MAT1320-Linear Algebra Lecture Notes

Vector Spaces, Subspaces

Mehmet E. KÖROĞLU

Fall 2024

Yildiz Technical University, Department of Mathematics

mkoroglu@yildiz.edu.tr

Table of contents

Vector Spaces

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers.

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$+ \quad : \quad V \times V \longrightarrow V,$$

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$\begin{array}{ccc} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) & \longmapsto & \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \ \text{(Addition)} \end{array}$$

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) \quad &\longmapsto \quad \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \ \ \text{(Addition)} \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \end{aligned}$$

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$\begin{array}{rcl} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) & \longmapsto & \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \ \text{(Addition)} \\ \cdot & : & \mathbb{R} \times V \longrightarrow V, \\ (\lambda, \overrightarrow{\mathbf{u}}) & \longmapsto & \lambda \overrightarrow{\mathbf{u}} \ \text{(Scalar Multiplication)} \end{array}$$

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$\begin{array}{ccl} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) & \longmapsto & \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \ \text{(Addition)} \\ \cdot & : & \mathbb{R} \times V \longrightarrow V, \\ (\lambda, \overrightarrow{\mathbf{u}}) & \longmapsto & \lambda \overrightarrow{\mathbf{u}} \ \text{(Scalar Multiplication)} \end{array}$$

be two binary operations on V. The triple-$(V, +, \cdot)$ is called a vector space over $\mathbb{R}$ if the following axioms satisfied.

Definition
Let $\varnothing \neq V$ be nonempty set and $\mathbb{R}$ (in general a field F) be the field of real numbers. Let

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) \quad &\longmapsto \quad \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \ \text{(Addition)} \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \\ (\lambda, \overrightarrow{\mathbf{u}}) \quad &\longmapsto \quad \lambda \overrightarrow{\mathbf{u}} \ \text{(Scalar Multiplication)} \end{aligned}$$

be two binary operations on V. The triple-$(V, +, \cdot)$ is called a vector space over $\mathbb{R}$ if the following axioms satisfied. The element of V are called vectors.

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.
6. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $r\overrightarrow{\mathbf{v}} \in V$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.
6. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $r\overrightarrow{\mathbf{v}} \in V$
7. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(r+s)\overrightarrow{\mathbf{v}} = r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{v}}$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.
6. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $r\overrightarrow{\mathbf{v}} \in V$
7. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(r + s)\overrightarrow{\mathbf{v}} = r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{v}}$
8. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V$, $r(\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) = r\overrightarrow{\mathbf{v}} + r\overrightarrow{\mathbf{w}}$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.
6. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $r\overrightarrow{\mathbf{v}} \in V$
7. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(r + s)\overrightarrow{\mathbf{v}} = r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{v}}$
8. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V$, $r(\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) = r\overrightarrow{\mathbf{v}} + r\overrightarrow{\mathbf{w}}$
9. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(rs)\overrightarrow{\mathbf{v}} = r(s\overrightarrow{\mathbf{v}})$

Vector Spaces

1. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in V$
2. For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{v}}$
3. For each $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V : (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) + \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{v}} + (\overrightarrow{\mathbf{w}} + \overrightarrow{\mathbf{u}})$
4. For each $\overrightarrow{\mathbf{v}} \in V : \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$ such that $\overrightarrow{\mathbf{0}} \in V$ exists.
5. For each $\overrightarrow{\mathbf{v}} \in V$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{0}}$ such that $\overrightarrow{\mathbf{w}} = -\overrightarrow{\mathbf{v}} \in V$ exists.
6. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $r\overrightarrow{\mathbf{v}} \in V$
7. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(r+s)\overrightarrow{\mathbf{v}} = r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{v}}$
8. For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in V$, $r(\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}}) = r\overrightarrow{\mathbf{v}} + r\overrightarrow{\mathbf{w}}$
9. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in V$, $(rs)\overrightarrow{\mathbf{v}} = r(s\overrightarrow{\mathbf{v}})$
10. For each $\overrightarrow{\mathbf{v}} \in V$, $1\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}}$

