2024-2025 FALL / MAT1071 MATHEMATICS 1 FINAL EXAM QUESTIONS

1) If ℓ is a nonzero real number and

$$\lim_{x\to 0} \frac{(1-\cos x)[\sin x - \ln(1+x)]}{x^n} = \ell$$

what is the natural number n?

$$\lim_{x\to 0} \frac{(1-\cos x)[\sin x - \ln(1+x)]}{x^n} = \lim_{x\to 0} \frac{\sin^2 x\,[\sin x - \ln(1+x)]}{x^2 \cdot x^{n-2} \cdot (1+\cos x)}$$

$$(1+\cos x)$$

$$= \lim_{x\to 0} \frac{\sin x - \ln(1+x)}{2 \cdot x^{n-2}} = \ell \neq 0 \text{ (Nominator} \to 0\text{, must be indeterminate)}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{\cos x - \frac{1}{1+x}}{2 \cdot (n-2) \cdot x^{n-3}} = \ell \text{ (Again!) } \overset{L}{=} \lim_{x\to 0} \frac{-\sin x + \frac{1}{(1+x^2)}}{2 \cdot (n-2) \cdot (n-3) \cdot x^{n-4}} = \ell$$

Nominator $\neq 0 \Rightarrow$ Denominator $\neq 0$ (If 0, limit $\to \infty$)

$$\lim_{x\to 0} x^{n-4} \neq 0 \Rightarrow n-4=0 \Rightarrow n=4$$

2) If y is a function of x and $x \cdot \cos y = \int_x^y \sqrt{1+t^4}\,dt$, find the equation of the tangent line to the curve $y=f(x)$ at the point $P(0,?)$.

$$x=0 \Rightarrow 0 = \int_0^y \sqrt{1+t^4}\,dt \Rightarrow y=0 \Rightarrow P(0,0)$$

$$\cos y - x \cdot \sin y \cdot y' = y' \cdot \sqrt{1+y^4} - \sqrt{1+x^4} \overset{P}{\Longrightarrow} \underbrace{\cos 0}_{1} - 0 \cdot \sin 0 \cdot y' = y' \cdot 1 - 1$$

$$\Rightarrow 1 = y' - 1 \Rightarrow m_T = y' = 2 \Rightarrow \text{Equation: } y = 2x$$

③ $\int_{-1}^{1}[\ln(2+x)-\ln(2-x)]\,dx = ?$

$f(x)=\ln(2+x)-\ln(2-x)$

$f(-x)=\ln(2-x)-\ln(2+x)=-f(x) \Rightarrow f(x)$ is odd function.

$$\int_{-1}^{1}[\ln(2+x)-\ln(2-x)]\,dx = 0$$

④ For a continuous function f, which of the below options for a satisfies the equation $\int_{a}^{x-a} f(t)\,dt = \sin x$?

A) 2 B) 1 C) 0 D) -1 E) -2

Take derivative: $f(x-a)=\cos x$. Let $F'(x)=f(x)$

$\Rightarrow F'(x-a)=\cos x \Rightarrow F(x-a)=\sin x + c$

Also, $\int_{a}^{x-a} f(t)\,dt = \int_{a}^{x-a} F'(t)\,dt = F(t)\Big|_{a}^{x-a} = F(x-a)-F(a)=\sin x$

$x=2a \Rightarrow F(2a-a)=F(a)=\sin 2a + c$

$\cancel{\sin x}+\cancel{c}-\sin 2a-\cancel{c}=\cancel{\sin x} \Rightarrow \sin 2a = 0 \Rightarrow a = 0$

⑤ $\lim\limits_{x\to 0^+}\left[\frac{1}{x}\left(\int_{\pi}^{2\pi} e^{-\sin^2 t}\,dt - \int_{x+\pi}^{2\pi} e^{-\sin^2 t}\,dt\right)\right] = ?$

$$\lim_{x\to 0^+}\frac{\int_{\pi}^{2\pi} e^{-\sin^2 t}\,dt - \int_{x+\pi}^{2\pi} e^{-\sin^2 t}\,dt}{x} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to 0^+}\frac{e^{-\sin^2(x+\pi)}}{1} = 1$$

