Limits at Infinity and Infinite Limits

Definition: We say that $f(x)$ has the limit L as x approaches infinity and write (minus)

$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ get arbitrarily close to L
(negative)

$\mathbb{R} = (-\infty, \infty)$

Ex.: $\displaystyle\lim_{x\to\mp\infty} \frac{2x^2 - x + 3}{3x^2+5}$

$$\lim_{x\to\mp\infty} \frac{\cancel{x^2}\left(2 + \frac{-1}{x} + \frac{3}{x^2}\right)}{\cancel{x^2}\left(3+\frac{5}{x^2}\right)} = \frac{2-0+0}{3+0} = \frac{2}{3}$$

Ex.: $\displaystyle\lim_{x\to\mp\infty} \frac{5x^1+2}{2x^3-1}$

$$\lim_{x\to\mp\infty} \frac{\cancel{x^3}\left(\frac{5}{x^2} + \frac{2}{x^3}\right)}{\cancel{x^3}\left(2-\frac{1}{x^3}\right)} = \frac{0+0}{2-0} = \frac{0}{2} = 0$$

Ex.: $\displaystyle\lim_{x\to\infty} \frac{x^3+1}{x^2+1}$

$$\lim_{x\to\infty} \frac{\cancel{x^3}\left(1+\frac{1}{x^3}\right)}{\cancel{x^3}\left(\underline{\frac{1}{x}} + \underline{\frac{1}{x^3}}\right)} = \lim_{x\to\infty} \frac{1+0}{0+0} = \infty$$

$$\lim_{x\to-\infty} \frac{x^3+1}{x^2+1} = -\infty$$

Rule: Let $P_m(x) = a_m x^m + \dots + a_0$ and

$Q_n(x) = b_n x^n + \dots + b_0$

be polynomials of degree m and n respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x \to \mp\infty} \frac{P_m(x)}{Q_n(x)}$$

1) equals 0 if $m < n$,
2) equals $\frac{a_m}{b_n}$ if $m = n$,
3) does not exist if $m > n$. (We find ∞ or $-\infty$)

Ex.: $\lim\limits_{x \to \infty} \left(\sqrt{x^2+x} - x\right) = \cancel{(\infty - \infty)}$

$$(\sqrt{x^2+x} + x)$$

$$\lim_{x \to \infty} \frac{(\sqrt{x^2+x} - x) \cdot (\sqrt{x^2+x} + x)}{(\sqrt{x^2+x} + x)} = \lim_{x \to \infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{(\sqrt{x^2+x} + x)}$$

$$\underline{x}\sqrt{1+\frac{1}{x}} + x$$

$$\lim_{x \to \underline{\infty}} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x}} + 1\right)} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

$$\sqrt{x^2 \cdot \left(1+\frac{1}{x}\right)} = \underline{|x|} \cdot \sqrt{1+\frac{1}{x}}$$

$$\lim_{x \to -\infty} \frac{\cancel{x}}{\underline{-}\cancel{x} \cdot \left(\sqrt{1+\frac{1}{x}} + 1\right)} = -\frac{1}{2}$$

Here $\sqrt{x^2} = |x| = x$ because $x > 0$ as $x \to \infty$

and $\sqrt{x^2} = |x| = -x$ if $x < 0$ as $x \to -\infty$.

Examples

1) $\lim\limits_{x\to 2^+} \frac{x-3}{x^2-4} = \lim\limits_{x\to 2^+} \frac{\overset{(-)}{x-3}}{\underbrace{\underbrace{(x-2)}_{0}}_{(+)} \cdot \underbrace{(x+2)}_{(+)}} = -\infty$

2) $\lim\limits_{x\to 2^-} \frac{x-3}{x^2-4} = \lim\limits_{x\to 2^-} \frac{\overset{(-)}{x-3}}{\underbrace{(x-2)}_{(-)} \cdot \underbrace{(x+2)}_{(+)}} = \infty$

