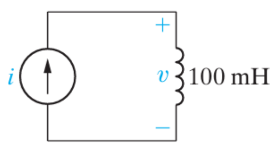
EXAMPLE 6.1 Determining the Voltage, Given the Current, at the Terminals of an Inductor

The independent current source in the circuit shown in Fig. 6.2 generates zero current for $t < 0$ and the pulse $10te^{-5t}$A for $t > 0$.



$$i = 0, \qquad t < 0$$

$$i = 10te^{-5t}\text{A}, \qquad t > 0$$

Figure 6.2 ▲ The circuit for Example 6.1.

a) Sketch the current waveform.

b) At what instant of time is the current maximum?

c) Express the voltage across the terminals of the 100 mH inductor as a function of time.

d) Sketch the voltage waveform.

e) Are the voltage and the current at a maximum at the same time?

f) At what instant of time does the voltage change polarity?

g) Is there ever an instantaneous change in voltage across the inductor? If so, at what time?

Solution

a) Figure 6.3 shows the current waveform.

b) $di/dt = 10(-5te^{-5t} + e^{-5t}) = 10e^{-5t}(1 - 5t)$ A/s; $di/dt = 0$ when $t = 0.2$ s. (See Fig. 6.3.)

c) $v = Ldi/dt = (0.1)10e^{-5t}(1 - 5t)$
$= e^{-5t}(1 - 5t)$ V, $t > 0$; $v = 0, t < 0$.

d) Figure 6.4 shows the voltage waveform.

e) No; the voltage is proportional to di/dt, not i.

f) At 0.2 s, which corresponds to the moment when di/dt is passing through zero and changing sign.

g) Yes, at $t = 0$. Note that the voltage can change instantaneously across the terminals of an inductor, even though the current in the inductor cannot change instantaneously.

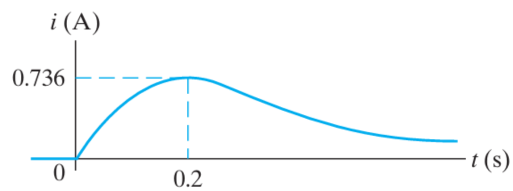


Figure 6.3 ▲ The current waveform for Example 6.1.

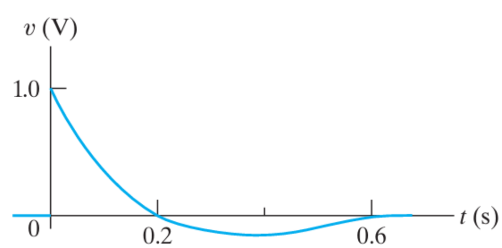


Figure 6.4 ▲ The voltage waveform for Example 6.1.

EXAMPLE 6.2 Determining the Current, Given the Voltage, at the Terminals of an Inductor

The voltage pulse applied to the 100 mH inductor shown in Fig. 6.5 is 0 for $t < 0$ and is given by the expression

$$v(t) = 20te^{-10t}\ \text{V}$$

for $t > 0$. Also assume $i = 0$ for $t \leq 0$.

a) Sketch the voltage as a function of time.

b) Find the inductor current as a function of time.

c) Sketch the current as a function of time.

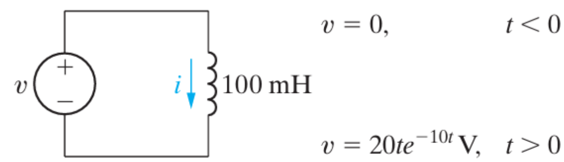


Figure 6.5 ▲ The circuit for Example 6.2.

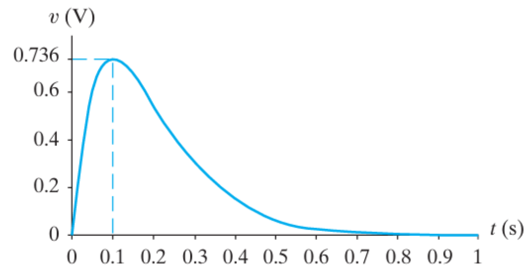


Figure 6.6 ▲ The voltage waveform for Example 6.2.

