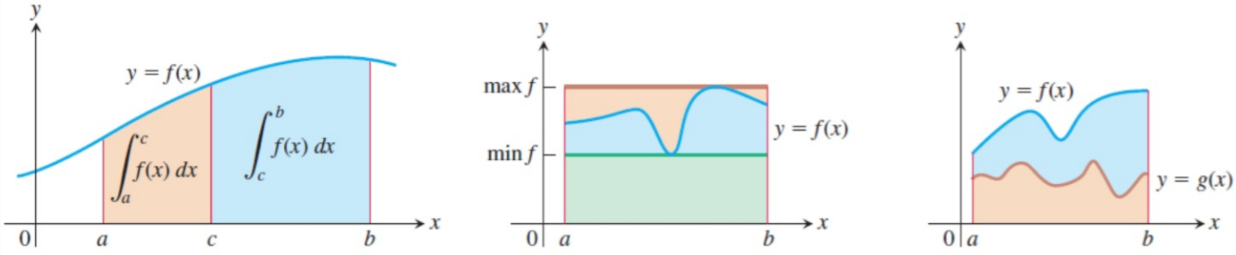
If $f(x)\geqslant 0$ on $[a,b]$, then $\int_a^b f(x)dx \geqslant 0$.



(a) *Zero Width Interval:*
$\int_a^a f(x)\,dx = 0$

(b) *Constant Multiple:* $(k = 2)$
$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(c) *Sum: (areas add)*
$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)$



(d) *Additivity for Definite Integrals:*
$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(e) *Max-Min Inequality:*
$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$

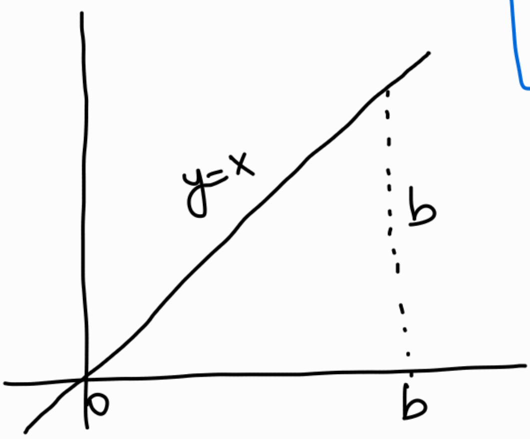
(f) *Domination:*
If $f(x) \ge g(x)$ on $[a, b]$, then
$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

Area Under the Graph of a Nonnegative Function

Definition: If $y = f(x)$ is a nonnegative and integrable over a closed interval $[a,b]$, then the area under the curve $y = f(x)$ over $[a,b]$ is the integral of f from a to b. $A = \int_a^b f(x)\,dx$.

Example: Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0,b]$, $b>0$.

$$\boxed{\text{Area } A = \frac{b^2}{2}.}$$



Choose $x_k = \frac{kb}{n}$ with right endpoints c_k.

$$I = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n}$$

$$\Rightarrow I = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb^2}{n^2} = \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{n.(n+1)}{2} = \frac{b^2}{2}$$

$$I = \int_0^b x\,dx = \frac{b^2}{2}.$$

Average Value of a Continuous Function

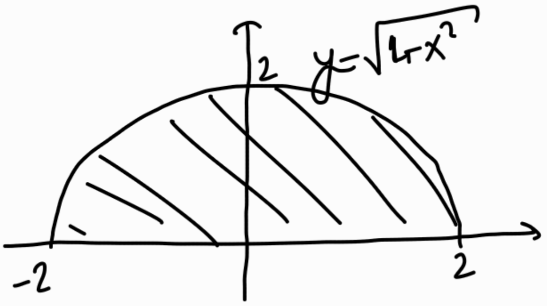
Definition: If f is integrable on $[a,b]$, then its average value on $[a,b]$, also called its meen, is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Theorem: (The M.V.T. for Definite Integrals)
If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example: Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$



$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi \cdot 4 = 2\pi$$

$$\int_{-2}^{-2} \sqrt{4-x^2}\,dx = 2\pi$$

$$av(f) = \frac{1}{2-(-2)} \cdot \int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}.$$

The Fundamental Theorem of Calculus

Part 1: If f is continuous $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Ex.: $\frac{d}{dx}\int_a^{x} \frac{1}{1+4t^3}\,dt = \frac{1}{1+4x^3}$

Ex.: $\frac{d}{dx}\int_2^{x^2} \cos t\,dt$: $y = \int_2^u \cos t\,dt$ with $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \underline{\left(\frac{d}{du}\int_2^u \cos t\,dt\right)}\cdot\frac{du}{dx} = \cos u \cdot 2x$$

$$= \cos x^2 \cdot 2x$$

$$\frac{d}{dx}\int_2^{x^2} \cos t\,dt = \cos x^2 \cdot 2x$$

Part 2: If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The usual notation for the difference $F(b)-F(a)$ is $F(x)\Big]_a^b$ or $F(x)\Big|_a^b$ or $F(x)\Big|_{x=a}^{x=b}$ or $\Big[F(x)\Big]_a^b$.

Ex.1

1) $\int_{-\pi/4}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1-\sqrt{2}$

2) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx = \cancel{\frac{3}{2}} \cdot \frac{x^{3/2}}{\cancel{\frac{3}{2}}} + \frac{4}{x}\Big|_1^4 = 4^{3/2} + \frac{4}{4} - \left(1^{3/2} + \frac{4}{1}\right) = 4$

3) $\int_1^2 \frac{1}{x}dx = \int_1^2 \frac{dx}{x} = \ln|x|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

4) $\int_0^1 \frac{dx}{x^2+1} = \arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Leibniz's Rule

If f is continuous on $[a,b]$ and if $a(x)$ and $b(x)$ are differentiable functions of x whose values lie in $[a,b]$ then

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

Ex.: $\frac{d}{dx}\left[\int_{\sin x}^{\cos x} t^2 dt\right] = \cos^2 x \cdot (-\sin x) - \sin^2 x \cdot \cos x$

$= -\sin x \cos^2 x - \sin^2 x \cdot \cos x$