3) $\lim\limits_{x\to 2} \frac{x-3}{x^2-4}$ does not exist

4) $\lim\limits_{x\to 2} \frac{2-x}{(x-2)^3} = \lim\limits_{x\to 2} \frac{-\cancel{(x-2)}}{(x-2)^{\cancel{3}\,2}} = \lim\limits_{x\to 2} \frac{\overset{(-)}{-1}}{\underbrace{(x-2)^2}_{(+)}} = -\infty$

$\lim\limits_{x\to 2^+} \frac{-\cancel{(x-2)}}{(x-2)^{\cancel{3}\,2}} = \lim\limits_{x\to 2^+} \frac{-1}{\underbrace{(x-2)^2}_{0^+}} = \frac{-1}{0^+} = -1 \cdot \infty = -\infty$

$\lim\limits_{x\to 2^-} \frac{-1}{\underbrace{(x-2)^2}_{0^-}} = -1 \cdot \infty = -\infty$

$(-) \cdot (-)$

$\lim\limits_{x\to 2^-} \frac{-1}{\underbrace{(x-2)}_{0^-} \cdot \underbrace{(x-2)}_{0^-}} = \frac{-1}{\underbrace{(0^-)}_{(-)} \cdot \underbrace{(0^-)}_{(-)}} = -1 \cdot (-\infty) \cdot (-\infty)$

$= -\infty$

$\infty^2 = \infty$

Limits Involving $\frac{\sin\theta}{\theta}$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad (\theta \text{ in radians})$$

$\sin\theta \to 0$, $\theta \to 0$

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\underbrace{\frac{\sin\theta}{\theta}}_{1}\cdot\underbrace{\frac{1}{\cos\theta}}_{1} = 1 \quad (\theta \text{ in radians})$$

$$(1)\ \lim_{\theta\to 0}\frac{\sin a\theta}{b\theta} = \lim_{\theta\to 0}\frac{a\theta}{\sin b\theta} = \lim_{\theta\to 0}\frac{\sin a\theta}{\sin b\theta} = \frac{a}{b}$$

$$(2)\ \lim_{\theta\to 0}\frac{\tan a\theta}{b\theta} = \lim_{\theta\to 0}\frac{a\theta}{\tan b\theta} = \lim_{\theta\to 0}\frac{\tan a\theta}{\tan b\theta} = \frac{a}{b}$$

$$(3)\ \lim_{\theta\to 0}\frac{\tan a\theta}{\sin a\theta} = \lim_{\theta\to 0}\frac{\sin a\theta}{\tan a\theta} = \frac{a}{b}$$

Ex.: $\lim_{x\to 0}\frac{1-\cos x}{x}$

$\frac{\sin a\theta}{\theta}$ $\quad a = \frac{1}{2}$ $\quad \frac{x}{2} = \frac{1}{2}\cdot x$

instead of x write $\frac{x}{2}$

$$\sin^2 x = \frac{1-\cos 2x}{2} \rightarrow 1-\cos 2x = 2\sin^2 x \Rightarrow 1-\cos x = 2\sin^2\frac{x}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0}2\cdot\underbrace{\frac{\sin\frac{x}{2}}{x}}_{\frac{1}{2}}\cdot\sin\frac{x}{2} = 2\cdot\frac{1}{2}\cdot 0 = 0$$

$\sin^2 x + \cos^2 x = 1$

$$\lim_{x\to 0}\frac{1-\cos x}{x}\,(1+\cos x) = \lim_{x\to 0}\frac{(1-\cos x)\cdot(1+\cos x)}{x\cdot(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x\underbrace{(1+\cos x)}_{1+1=2}}$$

$$\frac{1}{2}\cdot\lim_{x\to 0}\frac{\sin^2 x}{x} = \frac{1}{2}\lim_{x\to 0}\underbrace{\frac{\sin x}{x}}_{1}\cdot\underbrace{\sin x}_{0} = \frac{1}{2}\cdot 1\cdot 0 = 0$$

$$\left(\lim f(x)\cdot\lim g(x) = \lim_{x\to a}\big(f(x)\cdot g(x)\big)\right)$$

Ex.: $\lim_{x\to 0} \frac{5\sin 3x + \tan 7x}{3x+x^2}$ $\left(\frac{1}{x}\right)$ $= \lim_{x\to 0} \frac{\frac{5.\sin 3x}{x} + \frac{\tan 7x}{x}}{3+x} = \frac{5.3+7}{3+0}$

