2024-2025 FALL / MAT1071 MATHEMATICS 1
2nd MIDTERM

① For a differentiable function $g: \mathbb{R} \to \mathbb{R}$ where $g(0)=0$ and $a, b \in \mathbb{R}$, let

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \dfrac{g(x)}{x}, & x<0 \\ a+g(bx), & x \geq 0. \end{cases}$$

If the equation of the line tangent to the curve $y=f(x)$ at the point $P(0,a)$ is $y=a+x$, find the value of $a \cdot b$.

Start with the limit of $f(x)$ at $x=0$.

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}[a+g(bx)] = a+g(0) = a. \\ \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{g(x)}{x} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim\limits_{x\to 0^-} \dfrac{g'(x)}{1} = g'(0) \end{array}\right\} g'(0) = a.$$

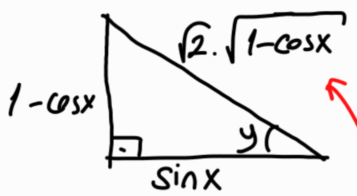
Tangent line: $y = a+x \Rightarrow m_T = 1 \quad (y'=1) \Rightarrow f'(0) = 1$

$f'(x) = (a+g(bx))' = b \cdot g'(bx)$ (We know that the derivative exists)

$f'(0) = b \cdot g'(0) = 1 \Rightarrow a \cdot b = 1$

② For the function $f: (0, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \operatorname{arccot}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$, what is the value of $f(1)+f'(1) = ?$

$$y = \operatorname{arccot}\sqrt{\frac{1+\cos x}{1-\cos x}} \Rightarrow \cot y = \sqrt{\frac{1+\cos x}{1-\cos x} \cdot \frac{(1-\cos x)}{(1-\cos x)}} = \sqrt{\frac{1-\overset{\sin^2 x}{\cos^2 x}}{(1-\cos x)^2}} = \frac{\sin x}{1-\cos x}$$



$\sqrt{\underbrace{\sin^2 x + \cos^2 x}_{1} - 2\cos x + 1}$

(Easier to evaluate)

$$\sin y = \frac{1-\cos x}{\sqrt{2}\cdot\sqrt{1-\cos x}} = \frac{\sqrt{1-\cos x}}{\sqrt{2}} \quad \text{(Take derivative)}$$

$$y' \cdot \cancel{\cos y} = \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{\sin x}}{2\cancel{\sqrt{1-\cos x}}} \Rightarrow y' = \frac{1}{2} \Rightarrow y = \frac{x}{2} + c$$

$\cos y = \dfrac{\sin x}{\sqrt{2}\cdot\sqrt{1-\cos x}}$

$f'(x)=\frac{1}{2} \Rightarrow f'(1)=\frac{1}{2}$. Now find c. (Pick an $x\in(0,\frac{\pi}{2})$)

Let $x=\frac{\pi}{3}$. $\sin y=\frac{\sqrt{1-\cos x}}{\sqrt{2}} \Rightarrow \sin(y(\frac{\pi}{3}))=\frac{\sqrt{1-\cos\frac{\pi}{3}}}{\sqrt{2}}=\frac{1}{2} \Rightarrow y=\frac{\pi}{6}$

$y=\frac{x}{2}+c \Rightarrow \frac{\pi}{6}=\frac{\pi}{3.2}+c \Rightarrow c=0 \Rightarrow f(x)=\frac{x}{2} \Rightarrow f(1)=\frac{1}{2}$

$\Rightarrow f(1)+f'(1)=\frac{1}{2}+\frac{1}{2}=1$

(3) If f is a continuously differentiable even function where $f(1)=2$ and $\int_0^1 \arctan x . f'(x)dx=\frac{\pi}{2}-1$ then what is the value of $\int_{-1}^{1}\frac{f(x)}{1+x^2}dx=?$

$$\int_{-1}^{1}\frac{f(x)dx}{1+x^2}=2.\underbrace{\int_0^1\frac{f(x)}{1+x^2}dx}_{A}=2.A$$ (f is even, $1+x^2$ is even)

$$\int_0^1 \arctan x . f'(x)dx=\frac{\pi}{2}-1$$

$\arctan x=u$ $\quad f'(x)dx=dv$

$\frac{1}{1+x^2}dx=du$ $\quad f(x)=v$

$$\Rightarrow \underbrace{\arctan x . f(x)\Big|_0^1}_{\arctan 1 . f(1)-0}-\underbrace{\int_0^1\frac{f(x)}{1+x^2}dx}_{A}=\frac{\pi}{2}-1 \Rightarrow 2.\frac{\pi}{4}-A=\frac{\pi}{2}-1 \Rightarrow A=1$$

