MAT1320 LINEAR ALGEBRA EXERCISES XI-XIV

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1. Let $V=\mathbb{R}^2$ and $A=\{(1,0),(0,1)\}$, $B=\{(2,0),(1,3)\}$, $C=\{(1,-3),(2,4)\}$ be subsets of V. The vector $v=(8,6)\in V$ can be written as the linear combination of ....

   a) Only A b) A and B c) Only C
   d) B and C e) A, B and C

A: $\begin{vmatrix}1&0\\0&1\end{vmatrix}=1\neq 0 \Rightarrow$ A is a basis for $\mathbb{R}^2$

$\Rightarrow$ A spans $\mathbb{R}^2$.

$\Rightarrow (8,6)$ is a linear combination for A.

B: $\begin{vmatrix}2&0\\1&3\end{vmatrix}=6\neq 0$. (Same argument)

C: $\begin{vmatrix}1&-3\\2&4\end{vmatrix}=14\neq 0$ (Same argument)

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\x+z\\z\end{bmatrix}=\begin{bmatrix}x\\x\\0\end{bmatrix}+\begin{bmatrix}0\\z\\z\end{bmatrix}$

$=x\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}+z\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}$

Let $B=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. See that

$\langle B\rangle=S$ and B is linearly independent

$\Rightarrow B$ is a basis for S. $\Rightarrow \dim S=|B|=2$.

2. Let $S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\mid y=x+z, \text{ where } x,y,z\in\mathbb{R}\right\}$ be the subspace of $\mathbb{R}^3$. What is the dimension of S ?

   a) 1 b) 2 c) 3 d) 4
   e) None of them

3. Let $M_{n\times n}$ denote the vector space of all $n\times n$ real matrices. Consider the subset

$$W=\left\{\begin{bmatrix}a&b\\c&0\end{bmatrix}\in M_{2\times 2}\mid a+b+c=0 \text{ where } a,b,c\in\mathbb{R}\right\}$$

Which of the following statements are always true?

✓ I. The set W is a subspace of $M_{2\times 2}$.

X II. $B=\left\{\begin{bmatrix}1&0\\-1&0\end{bmatrix},\begin{bmatrix}2&0\\-2&0\end{bmatrix}\right\}$ forms a basis for W.

✓ III. $\dim(W)=2$.

   a) Only I b) Only II c) Only III
   d) I and II e) I and III

I: See that W is closed under addition and multiplication by scalar. So, it is a subspace

II. While $\begin{bmatrix}0&1\\0&0\end{bmatrix}\in W$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ can not be written as a linear combination of B.

$\Rightarrow$ B is not a basis for W.

III. $a+b+c=0 \Rightarrow a=-b-c$

$\begin{bmatrix}a&b\\c&0\end{bmatrix}=\begin{bmatrix}-b-c&b\\c&0\end{bmatrix}=\begin{bmatrix}-b&b\\0&0\end{bmatrix}+\begin{bmatrix}-c&0\\c&0\end{bmatrix}$

$=b\begin{bmatrix}-1&1\\0&0\end{bmatrix}+c\cdot\begin{bmatrix}-1&0\\1&0\end{bmatrix}$.

See that $B=\left\{\begin{bmatrix}-1&1\\0&0\end{bmatrix},\begin{bmatrix}-1&0\\1&0\end{bmatrix}\right\}$ is a basis for W. $\Rightarrow \dim W=|B|=2$.

Since $\dim\mathbb{R}^3=3$, three vectors forms a basis if their determinant is not zero.

$\begin{vmatrix}1&0&2\\0&t&1\\t^2&0&2\end{vmatrix}=0\cdot A_{12}+t\cdot(-1)^{2+2}\cdot\begin{vmatrix}1&2\\t^2&2\end{vmatrix}+0\cdot A_{32}$

$=t\cdot(2-2t^2)=2\cdot t\cdot(1-t)\cdot(1+t)\neq 0$

$\Rightarrow t\neq 0,1,-1$

4. (C points) For what value(s) of t, the set $\{(1,0,2),(0,t,1),(t^2,0,2)\}$ forms a basis for $\mathbb{R}^3$?

   a) $t\in\mathbb{R}-\{0,1\}$ b) $t\in\mathbb{R}-\{0,-1\}$
   c) $t=-1$ d) $t\in\{-1,0,1\}$
   e) $t\in\mathbb{R}-\{-1,0,1\}$

5. Which of the following matrices is the transition matrix $[M]_S^T$ from basis S to basis T of $\mathbb{R}^2$ where

$$S = \{(-3,2),(4,-2)\},\ T = \{(-1,2),(2,-2)\}\ ?$$

a) $\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ b) $\begin{bmatrix} -1 & -2 \\ -2 & 3 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 2 \\ -2 & -3 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$(-3,2) = -1.(-1,2) + -2(2,-2)$

$(4,-2) = 2.(-1,2) + 3.(2,-2)$

$\Rightarrow [M]_S^T = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

$w_1 = 1.(1,0,1) + 2.(1,1,0) + -1.(0,0,1)$
$= (3,2,0)$

$w_2 = 1.(1,0,1) + 1.(1,1,0) + -1.(0,0,1)$
$= (2,1,0)$

$w_3 = 2.(1,0,1) + 1.(1,1,0) + 1.(0,0,1)$
$= (3,1,2)$

6. Let $S = \{(1,0,1),(1,1,0),(0,0,1)\}$ and $T = \{w_1,w_2,w_3\}$ be ordered bases for $\mathbb{R}^3$. Suppose that the transition matrix from T to S is $[M]_T^S = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$.

