Question: Which of the following points does line $\frac{x-6}{3}=\frac{y+2}{-6}=\frac{z-2}{4}$ intersect the ellipsoid $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$?

A) $(6,-2,2)$ and $(3,-4,-2)$ B) $(6,-2,2)$ and $(3,4,-2)$
C) $(6,-2,2)$ and $(-3,4,-2)$ D) $(6,-2,2)$ and $(-3,-4,-2)$
E) $(6,-2,2)$ and $(-1,-2,-3)$

If $\frac{x-6}{3}=\frac{y+2}{-6}=\frac{z-2}{4}=t$, then

$$\left.\begin{aligned} x&=3t+6\\ y&=-6t-2\\ z&=4t+2 \end{aligned}\right\} \Rightarrow \frac{x^2}{81}=\frac{(3t+6)^2}{81}=\frac{9(t^2+4t+4)}{81}=\frac{t^2+4t+4}{9}$$

$$\frac{y^2}{36}=\frac{4(9t^2+6t+1)}{36}=\frac{9t^2+6t+1}{9}$$

$$\frac{z^2}{9}=\frac{16t^2+16t+4}{9} \Rightarrow$$

$$\frac{26t^2+26t+9}{9}=1 \Rightarrow 26t^2+26t+9=9 \Rightarrow 26t(t+1)=0 \Rightarrow t=0 \text{ or } t=-1$$

- $t=0 \Rightarrow$
$$x=3.0+6=6$$
$$y=-6.0-2=-2$$
$$z=4.0+2=2$$
$$(6,-2,2)$$

- $t=-1$
$$x=3.(-1)+6=3$$
$$y=-6(-1)-2=4$$
$$z=4.(-1)+2=-2$$
$$(3,4,-2)$$

Question: Which of the following is the family of level curves of the function $f(x,y) = \frac{y}{x^2+y^2}$ for $(x,y) \neq (0,0)$?

A) $x^2 + \left(y - \frac{1}{c}\right)^2 = \frac{1}{c^2}$

B) $x^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{4c^2}$

C) $x^2 + y^2 = c(y+1)$

D) $x^2 + y^2 = 4c^2(y+1)$

E) $x^2 + \left(y^2 - \frac{1}{2c}\right)^2 = \frac{2}{c^2}$

If $f(x,y) = \frac{y}{x^2+y^2} = c$ for $c \in \mathbb{R} - \{0\}$, then

$$y = c(x^2+y^2) \Rightarrow y = cx^2 + cy^2 \Rightarrow$$

$$cx^2 + cy^2 - y = 0$$

$$cx^2 + c\left(y^2 - \frac{1}{c}y\right) = 0 \Rightarrow$$

$$cx^2 + c\left(y^2 - \frac{1}{c}y + \frac{1}{4c^2} - \frac{1}{4c^2}\right) = 0 \Rightarrow$$

$$cx^2 + c\left[\left(y - \frac{1}{2c}\right)^2 - \frac{1}{4c^2}\right] = 0 \Rightarrow$$

$$x^2 + \left(y - \frac{1}{2c}\right)^2 - \frac{1}{4c^2} = 0 \Rightarrow$$

$$x^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{4c^2}$$

Question : Which of the following is the domain of the function $f(x,y) = \ln(\ln(x)) + \ln(\ln(x-y))$?

A) $0 \le x \le 1$ and $x > y+1$

B) $x > 0$ and $x > y+1$

C) $x > 0$ and $x < y+1$

D) $x > 1$ and $x > y+1$

E) $0 \le x < 1$ and $x > y-1$

- $\ln(x) > 0$ for $\ln(\ln(x))$ so $x > 1$
- $\ln(x-y) > 0$ for $\ln(\ln(x-y))$ so $x-y > 1 \Rightarrow x > y+1$

Then, $x > 1$ and $x > y+1$.

Question : Let $f(x,y) = \arctan(x+2y)$.

What is the value of $f_y(1,0)$?

A) 2 B) 3 C) 1 D) -1 E) 0

$$f_y = \frac{2}{1+(x+2y)^2} \Rightarrow f_y(1,0) = \frac{2}{1+(1+2.0)^2}$$

$$= \frac{2}{1+1} = 1$$

Question: Equation of the plane

$D: 3x-4y+z=10$ with points $P(2,3,-1)$ and $Q(1,2,2)$. Let L be the line passing through point P and perpendicular to plane D. Let E be the plane passing through points P and Q and perpendicular to plane D. Which of the following options are the equations of line L and plane E given?

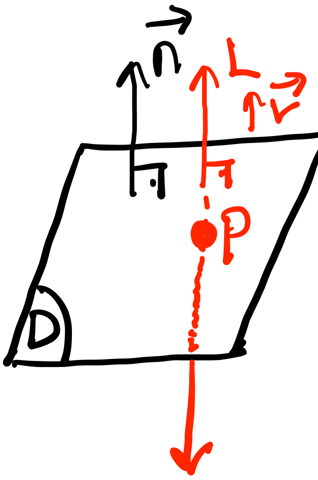
A) $L: x=3t+2,\ y=-4t+3,\ z=t-1;\ -\infty<t<\infty,$
$E: 11x+10y+7z=45$

B) $L: x=3t+2,\ y=-4t+3,\ z=t-1;\ -\infty<t<\infty,$
$E: 7x-4y+z=1$

C) $L: x=3t+12,\ y=-2t+13,\ z=t-1;\ -\infty<t<\infty$
$E: 11x+10y+7z=45$

D) $L: x=3t+12,\ y=-2t+13,\ z=t-1;\ -\infty<t<\infty$
$E: 7x-4y+z=1$

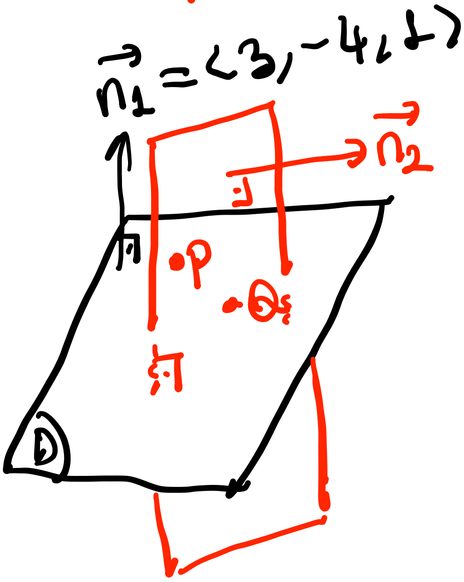
E) $L: x=3t+12,\ y=-2t+9,\ z=t-1;\ -\infty<t<\infty$
$E: 11x+10y+6z=25$



