



MAT1320-Linear Algebra

Lecture Notes

Change of Basis and Coordinate Transformations

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Coordinates

Theorem

Let V be vector space of dimension n and with basis

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Then any vector $\vec{w} \in V$ can be expressed uniquely as a linear combination of basis vectors in \mathcal{B} , say

$$\vec{w} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n.$$

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$$\vec{w} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n.$$

Note: These n scalars x_1, x_2, \dots, x_n are called the **coordinates** of \vec{w} relative to the basis \mathcal{B} , and they form a vector (x_1, x_2, \dots, x_n) in \mathbb{R}^n called the **coordinate vector** of \vec{w} relative to \mathcal{B} . We denote this vector by $[\vec{w}]_{\mathcal{B}}$, or simply $[\vec{w}]$, when \mathcal{B} is understood. Thus,

$$[\vec{w}]_{\mathcal{B}} = (x_1, x_2, \dots, x_n).$$

Example

Find coordinate vector of $\vec{\mathbf{w}} = (3, 2, 1) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Example

Find coordinate vector of $\vec{\mathbf{w}} = (3, 2, 1) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$

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we have $x_1 = x$, $x_2 = y$, $x_3 = z$.

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we have $x_1 = x$, $x_2 = y$, $x_3 = z$. Then

$$(3, 2, 1) = 3.(1, 0, 0) + 2.(0, 1, 0) + 1.(0, 0, 1)$$

and so $[\vec{\mathbf{w}}]_{\mathcal{B}} = (3, 2, 1)$.

Example

Find coordinate vector of $\vec{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$.

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Find coordinate vector of $\vec{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

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$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1)$$

$$x_1 + 2x_2 + x_3 = x$$

$$2x_1 + 2x_3 = y$$

$$x_2 + x_3 = z$$

Example

Find coordinate vector of $\vec{w} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

$$\begin{aligned}(x, y, z) &= x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1) \\ \begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} &\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array} \right)\end{aligned}$$

Coordinates

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Find coordinate vector of $\vec{w} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

$$\begin{aligned}(x, y, z) &= x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1) \\ \begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} &\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x+y-4z}{4} \\ 0 & 1 & 0 & \frac{2x-y}{4} \\ 0 & 0 & 1 & \frac{-2x+y+4z}{4} \end{array} \right)\end{aligned}$$

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Find coordinate vector of $\vec{w} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

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we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$.

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$$\begin{aligned} (x, y, z) &= x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1) \\ \begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} &\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x+y-4z}{4} \\ 0 & 1 & 0 & \frac{2x-y}{4} \\ 0 & 0 & 1 & \frac{-2x+y+4z}{4} \end{array} \right) \end{aligned}$$

we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$. Thus

$$(1, 2, 3) = -2 \cdot (1, 2, 0) + 0 \cdot (2, 0, 1) + 3 \cdot (1, 2, 1)$$

and $[\vec{w}]_{\mathcal{B}} = (-2, 0, 3)$.

Change of Basis and Coordinate Transformations Matrix

Change of Basis and Coordinate Transformations

Definition

Let V be an n -space with two ordered basis

$$\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \text{ and } \mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}.$$

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$$\vec{v}_1 = a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \dots + a_{n1}\vec{w}_n$$

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$$\vec{v}_n = a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \dots + a_{nn}\vec{w}_n$$

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$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

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The matrix A is called **coordinates transformation matrix** from

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basis \mathcal{B}_1 to basis \mathcal{B}_2 and denoted by $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$.

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$$\vec{w}_n = b_{1n}\vec{v}_1 + b_{2n}\vec{v}_2 + \dots + b_{nn}\vec{v}_n$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

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basis \mathcal{B}_2 to basis \mathcal{B}_1 and denoted by $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1}$.

Change of Basis and Coordinate Transformations

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$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$ and $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1}$ be the coordinates transformation

matrices from basis \mathcal{B}_1 to \mathcal{B}_2 and from basis \mathcal{B}_2 to \mathcal{B}_1 respectively.

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matrices from basis \mathcal{B}_1 to \mathcal{B}_2 and from basis \mathcal{B}_2 to \mathcal{B}_1 respectively.

Then we have the following assertions:

1. $AB = I_n$ or $B = A^{-1}$ i.e., $\left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right) \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = I_n$ or $\left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1}$,

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2. For all $\vec{u}, \vec{v} \in V$ we have $[\vec{u}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{u}]_{\mathcal{B}_2}$ and $[\vec{v}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\vec{v}]_{\mathcal{B}_1}$.

