IMPROPER INTEGRALS

Definition: Integrals with infinite limits of integration are improper integrals of Type I.

1) If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3) If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Example: Is the area under the curve $y=\frac{\ln x}{x^2}$ from $x=1$ to $x=\infty$ finite? If so, what is its value?

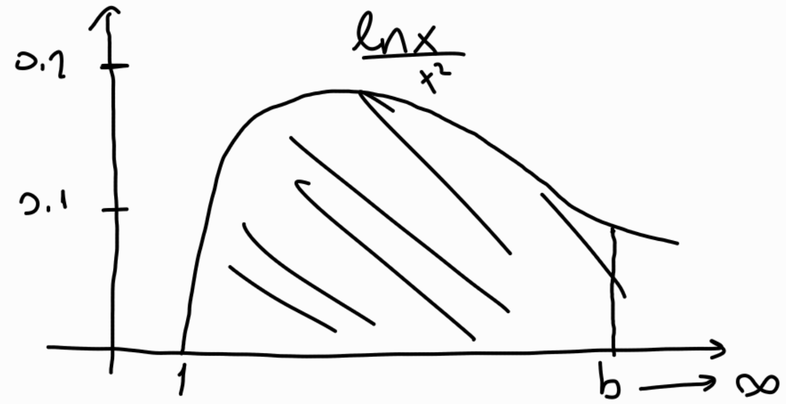
$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x}\Bigg|_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1$$

$\ln x = u \qquad \frac{dx}{x^2} = dv$

$\frac{dx}{x} = du \qquad -\frac{1}{x} = v$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\Bigg[-\frac{\ln b}{b} - \underbrace{\frac{1}{b}}_{0} + \underbrace{1}_{1}\Bigg] = 1 - \lim_{b\to\infty}\frac{\ln b}{b}$$

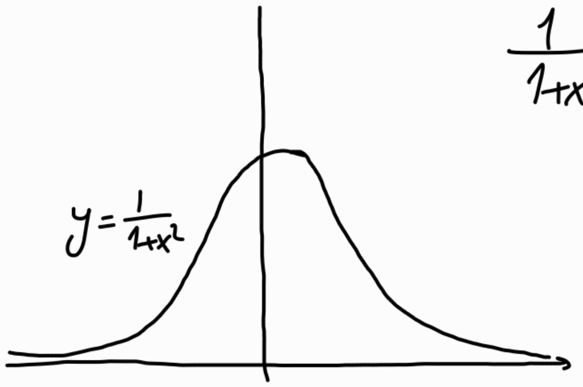
$$\xrightarrow[\left(\frac{\infty}{\infty}\right)]{L'} 1 - \underbrace{\lim_{b\to\infty}\frac{1}{b}}_{0} = 1 \;//$$



Example: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \arctan x\Big|_a^0 = \lim_{a\to-\infty} -\arctan a = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$



$\frac{1}{1+x^2} \rightarrow$ even symmetry w.r.t. y-axis $x = 0$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\cdot\int_{-\infty}^{0} \frac{dx}{1+x^2} = 2\cdot\frac{\pi}{2} = \pi \;//$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1) If $f(x)$ is continuous on $(\underline{a},\underline{b}]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

2) If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

3) If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c)\cup(c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit exists and is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Example: $\int_0^{1} \frac{dx}{1-x} = \lim_{b\to 1^-} \int_0^{b} \frac{dx}{1-x} = \lim_{b\to 1^-} -\ln|1-x| \Big|_0^{b}$

$$= -\lim_{b\to 1^-} \ln|\underbrace{1-b}_{0^+}| = \infty \Rightarrow \text{diverges.}$$

Example: $\int_0^{3} \frac{dx}{(x-1)^{2/3}}$ at $x=1$!

