INTEGRATION TECHNIQUES 3

Remark: $\sin^2 x + \cos^2 x = 1 \iff \boxed{1 - \sin^2 x = \cos^2 x}$

$\boxed{1 + \tan^2 x = \sec^2 x} \iff \boxed{\tan^2 x = \sec^2 x - 1}$

Caution: All substitutions will be made under following restrictions:

$\boxed{x = \sin t \quad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right)}$ $\boxed{x = \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)}$ $\boxed{x = \sec t \quad \left(0 \leq t < \frac{\pi}{2}\right)}$

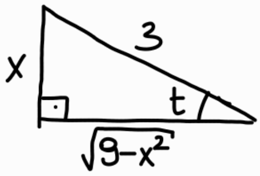
1) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = ?$

$x = 3\sin t$

$dx = 3\cos t\,dt$

$$I = \int \frac{\sqrt{9 - 9\sin^2 t}}{9\sin^2 t} \cdot 3\cos t\,dt = \int \frac{\cancel{3}\cos t \cdot \cancel{3}\cos t\,dt}{\cancel{9}\sin^2 t}$$

$$= \int \cot^2 t\,dt = \int (\cot^2 t + 1 - 1)\,dt = -\cot t - t + c$$



$$\Rightarrow I = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + c.$$

2) $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3\,dx}{(4x^2+9)^{3/2}} = ?$

$x = \frac{3}{2}\tan t$ $\qquad x = 0 \Rightarrow t = 0$

$dx = \frac{3}{2}\sec^2 t\,dt$ $\qquad x = \frac{3\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3}$

$$I = \int_0^{\pi/3} \frac{\frac{27}{8}\cdot\tan^3 t \cdot \frac{3}{2}\cdot\sec^2 t\,dt}{(9\tan^2 t + 9)^{3/2}}$$

$$= \int_0^{\pi/3} \frac{\frac{\cancel{27}}{8}\cdot\tan^3 t\cdot\frac{3}{2}\cdot\cancel{\sec^2 t}\,dt}{\cancel{27}\cdot\sec^{\cancel{3}} t} = \frac{3}{16}\int_0^{\pi/3} \frac{\overbrace{\tan^2 t}^{\sec^2 t - 1}\cdot\tan t}{\sec t}\,dt$$

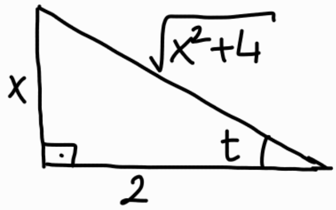
$$\frac{\tan t}{\sec t} = \frac{\sin t}{\cancel{\cos t}}\cdot\cancel{\cos t}$$

$$= \frac{3}{16}\int_0^{\pi/3}\left[\sec t\cdot\tan t - \frac{\tan t}{\sec t}\right]dt = \frac{3}{16}\Big[\sec t + \cos t\Big]_0^{\pi/3} = \frac{3}{16}\cdot\frac{1}{2} = \frac{3}{32}$$

③ $\int \frac{x\,dx}{\sqrt{x^2+4}} = ?$

$x = 2\tan t$
$dx = 2\sec^2 t\,dt$

$$I = \int \frac{2\tan t \cdot 2\sec^2 t\,dt}{\sqrt{4\tan^2 t+4}} = \int \frac{\cancel{2}\tan t \cdot 2\sec^{\cancel{2}} t\,dt}{\cancel{2}\,\cancel{\sec t}}$$

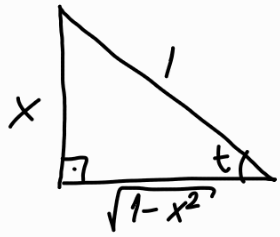


$$I = 2\sec t + c = \cancel{2} \cdot \frac{\sqrt{x^2+4}}{\cancel{2}} + c = \sqrt{x^2+4} + c.$$

④ $\int \frac{\sqrt{1-x^2}}{x^4}\,dx = ?$

$x = \sin t$
$dx = \cos t\,dt$

$$I = \int \frac{\sqrt{1-\sin^2 t} \cdot \cos t\,dt}{\sin^4 t} = \int \frac{\cos^2 t}{\sin^4 t}\,dt = \int \underbrace{\cot^2 t}_{u^2} \cdot \underbrace{\csc^2 t\,dt}_{-du}$$