$\vec{n} = \langle 3, -4, 1 \rangle \parallel \vec{v} \Rightarrow$

We can take $\vec{v} = \langle 3, -4, 1 \rangle$

$L: x = 2 + 3t,\ y = 3 - 4t,\ z = -1 + t$



$$\left.\begin{array}{l} n_2 \perp \vec{n_1} = \langle 3, -4, 1 \rangle \text{ and} \\ n_2 \perp \vec{PQ} = \langle -1, -1, 3 \rangle \end{array}\right\} \Rightarrow$$

$$\vec{n_2} = \vec{n_1} \times \vec{PQ} = \begin{vmatrix} i & j & k \\ 3 & -4 & 1 \\ -1 & -1 & 3 \end{vmatrix}$$

$$\begin{array}{ccc} i & j & k \\ 3 & -4 & 1 \end{array}$$

$$= (-12i - 3k - j) - (4k - i + 9j)$$

$$= -11i - 10j - 7k = \langle -11, -10, -7 \rangle$$

$$E: -11(x-2) - 10(y-3) - 7(z+1) = 0 \Rightarrow$$

$$-11x + 22 - 10y + 30 - 7z - 7 = 0 \Rightarrow$$

$$11x + 10y + 7z = 45$$

Question: Let $f(x,y) = x\ln y + y\ln x + \sin(xy)$ for $x>0$ and $y>0$. Which of the following is the function $x^2 f_{xx} - y^2 f_{yy} - x$?

A) x B) $y-x$ C) $-x$ D) $x+y$ E) $-y$

$$f_x = \ln y + \frac{y}{x} + y\cos(xy) \Rightarrow$$

$$f_{xx} = -\frac{y}{x^2} - y^2\sin(xy) \Rightarrow$$

$$x^2 f_{xx} = -y - x^2y^2\sin(xy)$$

$$f_y = \frac{x}{y} + \ln x + x\cos(xy) \Rightarrow$$

$$f_{yy} = -\frac{x}{y^2} - x^2\sin(xy) \Rightarrow$$

$$y^2 f_{yy} = -x - x^2y^2\sin(xy) \Rightarrow$$

$$x^2 f_{xx} - y^2 f_{yy} - x =$$

$$-y - x^2y^2\sin(xy) + x + x^2y^2\sin(xy) - x = -y$$

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Question: How should the value of the function

$$f(x,y)=\frac{x^2+y^2-2x^3y^3}{x^2+y^2},\ (x,y)\neq(0,0)$$ at the origin be defined so that it is continuous in the $\mathbb{R}^2$ plane?

A) $f(0,0)=0$ B) $f(0,0)=-1$ C) $f(0,0)=2$

D) $f(0,0)=-2$ E) $f(0,0)=1$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\frac{0}{0}$$ (indeterminated form)

Let we take $y=kx^n$ for $k\in\mathbb{R}-\{0\}$ and $n\geq1$.

Then, $$f=\frac{x^2+k^2x^{2n}-2k^3x^{3n+3}}{x^2+k^2x^{2n}}$$

$$=\frac{\cancel{x^2}[1+k^2x^{2n-2}-2k^3x^{3n+1}]}{\cancel{x^2}[1+k^2x^{2n-2}]}$$

so $$\lim_{x\to0} f=\lim_{x\to0}\frac{1+k^2x^{2n-2}-2k^3x^{3n+1}}{1+k^2x^{2n-2}}=\frac{1}{1}=1$$

If f is continuous, then it must be

$$f(0,0)=\lim_{(x,y)\to(0,0)} f(x,y)=1$$

Question: Let $w(x,y,z,t) = \frac{xy}{1+z}(1+t)$ be differentiable function with $x=t$, $y=2t$, $z=t-t^2$. What is the value of

$\left.\frac{\partial w}{\partial t}\right|_{t=-1}$?

A) 1 B) −1 C) 0 D) −2 E) 2

1. way:

$$w = \frac{t \cdot 2t}{1+(t-t^2)} \cdot (1+t) = \frac{2t^2+2t^3}{-t^2+t+1} \Rightarrow$$

$$\frac{\partial w}{\partial t} = \frac{[4t+6t^2][-t^2+t+1] - [-2t+1][2t^2+2t^3]}{(-t^2+t+1)^2}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=-1} = \frac{[2.(-1)]-[3.0]}{(-1)^2} = \frac{-2}{1} = -2$$

2. way: If $t=-1$, then $x=-1$, $y=z=-2$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{dt}{dt}$$

$$= \frac{y}{1+z}(1+t).1 + \frac{x}{1+z}(1+t).2 - \frac{xy}{(1+z)^2}(1+t)(1-2t)$$

$$+ \frac{xy}{1+z}.1 \Rightarrow$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=-1} = 0+0-0+\frac{(-1).(-2)}{1+(-2)}.1 = \frac{2}{-1} = -2$$

Question: Let $f(x,y)=\begin{cases} \dfrac{2x^3-y^3}{x^2+3y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

If $f_x(0,0)=a$ and $f_y(0,0)=b$, then what is the value of a and b?

A) $a=2, b=-\frac{1}{3}$ B) $a=2, b=\frac{1}{3}$

C) $a=-\frac{1}{2}, b=1$ D) $a=-\frac{1}{4}, b=1$

E) $a=-\frac{1}{2}, b=-1$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-\cancel{f(0,0)}^{\,0}}{h}=\lim_{h\to 0}\frac{\frac{2h^3}{h^2}}{h}$$

$$=\lim_{h\to 0}\frac{2h^3}{h^3}=2=a$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-\cancel{f(0,0)}^{\,0}}{h}=\lim_{h\to 0}\frac{\frac{-h^3}{3h^2}}{h}$$

$$=\lim_{h\to 0}-\frac{h^3}{3h^3}=-\frac{1}{3}=b$$

Question: Let $f(x,y)=\begin{cases} \dfrac{xy+3y-x-3}{y-1}, & (x,y)\neq(2,1) \\ 0, & (x,y)=(2,1) \end{cases}$.

Which of the following is true for the function f above?