Example

The set of n-tuples $V = \mathbb{R}^n = \left\{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \dots, x_n) \middle| \, x_i \in \mathbb{R} \right\}$ together with the following binary operations

$$+ \quad : \quad V \times V \longrightarrow V,$$

Example

The set of n-tuples $V = \mathbb{R}^n = \left\{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \dots, x_n) \middle| \, x_i \in \mathbb{R} \right\}$ together with the following binary operations

$$\begin{array}{ccc} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}) & \longmapsto & \overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \end{array}$$

Example

The set of n-tuples $V = \mathbb{R}^n = \{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R}\}$ together with the following binary operations

$$\begin{array}{ccl} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}) & \longmapsto & \overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \cdot & : & \mathbb{R} \times V \longrightarrow V, \end{array}$$

Example

The set of n-tuples $V = \mathbb{R}^n = \left\{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \middle| \, x_i \in \mathbb{R} \right\}$
together with the following binary operations

$$\begin{array}{ccc} + & : & V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}) & \longmapsto & \overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \cdot & : & \mathbb{R} \times V \longrightarrow V, \\ (\lambda, \overrightarrow{\mathbf{x}}) & \longmapsto & \lambda \overrightarrow{\mathbf{x}} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \end{array}$$

Example

The set of n-tuples $V = \mathbb{R}^n = \left\{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \middle| x_i \in \mathbb{R} \right\}$ together with the following binary operations

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \\ (\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}) \quad &\longmapsto \quad \overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \\ (\lambda, \overrightarrow{\mathbf{x}}) \quad &\longmapsto \quad \lambda \overrightarrow{\mathbf{x}} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \end{aligned}$$

is a vector space over $\mathbb{R}$.

Example

Let $V = \mathbb{R}^2 = \left\{ \overrightarrow{\mathbf{x}} = (a, b) \middle| \, a, b \in \mathbb{R} \right\}$ and

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \ \ (a, b) + (c, d) = (a + c, b + d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \ \ \lambda (a, b) = (a, \lambda b) \end{aligned}$$

be given.

Example

Let $V=\mathbb{R}^2=\left\{ \overrightarrow{\mathbf{x}}=(a,b) \middle| \, a,b\in\mathbb{R}\right\}$ and

$$\begin{aligned} + &: V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(a+c,b+d) \\ \cdot &: \mathbb{R}\times V\longrightarrow V,\ \lambda(a,b)=(a,\lambda b) \end{aligned}$$

be given. Then $(V,+,\cdot)$ is not a vector space. Because, the following axiom is not satisfied.

Example

Let $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\big|\, a,b\in\mathbb{R}\right\}$ and

$$\begin{aligned} + \quad &: \quad V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(a+c,b+d) \\ \cdot \quad &: \quad \mathbb{R}\times V\longrightarrow V,\ \lambda(a,b)=(a,\lambda b) \end{aligned}$$

be given. Then $(V,+,\cdot)$ is not a vector space. Because, the following axiom is not satisfied. For each $r,s\in\mathbb{R}$ and $\overrightarrow{\mathbf{u}}=(a,b)\in V$ we have

$$(r+s)\overrightarrow{\mathbf{u}} \quad = \quad (r+s)(a,b)=(a,(r+s)b)$$

Example
Let $V = \mathbb{R}^2 = \left\{ \overrightarrow{\mathbf{x}} = (a, b) \middle| \, a, b \in \mathbb{R} \right\}$ and

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \;\; (a, b) + (c, d) = (a + c, b + d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \;\; \lambda (a, b) = (a, \lambda b) \end{aligned}$$

be given. Then $(V, +, \cdot)$ is not a vector space. Because, the following axiom is not satisfied. For each $r, s \in \mathbb{R}$ and $\overrightarrow{\mathbf{u}} = (a, b) \in V$ we have

$$\begin{aligned} (r + s) \overrightarrow{\mathbf{u}} &= (r + s)(a, b) = (a, (r + s) b) \qquad (1) \\ r \overrightarrow{\mathbf{u}} + s \overrightarrow{\mathbf{u}} &= (a, rb) + (a, sb) = (2a, (r + s) b) \end{aligned}$$

