Question 1: Find the domain of the function f defined by $f(x,y) = \arccos\left(\frac{y}{x^2}\right) + \ln(1-x^2)$.

① $x^2 \neq 0 \Rightarrow x \neq 0$ for $\frac{y}{x^2}$

② If $\arccos\left(\frac{y}{x^2}\right) = z \Rightarrow \frac{y}{x^2} = \cos z$

$-1 \leq \cos z \leq 1 \Rightarrow -1 \leq \frac{y}{x^2} \leq 1$

$\Rightarrow -x^2 \leq y \leq x^2$

③ $1 - x^2 > 0$ for $\ln(1-x^2) \Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$

$D(f) = \{(x,y) : -x^2 \leq y \leq x^2,\ -1 < x < 1 \text{ and } x \neq 0\}$

Question 2: Find the domain of the function f defined by $f(x,y)=\sqrt{1-x^2}+\ln[x(y-1)]$.

Ⓐ $1-x^2 \geq 0$ for $\sqrt{1-x^2}$ $\Rightarrow x^2 \leq 1$
$\Rightarrow -1 \leq x \leq 1$

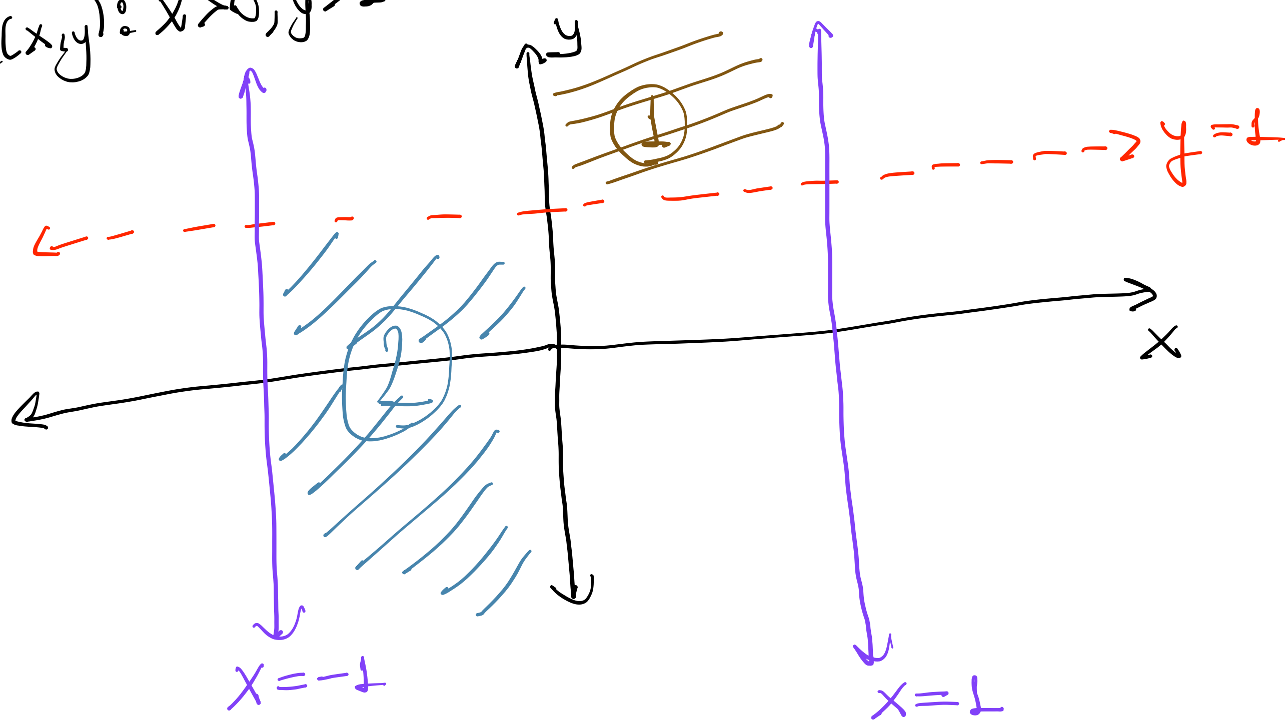
Ⓑ $x(y-1)>0$ for $\ln(x(y-1)) \Rightarrow x\,(y-1)>0$

$$\begin{array}{cc} + & + \\ - & - \end{array}$$

① If $x>0$, then $y-1>0 \Rightarrow y>1$ and $x\leq 1$

② If $x<0$, then $y-1<0 \Rightarrow y<1$ and $-1\leq x$

$D(f)=\{(x,y): x>0, y>1 \text{ and } x\leq 1\} \cup \{(x,y): x<0, y<1 \text{ and } -1\leq x\}$



Question 3: $\lim\limits_{(x,y)\to(1,1)} \dfrac{x-\sqrt{xy}}{2x^2-xy-y^2} = ?$ $\dfrac{1-\sqrt{1}}{2-1-1} = \dfrac{1-1}{2-2} = \dfrac{0}{0}$ indeterminated form