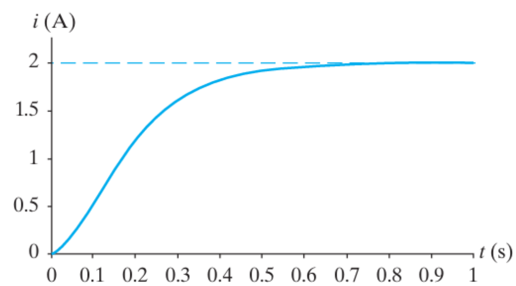


Figure 6.7 ▲ The current waveform for Example 6.2.

Solution

a) The voltage as a function of time is shown in Fig. 6.6.

b) The current in the inductor is 0 at $t = 0$. Therefore, the current for $t > 0$ is

$$\begin{aligned} i &= \frac{1}{0.1}\int_0^t 20\tau e^{-10\tau}d\tau + 0 \\ &= 200\left[\frac{-e^{-10\tau}}{100}(10\tau + 1)\right]\Bigg|_0^t, \\ &= 2(1 - 10te^{-10t} - e^{-10t})\ \text{A}, \quad t > 0. \end{aligned}$$

c) Figure 6.7 shows the current as a function of time.

EXAMPLE 6.3 Determining the Current, Voltage, Power, and Energy for an Inductor

a) For Example 6.1, plot i, v, p, and w versus time. Line up the plots vertically to allow easy assessment of each variable's behavior.

b) In what time interval is energy being stored in the inductor?

c) In what time interval is energy being extracted from the inductor?

d) What is the maximum energy stored in the inductor?

e) Evaluate the integrals

$$\int_0^{0.2} p\,dt \quad \text{and} \quad \int_{0.2}^{\infty} p\,dt,$$

and comment on their significance.

f) Repeat (a)–(c) for Example 6.2.

g) In Example 6.2, why is there a sustained current in the inductor as the voltage approaches zero?

Solution

a) The plots of i and v follow directly from the expressions for i and v obtained in Example 6.1. Applying Eq. 6.3,

$$\begin{aligned} p = vi &= \left[e^{-5t}(1 - 5t)\right](10te^{-5t}) \\ &= 10te^{-10t}(1 - 5t)\text{ W}. \end{aligned}$$

Applying Eq. 6.5,

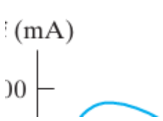
$$w = \frac{1}{2}Li = \frac{1}{2}(0.1)(10te^{-5t})^2 = 5t^2e^{-10t}\text{ J}.$$

The plots of i, v, p, and w are shown in Fig. 6.8.

b) When the energy curve increases, energy is being stored. Thus, from Fig. 6.8, energy is being stored in the time interval 0 to 0.2 s. This corresponds to the interval when $p > 0$.

c) When the energy curve decreases, energy is being extracted. Thus, from Fig. 6.8, energy is being extracted in the time interval 0.2 s to ∞. This corresponds to the interval when $p < 0$.

d) Equation 6.5 tells us that energy is at a maximum when current is at a maximum; the graphs in Fig. 6.8 confirm this. From Example 6.1, $i_{\max} = 0.736$ A. Therefore, $w_{\max} = 27.07$ mJ.



e) From part (a),

$p = 10te^{-10t}(1 - 5t) = 10te^{-10t} = 50t^2e^{-10t}$.

