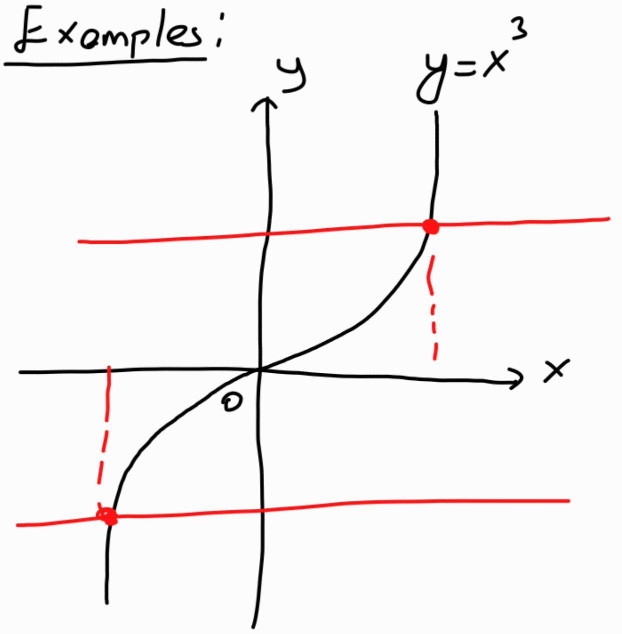
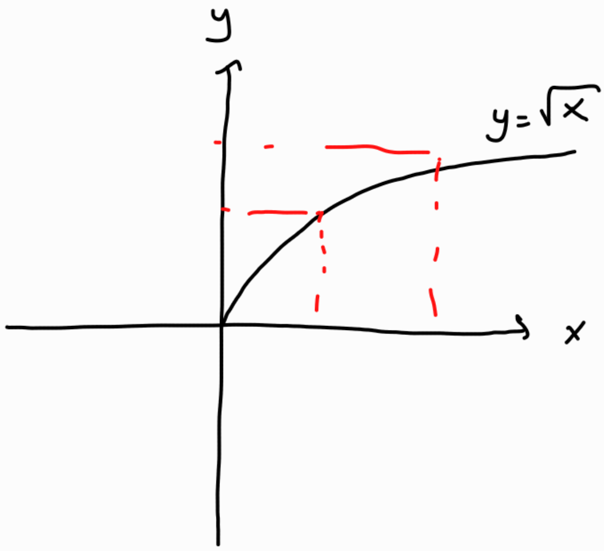
Inverse Functions

Definition: A function $f(x)$ is one-to-one (1-1) on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

These functions take on any value in their range exactly once.

Examples:



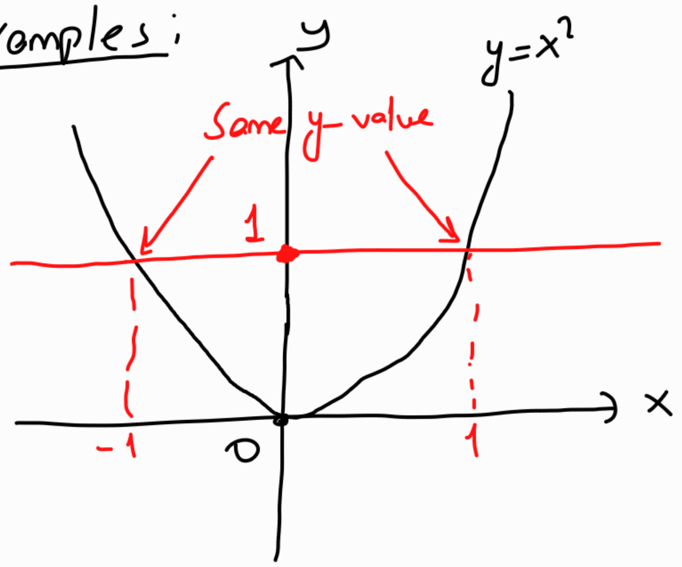


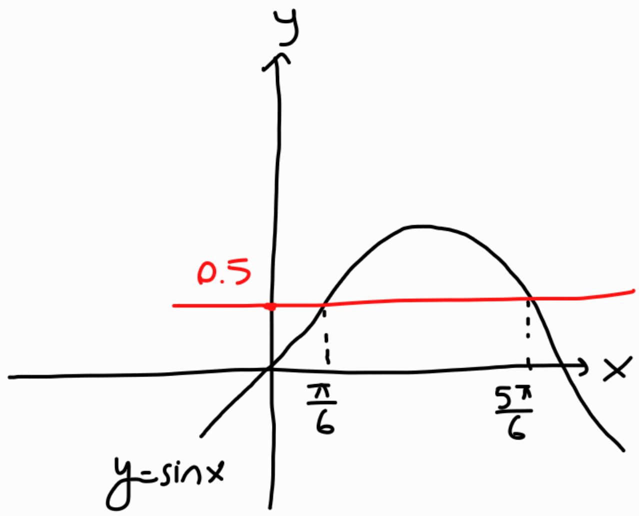
Both functions are one-to-one on $\mathbb{R}$, respectively on $\mathbb{R}_0^+$.

The Horizontal Line Test for One-to-One Functions

A function $y=f(x)$ is one-to-one if and only if (iff) its graph intersects each horizontal line at most once.

Examples:





$y=x^2$ is one-to-one on, e.g., $\mathbb{R}_0^+$ but not $\mathbb{R}$

$y=\sin x$ is one-to-one on, for example, $[0,\frac{\pi}{2}]$ but not $\mathbb{R}$

Definition: Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by

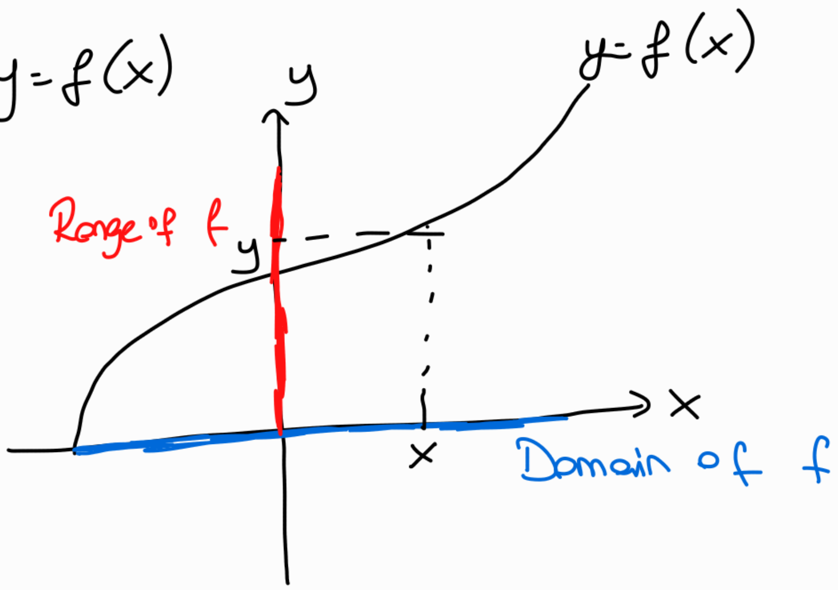
$$f^{-1}(b)=a \text{ if } f(a)=b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

