

Infinite Sequences and Series

Sequences

Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and n th term $a_n = 2n$. The integer n is called the **index** of a_n , and indicates where a_n occurs in the list. Order is important. The sequence $2, 4, 6, 8 \dots$ is not the same as the sequence $4, 2, 6, 8 \dots$.

We can think of the sequence

$$a_1, a_2, a_3, \dots, a_n, \dots$$

as a function that sends 1 to a_1 , 2 to a_2 , 3 to a_3 , and in general sends the positive integer n to the n th term a_n . More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers.

The function associated with the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is described by the formula $a_n = 2n$.

Definition 11.1.1. A sequence is a function whose domain is the set of positive integers.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

EXAMPLE 4 Find $\lim_{n \rightarrow \infty} \frac{n}{n + 1}$.

SOLUTION The method is similar to the one we used in Section 2.6: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{n + 1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1 + 0} = 1\end{aligned}$$

Here we used Equation 4 with $r = 1$. ■

EXAMPLE 5 Is the sequence $a_n = \frac{n}{\sqrt{10 + n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n :

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{10 + n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent. ■

EXAMPLE 6 Calculate $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n \rightarrow \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \dots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n \rightarrow \infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent. ■

EXAMPLE 8 Evaluate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, by Theorem 6,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$
 ■

Calculating Limits of Sequences

Since sequences are functions with domain restricted to the positive integers, it is not surprising that the theorems on limits of functions given in Chapter 2 have versions for sequences.

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$.

- | | |
|-----------------------------------|---|
| 1. <i>Sum Rule:</i> | $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$ |
| 2. <i>Difference Rule:</i> | $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$ |
| 3. <i>Constant Multiple Rule:</i> | $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$ (any number k) |
| 4. <i>Product Rule:</i> | $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$ |
| 5. <i>Quotient Rule:</i> | $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$ |

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.

An immediate consequence of Theorem 2 is that, if $|b_n| \leq c_n$ and $c_n \rightarrow 0$, then $b_n \rightarrow 0$ because $-c_n \leq b_n \leq c_n$. We use this fact in the next example.

EXAMPLE 4 Since $1/n \rightarrow 0$, we know that

$$(a) \frac{\cos n}{n} \rightarrow 0 \quad \text{because} \quad -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n};$$

$$(b) \frac{1}{2^n} \rightarrow 0 \quad \text{because} \quad 0 \leq \frac{1}{2^n} \leq \frac{1}{n};$$

$$(c) (-1)^n \frac{1}{n} \rightarrow 0 \quad \text{because} \quad -\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}. \quad \blacksquare$$

THEOREM 3—The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$. $(n \rightarrow \infty)$

EXAMPLE 6 The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \rightarrow f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 (Figure 10.5). \blacksquare

Using L'Hôpital's Rule

The next theorem formalizes the connection between $\lim_{n \rightarrow \infty} a_n$ and $\lim_{x \rightarrow \infty} f(x)$. It enables us to use l'Hôpital's Rule to find the limits of some sequences.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \rightarrow \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \rightarrow \infty} a_n = L.$$

EXAMPLE 8 Does the sequence whose n th term is

$$a_n = \left(\frac{n+1}{n-1} \right)^n$$

converge? If so, find $\lim_{n \rightarrow \infty} a_n$.

Solution The limit leads to the indeterminate form 1^∞ . We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n :

$$\begin{aligned}\ln a_n &= \ln \left(\frac{n+1}{n-1} \right)^n \\ &= n \ln \left(\frac{n+1}{n-1} \right).\end{aligned}$$

Then,

$$\begin{aligned}\lim_{n \rightarrow \infty} \ln a_n &= \lim_{n \rightarrow \infty} n \ln \left(\frac{n+1}{n-1} \right) && \text{$\infty \cdot 0$ form} \\ &= \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n+1}{n-1} \right)}{1/n} && \frac{0}{0} \text{ form} \\ &= \lim_{n \rightarrow \infty} \frac{-2/(n^2 - 1)}{-1/n^2} && \text{L'Hôpital's Rule: differentiate numerator and denominator.} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2}{n^2 - 1} = 2.\end{aligned}$$

Since $\ln a_n \rightarrow 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \rightarrow e^2.$$

The sequence $\{a_n\}$ converges to e^2 . ■

Commonly Occurring Limits

THEOREM 5 The following six sequences converge to the limits listed below:

$$1. \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2. \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3. \lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$$

$$4. \lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$$

$$5. \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$6. \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

In Formulas (3) through (6), x remains fixed as $n \rightarrow \infty$.

