MAT1320-Linear Algebra
Lecture Notes

Matrices

Mehmet E. KÖROĞLU

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Yildiz Technical University, Department of Mathematics
mkoroglu@yildiz.edu.tr

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$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

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The rows of such a matrix A are the m horizontal lists of scalars:

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and the columns of A are the n vertical lists of scalars:

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

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- A matrix with m rows and n columns is called an m by n matrix, written $m \times n$. The pair of numbers m and n is called the size of the matrix.
- Two matrices A and B are equal, written $A = B$, if they have the same size and if corresponding elements are equal. Thus, the equality of two $m \times n$ matrices is equivalent to a system of mn equalities, one for each corresponding pair of elements.

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- Matrices whose entries are all real numbers are called real matrices and are said to be matrices over $\mathbb{R}$.
- Analogously, matrices whose entries are all complex numbers are called complex matrices and are said to be matrices over $\mathbb{C}$. This text will be mainly concerned with such real matrices.

Matrices

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Example

The 2×4 zero matrix is the matrix $0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

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Find x, y, z, t such that $\begin{pmatrix} x+y & 3z+t \\ x-y & z-t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$.

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Solving the above system of equations yields

$$x=3, y=-1, z=1 \text{ and } t=-2.$$

Matrix Addition and Scalar Multiplication

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$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

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- The sum of matrices with different sizes is not defined.

Matrix Addition and Scalar Multiplication

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Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 6 & 8 \\ 1 & -3 & -7 \end{pmatrix}$.

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&= \begin{pmatrix} -10 & -22 & -18 \\ -3 & 17 & 31 \end{pmatrix}
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- $(kk')A=k(k'A)$

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- $A + B = B + A$
- $k(A + B) = kA + kB$
- $(k + k')\,A = kA + k'A$
- $(kk')\,A = k\,(k'A)$
- $1 \cdot A = A$

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- The letter k is called the index, and 1 and n are called, respectively, the lower and upper limits.
- We also generalize our definition by allowing the sum to range from any integer n_1 to any integer n_2. That is, we define
$$\sum_{k=n_1}^{n_2} f(k) = f(n_1) + f(n_1 + 1) + f(n_1 + 2) + \cdots + f(n_2)$$

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4. $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

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4. $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
5. $\sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \cdots + a_{ip} b_{pj}$

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- The product AB is not defined when A and B have different numbers of elements.

Example

1. $\begin{pmatrix} 7 & -4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 7(3) + (-4)(2) + 5(-1) =$
   $21 - 8 - 5 = 8.$

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2. $\begin{pmatrix} 6 & -1 & 8 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -9 \\ -2 \\ 5 \end{pmatrix} = 24 + 9 - 16 + 15 = 32.$

We are now ready to define matrix multiplication in general.

Matrix Multiplication

Suppose $A = [a_{ik}]$ and $B = [b_{kj}]$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix.

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$$\begin{pmatrix} a_{11} & \dots & a_{1p} \\ . & \dots & . \\ a_{i1} & \dots & a_{ip} \\ . & \dots & . \\ a_{m1} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ . & \dots & . \\ . & c_{ij} & . \\ . & \dots & . \\ c_{m1} & \dots & c_{mn} \end{pmatrix}$$

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- The product AB is not defined if A is an $m \times p$ matrix and B is a $q \times n$ matrix, where $p \neq q$.

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Find AB where $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{pmatrix}$.

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Note: The above example shows that matrix multiplication is not commutative that is, in general, $AB \neq BA$. However, matrix multiplication does satisfy the following properties.

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5. *We note that $0A = 0$ and $B0 = 0$, where 0 is the zero matrix.*

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- The next theorem lists basic properties of the transpose operation.

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We emphasize that, by (4), the transpose of a product is the product of the transposes, but in the reverse order.

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- Specifically, the operations of addition, multiplication, scalar multiplication, and transpose can be performed on any $n \times n$ matrices, and the result is again an $n \times n$ matrix.

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Then diagonal of $A = \{1, -4, 7\}$ and $Tr(A) = 1 - 4 + 7 = 4$
diagonal of $B = \{2, 3, -4\}$ and $Tr(B) = 2 + 3 - 4 = 1$

$$\begin{aligned} &Tr(A+B) = 3 - 1 + 3 = 5, && Tr(2A) = 2 - 8 + 14 = 8,\ Tr\left(A^T\right) = 4 \\ &Tr(AB) = 5 + 0 - 35 = -30, && Tr(BA) = 27 - 24 - 33 = -30 \end{aligned}$$

As expected from previous Theorem,

$$Tr(A+B) = Tr(A) + Tr(B),\ \ Tr\left(A^T\right) = Tr(A),\ \ Tr(2A) = 2\,Tr(A)$$

Furthermore, although $AB \neq BA$, the traces are equal.

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Identity Matrix, Scalar Matrices

The n -square identity or unit matrix, denoted by I_n, or simply I, is the n -square matrix with 1 's on the diagonal and 0's elsewhere. The identity matrix I is similar to the scalar 1 in that, for any n -square matrix A

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For any scalar k, the matrix k I that contains k 's on the diagonal and 0 's elsewhere is called the scalar matrix corresponding to the scalar k. Observe that

$$(kI)A = k(IA) = kA$$

That is, multiplying a matrix A by the scalar matrix kI is equivalent to multiplying A by the scalar k

Example

The following are the identity matrices of orders 3 and 4 and the corresponding scalar matrices for $k = 5$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 5 & & & \\ & 5 & & \\ & & 5 & \\ & & & 5 \end{pmatrix}$$

Powers of Matrices, Polynomials in Matrices

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Let A be an n -square matrix over a field K. Powers of A are defined as follows:

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Note that $f(A)$ is obtained from $f(x)$ by substituting the matrix A for the variable x and substituting the scalar matrix $a_0 I$ for the scalar a_0. If $f(A)$ is the zero matrix, then A is called a zero or root of $f(x)$.

Example

Suppose $A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \quad \text{and}$$

$$A^3 = A^2A = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} -11 & 38 \\ 57 & -106 \end{pmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$. Then

$$f(A) = 2\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix}$$

$$g(A) = \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, A is a zero of the polynomial $g(x)$

Invertible (Nonsingular) Matrices

Invertible (Nonsingular) Matrices

A square matrix A is said to be invertible or nonsingular if there exists a matrix B such that

$$AB = BA = I$$

where I is the identity matrix. Such a matrix B is unique. We will prove this fact by contradiction. Assume that the matrix A has two different inverses B_1 and B_2. That is, if $AB_1 = B_1A = I$ and $AB_2 = B_2A = I$ then

$$B_1 = B_1 I = B_1 (AB_2) = (B_1 A) B_2 = IB_2 = B_2$$

This is a contradiction. So, the inverse of a matrix, if exists, is unique. We call such a matrix B the inverse of A and denote it by A^{-1}. Observe that the above relation is symmetric; that is, if B is the inverse of A, then A is the inverse of B.

Example

Suppose that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$. Then

$$\begin{aligned} AB &= \begin{pmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ BA &= \begin{pmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, A and B are inverses. It is known that $AB = I$ if and only if $BA = I$. Thus, it is necessary to test only one product to determine whether or not two given matrices are inverses.

Invertible (Nonsingular) Matrices

Now suppose A and B are invertible. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. More generally, if $A_1, A_2, \dots, A_k$ are invertible, then their product is invertible and

$$(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$$

the product of the inverses in the reverse order.

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