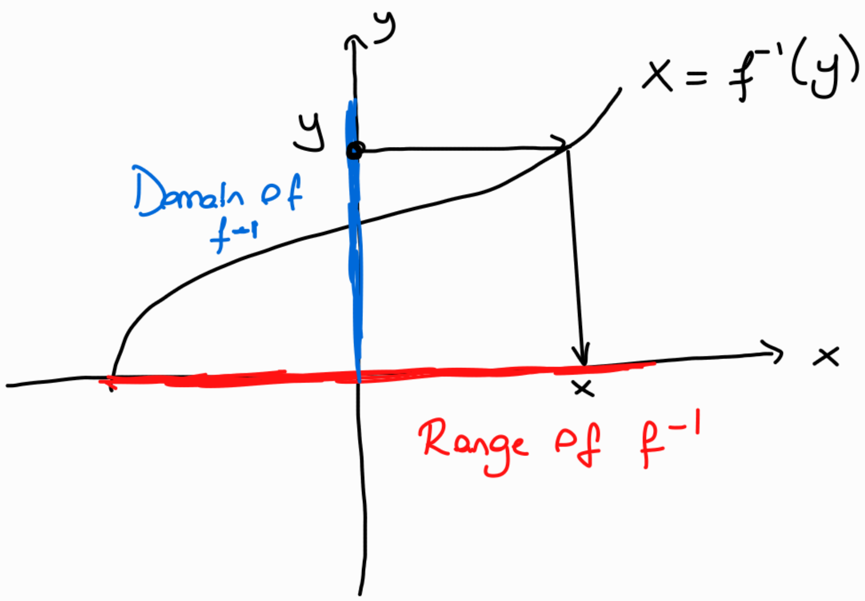
Remarks

* f^{-1} reads f inverse
* $f^{-1}(x) \neq (f(x))^{-1} = \dfrac{1}{f(x)}$ (f^{-1} is not an exponent)
* $(f^{-1}\circ f)(x)=x$ for all $x \in D(f)$
* $(f\circ f^{-1})(x)=x$ for all $x \in R(f)$

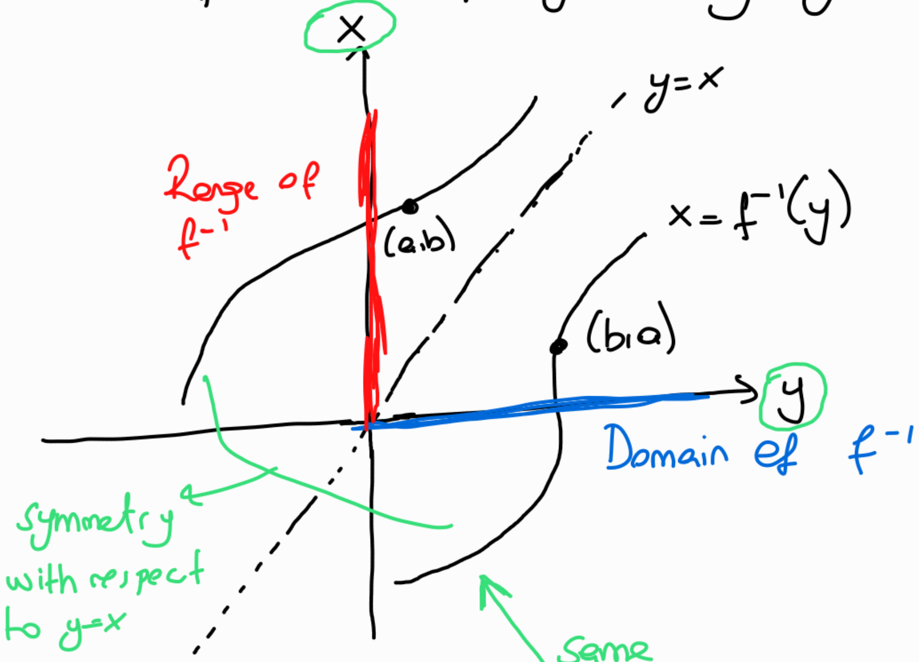
$x \longmapsto y=f(x)$



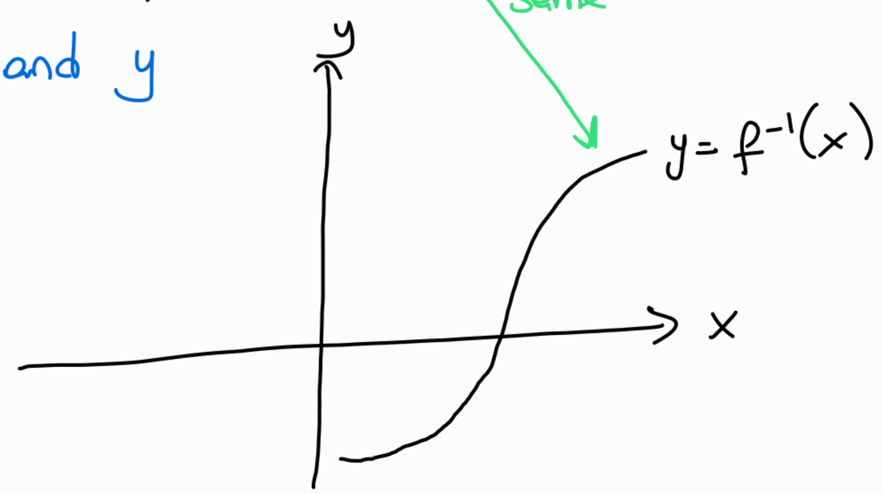
for inverse $y \mapsto x = f^{-1}(y)$



Note that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$, which suggests to reflect $x = f^{-1}(y)$ along $y = x$:



Swap x and y



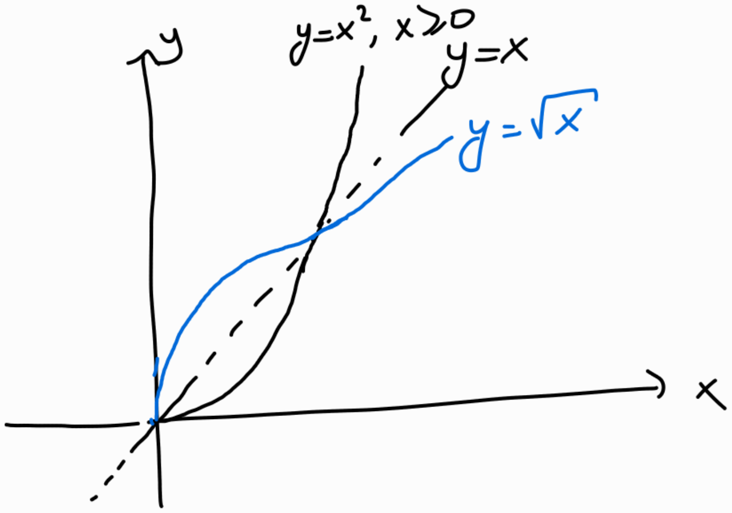
Method for finding inverses algebraically:

1. solve $y = f(x)$ for x: $x = f^{-1}(y)$
2. interchange x and y: $y = f^{-1}(x)$

Example: Find the inverse of $y = x^2$, $x \geq 0$.

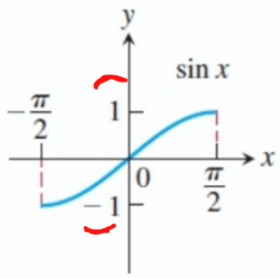
1. $y = x^2 \Longrightarrow \sqrt{y} = \sqrt{x^2} \Longrightarrow \sqrt{y} = |x| = x$, as $x \geq 0$
2. interchange x and y: $y = \sqrt{x}$.

(If $x \leq 0$, $\sqrt{y} = |x| = -x$)

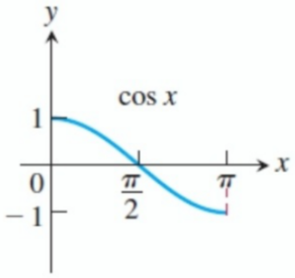


Inverse Trigonometric Functions

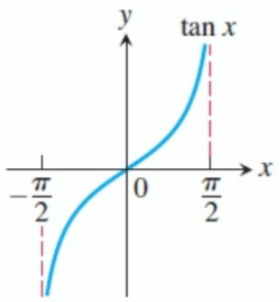
Domain restrictions that make the trigonometric functions one-to-one



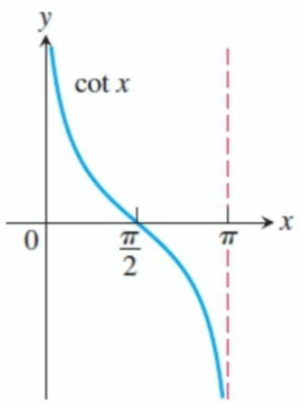
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



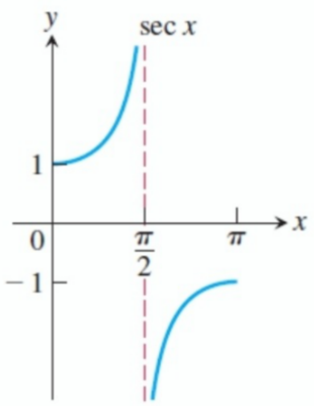
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