Change of Basis and Coordinate Transformations

Example

Let

$$\mathcal{B}_1 = \{ \vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1) \},$$

$$\mathcal{B}_2 = \{ \vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0) \}$$

be two ordered bases of V and $\vec{u} = (2, 3, 5)$ be the coordinates with respect to standard basis. Find each of the following.

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2},$

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1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2},$

2. $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1},$

3. $[\vec{u}]_{\mathcal{B}_1},$

4. $[\vec{u}]_{\mathcal{B}_2}.$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_1 = a_{11}\vec{\mathbf{w}}_1 + a_{21}\vec{\mathbf{w}}_2 + a_{31}\vec{\mathbf{w}}_3$$

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$$\begin{aligned}\vec{\mathbf{v}}_1 &= a_{11}\vec{\mathbf{w}}_1 + a_{21}\vec{\mathbf{w}}_2 + a_{31}\vec{\mathbf{w}}_3 \\ (1, 2, 0) &= a_{11}(0, 2, 1) + a_{21}(-1, 0, 1) + a_{31}(-1, 3, 0)\end{aligned}$$

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$$(1, 2, 0) = a_{11}(0, 2, 1) + a_{21}(-1, 0, 1) + a_{31}(-1, 3, 0)$$

$$-a_{21} - a_{31} = 1$$

$$2a_{11} + 3a_{31} = 2$$

$$a_{11} + a_{21} = 0$$

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$$(1, 2, 0) = a_{11}(0, 2, 1) + a_{21}(-1, 0, 1) + a_{31}(-1, 3, 0)$$

$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array} \right)$$

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$$(1, 2, 0) = a_{11}(0, 2, 1) + a_{21}(-1, 0, 1) + a_{31}(-1, 3, 0)$$

$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_1 = a_{11}\vec{\mathbf{w}}_1 + a_{21}\vec{\mathbf{w}}_2 + a_{31}\vec{\mathbf{w}}_3$$

$$(1, 2, 0) = a_{11}(0, 2, 1) + a_{21}(-1, 0, 1) + a_{31}(-1, 3, 0)$$

$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

$$a_{11} = 1, a_{21} = -1, a_{31} = 0$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_2 = a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}\vec{\mathbf{v}}_2 &= a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3 \\ (2, 0, 1) &= a_{12}(0, 2, 1) + a_{22}(-1, 0, 1) + a_{32}(-1, 3, 0)\end{aligned}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_2 = a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3$$

$$(2, 0, 1) = a_{12}(0, 2, 1) + a_{22}(-1, 0, 1) + a_{32}(-1, 3, 0)$$

$$-a_{22} - a_{32} = 2$$

$$2a_{12} + 3a_{32} = 0$$

$$a_{12} + a_{22} = 1$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_2 = a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3$$

$$(2, 0, 1) = a_{12}(0, 2, 1) + a_{22}(-1, 0, 1) + a_{32}(-1, 3, 0)$$

$$\begin{aligned} -a_{22} - a_{32} &= 2 \\ 2a_{12} + 3a_{32} &= 0 \\ a_{12} + a_{22} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right)$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_2 = a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3$$

$$(2, 0, 1) = a_{12}(0, 2, 1) + a_{22}(-1, 0, 1) + a_{32}(-1, 3, 0)$$

$$\begin{aligned} -a_{22} - a_{32} &= 2 \\ 2a_{12} + 3a_{32} &= 0 \\ a_{12} + a_{22} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & -\frac{6}{5} \end{array} \right)$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_2 = a_{12}\vec{\mathbf{w}}_1 + a_{22}\vec{\mathbf{w}}_2 + a_{32}\vec{\mathbf{w}}_3$$

$$(2, 0, 1) = a_{12}(0, 2, 1) + a_{22}(-1, 0, 1) + a_{32}(-1, 3, 0)$$

$$\begin{aligned} -a_{22} - a_{32} &= 2 \\ 2a_{12} + 3a_{32} &= 0 \\ a_{12} + a_{22} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & -\frac{6}{5} \end{array} \right)$$

$$a_{12} = \frac{9}{5}, a_{22} = -\frac{4}{5}, a_{32} = -\frac{6}{5}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{v}_3 = a_{13}\vec{w}_1 + a_{23}\vec{w}_2 + a_{33}\vec{w}_3$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}\vec{\mathbf{v}}_3 &= a_{13}\vec{\mathbf{w}}_1 + a_{23}\vec{\mathbf{w}}_2 + a_{33}\vec{\mathbf{w}}_3 \\ (1, 2, 1) &= a_{13}(0, 2, 1) + a_{23}(-1, 0, 1) + a_{33}(-1, 3, 0)\end{aligned}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_3 = a_{13}\vec{\mathbf{w}}_1 + a_{23}\vec{\mathbf{w}}_2 + a_{33}\vec{\mathbf{w}}_3$$