$\Rightarrow 2A=2$

(4) Which of the following is the result of the integral $\int_{-1}^{1}\frac{x^2.\sin x}{1+x^4}dx=?$

$$f(x)=\frac{x^2.\sin x}{1+x^4} \Rightarrow f(-x)=\frac{(-x)^2.\sin(-x)}{1+(-x)^4}=\frac{-x^2.\sin x}{1+x^4}=-f(x)$$

$\Rightarrow f(x)$ is odd. $\int_{-1}^{1}\frac{x^2.\sin x}{1+x^4}dx=0$

5) If $f: D(f) \to \mathbb{R}$ is a continuous and even function satisfying the following equation $\int_1^{\cos x} f(t)\,dt = 1 - \sin x + \int_0^{\pi + x} f(\cos t)\,dt$

then which of the following is equal to $f(x)$?

A) $\frac{1}{1+x}$ B) $\sqrt{1+x^2}$ C) $\frac{x}{1+\sqrt{1-x^2}}$ D) x E) 0

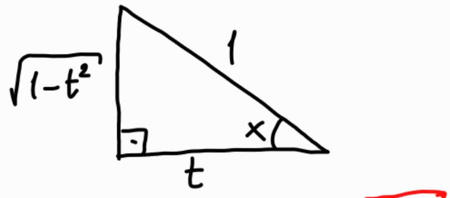
Take the derivative of both sides.

$$-\sin x \cdot f(\cos x) = -\cos x + \underbrace{f(\underbrace{\cos(\pi + x)}_{-\cos x})}_{f(\cos x)} \Rightarrow f(\cos x) = \frac{\cos x}{1+\sin x}$$

(f is even)

Let $x = \arccos t$ $(\cos x = t)$

$$f(t) = \frac{t}{1+\sin(\arccos t)} = \frac{t}{1+\sin(\arcsin\sqrt{1-t^2})}$$



$x = \arccos t = \arcsin\sqrt{1-t^2}$

$$f(t) = \frac{t}{1+\sqrt{1-x^2}} \Rightarrow f(x) = \frac{x}{1+\sqrt{1-x^2}}$$

6) If the continuous function $f: [1,4] \to \mathbb{R}$ satisfies the equation $\int_1^2 f(x)\,dx = \int_2^4 [3 - f(x)]\,dx$ then what is the average value of f in the interval $[1,4]$?

$$\int_1^2 f(x)\,dx = \int_2^4 [3-f(x)]\,dx = \int_2^4 3\,dx - \int_2^4 f(x)\,dx$$

$$\Rightarrow \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_2^4 3\,dx \Rightarrow \int_1^4 f(x)\,dx = 3x\Big|_2^4 = 3 \cdot 2 = 6$$

$$\bar{f} = \frac{1}{4-1} \cdot \int_1^4 f(x)\,dx = \frac{1}{3} \cdot 6 = 2$$

7) If a continuously differentiable function F is defined by $F(x)=\frac{1}{x}\int_{2x}^{x^3-4}[2t-3F'(t-2)]\,dt$ for $x\neq 0$, $F'(2)=?$

$$F'(x)=-\frac{1}{x^2}\cdot\int_{2x}^{x^3-4}[2t-3F'(t-2)]\,dt+\frac{1}{x}\left[3x^2\cdot\left(2(x^3-4)-3F'(x^3-6)\right)-2\cdot\left(4x-3F'(2x-2)\right)\right]$$

