1) Let $\{a_n\}$ be a sequence with positive terms, $\lim_{n\to\infty} a_n = a$ and $\dfrac{1+a_{n+1}}{2+a_n} = \dfrac{n+3}{2n+1}\cdot a_{n-1}$. What is the value of a?

A) $\sqrt{2}$ B) $\sqrt{3}$ C) 1 D) $\sqrt{6}$ E) 2

$$\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} a_{n+1}}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} a_n} = \lim_{n\to\infty} \frac{n+3}{2n+1} \cdot \lim_{n\to\infty} a_{n-1} \Rightarrow$$

$$\frac{\cancel{n}(1+3\cdot\cancel{\tfrac{1}{n}}^{\,0})}{\cancel{n}(2+\cancel{\tfrac{1}{n}}^{\,0})} = \frac{1}{2}$$

$$\frac{1+a}{2+a} \times \frac{1}{2}\cdot a \Rightarrow$$

$$2+2a = 2a+a^2 \Rightarrow a^2 = 2 \Rightarrow a = -\sqrt{2} \text{ or } \sqrt{2}$$

- $\{a_n\}$ is a sequence with positive terms so $\lim_{n\to\infty} a_n = a$ must be positive. Hence, $a = \sqrt{2}$.

2. Which of the following is true about the sequence $\{a_n\} = \{n - \sqrt{n^2-1}\}$?

A) $\{a_n\}$ is decreasing

B) $\{a_n\}$ is increasing

C) $\lim\limits_{n\to\infty} a_n = -1$

D) $\{a_n\}$ is not monotonic

E) $\{a_n\}$ is divergent

$$a_n = \left(n - \sqrt{n^2-1}\right) \cdot 1 = \left(n - \sqrt{n^2-1}\right) \frac{\left(n + \sqrt{n^2-1}\right)}{n + \sqrt{n^2-1}} \Rightarrow$$

$$a_n = \frac{n-(n^2-1)}{n+\sqrt{n^2-1}} = \frac{1}{n+\sqrt{n^2-1}} > \frac{1}{(n+1)+\sqrt{(n+1)^2-1}} = a_{n+1}$$

so $a_n > a_{n+1}$ for all n. Hence, $\{a_n\}$ is decreasing.

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3. Which of the following is true for the sequences $a_n = \frac{n^n}{n!}$ and $b_n = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2\cdot4\cdot6\cdots\cdots(2n)}$?

a) a_n is increasing, b_n is decreasing
b) a_n is decreasing, b_n is increasing
c) Both of them are decreasing
d) Both of them are increasing
e) Both of them are neither decreasing nor increasing

$$\frac{a_n}{a_{n+1}} = \frac{n^n / n!}{(n+1)^{(n+1)} / (n+1)!} = \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \frac{n^n}{n!} \cdot \frac{(n+1) \cdot n!}{(n+1)(n+1)^n}$$

$$= \frac{n^n}{(n+1)^n} < \frac{n^n}{n^n} = 1 \text{ so}$$

$\frac{a_n}{a_{n+1}} < 1$ for all n, i.e., $a_n < a_{n+1}$ for all n.

Hence, a_n is increasing.

$$\frac{b_n}{b_{n+1}} = \frac{1.3.5 \ldots (2n-1) / 2.4.6 \ldots (2n)}{1.3.5 \ldots (2n-1)(2n+1) / 2.4.6 \ldots (2n)(2n+2)} =$$

$$\frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots (2n)} \cdot \frac{2.4.6 \ldots (2n)(2n+2)}{1.3.5 \ldots (2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} > \frac{2n+1}{2n+1} = 1$$ so $\frac{b_n}{b_{n+1}} > 1$ for all n,

i.e., $b_n > b_{n+1}$ for all n. Hence, b_n is decreasing.

