DERIVATIVE

1) Let $f(x) = \begin{cases} (cx+1)^3, & x \leq 0 \\ x+1, & x > 0 \end{cases}$. Find all possible values for c that makes the function f(x) differentiable.

For a function to be differentiable, it must be continuous and derivative must exist for all domain.

$$\left.\begin{aligned} & f(0) = 1 \\ & \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (cx+1)^3 = 1 \\ & \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1 \end{aligned}\right\} f(x) \text{ is continuous } \forall x \in \mathbb{R}$$

(other than $x=0$, polynomials are continuous, no need to check.)

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(ch+1)^3-1}{h} = \lim_{h\to 0^-} \frac{c^3h^{\cancel{3}2}+3c^2h^2+3c\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0^-} \frac{c^3h^2+3c^2h+3c}{1} = 3c$$

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\cancel{h}+\cancel{1-1}}{\cancel{h}} = 1$$

$$f'_-(0) = f'_+(0) \Rightarrow 3c = 1 \Rightarrow \boxed{c = \frac{1}{3}}$$

2) Let $f(x) = \begin{cases} e^{ax}, & x \leq 0 \\ 1+\sin(bx), & x > 0 \end{cases}$. Find the possible values for a and b, such that f(x) is differentiable.

<u>Continuity</u>

$$\left.\begin{aligned} & \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{ax} = 1 \\ & \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1+\sin(bx)) = 1 \end{aligned}\right\} \begin{aligned} & \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 1 \\ & [f(0) = e^{a\cdot 0} = e^0 = 1] \end{aligned}$$

Derivative

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{e^{ah}-1}{h} = a$$

$\downarrow$ property (also, $\left(\frac{0}{0}\right)$)

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\cancel{1}+\sin(bh)-\cancel{1}}{h} = b \quad \left(\theta\to 0 \ \frac{\sin\theta}{\theta}=1\right)$$

Function $f(x)$ is differentiable at $x=0$ iff $f'_+(0)=f'_-(0)$ that is, $a=b$. Other than the point $x=0$, pieces of the function are differentiable as well (known fact).

③ Let $f(x)=\begin{cases} x^3, & x>0 \\ -x^3, & x\leq 0 \end{cases}$. a) $f'(0)=?$ b) $f''(0)=?$ c) $f'''(0)=?$

a) $f(0)=0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^3 = 0$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^3 = 0 \Rightarrow$ continuous.

$$f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0^+} \frac{\cancel{x^3}+3x^2\cancel{h}+3xh^{\cancel{2}}-h^{\cancel{3}\,2}-\cancel{x^3}}{\cancel{h}} = 3x^2$$

$$f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-(x+h)^3+x^3}{h} = \lim_{h\to 0^-} \frac{-\cancel{x^3}-3x^2\cancel{h}-3xh^{\cancel{2}}-h^{\cancel{3}\,2}+\cancel{x^3}}{\cancel{h}} = -3x^2$$

$$f'(x) = \begin{cases} 3x^2, & x>0 \\ -3x^2, & x\leq 0 \end{cases} . \quad f'(0) = f'_+(0) = f'_-(0) = 0.$$

b) $f'(0)=0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x^2 = 0$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -3x^2 = 0 \Rightarrow$ continuous.

$$f''_+(x) = \lim_{h\to 0^+} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0^+} \frac{3(x+h)^2-3x^2}{h} = \lim_{h\to 0^+} \frac{\cancel{3x^2}+6x\cancel{h}+h^{\cancel{2}}-\cancel{3x^2}}{\cancel{h}} = 6x$$

$$f''_-(x) = \lim_{h\to 0^-} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0^-} \frac{-3(x+h)^2+3x^2}{h} = \lim_{h\to 0^-} \frac{-\cancel{3x^2}-6x\cancel{h}-h^{\cancel{2}}+\cancel{3x^2}}{\cancel{h}} = -6x$$

$$f''(x) = \begin{cases} 6x, & x>0 \\ -6x, & x\leq 0 \end{cases} . \quad f''(0) = f''_+(0) = f''_-(0) = 0.$$

c) $f''(0)=0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 6x = 0$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -6x = 0 \Rightarrow$ continuous.

We won't be needing $f'''(x)$, we can directly work for $x=0$.

$$f'''_+(0) = \lim_{h\to 0^+} \frac{f''(0+h)-f''(0)}{h} = \lim_{h\to 0^+} \frac{6\cancel{h}}{\cancel{h}} = 6$$

$$f'''(0) = \lim_{h\to 0^-} \frac{f''(0+h)-f''(0)}{h} = \lim_{h\to 0^-} \frac{-6\cancel{h}}{\cancel{h}} = -6$$

Since $f'''_+(0) \neq f'''_-(0)$, $f'''(0)$ doesn't exist.