$= \frac{22}{3}$

Continuity

Definition: Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

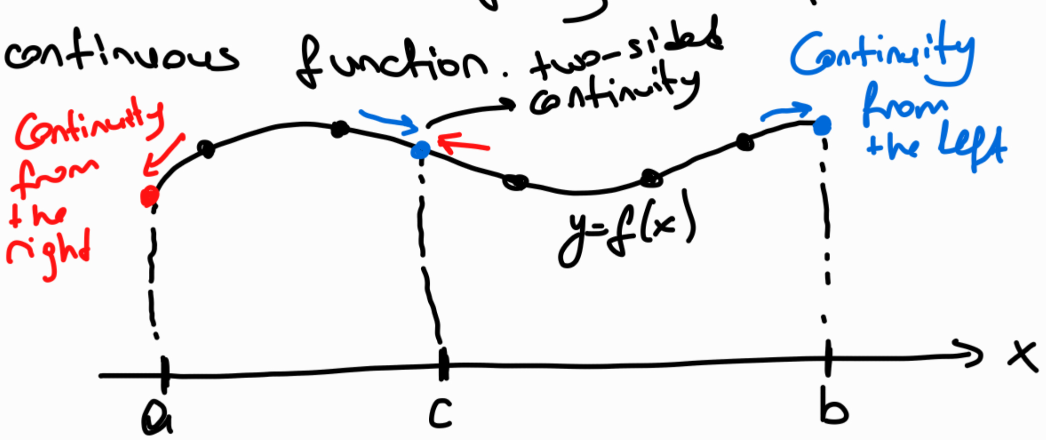
The function f is continuous at c if

$$\lim_{x\to c} f(x) = f(c).$$

The function f is (left) right-continuous at c (or continuous from the (left) right) if

$$\lim_{x\to c^+} f(x) = f(c). \qquad \lim_{x\to c^-} f(x) = f(c)$$

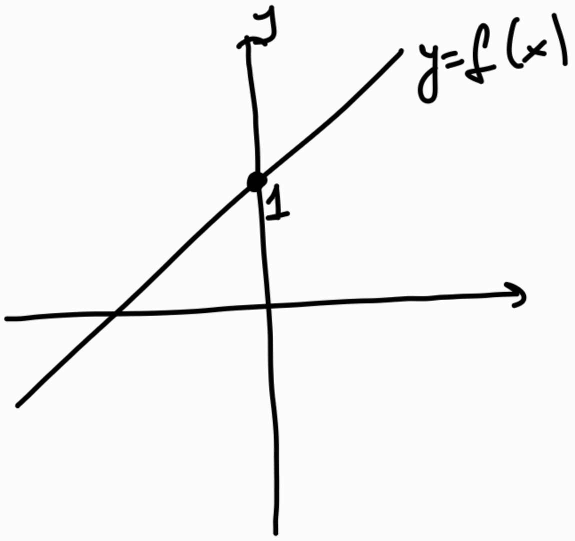
Definition (Informal): Any function whose graph can be sketched over its domain in one continuous motion, that is, without lifting the pen, is an example of a continuous function.



