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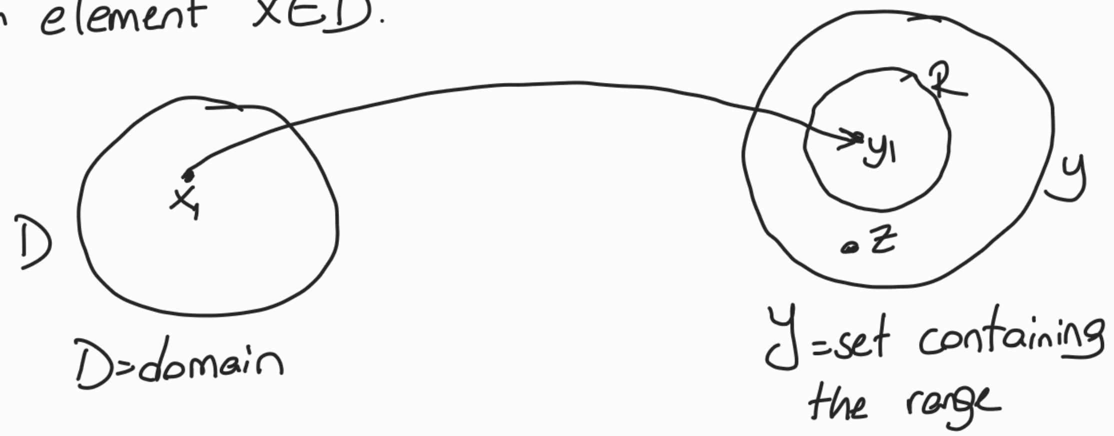
FUNCTIONS

"y is a function of x" $(y=f(x))$

* x is the independent variable (input)
* y is the dependent variable (output)
* f is a function

Remark: We have only one value $f(x)$ for every x.

Definition: A function from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.



The set D of all possible input values is called the domain of f.
The set R of all possible output values of $f(x)$ as x varies throughout D is called the range of f.

$$R \subseteq Y$$

$$f: D \to Y$$
$$x \mapsto y = f(x)$$

The natural domain is the largest set of real x which the rule f can be applied to.

<u>Examples</u>

Function	Domain $x \in D$	Range $y \in R$
$y = x^2$	$\mathbb{R} = (-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4-x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$

$4 - x \geqslant 0$
$4 \geqslant x$
$1 - x^2 \geqslant 0$
$1 \geqslant x^2$

<u>Remark</u>: A function is specified by the rule f and the domain D:

$f : x \longmapsto y = x^2, \quad D(f) = [0, \infty)$

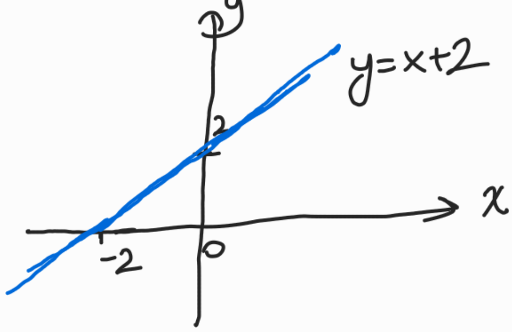
and

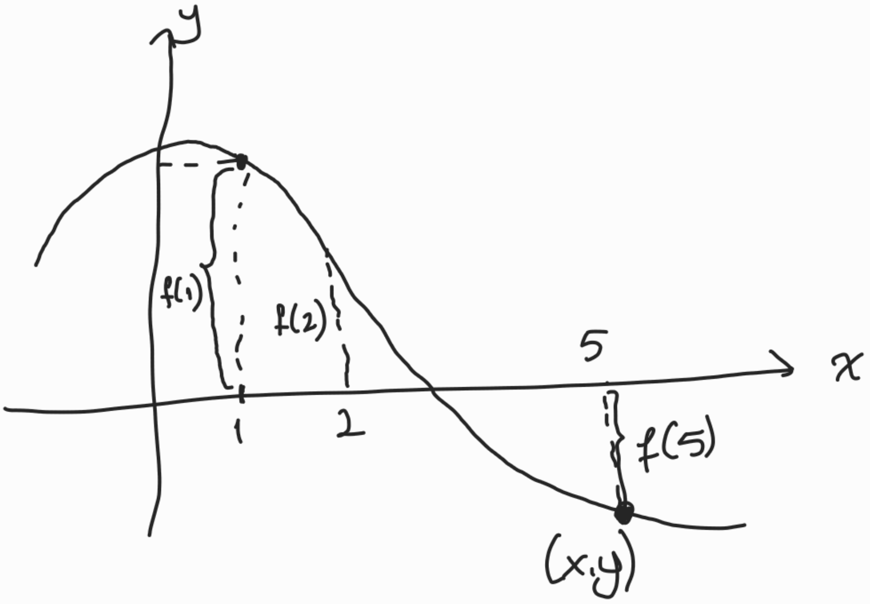
$f : x \longmapsto y = x^2, \quad D(f) = \mathbb{R}$

are different functions.

<u>Definition</u>: If f is a function with domain D, its graph consists of the points (x, y) whose coordinates are the input-output pairs for f: $\{(x, f(x)) \mid x \in D\}$

$y = x+2$



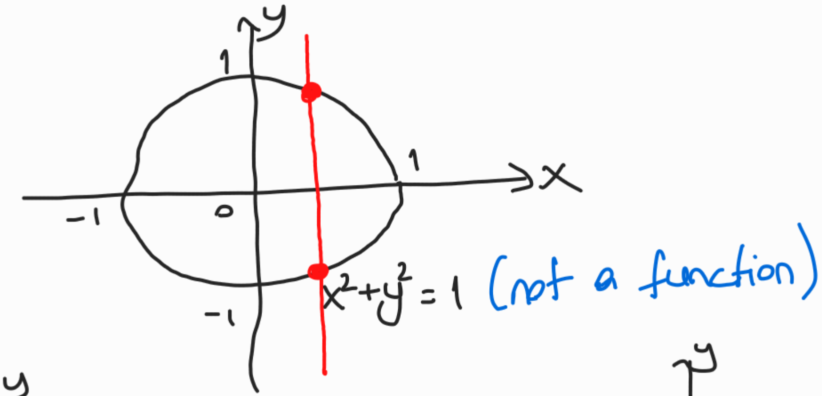


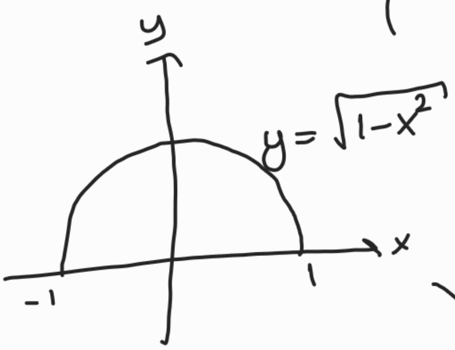
$y = f(x)$ is the height of the graph above/below x.