$$(1, 2, 1) = a_{13}(0, 2, 1) + a_{23}(-1, 0, 1) + a_{33}(-1, 3, 0)$$

$$-a_{23} - a_{33} = 1$$

$$2a_{13} + 3a_{33} = 2$$

$$a_{13} + a_{23} = 1$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_3 = a_{13}\vec{\mathbf{w}}_1 + a_{23}\vec{\mathbf{w}}_2 + a_{33}\vec{\mathbf{w}}_3$$

$$(1, 2, 1) = a_{13}(0, 2, 1) + a_{23}(-1, 0, 1) + a_{33}(-1, 3, 0)$$

$$\begin{aligned} -a_{23} - a_{33} &= 1 \\ 2a_{13} + 3a_{33} &= 2 \\ a_{13} + a_{23} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array} \right)$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_3 = a_{13}\vec{\mathbf{w}}_1 + a_{23}\vec{\mathbf{w}}_2 + a_{33}\vec{\mathbf{w}}_3$$

$$(1, 2, 1) = a_{13}(0, 2, 1) + a_{23}(-1, 0, 1) + a_{33}(-1, 3, 0)$$

$$\begin{aligned} -a_{23} - a_{33} &= 1 \\ 2a_{13} + 3a_{33} &= 2 \\ a_{13} + a_{23} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{2}{5} \end{array} \right)$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\vec{\mathbf{v}}_3 = a_{13}\vec{\mathbf{w}}_1 + a_{23}\vec{\mathbf{w}}_2 + a_{33}\vec{\mathbf{w}}_3$$

$$(1, 2, 1) = a_{13}(0, 2, 1) + a_{23}(-1, 0, 1) + a_{33}(-1, 3, 0)$$

$$\begin{aligned} -a_{23} - a_{33} &= 1 \\ 2a_{13} + 3a_{33} &= 2 \\ a_{13} + a_{23} &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{2}{5} \end{array} \right)$$

$$a_{13} = \frac{8}{5}, a_{23} = -\frac{3}{5}, a_{33} = -\frac{2}{5}$$

Change of Basis and Coordinate Transformations

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Change of Basis and Coordinate Transformations

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$B = \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1} \right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2} \right)^{-1} = A^{-1}$$

Change of Basis and Coordinate Transformations

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

2. $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\begin{aligned} B &= \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1} \right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2} \right)^{-1} = A^{-1} \\ &= \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix}. \end{aligned}$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

$$(2, 3, 5) = y_1 (1, 2, 0) + y_2 (2, 0, 1) + y_3 (1, 2, 1)$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

$$(2, 3, 5) = y_1 (1, 2, 0) + y_2 (2, 0, 1) + y_3 (1, 2, 1)$$

$$y_1 + 2y_2 + y_3 = 2$$

$$2y_1 + 2y_3 = 3$$

$$y_2 + y_3 = 5$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

$$(2, 3, 5) = y_1 (1, 2, 0) + y_2 (2, 0, 1) + y_3 (1, 2, 1)$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array} \right)$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

$$(2, 3, 5) = y_1 (1, 2, 0) + y_2 (2, 0, 1) + y_3 (1, 2, 1)$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{19}{4} \end{array} \right)$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$

$$\vec{u} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + y_3 \vec{v}_3$$

$$(2, 3, 5) = y_1 (1, 2, 0) + y_2 (2, 0, 1) + y_3 (1, 2, 1)$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{19}{4} \end{array} \right)$$

$$y_1 = -\frac{13}{4}, y_2 = \frac{1}{4}, y_3 = \frac{19}{4} \Rightarrow [\vec{u}]_{\mathcal{B}_1} = \left(-\frac{13}{4}, \frac{1}{4}, \frac{19}{4} \right).$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

$$(2, 3, 5) = x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0)$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

$$(2, 3, 5) = x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0)$$

$$-x_2 - x_3 = 1$$

$$2x_1 + 3x_3 = 3$$

$$x_1 + x_2 = 5$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

$$(2, 3, 5) = x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0)$$

$$\begin{array}{rcl} -x_2 - x_3 & = & 1 \\ 2x_1 + 3x_3 & = & 3 \\ x_1 + x_2 & = & 5 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array} \right)$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

$$(2, 3, 5) = x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0)$$

$$\begin{array}{rcl} -x_2 - x_3 = 1 \\ 2x_1 + 3x_3 = 3 \\ x_1 + x_2 = 5 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{24}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{array} \right)$$