Example

Let $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\middle|\, a,b\in\mathbb{R}\right\}$ and

$$\begin{array}{ccl} + & : & V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(a+c,b+d) \\ \cdot & : & \mathbb{R}\times V\longrightarrow V,\ \lambda(a,b)=(a,\lambda b) \end{array}$$

be given. Then $(V,+,\cdot)$ is not a vector space. Because, the following axiom is not satisfied. For each $r,s\in\mathbb{R}$ and $\overrightarrow{\mathbf{u}}=(a,b)\in V$ we have

$$\begin{aligned} (r+s)\overrightarrow{\mathbf{u}} &= (r+s)(a,b)=(a,(r+s)b) && (1) \\ r\overrightarrow{\mathbf{u}}+s\overrightarrow{\mathbf{u}} &= (a,rb)+(a,sb)=(2a,(r+s)b) && (2) \end{aligned}$$

Note that (1) $\neq$ (2).

Example

The nonempty set $V = \mathbb{R}^2 = \left\{ \overrightarrow{\mathbf{x}} = (a, b) \middle| \, a, b \in \mathbb{R} \right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \;\; (a, b) + (c, d) = (b, d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \;\; \lambda (a, b) = (\lambda a, \lambda b) \end{aligned}$$

is not a vector space.

Example
The nonempty set $V = \mathbb{R}^2 = \left\{ \overrightarrow{\mathbf{x}} = (a, b) \middle| \, a, b \in \mathbb{R} \right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \;\; (a, b) + (c, d) = (b, d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \;\; \lambda (a, b) = (\lambda a, \lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative.

Example

The nonempty set $V = \mathbb{R}^2 = \{ \overrightarrow{\mathbf{x}} = (a, b) |\, a, b \in \mathbb{R} \}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \ \ (a, b) + (c, d) = (b, d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \ \lambda (a, b) = (\lambda a, \lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}} = (a, b)$, $\overrightarrow{\mathbf{y}} = (c, d)$, $\overrightarrow{\mathbf{z}} = (e, f) \in V$. We need to show that $(\overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}}) + \overrightarrow{\mathbf{z}} = \overrightarrow{\mathbf{x}} + (\overrightarrow{\mathbf{y}} + \overrightarrow{\mathbf{z}})$ is not satisfied.

Example

The nonempty set $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\middle|\, a,b\in\mathbb{R}\right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(b,d) \\ \cdot \quad &: \quad \mathbb{R}\times V\longrightarrow V,\ \lambda\,(a,b)=(\lambda a,\lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}}=(a,b)\,,\ \overrightarrow{\mathbf{y}}=(c,d)\,,\ \overrightarrow{\mathbf{z}}=(e,f)\in V$. We need to show that $(\overrightarrow{\mathbf{x}}+\overrightarrow{\mathbf{y}})+\overrightarrow{\mathbf{z}}=\overrightarrow{\mathbf{x}}+(\overrightarrow{\mathbf{y}}+\overrightarrow{\mathbf{z}})$ is not satisfied.

$$((a,b)+(c,d))+(e,f)$$

Example

The nonempty set $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\middle|\, a,b\in\mathbb{R}\right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(b,d) \\ \cdot \quad &: \quad \mathbb{R}\times V\longrightarrow V,\ \lambda(a,b)=(\lambda a,\lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}}=(a,b)$, $\overrightarrow{\mathbf{y}}=(c,d)$, $\overrightarrow{\mathbf{z}}=(e,f)\in V$. We need to show that $(\overrightarrow{\mathbf{x}}+\overrightarrow{\mathbf{y}})+\overrightarrow{\mathbf{z}}=\overrightarrow{\mathbf{x}}+(\overrightarrow{\mathbf{y}}+\overrightarrow{\mathbf{z}})$ is not satisfied.

$$((a,b)+(c,d))+(e,f) \quad = \quad (b,d)+(e,f)=(d,f)$$

Example

The nonempty set $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\middle|\, a,b\in\mathbb{R}\right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V\times V\longrightarrow V,\ \ (a,b)+(c,d)=(b,d) \\ \cdot \quad &: \quad \mathbb{R}\times V\longrightarrow V,\ \ \lambda(a,b)=(\lambda a,\lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}}=(a,b)$, $\overrightarrow{\mathbf{y}}=(c,d)$, $\overrightarrow{\mathbf{z}}=(e,f)\in V$. We need to show that $(\overrightarrow{\mathbf{x}}+\overrightarrow{\mathbf{y}})+\overrightarrow{\mathbf{z}}=\overrightarrow{\mathbf{x}}+(\overrightarrow{\mathbf{y}}+\overrightarrow{\mathbf{z}})$ is not satisfied.

