EXTREMA AND CONCAVITY

The question is "Find the extreme values, intervals where the function increases and decreases, points of inflection, and concavity (if given, in the interval)" for all in this file.

① $f(x) = x^3 - 6x^2 - 15x + 4$

$D(f) = \mathbb{R}$

$f'(x) = 3x^2 - 12x - 15 = 0 \Rightarrow 3(x^2 - 4x - 5) = 3(x-5).(x+1) = 0$

$\Rightarrow$ Critical points: $x = 5$, $x = -1$

There is no $x \in D(f)$ such that $f'(x)$ is undefined.

$f''(x) = 6x - 12 = 0 \Rightarrow$ Inflection point: $x = 2$

There is no $x \in D(f)$ such that $f''(x)$ is undefined.

	$-\infty$	-1		2		5	$+\infty$
f'	$+$	0	$-$		$-$	0	$+$
f''	$-$		$-$	0	$+$		$+$
f	$\nearrow$ ⌒	local max	$\searrow$ ⌒	inflection	$\searrow$ ◡	local min	$\nearrow$ ◡

$f(-1) = 12 \Rightarrow$ local max. value

$f(5) = -96 \Rightarrow$ local min. value

$f(2) = -42 \Rightarrow$ inflection point

$(-\infty, -1) \cup (5, \infty)$: increasing

$(-1, 5)$: decreasing

$(-\infty, 2)$: concave down

$(2, \infty)$: concave up

(No absolute extrema!)

② $f(x) = x . \sqrt{1-x}$

$D(f) = (-\infty, 1]$ (endpoint $x = 1$!)

$f'(x) = \sqrt{1-x} + x . \dfrac{-1}{2\sqrt{1-x}} = \dfrac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2 - 3x = 0 \Rightarrow x = \dfrac{2}{3}$

$f'(x)$ is undefined at $x=1$ (Division by 0)

$\Rightarrow$ Critical points: $x=1$, $x=\frac{2}{3}$

$$f''(x) = \frac{-3\cdot 2\sqrt{1-x} + (2-3x)\cdot(1-x)^{-\frac{1}{2}}}{4\cdot(1-x)} = \frac{3x-4}{4(1-x)^{3/2}} = 0 \Rightarrow x = \frac{4}{3} \ (?)$$

There is no $x \in D(f)$ such that $f''(x)$ is undefined except $x=1$. But, $f(x)$ is not defined for $x>1$. Thus, change in concavity is not possible.

Inflection point: None. $\left(\frac{4}{3} \notin D(f)\right)$

	$-\infty$	$\frac{2}{3}$		1
f'	$+$	0	$-$	
f''	$-$		$-$	
f	$\nearrow$ $\frown$		$\searrow$ $\frown$	
		local max (absolute)		local min

$f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}} \Rightarrow$ global max. value

$f(1) = 0 \Rightarrow$ local min. value

$\left(-\infty, \frac{2}{3}\right)$: increasing

$\left(\frac{2}{3}, 1\right)$: decreasing

$(-\infty, 1)$: concave down

3) $f(x) = \frac{\sin x}{2-\cos x}$, $[0, 2\pi]$

$$f'(x) = \frac{\cos x(2-\cos x) - \sin x \cdot \sin x}{(2-\cos x)^2} = \frac{2\cos x - 1}{(2-\cos x)^2} = 0 \Rightarrow x = \frac{\pi}{3}, \ x = \frac{5\pi}{3}$$

There is no $x \in D(f)$ such that $f'(x)$ is undefined.

$(2-\cos x \neq 0, \ \cos x \leq 1)$

Critical points: $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

$$f''(x) = \frac{-2\sin x(2-\cos x)^2 - (2\cos x - 1)\cdot 2\cdot(2-\cos x)\cdot \sin x}{(2-\cos x)^4}$$

$$f''(x) = \frac{2\sin x(\cos^2 x - \cos x - 2)}{(2-\cos x)^4} = 0 \Rightarrow 2\sin x . (\cos^2 x - \cos x - 2) = 0$$

$\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$$\underset{(\cos x = t)}{\cos^2 x - \cos x - 2 = 0} \Rightarrow t^2 - t - 2 = 0 \Rightarrow \underset{\Downarrow \atop \cos x \neq 2}{(t-2)} . \underset{\Downarrow \atop \cos x = -1 \Rightarrow x = \pi \ (\star)}{(t+1)} = 0$$

There is no $x \in D(f)$ such that $f''(x)$ is undefined.

$(2 - \cos x \neq 0, \ \cos x \leq 1)$

Inflection points: $x = \cancel{0}, \pi, \cancel{2\pi}$ (Endpoints can't be inflection points.)