I. f is defined on $\mathbb{R}^2$.

II. The limit value of f at $(2,1)$ is 0.

III. f is continuous at $(2,1)$.

A) I, II, III B) Only I C) I and II

D) II and III E) I and III.

$$\lim_{(x,y)\to(2,1)} f(x,y) = \lim_{(x,y)\to(2,1)} \frac{xy+3y-x-3}{y-1} = \frac{0}{0} \text{ (indeterminated form)}$$

$$\lim_{(x,y)\to(2,1)} f(x,y) = \lim_{(x,y)\to(2,1)} \frac{xy-x+3y-3}{y-1}$$

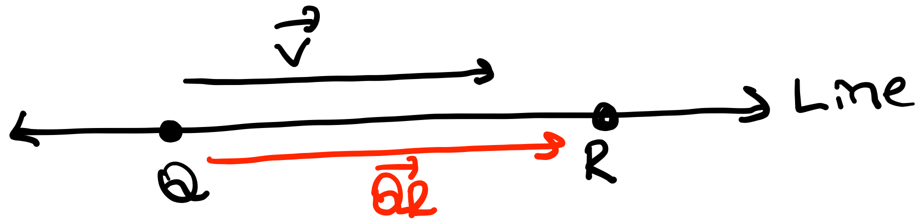
$$= \lim_{(x,y)\to(2,1)} \frac{x(y-1)+3(y-1)}{y-1} = \lim_{(x,y)\to(2,1)} \frac{\cancel{(y-1)}(x+3)}{\cancel{(y-1)}}$$

$$= \lim_{(x,y)\to(2,1)} x+3 = 2+3 = 5$$

$$0 = f(2,1) \neq 5 = \lim_{(x,y)\to(2,1)} f(x,y)$$ so f is discontinuous (is not continuous) at $(2,1)$.

Question : Which of the following is the distance from the point $P(1,-1,2)$ to the line passing through $Q(3,1,4)$ and $R(1,-3,0)$?

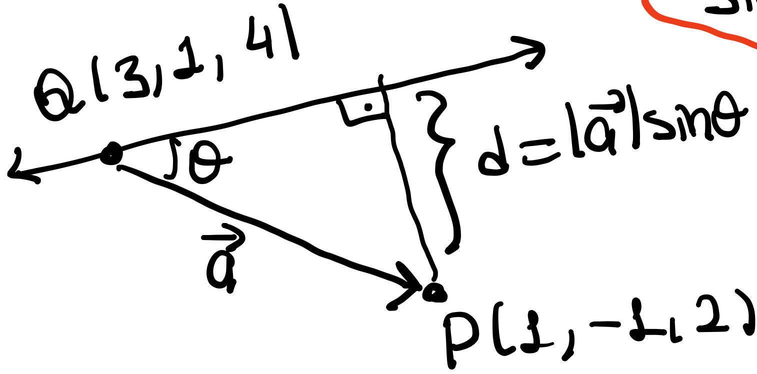
A) $\frac{2\sqrt{3}}{3}$ B) $\frac{2}{3}$ C) $\frac{2\sqrt{2}}{3}$ D) $\frac{2\sqrt{2}}{5}$ E) $\sqrt{2}$



We can take $\vec{v}=\vec{QR}=\langle -2,-4,-4\rangle$

Line: $x=3-2t,\ y=1-4t,\ z=4-4t$

Line: $Q+\vec{QR}t$



$\sin\theta=\frac{d}{|\vec{a}|} \Rightarrow$

$d=|\vec{a}|\sin\theta$

$\vec{a}=\vec{QP}=\langle -2,-2,-2\rangle$

$|\vec{QR}\times\vec{a}|=|\vec{QR}|\underbrace{|\vec{a}|\sin\theta}_{=d} \Rightarrow d=\frac{|\vec{QR}\times\vec{a}|}{|\vec{QR}|}=\frac{4\sqrt{2}}{6}=\frac{2\sqrt{2}}{3}$

$\vec{QR}\times\vec{a}=\begin{vmatrix} i & j & k \\ -2 & -4 & -4 \\ -2 & -2 & -2 \end{vmatrix}=\langle 0,4,-4\rangle$, $|\vec{QR}|=\sqrt{(-2)^2+(-4)^2+(-4)^2}$

$\begin{matrix} i & j & k \\ -2 & -4 & -4 \end{matrix}$

$=\sqrt{4+16+16}$

$=\sqrt{36}=6$

$\Rightarrow |\vec{QR}\times\vec{a}|=\sqrt{0^2+4^2+(-4)^2}$

$=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$

Question: Let $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{-x^5 y^4}{\sin(x^2y^2)(x^3y^2)} = L.$

Which of the following can be said for L?

A) $L>2$ B) $L=0$ C) $L=-1$ D) $L=1$ E) $L=-\infty$

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sin(x^2y^2)}}_{=1} \cdot \underbrace{\lim_{(x,y)\to(0,0)} \frac{-x^3y^2}{x^3y^2}}_{=\lim\limits_{(x,y)\to(0,0)} -1 = -1} = 1\cdot(-1) = -1$$

If we say $x^2y^2=\theta$,

then $\theta\to 0$ when $x,y\to 0$.

Then, $\lim\limits_{\theta\to 0} \dfrac{\theta}{\sin\theta} = 1$

$L=-1$

Question: Which of the following is true for the local extrema of the function

$$f(x,y) = -\frac{1}{2}xy + \frac{2}{x} - \frac{1}{y}?$$

A) There exists a local maximum at the $(2,-1)$.
B) There exists a local minumum at the $(2,-1)$.
C) There exists a local maximum at the $(-2,1)$.
D) There exists a local minumum at the $(-2,1)$.
E) The point $(-2,1)$ is saddle.

$f_x = -\frac{1}{2}y - \frac{2}{x^2}$ and $f_y = -\frac{1}{2}x + \frac{1}{y^2}$

If $f_x = 0$, then $y = -\frac{4}{x^2}$

If $f_y = 0$, then, $x = \frac{2}{y^2}$ $\Rightarrow x = \frac{2}{\left(-\frac{4}{x^2}\right)^2} \Rightarrow$

$x = \frac{2}{\frac{16}{x^4}} \Rightarrow x = \frac{x^4}{8} \Rightarrow x^4 - 8x = 0$

$\Rightarrow x(x^3-8) = 0$

$\Rightarrow x = 0$ or $x = 2$

- We can not take $x=0$ since $x=0 \notin D(f)$
- If $x=2$, then $y = -\frac{4}{x^2} = -\frac{4}{2^2} = -1$. $P(2,-1)$

$f_{xx} = \frac{4}{x^3}$, $f_{xy} = -\frac{1}{2}$, $f_{yy} = -\frac{2}{y^3}$

$A = f_{xx}|_P = 1/2$, $B = f_{xy}|_P = -\frac{1}{2}$, $C = f_{yy}|_P = 2$

$A = 11_2 > 0$ and $B^2 - AC = \frac{1}{4} - 1 = -\frac{3}{4} < 0$

so there exists a local minumum

at the $(2,-1)$.