$$\int_0^{3} \frac{dx}{(x-1)^{2/3}} = \underbrace{\int_0^{1} \frac{dx}{(x-1)^{2/3}}}_{I_1} + \underbrace{\int_1^{3} \frac{dx}{(x-1)^{2/3}}}_{I_2} \qquad \frac{1}{(x-1)^{2/3}} = (x-1)^{-2/3}$$

$$I_1 = \lim_{b\to 1^-} \int_0^{b} \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{1/3} \Big|_0^{b} = \lim_{b\to 1^-} 3\left[\underbrace{(b-1)^{1/3}}_{0} + 1\right] = 3$$

$$I_2 = \lim_{a\to 1^+} \int_a^{3} \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} 3.(x-1)^{1/3} \Big|_a^{3} = \lim_{a\to 1^+} \left[3\sqrt[3]{2} - 3\underbrace{(a-1)^{1/3}}_{0}\right] = 3\sqrt[3]{2}$$

$$I = I_1 + I_2 = 3 + 3\sqrt[3]{2}$$

Ex.: $\int_1^{\infty} \frac{dx}{(3x+1)^2} = ?$

$$I = \lim_{c\to\infty} \int_1^{c} \frac{dx}{(3x+1)^2} = \lim_{c\to\infty} \frac{-1}{3(3x+1)} \Big|_1^{c} = \lim_{c\to\infty} \underbrace{\frac{-1}{3(3c+1)}}_{0} + \frac{1}{12} = \frac{1}{12}$$

Convergent.

Ex.: $\displaystyle\int_{-\infty}^{-1} \frac{dx}{\sqrt{2-x}} = ?$

$$I = \lim_{c\to-\infty} \int_{c}^{-1} \frac{dx}{\sqrt{2-x}} = \lim_{c\to-\infty} -2\sqrt{2-x}\,\Big|_{c}^{-1} = \lim_{c\to-\infty} -2\sqrt{3} + 2\underbrace{\sqrt{2-c}}_{\infty} = \infty$$

$\Rightarrow$ Divergent.

Ex.: $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{9+x^6} \longrightarrow$ <u>even</u>

$$\int \frac{x^2\,dx}{9+x^6} \qquad \begin{array}{l} x^3 = t \\ 3x^2\,dx = dt \end{array} \quad \Rightarrow \int \frac{dt}{3(9+t^2)} = \frac{1}{\cancel{3}}\cdot\frac{1}{9}\cdot\cancel{3}\arctan\left(\frac{t}{3}\right) + c$$

$$9\left(1+\frac{t^2}{9}\right) \qquad \frac{1}{3}\cdot\frac{1}{1+\frac{t^2}{9}}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{9+x^6} = 2\cdot\underbrace{\int_{0}^{\infty} \frac{x^2\,dx}{9+x^2}}_{A}$$

$$\underline{\underline{A}} = \lim_{c\to\infty} \int_{0}^{c} \frac{x^2\,dx}{9+x^2} = \lim_{c\to\infty} \frac{1}{9}\arctan\left(\frac{x^3}{3}\right)\Big|_{0}^{c} = \lim_{c\to\infty} \frac{1}{9}\arctan\left(\frac{c^3}{3}\right) = \frac{1}{9}\,\frac{\pi}{2}$$

$$I = 2A = 2\cdot\frac{1}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}$$

Ex.: $\int_e^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{c\to\infty} \int_e^{c} \frac{dx}{x\underbrace{(\ln x)^3}_{\ln^3 x}}$ $\quad \ln x = t \quad x=c \Rightarrow t=\ln c$

$\frac{dx}{x} = dt \quad x=e \Rightarrow t=1$

$$I = \lim_{c\to\infty} \int_1^{\ln c} \frac{dt}{t^3} = \lim_{c\to\infty} \frac{-1}{2t^2}\Big|_1^{\ln c} = \lim_{c\to\infty} \underbrace{\frac{-1}{2\ln^2 c}}_{0} + \frac{1}{2} = \frac{1}{2} \text{ (Convergent)}$$

t^{-3}

Ex.: $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{a\to -2^+} \int_a^{14} (x+2)^{-1/4} dx = \lim_{a\to -2^+} \frac{4}{3}(x+2)^{3/4}\Big|_a^{14}$

$$= \lim_{a\to -2^+} \frac{4}{3}\left[8 - \underbrace{(a+2)^{3/4}}_{0}\right] = \frac{32}{3} \text{ (Convergent)}$$