[If the domain wasn't restricted, 0 and 2π would also be inflection points. But for this example, they are endpoints. Hence, the behavior of the function can't be investigated.]

	0		$\frac{\pi}{3}$		π		$\frac{5\pi}{3}$		2π
f'		$+$	0	$-$		$-$	0	$+$	
f''		$-$		$-$	0	$+$		$+$	
f	local min	↗	local max	↘	inflection point	↘	local min	↗	local max

$f(0) = 0 \Rightarrow$ local min. value

$f\left(\frac{5\pi}{3}\right) = -\frac{1}{\sqrt{3}} \Rightarrow$ global min. value

$f(2\pi) = 0 \Rightarrow$ local max. value

$f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}} \Rightarrow$ global max. value

$\left(0, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)$: increasing

$\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$: decreasing

$(0, \pi)$: concave down

$(\pi, 2\pi)$: concave up

$(\star)$ $x = \pi$ is a root for both $\sin x$ and $\cos^2 x - \cos x - 2$. But it is not considered as even root, because at $x = \pi$, the sign of $\sin x$ changes but the sign of $\cos^2 x - \cos x - 2$ doesn't change. In fact, $\cos^2 x - \cos x - 2 \leq 0$ for all $x \in \mathbb{R}$. ($\cos x \leq 1$!)

④ $f(x)=\dfrac{x^2}{x-1}$.

$D(f)=\mathbb{R}\setminus\{1\}$

$f'(x)=\dfrac{2x\cdot(x-1)-x^2}{(x-1)^2}=\dfrac{x^2-2x}{(x-1)^2}=0 \Rightarrow x=0,\ x=2$

There is no $x\in D(f)$ such that $f'(x)$ is undefined. $(1\notin D(f))$

Critical points: $x=0,\ x=2$

$f''(x)=\dfrac{(2x-2)\cdot(x-1)^2-(x^2-x)\cdot 2(x-1)}{(x-1)^4}=\dfrac{-2}{(x-1)^2}\neq 0$

There is no $x\in D(f)$ such that $f''(x)$ is undefined.

Inflection point: None.

	$-\infty$		0		1		2		∞
f'		$+$	0	$-$		$-$	0	$+$	
f''		$-$		$-$		$-$		$-$	
f		↗	local max	↘		↘	local min	↗	

$f(0)=0 \Rightarrow$ local max. value

$f(2)=4 \Rightarrow$ local min. value

$(-\infty,0)\cup(2,\infty)$: increasing

$(0,2)$: decreasing

$(-\infty,\infty)$: concave down

[Remark: If $x=1$ wasn't a double root for denominators of f' and f'', it would change the sign!]

⑤ $f(x)=x\cdot\sqrt{4-x^2}$

$D(f)=[-2,2]$

$f'(x)=\sqrt{4-x^2}+x\cdot\dfrac{-2x}{2\sqrt{4-x^2}}=\dfrac{4-2x^2}{\sqrt{4-x^2}}=0 \Rightarrow x=\mp\sqrt{2}$

$f'(x)$ is undefined : $x=\mp 2$

Critical points: $x=\mp\sqrt{2},\ \mp 2$

$$f''(x)=\frac{-4x\cdot\sqrt{4-x^2}-(4-2x^2)\cdot\frac{-2x}{2\sqrt{4-x^2}}}{4-x^2}=\frac{2x(x^2-6)}{(4-x^2)^{3/2}}=0$$

$\Rightarrow x=0$: inflection point $\quad (x^2-6=0 \Rightarrow x=\mp\sqrt{6}$ <u>outside</u> of $D(f'))$

$f''(x)$ is undefined : $x=\mp 2$ (endpoints are not inflection points)

	-2		$-\sqrt{2}$		0		$\sqrt{2}$		2
f'		$-$	0	$+$		$+$	0	$-$	
f''		$+$		$+$	0	$-$		$-$	
f		↘		↗		↗		↘	
	local max		local min		inflection point		local max		local min

$f(2)=0 \Rightarrow$ local min. value

$f(-\sqrt{2})=-2 \Rightarrow$ absolute min. value

$f(-2)=0 \Rightarrow$ local max. value

$f(\sqrt{2})=2 \Rightarrow$ absolute max. value

$f(0)=0 \Rightarrow$ inflection point

$(-2,-\sqrt{2})\cup(\sqrt{2},2)$: decreasing

$(-\sqrt{2},\sqrt{2})$: increasing

$(-2,0)$: concave up

$(0,2)$: concave down

6) $f(x)=\arcsin(x^2-1)$

$D(f)=[-\sqrt{2},\sqrt{2}]$

$$f'(x)=\frac{2x}{\sqrt{1-(x^2-1)^2}}=\frac{2x}{|x|\cdot\sqrt{2-x^2}}=0 \Rightarrow x=0$$