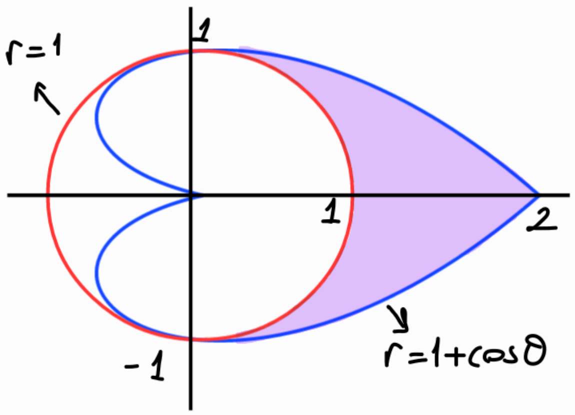
6) If a continuously differentiable function F is defined for $x \neq 0$, $F(x) = \frac{1}{x} \cdot \int_{2x}^{x^3-4} [2t - 3F'(t-2)]\,dt$, what is the value of $F'(2)$?

$$F'(x) = \frac{-1}{x^2} \cdot \int_{2x}^{x^3-4} [2t - 3F'(t-2)]\,dt + \frac{1}{x}\left[3x^2 \cdot \left(2(x^3-4) - 3F'(x^3-6)\right) - 2 \cdot \left(4x - 3F'(2x-2)\right)\right]$$

$$F'(2) = -\frac{1}{4} \underbrace{\cancel{\int_{4}^{4} [2t - 3F'(t-2)]\,dt}}_{0} + \cancel{\frac{1}{2}}\left[\cancel{12}^{\,6}\left(8 - 3F'(2)\right) - \cancel{2}\left(8 - 3F'(2)\right)\right]$$

$$F'(2) = 48 - 18F'(2) - 8 + 3F'(2) \Rightarrow 16F'(2) = 40 \Rightarrow F'(2) = \boxed{\frac{5}{2}}$$

7) Find the area of the region inside of the cardioid $r = 1 + \cos\theta$ and outside of the circle $r = 1$ in unit square.



$$A = \frac{1}{2} \cdot \underset{\text{symmetry}}{2} \int_0^{\pi/2} \left[\underbrace{(1+\cos\theta)^2}_{\cancel{1}+2\cos\theta+\cos^2\theta} - \cancel{1}\right] d\theta$$

$$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= 2\sin\theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \Big|_0^{\pi/2} = 2 \cdot 1 + \frac{\pi}{4} = \boxed{2 + \frac{\pi}{4}}$$

8) If a continuous function $f: \mathbb{R} \to \mathbb{R}$ satisfies the equation $\int_0^{x^2} f(t)\,dt = x^2 \cdot e^{x^2}$, what is the value of $f(-1)$?

Take derivative: $\cancel{2x} \cdot f(x^2) = \cancel{2x} \cdot e^{x^2} + x^2 \cdot \cancel{2x} \cdot e^{x^2} \Rightarrow f(x^2) = e^{x^2} + x^2 \cdot e^{x^2}$

$\Rightarrow f(x) = e^x + x \cdot e^x \Rightarrow f(-1) = \cancel{e^{-1}} - \cancel{e^{-1}} = \boxed{0}$

9) If the continuous function $f:[0,\infty)\to \mathbb{R}$ satisfies the equation $\int_0^x f(t)dt = x + \int_x^1 t^2 \cdot f(t)dt$, what is the result of the integral $\int_0^\infty f(x)dx$?