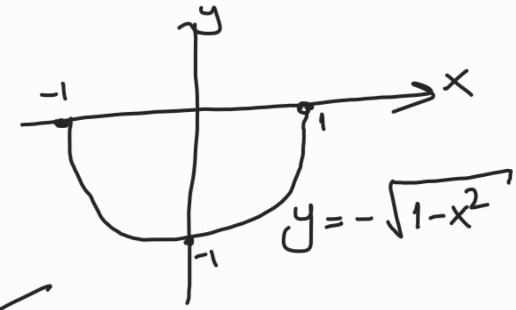
<u>Recall</u>: A function f can have only one value $f(x)$ for each x in its domain.

<u>Vertical line test</u>: No vertical line can intersect the graph of a function more than once.

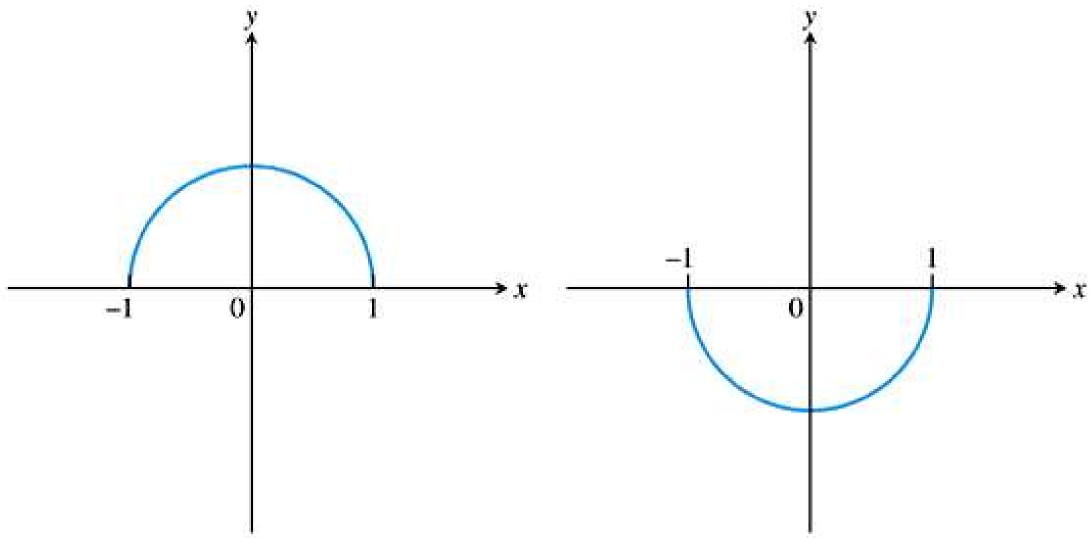
<u>Example</u>:







functions



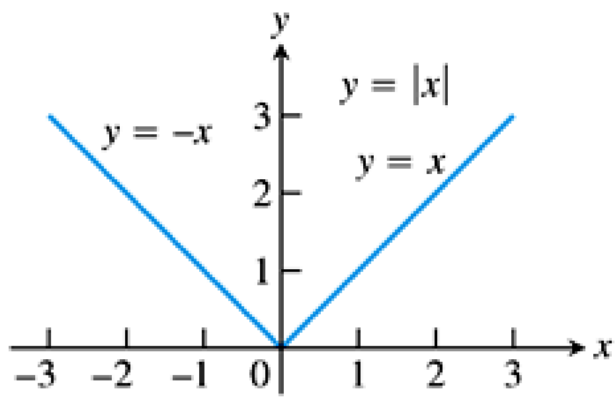
(b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

A **piecewise defined function** is a function that is is described by using *different formulas on different parts of its domain.*

examples:

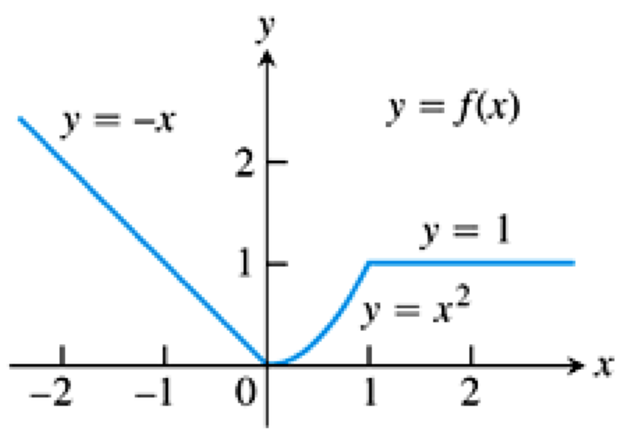
- the *absolute value function*

$$f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



- some other function

$$f(x) = \begin{cases} -x & , x < 0 \\ x^2 & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$

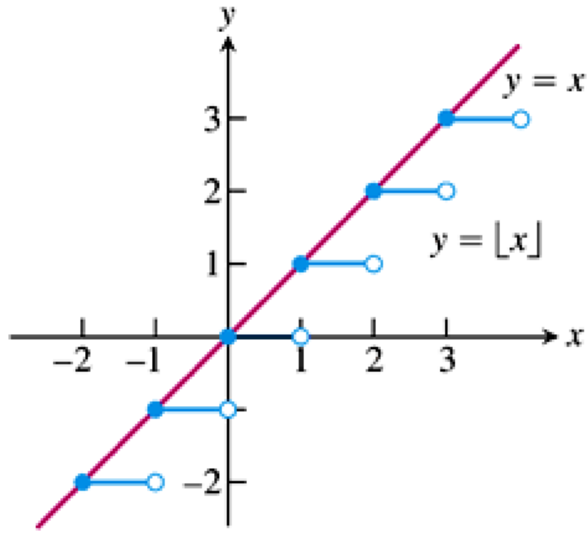


- the *floor function*

$$f(x) = \lfloor x \rfloor$$

is given by the greatest integer less than or equal to x:

$$\lfloor 1.3 \rfloor = 1, \ \lfloor -2.7 \rfloor = -3$$

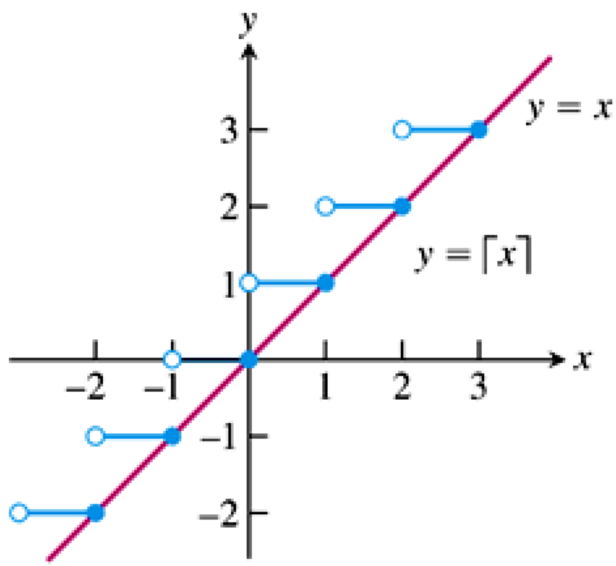


- the *ceiling function*

$$f(x) = \lceil x \rceil$$

is given by the smallest integer greater than or equal to x:

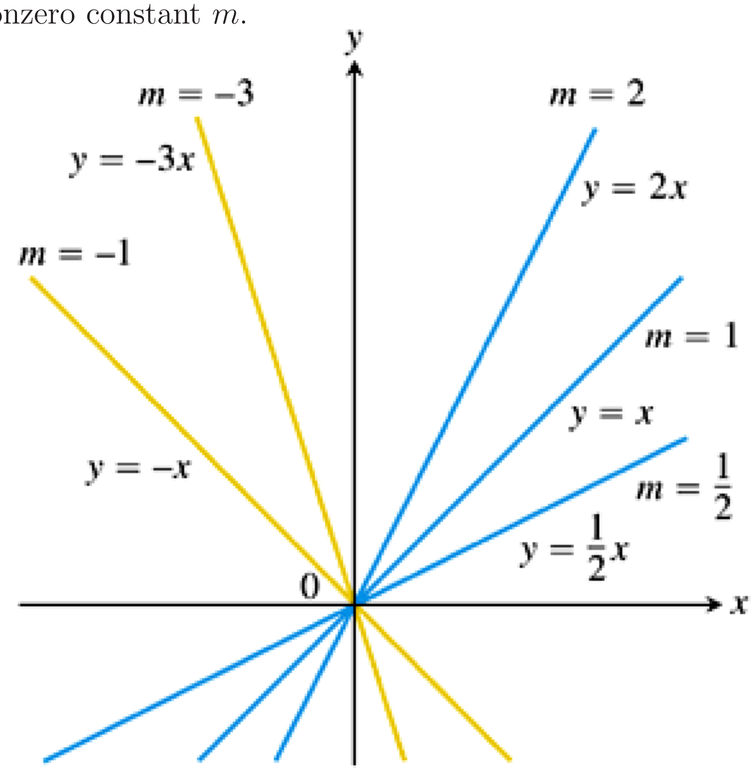
$$\lceil 3.5 \rceil = 4, \ \lceil -1.8 \rceil = -1$$



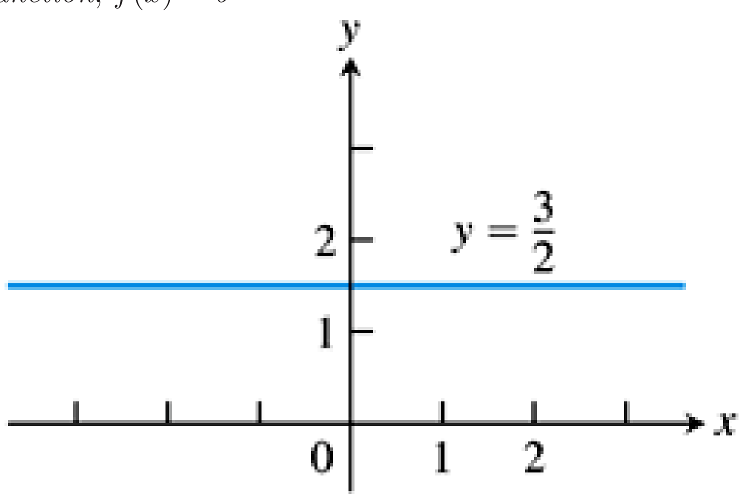
Some fundamental types of functions

- **linear function:** $\boxed{f(x) = mx + b}$

$b = 0$: all lines pass through the origin, $f(x) = mx$. One also says "$y = f(x)$ is proportional to x" for some nonzero constant m.



