IMPROPER INTEGRALS

① $\displaystyle\int_0^{\pi/2} \tan x\,dx = ?$ (Discontinuous at $x=\frac{\pi}{2}$)

$$\lim_{b\to\frac{\pi}{2}^-} \int_0^b \tan x\,dx = \lim_{b\to\frac{\pi}{2}^-} -\ln|\cos x|\Big|_0^b = \lim_{b\to\frac{\pi}{2}^-}\Big[\underbrace{\ln|\underbrace{\cos 0}_{1}|}_{0} - \underbrace{\ln|\underbrace{\cos b}_{0^+}|}_{-\infty}\Big] = \infty$$

$\Rightarrow$ Integral is divergent

② $\displaystyle\int_0^1 \frac{x}{(1-x^2)^{1/4}}\,dx = ?$ (Discontinuous at $x=1$)

$$\lim_{b\to1^-} \int_0^b \frac{x}{(1-x^2)^{1/4}}\,dx \qquad \begin{aligned} x&=\sin t\\ dx&=\cos t\,dt\end{aligned} \qquad \begin{aligned} x=0 &\Rightarrow t=0\\ x=b &\Rightarrow t=\arcsin b\end{aligned}$$

$$I = \lim_{b\to1^-} \int_0^{\arcsin b} \frac{\sin t.\cos t\,dt}{(1-\sin^2 t)^{1/4}} = \lim_{b\to1^-}\int_0^{\arcsin b}\frac{\sin t.\cos t\,dt}{\sqrt{\cos t}} \qquad \begin{aligned} \cos t&=u\\ -\sin t\,dt&=du\end{aligned} \qquad \begin{aligned} t=0 &\Rightarrow u=1\\ t=\arcsin b &\Rightarrow u=\cos(\arcsin b)\end{aligned}$$

$$I = \lim_{b\to1^-}\int_1^{\cos(\arcsin b)} -u^{1/2}\,du = \lim_{b\to1^-} -\frac{2}{3}.u^{3/2}\Big|_1^{\cos(\arcsin b)} = \lim_{b\to1^-}\Big[\frac{2}{3}.1-\frac{2}{3}\big(\underbrace{\cos(\underbrace{\arcsin b}_{\frac{\pi}{2}^-})}_{0^+}\big)^{3/2}\Big] = \frac{2}{3}$$

③ $\displaystyle\int_0^1 \frac{dx}{(1+x^2)\sqrt{\arctan x}} = ?$ (Discontinuous at $x=0$)

$$\lim_{a\to0^+}\int_a^1 \frac{dx}{(1+x^2)\sqrt{\arctan x}} \qquad \begin{aligned} \arctan x&=u\\ \frac{dx}{1+x^2}&=du\end{aligned} \qquad \begin{aligned} x=a &\Rightarrow u=\arctan a\\ x=1 &\Rightarrow u=\frac{\pi}{4}\end{aligned}$$

$$I = \lim_{a\to0^+}\int_a^1 \frac{du}{\underbrace{\sqrt{u}}_{u^{-1/2}}} = \lim_{a\to0^+} 2.\sqrt{u}\Big|_a^1 = \lim_{a\to0^+}\big[2-2\sqrt{a}\big] = 2$$

④ $\int_0^1 \frac{dx}{x.\ln x} = ?$ (Discontinuous at $x=0$ and $x=1$)

$$I = \underbrace{\int_0^{e^{-1}} \frac{dx}{x\ln x}}_{I_1} + \underbrace{\int_{e^{-1}}^{1} \frac{dx}{x\ln x}}_{I_2}$$

$$I_1 = \lim_{a\to 0^+} \int_a^{e^{-1}} \frac{dx}{x\ln x} \qquad \ln x = u \quad \frac{dx}{x} = du \qquad x=a \Rightarrow u=\ln a \quad x=e^{-1} \Rightarrow u=-1$$

$$I_1 = \lim_{a\to 0^+} \int_{\ln a}^{-1} \frac{du}{u} = \lim_{a\to 0^+} \ln|u| \Big|_{\ln a}^{-1} = \lim_{a\to 0^+}\Big[\underbrace{\ln 1}_{0} - \underbrace{\ln\underbrace{|\ln a|}_{|-\infty|}}_{\infty}\Big] = -\infty$$ (divergent)

$$I_2 = \lim_{b\to 1^-} \int_{e^{-1}}^{b} \frac{dx}{x.\ln x} \qquad \ln x = u \quad \frac{dx}{x} = du \qquad x=e^{-1} \Rightarrow u=-1 \quad x=b \Rightarrow u=\ln b$$

