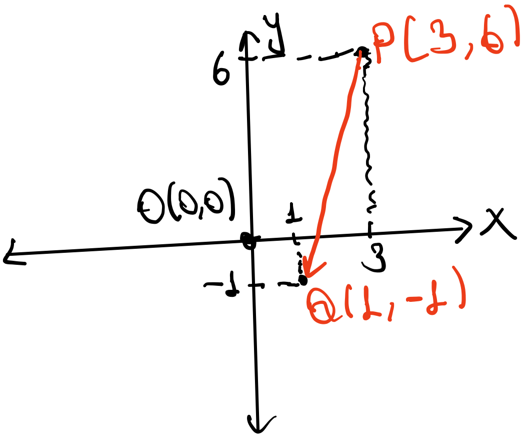
Q1: What is the value of $\int_C (3x^2-2y)\,ds$ where C is the line segment from $(3,6)$ to $(1,-1)$?

A) $8\sqrt{53}$ B) $7\sqrt{53}$ C) $\sqrt{47}$ D) $8\sqrt{47}$ E) $7\sqrt{47}$



$$C: \vec{r}(t) = \vec{OP} + t\,\vec{PQ}$$
$$= \langle 3-0, 6-0\rangle + t\langle 1-3, -1-6\rangle$$
$$= \langle 3, 6\rangle + t\langle -2, -7\rangle$$

$$\Rightarrow \begin{cases} x = 3-2t \\ y = 6-7t \end{cases} \quad \begin{aligned} &\to t=0 \Rightarrow (3,6) \\ &\to t=1 \Rightarrow (1,-1) \end{aligned}$$

x, y

$0 \le t \le 1$

$$\int_C f\,ds = \int_a^b \left[f(g(t), h(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\right] dt, \quad a \le t \le b$$

$$\int_C (3x^2-2y)\,ds = \int_0^1 \left[3(3-2t)^2 - 2(6-7t)\right] \underbrace{\sqrt{(-2)^2+(-7)^2}}_{=\sqrt{53}}\, dt$$

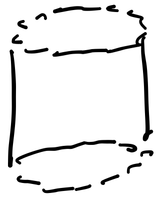
$$= \sqrt{53}\int_0^1 \left[12t^2 - 22t + 15\right] dt = \sqrt{53}\left[4t^3 - 11t^2 + 15t \Big|_0^1\right]$$

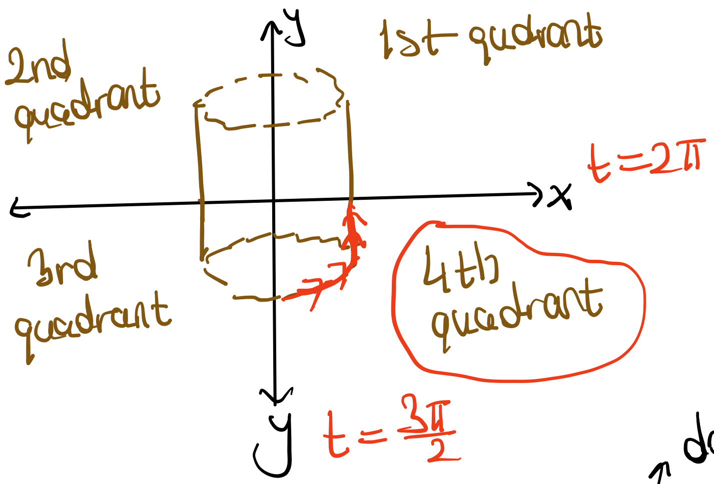
$$= \sqrt{53}\left[4 - 11 + 15\right] = 8\sqrt{53}$$

Q1: What is the value of $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = (x+y)\vec{i} + (1-x)\vec{j}$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4th quadrant with the counter clockwise rotation?

A) $5-\pi$ B) $4-3\pi$ C) $5-2\pi$ D) $5-3\pi$ E) $4-2\pi$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ (Elliptic cylinder)





$C:\ \begin{aligned} x &= 2\cos t \\ y &= 3\sin t \end{aligned} \Bigg\} \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} =$

$\frac{\cancel{4}\cos^2 t}{\cancel{4}} + \frac{\cancel{9}\sin^2 t}{\cancel{9}} = 1$

dot product

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\right) dt$$

__1st step__: $\vec{F}(\vec{r}(t)) = \langle 2\cos t + 3\sin t,\ 1-2\cos t \rangle$

__2nd step__: $\frac{d\vec{r}}{dt} = \langle -2\sin t,\ 3\cos t \rangle$

__3rd step__: $\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = (2\cos t + 3\sin t)(-2\sin t) + (1-2\cos t)(3\cos t)$

$$= \underbrace{-4\cos t \sin t}_{\substack{\shortparallel \\ -2\cdot \underbrace{2\cos t \sin t}_{=\sin(2t)}}} - 6\sin^2 t + 3\cos t - 6\cos^2 t$$

$$-6\underbrace{(\sin^2 t + \cos^2 t)}_{=1}$$

$$= -2\sin(2t) + 3\cos t - 6$$

4th step: $\displaystyle\int_{\frac{3\pi}{2}}^{2\pi} (-2\sin(2t) + 3\cos t - 6)\, dt$

$$= \left[\cos(2t) + 3\sin t - 6t\right]\Big|_{\frac{3\pi}{2}}^{2\pi}$$

$$= (\cos(4\pi) + 3\sin(2\pi) - 12\pi) - (\cos(3\pi) + 3\sin(\tfrac{3\pi}{2}) - 9\pi)$$

$$= (1 + 0 - 12\pi) - (-1 - 3 - 9\pi)$$

$$= (1 - 12\pi) - (-4 - 9\pi) = 5 - 3\pi$$

Q3: What is the value of $\int_C \vec{\nabla} f \cdot d\vec{r}$ where

$f(x,y) = x^3(3-y^2) + 4y$ and C is given by

$\vec{r}(t) = \langle 3-t^2, 5-t \rangle$ with $-2 \leq t \leq 3$?