EXAMPLE 9 These are examples of the limits in Theorem 5.

$$(a) \frac{\ln(n^2)}{n} = \frac{2\ln n}{n} \rightarrow 2 \cdot 0 = 0 \quad \text{Formula 1}$$

$$(b) \sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \rightarrow (1)^2 = 1 \quad \text{Formula 2}$$

$$(c) \sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \rightarrow 1 \cdot 1 = 1 \quad \text{Formula 3 with } x = 3 \text{ and Formula 2}$$

$$(d) \left(-\frac{1}{2}\right)^n \rightarrow 0 \quad \text{Formula 4 with } x = -\frac{1}{2}$$

$$(e) \left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \rightarrow e^{-2} \quad \text{Formula 5 with } x = -2$$

$$(f) \frac{100^n}{n!} \rightarrow 0 \quad \text{Formula 6 with } x = 100 \quad \blacksquare$$

Recursive Definitions

So far, we have calculated each a_n directly from the value of n . But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

EXAMPLE 10

- (a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$ define the sequence $1, 2, 3, \dots, n, \dots$ of positive integers. With $a_1 = 1$, we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.
- (b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ for $n > 1$ define the sequence $1, 2, 6, 24, \dots, n!, \dots$ of factorials. With $a_1 = 1$, we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 4 \cdot a_3 = 24$, and so on.
- (c) The statements $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ define the sequence $1, 1, 2, 3, 5, \dots$ of **Fibonacci numbers**. With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, and so on.
- (d) As we can see by applying Newton's method (see Exercise 135), the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$ for $n > 0$ define a sequence that, when it converges, gives a solution to the equation $\sin x - x^2 = 0$. ■

Bounded Monotonic Sequences

Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequence and a *monotonic* sequence.

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n . The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n . The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

EXAMPLE 11

- (a) The sequence $1, 2, 3, \dots, n, \dots$ has no upper bound because it eventually surpasses every number M . However, it is bounded below by every real number less than or equal to 1. The number $m = 1$ is the greatest lower bound of the sequence.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is bounded above by every real number greater than or equal to 1. The upper bound $M = 1$ is the least upper bound (Exercise 127). The sequence is also bounded below by every number less than or equal to $\frac{1}{2}$, which is its greatest lower bound. ■

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n . That is, $a_1 \leq a_2 \leq a_3 \leq \dots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n . The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE 12

- (a) The sequence $1, 2, 3, \dots, n, \dots$ is nondecreasing.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is nondecreasing.
- (c) The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$ is nonincreasing.
- (d) The constant sequence $3, 3, 3, \dots, 3, \dots$ is both nondecreasing and nonincreasing.
- (e) The sequence $1, -1, 1, -1, 1, -1, \dots$ is not monotonic. ■

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Infinite Series

1

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

2 **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{2n}{3n+5} = \lim_{n \rightarrow \infty} \frac{2}{3 + \frac{5}{n}} = \frac{2}{3}$$

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r .

If $r = 1$, then $s_n = a + a + \cdots + a = na \rightarrow \pm\infty$. Since $\lim_{n \rightarrow \infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and $rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

$$\boxed{3} \quad s_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \rightarrow 0$ as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \rightarrow \infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n \rightarrow \infty} s_n$ does not exist. Therefore the geometric series diverges in those cases. ■

We summarize the results of Example 2 as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

V EXAMPLE 3 Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by **[4]** and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$
■

The n th-Term Test for a Divergent Series

Theorem: Test for Divergence

(a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n \rightarrow \infty} a_n = 0$.

(b) If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 7 The following are all examples of divergent series.

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \rightarrow \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \rightarrow 1$. $\lim_{n \rightarrow \infty} a_n \neq 0$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n \rightarrow \infty} (-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$. ■

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

2. *Difference Rule:* $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (any number k).

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Caution Remember that $\sum(a_n + b_n)$ can converge when $\sum a_n$ and $\sum b_n$ both diverge. For example, $\sum a_n = 1 + 1 + 1 + \dots$ and $\sum b_n = (-1) + (-1) + (-1) + \dots$ diverge, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \dots$ converges to 0.

EXAMPLE 9 Find the sums of the following series.

$$\begin{aligned}
 \text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} && \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6 \\
 &= 2 - \frac{6}{5} = \frac{4}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \frac{1}{2^n} && \text{Constant Multiple Rule} \\
 &= 4 \left(\frac{1}{1 - (1/2)} \right) && \text{Geometric series with } a = 1, r = 1/2 \\
 &= 8
 \end{aligned}$$

■

Adding or Deleting Terms

REMARK(4):

A finite number of terms doesn't affect the divergence of a series.