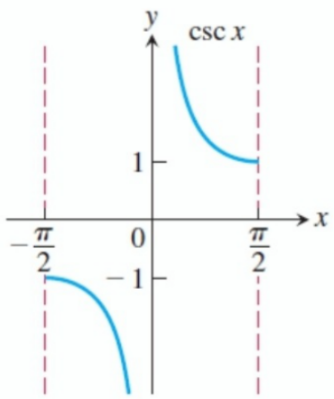
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \sin^{-1} x$ or $y = \arcsin x$

$y = \cos^{-1} x$ or $y = \arccos x$

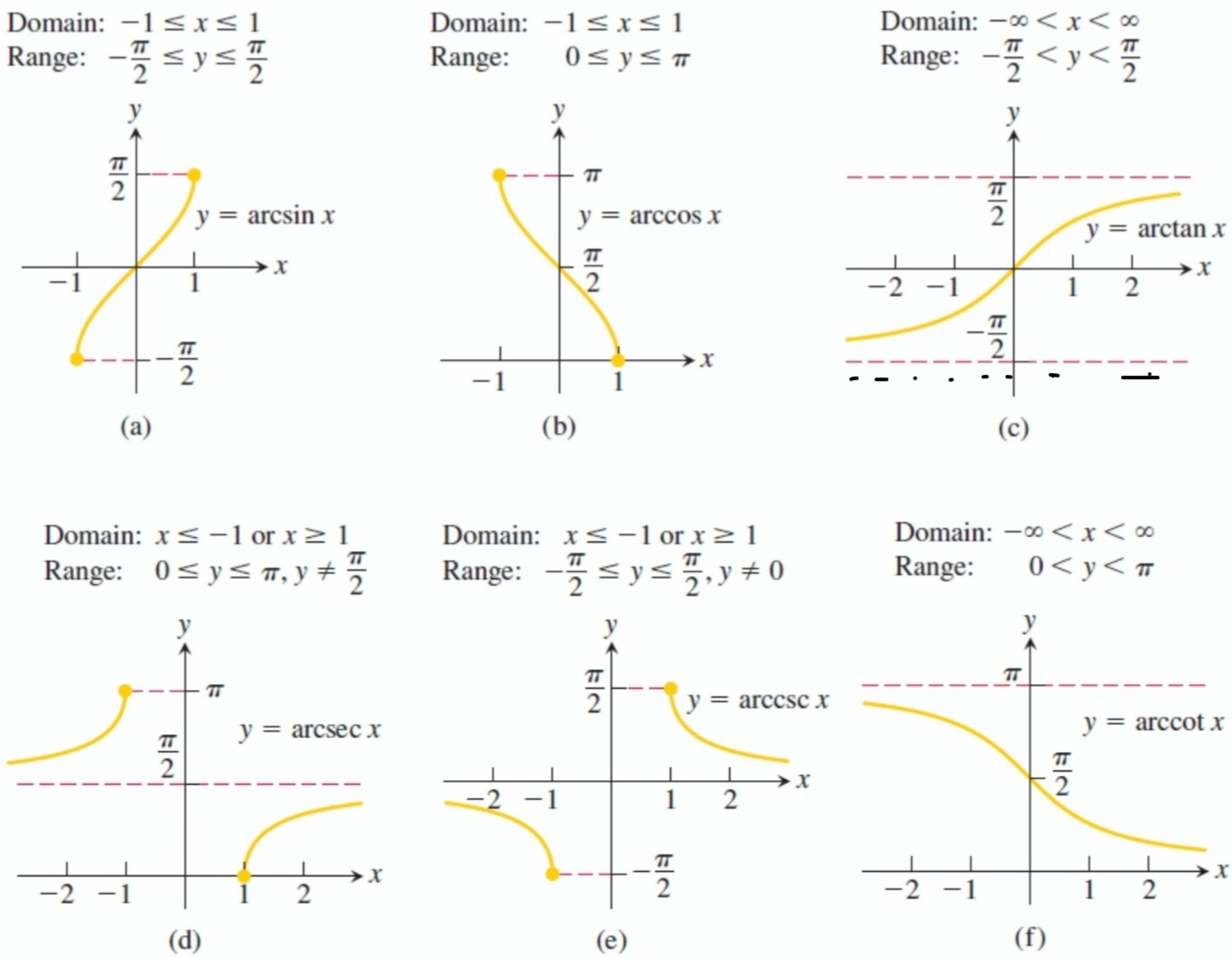
$y = \tan^{-1} x$ or $y = \arctan x$

$y = \cot^{-1} x$ or $y = \operatorname{arccot} x$

$y = \sec^{-1} x$ or $y = \operatorname{arcsec} x$

$y = \csc^{-1} x$ or $y = \operatorname{arccsc} x$

Remark: The -1 in the expressions for the inverse means "inverse", not reciprocal. For example, the reciprocal of $\sin x$ is $(\sin x)^{-1} = \dfrac{1}{\sin x} = \csc x$



Graphs of the six basic inverse trigonometric functions.