Thus

$$\begin{aligned} \int_0^{0.2} p\,dt &= 10\left[\frac{e^{-10t}}{100}(-10t - 1)\right]_0^{0.2} \\ &\quad -50\left\{\frac{t^2e^{-10t}}{-10} + \frac{2}{10}\left[\frac{e^{-10t}}{100}(-10t - 1)\right]\right\}_0^{0.2} \\ &= 0.2e^{-2} = 27.07\text{ mJ}, \end{aligned}$$

$$\begin{aligned} \int_{0.2}^{\infty} p\,dt &= 10\left[\frac{e^{-10t}}{100}(-10t - 1)\right]_{0.2}^{\infty} \\ &\quad -50\left\{\frac{t^2e^{-10t}}{-10} + \frac{2}{10}\left[\frac{e^{-10t}}{100}(-10t - 1)\right]\right\}_{0.2}^{\infty} \\ &= -0.2e^{-2} = -27.07\text{ mJ}. \end{aligned}$$

Based on the definition of p, the area under the plot of p versus t represents the energy expended over the interval of integration. Hence, integrating the power between 0 and 0.2 s represents the energy

EXAMPLE 6.4 Determining Current, Voltage, Power, and Energy for a Capacitor

The voltage pulse across the terminals of a 0.5 μF capacitor is:

$$v(t) = \begin{cases} 0, & t \le 0\text{ s}; \\ 4t\text{ V}, & 0\text{ s} \le t \le 1\text{ s}; \\ 4e^{-(t-1)}\text{ V}, & t \ge 1\text{ s}. \end{cases}$$

a) Derive the expressions for the capacitor current, power, and energy.

b) Sketch the voltage, current, power, and energy as functions of time. Line up the plots vertically.

c) Specify the time interval when energy is being stored in the capacitor.

d) Specify the time interval when energy is being delivered by the capacitor.

e) Evaluate the integrals

$$\int_0^1 p\,dt \qquad \text{and} \qquad \int_1^\infty p\,dt$$

and comment on their significance.

Solution

a) From Eq. 6.6,

$$i = \begin{cases} (0.5\,\mu)(0) = 0, & t < 0\text{ s}; \\ (0.5\,\mu)(4) = 2\,\mu\text{A}, & 0\text{ s} < t < 1\text{ s}; \\ (0.5\,\mu)(-4e^{-(t-1)}) = -2e^{-(t-1)}\,\mu\text{A}, & t > 1\text{ s}. \end{cases}$$

The expression for the power is derived from Eq. 6.8:

$$p = \begin{cases} 0, & t \le 0\text{ s}; \\ (4t)(2\,\mu) = 8t\,\mu\text{W}, & 0\text{ s} \le t < 1\text{ s}; \\ (4e^{-(t-1)})(-2\,\mu e^{-(t-1)}) = -8e^{-2(t-1)}\,\mu\text{W}, & t > 1\text{ s}. \end{cases}$$

The energy expression follows directly from Eq. 6.9:

$$w = \begin{cases} 0, & t \le 0\text{ s}; \\ \frac{1}{2}(0.5\,\mu)16t^2 = 4t^2\mu\text{J}, & 0\text{ s} \le t < 1\text{ s}; \\ \frac{1}{2}(0.5\,\mu)16e^{-2(t-1)} = 4e^{-2(t-1)}\,\mu\text{J}, & t \ge 1\text{ s}. \end{cases}$$

b) Figure 6.11 shows the voltage, current, power, and energy as functions of time.

c) Energy is being stored in the capacitor whenever the power is positive. Hence, energy is being stored in the interval from 0 to 1 s.

d) Energy is being delivered by the capacitor whenever the power is negative. Thus, energy is being delivered for all t greater than 1 s.

e) The integral of $p\;dt$ is the energy associated with the time interval corresponding to the integral's limits. Thus, the first integral represents the energy stored in the capacitor between 0 and 1 s, whereas the second integral represents the energy returned, or delivered, by the capacitor in the interval 1 s to ∞:

$$\int_0^1 p\,dt = \int_0^1 8t\,dt = 4t^2\Big|_0^1 = 4\,\mu\text{J},$$

$$\int_1^\infty p\,dt = \int_1^\infty (-8e^{-2(t-1)})dt = (-8)\frac{e^{-2(t-1)}}{-2}\Bigg|_1^\infty = -4\,\mu\text{J}.$$

The voltage applied to the capacitor returns to zero as time increases, so the energy returned by this ideal capacitor must equal the energy stored.