$m = 0$: *constant function*, $f(x) = b$

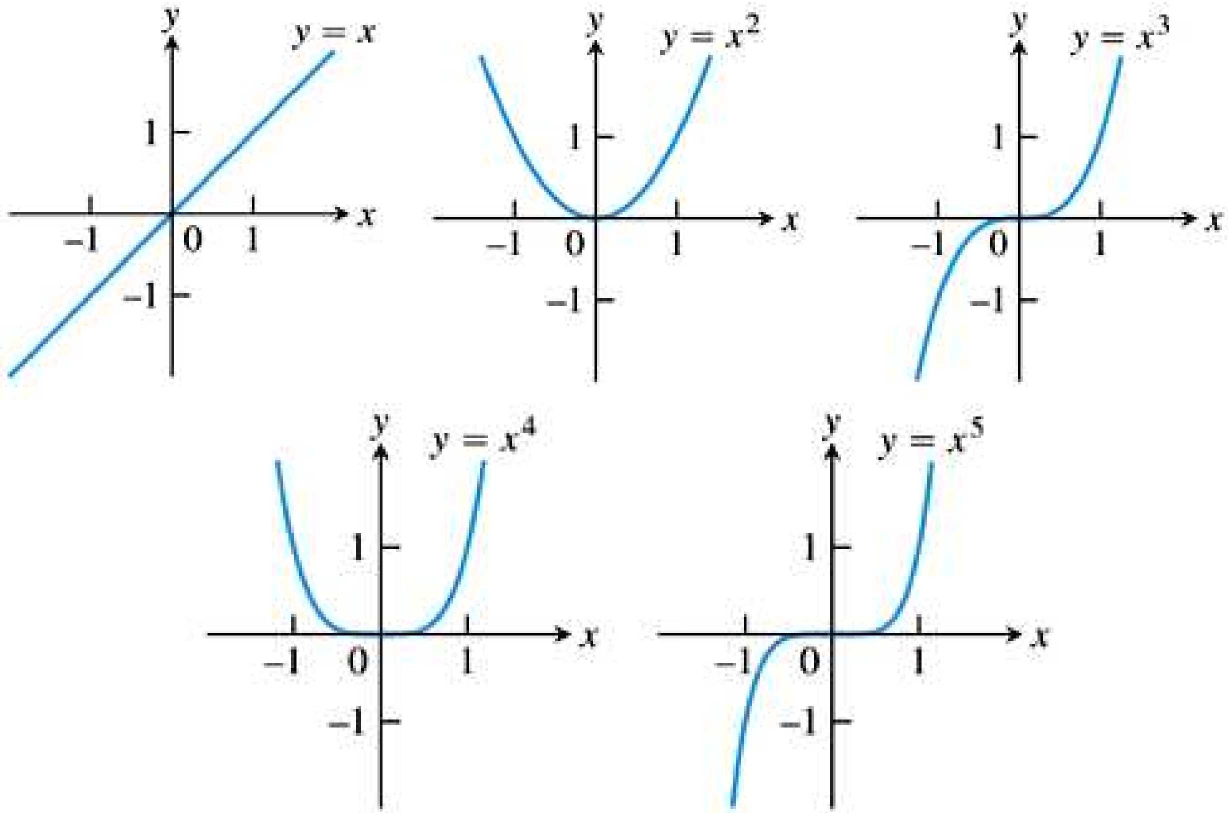


• **power function:** $\boxed{f(x) = x^a}$

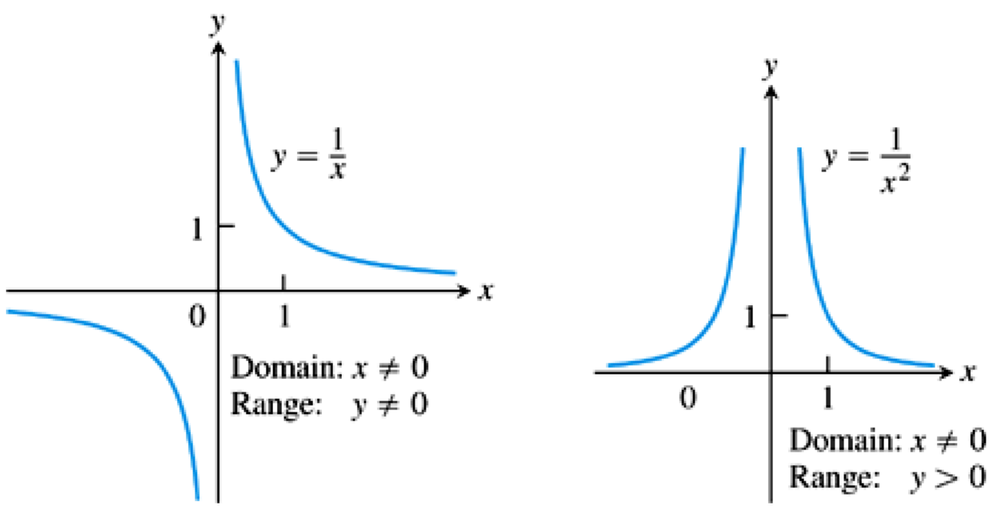
n=2 : quadratic function

n=3 : cubic function

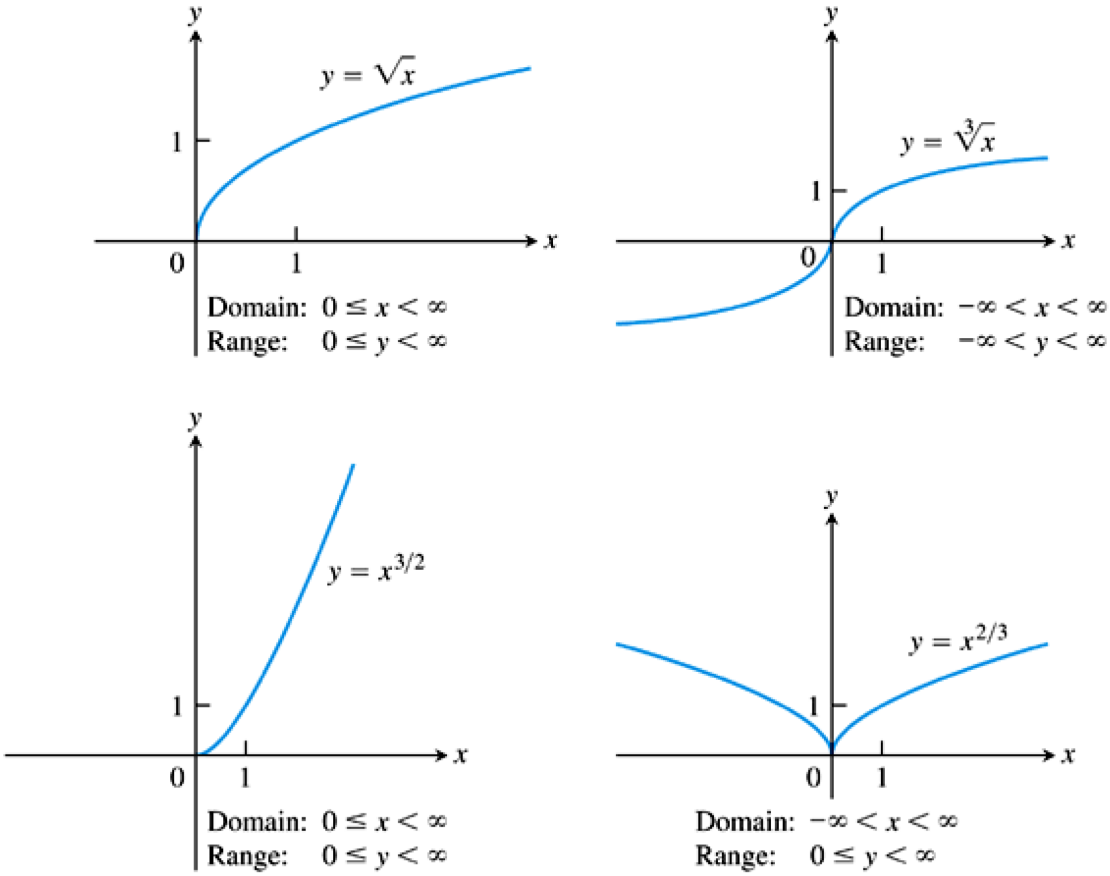
$a = n \in \mathbb{N}$: graphs of $f(x)$ for $n = 1, 2, 3, 4, 5$



$a = -n$, $n \in \mathbb{N}$: graphs of $f(x)$ for $n = -1, -2$



$a \in \mathbb{Q}$: graphs of $f(x)$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



• **polynomials:** $\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\ ,\ n \in \mathbb{N}_0}$
with *coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ and domain $\mathbb{R}$
If the leading coefficient $a_n \neq 0$, $n > 0$, n is called the *degree* of the polynomial.