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7. Let $a_n = \int_{n-3}^{n+4} \left(\frac{x^2+5}{x^2+1}\right) dx$ for $n = 4, 5, 6, \cdots$ .Which of the following is equal to the limit of the sequence (a_n) ?

a) 1 b) 7 c) -7

d) -1 e) $7+\pi$

$$a_n = \int_{n-3}^{n+4} \left(\frac{x^2+5}{x^2+1}\right) dx = \int_{n-3}^{n+4} \left(\frac{x^2+1+4}{x^2+1}\right) dx = \int_{n-3}^{n+4} \left(1+4 \cdot \frac{1}{x^2+1}\right) dx$$

$$\frac{x^2+1+4}{x^2+1} \downarrow \quad \underbrace{\frac{x^2+1}{x^2+1}}_{=1} + 4 \cdot \frac{1}{x^2+1}$$

$$a_n = \left[x + 4 \cdot \arctan x\right] \Big|_{n-3}^{n+4}$$

$$a_n = \left[(n+4) + 4 \cdot \arctan(n+4)\right] - \left[(n-3) + 4 \cdot \arctan(n-3)\right]$$

$$a_n = \cancel{n} + 4 - \cancel{n} + 3 + 4 \cdot \left[\arctan(n+4) - \arctan(n-3)\right]$$

$$a_n = 7 + 4 \cdot \left[\arctan(n+4) - \arctan(n-3)\right]$$

$$\lim_{n\to\infty} a_n = \underbrace{\lim_{n\to\infty} 7}_{=7} + 4 \cdot \lim_{n\to\infty} \left[\arctan(n+4) - \arctan(n-3)\right]$$

$$7 + 4 \cdot \left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 7 + 4 \cdot 0 = 7$$

$$7$$

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19. Let $b_n = \left(1+\frac{1}{5n}\right)^{10n}$ $(n = 1, 2, 3, \cdots)$ and $\lim_{n\to\infty} b_n = b$. Then which of the following is equal to b ?

a) e^2 b) e^{-2} c) $2e$

d) $\sqrt{2}e^2$ e) $e^{\sqrt{2}}$

$$\ln b_n = 10n.\ln\left(1+\frac{1}{5n}\right) \Rightarrow \lim_{n\to\infty} \ln b_n = \ln\left(\lim_{n\to\infty} b_n\right)$$

$$= \ln b = \lim_{n\to\infty} 10n.\ln\left(1+\frac{1}{5n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{5n}\right)}{\frac{1}{10n}} \left(\frac{0}{0}\right)$$

$$\underset{L'H}{=} \lim_{n\to\infty} \frac{\left(-1/5n^2\right) / \left(1+1/5n\right)}{\frac{-1}{10n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{-10n^2}{-5n^2}}{1+\frac{1}{5n}} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{5}\cdot\cancel{\frac{1}{n}}^{\,0}} = \frac{2}{1} = 2$$

$$\Rightarrow \ln b = 2 \Rightarrow b = e^2$$

6 Which of the following is equal to the sum

$$\sum_{n=2}^{\infty} \frac{(-1)^n+2^n+3^n}{4^n} ?$$

A) $\frac{14}{5}$ B) $\frac{9}{5}$ C) $\frac{7}{2}$ D) 1 E) $\frac{7}{3}$

$$\sum_{n=2}^{\infty} \frac{(-1)^n+2^n+3^n}{4^n} = \underbrace{\sum_{n=2}^{\infty}\left(-\frac{1}{4}\right)^n + \sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n}_{\text{Geometric series}}$$

$$\left(\frac{1}{2}\right)^n = \left(\frac{2}{4}\right)^n$$

$$= \frac{(-1/4)^2}{1-\left(-\frac{1}{4}\right)} + \frac{(1/2)^2}{1-1/2} + \frac{(3/4)^2}{1-\frac{3}{4}}$$

$$= \frac{1}{16}\cdot\frac{4}{5} + \frac{1}{4}\cdot 2 + \frac{9}{16}\cdot 4 = \frac{1}{20} + \underset{(10)}{\frac{1}{2}} + \underset{(5)}{\frac{9}{4}}$$

$$= \frac{1+10+45}{20} = \frac{56}{20} = \frac{14}{5}$$

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Note: $\sum \overset{a_1}{\overset{\shortparallel}{a}} r^{n-1}$ Geometric series

If $|r|<1$, then $\sum a r^{n-1} = \frac{a}{1-r}$

7) Let $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}b_n=2$ where $a_n>0,\ b_n>0$ for all $n\in\mathbb{Z}^+$. Then which of the following is correct for the series given by I. $\sum_{n=1}^{\infty}a_n$ II. $\sum_{n=1}^{\infty}b_n$?