When calculating $f'(x)$, $f(x)$ must be continuous and function inside the limit for derivative is $f(x)$. But, for $f''(x)$, $f'(x)$ must be continuous and we find limit for derivative by $f'(x)$. Similarly, for $f'''(x)$, $f''(x)$ is the handled function.

4) If f is a differentiable function, evaluate

$$\lim_{h\to 0} \frac{f(h^2)-f(h)}{h}.$$

Remember the definition of derivative!

$$\lim_{h\to 0} \frac{f(h^2)-f(0)+f(0)-f(h)}{h} \quad \text{(add and subtract } f(0))$$

$$\lim_{h\to 0} \frac{f(h^2)-f(0)}{h} - \underbrace{\lim_{h\to 0} \frac{f(h)-f(0)}{h}}_{f'(0)} \quad \text{(write as separate limits)}$$

$$\underbrace{\lim_{h\to 0} \frac{f(h^2)-f(0)}{h^2}}_{f'(0)} \cdot h - f'(0)$$

(multiplicate and divide by h)

Let $h^2 = t$, $h\to 0$, $t\to 0$: $\lim_{t\to 0} \frac{f(t)-f(0)}{h} = f'(0)$

$$f'(0)\cdot \underbrace{\lim_{h\to 0} h}_{0} - f'(0) = -f'(0) \quad \text{(don't forget the limit of } h.)$$

5) Let f be a differentiable function. Compute

$$\lim_{h\to 0} \frac{f(1+h)-f(1-h^2)}{h}.$$

$$\lim_{h\to 0} \frac{f(1+h)-f(1-h^2)}{h} = \lim_{h\to 0} \frac{f(1+h)-f(1)+f(1)-f(1-h^2)}{h}$$

$$= \underbrace{\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}}_{f'(1)} + \underbrace{\lim_{h\to 0} \frac{f(1-h^2)-f(1)}{-h}\cdot\frac{h}{h}}_{f'(1).\lim_{h\to 0} h = f'(1).0=0} = f'(1)$$

6) Let $f(x) = \begin{cases} \dfrac{\sin^2 x}{x} & , x<0 \\ \sqrt{ax+1}-1 & , x\geq 1 \end{cases}$ where $a>0$.

Find the values of a and $f'(0)$, if possible.

Before the derivative, we need to check for the continuity of the function at $x=0$.

$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{\sin^2 x}{x} = \lim\limits_{x\to 0^-} \underbrace{\dfrac{\sin x}{x}}_{1}\cdot\sin x = 0 \\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left(\sqrt{ax+1}-1\right) = 0 \end{array}\right\} \text{Both equal}$$

Also, $\lim\limits_{x\to 0} f(x) = f(0) = \sqrt{a.0+1}-1 = 0$. Thus, f is continuous at $x=0$.

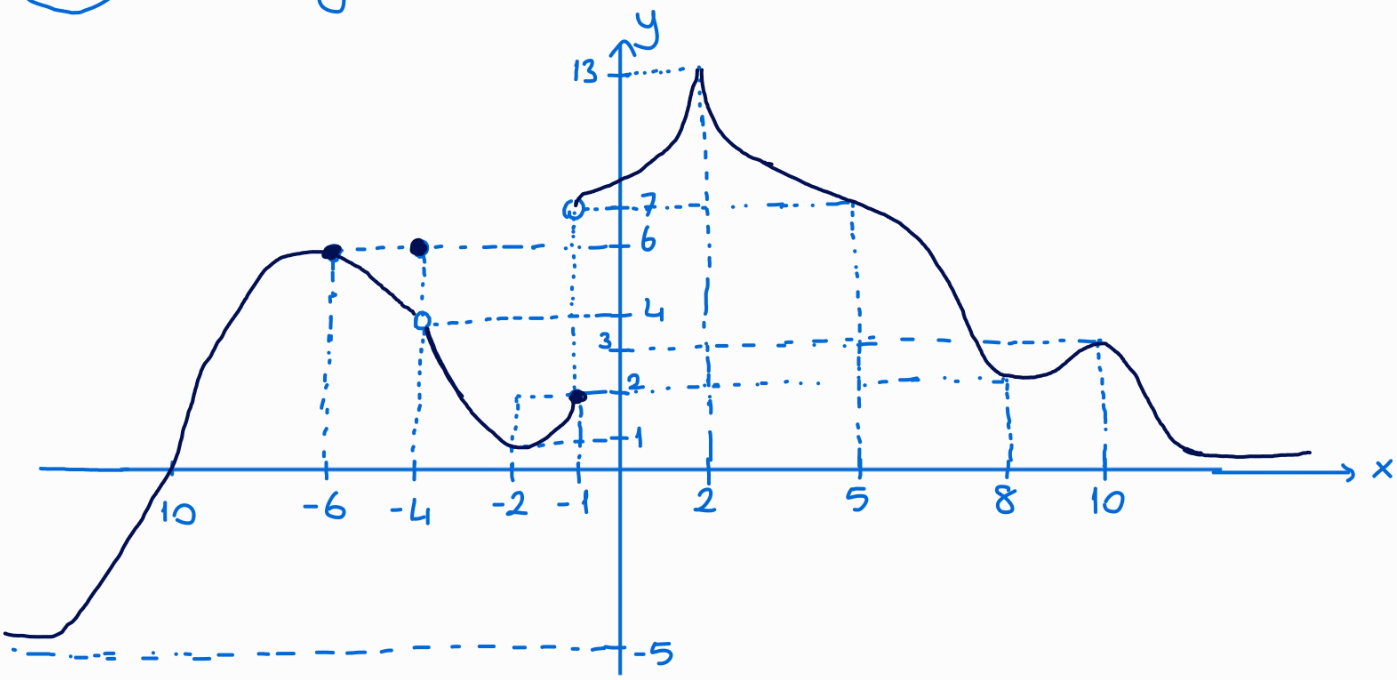
$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\sin^2 h}{h.h} = 1$$

