• Sequences

★ A sequence is a list numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

a_i : term

$\downarrow$

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★ A sequence is a function whose domain is the set of positive integers.

If $a_n = f(n)$, then $f: \mathbb{Z}^+ \to \mathbb{R}$
$n \mapsto f(n) = a_n$

is a sequence of real numbers.

• Example : $1, 2, 4, 8, 16, \ldots, 2^{n-1}, \ldots$

$$\begin{array}{cccccc} 1, & 2, & 4, & 8, & 16, \ldots, & 2^{n-1}, \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_n \end{array}$$

$$a_n = 2^{n-1} \quad , n \geq 1$$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

$\lim_{n\to\infty} a_n = L$, $L \in \mathbb{R}$ convergent

① $\lim_{n\to\infty} a_n = \mp\infty$, or
② does not exist

$\Big\}$ divergent

Example: Let $\{a_n\} = \left\{ \frac{n}{n+1} \right\}$. Then,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$ so this sequence is convergent (or converges to 1).

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Example: Let $a_n = \frac{\cos n}{n}$, then we know that $-1 \leq \cos n \leq 1$ so

$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. This implies that,

$$0 = \lim_{n\to\infty}\left(-\frac{1}{n}\right) \leq \lim_{n\to\infty} \frac{\cos n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence, $\lim_{n\to\infty} \frac{\cos n}{n} = 0$ by the Sandwich Theorem for sequences.

Using L'Hôpital's Rule

The next theorem formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$. It enables us to use l'Hôpital's Rule to find the limits of some sequences.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

Example: Let $a_n = \frac{\ln n}{n}$, then

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \left(\frac{\infty}{\infty}\right)$. If we apply L'Hopital's rule, then $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$

THEOREM 3—The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$. $(n\to\infty)$

Example: Let $a_n = \frac{1}{n}$ and $f(x) = 2^x$.

$a_n \to 0 \quad (L=0)$.

$$\left.\begin{array}{l} ① \ f(L) = f(0) = 2^0 = 1 \\ ② \ \lim_{x\to 0} f(x) \text{ exists} \\ ③ \ \lim_{x\to 0} f(x) = f(0) = 2^0 = 1 \end{array}\right\} f \text{ is continuous at } x=0 \ (L=0)$$

so $f(a_n) = f\left(\frac{1}{n}\right) \to f(L) = f(0) = 2^0 = 1$ by the continuous function theorem for sequences.

Commonly Occurring Limits

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Example: $\sqrt[n]{5n} = 5^{1/n} \cdot n^{1/n} \to 1 \cdot 1 = 1$

$5^{1/n} \to 1$ Formula 3

$n^{1/n} \to 1$ Formula 2

★ A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

Example: $\underset{a_1}{2}, \underset{a_2}{3}, \underset{a_3}{5}, \underset{a_4}{9}, \underset{a_5}{17}, \ldots$

- Recursion formula: $a_n = 2 \cdot a_{n-1} - 1$, $n \geq 2$ for $a_1 = 2$.

$a_2 = 2 \cdot a_1 - 1 = 2 \cdot 2 - 1 = 3$

$a_3 = 2 \cdot a_2 - 1 = 2 \cdot 3 - 1 = 5$

$a_4 = 2 \cdot a_3 - 1 = 2 \cdot 5 - 1 = 9$

$a_5 = 2 \cdot a_4 - 1 = 2 \cdot 9 - 1 = 17$

$\vdots$

- $a_n \leq M$ for all n ($\{a_n\}$ bounded from above)
  ↳ upper bound for $\{a_n\}$

- $m \leq a_n$ for all n ($\{a_n\}$ bounded from below)
  ↳ lower bound for $\{a_n\}$

- $m \leq a_n \leq M$ for all n ($\{a_n\}$ bounded sequence)

Example: Let $\{a_n\} = \left\{\frac{n}{n+1}\right\}$. Then,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots$$

- $\frac{1}{2} \leq a_n$ for all n so $\frac{1}{2}$ is a lower bound
  $a_n < 1$ for all n so 1 is an upper bound.

Hence, $\{a_n\}$ is a bounded sequence since
$\frac{1}{2} \leq a_n < 1$ for all n.

★ $a_1 \leq a_2 \leq a_3 \leq \dots$ nondecreasing sequence
$a_1 \geq a_2 \geq a_3 \geq \dots$ nonincreasing sequence

★ $a, a, a, a, \dots$ constant sequence (bot nondecreasing and nonincreasing)

★

• Monotonic Sequences

Nondecreasing | Nonincreasing | Decreasing (nonincreasing) | Increasing (nondecreasing)

Monotonic sequences

Examples: (1) $5, 5, 5, \dots, 5, \dots$ the constant sequence (both nondecreasing and nonincreasing)

(2) $1, 3, 5, 7, 9, \dots$ increasing (nondecreasing)

(3) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ decreasing (nonincreasing)

(4) $-1, 1, -1, 1, -1, \dots$ is not monotonic.

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges. $\to \lim_{n\to\infty} a_n = L,\ L \in \mathbb{R}$.

Example: Let $\{a_n\}$ be a decreasing and bounded sequence with positive terms, $a_1 = 14$ and $a_n = \frac{a_{n-1}}{2} + \frac{7}{a_{n-1}}$, $n \geq 2$. What is the value of $\lim_{n\to\infty} a_n = ?$

• Decreasing: monotonic and bounded $\Big\}$ then, $\{a_n\}$ is convergent by the monotonic sequence theorem.

Let $\lim_{n\to\infty} a_n = L$, $L \in \mathbb{R}$. Then, $\lim_{n\to\infty} a_{n-1} = L$.

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} a_{n-1}}{\lim_{n\to\infty} 2} + \frac{\lim_{n\to\infty} 7}{\lim_{n\to\infty} a_{n-1}} \Rightarrow$$

$$L = \frac{L}{2} + \frac{7}{L} \Rightarrow \frac{L}{1} = \frac{L^2+14}{2L} \Rightarrow 2L^2 = L^2 + 14$$

$$\Rightarrow L^2 = 14$$

$$\Rightarrow L = \mp\sqrt{14}$$

In here, $L = \lim_{n\to\infty} a_n = \sqrt{14}$ since terms of $\{a_n\}$ are positive.