Question: If the temperature function is given as $f(x,y,z) = 3x^2 - 5y^2 + 2z^2$ according to your location, and you are at location $\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{2}\right)$ and want to cool down as soon as possible, which of the following directions should you head towards?

A) $-\vec{i} - 2\vec{j} - \vec{k}$ B) $\vec{i} + \vec{j} + \vec{k}$ C) $-\vec{i} + \vec{j} - \vec{k}$

D) $\vec{i} - \vec{j} + \vec{k}$ E) $-\vec{i} - \vec{j} - \vec{k}$

$\vec{\nabla f} = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = (6x)\vec{i} + (-10y)\vec{j} + (4z)\vec{k} \Rightarrow$

$\vec{\nabla f}\Big|_{\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{2}\right)} = 2\vec{i} - 2\vec{j} + 2\vec{k} = \langle 2, -2, 2 \rangle$

When f decreases most rapidly, direction $\vec{v} \parallel -\vec{\nabla f}$. Then, we can write $\vec{\nabla f} = a\vec{v}$ for $a \in \mathbb{R}, a < 0$.

$\langle 2, -2, 2 \rangle = -2\langle -1, 1, -1 \rangle$ so

$\vec{v} = -\vec{i} + \vec{j} - \vec{k}$

Question : Let g be a differentiable function and $f(x,y) = g(2xy^3)$, $g(6) = 2$, $g'(6) = 1$. What is the approximate value of $f(3.1, 1.2)$?

A) 4.8 B) 5.8 C) 3.8 D) 2.8 E) 1.8

$$L(x,y) \approx f(3,1) + f_x(3,1).(x-3) + f_y(3,1)(y-1)$$

$$f(3,1) = g(6) = 2$$

$$f_x = 2y^3 g'(2xy^3) \Rightarrow f_x(3,1) = 2g'(6) = 2$$

$$f_y = 6xy^2 g'(2xy^3) \Rightarrow f_y(3,1) = 18g'(6) = 18$$

$$L(x,y) \simeq 2 + 2(x-3) + 18(y-1) \Rightarrow$$

$$f(3.1, 1.2) = 2 + 2(3.1-3) + 18(1.2-1)$$

$$= 2 + 2(0.1) + 18(0.2)$$

$$= 2 + 0{,}2 + 3{,}6$$

$$= 5.8$$

Question: Which of the following is the equation of the tangent plane of the function $f(x,y) = \arctan\frac{y}{x}$ at the point $(1,-1)$?

A) $x+y-2z=-\frac{\pi}{2}$ B) $x-y-2z=\frac{\pi}{2}$

C) $x+y+2z=\frac{\pi}{2}$ D) $x+y-2z=\frac{\pi}{2}$

E) $x-y-2z=-\frac{\pi}{2}$

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0.$$

$$f_x = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} \Rightarrow f_x(1,-1)=\frac{1}{2}$$

$$f_y = \frac{1/x}{1+\left(\frac{y}{x}\right)^2} \Rightarrow f_y(1,-1)=\frac{1}{2}$$

$$z_0 = f(1,-1) = \arctan(-1) = -\frac{\pi}{4}$$

$$\frac{1}{2}(x-1)+\frac{1}{2}(y-(-1))-\left(z-\left(-\frac{\pi}{4}\right)\right)=0 \quad \times 2$$

$$x-1+y+1-2z-\frac{\pi}{2}=0 \Rightarrow$$

$$x+y-2z=\frac{\pi}{2}$$

Question : Let $x^2y+xz+3y^3+yz^2+2z=8$ and $z=f(x,y)$. What is the value of

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = ?$$

A) $-\frac{3}{4}$ B) $-\frac{5}{2}$ C) $-\frac{3}{5}$ D) $-\frac{2}{5}$ E) $\frac{5}{2}$

$$x^2y+xz+3y^3+yz^2+2z-8=0 \Rightarrow$$

$$2xy+z+x\frac{\partial z}{\partial x}+2yz\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial x}=0$$

$\downarrow x=y=1$

$$2+1+\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}+2\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}+2\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}=0$$

$$3+5\cdot\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}=0 \Rightarrow$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}=-\frac{3}{5}$$

Question: Let $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$.

What is the value of $f_{xy}(0,0)$?

A) -2 B) 1 C) 2 D) -1 E) 0

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}.$$

$$=\lim_{h\to 0}\frac{0-0}{h}=0.$$

$$f_x=\frac{(3x^2y-y^3)\cdot(x^2+y^2)-2x\cdot(x^3y-xy^3)}{(x^2+y^2)^2}$$

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_x(0,0+h)-f_x(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-h^5}{h^4}-0}{h}=-1.$$

Question: What is the rate of change of function $f(x,y,z) = x^3 + y^3 - z$ at point $(1,1,2)$ along line $\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-2}{-2}$?

A) $\sqrt{17}$ B) $\sqrt{19}$ C) $\sqrt{15}$ D) $\sqrt{13}$ E) 1

$$(D_u f)_p = \vec{\nabla} f|_p \cdot \vec{u}$$

$$\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-2}{-2} = t \Rightarrow \left.\begin{aligned} x &= 1+3t \\ y &= 1+2t \\ z &= 2-2t \end{aligned}\right\} \begin{aligned} &\Rightarrow \vec{V} = \langle 3,2,-2 \rangle \\ &\Rightarrow |\vec{V}| = \sqrt{9+4+4} \\ &= \sqrt{17} \end{aligned}$$

$$\vec{u} = \frac{\vec{V}}{|\vec{V}|} = \left\langle \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}} \right\rangle$$

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (3x^2)\vec{i} + (3y^2)\vec{j} - 1 \cdot \vec{k} \Rightarrow$$

$$\vec{\nabla} f|_{(1,1,2)} = 3\vec{i} + 3\vec{j} - \vec{k} = \langle 3,3,-1 \rangle$$

$$(D_u f)|_{(1,1,2)} = \left\langle \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}} \right\rangle \cdot \langle 3,3,-1 \rangle$$