Inverse Function–Inverse Cofunction Identities

* $\arccos x = \frac{\pi}{2} - \arcsin x$ $\left(\sin\frac{\pi}{6} = \cos\frac{\pi}{3}\right)$

* $\text{arccot}\, x = \frac{\pi}{2} - \arctan x$

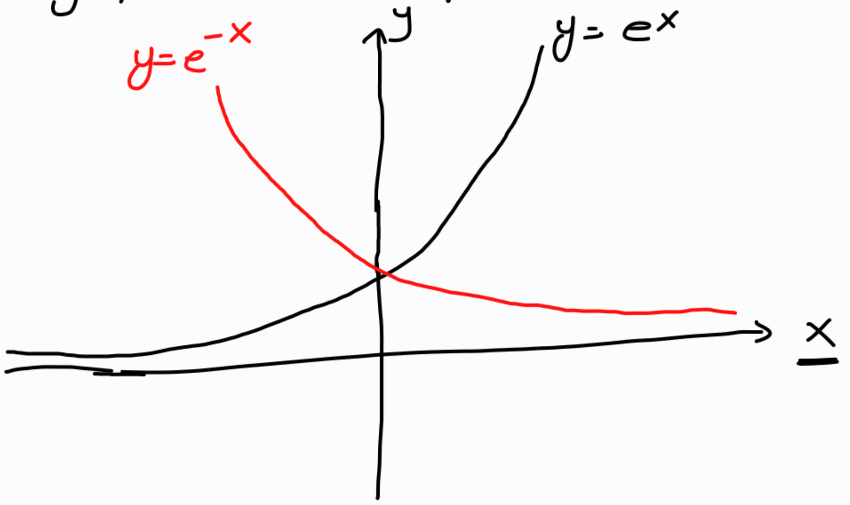
* $\text{arccsc}\, x = \frac{\pi}{2} - \text{arcsec}\, x$

Exponential and Logarithmic Functions

Definition: The exponential function f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number. The domain of the exponential function $f(x) = a^{g(x)}$ is the same as the domain of $g(x)$.

The natural base e: For many applications, the convenient choice for a base is the irrational number $e \approx 2.7\overline{182}$. This number is called the natural base. The function $f(x) = e^x$ is called the natural exponential function.

The graph of the e^x has the same basic characteristics as the graph of the function $f(x) = a^x$.



(never touches to axis -x)

Properties

$a^x a^y = a^{x+y}$

$\frac{a^x}{a^y} = a^{x-y}$

$a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$

$a^0 = 1$

$\sqrt[n]{a^m} = a^{m/n}$

$(ab)^x = a^x b^x$

$(a^x)^y = a^{xy}$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$|a^2| = |a|^2 = a^2$

<u>Definition</u>: For $x>0$, $a>0$, and $a\neq 1$,

$y=\log_a x$ if and only if $x=a^y$.

The function given by $f(x)=\log_a x$ is called the logarithmic function with base a. The domain of the logarithmic function $f(x)=\log_a g(x)$ is $\{x\in\mathbb{R} : g(x)>0\} \cap$ the domain of $g(x)$

The equations $y=\log_a x$ and $x=a^y$ are equivalent. The first equation is in logaritmic form and the second is in exponentiel form. When evaluating logerithms, remember that a logarithm is an exponent.

