Ex.: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t the curve $x=t^2+1$, $y=t^2+t$ concave upward?

$\frac{dy}{dt}=2t+1$ $\quad \frac{dy}{dx}=\frac{2t+1}{2t}=1+\frac{1}{2t}=1+(2t)^{-1}$

$\frac{dx}{dt}=2t$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-1\cdot(2t)^{-2}\cdot 2=\frac{-2}{4t^2}=-\frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\frac{1}{2t^2}}{2t}=\frac{-1}{4t^3}$$

$\downarrow$

$t=0$

The curve is concave upward when $\frac{d^2y}{dx^2}>0$, $t<0$

$(-\infty,0)$: concave upward.

Ex.: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t the curve $x=e^t$, $y=t\cdot e^{-t}$ concave upward?

$\frac{dy}{dt}=e^{-t}-t\cdot e^{-t}=e^{-t}\cdot(1-t)$ $\quad \frac{dx}{dt}=e^t$

$$\frac{dy}{dx}=\frac{e^{-t}\cdot(1-t)}{e^t}=e^{-2t}(1-t)$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\underbrace{-2\cdot e^{-2t}(1-t)-e^{-2t}}_{e^{-2t}(-2+2t)}=e^{-2t}(-3+2t)$$

$$\frac{d^2y}{dx^2}=\frac{e^{-2t}\cdot(2t-3)}{e^t}=\underbrace{e^{-3t}}_{\neq 0}\cdot\underbrace{(2t-3)}_{t=\frac{3}{2}\to \text{p.o.i}}=0$$

C.U: $\left(\frac{3}{2},\infty\right)$

Ex.: Find the slope of the tangent line of the curve $x = \ln t$, $y = 1 + t^2$ at $t = 1$.

$$\frac{dy}{dt} = 2t \qquad \frac{dy}{dx} = \frac{2t}{\frac{1}{t}} = 2t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = 2.$$

$$\frac{dx}{dt} = \frac{1}{t}$$

Ex.: Find the coordinates of the highest point of the curve $x = 96t$, $y = 96t - 16t^2$.

$\Rightarrow$ Find the abs. max.

$$\frac{dy}{dt} = 96 - 32t \qquad \frac{dy}{dx} = \frac{96 - 32t}{96} = 0 \Rightarrow 96 - 32t = 0$$

$$\frac{dx}{dt} = 96 \qquad t = 3$$

$\Downarrow$

critical point.

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-32}{96} = -\frac{1}{3} \qquad \frac{d^2y}{dx^2} = \frac{-1}{3 \cdot 96} < 0$$

$\underline{\underline{t = 3}} \Rightarrow$ abs. max. $x = 288$, $y = 144$

Ex.: At what point(s) on the curve $x = 3t^2 + 1$, $y = t^3 - 1$ does the tangent line have slope $\frac{1}{2}$?

$$\frac{dy}{dt} = 3t^2 \qquad \frac{dy}{dx} = \frac{3t^2}{6t} = \frac{1}{2}t \Rightarrow \frac{1}{2}t = \frac{1}{2} \Rightarrow t = 1$$

$$\frac{dx}{dt} = 6t \qquad t = 1 \Rightarrow x = 4, \; y = 0$$

Ex.: Find the length of the curve $x = e^t . \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi$

$$\frac{dy}{dt} = e^t . \sin t + e^t . \cos t = e^t (\sin t + \cos t)$$

$$\frac{dx}{dt} = e^t . \cos t - e^t . \sin t = e^t (\cos t - \sin t)$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = e^{2t} (\underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{2 \sin t \cos t})$$
$$+ e^{2t} (\underbrace{\cos^2 t + \sin^2 t}_{1} - \cancel{2 \sin t \cos t}) = 2 . e^{2t}$$

$$L = \int_0^{\pi} \sqrt{2 . e^{2t}}\, dt = \int_0^{\pi} \sqrt{2} . e^t \, dt = \sqrt{2}\, e^t \Big|_0^{\pi} = \sqrt{2}\,(e^{\pi} - 1)$$