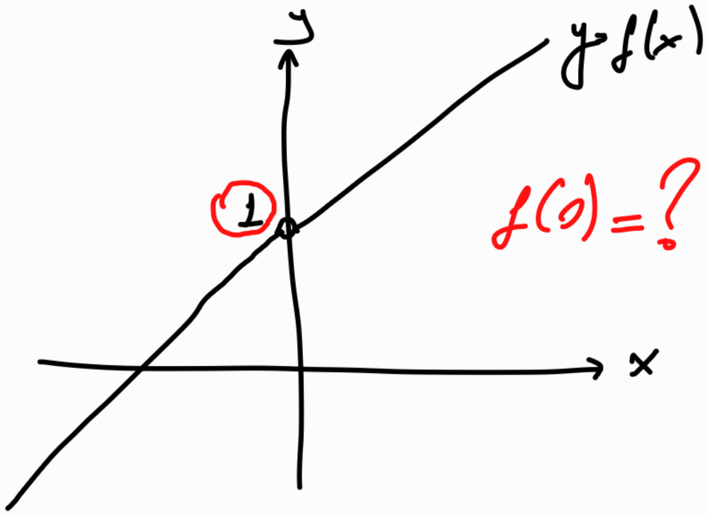
Continuity Test

A function $f(x)$ is continuous at a point $x=c$ iff it meets the following three conditions.

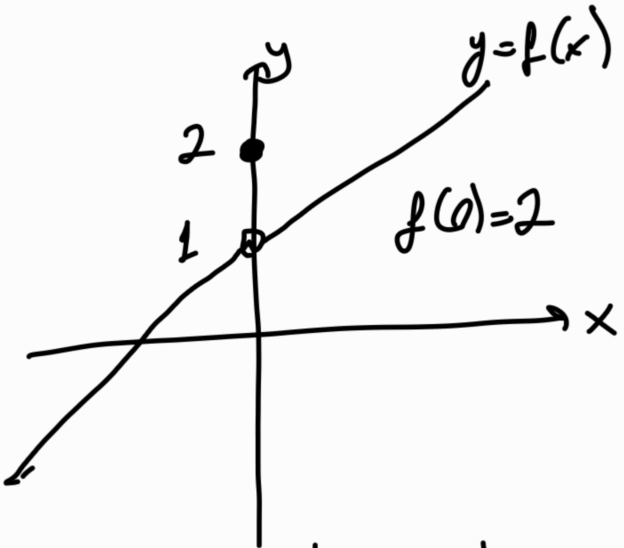
1) $f(c)$ exists (x lies in the domain of f)
2) $\lim\limits_{x\to c} f(x)$ exists (f has a limit as $x\to c$)
3) $\lim\limits_{x\to c} f(x) = f(c)$ (the limit equals the function value)



continuous function



removable discontinuity



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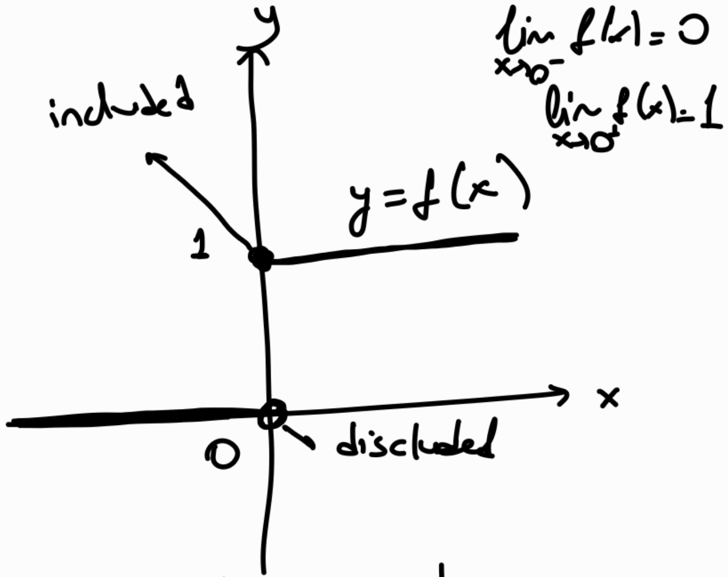
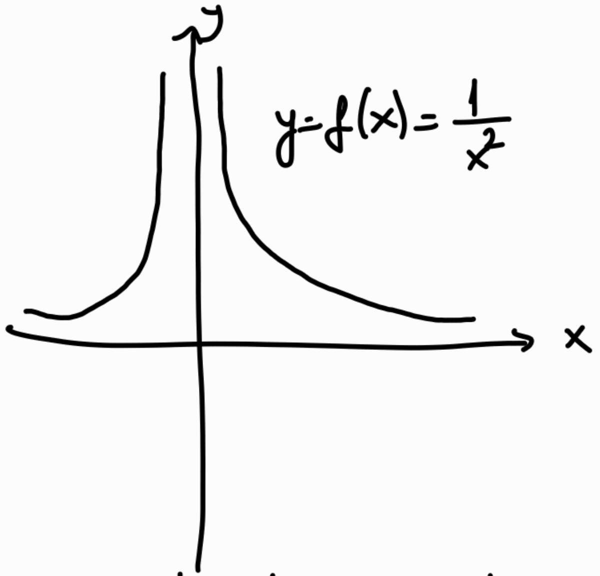
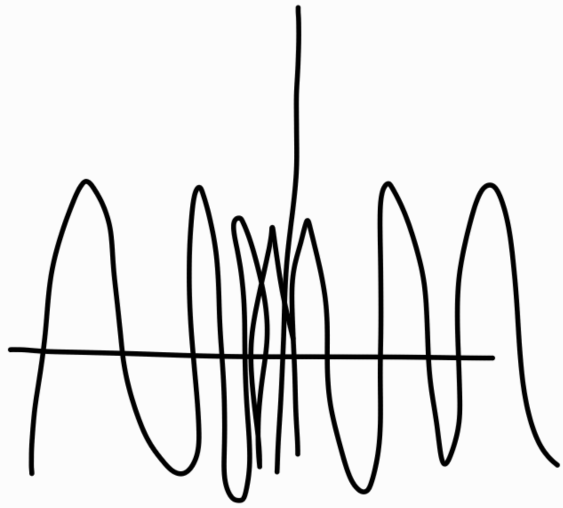