Properties (base is a)

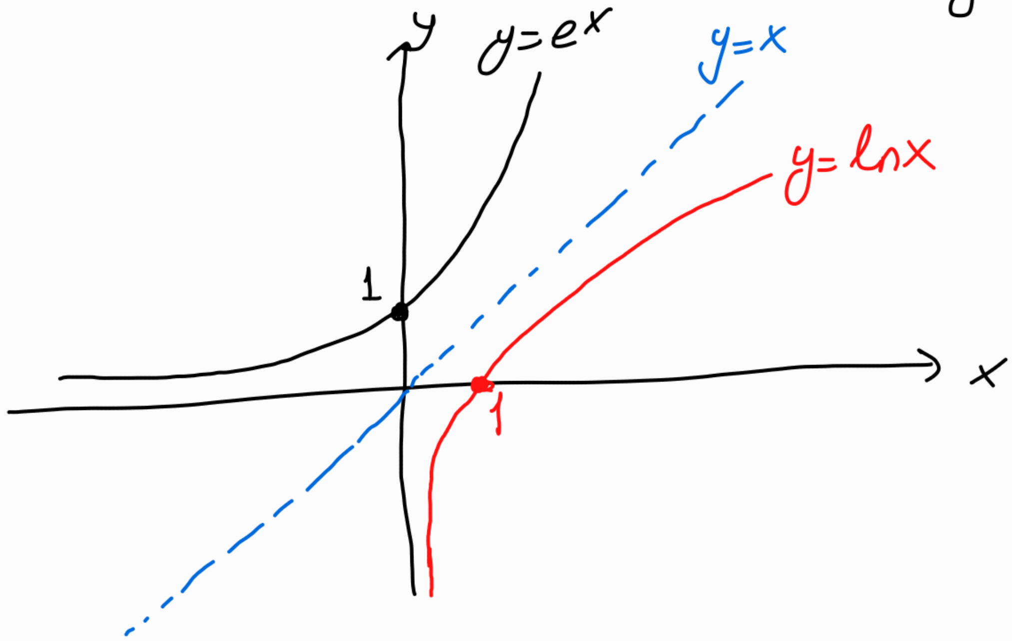
1) $\log_a 1=0$ because $a^0=1$
2) $\log_a a=1$ because $a^1=a$
3) $\log_a b^x = x\cdot\log_a b$
4) $\log_a a^x=x$ and $a^{\log_a x}=x$
5) $\log_a (x\cdot y)=\log_a x+\log_a y$
6) $\log_a \left(\frac{x}{y}\right)=\log_a x-\log_a y$
7) If $\log_a x=\log_a y$, then $x=y$

__Definition__: For $x>0$,

$$y=\ln x \text{ if and only if } x=e^y.$$

The function given by $f(x)=\log_e x=\ln x$ is called the natural logarithmic function. The domain of the natural logarithmic function $f(x)=\ln g(x)$ is
$\{x\in\mathbb{R} : g(x)>0\}\cap$ the domain of $g(x)$

Because of the functions $f(x)=e^x$ and $g(x)=\ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y=x$.



__Remark__: The properties of logarithms previously listed are also valid for natural logarithms (take $a=e$).

Example: Find the domain of $f(x) = \log_4(9 - 16x^2)$.

$9 - 16x^2 > 0$ and $\underbrace{\text{Domain of } 9-16x^2}_{\mathbb{R}}$

$9 > 16x^2$

$\sqrt{\frac{9}{16}} > \sqrt{x^2}$

$|x| < \frac{3}{4}$

$-\frac{3}{4} < x < \frac{3}{4}$

$D(f) = \left(-\frac{3}{4}, \frac{3}{4}\right) \cap \mathbb{R}$

$= \left(-\frac{3}{4}, \frac{3}{4}\right)$

Example: Find the domain of $f(x) = \log_3(1 - \sqrt{x})$.

$1 - \sqrt{x} > 0$

$1^2 > (\sqrt{x})^2$

$1 > x$

$[0, 1)$

Domain of $(1 - \sqrt{x})$

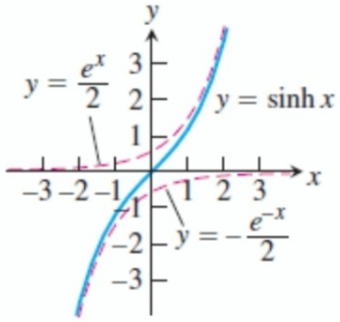
$\sqrt{x}$ $\quad x \geq 0$

$D(f) = [0,1) \cap [0, \infty) = [0, 1)$

Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

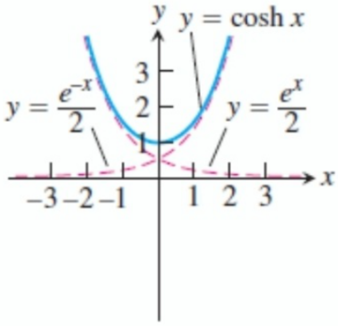
$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$