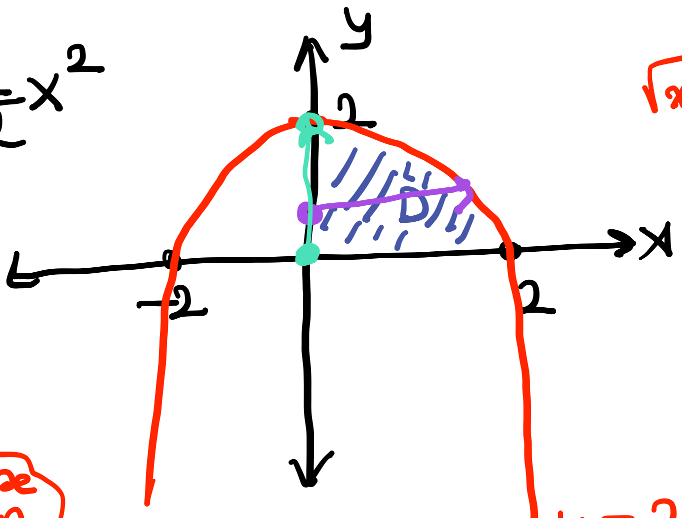
$$= \frac{9+6+2}{\sqrt{17}} = \frac{17}{\sqrt{17}} = \frac{\cancel{17}\sqrt{17}}{\cancel{17}} = \sqrt{17}$$

Question: If D is a closed region bounded by the $x^2 = 4-2y$ curve in the first quadrant. What is the value of the integral $\iint_D \frac{1}{\sqrt{2y-y^2}}\,dA$ over this region?

A) 1 B) 4 C) −4 D) −1 E) $\frac{3}{2}$

$$y = \frac{4-x^2}{2} = 2 - \frac{1}{2}x^2$$

- $x=0 \Rightarrow y=2$
- $y=0 \Rightarrow x=\mp 2$
- coefficient of x^2 is $-\frac{1}{2}<0$ so graph must be concave down



$\sqrt{x^2}=\sqrt{4-2y}$

$x=\mp\sqrt{4-2y}$

1st quadrant

$x>0$

$x=\sqrt{4-2y}$

$y=2-\frac{1}{2}x^2$

$$\iint_D \frac{1}{\sqrt{2y-y^2}}\,dA = \int_0^2\int_0^{\sqrt{4-2y}} \frac{1}{\sqrt{2y-y^2}}\,dx\,dy$$

→ type 2

$0=h_1(y)\le x\le h_2(y)=\sqrt{4-2y}$

$0=c\le y\le d=2$

$$= \int_0^2 \left(\frac{x}{\sqrt{2y-y^2}} \Bigg|_0^{\sqrt{4-2y}} \right) dy$$

$$\frac{\sqrt{2\cdot(2-y)}}{\sqrt{y(2-y)}}$$

$$= \int_0^2 \frac{\sqrt{4-2y}}{\sqrt{2y-y^2}}\,dy = \int_0^2 \frac{\sqrt{2}\cdot\cancel{\sqrt{2-y}}}{\sqrt{y}\,\cancel{\sqrt{2-y}}}\,dy$$

$$= \sqrt{2}\int_0^2 y^{-1/2}\,dy = 2\sqrt{2}\sqrt{y}\Big|_0^2$$

$$= 2\sqrt{2}\cdot\sqrt{2} = 4$$

x

$x=0$ $x=\sqrt{4-2y}$

y

$y=2$

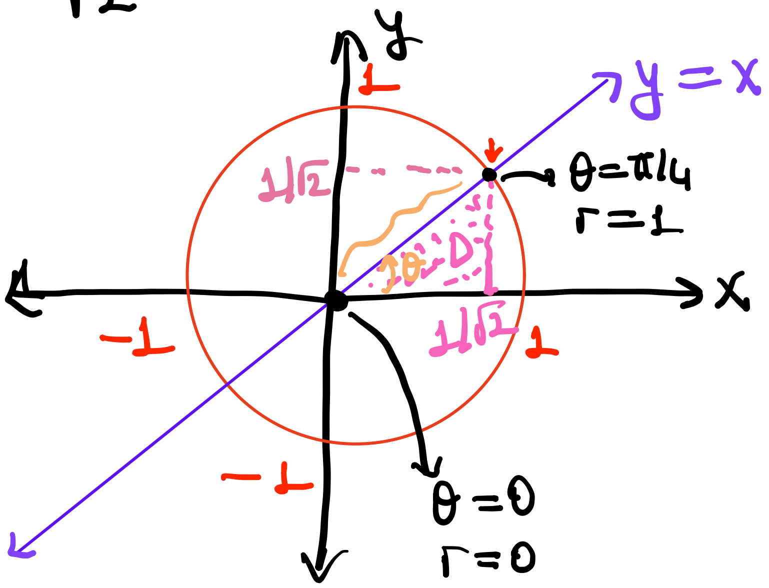
$y=0$

Question: $\displaystyle\int_0^{1/\sqrt{2}}\int_x^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx = ?$

A) $\pi - e$ B) $-\frac{\pi}{6}$ C) $\frac{\pi}{6}$ D) $-\frac{\pi}{12}$ E) $\frac{\pi}{12}$

$x=0 \qquad y=x$

$x=\frac{1}{\sqrt{2}} \qquad y=\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2=1$



• intersection

$x=\sqrt{1-x^2} \Rightarrow$

$x^2 = 1-x^2 \Rightarrow$

$2x^2 = 1 \Rightarrow$

$x^2 = \frac{1}{2} \Rightarrow$ $x>0$

$\Rightarrow x = \frac{1}{\sqrt{2}}$

• Polar form

• $\theta=0$, $r=0$

• If $y=x$, then $r\sin\theta = r\cos\theta \Rightarrow \tan\theta = 1$
$\Rightarrow \theta = \pi/4$

and $r^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{2}{2} = 1$

$r^2 = x^2+y^2 \Rightarrow \sqrt{r^2} = \sqrt{x^2+y^2} \Rightarrow r = \sqrt{x^2+y^2}$