[Don't cancel x and |x| when looking for domain or critical points]

$f'(x)$ is undefined : $x=0,\ x=\mp\sqrt{2}$

Critical points : $x=0, \mp\sqrt{2}$ $\qquad \left[\frac{d}{dx}|x|=\frac{x}{|x|}\right]$

$$f''(x)=\frac{2\cdot|x|\cdot\sqrt{2-x^2}-2x\cdot\left(\frac{x}{|x|}\cdot\sqrt{2-x^2}+|x|\cdot\frac{-2x}{2\sqrt{2-x^2}}\right)}{x^2\cdot(2-x^2)}=\frac{2x^2}{|x|\cdot(2-x^2)^{3/2}}=0 \Rightarrow x=0$$

(double!)

(|x| in denominator won't change the sign!)

Inflection point : None
(x=0 is not a point which sign changes)

($x=\mp\sqrt{2}$ makes f'' undefined, but they are endpoints of the domain)

	$-\sqrt{2}$		0		$\sqrt{2}$
f'		$-$	0	$+$	
f''		$+$	0	$+$	
f		↘		↗	
	local max		local min		local max

$f(0)=-\frac{\pi}{2} \Rightarrow$ absolute min. value

$f(-\sqrt{2})=f(\sqrt{2})=\frac{\pi}{2} \Rightarrow$ absolute max. value

$(-\sqrt{2},0)$: decreasing / $(0,\sqrt{2})$: increasing

$(-\sqrt{2},\sqrt{2})$: concave up

7) $f(x)=\sqrt{1+x}+\sqrt{1-x}$

$D(f)=[-1,1]$

$$f'(x)=\frac{1}{2\sqrt{1+x}}+\frac{-1}{2\sqrt{1-x}}=\frac{\sqrt{1-x}-\sqrt{1+x}}{2\sqrt{1-x^2}}=0 \Rightarrow x=0$$

$f'(x)$ is undefined : $x=\mp 1$

Critical points : $x=0,\mp 1$ $\qquad \left(f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}(1-x)^{-\frac{1}{2}}\right)$

$$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}-\frac{1}{4}(1-x)^{-\frac{3}{2}}=-\frac{1}{4}\left(\frac{(1-x)^{3/2}+(1+x)^{3/2}}{(1-x^2)^{3/2}}\right)\neq 0$$

Point of inflection : None. ($x=\mp 1$ are endpoints, can't be inflection point)

	-1		0		1
f'		+	0	—	
f''		—		—	
f	local min	↗	local max	↘	local min

$f(-1)=f(1)=\sqrt{2} \Rightarrow$ absolute min. value

$f(0)=2 \Rightarrow$ absolute max. value

$(-1,0)$: increasing

$(0,1)$: decreasing

$(-1,1)$: concave down

8) $f(x)=\ln(1-\ln x)+\ln x^2$

$D(f)=(0,e)$, $\quad f(x)=\ln(1-\ln x)+2\ln x$

$$f'(x)=\frac{-\frac{1}{x}}{1-\ln x}+\frac{2}{x}=\frac{2\ln x-1}{x(\ln x-1)}=0 \Rightarrow 2\ln x=1 \Rightarrow x=e^{1/2}=\sqrt{e}$$

There is no $x\in D(f)$ such that $f'(x)$ is undefined. ($0,e\notin D(f)$)

Critical point : $x=\sqrt{e}$

$$f''(x)=\frac{\frac{2}{x}\cdot x\cdot(\ln x-1)-(2\ln x-1)\cdot\left((\ln x-1)+x\cdot\frac{1}{x}\right)}{x^2(\ln x-1)^2}=\frac{-2\ln^2 x+3\ln x-2}{x^2(\ln x-1)^2}\neq 0$$

There is no $x\in D(f)$ such that $f''(x)$ is undefined. ($0,e\notin D(f)$)

Inflection points : None

	0		$\sqrt{e}$		e
f'		+	0	—	
f''		—		—	
f		↗	local max	↘	

$f(\sqrt{e})=1-\ln 2 \Rightarrow$ absolute max. value

$(0,\sqrt{e})$: increasing / $(\sqrt{e},e)$: decreasing

$(0,e)$: concave down

[Endpoints can't be local extrema, because the domain is open interval $\Rightarrow$ endpoints are excluded.]