CONTINUITY

(1) Find an extension for the function $f(x)=\frac{x^2+x-2}{1-\sqrt{x}}$ to make it continuous at $x=1$.

For the function to be continuous at $x=1$, we need to find a value such that $\lim_{x\to 1} f(x)=F(1)$. (F: extension)

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2+x-2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\overset{3}{(x+2)}\cdot(x-1)}{1-\sqrt{x}}=\lim_{x\to 1}\frac{3\cancel{(\sqrt{x}-1)}\cdot(\sqrt{x}+1)}{-\cancel{(\sqrt{x}-1)}}=-6$$

$$F(x)=\begin{cases}\dfrac{x^2+x-2}{1-\sqrt{x}}, & \text{if } x\neq 1\\ -6, & \text{if } x=1\end{cases}$$

(2) Find the multiplication of the values for k such that the function $f(x)=\begin{cases}\dfrac{1-\cos kx}{x\cdot\sin x}, & x\neq 0\\ 2, & x=0\end{cases}$ is continuous at $x=0$.

If the function is continuous at $x=0$, then $\lim_{x\to 0} f(x)=f(0)=2$.

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1-\cos kx}{x\cdot\sin x}\cdot\frac{(1+\cos kx)}{(1+\cos kx)}=\lim_{x\to 0^+}\frac{1-\cos^2 kx}{x\cdot\sin x\cdot\underset{2}{(1+\cos kx)}}$$

$$=\frac{1}{2}\lim_{x\to 0^+}\frac{\sin^2 kx}{x\cdot\sin x}=\frac{1}{2}\lim_{x\to 0^+}\underset{k}{\frac{\sin kx}{x}}\cdot\underset{k}{\frac{\sin kx}{\sin x}}=\frac{1}{2}k^2=2$$

$$\Rightarrow k^2=4 \Rightarrow k_1=2,\ k_2=-2 \Rightarrow k_1\cdot k_2=-4$$

If $\lim_{x\to 0} f(x)$ exists, then it means that $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)$ and continuity states that this limit is $f(0)=2$. We don't need to check for $x\to 0^-$. Because, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1-\cos kx}{x\cdot\sin x}$ and both x and sinx is negative, thus, sign won't be changed.

3) Find the value(s) of k that makes the function

$$f(x)=\begin{cases}\dfrac{\tan kx}{x} & , \text{ if } x<0 \\ 3x+2k^2 & , \text{ if } x\geq 0\end{cases}$$ continuous at $x=0$.

If function is continuous at $x=0$, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=f(0)$.

$$\left.\begin{aligned} & f(0)=3.0+2k^2=2k^2 \\ & \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(3x+2k^2)=2k^2 \\ & \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\underbrace{\frac{\tan kx}{x}}_{k}=k \end{aligned}\right\} \quad 2k^{\cancel{2}}=\cancel{k} \Rightarrow 2k=1 \Rightarrow \underline{k_1=\frac{1}{2}}$$

and $\underline{k_2=0}$

(In case of cancellation, cancelling value $(k=0)$ also must be considered. $(2k^2=k \Rightarrow 2k^2-k=0 \Rightarrow k(2k-1)=0 \Rightarrow k=0,\ k=\frac{1}{2})$)

4) Find an extension $F(x)$ for the function

$f(x)=\dfrac{\cos 7x-\cos 3x}{\ln x^2}$ such that $F(x)$ is continuous.

cos is continuous $\forall x\in\mathbb{R}$. However, for $x=0$, $\ln x^2$ is not defined. We have to find a value for extension F such that

$\lim\limits_{x\to 0} f(x)=F(0)$.

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\cos 7x-\cos 3x}{\ln x^2} \underset{(*)}{=} \lim_{x\to 0}\frac{-2.\sin\left(\frac{7x+3x}{2}\right).\sin\left(\frac{7x-3x}{2}\right)}{\ln x^2}$$

$$=\lim_{x\to 0}\frac{-\cancel{2}.\sin 5x.\sin 2x}{\cancel{2}.\ln x}=\lim_{x\to 0}-\underbrace{\frac{\sin 5x}{x}}_{5}.x.\underbrace{\frac{\sin 2x}{x}}_{2}.x.\frac{1}{\ln x}$$

$$=\lim_{x\to 0}\frac{-10x^2}{\ln x} \underset{(**)}{=} \lim_{x\to 0}\underbrace{-10x^2}_{0}.\underbrace{\frac{1}{\ln x}}_{0}=0$$

$$F(x)=\begin{cases}\dfrac{\cos 7x-\cos 3x}{\ln x^2} & , \text{ if } x\neq 0 \\ 0 & , \text{ if } x=0\end{cases}$$

$(*)$ $\cos a - \cos b = -2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right)$

$(**)$ $\lim_{x\to 0} \frac{-10x^2}{\ln x} = \frac{0}{\infty}$, but $\lim_{x\to 0} -10x^2 \cdot \frac{1}{\ln x} = 0.0 = 0$

⑤ Investigate the continuity of the function

$$f(x) = \begin{cases} \frac{e^{1/x}}{1+e^{1/x}} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$$ at the point $x=0$.