Ex.: $\int_{-2}^{3} \frac{1}{x^4} dx = ?$ $\quad \underbrace{\int_{-2}^{0} \frac{dx}{x^4}}_{I_1} + \underbrace{\int_{0}^{3} \frac{dx}{x^4}}_{I_2}$

$$I_1 = \lim_{b\to 0^-} \int_{-2}^{b} x^{-4} dx = \lim_{b\to 0^-} \frac{x^{-3}}{-3}\Big|_{-2}^{b} = \lim_{b\to 0^-} \frac{-1}{3b^3} + \frac{1}{-24} = \infty$$

$$I_2 = \lim_{a\to 0^+} \frac{-1}{3x^3}\Big|_a^3 = \frac{-1}{81} + \frac{1}{3a^3} = \infty \qquad I\to\infty \text{ (Divergent)}$$

Ex.: $\int_0^{\pi/2} \sec x\, dx = \lim_{b\to \frac{\pi}{2}^-} \int_0^b \sec x\, dx = \lim_{b\to \frac{\pi}{2}^-} \ln|\sec x + \tan x|\Big|_0^b$

$$= \lim_{b\to \frac{\pi}{2}^-} \ln\underbrace{|\sec b + \tan b|}_{\frac{1+\sin b}{\cos b}} - \underbrace{\ln 1}_{0} = \infty \text{ (Divergent)}$$

Ex.: $\displaystyle\int_0^3 \frac{dx}{x^2-6x+5}$ $\qquad x^2-6x+5=(x-1).(x-5)$

$\begin{array}{ll} x & -5 \\ x & -1 \end{array}$

$$\frac{1}{x^2-6x+5}=\frac{A}{\underset{\downarrow}{x-1}}+\frac{B}{\underset{\downarrow}{x-5}}$$

$\qquad\qquad x=1 \qquad x=5$

$$A=\frac{1}{x-5}\Big|_{x=1}=\frac{-1}{4} \qquad B=\frac{1}{x-1}\Big|_{x=5}=\frac{1}{4}$$

$$I=I_1+I_2=\int_0^1\frac{dx}{x^2-6x+5}+\int_1^3\frac{dx}{x^2-6x+5}$$

$$I_1=\lim_{b\to1^-}\frac{1}{-4}\int_0^b\left[\frac{1}{x-5}-\frac{1}{x-1}\right]dx=\frac{-1}{4}\lim_{b\to1^-}\ln\left|\frac{x-5}{x-1}\right|\Big|_0^b=\frac{1}{4}\lim_{b\to1^-}\ln\left|\frac{b-5}{b-1}\right|-\ln 4$$

$$=\infty$$

$$I_2=\frac{1}{4}\lim_{a\to1^+}\ln\left|\frac{x-5}{x-1}\right|\Big|_a^3=\frac{1}{4}\lim_{a\to1^+}\underbrace{\ln 1}_{0}-\Big[\underbrace{\ln|a-5|}_{\text{constant}}-\underbrace{\ln|a-1|}_{-\infty}\Big]=-\infty$$

ignore

Divergent

Ex.: $\displaystyle\int_0^1\frac{\ln x}{\sqrt{x}}\,dx$ $\qquad \sqrt{x}=t \qquad x=0\Rightarrow t=0$

$\qquad \frac{1}{2\sqrt{x}}dx=dt \qquad x=1\Rightarrow t=1$

$$\int\frac{\ln(\sqrt{x})^2}{\sqrt{x}}dx=\int\ln t^2\cdot 2\,dt=\int 4\ln t\,.\,dt$$

$$\lim_{a\to0^+}\int_a^1 4\ln t\,dt=4\lim_{a\to0^+}(t.\ln t-t)\Big|_a^1=4.\lim_{a\to0^+}\underbrace{(-1)}_{-1}-(\underset{\substack{\downarrow\\0.(-\infty)}}{a\ln a}-\underset{\substack{\downarrow\\0}}{a})$$

$$-4-4\lim_{a\to0^+}\frac{\ln a}{\frac{1}{a}}\overset{L}{\underset{(\frac{\infty}{\infty})}{=}}-4-4\lim_{a\to0^+}\frac{\frac{1}{\cancel{a}}}{-\frac{1}{a^{\cancel{2}}}}=-4+4\underbrace{\lim_{a\to0^+}a}_{0}=-4$$

H.W.: $\displaystyle\int_0^\infty\frac{e^x}{e^{2x}+3}dx=?\qquad\left(\frac{\pi}{3\sqrt{3}}\right)$