Example

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$$

$$\sum_{n=10}^{\infty} \frac{n^2}{5n^2 + 4}$$

REMARK(5):

A finite number of terms doesn't affect the convergence of a series.

Example

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^n = 5$$

$$\sum_{n=3}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{4}$$

REMARK(6):

A finite number of terms doesn't affect the convergence of a series but it affects the sum.

Reindexing

Example

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

We can write this geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\sum_{m=6}^{\infty} \frac{1}{2^{m-5}}$$

$$\sum_{m=-2}^{\infty} \frac{1}{2^{m+3}}$$

$$m = n + 5$$

$$n = m - 5$$

$$m = n - 3$$

$$n = m + 3$$

Convergence tests for positive series

Definition 9.1 An infinite series is called positive if all its terms are positive; more precisely $\sum a_n$ is positive if $a_n \geq 0$ for all n .

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

- (i) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- (ii) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 1 Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1} x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\tan^{-1} t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

Thus $\int_1^{\infty} 1/(x^2 + 1) dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2 + 1)$ is convergent.

EXAMPLE 3Show that the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x . Since

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left(\frac{1}{b^{p-1}} - 1 \right) \\ &= \frac{1}{1-p} (0 - 1) = \frac{1}{p-1},\end{aligned}$$

$b^{p-1} \rightarrow \infty$ as $b \rightarrow \infty$
because $p-1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p -series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p \leq 0$, the series diverges by the n th-term test. If $0 < p < 1$, then $1-p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b \rightarrow \infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

We have convergence for $p > 1$ but divergence for all other values of p . ■

Definition 9.6 *The harmonic series is the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

1 The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p -series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p -series with $p = \frac{1}{3} < 1$. ■

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

V EXAMPLE 1 Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p -series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test. ■

V EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 4 of Section 11.3, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geq 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \quad k \geq 3$$

We know that $\sum 1/k$ is divergent (p -series with $p = 1$). Thus the given series is divergent by the Comparison Test. ■

THEOREM 11—Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \quad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test. ■

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$\begin{aligned} a_n &= \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \quad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{\frac{2}{n} + 0}{2\sqrt{0 + 1}} = 1 \end{aligned}$$

Since $\sum b_n = 2 \sum 1/n^{1/2}$ is divergent (p -series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test. ■

Absolute Convergence; The Ratio and Root Tests

DEFINITION A series $\sum a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

EXAMPLE 1 This example gives two series that converge absolutely.

- (a) For $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

The original series converges because it converges absolutely.

- (b) For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \dots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \dots,$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ because $|\sin n| \leq 1$ for every n . The original series converges absolutely; therefore it converges. ■

The Ratio Test

The Ratio Test

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

EXAMPLE 2

Investigate the convergence of the following series.

$$(a) \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

$$(b) \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

$$(c) \sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}} \right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \rightarrow 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n! n! / (2n)!$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n! n!} \\ &= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \rightarrow 1. \end{aligned}$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the n th term does not approach zero as $n \rightarrow \infty$. The series diverges. ■

The Root Test

The Root Test

- (i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

V EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2} \right)^n$$
$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{\frac{2}{n} + \frac{3}{n}}{\frac{3}{n} + \frac{2}{n}} \rightarrow \frac{2}{3} < 1$$

Thus the given series converges by the Root Test. ■

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$
$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the n th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \quad b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n \rightarrow \infty} b_n = 0$$

then the series is convergent.

V EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

$$(i) \quad b_{n+1} < b_n \quad \text{because} \quad \frac{1}{n+1} < \frac{1}{n}$$

$$(ii) \quad \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test. ■

V EXAMPLE 2 The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3n}{4n-1} = \lim_{n \rightarrow \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the n th term of the series:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence. ■

Conditional Convergence

DEFINITION A convergent series that is not absolutely convergent is **conditionally convergent**.

EXAMPLE 4 If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore the alternating p -series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely as an ordinary p -series. If $0 < p \leq 1$, the series converges conditionally by the alternating series test. For instance,

$$\text{Absolute convergence } (p = 3/2): \quad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \dots$$

$$\text{Conditional convergence } (p = 1/2): \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots \quad \blacksquare$$

Power Series

Power Series and Convergence

DEFINITIONS A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

EXAMPLE 1 Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x . It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1. \quad (3)$$



EXAMPLE 2 The power series

$$1 - \frac{1}{2} (x - 2) + \frac{1}{4} (x - 2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x - 2)^n + \cdots \quad (4)$$

matches Equation (2) with $a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \dots, c_n = (-1/2)^n$.