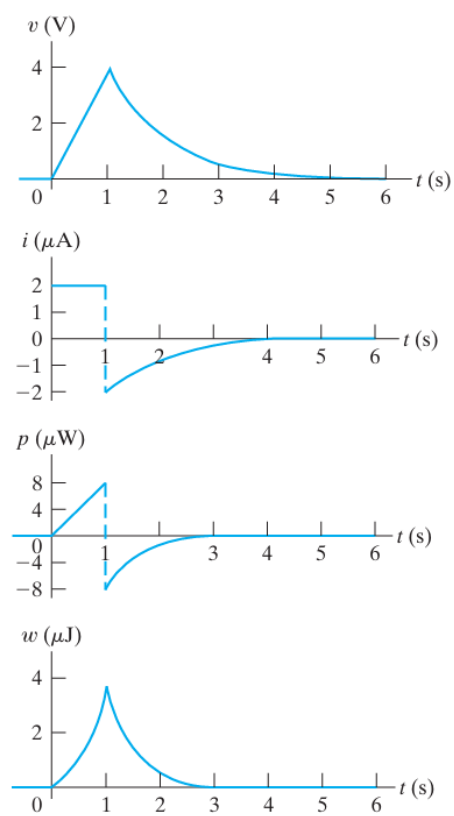


Figure 6.11 ▲ The variables v, i, p, and w versus t for Example 6.4.

EXAMPLE 6.5 Finding v, p, and w Induced by a Triangular Current Pulse for a Capacitor

An uncharged 0.2 μF capacitor is driven by a triangular current pulse. The current pulse is described by

$$i(t) = \begin{cases} 0, & t \leq 0; \\ 5000t \text{ A}, & 0 \leq t \leq 20\ \mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \leq t \leq 40\ \mu\text{s}; \\ 0, & t \geq 40\ \mu\text{s}. \end{cases}$$

a) Derive the expressions for the capacitor voltage, power, and energy for each of the four time intervals needed to describe the current.

b) Plot i, v, p, and w versus t. Align the plots as specified in the previous examples.

c) Why does a voltage remain on the capacitor after the current returns to zero?

Solution

a) For $t \leq 0$, v, p, and w all are zero.
For $0 \leq t \leq 20\ \mu$s,

$$v = \frac{1}{0.2 \times 10^{-6}} \int_0^t (5000\tau)\, d\tau + 0 = 12.5 \times 10^9 t^2 \text{ V},$$

$$p = vi = 62.5 \times 10^{12} t^3 \text{ W},$$

$$w = \frac{1}{2} Cv^2 = 15.625 \times 10^{12} t^4 \text{ J}.$$

For $20\ \mu\text{s} \leq t \leq 40\ \mu$s,

$$v(20 \times 10^{-6}) = 12.5 \times 10^9 (20 \times 10^{-6})^2 = 5 \text{ V}.$$

Then,

$$v = \frac{1}{0.2 \times 10^{-6}} \int_{20\mu s}^t (0.2 - 5000\tau)\, d\tau + 5$$

$$= (10^6 t - 12.5 \times 10^9 t^2 - 10) \text{ V},$$

$$p = vi,$$

$$= (62.5 \times 10^{12} t^3 - 7.5 \times 10^9 t^2 + 2.5 \times 10^5 t - 2) \text{ W},$$

$$w = \frac{1}{2} Cv^2,$$

$$= (15.625 \times 10^{12} t^4 - 2.5 \times 10^9 t^3 + 0.125 \times 10^6 t^2$$

$$-2t + 10^{-5}) \text{ J}.$$

For $t \geq 40\ \mu$s,

$$v = 10 \text{ V},$$

$$p = vi = 0,$$

$$w = \frac{1}{2} Cv^2 = 10\ \mu\text{J}.$$

b) The excitation current and the resulting voltage, power, and energy are plotted in Fig. 6.12.

c) Note that the power is always positive for the duration of the current pulse, which means that energy is continuously being stored in the capacitor. When the current returns to zero, the stored energy is trapped because the ideal capacitor cannot dissipate energy. Thus, a voltage remains on the capacitor after its current returns to zero.