Ex.: Find the arc length of $x = 2\cos t + \cos 2t + 1$, $y = 2 \sin t + \sin 2t$. $0 \leq t \leq 2\pi$

$$\frac{dy}{dt} = 2\cos t + 2 \cos 2t \qquad \frac{dx}{dt} = -2 \sin t - 2 \sin 2t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = 4\cos^2 t + 4 \cos^2 2t + 8 \cos t . \cos 2t$$
$$+ 4 \sin^2 t + 4 \sin^2 2t + 8 . \sin t . \sin 2t$$

$2\cos^2 \frac{t}{2} = 1 + \cos t$

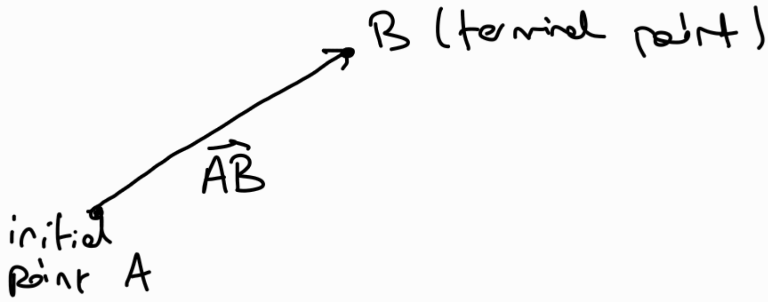
$$8 + 8(\underbrace{\cos 2t . \cos t + \sin 2t \sin t}_{\cos(2t - t) = \cos t})$$

$$L = \int_0^{2\pi} \sqrt{8(1 + \cos t)}\, dt = \int_0^{2\pi} \sqrt{16 \cos^2 \frac{t}{2}}\, dt = \int_0^{2\pi} 4 \left| \cos \frac{t}{2} \right| dt$$

$$= \int_0^{\pi} 4\cos\frac{t}{2}\,dt - \int_{\pi}^{2\pi} 4\cos\frac{t}{2}\,dt = \underbrace{8\sin\frac{t}{2}\Big|_0^{\pi}}_{8.1} - \underbrace{8\sin\frac{t}{2}\Big|_{\pi}^{2\pi}}_{+8.1} = 16$$

VECTORS

<u>Definition</u>: The vector represented by the directed line segment $\vec{AB}$ has initial point A and terminal point B, and its length is denoted by $|\vec{AB}|$. Two vectors are equal if they have the same length and direction.



<u>Definition</u>: If $\vec{v}$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the component form of $\vec{v}$ is

$$\vec{v} = \langle v_1, v_2 \rangle.$$

If $\vec{v}$ is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of $\vec{v}$ is

$$\vec{v} = \langle v_1, v_2, v_3 \rangle.$$

The magnitude or length of the vector $v = \langle v_1, v_2, v_3 \rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the nonnegative number

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Vecter Algebra Operetions

Definitions: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

- Addition: $\vec{u} + \vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$
- Scalar multiplication: $k.\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

Unit Veetors

A vector $\vec{v}$ of length 1 is called a unit vector. The standert unit vectors are

$$\vec{i} = (1,0,0), \quad \vec{j} = (0,1,0), \quad \vec{k} = (0,0,1) \qquad \langle \ \rangle$$

Any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a lineor combination of the standard unit vectors as follows:

$$\begin{aligned}
\vec{v} = \langle v_1, v_2, v_3 \rangle &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\
&= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\
&= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}
\end{aligned}$$

If $\vec{v} \neq \vec{0}$, then its length $|\vec{v}|$ is not zero and

$$\left| \frac{\vec{v}}{|\vec{v}|} \right| = \frac{1}{|\vec{v}|} \cdot |\vec{v}| = 1$$

That is, if the vector $\vec{v}$ is not the zero veetor, then $\dfrac{\vec{v}}{|\vec{v}|}$ is a unit vecter in the direction of $\vec{v}$.

The Dot Product

Definition: The dot product $\vec{u}\cdot\vec{v}$ of vectors $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$ is the scaler

$$\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3 .$$

The dot products of two vectors $\vec{u}$ and $\vec{v}$ is given by $\vec{u}\cdot\vec{v}=|\vec{u}|\cdot|\vec{v}|\cdot\cos\theta$.