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt{ah+1}-1}{h} \stackrel{L}{=} \lim_{h\to 0^+} \frac{\frac{a}{2\sqrt{ah+1}}}{1}$$

$$= \lim_{h\to 0^+} \frac{a}{2\sqrt{ah+1}} = \frac{a}{2} \quad (\text{must be equal to } f'_-(0)=1)$$

$$\Rightarrow \frac{a}{2}=1 \Rightarrow a=2,\quad f'(0)=1.$$

7) The graph of the function f is given below.



Considering the graph, answer the following questions.

a) $\lim\limits_{x \to 2} f(x) = ?$ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = 13$.

b) $\lim\limits_{x \to -1} f(x) = ?$ $\lim\limits_{x \to -1^-} f(x) = 2$, $\lim\limits_{x \to -1^+} f(x) = 7$ $\Rightarrow$ not equal $\Rightarrow$ no limit.

c) $\lim\limits_{x \to 2^-} f'(x) = ?$ f is not differentiable at $x=2$ (due to cusp. infinitely many tangents). However, limit is $\underline{\underline{\infty}}$.

d) $\lim\limits_{x \to -\infty} f(x) = ?$ -5 (horizontal asymptote)

e) Classify the discontinuity points of f, if any.

At $x=-1$, $\lim\limits_{x \to -1^-} f(x) \neq \lim\limits_{x \to -1^+} f(x) \Rightarrow$ jump discontinuity

At $x=-4$, $\lim\limits_{x \to -4^-} f(x) = \lim\limits_{x \to -4^+} f(x) = 4 \neq f(-4) = 6 \Rightarrow$ removable dis.

f) Find the points where f is not differentiable.

At $x=-1$ and $x=-4$ (due to discontinuity) and at $x=2$ (due to cusp).

8) If the function $f(x)=|x^3-3x+a|$ doesn't have a derivative at $x=2$, calculate the value of a.

The derivative of an absolute value function doesn't exist at its root. Hence, $x=2$ is a root of $f(x)$ that is, $f(2)=0$.

$$f(2)=|8-6+a|=0 \Rightarrow |2+a|=0 \Rightarrow 2+a=0 \Rightarrow a=-2.$$

9) Determine the value of $f'(-1)$ if $f(x)$ is given by.

$$f(x)=\frac{(x+1).h(x)}{(2x+1).h(2x+1)}, \quad h(-1)\neq 0.$$

$$f'(-1)=\lim_{x\to -1}\frac{f(x)-f(1)}{x-(-1)} \quad \text{where } f(-1)=\frac{0.h(-1)}{(-1).h(-1)}=0.$$

$$f'(-1)=\lim_{x\to -1}\frac{\frac{\cancel{(x+1)}.h(x)}{(2x+1).h(2x+1)}-0}{\cancel{x+1}}=\lim_{x\to -1}\frac{h(x)}{(2x+1).h(2x+1)}=\frac{\cancel{h(-1)}}{-\cancel{h(-1)}}=-1.$$

10) Let $f(0)=5$, $\lim\limits_{x\to 0}\frac{f(x)-5}{x}=4$ and $g(x)=(x^2+2x+3).f(x)$.