$$I_2 = \lim_{b\to 1^-} \int_{-1}^{\ln b} \frac{du}{u} = \lim_{b\to 1^-} \ln|u| \Big|_{-1}^{\ln b} = \lim_{b\to 1^-}\Big[\underbrace{\ln\underbrace{|\ln b|}_{|0^-|}}_{-\infty} - \underbrace{\ln|-1|}_{0}\Big] = -\infty$$ (divergent)

$$I = I_1 + I_2 = -\infty - \infty = -\infty \Rightarrow \text{Divergent}$$

⑤ $\int_0^1 \frac{\arcsin x\, dx}{\sqrt{1-x^2}} = ?$ (Discontinuous at $x=1$)

$$\lim_{b\to 1^-} \int_0^b \frac{\arcsin x\, dx}{\sqrt{1-x^2}} \qquad \arcsin x = u \quad \frac{dx}{\sqrt{1-x^2}} = du \qquad x=0 \Rightarrow u=0 \quad x=b \Rightarrow u=\arcsin b$$

$$I = \lim_{b\to 1^-} \int_0^{\arcsin b} u\,du = \lim_{b\to 1^-} \frac{u^2}{2} \Big|_0^{\arcsin b} = \lim_{b\to 1^-}\left[\frac{(\arcsin b)^2}{2} - 0\right] = \frac{\pi^2}{8}$$

(6) $\int_0^{\infty} \frac{e^x dx}{e^{2x}+1} = ?$ (Upper limit is infinity)

$\lim_{b\to\infty} \int_0^b \frac{e^x dx}{e^{2x}+1}$ $\quad e^x = t \quad e^x dx = dt \quad x=0 \Rightarrow t=1 \quad x=b \Rightarrow t=e^b$

$\Rightarrow I = \lim_{b\to\infty} \int_1^{e^b} \frac{dt}{t^2+1} = \lim_{b\to\infty} \arctan t \Big|_1^{e^b} = \lim_{b\to\infty} [\underbrace{\arctan e^b}_{\frac{\pi}{2}} - \underbrace{\arctan 1}_{\frac{\pi}{4}}] = \frac{\pi}{4}$

(7) $\int_{-1}^{\infty} \frac{dx}{x^2+5x+6} = ?$ (Upper limit is infinity)

$\frac{1}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2} \quad A = \frac{1}{x+2}\Big|_{x=-3} = -1, \quad B = \frac{1}{x+3}\Big|_{x=-2} = 1$

$I = \lim_{b\to\infty} \int_{-1}^b \left[\frac{1}{x+2} - \frac{1}{x+3}\right] dx = \lim_{b\to\infty} [\ln|x+2| - \ln|x+3|]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{x+2}{x+3}\right|\right]_{-1}^b$

$= \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right| - \ln\left|\frac{1}{2}\right|\right] = \ln 1 - \ln 2^{-1} = \ln 2$

(8) $\int_0^{\infty} \left(\frac{2}{3}\right)^x dx = ?$ (Upper limit is infinity)

$\lim_{b\to\infty} \int_0^b \left(\frac{2}{3}\right)^x dx = \lim_{b\to\infty} \left(\frac{2}{3}\right)^x \cdot \frac{1}{\ln\left(\frac{2}{3}\right)} \Big|_0^b = \frac{1}{\ln\left(\frac{2}{3}\right)} \cdot \lim_{b\to\infty} \left[\underbrace{\left(\frac{2}{3}\right)^b}_{0} - 1\right] = \frac{-1}{\ln\left(\frac{2}{3}\right)}$

(9) $$\int_{-\infty}^{\infty} \frac{dx}{4+x^2} = ?$$ (Upper and lower limits are infinity)

$$I = \underbrace{\int_{-\infty}^{0} \frac{dx}{4+x^2}}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{dx}{4+x^2}}_{I_2}$$ (Seperation point $x=0$ is arbitrary)

$$I_1 = \lim_{k\to-\infty} \int_{k}^{0} \frac{dx}{4+x^2} = \lim_{k\to-\infty} \arctan\left(\frac{x}{2}\right)\cdot\frac{1}{2}\Big|_{k}^{0} = \frac{1}{2}\cdot\lim_{k\to-\infty}\Big[\underbrace{\arctan 0}_{0} - \underbrace{\arctan\tfrac{k}{2}}_{-\frac{\pi}{2}}\Big] = \frac{\pi}{4}$$