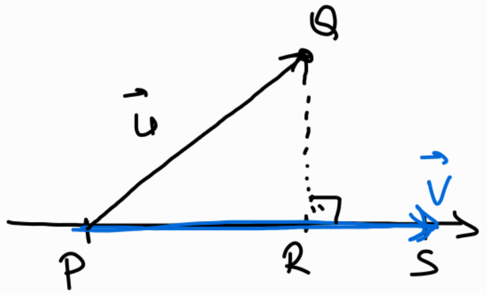
Definition: Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v}=0$.

Properties of the Dot Product

If $\vec{u},\vec{v}$, and $\vec{w}$ are any vectors and c is a scalar, then

1) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
2) $(c\cdot\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\cdot\vec{v})$
3) $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
4) $\vec{u}\cdot\vec{u}=|u|^2$
5) $\vec{0}\cdot\vec{u}=0$

Definition: The vector projection of $\vec{u}=\overrightarrow{PQ}$ onto a nonzero vector $\vec{v}=\overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.



The notation for this vector is $\text{proj}_{\vec{v}}\vec{u}$ "the projection of $\vec{u}$ onto $\vec{v}$"

If $\vec{u}$ represents a force, the $\text{proj}_{\vec{v}}\vec{u}$ represents the effective force in the direction of $\vec{v}$

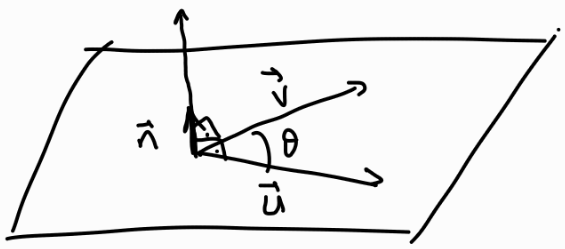
$$\text{proj}_{\vec{v}}\vec{u} = (|u| \cdot \cos\theta) \cdot \frac{v}{|\vec{v}|} \qquad |\vec{u}| \cdot \cos\theta = \frac{|\vec{u}| \cdot |\vec{v}| \cdot \cos\theta}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) \cdot \frac{\vec{v}}{|\vec{v}|} = \underbrace{\left(\frac{\vec{u} \cdot \vec{v}}{|v|^2}\right)}_{\text{scaler}} \vec{v}$$

The Cross Product

Definition: The cross product $\vec{u} \times \vec{v}$ is the vector

$\vec{u} \times \vec{v} = |\vec{u}| . |\vec{v}| . \sin\theta . \vec{n} \qquad \vec{n} \perp \vec{u} \ , \ \vec{n} \perp \vec{v} \qquad (\vec{n}$ is unit$)$



Parallel Vectors

Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel $\iff \vec{u} \times \vec{v} = \vec{0}$

Properties of the Cross Product

If $\vec{u}, \vec{v}$, and $\vec{w}$ are any vectors and r, s are scalers, then

1) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$
2) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
3) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$
4) $(\vec{v} + \vec{w}) \times \vec{u} = (\vec{v} \times \vec{u}) + (\vec{w} \times \vec{u})$
5) $\vec{0} \times \vec{u} = \vec{0}$
6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} . \vec{w}) . \vec{v} - (\vec{u} . \vec{v}) . \vec{w}$

$|\vec{u} \times \vec{v}|$ is the area of a parallelogrem.

$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot |\sin\theta| \cdot |\vec{n}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta \quad \left(\text{Triangle: } \tfrac{1}{2}\right)$$

Calculating the Cross Product as a Determinant

If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Triple Scaler or Box Product

The product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is called the triple scalar product of $\vec{u}, \vec{v}$, and $\vec{w}$ (in that order)

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \rightarrow \text{scaler}$$

The number $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$ is the volume of a parallelepiped.

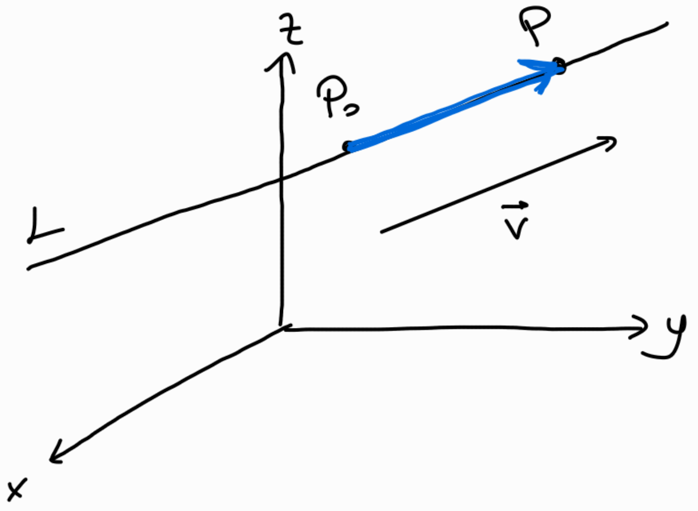
abs. value.