$$F'(2)=-\frac{1}{4}\cancelto{0}{\int_{4}^{4}[2t-3F'(t-2)]\,dt}+\frac{1}{\cancel{2}}\left[\cancelto{6}{12}\left(8-3F'(2)\right)-\cancel{2}\left(8-3F'(2)\right)\right]$$

$$F'(2)=48-18F'(2)-8+3F'(2)\Rightarrow 16F'(2)=40\Rightarrow F'(2)=\frac{5}{2}$$

8) $\int_{-1}^{1}\frac{x^2\,dx}{1+e^x}=?$

$(e^x+1)^{-1}=u$ $\qquad$ $x^2\,dx=dv$

$-1(e^x+1)^{-2}dx=du$ $\qquad$ $\frac{x^3}{3}=v$

$$I=\underbrace{\frac{x^3}{3}\cdot(e^x+1)^{-1}\Big|_{-1}^{1}}_{A}+\underbrace{\int_{-1}^{1}\frac{x^3}{3}\cdot\frac{e^x}{(e^x+1)^2}\,dx}_{B}=A+B$$

$$A=\frac{1}{3}\cdot\left[\frac{1}{e+1}+\frac{1}{\frac{1}{e}+1}\right]=\frac{1}{3}\left[\cancelto{1}{\frac{1}{e+1}+\frac{e}{e+1}}\right]=\frac{1}{3}$$

$$B=\frac{1}{3}\int_{-1}^{1}\frac{x^3.e^x}{(e^x+1)^2}\,dx \qquad f(x)=\frac{x^3.e^x}{(e^x+1)^2}$$

$$f(-x)=\frac{-x^3.e^{-x}}{(e^{-x}+1)^2}=\frac{-x^3.e^{-x}}{e^{-2x}+2e^{-x}+1}=\frac{-x^3.\cancel{e^{-x}}}{e^{-\cancel{2}x}(1+2e^x+e^{2x})}=\frac{-x^3.e^x}{(e^x+1)^2}=-f(x)$$

$\Rightarrow f(x)$ is an odd function $\Rightarrow B=0$

$$I=\frac{1}{3}+0=\frac{1}{3}$$

9) Find the area of the region bounded by the curve $y = x^2 + x - 2$ and the line $y = 1 - x$ where $x \geq 0$, as unit square.

Intersection points: $x^2 + x - 2 = 1 - x \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x = -3, 1$

$$\begin{array}{cc} x & 3 \\ x & -1 \end{array}$$

Which one is above and which one is below?

$x = 0 \Rightarrow$ (curve) $y_1 = -2$ and (line) $y_2 = 1$ above: line
below: curve

$$A = \int_{0}^{1} \left[(1-x) - (x^2 + x - 2)\right] dx = \int_0^1 (3 - 2x - x^2)\,dx = 3x - x^2 - \frac{x^3}{3} \Big|_0^1 = \boxed{\frac{5}{3}}$$

given as $x \geq 0$

10) If two functions $f, g: \mathbb{R} \to \mathbb{R}$ have the properties $f(x) = 1 - f(\pi - x)$, $g(x) = g(\pi - x)$ and $\int_0^{\pi} g(x)dx = 2$, then $\int_0^{\pi} f(x) \cdot g(x) dx = ?$

$$I = \int_0^{\pi} \underset{\downarrow\, 1 - f(\pi - x)}{f(x)} \cdot g(x)\,dx = \int_0^{\pi} \left[1 - f(\pi - x)\right] \cdot g(x)\,dx = \underbrace{\int_0^{\pi} g(x)\,dx}_{2} + \int_0^{\pi} f(\pi - x) \cdot \overbrace{g(x)}^{= g(\pi - x)}\,dx$$

$$\left.\begin{array}{ll} \pi - x = t & x = 0 \Rightarrow t = \pi \\ -dx = dt & x = \pi \Rightarrow t = 0 \end{array}\right\} \quad I = 2 + \int_{\pi}^{0} f(t) \cdot g(t)\,dt$$ (swap boundaries)

$$I = 2 - \underbrace{\int_0^{\pi} f(t) \cdot g(t)\,dt}_{I} \Rightarrow 2I = 2 \Rightarrow I = \boxed{1}$$