$$\begin{aligned} ((a,b)+(c,d))+(e,f) \quad &= \quad (b,d)+(e,f)=(d,f) \qquad (3) \\ (a,b)+((c,d)+(e,f)) \end{aligned}$$

Example

The nonempty set $V = \mathbb{R}^2 = \left\{ \overrightarrow{\mathbf{x}} = (a, b) \middle| \, a, b \in \mathbb{R} \right\}$ together with the binary operations

$$\begin{aligned} + \quad &: \quad V \times V \longrightarrow V, \;\; (a, b) + (c, d) = (b, d) \\ \cdot \quad &: \quad \mathbb{R} \times V \longrightarrow V, \;\; \lambda (a, b) = (\lambda a, \lambda b) \end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}} = (a, b)$, $\overrightarrow{\mathbf{y}} = (c, d)$, $\overrightarrow{\mathbf{z}} = (e, f) \in V$. We need to show that $(\overrightarrow{\mathbf{x}} + \overrightarrow{\mathbf{y}}) + \overrightarrow{\mathbf{z}} = \overrightarrow{\mathbf{x}} + (\overrightarrow{\mathbf{y}} + \overrightarrow{\mathbf{z}})$ is not satisfied.

$$\begin{aligned} ((a, b) + (c, d)) + (e, f) &= (b, d) + (e, f) = (d, f) \qquad (3) \\ (a, b) + ((c, d) + (e, f)) &= (a, b) + (d, f) = (b, f) \end{aligned}$$

Example
The nonempty set $V=\mathbb{R}^2=\left\{\overrightarrow{\mathbf{x}}=(a,b)\,\middle|\,a,b\in\mathbb{R}\right\}$ together with the binary operations

$$\begin{aligned} + &: V\times V\longrightarrow V,\ (a,b)+(c,d)=(b,d)\\ \cdot &: \mathbb{R}\times V\longrightarrow V,\ \lambda(a,b)=(\lambda a,\lambda b)\end{aligned}$$

is not a vector space. Because, V is not associative. Let $\overrightarrow{\mathbf{x}}=(a,b)$, $\overrightarrow{\mathbf{y}}=(c,d)$, $\overrightarrow{\mathbf{z}}=(e,f)\in V$. We need to show that $(\overrightarrow{\mathbf{x}}+\overrightarrow{\mathbf{y}})+\overrightarrow{\mathbf{z}}=\overrightarrow{\mathbf{x}}+(\overrightarrow{\mathbf{y}}+\overrightarrow{\mathbf{z}})$ is not satisfied.

$$((a,b)+(c,d))+(e,f) = (b,d)+(e,f)=(d,f) \quad (3)$$
$$(a,b)+((c,d)+(e,f)) = (a,b)+(d,f)=(b,f) \quad (4)$$

Note that (3) $\neq$ (4).

Example

The set of column vectors $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} x + y + z = 0 \\ x, y, z \in \mathbb{R} \end{array} \right\}$

together with binary operations

Example

The set of column vectors $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} x + y + z = 0 \\ x, y, z \in \mathbb{R} \end{array} \right\}$

together with binary operations

$$+ \quad : \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Example

The set of column vectors $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} x + y + z = 0 \\ x, y, z \in \mathbb{R} \end{array} \right\}$

together with binary operations

$$+ \quad : \quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$\cdot \quad : \quad \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \end{pmatrix}$$

Example

The set of column vectors $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} x+y+z=0 \\ x,y,z \in \mathbb{R} \end{array} \right\}$

together with binary operations

$$+ \quad : \quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

$$\cdot \quad : \quad \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \end{pmatrix}$$

is a vector space.

Example

Let $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, \ldots, a_3 \in \mathbb{R}\}$ be the set of all polynomials of degree at most three in one variable.