jump discontinuity



infinite discontinuity

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$



oscillating discontinuity

Theorem (Properties of Continuous Function)

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at c.

1) Sums: $f+g$
2) Differences: $f-g$
3) Constant multiples: $k \cdot f$, for any number k
4) Products: $f \cdot g$
5) Quotients: $\frac{f}{g}$, provided $g(c) \neq 0$
6) Powers: f^n, n a positive integer
7) Roots: $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer.

Theorem (Compositions of Continuous Functions)

If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Example: Find the points of discontinuity of the following.

1) $f(x) = \dfrac{1}{3-|x|}$ $\quad D(f) = \mathbb{R} \setminus \{-3,3\}$ (f is cont.)

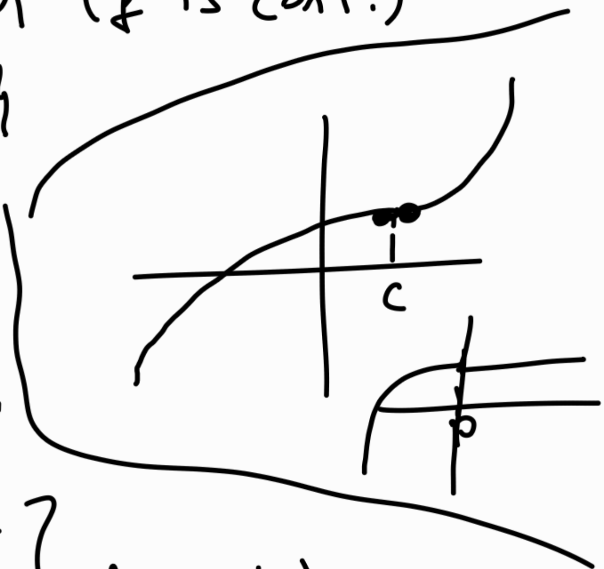
Discontinuous on $\{-3,3\}$

2) $f(x) = 2\ln\sqrt{1-x}$ $\quad D(f) = (-\infty,1)$ (f is cont.)

Discontinuous on $[1,\infty)$

3) $f(x) = 5 + 2^{1/x}$ $\quad D(f) = \mathbb{R} \setminus \{0\}$ (f is cont.)