A) 170 B) 152 C) 150 D) 100 E) 98

$$\int \vec{\nabla} f \cdot d\vec{r} = \int \left(\vec{\nabla} f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \right) dt$$

dot product

$\vec{\nabla} f = \langle f_x(x,y), f_y(x,y) \rangle \Rightarrow$

$\vec{\nabla} f(\vec{r}(t)) = \langle f_x(g(t), h(t)), f_y(g(t), h(t)) \rangle$

where $\vec{r}(t) = \langle \underset{x}{\underset{\shortparallel}{g(t)}}, \underset{y}{\underset{\shortparallel}{h(t)}} \rangle$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dg}{dt}, \frac{dh}{dt} \right\rangle = \left\langle \frac{dx}{dt}, \frac{dh}{dt} \right\rangle$$

Then, $\vec{\nabla} f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = f_x(g(t), h(t)) \cdot \frac{dx}{dt} + f_y(g(t), h(t)) \frac{dy}{dt}$

dot product $= \dfrac{d f(\vec{r}(t))}{dt}$ by the chain rule.

Hence, $\int_C \vec{\nabla} f \cdot dr = \int_C \left(\frac{d}{dt} \vec{f}(\vec{r}(t)) \right) dt = \vec{f}(\vec{r}(t))$

$$\Rightarrow \int_{-2}^{3} \left(\frac{d}{dt} \vec{f}(\vec{r}(t)) \right) dt =$$

$$\vec{f}(r(3)) - \vec{f}(r(-2))$$

$r(3) = \langle 3 - 3^2,\ 5 - 3 \rangle = \langle -6, 2 \rangle$

$r(-2) = \langle 3 - (-2)^2,\ 5 - (-2) \rangle = \langle -1, 7 \rangle$

$$\left.\begin{aligned} f(r(3)) &= f(-6,2) = (-6)^3 (3 - 2^2) + 4.2 \\ &= (-216)(-1) + 8 \\ &= 216 + 8 = 224 \\ f(r(-2)) &= f(-1,7) = (-1)^3 (3 - 7^2) + 4.7 \\ &= (-1)(-46) + 28 \\ &= 46 + 28 = 74 \end{aligned}\right\} \Rightarrow$$

$$f(r(3)) - f(r(-2)) = 224 - 74$$
$$= \underline{150}$$

Q4: What is the value of $\int_C 2y\,dx + (1-x)\,dy$

where C is portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$?

A) $\frac{1}{4}$ B) $\frac{2}{3}$ C) $\frac{1}{3}$ D) $\frac{4}{7}$ E) $\frac{3}{4}$

$$C:\ \left.\begin{aligned} x &= t \\ y &= 1 - t^3, \quad -1 \le t \le 2 \end{aligned}\right\} \Rightarrow \begin{aligned} dx &= dt \\ dy &= -3t^2\,dt \end{aligned}$$

$$\int_C 2y\,dx + (1-x)\,dy = \int_{-1}^{2} \left[2(1-t^3) + (1-t)(-3t^2)\right] dt$$

$$= \int_{-1}^{2} \left[2 - 2t^3 - 3t^2 + 3t^3\right] dt$$

$$= \int_{-1}^{2} \left[t^3 - 3t^2 + 2\right] dt = \left[\frac{t^4}{4} - t^3 + 2t\right]_{-1}^{2}$$

$$= [\underbrace{4 - 8 + 4}_{=0}] - [\underbrace{\tfrac{1}{4} + 1 - 2}_{-3/4}]$$

$$= 0 - \left(-\frac{3}{4}\right) = \frac{3}{4}$$

• Q5 : Let $f(x,y)=\frac{x^3}{y}$ and $C: y=\frac{x^2}{2}$, $0\le x\le 2$.

What is the value of $\int_C f\,ds = ?$

A) $\frac{5\sqrt{5}+2}{3}$ B) $\frac{10\sqrt{5}-2}{3}$ C) $\frac{7\sqrt{5}-1}{3}$ D) $\frac{5\sqrt{5}-2}{3}$ E) $\frac{5\sqrt{5}+4}{3}$

$$C:\ \left.\begin{aligned} x&=t \\ y&=\frac{t^2}{2}, \quad 0\le t\le 2\end{aligned}\right\} \Rightarrow \frac{dx}{dt}=1,\ \frac{dy}{dt}=t$$

$$\int_C f\,ds = \int_a^b f(g(t),h(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt, \quad a\le t\le b,\ x=g(t),\ y=h(t)$$

$$\int_C \frac{x^3}{y}\,ds = \int_0^2 \frac{t^3}{\left(\frac{t^2}{2}\right)}\cdot\sqrt{1^2+t^2}\,dt$$

$$=\int_0^2 2t\cdot\sqrt{1+t^2}\,dt, \quad \begin{aligned} 1+t^2&=u \\ 2t\,dt&=du\end{aligned} \left\{\begin{aligned} &\to t=0\Rightarrow u=1 \\ &t=2\Rightarrow u=5\end{aligned}\right.$$

$$=\int_1^5 \underbrace{\sqrt{u}}_{u^{1/2}}\cdot du = \frac{2}{3}u^{3/2}\Big|_1^5$$

$$=\frac{2}{3}[\underbrace{5^{3/2}}_{5\sqrt{5}}-1]=\frac{10\sqrt{5}-2}{3}$$