Check for the equalities $\lim_{x\to 0^-} f(x) \overset{?}{=} \lim_{x\to 0^+} f(x) \overset{?}{=} f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^-} \frac{\cancel{e^{1/x}}}{\cancel{e^{1/x}} \cdot (\underbrace{e^{-1/x}}_{e^{\infty}=\infty}+1)} = 0 \quad \checkmark$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+} \frac{\cancel{e^{1/x}}}{\cancel{e^{1/x}} (\underbrace{e^{-1/x}}_{e^{-\infty}=0}+1)} = 1 \quad \times$$

While $\lim_{x\to 0^-} f(x) = f(0)$, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$. Hence, the function $f(x)$ is discontinuous at $x=0$.

Discontinuity type: Jump discontinuity.

⑥ Find the point(s) which makes the function $f(x) = \frac{|x-3|}{x.(x^2-9)}$ discontinuous. Determine the discontinuity types for corresponding point(s).

i) Check for the zeros of denominator; $x=0$, $x=\mp 3$.

ii) Check for the point that absolute value changes its sign: $x=3$.

For $x=0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{-\cancel{(x-3)}}{\underset{(+)}{x}.\cancel{(x-3)}.\underset{(+)}{(x+3)}} = -\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-\cancel{(x-3)}}{\underset{(-)}{x}.\cancel{(x-3)}\underset{(+)}{(x+3)}} = \infty$$

Discontinuous at $x=0$
Type: infinite discontinuity

For $x=-3$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{-\cancel{(x-3)}}{\underset{(-)}{x}.\cancel{(x-3)}.\underset{(+)}{(x+3)}} = \infty$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{-\cancel{(x-3)}}{\underset{(-)}{x}.\cancel{(x-3)}.\underset{(-)}{(x+3)}} = -\infty$$

Discontinuous at $x=-3$
Type: infinite discontinuity

For $x=3$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{\cancel{x-3}}{x.\cancel{(x-3)}.(x+3)} = \frac{1}{18}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{-\cancel{(x-3)}}{x.\cancel{(x-3)}.(x+3)} = \frac{-1}{18}$$

Discontinuous at $x=3$
Type: jump discontinuity

7) Find the point(s) where the function $f(x) = \dfrac{|x^2-1|.(x-3)}{(x+1).x}$ is discontinuous. Determine the types of discontinuity.

i) Check for the zeros of denominator: $x=0$, $x=-1$.

ii) Check for the point that absolute value changes its sign: $x=\mp 1$.

For $x=0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{-\overset{(-)}{(x-1)}.\cancel{(x+1)}.\overset{(-)}{(x-3)}}{\cancel{(x+1)}.\underset{(+)}{x}} = -\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-\overset{(-)}{(x-1)}.\cancel{(x+1)}.\overset{(-)}{(x-3)}}{\cancel{(x+1)}.\underset{(-)}{x}} = \infty$$

Discontinuous at $x=0$
Type: infinite discontinuity

For $x=1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1).\cancel{(x+1)}.(x-3)}{\cancel{(x+1)}.x} = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-(x-1).\cancel{(x+1)}.(x-3)}{\cancel{(x+1)}.x} = 0$$

$f(1)=0$. Thus, the function is continuous at $x=1$

For $x=-1$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{-(x-1).\cancel{(x+1)}.(x-3)}{\cancel{(x+1)}.x} = \frac{-8}{-1} = 8$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{(x-1).\cancel{(x+1)}.(x-3)}{\cancel{(x+1)}.x} = \frac{8}{-1} = -8$$

Discontinuous at $x=-1$

Type: jump discontinuity

8) Investigate the discontinuity points of the function $f(x)=\frac{(x+1).(x-2)}{|x^2-4|.|x+1|}$ and classify them.

i) Check for the zeros of denominator; $x=\mp 2$, $x=-1$.

ii) Check for the point that absolute value changes sign; same as (i).

For $x=-1$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{-\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \frac{1}{-1} = -1$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{+\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \frac{1}{1} = 1$$

Discontinuous at $x=-1$

Type: jump discontinuity

For $x=-2$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{+\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \infty$$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{-\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \infty$$

Discontinuous at $x=-2$

Type: infinite discontinuity

For $x=2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \frac{1}{4}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{\cancel{(x+1)}.\cancel{(x-2)}}{-\cancel{(x-2)}.(x+2).\cancel{(x+1)}} = \frac{1}{-4}$$

Discontinuous at $x=2$

Type: jump discontinuity

9) Let $f(x)$ be a continuous function. Find a and b, if

$$f(x) = \begin{cases} 3\operatorname{arcsec}(x-2) & , \text{ if } x \leq 0 \\ \dfrac{\sin(ax)}{x} & , \text{ if } 0 < x \leq 1 \\ \ln(bx) + \cos\left(\dfrac{ax}{4}\right) & , \text{ if } x > 1 \end{cases}$$

i) We must have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ $[f(0) = 2\pi]$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3\operatorname{arcsec}(x-2) = 2\pi$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(ax)}{x} = a$$