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}$$

LINES AND PLANES IN SPACE

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a vector giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$. Thus, $\overrightarrow{P_0P} = t\vec{v}$ for some scalar parameter t.



A point P lies on L through P_0 parallel to $\vec{v}$ iff $\overrightarrow{P_0P}$ is a scalar multiple of $\vec{v}$.

Vector Equation for a Line

A vector eq. for the line L through $P_0(x_0, y_0, z_0)$ parallel to a nonzero vector $\vec{v}$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations

Parametric eqs. for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Example: Find parametric eqs. for the line through $(-2, 0, 4)$ parallel to $\vec{v} = \underline{2}\vec{i} + \underline{4}\vec{j} - \underline{2}\vec{k}$.

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$$

Example: Find parametric eqs. for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

$\vec{PQ} = \langle 4, -3, 7 \rangle \rightarrow$ direction vector

By P : $x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$

By Q : $x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t$

Distance From a Point S to a Line Through P Parallel to $\vec{v}$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Ex.: Find the distance from the point $S(1,1,5)$ to the line $L: x=1+t,\ y=3-t,\ z=2t$

$$\vec{v} = \langle 1, -1, 2 \rangle$$

Point P from L $(t=0)$ $P(1,3,0)$

$\vec{PS} = \langle 0,-2,5 \rangle$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \langle 1,5,2 \rangle$$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \sqrt{\frac{30}{6}} = \sqrt{5}$$

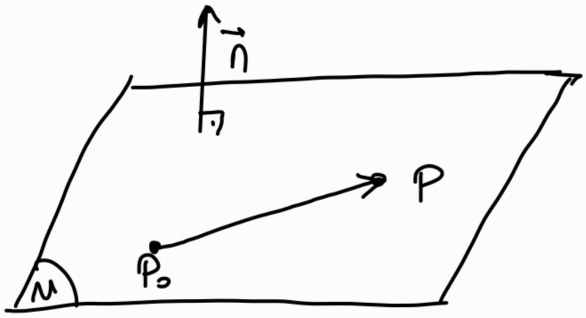
An Equation for a Plane in Space

Suppose that plane M passes through a point $P_0(x_0,y_0,z_0)$ and is normal to the nonzero vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$. A vector from P_0 to any point P on the plane is orthogonal to $\vec{n}$. Then M is the set of all points $P(x,y,z)$ for which $\vec{P_0P}$ is orthogonal to $\vec{n}$.

Thus, $\vec{n} \cdot \vec{P_0P} = 0$. This eq. is equivalent to

$$(A\vec{i} + B\vec{j} + C\vec{k}) \cdot [(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}] = 0$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \rightarrow \text{Plane.}$$



$\vec{n} \cdot \overrightarrow{P_0P} = 0$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to a nonzero vector $\vec{n} = \langle A, B, C \rangle$ has

Vector eq.: $\vec{n} \cdot \overrightarrow{P_0P} = 0$
Component eq.: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component eq. simplified: $Ax + By + Cz = D$ where
$D = Ax_0 + By_0 + Cz_0$

Ex.: Find an eq. for the plane through $P_0(-3, 0, 7)$ perpendicular to $\vec{n} = \langle 5, 2, -1 \rangle$

$$5(x+3) + 2(y) - 1 \cdot (z-7) = 0 \Rightarrow 5x + 2y - z = -22$$

Ex.: Find an eq. for the plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$.

$\left.\begin{array}{l} \overrightarrow{AB} = (2, 0, -1) \\ \overrightarrow{AC} = (0, 3, -1) \end{array}\right\}$ on the plane

$\Downarrow$

not parallel

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \langle 3, 2, 6 \rangle$$

(Using A): $3(x-0) + 2(y-0) + 6(z-1) = 0$

$$3x + 2y + 6z = 6.$$