Find $g'(0)$.

$$g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\frac{(x^2+2x+3).f(x)-\overbrace{3.f(0)}^{-15}}{x}$$

$$=\lim_{x\to 0}\frac{(x^2+2\cancel{x}).f(x)}{\cancel{x}}+\lim_{x\to 0}3\Big(\underbrace{\frac{f(x)-5}{x}}_{4}\Big)=2\underbrace{f(0)}_{5}+12=22$$

Alternative solution

$$g'(x)=(2x+2).f(x)+(x^2+2x+3).f'(x)$$

$$g'(0)=\underbrace{2.f(0)}_{2.5}+\underbrace{3.f'(0)}_{3.4}=10+12=22$$

$$\left[\;\lim_{x\to 0}\frac{f(x)-5}{x}\underset{\left(\frac{0}{0}\right)}{\overset{L}{=}}\lim_{x\to 0}\frac{f'(x)}{1} \;\Rightarrow\; f'(0)=4\;\right]$$

(11) Let f be a function such that $f(x) \neq 0, \forall x$.
Find $g'(x)$ for $g(x) = \dfrac{1}{f\left(\frac{1}{x}\right)}$ where $x \neq 0$.

$$g'(x) = \frac{0 \cdot f\left(\frac{1}{x}\right) - 1 \cdot f'\left(\frac{1}{x}\right)}{f^2\left(\frac{1}{x}\right)} \cdot \left(-\frac{1}{x^2}\right) = \frac{f'\left(\frac{1}{x}\right)}{x^2 \cdot f^2\left(\frac{1}{x}\right)}.$$

(12) Let f be a continuous and not differentiable for the point $x=1$ and $f(1)=2$. Find the value of $g'(1)$ where $g(x) = f(x) \cdot \sin(\pi x)$.

Since f is not differentiable for $x=1$, we must use the definition of derivative.

$$g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{f(1+h) \cdot \sin(\pi(1+h)) - f(1) \cdot \overbrace{\sin \pi}^{0}}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) \cdot \sin(\pi + \pi h)}{h} = \lim_{h \to 0} f(1+h) \cdot \underbrace{\frac{\sin(-\pi h)}{h}}_{-\pi} = -\pi \cdot f(1) = -2\pi$$

(13) If $f\left(\frac{1}{t}\right) + g(\sqrt{t}) = t^2 + 1$ and $g'(1) = 5$, calculate the value of $f'(1)$.

Take derivative.

$$f'\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right) + g'(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = 2t \qquad \frac{1}{t} = 1 \text{ if } t = 1.$$

$$f'(1) \cdot (-1) + \underbrace{g'(1)}_{5} \cdot \frac{1}{2} = 2 \Rightarrow -f'(1) + \frac{5}{2} = 2$$

$$\Rightarrow f'(1) = \frac{1}{2}.$$

(14) On a curve with the function of $y=x^3-6x+12$, two tangent lines, parallel to x-axis, have been drawn. Determine the distance between these two lines.

Line is parallel to x-axis $\Leftrightarrow$ slope is 0.

$y' = 3x^2-6=0 \Rightarrow x^2=2 \Rightarrow x=\mp\sqrt{2}$

For $x_1=\sqrt{2} \Rightarrow y_1 = 2\sqrt{2}-6\sqrt{2}+12 = 12-4\sqrt{2}$

For $x_2=-\sqrt{2} \Rightarrow y_2=-2\sqrt{2}+6\sqrt{2}+12 = 12+4\sqrt{2}$

Distance: $y_2-y_1 = 8\sqrt{2}$.

(Only y values are used for distance, because we have two lines, not two points. They are parellel <u>through</u> x-axis)

(15) Determine the derivative of $f(\sqrt{|-x|+3})$ for $x=-1$, if the relation $\lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2} = -\dfrac{1}{3}$ is known.

Clearly, $\lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2} = f'(2)$. That is, $f'(2) = -\dfrac{1}{3}$.

For $x=-1 \Rightarrow \sqrt{|-1|+3} = \sqrt{1+3} = 2$.

$$\frac{d}{dx}f\left(\sqrt{-x+3}\right) = \underbrace{\frac{-1}{2\sqrt{-x+3}}}_{-\frac{1}{4}} \cdot \underbrace{f'\left(\sqrt{-x+3}\right)}_{-\frac{1}{3}} = \frac{1}{12}.$$

(16) Evaluate y' for $x.\arctan(xy)+\tan(\sqrt{y}) = \sin(x^2-y^2)$.