This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1 - r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \quad 0 < x < 4.$$

EXAMPLE 3

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

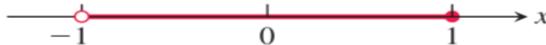
(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \dots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the n th term of the power series in question.

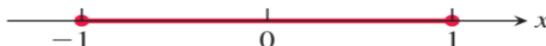
(a) $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x} \right| = \frac{n}{n+1}|x| \rightarrow |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the n th term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \dots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \dots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \leq 1$ and diverges elsewhere.



(b) $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right| = \frac{2n-1}{2n+1}x^2 \rightarrow x^2.$ $2(n+1)-1 = 2n+1$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the n th term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \dots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \leq x \leq 1$ and diverges elsewhere.



(c) $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0$ for every x . $\frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}$

The series converges absolutely for all x .



(d) $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = (n+1)|x| \rightarrow \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.



The Radius of Convergence of a Power Series

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ there are only three possibilities:

- (i) The series converges only when $x = a$.
- (ii) The series converges for all x .
- (iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a . In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \quad (a - R, a + R] \quad [a - R, a + R) \quad [a - R, a + R]$$

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the n th term does not approach zero for those values of x .

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n / \sqrt{n+1}$. Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \left| -3x \sqrt{\frac{n+1}{n+2}} \right| \\ &= 3 \sqrt{\frac{1 + (1/n)}{1 + (2/n)}} |x| \rightarrow 3|x| \quad \text{as } n \rightarrow \infty \end{aligned}$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$$

which diverges. (Use the Integral Test or simply observe that it is a p -series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore the given power series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

V EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n / 3^{n+1}$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right| \\ &= \left(1 + \frac{1}{n} \right) \frac{|x+2|}{3} \rightarrow \frac{|x+2|}{3} \quad \text{as } n \rightarrow \infty \end{aligned}$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1 . When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.

Operations on Power Series

THEOREM 19—The Series Multiplication Theorem for Power Series If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^n a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^n \right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \right) \\ &= (1 + x + x^2 + \cdots) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right) \quad \text{Multiply second series...} \\ &= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right)}_{\text{by 1}} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots \right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots \right)}_{\text{by } x^2} + \cdots \\ &= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6} \cdots \quad \text{and gather the first four powers.} \end{aligned}$$

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

Since $1/(1-x) = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows from Theorem 20 that $1/(1-4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely for $|4x^2| < 1$ or $|x| < 1/2$.

Differentiation and Integration of Power Series

2 Theorem If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

$$\begin{aligned} \text{(i)} \quad f'(x) &= c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1} \\ \text{(ii)} \quad \int f(x) dx &= C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \cdots \\ &= C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} \end{aligned}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R .

This is called **term-by-term differentiation and integration**.

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$\begin{aligned} f(x) &= \frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \end{aligned}$$

Solution We differentiate the power series on the right term by term:

$$\begin{aligned} f'(x) &= \frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1; \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{2}{(1 - x)^3} = 2 + 6x + 12x^2 + \cdots + n(n - 1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n - 1)x^{n-2}, \quad -1 < x < 1. \end{aligned}$$

Caution Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x . But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n! \cos(n!x)}{n^2},$$

which diverges for all x . This is not a power series since it is not a sum of positive integer powers of x .

V EXAMPLE 7 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\begin{aligned}\tan^{-1}x &= \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1. ■

EXAMPLE 6 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots \quad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + \dots) dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \quad |x| < 1\end{aligned}$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1 + 0) = C$. Thus $C = 0$ and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$. ■

Taylor and Maclaurin Series

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.\end{aligned}$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The Maclaurin series generated by f is often just called the Taylor series of f .

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \dots$. Taking derivatives we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad \dots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

so that

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \quad \dots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$\begin{aligned}f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\ = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots\end{aligned}$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

Taylor Polynomials

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

EXAMPLE 2 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \dots$, the Taylor series generated by f at $x = 0$ (see Figure 10.17) is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots \\ = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This is also the Maclaurin series for e^x . In the next section we will see that the series converges to e^x at every x .

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}.$$

EXAMPLE 3 Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{array}{lll} f(x) = \cos x, & f'(x) = -\sin x, \\ f''(x) = -\cos x, & f^{(3)}(x) = \sin x, \\ \vdots & \vdots \\ f^{(2n)}(x) = (-1)^n \cos x, & f^{(2n+1)}(x) = (-1)^{n+1} \sin x. \end{array}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for $\cos x$. Notice that only even powers of x occur in the Taylor series generated by the cosine function, which is consistent with the fact that it is an even function. In Section 10.9, we will see that the series converges to $\cos x$ at every x .