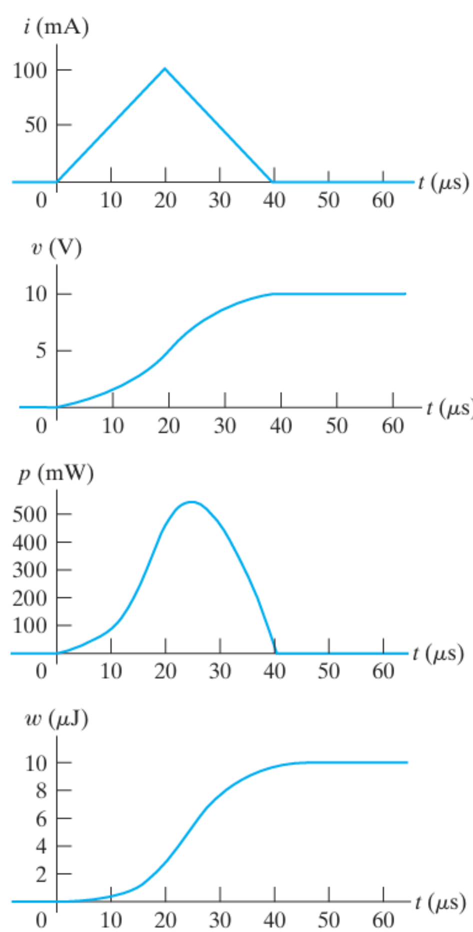


Figure 6.12 ▲ The variables i, v, p, and w versus t for Example 6.5.

EXAMPLE 6.6 Finding the Equivalent Inductance

Figure 6.19 shows four interconnected inductors. The initial currents for two of the inductors are also shown in Fig. 6.19. A single equivalent inductor, together with its initial current, is shown in Fig. 6.20.

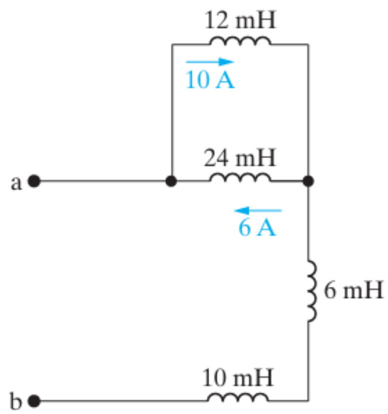


Figure 6.19 ▲ Interconnected inductors for Example 6.6.

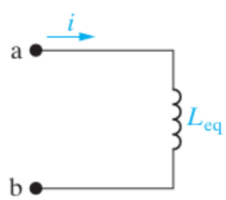


Figure 6.20 ▲ The equivalent inductor for the inductors in Fig. 6.19

a) Find the equivalent inductance, L_{eq}.

b) Find the initial current in the equivalent inductor.

Solution

a) Begin by replacing the parallel-connected 12 mH and 24 mH inductors with a single equivalent inductor whose inductance is

$$\left(\frac{1}{0.012} + \frac{1}{0.024}\right)^{-1} = 0.008 = 8 \text{ mH}.$$

Now the 8 mH, 6 mH, and 10 mH inductors are in series. Combining them gives

$$L_{eq} = 0.008 + 0.006 + 0.010 = 0.024 = 24 \text{ mH}.$$

b) The initial current in the equivalent inductor, i, is the same as the current entering the node to the left of the 24 mH inductor. The KCL equation at that node, summing the currents entering the node, is

$$i - 10 + 6 = 0.$$

Therefore, the initial current in the equivalent inductor is $i = 4$ A.