Rational Functions

Definition 1.2.1. A function in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials, is called a rational function. The domain of the rational function $f(x)$ is the set

$$D = \mathbb{R} - \{x \in \mathbb{R} : Q(x) = 0\}$$

Informally,

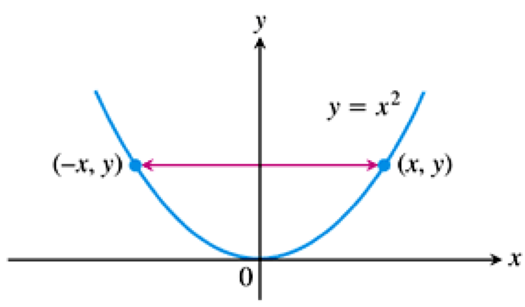
- a function is called **increasing** if the graph of the function "climbs" or "rises" as you move *from left to right.*
- a function is called **decreasing** if the graph of the function "descends" or "falls" as you move *from left to right.*

examples:

function	where increasing	where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = 1/x$	nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

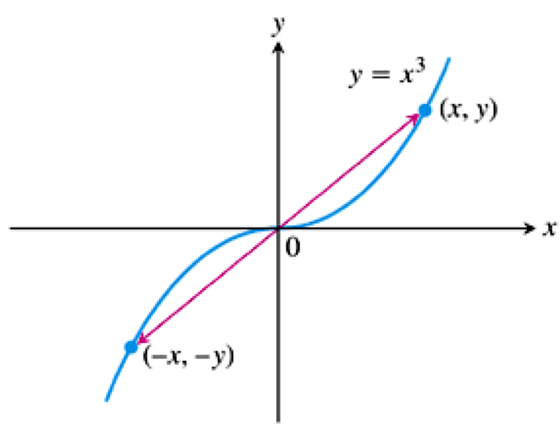
Definition 3 *A function $y = f(x)$ is an*
even function of x *if* $f(-x) = f(x)$,
odd function of x *if* $f(-x) = -f(x)$,
for every x in the function's domain.

examples:



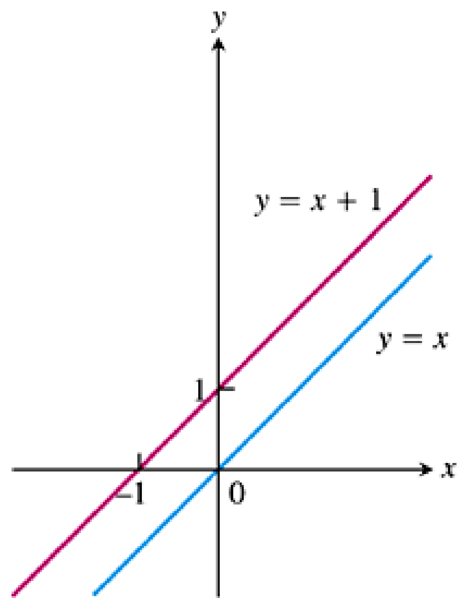
(a)

$f(-x) = (-x)^2 = x^2 = f(x)$: even function; graph is *symmetric about the y-axis*



(b)

$f(-x) = (-x)^3 = -x^3 = -f(x)$: odd function; graph is *symmetric about the origin*



1. $f(-x) = -x = -f(x)$: odd function

2. $f(-x) = -x + 1 \neq f(x)$ and $-f(x) = -x - 1 \neq f(-x)$: *neither even nor odd!*

Combining functions

If f and g are functions, then for every $x \in D(f) \cap D(g)$ (that is, for every x that belongs to the domains of *both* f and g) we define *sums, differences, products and quotients:*

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) \\
(f-g)(x) &= f(x) - g(x) \\
(fg)(x) &= f(x)g(x) \\
(f/g)(x) &= f(x)/g(x) \quad \text{if } g(x) \neq 0
\end{aligned}$$

algebraic operation on *functions* = algebraic operation on function *values*

Special case - multiplication by a constant $c \in \mathbb{R}$: $(cf)(x) = c\,f(x)$ (take $g(x) = c$ constant function)

examples: combining functions algebraically

$$f(x) = \sqrt{x} \quad , \quad g(x) = \sqrt{1-x}$$

(natural) domains:

$$D(f) = [0, \infty) \qquad D(g) = (-\infty, 1]$$

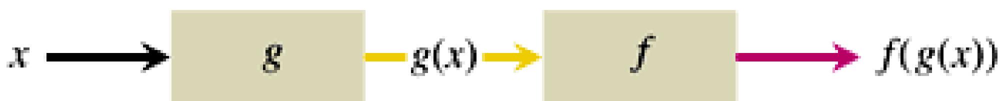
intersection of both domains:

$$D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

function	formula	domain
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]=D(f)\cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

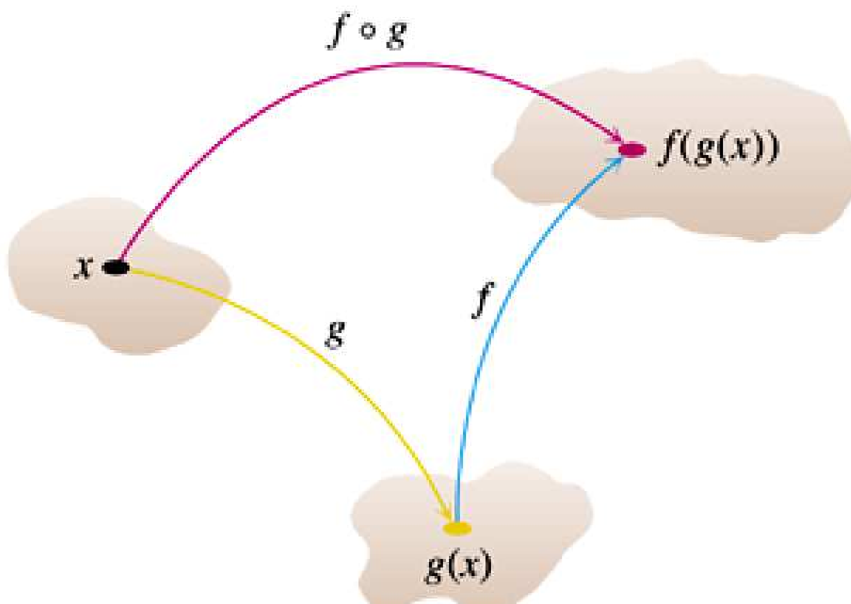
Definition 4 (Composition of functions) *If f and g are functions, the* **composite** *function $f\circ g$ ("f composed with g") is defined by*