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 10.18 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y -axis.

Common/Known Maclaurin Series

Here is a list of Maclaurin series representations of common functions

Common Functions	Maclaurin series	Radius of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n, -1 < x < 1$	$R = 1$
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n, \forall x \in \mathbb{R}$	$R = \infty$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, -1 < x \leq 1$	$R = 1$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n, -1 < x < 1$	$R = 1$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \forall x \in \mathbb{R}$	$R = \infty$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \forall x \in \mathbb{R}$	$R = \infty$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}, -1 \leq x \leq 1$	$R = 1$

Applications of Taylor Series

Taylor Series Approximation

A **Taylor series approximation** uses a [Taylor series](#) to represent a number as a polynomial that has a very similar value to the number in a neighborhood around a specified x value:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

Taylor series are extremely powerful tools for approximating functions that can be difficult to compute otherwise, as well as evaluating infinite sums and integrals by recognizing Taylor series.

Suggested steps for approximating values:

1. Identify a function to resemble the operation on the number in question.
2. Choose a to be a number that makes $f(a)$ easy to compute.
3. Select x to make $f(x)$ the number being approximated.

EXAMPLE

Using the first three terms of the Taylor series expansion of $f(x) = \sqrt[3]{x}$ centered at $x = 8$, approximate $\sqrt[3]{8.1}$:

$$f(x) = \sqrt[3]{x} \approx 2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{288}.$$

The first three terms shown will be sufficient to provide a good approximation for $\sqrt[3]{x}$. Evaluating this sum at $x = 8.1$ gives an approximation for $\sqrt[3]{8.1}$:

$$\begin{aligned}f(8.1) &= \sqrt[3]{8.1} \approx 2 + \frac{(8.1-8)}{12} - \frac{(8.1-8)^2}{288} \\&= 2.00829861111\dots\end{aligned}$$

$$\sqrt[3]{8.1} = 2.00829885025\dots$$

With just three terms, the formula above was able to approximate $\sqrt[3]{8.1}$ to six decimal places of accuracy. □

Evaluating Nonelementary Integrals

Sometimes we can use a familiar Taylor series to find the sum of a given power series in terms of a known function. For example,

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = \sin x^2.$$

Additional examples are provided in Exercises 59–62.

Taylor series can be used to express nonelementary integrals in terms of series. Integrals like $\int \sin x^2 dx$ arise in the study of the diffraction of light.

EXAMPLE 3 Express $\int \sin x^2 dx$ as a power series.

Solution From the series for $\sin x$ we substitute x^2 for x to obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots. \quad \blacksquare$$

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

EXAMPLE 5 Evaluate

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}.$$

Solution We represent $\ln x$ as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x - 1$ in the series for $\ln(1 + x)$ in Section 10.7, Example 6. Either way, we obtain

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

from which we find that

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \left(1 - \frac{1}{2}(x - 1) + \cdots \right) = 1. \quad \blacksquare$$

Of course, this particular limit can be evaluated using l'Hôpital's Rule just as well.

EXAMPLE 6 Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5 , are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots.$$

Subtracting the series term by term, it follows that

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \dots = x^3 \left(-\frac{1}{2} - \frac{x^2}{8} - \dots \right).$$

Division of both sides by x^3 and taking limits then gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} &= \lim_{x \rightarrow 0} \left(-\frac{1}{2} - \frac{x^2}{8} - \dots \right) \\ &= -\frac{1}{2}. \end{aligned}$$
■

If we apply series to calculate $\lim_{x \rightarrow 0} ((1/\sin x) - (1/x))$, we not only find the limit successfully but also discover an approximation formula for $\csc x$.

EXAMPLE 7 Find $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

Solution Using algebra and the Taylor series for $\sin x$, we have

$$\begin{aligned} \frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)} \\ &= \frac{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \dots \right)}{x^2 \left(1 - \frac{x^2}{3!} + \dots \right)} = x \cdot \frac{\frac{1}{3!} - \frac{x^2}{5!} + \dots}{1 - \frac{x^2}{3!} + \dots}. \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x \cdot \frac{\frac{1}{3!} - \frac{x^2}{5!} + \dots}{1 - \frac{x^2}{3!} + \dots} \right) = 0.$$

From the quotient on the right, we can see that if $|x|$ is small, then

$$\frac{1}{\sin x} - \frac{1}{x} \approx x \cdot \frac{1}{3!} = \frac{x}{6} \quad \text{or} \quad \csc x \approx \frac{1}{x} + \frac{x}{6}.$$
■