$\boxed{a = 2\pi}$

ii) We must have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ $[f(1) = \sin(2\pi) = 0]$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\sin(2\pi x)}{x} = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left[\ln(bx) + \cos\left(\frac{\pi}{2}x\right)\right] = \ln b + \cos\left(\frac{\pi}{2}\right) = \ln b$$

$\Rightarrow \ln b = 0 \Rightarrow \boxed{b=1}$

(10) For which values of a and b, the function

$$f(x)=\begin{cases} x\cdot\sin\left(\frac{1}{x}\right)+ab, & \text{if } x<0 \\ \arccos\left(\frac{x-1}{2}\right), & \text{if } 0\leq x\leq 1 \\ a\cdot e^{\frac{(x-1)^2}{\sin(x-1)}}, & \text{if } 1<x<3 \end{cases}$$

is continuous in the domain $(-\infty,3)$?

i) We must have $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$ $\left[f(0)=\frac{2\pi}{3}\right]$

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right)+ab = ab \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \arccos\left(\frac{x-1}{2}\right)=\frac{2\pi}{3} \end{aligned}\right\} \quad ab=\frac{2\pi}{3} \quad \dots (*)$$

ii) We must have $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$ $\left[f(1)=\frac{\pi}{2}\right]$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\arccos\left(\frac{x-1}{2}\right)=\frac{\pi}{2}$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} a\cdot e^{\frac{(x-1)^2}{\sin(x-1)}} \underset{\substack{\downarrow \\ e \text{ is continuous}}}{=} a\cdot e^{\lim\limits_{x\to 1^+}\overbrace{\frac{(x-1)}{\sin(x-1)}}^{1}\cdot(x-1)} = a\cdot e^0 = a$$

$$\Rightarrow \boxed{a=\frac{\pi}{2}}$$

By writing the value of $a=\frac{\pi}{2}$ into equation $(*)$, we get

$$\frac{\cancel{\pi}}{2}\cdot b=\frac{2\cancel{\pi}}{3} \Rightarrow \boxed{b=\frac{4}{3}}$$

(11) Let f be a function defined by

$$f(x)=\begin{cases} \dfrac{1-\cos x^2}{x^4} & , \quad x<0 \\ \dfrac{2}{\pi}\arcsin x+\dfrac{3}{\pi}\arccos x & , \quad 0\leq x\leq 1 \\ (x-1)\sin\dfrac{1}{x-1} & , \quad x>1 \end{cases}$$

Investigate the continuity of f at the points $x=0$ and $x=1$. If any, classify the type of discontinuity.

<u>For $x=0$</u>

We must have $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$ for continuity.

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1-\cos x^2}{x^4}\cdot\frac{(1+\cos x^2)}{(1+\cos x^2)}=\lim_{x\to 0^-}\frac{1-\cos^2 x^2}{x^4\cdot\underbrace{(1+\cos x^2)}_{2}}=\lim_{x\to 0^-}\frac{\sin^2 x^2}{2x^4}$$

$$=\lim_{x\to 0^-}\frac{\overbrace{\sin x^2}^{1}\cdot\sin x^2}{2\cdot x^2\cdot\underbrace{x^2}_{1}}=\frac{1}{2}$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(\frac{2}{\pi}\underbrace{\arcsin x}_{0}+\frac{3}{\pi}\underbrace{\arccos x}_{\frac{\pi}{2}}\right)=\frac{3}{\pi}\cdot\frac{\pi}{2}=\frac{3}{2}$$

Since $\lim\limits_{x\to 0^-} f(x)\neq\lim\limits_{x\to 0^+} f(x)$, the function is discontinuous (doesn't have a limit) and its type is jump discontinuity.

<u>For $x=1$</u>

We must have $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$ for continuity.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(\frac{2}{\pi}\underbrace{\arcsin x}_{\frac{\pi}{2}}+\frac{3}{\pi}\underbrace{\arccos x}_{0}\right)=\frac{2}{\pi}\cdot\frac{\pi}{2}=1$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+}\underbrace{(x-1)}_{0}\,\underbrace{\sin\frac{1}{x-1}}_{\text{constant in }[-1,1]}=0$$

$\Rightarrow$ Same as $x=0$, the function has jump discontinuity at $x=1$.

12) Find the points where the function defined by $f(x) = \dfrac{|x^2-9|}{x^2-4x+3}$ is discontinuous. Classify the points of discontinuity.

$x^2 - 4x + 3 = (x-3).(x-1) \Rightarrow x=1,\ x=3$ are the points of interest.

For $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-(\cancel{x-3}).(x+3)}{(\cancel{x-3}).(x-1)} = \infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{-(\cancel{x-3}).(x+3)}{(\cancel{x-3}).(x-1)} = -\infty$$

Infinite discontinuity at $x=1$.

For $x=3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{-(\cancel{x-3}).(x+3)}{(\cancel{x-3}).(x-1)} = -3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{(x-3).(x+3)}{(x-3).(x-1)} = 3$$

Jump discontinuity at $x=3$.

We didn't check for $x=-3$ (root of absolute value) because it is not a root for denominator and absolute value is a continuous function (as seen from previous questions.)