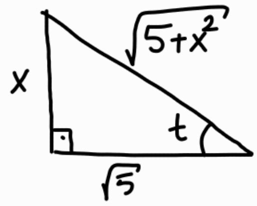


$$I = -\frac{\cot^3 t}{3} + c = -\frac{(1-x^2)^{3/2}}{3x^3} + c.$$

⑤ $\int \frac{dx}{(5+x^2)^{3/2}} = ?$

$\left.\begin{array}{l} x = \sqrt{5}\tan t \\ dx = \sqrt{5}\sec^2 t\,dt \end{array}\right\}$

$$I = \int \frac{\cancel{\sqrt{5}}\sec^2 t\,dt}{\underbrace{(5+5\tan^2 t)^{3/2}}_{5\cancel{\sqrt{5}}(1+\tan^2 t)^{3/2}}} = \int \frac{\cancel{\sec^2} t\,dt}{5\sec^{\cancel{3}} t} = \frac{1}{5}\int \cos t\,dt$$

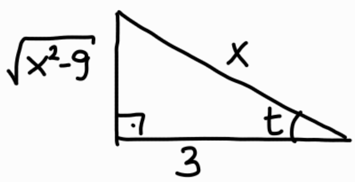


$$I = \frac{1}{5}\sin t + c = \frac{1}{5} \cdot \frac{x}{\sqrt{5+x^2}} + c.$$

⑥ $\int \frac{dx}{x^2\sqrt{x^2-9}} = ?$

$x = 3\sec t$
$dx = 3\sec t \cdot \tan t\,dt$

$$I = \int \frac{3 \cdot \cancel{\sec t} \cdot \tan t\,dt}{9\sec^{\cancel{2}} t\,\sqrt{9\sec^2 t - 9}} = \int \frac{\cancel{3\tan t}\,dt}{9 \cdot \sec t \cdot \cancel{3 \cdot \tan t}}$$



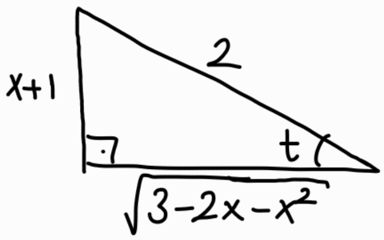
$$I = \frac{1}{9}\int \cos t\,dt = \frac{1}{9}\sin t + c = \frac{1}{9}\,\frac{\sqrt{x^2-9}}{x} + c$$

$$⑦ \int \frac{x\,dx}{\sqrt{3-2x-x^2}} = ? \qquad 3-2x-x^2 = 4-(x+1)^2$$

$x+1 = 2\sin t$

$dx = 2\cos t\,dt$

$$I = \int \frac{(2\sin t-1).2\cos t\,dt}{\sqrt{4-4\sin^2 t}} = \int \frac{(2\sin t-1).2\cancel{\cos t}\,dt}{2\cancel{\cos t}}$$



$$\Rightarrow I = -2\cos t - t + C$$

$$= -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C.$$

$$⑧ \int x^3\sqrt{9-x^2}\,dx = ?$$

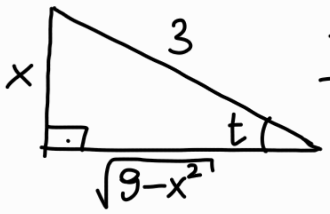
$x = 3\sin t$

$dx = 3\cos t\,dt$

$$I = \int 27\underbrace{\sin^3 t}_{\sin^2 t.\sin t = (1-\cos^2 t).\sin t}\sqrt{9-9\sin^2 t}\,.3\cos t\,dt$$

$$I = \int 243.(1-\cos^2 t).\cos^2 t.\sin t\,dt \qquad \begin{aligned}\cos t &= u\\ -\sin t\,dt &= du\end{aligned}$$

$$I = 243\int -(1-u^2).u^2.du = 243\int (u^4-u^2)du = 243\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C$$



$$I = 243\left(\frac{\cos^5 t}{5}-\frac{\cos^3 t}{3}\right)+C = 243\left(\frac{(9-x^2)^{5/2}}{243.5}-\frac{(9-x^2)^{3/2}}{81}\right)+C$$

$$\Rightarrow I = \frac{(9-x^2)^{5/2}}{5} - 3(9-x^2)^{3/2} + C.$$

$$⑨ \int \frac{x^3\,dx}{\sqrt{9-25x^2}} = ?$$

$x = \frac{3}{5}\sin t$

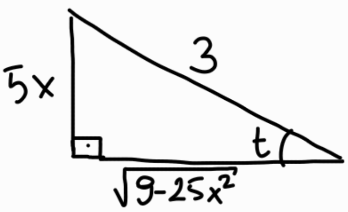
$dx = \frac{3}{5}\cos t\,dt$

$$I = \int \frac{\frac{27}{125}\sin^3 t.\frac{3}{5}\cos t\,dt}{\sqrt{9-9\sin^2 t}} = \int \frac{\frac{27}{125}\overbrace{\sin^3 t}^{\sin^2 t.\sin t = (1-\cos^2 t)\sin t}.\frac{3}{5}\cancel{\cos t}\,dt}{3.\cancel{\cos t}}$$