Disc. on. $\{0\}$



4) $f(x) = \begin{cases} 2x-3 & \text{if } x \leq 4 \\ 1+\dfrac{16}{x} & \text{if } x > 4 \end{cases}$

a) $f(4) = 2\cdot 4 - 3 = 5$ is defined

b) $\left.\begin{array}{l} \lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+}\left(1+\dfrac{16}{x}\right) = 5 \\ \lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-}(2x-3) = 5 \end{array}\right\}$ $\lim\limits_{x\to 4} f(x) = 5$

c) $f(4) = \lim\limits_{x\to 4} f(x) = 5$ $\quad \checkmark$ $\quad f(x)$ is cont.

Theorem: If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(b).$$

In other words, $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right).$

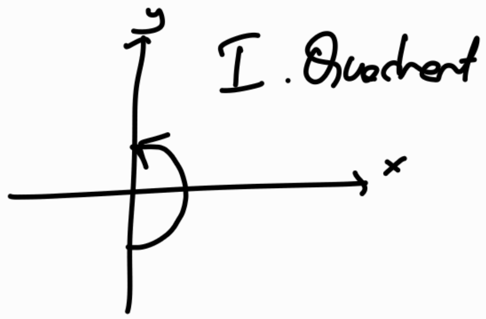
Ex.: $\lim\limits_{x\to 1} \sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right) = \lim\limits_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

arcsin is continuous

$$\arcsin\left(\lim_{x\to 1}\left(\frac{1-\sqrt{x}}{1-x}\right)\right) = \arcsin\left(\lim_{x\to 1}\frac{\cancel{1-x}}{(\cancel{1-x})(1+\sqrt{x})}\right)$$

$(1+\sqrt{x})$

$$\arcsin\left(\frac{1}{2}\right) = \underline{\frac{\pi}{6}} \qquad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$$

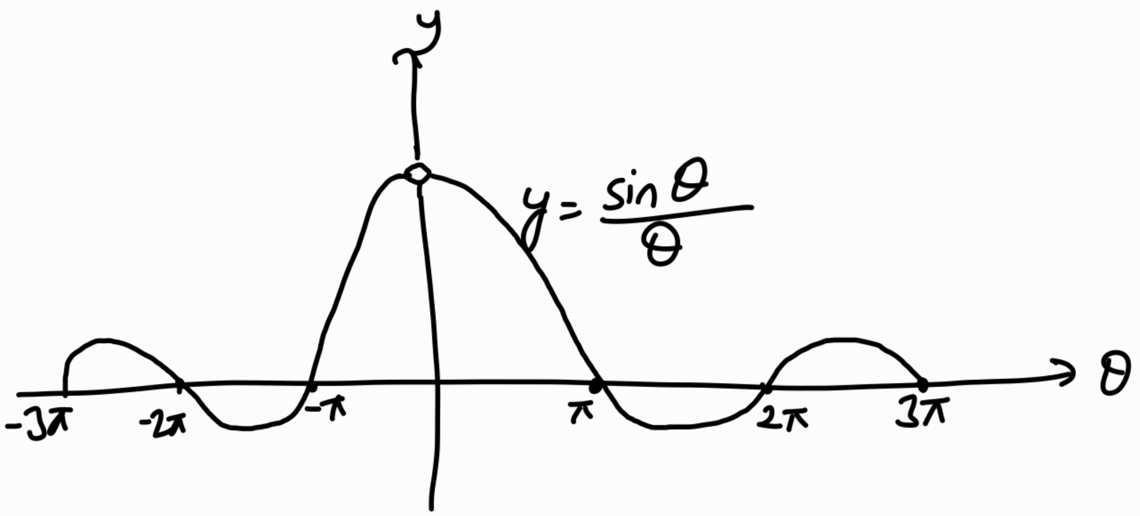


Definition: If $\lim\limits_{x\to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases}$$

which is continuous at c. It is called the continuous extension of $f(x)$ to c.

Ex.:



$y = \frac{\sin\theta}{\theta}$ is defined and continuous for all $x \neq 0$.