Derivative: $\arctan(xy) + x.\dfrac{y+xy'}{1+x^2y^2} + \dfrac{\sec^2(\sqrt{y}).y'}{2\sqrt{y}} = \cos(x^2-y^2).(2x-2yy')$

$$y' = \left[2x.\cos(x^2-y^2) - \arctan(xy) - \frac{xy}{1+x^2y^2}\right].\left[\frac{x^2}{1+x^2y^2} + \frac{\sec^2(\sqrt{y})}{2\sqrt{y}} + 2y\cos(x^2-y^2)\right]^{-1}$$

(17) Let $y.\ln x = x.e^y - 1$. Find $\frac{dy}{dx}$.

$$y'.\ln x + y.\frac{1}{x} = e^y + x.e^y.y'$$

$$y'(\ln x - x.e^y) = e^y - \frac{y}{x} = \frac{x.e^y - y}{x}$$

$$\frac{dy}{dx} = \frac{x.e^y - y}{x(\ln x - x.e^y)}.$$

(18) Find the tangent and normal lines of the function given as $x.\sin y + \cos y^2 = \ln x^2$ at the point $P(1,0)$.

$$x.\sin y + \cos y^2 = 2\ln x. \quad \text{(Take derivative)}$$

$$\sin y + x.\cos y.y' + 2y.y'.(-\sin y^2) = \frac{2}{x}. \quad \text{(Put point)}$$

$$\underbrace{\sin 0}_{0} + \underbrace{1.\cos 0.y'}_{y'} + \underbrace{2.0.y'(-\sin 0)}_{0} = \frac{2}{1} \Rightarrow y' = 2 = m_T$$

$$m_T.m_N = -1 \Rightarrow 2.m_N = -1 \Rightarrow m_N = -\frac{1}{2}$$

Tangent line: $y = 2.(x-1) = 2x - 2$

Normal line: $y = -\frac{1}{2}(x-1) = \frac{1-x}{2}$

(19) Find the equation of the tangent line for the function $x.e^y + y.\sin x = \ln(x+y)$ at $x=0$.

$$x=0 \Rightarrow 0.e^y + y.\sin 0 = \ln(0+y) \Rightarrow \ln y = 0 \Rightarrow y = 1. \quad P(0,1)$$

Derivative: $e^y + x.e^y.y' + y'.\sin x + y.\cos x = \frac{1+y'}{x+y}$

$$\Rightarrow e^1 + \underbrace{0.e^1.y'|_P}_{0} + \underbrace{y'|_P.\sin 0}_{0} + 1.\cos 0 = \frac{1+y'|_P}{0+1}$$

$$\Rightarrow e + \underline{1} = y'|_P + \underline{1} \Rightarrow m_T = e$$

Tangent line: $y - 1 = ex \Rightarrow y = ex + 1.$

20) Find the point(s) on the curve $y=\frac{x}{x+1}+x$ such that the slope of the tangent line is $\frac{5}{4}$.

$$y'\big|_{?}=\frac{5}{4}.\quad y'=\frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}+1=\frac{1}{(1+x^2)}+1=\frac{5}{4}\Rightarrow\frac{1}{(1+x^2)}=\frac{1}{4}$$

$$(x+1)^2=4\Rightarrow x_1=1,\quad x_2=-3.$$

For $x_1=1\Rightarrow y_1=\frac{1}{1+1}+1=\frac{3}{2}.\quad P_1\left(1,\frac{3}{2}\right)$

For $x_2=-3\Rightarrow y_2=\frac{-3}{-3+1}-3=\frac{3}{2}-3=-\frac{3}{2}.\quad P_2\left(-3,-\frac{3}{2}\right)$

21) Let $h(2)=2$, $h'(2)=1$, $f'(8)=-1$. Find $g'(2)$ for the function $g(x)=f(x^2+x.h(x))$.

$$g'(x)=f'(x^2+x.h(x)).(2x+h(x)+x.h'(x))$$

$$g'(2)=f'(\underbrace{4+2.\underbrace{h(2)}_{2}}_{8}).(4+\underbrace{h(2)}_{2}+2.\underbrace{h'(2)}_{1})=f'(8).8=-8$$

22) Find the points which the curve $\sin(xy)=1-x^2-y^2+x^2y^3$ intersects with the x-axis. Determine the relations of the tangent lines at these points.

Intersecting with the x-axis $\Leftrightarrow y=0$

$$y=0\Rightarrow 0=1-x^2\Rightarrow x=\mp1.$$

$$F(x,y)=1-x^2-y^2+x^2y^3-\sin(xy)=0$$

$$y'=-\frac{F_x}{F_y}=-\frac{-2x+2xy^3-y\cos(xy)}{-2y+3x^2y^2-x\cos(xy)}\qquad m_1=y'\big|_{(1,0)}=-2\quad m_2=y'\big|_{(-1,0)}=-2$$

$\Rightarrow$ Since the two tangent lines have the same slope, they are parallel to each other.

23) Let g be a continuous but not differentiable function at $x=0$ and $g(0)=8$. Consider the function f is given by $f(x)=x.g(x)$. Find the value of $f'(0)$.

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\cancel{h}.g(h)-0}{\cancel{h}}=g(0)=8.$$

24) Find the point which the curve intersects with the y-axis, tangent, and normal lines (at this point) for the curve $yx^2+e^y\cos x+\arctan x=e$.