$\cos t = u$

$-\sin t\,dt = du$

$$I = \frac{27}{625}\int (u^2-1)\,du = \frac{27}{625}\left[\frac{u^3}{3}-u\right]+C$$



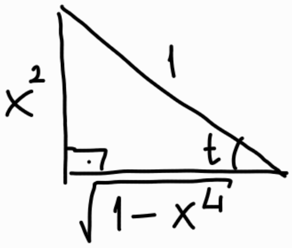
$$I = \frac{27}{625}\left[\frac{\cos^3 t}{3}-\cos t\right]+C = \frac{27}{625}\left[\frac{(9-25x^2)^{3/2}}{81}-\frac{\sqrt{9-25x^2}}{3}\right]+C$$

(10) $\int x\sqrt{1-x^4}\,dx = ?$

$x^2 = \sin t$

$2x\,dx = \cos t\,dt$

$$I = \int \frac{1}{2}\sqrt{1-\sin^2 t}\,.\cos t\,dt = \frac{1}{2}\int \underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\,dt$$



$$I = \frac{1}{4}\left[t + \frac{\sin 2t}{2}\right] + c = \frac{1}{4}\left[t + \sin t.\cos t\right] + c$$

$$= \frac{1}{4}\left[\arcsin(x^2) + x^2\sqrt{1-x^4}\right] + c.$$

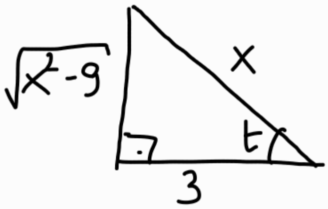
(11) $\int \frac{\sqrt{x^2-9}}{x^3}\,dx = ?$

$x = 3\sec t$

$dx = 3\sec t\tan t\,dt$

$$I = \int \frac{\sqrt{9\sec^2 t - 9}}{27\sec^{\cancel{3}\,2} t}\,.3.\cancel{\sec t}.\tan t\,dt$$

$$I = \int \frac{\cancel{3}.\tan t.\cancel{3}.\tan t\,dt}{{}^{3}\cancel{27}\sec^2 t} = \frac{1}{3}\int \sin^2 t\,dt = \frac{1}{6}\int (1-\cos 2t)\,dt$$



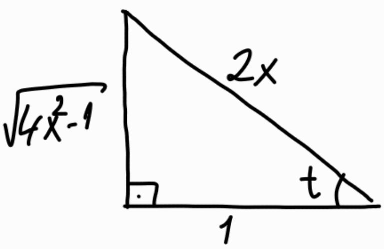
$$I = \frac{1}{6}\Big(t - \underbrace{\frac{\sin 2t}{2}}_{\sin t.\cos t}\Big) + c = \underbrace{\frac{1}{6}\operatorname{arcsec}\left(\frac{x}{3}\right)}_{\text{OR } \arccos\left(\frac{3}{x}\right)} - \frac{1}{2}.\frac{\sqrt{x^2-9}}{x^2} + c$$

(12) $\int \frac{dx}{x^3\sqrt{4x^2-1}} = ?$

$x = \frac{1}{2}\sec t$

$dx = \frac{1}{2}\sec t\tan t\,dt$

$$I = \int \frac{\cancel{\tfrac{1}{2}}\,\cancel{\sec t}.\cancel{\tan t}\,dt}{{}_{4}\tfrac{1}{\cancel{8}}\sec^{\cancel{3}\,2} t.\underbrace{\cancel{\sqrt{\sec^2 t-1}}}_{\tan^2 t}} = \int \frac{4\,dt}{\sec^2 t} = 4\int \cos^2 t\,dt$$



$$I = 2\int (1+\cos(2t))\,dt = 2t + \underbrace{\sin(2t)}_{2.\sin t.\cos t} + c$$

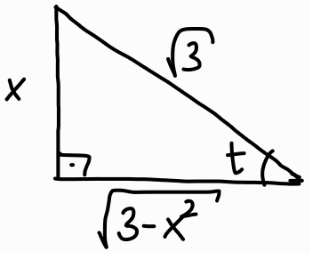
$$= 2\operatorname{arcsec}(2x) + 2.\frac{\sqrt{4x^2-1}}{4x^2} + c$$