$$(f\circ g)(x)=f(g(x))$$



The domain *of $f\circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e.*

$$D(f\circ g)=\{x|x\in D(g)\ \textit{and}\ g(x)\in D(f)\}$$



examples: finding formulas for composites

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x + 1 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$(g \circ g)(x) = g(g(x)) = g(x) + 1 = x + 2$	$(-\infty, \infty)$

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x^2 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = \lvert x \rvert$	$(-\infty, \infty)$
$(g \circ f)(x) = x$	$[0, \infty)$

Shifting a graph of a function:

Shift Formulas

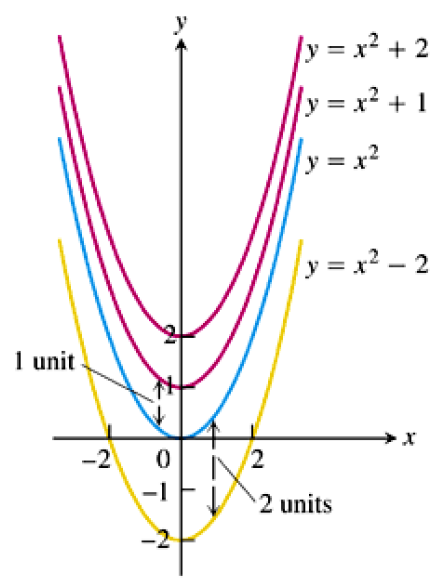
Vertical Shifts

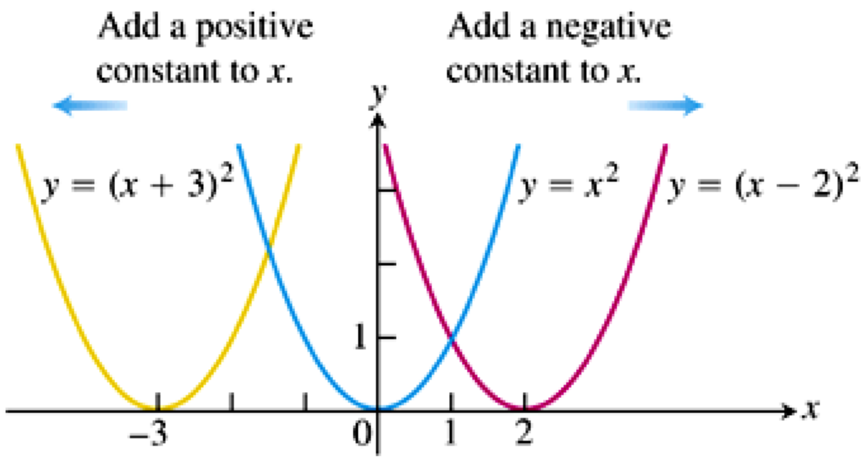
$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

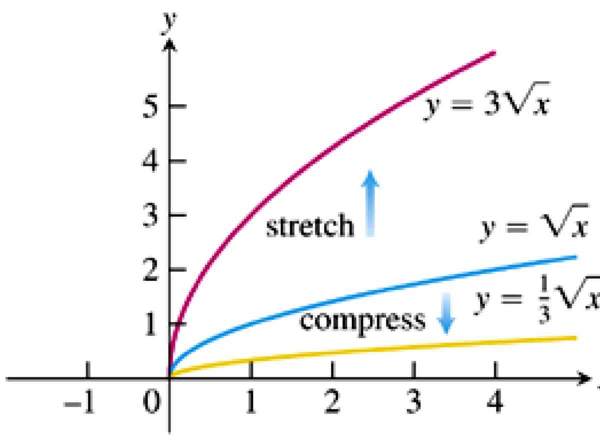
examples:



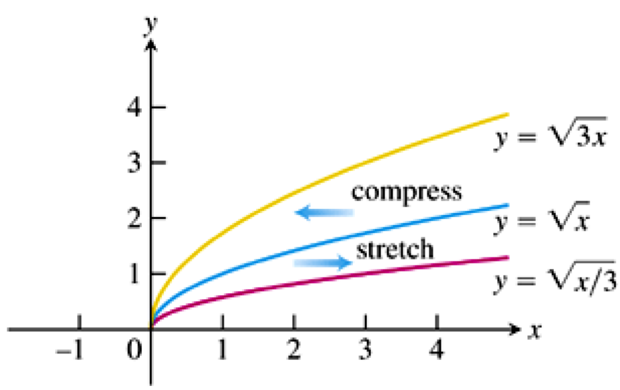


Scaling and reflecting a graph of a function:
For $c > 1$,

$y = cf(x)$ *stretches* the graph of f *along the y-axis* by a factor of c
$y = \frac{1}{c}f(x)$ *compresses* the graph of f *along the y-axis* by a factor of c

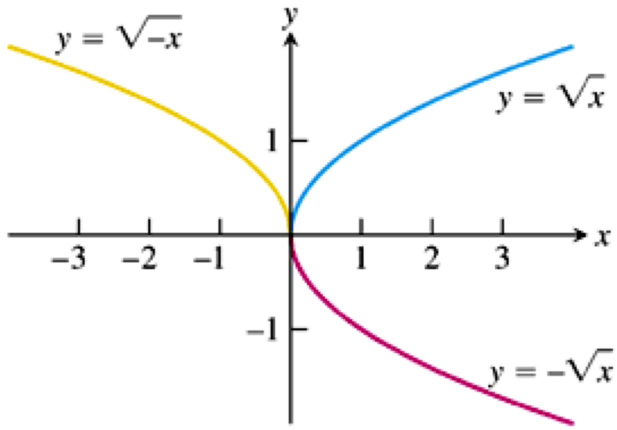


$y = f(cx)$ *compresses* the graph of f *along the x-axis* by a factor of c
$y = f(x/c)$ *stretches* the graph of f *along the x-axis* by a factor of c