Take derivative: $f(x) = 1 - x^2 f(x) \Rightarrow f(x) = \frac{1}{1+x^2}$

$$\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan x \Big|_0^b = \lim_{b\to\infty}(\underbrace{\arctan b}_{\frac{\pi}{2}} - \underbrace{\arctan 0}_{0}) = \boxed{\frac{\pi}{2}}$$

10) For $x>0$, if $f(x) = \int_1^x \frac{\ln t}{1+t}dt$ and $F(x) = f(x) + f\left(\frac{1}{x}\right)$, which of the following gives F as a function of x?

A) $\ln x$ B) $x-1$ C) $\frac{1}{2}\ln^2 x$ D) x^2 E) x

$$f'(x) = \frac{\ln x}{1+x} \Rightarrow f'\left(\frac{1}{x}\right) = \frac{\ln x^{-1}}{1+\frac{1}{x}} = \frac{-x\cdot \ln x}{1+x}$$

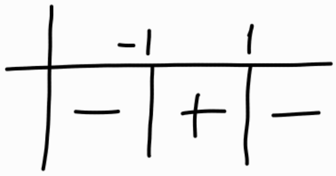
$$F'(x) = f'(x) - \frac{1}{x^2}\cdot f'\left(\frac{1}{x}\right) = \frac{\ln x}{1+x} + \frac{1}{x^{\cancel{2}}}\cdot \cancel{x}\cdot \frac{\ln x}{1+x} = \frac{x\cdot \ln x + \ln x}{x\cdot(1+x)}$$

$$= \frac{\ln x\cancel{(1+x)}}{x\cdot\cancel{(1+x)}} = \frac{\ln x}{x} \Rightarrow F(x) = \int \frac{\ln x}{x}dx \qquad \ln x = t, \ \frac{dx}{x} = dt$$

$$= \int t\,dt = \frac{t^2}{2} + c = \frac{1}{2}\ln^2 x + c$$

$$F(1) = f(1) + f(1) = 0 + 0 = 0 \Rightarrow F(x) = \boxed{\frac{1}{2}\ln^2 x}$$

11) $\int_{-2}^{2} |1-x^2|\,dx = ?$



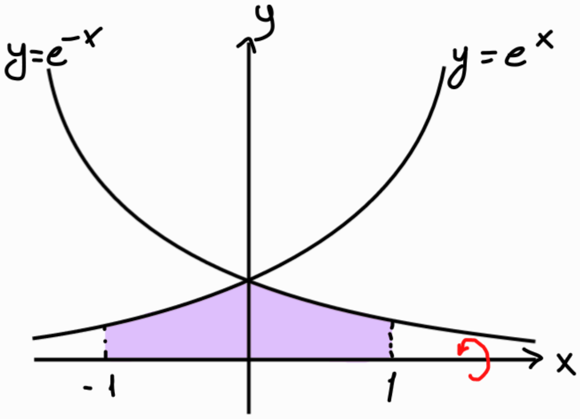
$$I = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (1-x^2)\,dx + \int_{1}^{2} (x^2-1)\,dx$$

Note that x^2-1 is even function and intervals $[-2,-1]$ and $[1,2]$ have symmetry. Also $[-1,1] \to [0,1].2$

$$I = 2 \cdot \int_{0}^{1} (1-x^2)\,dx + 2 \cdot \int_{1}^{2} (x^2-1)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 + 2 \cdot \left(\frac{x^3}{3} - x\right)\Big|_1^2$$

$$= 2 \cdot \left(1 - \frac{1}{3}\right) + 2\left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right) = 2 \cdot \frac{6}{3} = 4$$

12) For $x \in [-1,1]$, if the region bounded by the curves $y=e^x$, $y=e^{-x}$ and the line $y=0$ is rotated about the x-axis, what is the volume of the obtained solid in unit cube?