Remark: $\operatorname{arcsec}(2x) = \arccos\left(\frac{1}{2x}\right) = \arcsin\left(\frac{\sqrt{4x^2-1}}{2x}\right) = \arctan\sqrt{4x^2-1}$

13 $\displaystyle\int \frac{x^2\,dx}{(3-x^2)^{3/2}} = ?$

$x=\sqrt{3}\sin t$

$dx=\sqrt{3}\cos t\,dt$



$$I=\int \frac{\cancel{3}\sin^2 t\cdot\cancel{\sqrt{3}}\cos t\,dt}{\cancel{3\sqrt{3}}\,\underbrace{(1-\sin^2 t)}_{\cos^2 t}{}^{3/2}} = \int \frac{\sin^2 t\cdot\cancel{\cos t}}{\cos^{\cancel{3}\,2} t}\,dt = \int \tan^2 t\,dt$$

$$I=\int(\tan^2 t+1-1)\,dt = \tan t - t + c$$

$$= \frac{x}{\sqrt{3-x^2}} - \arcsin\left(\frac{x}{\sqrt{3}}\right) + c.$$

14 $\displaystyle\int_0^{1/2} \frac{x^2\,dx}{\sqrt{(1-x^2)^5}} = ?$

$x=\sin t$ $\quad x=0 \Rightarrow t=0$

$dx=\cos t\,dt$ $\quad x=\frac{1}{2} \Rightarrow t=\frac{\pi}{6}$

$$I=\int_0^{\pi/6} \frac{\sin^2 t\cdot\cos t\,dt}{\sqrt{(\underbrace{1-\sin^2 t}_{\cos^2 t})^5}} = \int_0^{\pi/6}\frac{\sin^2 t\cos t\,dt}{\cos^5 t}$$

$$\Rightarrow I=\int_0^{\pi/6}\tan^2 t\cdot\sec^2 t\,dt = \frac{\tan^3 t}{3}\Bigg|_0^{\pi/6} = \frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{1}{9\sqrt{3}} = \frac{\sqrt{3}}{27}$$

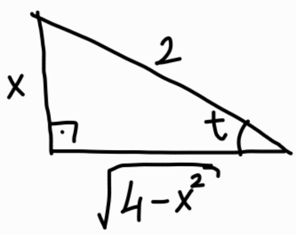
$(\tan t = u \Rightarrow \sec^2 t\,dt = du)$

15 $\displaystyle\int \frac{x^2+x}{\sqrt{1-\frac{x^2}{4}}}\,dx = ?$

$x=2\sin t$

$dx=2\cos t\,dt$

$$I=\int\frac{4\sin^2 t+2\sin t}{\underbrace{\sqrt{1-\sin^2 t}}_{\cancel{\sqrt{\cos^2 t}}}}\,2\cdot\cancel{\cos t}\,dt$$



$$I=\int\left[4\cdot(1+\cos 2t)+4\sin t\right]dt = 4t+2\underbrace{\sin 2t}_{2\sin t\cdot\cos t}-4\cos t+c$$

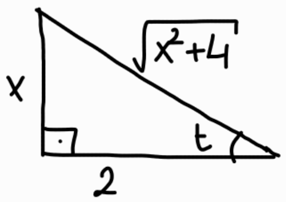
$$=4\arcsin\left(\frac{x}{2}\right)+\cancel{4}\cdot\frac{x\sqrt{4-x^2}}{\cancel{4}}-\overset{2}{\cancel{4}}\,\frac{\sqrt{4-x^2}}{\cancel{2}}+c.$$

(16) $\int \frac{dx}{x^2\sqrt{x^2+4}} = ?$

$x = 2\tan t$
$dx = 2\sec^2 t\,dt$

$$I = \int \frac{2\sec^2 t\,dt}{4\tan^2 t\sqrt{4\tan^2 t+4}} = \int \frac{\cancel{2}\sec^{\cancel{2}} t\,dt}{4\tan^2 t\cdot\cancel{2}\cdot\cancel{\sec} t}$$

$$= \frac{1}{4}\int \frac{\cos^{\cancel{2}} t}{\cancel{\cos} t\cdot\sin^2 t}dt = \frac{1}{4}\int \cot t\cdot\csc t\,dt = -\frac{1}{4}\csc t + c$$

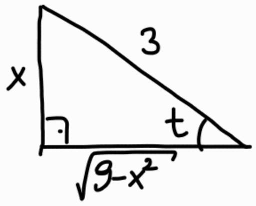


$$\Longrightarrow I = -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + c.$$