For $c = -1$,

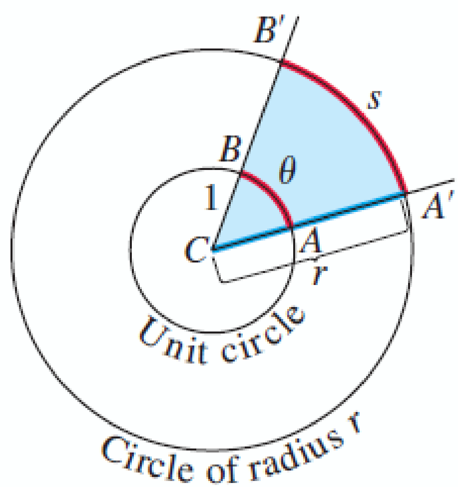
$y = -f(x)$ *reflects* the graph of f *across the x-axis*



$y = f(-x)$ *reflects* the graph of f *across the y-axis*

Combining scalings and reflections: *see next exercise sheet for examples!*

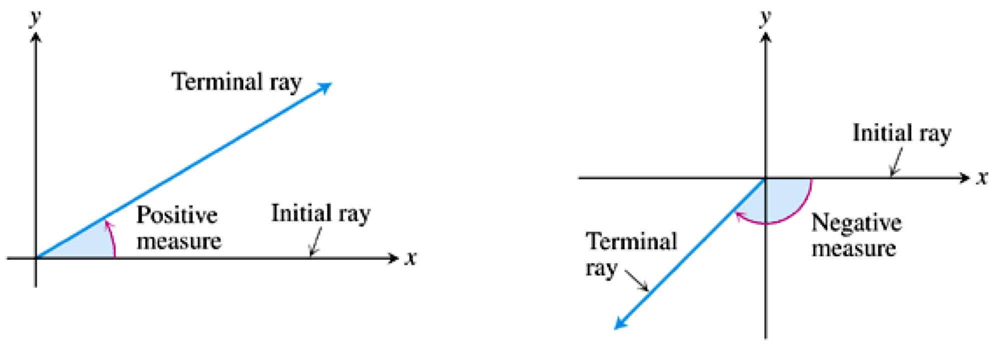
Trigonometric functions



The **radian measure** of the angle ACB is the length θ of arc AB on the unit circle.
$s = r\theta$ is the *length of arc* on a circle of radius r when θ is measured in radians.

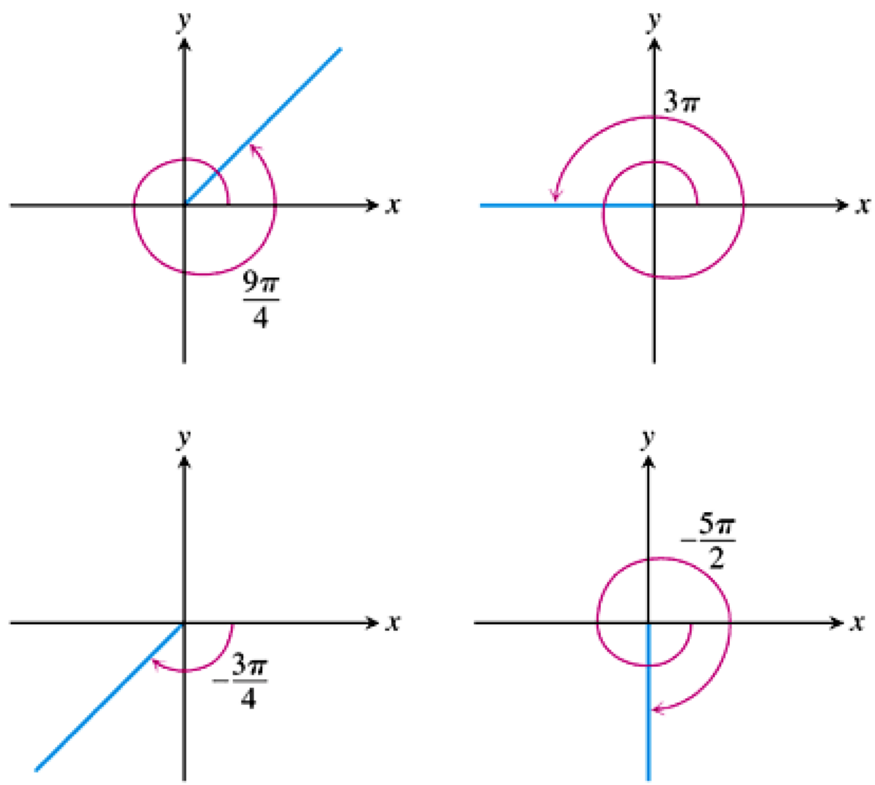
conversion formula degrees $\leftrightarrow$ radians:

$$360^\circ \text{ corresponds to } 2\pi \;\Rightarrow\; \boxed{\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}}$$

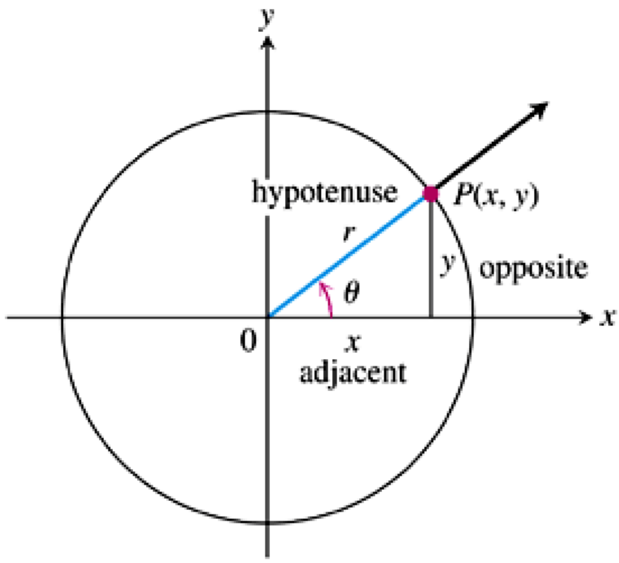


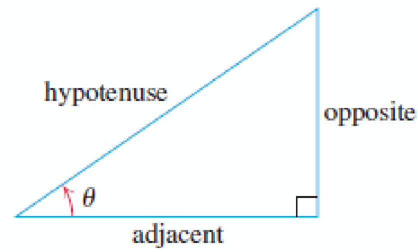
- angles are **oriented**
- *positive angle*: counter-clockwise
- *negative angle*: clockwise