EXAMPLE 6.7 Finding the Equivalent Capacitance

Figure 6.21 shows four interconnected capacitors. The initial voltages for three of the capacitors are also shown in Fig. 6.21. A single equivalent capacitor, together with its initial voltage, is shown in Fig. 6.22.

a) Find the equivalent capacitance, C_{eq}.

b) Find the initial voltage across the equivalent capacitor.

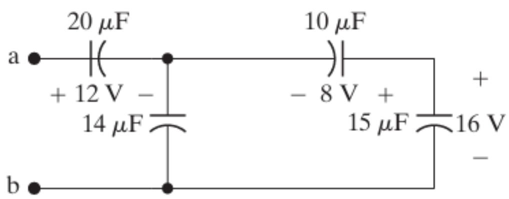


Figure 6.21 ▲ Interconnected capacitors for Example 6.7.

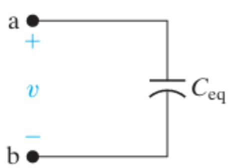


Figure 6.22 ▲ The equivalent capacitor for the capacitors in Fig. 6.21.

Solution

a) Begin by replacing the $10\,\mu$F and $15\,\mu$F capacitors with a single equivalent capacitor whose capacitance is

$$\left(\frac{1}{10 \times 10^{-6}} + \frac{1}{15 \times 10^{-6}}\right)^{-1} = 6 \times 10^{-6} = 6\,\mu\text{F}.$$

Next, combine the $6\,\mu$F capacitor from the first simplification with the $14\,\mu$F capacitor to give

$$6 \times 10^{-6} + 14 \times 10^{-6} = 20 \times 10^{-6} = 20\,\mu\text{F}.$$

Finally, combine the $20\,\mu$F from the previous simplification with the $20\,\mu$F on the left side of the circuit to give

$$C_{eq} = \left(\frac{1}{20 \times 10^{-6}} + \frac{1}{20 \times 10^{-6}}\right)^{-1} = 10 \times 10^{-6} = 10\,\mu\text{F}.$$

b) To find the initial voltage from a to b, use KVL to sum the initial voltages for the capacitors on the perimeter of the circuit. This gives

$$12 - 8 + 16 = 20\text{ V}.$$

Therefore, the initial voltage across the equivalent capacitor is 20 V.

EXAMPLE 6.8 Finding Mesh-Current Equations for a Circuit with Magnetically Coupled Coils

a) Use the mesh-current method to write equations for the circuit in Fig. 6.29 in terms of the currents i_1 and i_2.

b) Verify that if there is no energy stored in the circuit at $t = 0$ and if $i_g = 16 - 16e^{-5t}$ A, the solutions for i_1 and i_2 are

$$i_1 = 4 + 64e^{-5t} - 68e^{-4t}\ \text{A},$$

$$i_2 = 1 - 52e^{-5t} + 51e^{-4t}\ \text{A}.$$

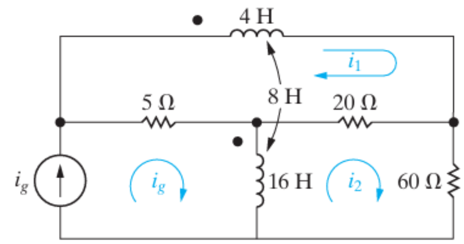


Figure 6.29 ▲ The circuit for Example 6.8.

Solution

a) Follow the steps in Analysis Method 4.6. Steps 1 and 2 identify the meshes and label the mesh currents, as shown in Fig. 6.29. In Step 3, we write a KVL equation for each mesh where the current is unknown. Summing the voltages around the i_1 mesh yields

$$4\frac{di_1}{dt} + 8\frac{d}{dt}(i_g - i_2) + 20(i_1 - i_2) + 5(i_1 - i_g) = 0.$$

[1]See discussion of Faraday's law on page 231.