A) Both I and II diverge B) Both I and II converge C) I diverges, II converges D) I converges, II diverges E) None of them

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|2|=2>1$ so $\{a_n\}$ is divergent by the ratio test.
- $\lim_{n\to\infty}b_n=2\neq 0$ so $\{b_n\}$ is divergent by the nth term test.

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6. If a_n is the general term of the sequence, which of the following matches is correct?

I. If the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent, then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is also divergent.

II. If the series $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is also divergent.

III. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent.

$M : a_n = \sqrt{n}$ $\quad K : a_n = \frac{1}{2^n}$ $\quad L : a_n = \frac{1}{n}$

a) I-L, II-M, III-K
b) I-M, II-K, III-L
c) I-M, II-L, III-K
d) I-L, II-K, III-M
e) I-K, II-M, III-L

• K: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ is Geometric series with $r = 1/2$ so $|r| = 1/2 < 1$. Hence, $\sum_{n=1}^{\infty} a_n$ is convergent.

$\Rightarrow$ III $-$ K

• L: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is Harmonic series divergent

$\lim_{n\to\infty} \frac{1}{a_n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$ so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ series is divergent by the nth term test.

$\Rightarrow$ II $-$ L

I $-$ M $\rightarrow$ $\lim_{n\to\infty} n^{1/2} = \infty$ so $\{a_n\}$ is divergent by the nth term test

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series $p = 1/2 < 1$ so series $\sum \frac{1}{a_n}$ is divergent

• As a result

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1. Which of the following is equal to the sum $\sum_{n=1}^{\infty} \underbrace{\left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right)}_{=a_n}$?

a) 1   b) -1 c) 2   d) -2 e) $\frac{1}{2}$

$S_n = a_1 + a_2 + \dots + a_n$ and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$$S_n = (2 - \sqrt{2}) + (\sqrt{2} - \sqrt[3]{2}) + (\sqrt[3]{2} - \sqrt[4]{2}) + \dots + \left(2^{\frac{1}{n-1}} - 2^{\frac{1}{n}}\right) + \left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right) \text{ so}$$

$$S_n = 2 - 2^{\frac{1}{n+1}}$$

$$\sum_{n=1}^{\infty}\left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2 - 2^{\frac{1}{n+1}}\right)$$

$$= 2 - 2^0 = 2 - 1 = 1$$

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4. Which of the following is true about the convergence of the following series, respectively?

$$\sum_{n=2}^{\infty} \frac{n \ln n}{3^n}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n, \quad \sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$$

a) convergent, convergent, conditionally convergent

b) convergent, convergent, divergent

c) divergent, convergent, absolutely convergent

d) convergent, divergent, conditionally convergent

e) divergent, convergent, divergent

$$\bullet \lim_{n\to\infty} \frac{\frac{(n+1)\ln(n+1)}{3^{n+1}}}{\frac{n\ln n}{3^n}} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{\cancel{3^n}\cdot 3} \cdot \frac{\cancel{3^n}}{n \ln n}$$

$$= \frac{1}{3} \cdot \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{= \frac{\cancel{n}(1+\cancel{\frac{1}{n}}^{\,0})}{\cancel{n}} = 1} \cdot \underbrace{\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n}}_{=1}$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \left(\frac{\infty}{\infty}\right) \underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\cancel{\frac{1}{n}}^{\,0})}$$

$$= \lim_{n\to\infty} \frac{1}{1} = 1$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$. Hence,

$\sum_{n=2}^{\infty} \frac{n \ln n}{3^n}$ is absolutely convergent (so convergent) by the ratio test.

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{n} = \frac{1}{2} + 0 = \frac{1}{2} < 1$ so

$\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$ is absolutely convergent (so convergent) by the root test.

- $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1 \neq 0$ so we can not use Alternating Series test.

$\lim_{n\to\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$ does not exist (It oscillates between -1 and 1).

Hence, $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \cos\left(\frac{\pi}{n}\right)$ is divergent.