Intersecting the y-axis $\Leftrightarrow$ $x=0$.

$x=0 \Rightarrow 0+e^y+0=e \Rightarrow y=1 \quad P(0,1)$.

$$y'.x^2+2xy+e^y.y'.\cos x-e^y.\sin x+\frac{1}{1+x^2}=0$$

$$\underbrace{y'|_P.0}_{0}+\underbrace{2.0.1}_{0}+\underbrace{e^1.y'|_P.\cos 0}_{e.y'|_P}-\underbrace{e^1.\sin 0}_{0}+\underbrace{\frac{1}{1+0}}_{1}=0$$

$\Rightarrow y'|_P=-\frac{1}{e}=m_T \qquad m_T.m_N=-1 \Rightarrow m_N=e$

Tangent line: $y-1=-\frac{1}{e}.x \Rightarrow y=\frac{e-x}{e}$

Normal line: $y-1=ex \Rightarrow y=ex+1$

25) Let $y=\left(\frac{1}{x}\right)^{\ln x}$. Find y'.

$\ln y=\ln x.\ln\left(\frac{1}{x}\right)=\ln x.\ln x^{-1}=-\ln^2 x$. (or $-(\ln x)^2$)

$$\frac{y'}{y}=-2.\ln x.\frac{1}{x} \Rightarrow y'=\frac{-2\ln x}{x}.y=-\frac{2\ln x}{x}.\left(\frac{1}{x}\right)^{\ln x}.$$

26) Let $y = (\cos x)^x - x^{\cos x}$. $y' = ?$

$$y = \underbrace{(\cos x)^x}_{A} - \underbrace{x^{\cos x}}_{B}$$

For $A = (\cos x)^x$

$\ln A = x \cdot \ln(\cos x)$ (Take ln)

$\dfrac{A'}{A} = \ln(\cos x) - \dfrac{x \cdot \sin x}{\cos x}$ (Take derivative)

$A' = [\ln(\cos x) - x \tan x] \cdot (\cos x)^x$ (Multiplicate both by A)

For $B = x^{\cos x}$

$\ln B = \cos x \cdot \ln x$

$\dfrac{B'}{B} = -\sin x \cdot \ln x + \cos x \cdot \dfrac{1}{x} \Rightarrow B' = \left[\dfrac{\cos x}{x} - \sin x \cdot \ln x\right] \cdot x^{\cos x}$

$y' = A' - B' = [\ln(\cos x) - x \cdot \tan x] \cdot (\cos x)^x - \left[\dfrac{\cos x}{x} - \sin x \cdot \ln x\right] \cdot x^{\cos x}$.

27) Find $f'(x)$ where $f(x) = \sqrt[3]{\dfrac{(x+1)^2 \cdot (x-2)}{x^2+4}}$.

Use logarithmic differentiation.

$$\ln[f(x)] = \ln\left[\frac{(x+1)^2 \cdot (x-2)}{x^2+4}\right]^{\frac{1}{3}} = \frac{1}{3}\left[2\ln(x+1) + \ln(x-2) - \ln(x^2+4)\right]$$

$$\frac{f'(x)}{f(x)} = \frac{1}{3}\left[\frac{2}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right]$$

$$f'(x) = \frac{1}{3}\left[\frac{2}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right] \cdot \sqrt[3]{\frac{(x+1)^2 \cdot (x-2)}{x^2+4}}.$$

(28) Let $f(0)=1$ and $f'(0)=2$. Evaluate $g'(0)$ if $g(x)=[f(x)]^{\arccos x+x}$.

$$\ln[g(x)]=(\arccos x+x).\ln[f(x)]$$

$$\frac{g'(x)}{g(x)}=\left(\frac{-1}{\sqrt{1-x^2}}+1\right).\ln[f(x)]+(\arccos x+x).\frac{f'(x)}{f(x)}$$

$$g(0)=[f(0)]^{\arccos 0+0}=1^{\pi/2}=1$$

$$\underbrace{\frac{g'(0)}{g(0)}}_{g'(0)}=\underbrace{\left(\underbrace{\frac{-1}{\sqrt{1-0}}}_{-1}+1\right)}_{0}.\underbrace{\ln[\underbrace{f(0)}_{1}]}_{0}+\underbrace{(\arccos 0+0)}_{\frac{\pi}{2}}.\underbrace{\frac{f'(0)}{f(0)}}_{2}$$

$$\Rightarrow g'(0)=2.\frac{\pi}{2}=\pi.$$

(29) Let f be a differentiable function defined by $f(x)=(\cos x^4)^{\arctan x^2}$. Find $f'(0)$.