Change of Basis and Coordinate Transformations

4. $[\vec{u}]_{\mathcal{B}_2}$:

$$\mathcal{B}_2 = \{\vec{w}_1 = (0, 2, 1), \vec{w}_2 = (-1, 0, 1), \vec{w}_3 = (-1, 3, 0)\}$$

$$\vec{u} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + x_3 \vec{w}_3$$

$$(2, 3, 5) = x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0)$$

$$\begin{array}{rcl} -x_2 - x_3 = 1 \\ 2x_1 + 3x_3 = 3 \\ x_1 + x_2 = 5 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{24}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{array} \right)$$

$$x_1 = \frac{24}{5}, x_2 = \frac{1}{5}, x_3 = -\frac{11}{5} \Rightarrow [\vec{u}]_{\mathcal{B}_2} = \left(\frac{24}{5}, \frac{1}{5}, -\frac{11}{5} \right).$$

Change of Basis and Coordinate Transformations

Notice that

$$[\vec{u}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{u}]_{\mathcal{B}_2}$$

Change of Basis and Coordinate Transformations

Notice that

$$[\vec{u}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{u}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix}$$

Change of Basis and Coordinate Transformations

Notice that

$$[\vec{u}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{u}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix}$$

$$[\vec{u}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\vec{u}]_{\mathcal{B}_1}$$

Change of Basis and Coordinate Transformations

Notice that

$$\begin{aligned} [\vec{u}]_{\mathcal{B}_1} &= [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{u}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix} \\ [\vec{u}]_{\mathcal{B}_2} &= [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\vec{u}]_{\mathcal{B}_1} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} \end{aligned}$$

Change of Basis and Coordinate Transformations

Example

Let $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two ordered bases of V such that

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2 - \vec{w}_3$$

Change of Basis and Coordinate Transformations

Example

Let $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two ordered bases of V such that

$$\begin{aligned}\vec{v}_1 &= \vec{w}_1 + \vec{w}_2 - \vec{w}_3 \\ \vec{v}_2 &= 2\vec{w}_1 - \vec{w}_2 + \vec{w}_3\end{aligned}$$

Change of Basis and Coordinate Transformations

Example

Let $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two ordered bases of V such that

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2 - \vec{w}_3$$

$$\vec{v}_2 = 2\vec{w}_1 - \vec{w}_2 + \vec{w}_3$$

$$\vec{v}_3 = \vec{w}_1 + 2\vec{w}_3.$$

Change of Basis and Coordinate Transformations

Example

Let $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two ordered bases of V such that

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2 - \vec{w}_3$$

$$\vec{v}_2 = 2\vec{w}_1 - \vec{w}_2 + \vec{w}_3$$

$$\vec{v}_3 = \vec{w}_1 + 2\vec{w}_3.$$

1. $[M]_{\mathcal{B}_1}^{\mathcal{B}_2} = ?$

Change of Basis and Coordinate Transformations

Example

Let $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two ordered bases of V such that

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2 - \vec{w}_3$$

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Change of Basis and Coordinate Transformations

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3. $[\vec{u}]_{\mathcal{B}_1} = (1, 0, 3) \Rightarrow [\vec{u}]_{\mathcal{B}_2} = ?$

Change of Basis and Coordinate Transformations

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4. $[\vec{v}]_{\mathcal{B}_2} = (2, -1, -2) \Rightarrow [\vec{v}]_{\mathcal{B}_1} = ?$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$[\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

Change of Basis and Coordinate Transformations

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2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\left([\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = \left([\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$[\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\begin{aligned} ([\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}) &= ([\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2})^{-1} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

3. $[\vec{u}]_{\mathcal{B}_2}$:

$$[\vec{u}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\vec{u}]_{\mathcal{B}_1}$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_2}$:

$$[\vec{u}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\vec{u}]_{\mathcal{B}_1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

Change of Basis and Coordinate Transformations

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4. $[\vec{v}]_{\mathcal{B}_1}$:

$$[\vec{v}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{v}]_{\mathcal{B}_2}$$

Change of Basis and Coordinate Transformations

3. $[\vec{u}]_{\mathcal{B}_2}$:

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4. $[\vec{v}]_{\mathcal{B}_1}$:

$$[\vec{v}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\vec{v}]_{\mathcal{B}_2} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

?