• **Integral**:

$$\int_0^{\pi/4}\int_0^{1} r\cdot r\cdot dr\,d\theta=\int_0^{\pi/4}\int_0^{1} r^2\,dr\,d\theta$$

$$=\int_0^{\pi/4}\left(\frac{r^3}{3}\Big|_0^1\right)d\theta=\int_0^{\pi/4}\frac{1}{3}\,d\theta=\frac{1}{3}\theta\Big|_0^{\pi/4}$$

$$=\frac{1}{3}\cdot\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

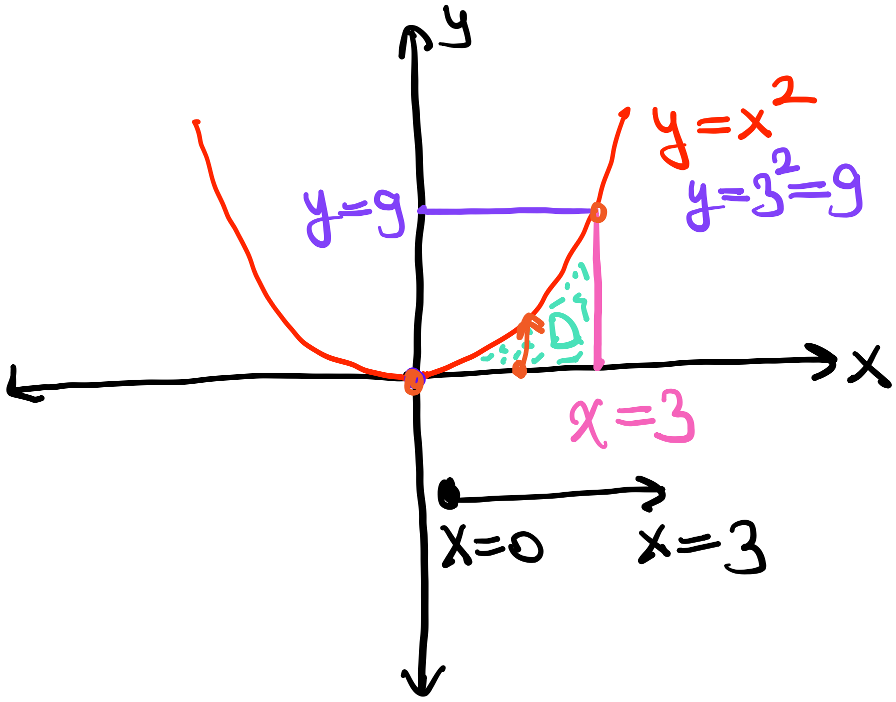
Question: $\int_0^{9}\int_{\sqrt{y}}^{3} e^{x^3}\,dx\,dy=?$

A) $\frac{1}{3}(e^9-1)$ B) $\frac{1}{2}(e^{27}-1)$ C) $\frac{1}{3}(e^{27}-1)$

D) $\frac{1}{3}(e^3-1)$ E) $\frac{1}{3}(e^{27}+1)$

$y=0$ $x=\sqrt{y} \Rightarrow y=x^2$

$y=9$ $x=3$



- Let we change order of the double integral.

$$\int_0^3\int_0^{x^2} e^{x^3}\,dy\,dx$$

type 1 ; $\overset{=0}{a} \le x \le \overset{=3}{b}$

$g_1(x) \le y \le g_2(x)$, $g_1(x) = 0$, $g_2(x) = x^2$

$$= \int_0^3 \left(y \cdot e^{x^3} \Big|_0^{x^2} \right) dx$$

$$= \int_0^3 e^{x^3} x^2\,dx = \frac{1}{3}\int_0^{27} e^u\,du = \frac{1}{3} e^u \Big|_0^{27} = \frac{1}{3}\left(e^{27} - e^0\right)$$

$$= \frac{1}{3}\left(e^{27} - 1\right)$$

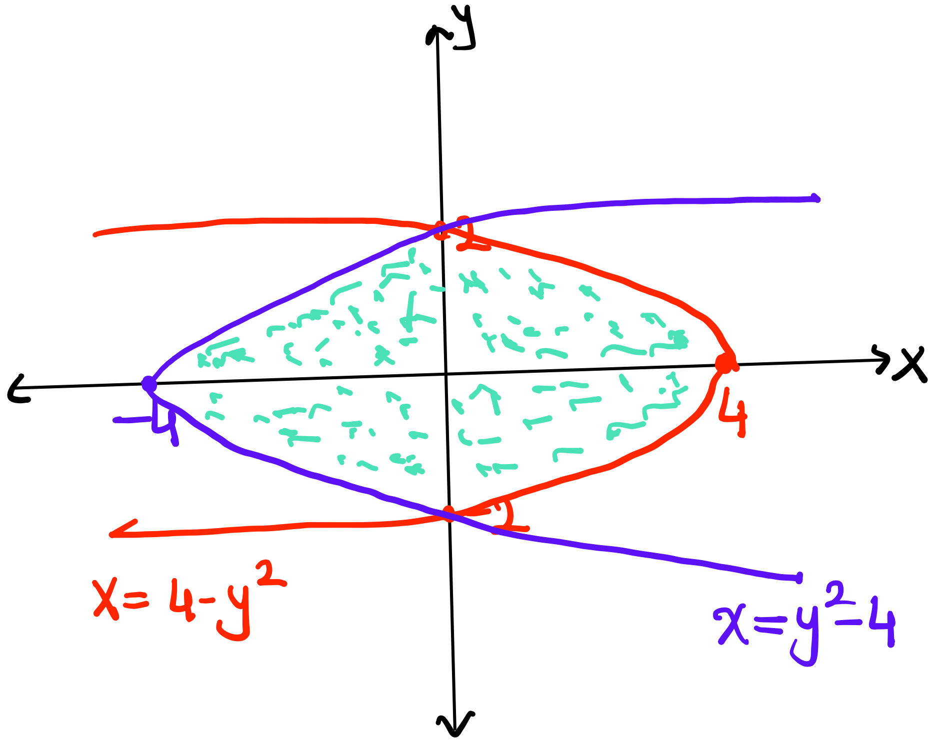
If $u = x^3$, then $du = 3x^2\,dx \Rightarrow \frac{1}{3}\,du = x^2\,dx$

$x = 3 \Rightarrow u = 3^3 = 27$

$x = 0 \Rightarrow u = 0^3 = 0$

3) Let D be the region bounded by the curves $x = 4 - y^2$ and $x = y^2 - 4$. Then, which of the following double integral gives the area of the region D ?