A) $\dfrac{1}{2}$ B) $\dfrac{1}{3}$ C) $\dfrac{1}{6}$ D) $\dfrac{1}{12}$ E) $\dfrac{1}{15}$

$x = \sqrt{x}\cdot\sqrt{x}$

$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y}$

$2x^2-xy-y^2 = (2x+y)(\underbrace{x-y}_{(\sqrt{x})^2-(\sqrt{y})^2}) = (2x+y)(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$

$$\begin{array}{cc} 2x & y \\ x & -y \end{array}$$

$$\frac{\sqrt{x}\cdot\sqrt{x}-\sqrt{x}\cdot\sqrt{y}}{(2x+y)(x-y)} = \frac{\sqrt{x}(\sqrt{x}-\sqrt{y})}{(2x+y)(x-y)} = \frac{\sqrt{x}\cancel{(\sqrt{x}-\sqrt{y})}}{(2x+y)\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}$$

$$= \frac{\sqrt{x}}{(2x+y)(\sqrt{x}+\sqrt{y})}$$

$$\lim_{(x,y)\to(1,1)} \frac{\sqrt{x}}{(2x+y)(\sqrt{x}+\sqrt{y})} = \frac{1}{3\cdot 2} = \frac{1}{6}$$

Question 4: Let $f(x,y)=\begin{cases} \dfrac{5xy^2}{x^2+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Which of the following(s) is/are true about the function $f(x,y)$?

I. f is defined on the entire $\mathbb{R}^2$ plane. ✓

II. $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ ✗

III. f is continuous at the point $(0,0)$. ✗

I. If $x=y=0 \Rightarrow f(x,y)=0$
If $x,y\neq0 \Rightarrow x^2, y^4\neq0 \Rightarrow x^2+y^2\neq0$ $\Big\}\Rightarrow$ f is defined on the entire $\mathbb{R}^2$ plane

II. If $x=y \Rightarrow \lim\limits_{y\to0}\dfrac{5y^3}{\underbrace{y^2+y^4}_{y^2(1+y^2)}}=\lim\limits_{y\to0}\dfrac{5y}{1+y^2}=\dfrac{0}{1}=0$

If $x=y^2 \Rightarrow \lim\limits_{y\to0}\dfrac{5y^4}{\underbrace{y^4+y^4}_{=2y^4}}=\dfrac{5}{2}$ $\rightarrow 0\neq 5/2$ so there is no limit at the point $(0,0)$.

III. f is not continuous at the point $(0,0)$ since there is no limit at the point $(0,0)$.

$\boxed{\text{Only I}}$

Question 5: Let $f(x,y) = \ln(x^2+y^2-4) + \arccos\left(\frac{1-x^2-y^2}{8}\right)$. What is the value of the area of the region in the plane specified by the domain of the function f?

A) 3π B) 5π C) 8π D) 10π E) 12π

① $x^2+y^2-4>0$ for $\ln(x^2+y^2-4) \Rightarrow 4 < x^2+y^2$

② If $\arccos\left(\frac{1-x^2-y^2}{8}\right) = z \Rightarrow \frac{1-x^2-y^2}{8} = \cos z$

$-1 \le \cos z \le 1$

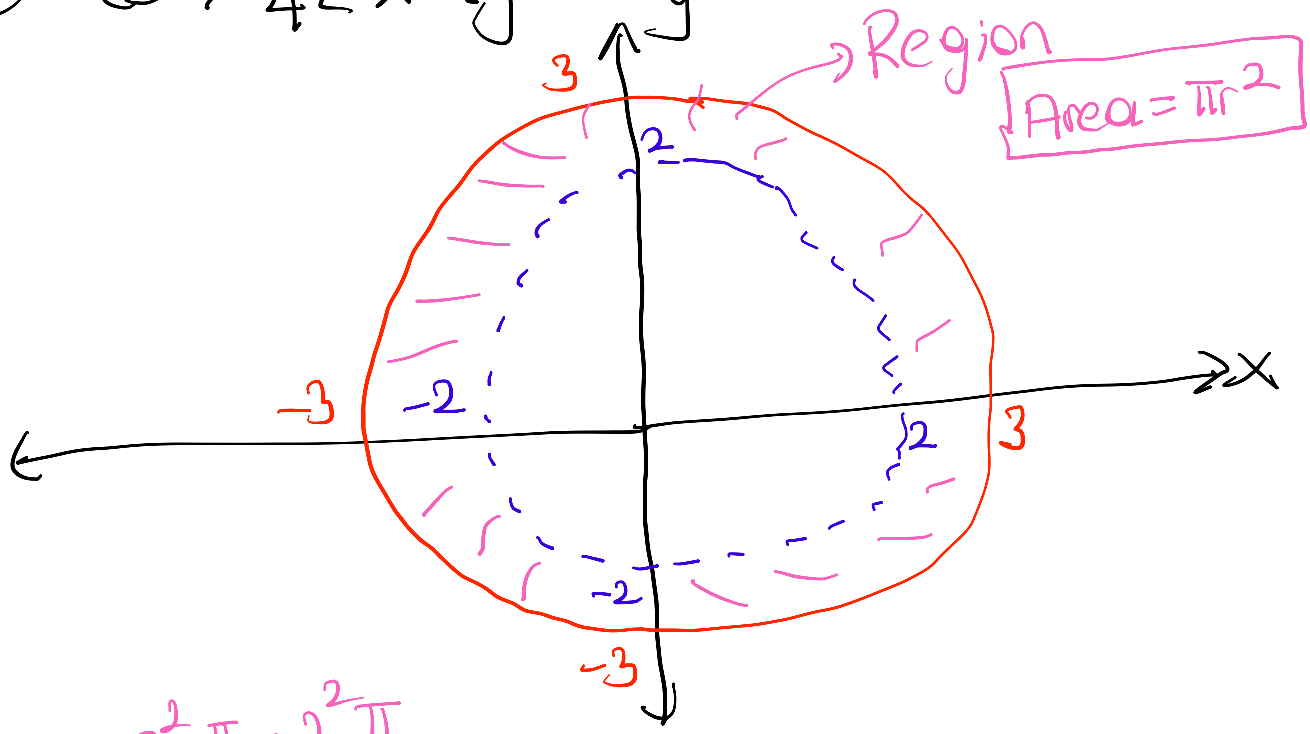
$\Rightarrow -1 \le \frac{1-x^2-y^2}{8} \le 1$