(17) $\int \frac{x^2+1}{(9-x^2)^{3/2}}dx = ?$

$x = 3\sin t$
$dx = 3\cos t\,dt$

$$I = \int \frac{9\sin^2 t+1}{\underbrace{(9-9\sin^2 t)}_{9(1-\sin^2 t)=9\cos^2 t}^{3/2}} 3\cos t\,dt = \int \frac{(9\sin^2 t+1).\cancel{3}\cancel{\cos} t\,dt}{\underset{9}{\cancel{27}}\cos^{\overset{2}{\cancel{3}}} t}$$

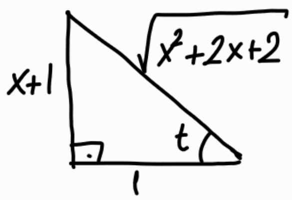


$$I = \int\left[\underbrace{\tan^2 t}_{\tan^2 t+1-1} + \frac{\sec^2 t}{9}\right]dt = \tan t - t + \frac{\tan t}{9} + c = \frac{10\tan t}{9} - t + c$$

$$= \frac{10}{9}\frac{x}{\sqrt{9-x^2}} - \arcsin\left(\frac{x}{3}\right) + c.$$

(18) $\int \frac{x\,dx}{\sqrt{x^2+2x+2}} = ?$

$x^2+2x+2 = (x+1)^2+1$ $\qquad I = \int \frac{x\,dx}{\sqrt{(x+1)^2+1}}$ $\qquad x+1 = \tan t$, $dx = \sec^2 t\,dt$



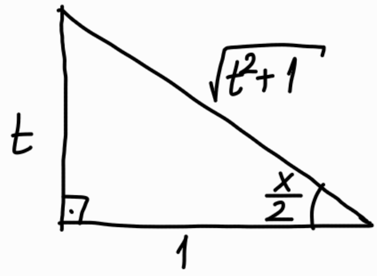
$$I = \int \frac{(\tan t-1).\sec^{\cancel{2}} t\,dt}{\underbrace{\cancel{\sqrt{\tan^2 t+1}}}_{\sec^2 t}} = \int(\tan t.\sec t - \sec t)dt = \sec t - \ln|\sec t+\tan t| + c$$

$$= \sqrt{x^2+2x+2} - \ln\left|\sqrt{x^2+2x+2} + x+1\right| + c.$$

Remark: Let $\boxed{\tan\frac{x}{2}=t}$

$$\frac{1}{2}\cdot\sec^2\frac{x}{2}\,dx=dt$$

$$\boxed{dx=\frac{2}{1+t^2}dt}$$



$$\sin x=2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}\Rightarrow\boxed{\sin x=\frac{2t}{1+t^2}}\qquad \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\Rightarrow\boxed{\cos x=\frac{1-t^2}{1+t^2}}$$

19 $\displaystyle\int\frac{dx}{1+\cos x}=?$

$$\tan\frac{x}{2}=t\ ,\quad dx=\frac{2dt}{1+t^2}\ ,\quad \sin x=\frac{2t}{1+t^2}\ ,\quad \cos x=\frac{1-t^2}{1+t^2}$$

$$I=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}=\int\frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1+\cancel{t^2}+1-\cancel{t^2}}{\cancel{1+t^2}}}=\int\frac{\cancel{2}dt}{\cancel{2}}=t+c=\tan\frac{x}{2}+c.$$

20 $\displaystyle\int\frac{dx}{1-\sin x+\cos x}=?$

$$\tan\frac{x}{2}=t\ ,\quad dx=\frac{2dt}{1+t^2}\ ,\quad \sin x=\frac{2t}{1+t^2}\ ,\quad \cos x=\frac{1-t^2}{1+t^2}$$

$$I=\int\frac{\frac{2dt}{1+t^2}}{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1+\cancel{t^2}-2t+1-\cancel{t^2}}{\cancel{1+t^2}}}=\int\frac{\cancel{2}dt}{\cancel{2}(1-t)}=\ln|1-t|+c$$

$$=\ln\left|1-\tan\frac{x}{2}\right|+c$$

21 $\displaystyle\int\frac{dx}{1+\sin x}=?$

$$\tan\frac{x}{2}=t\ ,\quad dx=\frac{2dt}{1+t^2}\ ,\quad \sin x=\frac{2t}{1+t^2}\ ,\quad \cos x=\frac{1-t^2}{1+t^2}$$

$$I=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}=\int\frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1+2t+t^2}{\cancel{1+t^2}}}=\int\frac{2dt}{(t+1)^2}=\frac{-2}{t+1}+c=\frac{-2}{\tan\frac{x}{2}+1}+c$$