(a)

Hyperbolic sine:

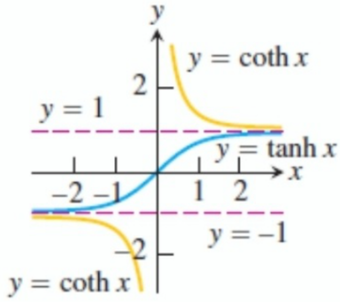
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



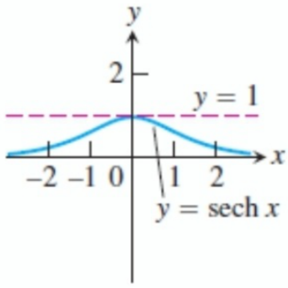
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

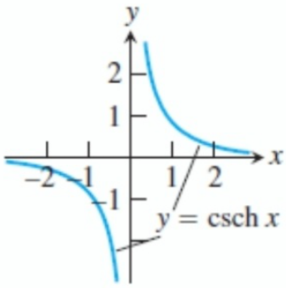
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

The six basic hyperbolic functions

Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cdot \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Example: Prove the identity $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4} = \frac{4}{4} = 1$$

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

LIMITS AND CONTINUITY

Definition: Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. If $f(x)$ get arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches to x_0, and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches to x_0."

Limit Laws

If L, M, c, and k are real numbers and

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1) Sum Rule: $\lim_{x \to c} (f(x) + g(x)) = L + M$

2) Difference Rule: $\lim_{x \to c} (f(x) - g(x)) = L - M$

3) Constant Multiple Rule: $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

4) Product Rule: $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

5) Quotient Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

6) Power Rule: $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer

7) Root Rule: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive intege

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

Ex.: $$\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{\lim\limits_{x\to c}(x^4+x^2-1)}{\lim\limits_{x\to c}(x^2+5)}$$

$$=\frac{\lim\limits_{x\to c}x^4+\lim\limits_{x\to c}x^2-\lim\limits_{x\to c}1}{\lim\limits_{x\to c}x^2+\lim\limits_{x\to c}5}=\frac{c^4+c^2-1}{c^2+5}$$

Ex.: $$\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}(4x^2-3)}=\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$$

$$=\sqrt{4\cdot(-2)^2-3}=\sqrt{16-3}=\sqrt{13}$$

Ex.: $$\lim_{x\to 1}\frac{\overset{x,\ x}{x^2}+x\overset{-1,\ +2}{-2}}{x^2-x}$$ Domain does not include $\underline{\underline{x=1}}$

$$\lim_{x\to 1}\frac{\cancel{(x-1)}\cdot(x+2)}{x\cdot\cancel{(x-1)}}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3$$

<u>Function didn't need to be defined on $x=1$</u>

<u>Theorem - The Sandwich Theorem</u>

Suppose that $g(x)\leq f(x)\leq h(x)$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that

$$\lim_{x\to c}g(x)=\lim_{x\to c}h(x)=L.$$

Then $\lim\limits_{x\to c}f(x)=L$.

Remark: The Sandwich Theorem is also called Squeeze Theorem or the Pinching Theorem.

__Example:__ Given a function u that satisfies

$$1-\frac{x^2}{4} \leq u(x) \leq 1+\frac{x^2}{2} \quad \text{for all } x \neq 0.$$

find $\lim_{x \to 0} u(x)$ no matter how complicated u is.

__Soln.:__ $\lim_{x \to 0} 1-\frac{x^2}{4} = 1$ and $\lim_{x \to 0} 1+\frac{x^2}{2} = 1$

$\Rightarrow$ From the S.T. $\lim_{x \to 0} u(x) = 1$

__Homeworks:__ $\lim_{\theta \to 0} \sin\theta = 0$ and $\lim_{\theta \to 0} \cos\theta = 1$

Prove this with the Sandwich Theorem.