Disk

$$V = \pi \int_{-1}^{0} e^{2x}\,dx + \pi \int_{0}^{1} e^{-2x}\,dx \quad \text{OR}$$

symmetry: $V = 2\pi \int_{0}^{1} e^{-2x}\,dx = 2\pi \frac{e^{-2x}}{-2}\Big|_0^1$

$$= \pi\left(\frac{-1}{e^2} + 1\right) = \pi \cdot \frac{e^2-1}{e^2}$$

13) Which of the following is the angle between the plane $x+2y+3z=5$ and the line $\vec{r}(t) = \langle 1+2t, 3+4t, 6t \rangle$?

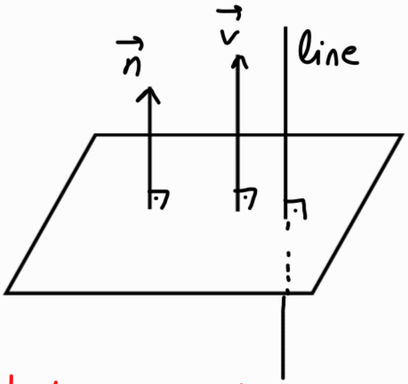
A) π B) $\frac{\pi}{2}$ C) $\frac{\pi}{3}$ D) $\frac{3\pi}{4}$ E) $\frac{\pi}{5}$

normal vector of the plane : $\vec{n} = \langle 1,2,3 \rangle$

direction vector of the line : $\vec{v} = \langle 2,4,6 \rangle$

$\vec{v} = 2 \cdot \vec{n} \Rightarrow \vec{v}$ and $\vec{n}$ are parallel.

$\Rightarrow$ line is verticle to the plane



If $\frac{3\pi}{2}$ was an option instead of $\frac{\pi}{2}$, it would be correct.

14 $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx = ?$

$$\frac{\sqrt{\cos x} + \sqrt{\sin x} - \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} = \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} - \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} = 1 - \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} = \underbrace{\int_0^{\pi/2} 1 \, dx}_{x \big|_0^{\pi/2} = \frac{\pi}{2}} - \underbrace{\int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}}_{J}$$

For J : $\sin x = \cos\left(\frac{\pi}{2} - x\right)$
$\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$J = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\, dx}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}}$$

$\frac{\pi}{2} - x = t \qquad x=0 \Rightarrow t = \frac{\pi}{2}$

$-dx = dt \qquad x = \frac{\pi}{2} \Rightarrow t = 0$

$$J = -\int_{\pi/2}^{0} \frac{\sqrt{\cos t}\, dt}{\sqrt{\sin t} + \sqrt{\cos t}}$$

$$J = + \int_0^{\pi/2} \frac{\sqrt{\cos t}\, dt}{\sqrt{\cos t} + \sqrt{\sin t}} = I \Rightarrow I = \frac{\pi}{2} - I \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

15 $\displaystyle\int \frac{2x \cdot e^{2x}}{(x+1)^3} \, dx = ?$

$$\frac{(2x+2-2) \cdot e^{2x}}{(x+1)^3} = \frac{2\cancel{(x+1)} \cdot e^{2x}}{\cancel{(x+1)} \cdot (x+1)^2} - \frac{2 \cdot e^{2x}}{(x+1)^3} \Rightarrow I = \underbrace{\int \frac{2e^{2x}\, dx}{(x+1)^2}}_{J} - \underbrace{\int \frac{2e^{2x}\, dx}{(x+1)^3}}_{K}$$

For J: $\frac{1}{(x+1)^2} = u \qquad 2 \cdot e^{2x}dx = dv \qquad J = \frac{e^{2x}}{(x+1)^2} + \int \frac{2e^{2x}dx}{(x+1)^3}$

$\frac{-2dx}{(x+1)^3} = du \qquad e^{2x} = v$

$$I = J - K = \frac{e^{2x}}{(x+1)^2} + \cancel{\int \frac{2e^{2x}dx}{(x+1)^3}} - \cancel{\int \frac{2e^{2x}dx}{(x+1)^3}} = \boxed{\frac{e^{2x}}{(x+1)^2} + c}$$