A) $\int_{-2}^{2}\int_{4-y^2}^{y^2-4} dx\,dy$ B) $\int_{-2}^{2}\int_{4-y^2}^{y^2-4} dy\,dx$ C) $\int_{0}^{4}\int_{4-y^2}^{y^2-4} dy\,dx$ D) $\int_{-2}^{2}\int_{y^2-4}^{4-y^2} dy\,dx$ E) $\int_{-2}^{2}\int_{y^2-4}^{4-y^2} dx\,dy$



$$\text{Area} = \iint_D dx\,dy$$

type 2: $h_1(y) \le x \le h_2(y)$

$c \le y \le d$

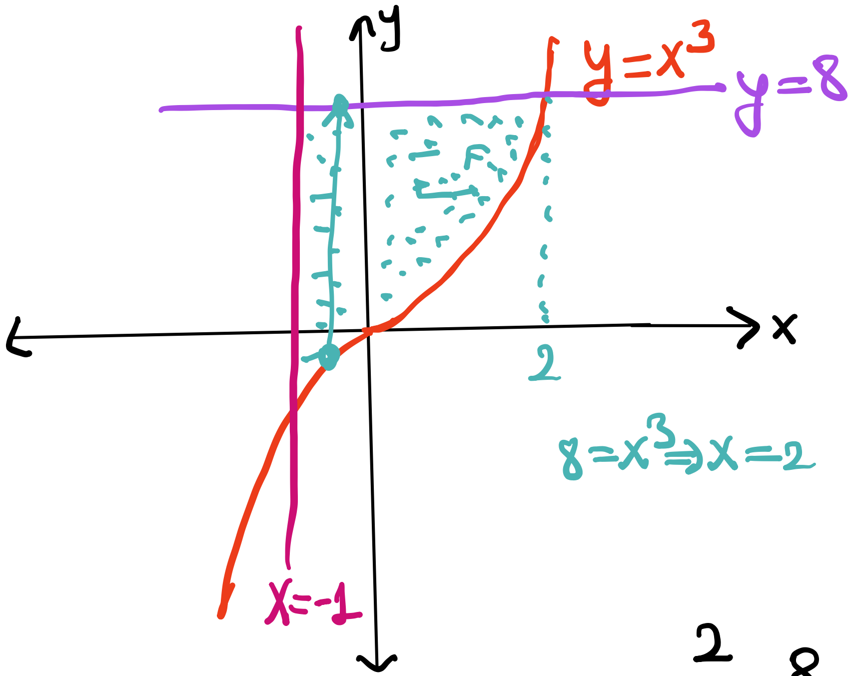
x: $x = y^2 - 4 \rightarrow x = 4 - y^2$

y: $y = -2 \rightarrow y = 2$

$$\text{Area} = \int_{-2}^{2}\int_{y^2-4}^{4-y^2} dx\,dy$$

15) Let D be the region bounded by the curve $y = x^3$ and the lines $y = 8$ and $x = -1$. Then, which of the following double integral gives the area of the region D?

A) $\int_{-1}^{2}\int_{x^3}^{-1} dy\,dx$ B) $\int_{-1}^{2}\int_{-8}^{x^3} dy\,dx$ C) $\int_{-1}^{2}\int_{\sqrt[3]{y}}^{8} dy\,dx$ D) $\int_{-1}^{2}\int_{x^3}^{8} dy\,dx$ E) $\int_{-1}^{2}\int_{x^3}^{0} dy\,dx$



$8=x^3 \Rightarrow x=2$

$\iint_D dy\,dx$

type 1:

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

$$\int_{-1}^{2}\int_{x^3}^{8} dy\,dx$$

x

$x=-1$ $x=2$

$\| \quad \|$

a b

y

$y=8=g_2(x)$

$g_1(x)=x^3$

x: $x=0$, $x=2/3$ $\qquad$ y: $y=2x$, $y=2-x$

17) If we rewrite the integral $\displaystyle\int_0^{\frac{2}{3}}\int_{2x}^{2-x} f(x,y)\,dy\,dx$ by changing the integration order, which of the following will we get?

A) $\displaystyle\int_0^{\frac{2}{3}}\int_0^{\frac{y}{2}} f(x,y)\,dx\,dy+\int_{\frac{2}{3}}^{2}\int_{\frac{y}{2}}^{2-y} f(x,y)\,dx\,dy$

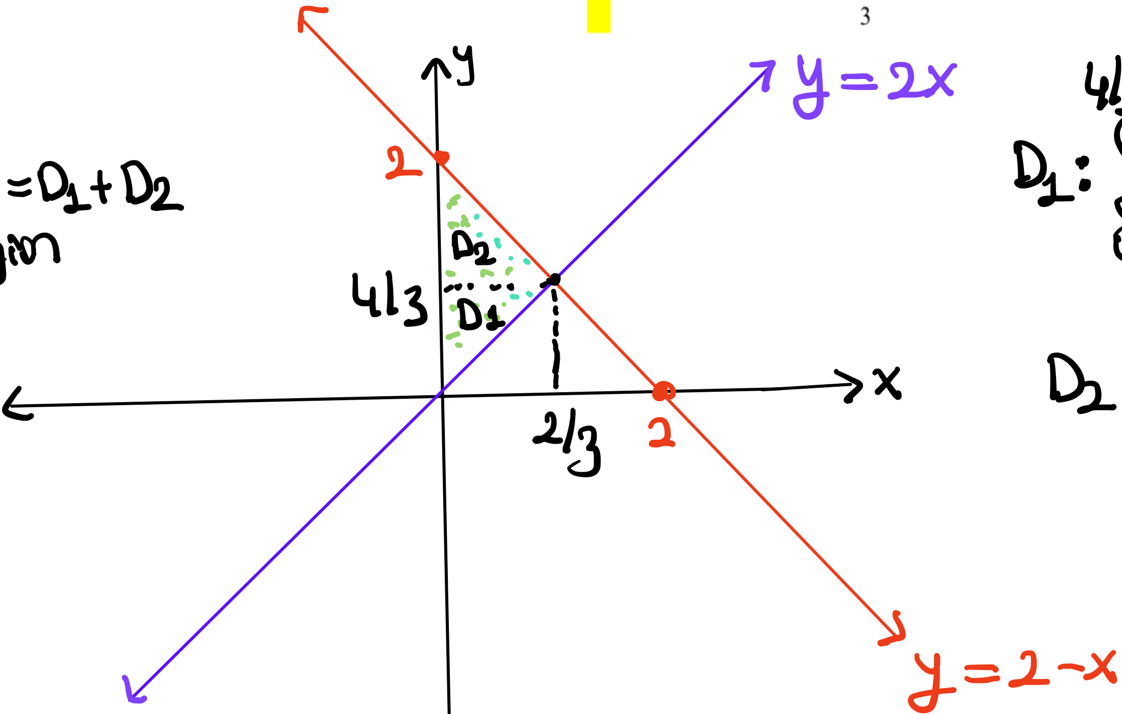
B) $\displaystyle\int_{2x}^{2-x}\int_0^{\frac{2}{3}} f(x,y)\,dx\,dy$

C) $\displaystyle\int_0^{\frac{4}{3}}\int_0^{\frac{y}{2}} f(x,y)\,dx\,dy+\int_{\frac{4}{3}}^{2}\int_{\frac{y}{2}}^{2-y} f(x,y)\,dx\,dy$