$$\ln[f(x)]=\arctan x^2.\ln[\cos x^4]$$

$$\frac{f'(x)}{f(x)}=\frac{2x}{1+x^2}.\ln[\cos x^4]-\arctan x^2.\frac{4x^3.\sin x^4}{\cos x^4}$$

$$f(0)=(\cos 0)^{\arctan 0}=1^0=1$$

$$\underbrace{\frac{f'(0)}{f(0)}}_{f'(0)}=\underbrace{\frac{2.0}{1+0}}_{0}.\underbrace{\ln[\cos 0]}_{\ln 1=0}-\underbrace{\arctan 0}_{0}.\underbrace{\frac{4.0.\sin 0}{\cos 0}}_{0}$$

$$\Rightarrow f'(0)=0.$$

30) Let a function be given such that $f\left(\frac{\pi}{2}\right)=6$ and $f'\left(\frac{\pi}{2}\right)=3$. If another function g is defined by $g(x)=[f(x)]^{\sin x}$, find $g'(x)$ at $x=\frac{\pi}{2}$.

$$\ln[g(x)]=\sin x.\ln[f(x)]$$

$$\frac{g'(x)}{g(x)}=\cos x.\ln[f(x)]+\sin x.\frac{f'(x)}{f(x)}$$

$$g\left(\frac{\pi}{2}\right)=\left[f\left(\frac{\pi}{2}\right)\right]^{\sin\frac{\pi}{2}}=6^1=6$$

$$\frac{g'\left(\frac{\pi}{2}\right)}{g\left(\frac{\pi}{2}\right)}=\underbrace{\cos\frac{\pi}{2}}_{0}.\ln\Big[\underbrace{f\left(\frac{\pi}{2}\right)}_{6}\Big]+\underbrace{\sin\frac{\pi}{2}}_{1}.\underbrace{\frac{f'\left(\frac{\pi}{2}\right)}{f\left(\frac{\pi}{2}\right)}}_{3/6}$$

$$\Rightarrow g'\left(\frac{\pi}{2}\right)=3.$$

31) Find the derivative of the function $f(x)=x^{\sin x}$ at the point $x=\frac{\pi}{2}$.

$$\ln[f(x)]=\sin x.\ln x$$

$$\frac{f'(x)}{f(x)}=\cos x.\ln x+\sin x.\frac{1}{x}$$

$$f\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)^{\sin\frac{\pi}{2}}=\left(\frac{\pi}{2}\right)^1=\frac{\pi}{2}$$

$$\frac{f'\left(\frac{\pi}{2}\right)}{f\left(\frac{\pi}{2}\right)}=\underbrace{\cos\frac{\pi}{2}}_{0}.\ln\frac{\pi}{2}+\underbrace{\sin\frac{\pi}{2}}_{1}.\frac{2}{\pi}\Rightarrow f'\left(\frac{\pi}{2}\right)=\frac{2}{\pi}.\frac{\pi}{2}=1.$$

(32) Let g and h be positive valued and differentiable functions and $g(1)=h'(1)=1$, $g'(1)=h(1)=2$. Find $f'(1)$ where $f(x)=[g(x^2)]^{h(x)}$.

$$\ln[f(x)]=h(x).\ln[g(x^2)]$$

$$\frac{f'(x)}{f(x)}=h'(x).\ln[g(x^2)]+h(x).\frac{g'(x^2).2x}{g(x^2)}$$

$$\frac{f'(1)}{f(1)}=\underbrace{h'(1)}_{1}.\underbrace{\ln[\underbrace{g(1)}_{1}]}_{0}+\underbrace{h(1)}_{2}.\underbrace{\frac{g'(1).2}{g(1)}}_{4}$$

$$f(1)=[g(1)]^{h(1)}=1^2=1$$

$$\Rightarrow f'(1)=2.4=8.$$

(33) Let $f(x)=x.\sqrt{3+x^2}$ be an invertible function. Evaluate $(f^{-1})'(-2)$.

$$f(a)=b=-2 \Rightarrow a.\underbrace{\sqrt{3+a^2}}_{\geq 0}=-2 \Rightarrow a=-1$$

$$f'(x)=\sqrt{3+x^2}+x.\frac{\cancel{2}x}{\cancel{2}\sqrt{3+x^2}} \Rightarrow f'(-1)=2+\frac{1}{2}=\frac{5}{2}$$

$$(f^{-1})'(b)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(-2)=\frac{1}{f'(-1)}=\frac{1}{\frac{5}{2}}=\frac{2}{5}.$$

(34) Let $f(x) = \ln(x+1) + \arctan(x^2+1)$ $(x \geq 0)$ be an invertible function. Find $(f^{-1})'(\frac{\pi}{4})$.