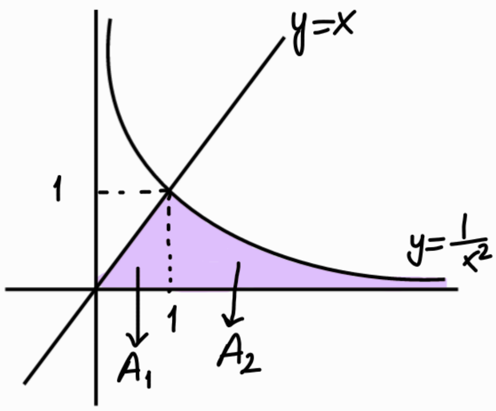
16) If for a continuous function $f: \mathbb{R} \to \mathbb{R}$, $\int_0^2 f(x)dx = 2$ and $\int_{-1}^2 x \cdot f(2x^2)dx = \frac{15}{2}$, what is the value of $\int_0^4 f(2x)dx$?

$\int_0^2 f(x)dx = 2 \qquad \begin{matrix} x = 2t \\ dx = 2dt \end{matrix} \qquad \begin{matrix} x=0 \Rightarrow t=0 \\ x=2 \Rightarrow t=1 \end{matrix} \qquad \int_0^1 2f(2t)dt = 2 \Rightarrow \int_0^1 f(2t)dt = 1 = A$

$\int_{-1}^2 x \cdot f(2x^2)dx = \frac{15}{2} \qquad \begin{matrix} x^2 = t \\ 2xdx = dt \end{matrix} \qquad \begin{matrix} x=-1 \Rightarrow t=1 \\ x=2 \Rightarrow t=4 \end{matrix} \qquad \int_1^4 \frac{f(2t)}{\cancel{2}}dt = \frac{15}{\cancel{2}} \Rightarrow \int_1^4 f(2t)dt = 15 = B$

$$\int_0^4 f(2x)dx = \int_0^1 f(2x)dx + \int_1^4 f(2x)dx = A + B = \boxed{16}$$

17) What is the area of the region bounded by the lines $y=0$, $y=x$ and the curve $y = \frac{1}{x^2}$ in unit square?

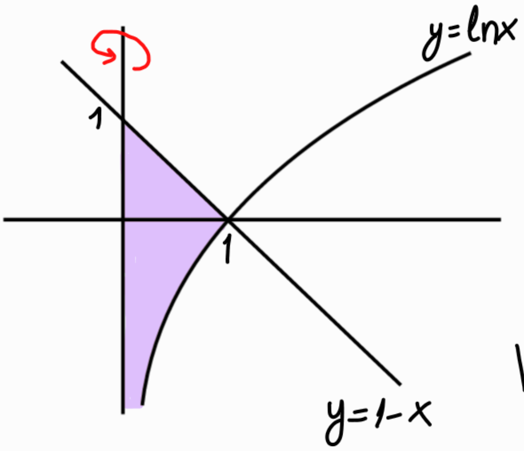


$x = \frac{1}{x^2} \Rightarrow x = 1$ (intersection) $\quad \underbrace{A_1}_{\text{triangle}} = \frac{1 \cdot 1}{2} = \frac{1}{2}$

$$A_2 = \int_1^\infty \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to \infty} \frac{-1}{x}\Big|_1^b$$

$$= \lim_{b \to \infty} \Big(-\underbrace{\frac{1}{b}}_{0} + 1\Big) = 1 \Rightarrow A = A_1 + A_2 = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

18 If the region bounded by the lines $x=0$, $y=1-x$ and the curve $y=\ln x$ is rotated about the y-axis, what is the volume of this obtained solid in unit cube?