Example

Let $\mathcal{P}_3 = \left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 | a_0, \ldots, a_3 \in \mathbb{R}\right\}$ be the set of all polynomials of degree at most three in one variable. $\mathcal{P}_3$ is a vector space with respect to polynomial addition

$$\begin{aligned} & \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3\right) \\ = \; & (a_0 + b_0) + (a_1 + b_1)\, x + (a_2 + b_2)\, x^2 + (a_3 + b_3)\, x^3 \end{aligned}$$

Example

Let $\mathcal{P}_3 = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, \ldots, a_3 \in \mathbb{R}\right\}$ be the set of all polynomials of degree at most three in one variable. $\mathcal{P}_3$ is a vector space with respect to polynomial addition

$$\begin{aligned} & \left(a_0 + a_1x + a_2x^2 + a_3x^3\right) + \left(b_0 + b_1x + b_2x^2 + b_3x^3\right) \\ = \quad & \left(a_0 + b_0\right) + \left(a_1 + b_1\right)x + \left(a_2 + b_2\right)x^2 + \left(a_3 + b_3\right)x^3 \end{aligned}$$

and

$$r\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \left(ra_0\right) + \left(ra_1\right)x + \left(ra_2\right)x^2 + \left(ra_3\right)x^3$$

Example

Let $\mathcal{P}_3 = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, \ldots, a_3 \in \mathbb{R}\right\}$ be the set of all polynomials of degree at most three in one variable. $\mathcal{P}_3$ is a vector space with respect to polynomial addition

$$\begin{aligned} & \left(a_0 + a_1x + a_2x^2 + a_3x^3\right) + \left(b_0 + b_1x + b_2x^2 + b_3x^3\right) \\ = \; & \left(a_0 + b_0\right) + \left(a_1 + b_1\right)x + \left(a_2 + b_2\right)x^2 + \left(a_3 + b_3\right)x^3 \end{aligned}$$

and

$$r\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \left(ra_0\right) + \left(ra_1\right)x + \left(ra_2\right)x^2 + \left(ra_3\right)x^3$$

scalar multiplication.

Example
The set of 2×2 square matrices
$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$ is a vector space with respect to binary operations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$$

Example
The set of 2×2 square matrices
$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$ is a vector space with respect to binary operations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$$

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

Subspaces

Definition
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V, if U itself is a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication on V.

Subspaces

Definition
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V, if U itself is a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication on V.

Theorem
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V if the following three conditions hold:

Subspaces

Definition
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V, if U itself is a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication on V.

Theorem
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V if the following three conditions hold:

1. *The zero vector* 0 *belong to* U.
2. *For each* $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in U$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in U$.
3. *For each* $r \in \mathbb{R}$ *and* $\overrightarrow{\mathbf{v}} \in U$, $r\overrightarrow{\mathbf{v}} \in U$.

Subspaces

Definition
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V, if U itself is a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication on V.

Theorem
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V if the following three conditions hold:

1. *The zero vector 0 belong to U.*
2. *For each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in U$, $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}} \in U$.*
3. *For each $r \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}} \in U$, $r\overrightarrow{\mathbf{v}} \in U$.*

Note: The three conditions given in above Theorem can be combined as a single one. That is, U is a subspace of V if for each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in U$, and $r, s \in \mathbb{R}$, $r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{w}} \in U$.

Example

Let $\mathcal{B} = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \middle| \, c = a + b \text{ and } a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$. Is $\mathcal{B}$ a subspace of $\mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$?

Example

Let $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, c = a + b \text{ and } a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$. Is $\mathcal{B}$ a subspace of $\mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$?

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \underbrace{\begin{pmatrix} a+x & b+y \\ 0 & c+z \end{pmatrix}}_{c+z=a+b+x+y} \in \mathcal{B}$$

Example

Let $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, c = a + b \text{ and } a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 2}(\mathbb{R})$. Is $\mathcal{B}$ a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$?

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \underbrace{\begin{pmatrix} a+x & b+y \\ 0 & c+z \end{pmatrix}}_{c+z=a+b+x+y} \in \mathcal{B}$$

$$r \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \underbrace{\begin{pmatrix} ra & rb \\ 0 & rc \end{pmatrix}}_{rc=ra+rb} \in \mathcal{B}$$

Example

Let $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| c = a + b \text{ and } a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 2}(\mathbb{R})$. Is $\mathcal{B}$ a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$?