Look carefully at the second term in this equation and make certain you understand how the dot convention was used. Note that the voltage across the 4 H coil due to the current $(i_g - i_2)$, that is, $8d(i_g - i_2)/dt$, is a voltage drop in the direction of i_1, so this term has a positive sign. The KVL equation for the i_2 mesh is

$$20(i_2 - i_1) + 60i_2 + 16\frac{d}{dt}(i_2 - i_g) - 8\frac{di_1}{dt} = 0.$$

Look carefully at the fourth term in this equation and make certain you understand how the dot convention was used. The voltage induced in the 16 H coil by the current i_1, that is, $8\,di_1/dt$, is a voltage rise in the direction of i_2, so this term has a negative sign.

b) To check the validity of i_1 and i_2, we begin by testing the initial and final values of i_1 and i_2. We know by hypothesis that $i_1(0) = i_2(0) = 0$. From the given solutions we have

$$i_1(0) = 4 + 64 - 68 = 0,$$

$$i_2(0) = 1 - 52 + 51 = 0.$$

Now we observe that as t approaches infinity, the source current (i_g) approaches a constant value of 16 A, and therefore the magnetically coupled coils behave as short circuits. Hence, at $t = \infty$ the circuit reduces to that shown in Fig. 6.30. From Fig. 6.30 we see that at $t = \infty$ the three resistors are in parallel across the 16 A source. The equivalent resistance is 3.75 Ω, and thus the voltage across the 16 A current source is 60 V. Write a KCL equation at node a, using Ohm's law to find the currents in the 20 Ω and 60 Ω resistors to give

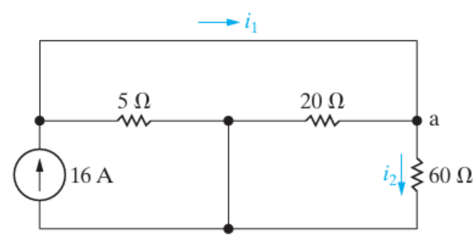


Figure 6.30 ▲ The circuit of Example 6.8 when $t = \infty$.

$$i_1(\infty) = \frac{60}{20} + \frac{60}{60} = 4\ \text{A}.$$

Using Ohm's law,

$$i_2(\infty) = \frac{60}{60} = 1\ \text{A}.$$

These values agree with the final values predicted by the solutions for i_1 and i_2:

$$i_1(\infty) = 4 + 64(0) - 68(0) = 4\ \text{A},$$

$$i_2(\infty) = 1 - 52(0) + 51(0) = 1\ \text{A}.$$

Finally, we check the solutions to see if they satisfy the differential equations derived in (a). We will leave this final check to the reader via Problem 6.36.

EXAMPLE 6.9 Calculating the Coupling Coefficient and Stored Energy for Magnetically Coupled Coils

The mutual inductance and self-inductances of the coils in Fig. 6.34 are $M = 40$ mH, $L_1 = 25$ mH, and $L_2 = 100$ mH.

a) Calculate the coupling coefficient.

b) Calculate the energy stored in the coupled coils when $i_1 = 10$ A and $i_2 = 15$ A.

c) If the coupling coefficient is increased to 1 and $i_1 = 10$ A, what value of i_2 results in zero stored energy?

Solution

a) $$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.04}{\sqrt{(0.025)(0.1)}} = 0.8.$$

b) $$w = \frac{1}{2}(0.025)(10)^2 + \frac{1}{2}(0.1)(15)^2 + (0.04)(10)(15) = 18.5 \text{ J}.$$

c) When $k = 1, M = \sqrt{(0.025)(0.1)} = 0.05 = 50$ mH. The energy in the coils is now

$$\frac{1}{2}(0.025)(10)^2 + \frac{1}{2}(0.1)(i_2)^2 + (0.05)(10)(i_2) = 0$$

so i_2 must satisfy the quadratic equation

$$0.05 i_2^2 + 0.5 i_2 + 1.25 = 0.$$

Use the quadratic formula to find i_2:

$$i_2 = \frac{-0.5 \pm \sqrt{0.5^2 - 4(0.05)(1.25)}}{2(0.05)} = -5 \text{ A}.$$

You should verify that the energy is zero for this value of i_2, when $k = 1$.