$$f(a) = b = \frac{\pi}{4} \Rightarrow \ln(a+1) + \arctan(a^2+1) = \frac{\pi}{4} \Rightarrow a = 0$$

$$f'(x) = \frac{1}{x+1} + \frac{2x}{1+(x^2+1)^2} \Rightarrow f'(0) = 1 + 0 = 1$$

$$(f^{-1})'(\frac{\pi}{4}) = \frac{1}{f'(f^{-1}(\frac{\pi}{4}))} = \frac{1}{f'(0)} = 1.$$

(35) If $f(x) = \frac{4x^3}{x^2+1}$, $(f^{-1})'(2) = ?$

$$f(a) = b = 2 \Rightarrow \frac{4a^3}{a^2+1} = 2 \Rightarrow 4a^3 = 2a^2+1 \Rightarrow a = 1$$

$$f'(x) = \frac{12x^2(x^2+1) - 4x^3 . 2x}{(x^2+1)^2} \Rightarrow f'(1) = \frac{12 . 2 - 4 . 2}{4} = \frac{24-8}{4} = 4.$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}.$$

(36) Let $f : (-\infty, 1] \to \mathbb{R}$, $f(x) = \frac{1+x}{\sqrt{1+x^2}}$. Find $(f^{-1})'(0)$.

$$\frac{1+a}{\sqrt{1+a^2}} = 0 \Rightarrow 1+a = 0 \Rightarrow a = -1 \quad (f^{-1}(0) = -1)$$

$$f'(x) = \frac{\sqrt{1+x^2} - (1+x) . \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}}{1+x^2} \Rightarrow f'(-1) = \frac{\sqrt{2} - 0}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$(f^{-1})'(0) = \frac{1}{f'(-1)} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

37) If $f(x)=2+\arctan x+e^{2x}$ is an invertible function, find $(f^{-1})'(3)$.

$$2+\arctan a+e^{2a}=3 \Rightarrow \arctan a+e^{2a}=1 \Rightarrow a=0.$$

$$f'(x)=\frac{1}{1+x^2}+2\cdot e^{2x} \Rightarrow f'(0)=1+2=3$$

$$(f^{-1})'(3)=\frac{1}{f'(0)}=\frac{1}{3}.$$

38) Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=x.\sec x$ be an invertible function. Find $(f^{-1})'\left(\frac{2\pi}{3}\right)$.

$$a.\sec a=\frac{2\pi}{3} \Rightarrow a=\frac{\pi}{3}$$

$$f'(x)=\sec x+x.\sec x.\tan x \Rightarrow f'\left(\frac{\pi}{3}\right)=\sec\frac{\pi}{3}+\frac{\pi}{3}\cdot\sec\frac{\pi}{3}.\tan\frac{\pi}{3}$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=2+\frac{\pi}{3}.2.\sqrt{3}=\frac{6+2\sqrt{3}\pi}{3}$$

$$(f^{-1})'\left(\frac{2\pi}{3}\right)=\frac{1}{f'\left(\frac{\pi}{3}\right)}=\frac{3}{6+2\sqrt{3}\pi}.$$

39) The equation of the normal line to the curve $y=f(x)$ at $x=1$ is $2x+y-1=0$. If $f^{-1}(x)$ exists at $x=-1$, find $(f^{-1})'(-1)$.

Since we have the equation of the normal line at $x=1$, we can find the other part of the pair (x,y) from this equation. That is,

$$2.1+y-1=0 \Rightarrow y=-1. \text{ Hence, } f(1)=-1.$$

Equation of the normal line: $2x+y-1=0 \Rightarrow y=1-2x$.

$\Rightarrow m_N=-2$ (slope) and $m_T \cdot m_N=-1 \Rightarrow m_T=\frac{1}{2}=f'(1)$

$$(f^{-1})'(-1)=\frac{1}{f'(f^{-1}(-1))}=\frac{1}{f'(1)}=\frac{1}{\frac{1}{2}}=2.$$

40) Let f be an invertible function and its equation of the normal line at $x=1$ be given by $3x+y+1=0$. Find the equation of the tangent line at $x=1$ and the value of $(f^{-1})'(-4)$.

$3.1+y+1=0 \Rightarrow y=-4$. Hence $f(1)=-4$.

$3x+y+1=0 \Rightarrow y=-3x-1 \Rightarrow m_N=-3$ (slope of the normal line)

$m_T \cdot m_N=-1 \Rightarrow m_T=\frac{1}{3}=f'(1)$ (slope of the tangent line)

Since at the point $(1,-4)$ the normal line and the tangent line intersect, we can use this point and the slope of the tangent line to find the equation of the tangent line.

Equation of the tangent line: $y-(-4)=\frac{1}{3}(x-1)$

$$\Rightarrow y=\frac{x-13}{3}.$$

$$(f^{-1})'(-4)=\frac{1}{f'(1)}=\frac{1}{\frac{1}{3}}=3.$$