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \underbrace{\begin{pmatrix} a+x & b+y \\ 0 & c+z \end{pmatrix}}_{c+z=a+b+x+y} \in \mathcal{B}$$

$$r \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \underbrace{\begin{pmatrix} ra & rb \\ 0 & rc \end{pmatrix}}_{rc=ra+rb} \in \mathcal{B}$$

Also the zero of $\mathcal{M}_{2\times 2}(\mathbb{R})$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}$. Then $\mathcal{B}$ is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

Example

Let $\mathcal{A} = \{(x, 0) | \, x \in \mathbb{R}\} \subset \mathbb{R}^2$. Is $\mathcal{A}$ a subspace of $\mathbb{R}^2$?

Example

Let $\mathcal{A} = \{(x,0) | x \in \mathbb{R}\} \subset \mathbb{R}^2$. Is $\mathcal{A}$ a subspace of $\mathbb{R}^2$? For each $(a,0)$, $(b,0) \in \mathcal{A}$ and $r \in \mathbb{R}$

$$(a,0) + (b,0) = (a+b,0) \in \mathcal{A}$$

Example

Let $\mathcal{A} = \{(x,0) | x \in \mathbb{R}\} \subset \mathbb{R}^2$. Is $\mathcal{A}$ a subspace of $\mathbb{R}^2$? For each $(a,0)\,,\ (b,0) \in \mathcal{A}$ and $r \in \mathbb{R}$

$$(a,0) + (b,0) = (a+b,0) \in \mathcal{A}$$

$$r\,(a,0) = (ra,0) \in \mathcal{A}$$

Example

Let $\mathcal{A} = \{(x,0) | \, x \in \mathbb{R}\} \subset \mathbb{R}^2$. Is $\mathcal{A}$ a subspace of $\mathbb{R}^2$? For each $(a,0)$, $(b,0) \in \mathcal{A}$ and $r \in \mathbb{R}$

$$(a,0) + (b,0) = (a+b,0) \in \mathcal{A}$$

$$r\,(a,0) = (ra,0) \in \mathcal{A}$$

Also the zero of $\mathbb{R}^2$, $(0,0) \in \mathcal{A}$. Then $\mathcal{A}$ is a subspace of $\mathbb{R}^2$.

Example
$\mathcal{W} = \{ (a, b, c) | \, a \geqslant 0 \text{ and } a, b, c \in \mathbb{R} \} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

Example
$\mathcal{W} = \{ (a, b, c) | \, a \geqslant 0$ and $a, b, c \in \mathbb{R} \} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$. Since $\mathcal{W}$ is not closed under scalar multiplication. That is

$$(a, b, c) + (x, y, z) = (\underbrace{a + x}_{a+x \geqslant 0}, b + y, c + z) \in \mathcal{W}$$

Example
$\mathcal{W} = \{(a, b, c) \mid a \geqslant 0 \text{ and } a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$. Since $\mathcal{W}$ is not closed under scalar multiplication. That is

$$(a, b, c) + (x, y, z) = (\underset{a+x \geqslant 0}{a + x}, b + y, c + z) \in \mathcal{W}$$

$$r\,(a, b, c) = \underbrace{(ra, rb, rc)}_{r<0 \Rightarrow ra<0} \notin \mathcal{W}.$$

Example
$\mathcal{W}' = \{\, (a, b, c) |\, a^2 + b^2 + c^2 \leqslant 1 \text{ and } a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

Example
$\mathcal{W}' = \{ (a, b, c) | \, a^2 + b^2 + c^2 \leqslant 1 \text{ and } a, b, c \in \mathbb{R} \} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$. Because $\mathcal{W}'$ is not closed under addition. For example,

$$\overrightarrow{\mathbf{u}} = (1, 0, 0) \, , \overrightarrow{\mathbf{v}} = (0, 1, 0) \in \mathcal{W}'$$

Example
$\mathcal{W}' = \{ (a, b, c) | \, a^2 + b^2 + c^2 \leqslant 1 \text{ and } a, b, c \in \mathbb{R} \} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$. Because $\mathcal{W}'$ is not closed under addition. For example,

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= (1, 0, 0), \overrightarrow{\mathbf{v}} = (0, 1, 0) \in \mathcal{W}' \\ \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} &= \underbrace{(1, 1, 0)}_{1^2+1^2=2>1} \notin \mathcal{W}'. \end{aligned}$$

?