Which of the following is T ?

a) $\{(3,2,0),(2,1,0),(3,1,2)\}$
b) $\{(1,0,1),(2,1,3),(3,0,1)\}$
c) $\{(1,1,1),(1,1,3),(3,3,1)\}$
d) $\{(1,2,1),(1,1,2),(2,2,1)\}$
e) $\{(2,0,2),(1,3,0),(3,0,1)\}$

7. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. Which of the following can be the eigenvector associated with the largest eigenvalue of the matrix A ?

a) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 15 \\ 6 \\ 1 \end{bmatrix}$

d) $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ e) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$A - \lambda.I_3 = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{pmatrix}$

$|A-\lambda.I_3| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)(-1-\lambda).(2-\lambda) \equiv 0 \Rightarrow \lambda = -1, 1, 2$

$\Rightarrow$ The largest eigenvalue is $\lambda = 2$.

$\Rightarrow (A-2.I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow -x_2 + x_3 = 0 \Rightarrow x_2 = x_3$, and $-x_1 + 2x_2 + 3x_3 = 0$
$\Rightarrow x_1 = 5x_2 = 5x_3$.

$\Rightarrow \begin{bmatrix} 5x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \ldots$

For a 2×2 matrix A, its characteristic polynomial is $P_A(\lambda) = \lambda^2 - Tr(A)\lambda + |A|$

Since $\lambda = 1$ is an eigenvalue of A, $P_A(1) = 0$.

See that $Tr(A) = -2$, $|A| = -15 - a.b$.

$\Rightarrow P_A(\lambda) = \lambda^2 + 2\lambda - 15 - a.b$

$\Rightarrow P_A(1) = 1 + 2 - 15 - a.b = 0 \Rightarrow -15 - a.b = -3$

$\Rightarrow P_A(\lambda) = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1)$

$\Rightarrow \lambda = -3$ is another eigenvalue.

8. If $\lambda = 1$ is one of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & a \\ b & -5 \end{bmatrix}$, which of the following might be another eigenvalue for A ?

a) 2 b) 3 c) -1 d) -3 e) -2

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc = 5$ and $a + d = 6$, which of the following is the characteristic polynomial of A ?

a) $p(\lambda) = \lambda^2 - 6\lambda + 5$ b) $p(\lambda) = 3\lambda^2 - 4\lambda + 6$
c) $p(\lambda) = \lambda^2 - 5\lambda + 6$ d) $p(\lambda) = 2\lambda^2 - 3\lambda + 6$
e) $p(\lambda) = \lambda^2 + 5\lambda - 6$

For a 2x2 square matrix A,

$P_A(\lambda) = \lambda^2 - Tr(A).\lambda + |A|$.

See that $Tr(A) = a+d = 6$ and

$|A| = ad - bc = 5$.

$\Rightarrow P_A(\lambda) = \lambda^2 - 6\lambda + 5$

10. Let B be an invertible matrix with an appropriate size and $A = \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix}$. If the equation $A^{-1}B^2 = A^3B$ holds, what is B ? (Hint: Cayley-Hamilton theorem can be used.)

a) $16A + 24I_2$ b) $32A + 34I_2$ c) $44A + 117I_2$
d) $76A + 184I_2$ e) $96A + 196I_2$

$A^{-1}B^2 = A^3B \Rightarrow A^{-1}.\underbrace{B^2.B^{-1}}_{B} = A^3.\underbrace{B.B^{-1}}_{I_2}$

$\Rightarrow A^{-1}B = A^3 \Rightarrow \underbrace{A.A^{-1}}_{I_2}.B = \underbrace{A.A^3}_{A^4}$

$\Rightarrow \boxed{B = A^4}$

See that $P_A(\lambda) = \lambda^2 - 2\lambda - 9$ and

so $P_A(A) = A^2 - 2A - 9.I_2 = 0$

$\Rightarrow A^2 = 2A + 9.I_2$.

$\Rightarrow B = A^4 = (A^2)^2 = (2A + 9.I_2)^2$

$= 4A^2 + 36A + 81I_2$

$= 4(2A + 9I_2) + 36A + 81I_2$

$\boxed{B = 44A + 117I_2}$