$$I_2 = \lim_{k\to\infty} \int_{0}^{k} \frac{dx}{4+x^2} = \lim_{k\to\infty} \frac{1}{2}\cdot\arctan\left(\frac{x}{2}\right)\Big|_{0}^{k} = \frac{1}{2}\cdot\lim_{k\to\infty}\Big[\underbrace{\arctan\tfrac{k}{2}}_{\frac{\pi}{2}} - \underbrace{\arctan 0}_{0}\Big] = \frac{\pi}{4}$$

$$I = I_1 + I_2 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

(10) $$\int_{-\infty}^{\infty} x.e^{-x^2}dx = ?$$ (Upper and lower limits are infinity)

$$I = \underbrace{\int_{-\infty}^{0} x.e^{-x^2}dx}_{I_1} + \underbrace{\int_{0}^{\infty} x.e^{-x^2}dx}_{I_2}$$

$x^2 = t$
$2xdx = dt$

$$\int x.e^{-x^2}dx = \int \frac{e^{-t}}{2}dt = -\frac{e^{-t}}{2} + c = -\frac{e^{-x^2}}{2} + c$$

$$I_1 = \lim_{k\to-\infty} \int_{k}^{0} x.e^{-x^2}dx = \lim_{k\to-\infty} -\frac{e^{-x^2}}{2}\Big|_{k}^{0} = \lim_{k\to-\infty}\Big[\underbrace{\frac{-e^0}{2}}_{-\frac{1}{2}} + \underbrace{\frac{e^{-k^2}}{2}}_{0}\Big] = -\frac{1}{2}$$

$$I_2 = \lim_{k\to\infty} \int_{0}^{k} x.e^{-x^2}dx = \lim_{k\to\infty} \frac{-e^{-x^2}}{2}\Big|_{0}^{k} = \lim_{k\to\infty}\Big[\underbrace{\frac{-e^{-k^2}}{2}}_{0} + \underbrace{\frac{e^0}{2}}_{\frac{1}{2}}\Big] = \frac{1}{2}$$

$$I = I_1 + I_2 = -\frac{1}{2} + \frac{1}{2} = 0$$

(11) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)} = ?$ (Discontinuous at $x=0$, upper limit is infinity)

$$I = \underbrace{\int_0^1 \frac{dx}{\sqrt{x}(1+x)}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)}}_{I_2}$$

$\sqrt{x} = u$, $\dfrac{dx}{2\sqrt{x}} = du$ $\qquad \displaystyle\int \frac{2du}{1+u^2} = 2\arctan \underset{\sqrt{x}}{u} + C$

$$I_1 = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}(1+x)} = \lim_{a\to 0^+} 2\arctan\sqrt{x}\Big|_a^1 = \lim_{a\to 0^+} 2\Big[\underbrace{\arctan 1}_{\frac{\pi}{4}} - \underbrace{\arctan a}_{0}\Big] = \frac{\pi}{2}$$

$$I_2 = \lim_{k\to\infty} \int_1^k \frac{dx}{\sqrt{x}(1+x)} = \lim_{k\to\infty} 2\arctan\sqrt{x}\Big|_1^k = \lim_{k\to\infty} 2\Big[\underbrace{\arctan\sqrt{k}}_{\frac{\pi}{2}} - \underbrace{\arctan 1}_{\frac{\pi}{4}}\Big] = \frac{\pi}{2}$$

$$I = I_1 + I_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(12) $\displaystyle\int_0^{\pi} \tan^2 x\,dx = ?$ (Discontinuous at $x = \frac{\pi}{2} \in [0,\pi]$)

$$I = \underbrace{\int_0^{\pi/2} \tan^2 x\,dx}_{I_1} + \underbrace{\int_{\pi/2}^{\pi} \tan^2 x\,dx}_{I_2}$$

$\displaystyle\int (\tan^2 x + 1 - 1)\,dx = \tan x - x + C$

$$I_1 = \lim_{k\to\frac{\pi}{2}^-} \int_0^k \tan^2 x\,dx = \lim_{k\to\frac{\pi}{2}^-} \Big[\tan x - x\Big]_0^k = \lim_{k\to\frac{\pi}{2}^-} \Big[\tan k - k - 0\Big] = \infty$$

$$I_2 = \lim_{k\to\frac{\pi}{2}^+} \int_k^{\pi} \tan^2 x\,dx = \lim_{k\to\frac{\pi}{2}^+} \Big[\tan x - x\Big]_k^{\pi} = \lim_{k\to\frac{\pi}{2}^+} \Big[0 - \pi \underbrace{- \underbrace{\tan k}_{-\infty}}_{\infty} + k\Big] = \infty$$

$I = I_1 + I_2 = \infty + \infty = \infty \Rightarrow$ Integrel is divergent