$\Rightarrow -8 \le 1-x^2-y^2 \le 8$

$\Rightarrow -9 \le -x^2-y^2 \le 7$

$\Rightarrow -7 \le x^2+y^2 \le 9$

① + ② $\Rightarrow 4 < x^2+y^2 \le 9$



Area $= 3^2\pi - 2^2\pi$
$= 9\pi - 4\pi = 5\pi$

Question 6: $\lim_{(x,y)\to(1,1)} \dfrac{\sqrt{1+3x^2y^2}-2}{xy-1} = ?$ $\quad \dfrac{\sqrt{4}-2}{1-1} = \dfrac{2-2}{1-1} = \dfrac{0}{0}$ indeterminated form

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $\frac{4}{3}$ D) $\frac{5}{2}$ E) 0

$$\frac{\sqrt{1+3x^2y^2}-2}{xy-1} \cdot \frac{\sqrt{1+3x^2y^2}+2}{\sqrt{1+3x^2y^2}+2} = \frac{1+3x^2y^2-4}{(xy-1)(\sqrt{1+3x^2y^2}+2)}$$

$$= \frac{3x^2y^2-3}{(xy-1)(\sqrt{1+3x^2y^2}+2)} = \frac{3(x^2y^2-1)}{(xy-1)(\sqrt{1+3x^2y^2}+2)}$$

$$x^2y^2-1 = (xy)^2-1^2 = (xy-1)(xy+1)$$

$$= \frac{3\cancel{(xy-1)}(xy+1)}{\cancel{(xy-1)}(\sqrt{1+3x^2y^2}+2)} \Rightarrow$$

$$\lim_{(x,y)\to(1,1)} \frac{3(xy+1)}{\sqrt{1+3x^2y^2}+2} = \frac{3(1+1)}{\sqrt{4}+2} = \frac{6}{4} = \frac{3}{2}$$

Question 7: $\lim_{(x,y)\to(0,0)} \dfrac{1-\cos(xy)}{xy} = ?$ $\quad \dfrac{1-\cos 0}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$

indeterminated form

A) -1 B) 0 C) 1 D) $\frac{1}{2}$ E) $\frac{1}{3}$

$$\left.\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}\right\} \Rightarrow \cos(xy) = 1 - 2\sin^2\left(\frac{xy}{2}\right)$$

$$\frac{1-\cos(xy)}{xy} = \frac{1-\left(1-2\sin^2\left(\frac{xy}{2}\right)\right)}{xy}$$

$$= \frac{1-1+2\sin^2\left(\frac{xy}{2}\right)}{xy} = \frac{\cancel{2}\cdot\sin^2\left(\frac{xy}{2}\right)}{\cancel{2}\cdot\frac{xy}{2}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin\left(\frac{xy}{2}\right)\cdot\sin\left(\frac{xy}{2}\right)}{\frac{xy}{2}} =$$

$$\underbrace{\lim_{(x,y)\to(0,0)} \sin\left(\frac{xy}{2}\right)}_{\substack{=\sin 0 \\ =0}} \cdot \underbrace{\lim_{(x,y)\to(0,0)} \frac{\sin\left(\frac{xy}{2}\right)}{\left(\frac{xy}{2}\right)}}_{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1} =$$

$$0\cdot 1 = \boxed{0}$$