D) $\displaystyle\int_0^{\frac{2}{3}}\int_0^{\frac{y}{2}} f(x,y)\,dx\,dy+\int_0^{\frac{4}{3}}\int_0^{2-y} f(x,y)\,dx\,dy$

E) $\displaystyle\int_0^{\frac{4}{3}}\int_0^{\frac{y}{2}} f(x,y)\,dx\,dy+\int_{\frac{4}{3}}^{2}\int_0^{2-y} f(x,y)\,dx\,dy$

$D=D_1+D_2$ Region



$\Rightarrow x=2-y$

$y=2x \Rightarrow x=y/2$

$D_1: \displaystyle\int_0^{4/3}\int_0^{y/2} f(x,y)\,dx\,dy$

$D_2: \displaystyle\int_{4/3}^{2}\int_0^{2-y} f(x,y)\,dx\,dy$

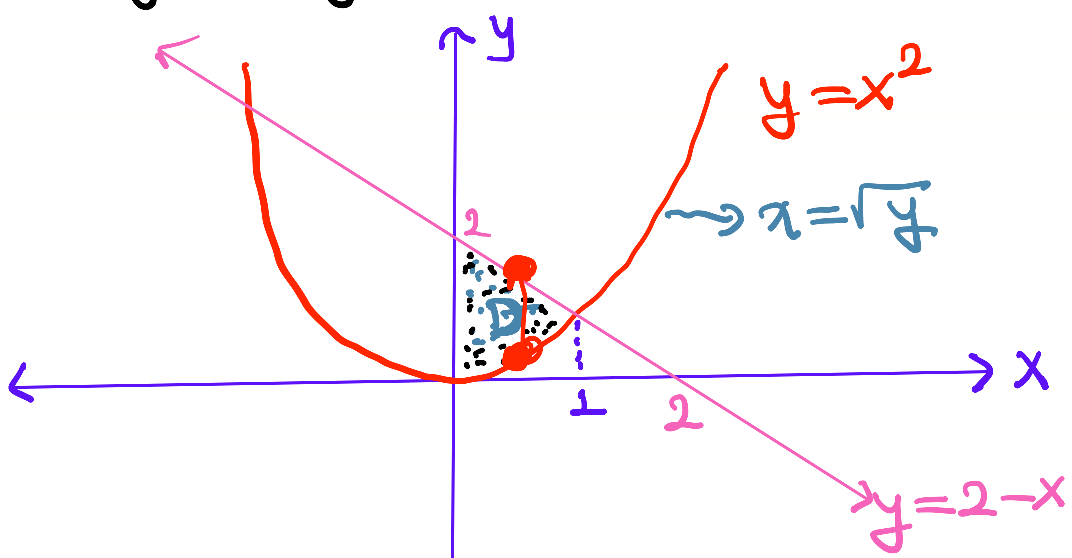
- Intersection: $2-x=2x \Rightarrow 3x=2$
  $\Rightarrow x=2/3$
  $\Rightarrow y=2\cdot\frac{2}{3}=2-\frac{2}{3}$
  $=4/3$

- $dx\,dy \to$ type 2:
  $h_1(y)\le x\le h_2(y)$
  $c\le y\le d$

Question: Let D be the region bounded by the curve $x=\sqrt{y}$ and lines $x+y=2$, $x=0$. Then, which of the following double integral gives the area of the region D?

A) $\int_0^1\int_{\sqrt{x}}^{2-x} dxdy$, B) $\int_0^1\int_{y^2}^{2-y} dydx$ C) $\int_0^1\int_{x^2}^{2-x} dydx$ D) $\int_0^1\int_{\sqrt{y}}^{2-y} dxdy$ E) $\int_0^1\int_{2-x}^{x^2} dydx$

$$x=\sqrt{y} \Rightarrow y=x^2 \text{ and } x+y=2 \Rightarrow y=2-x$$



x: $a=0 \to b=1$

y: from $y=x^2$ to $y=2-x$

• Intersection: $x=\sqrt{2-x} \Rightarrow$

$x^2=2-x \Rightarrow x^2+x-2=0$

$\Rightarrow (x+2)(x-1)=0$

$\Rightarrow x=-2$ or $x=1$

$$\text{Area}=\iint_D dydx=\int_0^1\int_{x^2}^{2-x} dydx$$

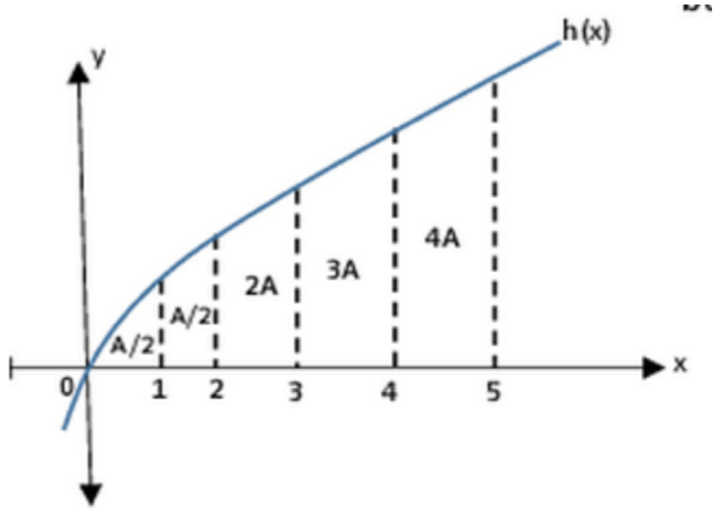
type I:

$a\le x\le b$

$g_1(x)\le y\le g_2(x)$

$g_1(x)=x^2$, $g_2(x)=2-x$

• **Question:** Let f and g be two integrable functions with $\int_1^5 f(x,y)\,dy = h(x)$ and $\int_0^4 h(x)\,dx = 12$.



If $\frac{A}{2}$, $2A$, $3A$ and $4A$ are the areas of the region where they are located, then what is the value of $\int_1^5\int_2^4 f(x,y)\,dx\,dy$?

A) 9 B) 10 C) 18 D) 19 E) 20

$$\int_0^4 h(x)\,dx = \frac{A}{2} + \frac{A}{2} + 2A + 3A = 6A = 12 \Rightarrow A = 2$$

$$\int_1^5\int_2^4 f(x,y)\,dx\,dy = \int_2^4 \underbrace{\int_1^5 f(x,y)\,dy}_{=h(x)}\,dx$$

$$= \int_2^4 h(x)\,dx = 2A + 3A$$
$$= 5A$$
$$= 5.2 = 10$$