angles can be *larger* (counter-clockwise) *smaller* (clockwise) than 2π:



reminder: the six **basic trigonometric functions**





$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

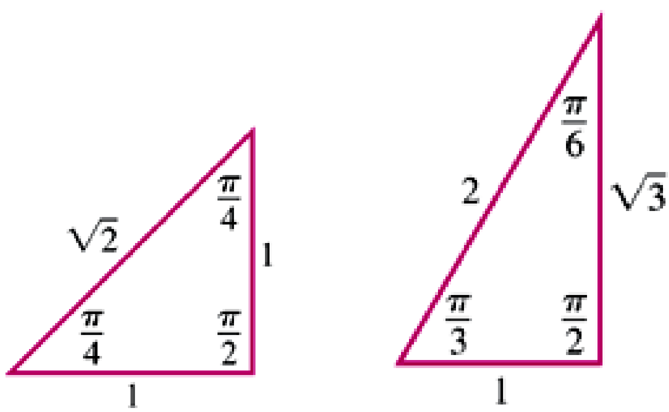
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.39 Trigonometric ratios of an acute angle.

sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$

note: These definitions hold not only for $0 \le \theta \le \pi/2$ but also for $\theta < 0$ and $\theta > \pi/2$. recommended to memorize the following two triangles:

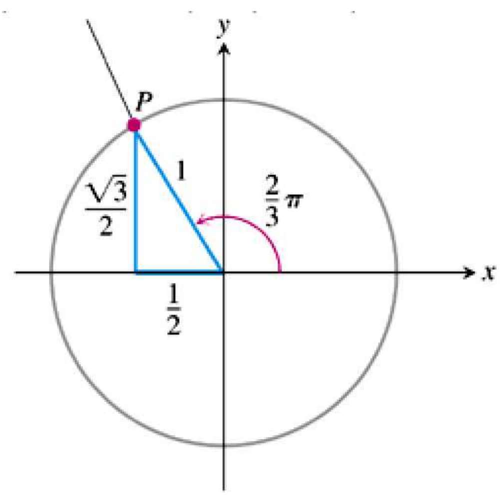


because *exact values* of trigonometric ratios in the *surds form* can be read from them

example:

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

a more non-trivial **example:**



From the above triangle and with $r = 1$, $x = -1/2$ and $y = \sqrt{3}/2$ we can read off the values of all trigonometric functions:

$$\sin\left(\frac{2}{3}\pi\right) = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{2}{3}\pi\right) = \frac{r}{y} = \frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{2}{3}\pi\right) = \frac{x}{r} = -\frac{1}{2} \qquad \sec\left(\tfrac{2}{3}\pi\right) = \tfrac{r}{x} = -2$$

$$\tan\left(\frac{2}{3}\pi\right) = \frac{y}{x} = -\sqrt{3} \qquad \cot\left(\frac{2}{3}\pi\right) = \frac{x}{y} = -\frac{1}{\sqrt{3}}$$

note: For an angle of measure θ and an angle of measure $\theta + 2\pi$ we have the *very same* trigonometric function values (why?)

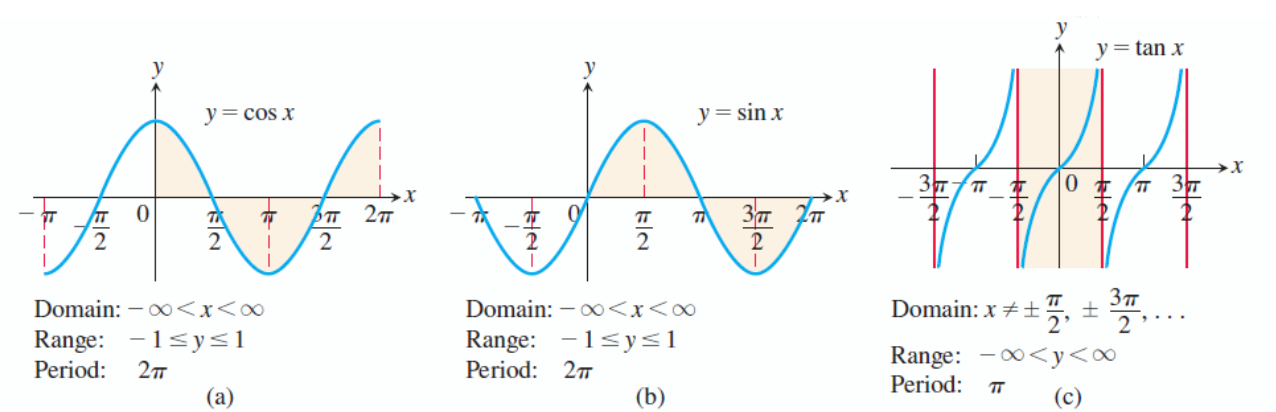
$$\sin(\theta + 2\pi) = \sin\theta \quad ; \quad \cos(\theta + 2\pi) = \cos\theta \quad ; \quad \tan(\theta + 2\pi) = \tan\theta$$

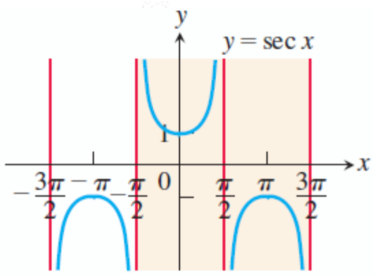
and so on.

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

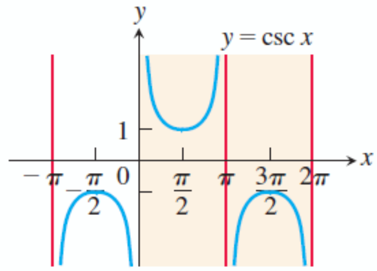
Graphs of trigonometric functions:





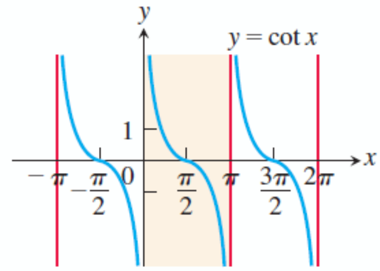
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \leq -1$ or $y \geq 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \leq -1$ or $y \geq 1$
Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

Even
$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$