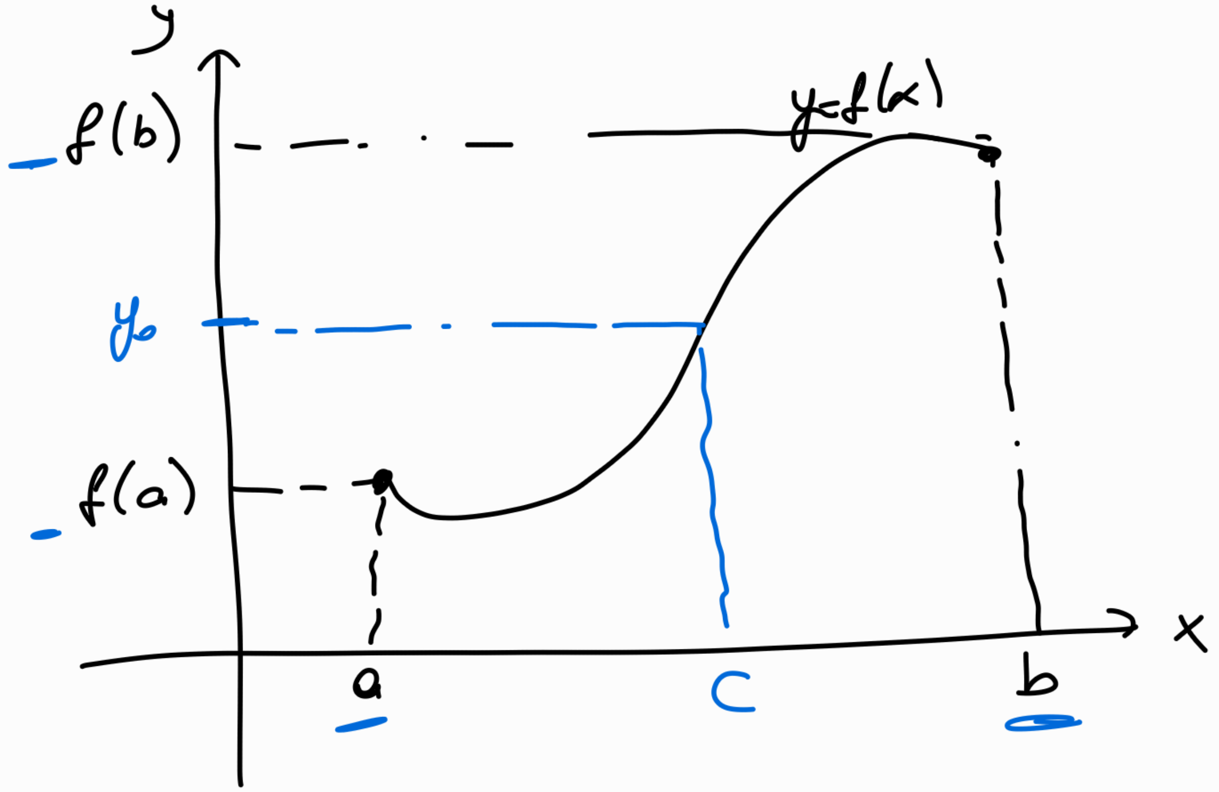
As $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, it makes sense to define a new function

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

continuous at everywhere ($\mathbb{R}$)

Theorem (The Intermediate Value Theorem for Cont. Func.)

If f is a continuous function on a closed interval $[a,b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a,b]$



Ex.: Show that there is a root of the equation

$x^3 - x - 1 = 0$ between 1 and 2.

$[a,b] = [1,2]$ $\quad y_0 = 0$ $\quad y(x) = x^3 - x - 1$

(+) ↑ 0 ↓ (−)

$y(1) = 1 - 1 - 1 = -1 < 0$ $\qquad -1 \leq 0 \leq 5$

$y(2) = 8 - 2 - 1 = 5 > 0$ $\qquad c \in [1,2] \Rightarrow f(c) = 0$

Since the function is cont. from the I.V.T says there is a zero (root) of between 1 and 2

That root is $x = 1.32$