Shell ; $V=2\pi\int_0^1 x\cdot\underbrace{[(1-x)-\ln x]}_{x-x^2-x\ln x}\,dx$

improper! (for $\ln x$)

$$V=2\pi\left[\int_0^1 (x-x^2)\,dx-\lim_{a\to0}\int_a^1 x\ln x\,dx\right]$$

$\ln x=u$ $\quad x\,dx=dv$

$\frac{dx}{x}=du$ $\quad \frac{x^2}{2}=v$

$$V=2\pi\left[\underbrace{\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1}_{\frac12-\frac13=\frac16}-\lim_{a\to0}\left[\frac{x^2}{2}\cdot\ln x-\underbrace{\int\frac{x^{\cancel{2}}}{2}\cdot\frac{dx}{\cancel{x}}}_{\frac{x^2}{4}}\right]_a^1\right]$$

$$V=2\pi\left[\frac16-\lim_{a\to0}\left(\underbrace{\frac12\cdot\ln1}_{0}-\frac14-\underbrace{\frac{a^2}{2}\ln a}+\underbrace{\frac{a^2}{4}}_{0}\right)\right]=2\pi\left(\underbrace{\frac16+\frac14}_{\frac{5}{12}}\right)=\boxed{\frac{5\pi}{6}}$$

$$\lim_{a\to0}a^2\cdot\ln a\overset{(0\cdot\infty)}{=}\lim_{a\to0}\frac{\ln a}{\frac{1}{a^2}}\underset{(\frac{\infty}{\infty})}{\overset{L}{=}}\lim_{a\to0}\frac{\frac{1}{\cancel{a}}}{\frac{-2}{a^{\cancel{3}2}}}=0$$

2nd way (Disk) ; $V=\pi\left[\int_{-\infty}^0 e^{2y}\,dy+\int_0^1\underbrace{(1-y)^2}_{1-2y+y^2}\,dy\right]$

$$V=\pi\left[\underbrace{\lim_{a\to-\infty}\int_a^0 e^{2y}\,dy}_{\frac{e^{2y}}{2}\Big|_a^0=\frac12-\underbrace{\frac{e^{2a}}{2}}_{\to0}}+\left(y-y^2+\frac{y^3}{3}\right)\Big|_0^1\right]=\pi\left(\frac12+1-1+\frac13\right)=\boxed{\frac{5\pi}{6}}$$

19) Which of the following is a curve whose arc length integral is $S=\int_0^{\pi}\sqrt{1+4x^2\cdot\sin^2(x^2)}\,dx$ and passes through the point $P(\sqrt{\pi},1)$?

A) $y=1$ B) $y=1+\sqrt{\pi}-x$ C) $y=-\cos x^2$ D) $y=3+2\cos x^2$ E) $y=1+\sin x^2$

$$(y')^2=4x^2\cdot\sin^2 x^2 \Rightarrow y'=2x\sin x^2 \Rightarrow y=\int 2x\sin x^2\,dx \qquad \begin{array}{l} x^2=t \\ 2x\,dx=dt \end{array}$$

$$y=\int \sin t\,dt=-\cos t+C=-\cos x^2+C \Rightarrow \underbrace{-\underbrace{\cos\pi}_{-1}}_{1}+C=1 \Rightarrow C=0$$

$$y=-\cos x^2$$

20) For $x>0$, what is the partial fraction decomposition of the integral $\int\frac{2\,dx}{x(\ln^2x-\ln x^2)}$?

$$\frac{2}{\ln^2x-2\ln x}\overset{\ln x=t}{\Longrightarrow}\frac{2}{t^2-2t}=\frac{2}{t(t-2)}=\frac{A}{t}+\frac{B}{t-2}$$

$$A=\frac{2}{t-2}\Big|_{t=0}=-1\ ,\quad B=\frac{2}{t}\Big|_{t=2}=1 \Rightarrow \frac{1}{\ln x-2}-\frac{1}{\ln x}$$

$$\Rightarrow \int\left(\frac{1}{x(\ln x-2)}-\frac{1}{x\ln x}\right)dx=\int\left(\frac{1}{x\ln x-2x}-\frac{1}{x\ln x}\right)dx$$