EXAMPLE 7.1 Determining the Natural Response of an *RL* Circuit

The switch in the circuit shown in Fig. 7.6 has been closed for a long time before it is opened at $t = 0$. Find

a) $i_L(t)$ for $t \geq 0$,

b) $i_o(t)$ for $t \geq 0^+$,

c) $v_o(t)$ for $t \geq 0^+$,

d) the percentage of the total energy stored in the 2 H inductor that is dissipated in the 10 Ω resistor.

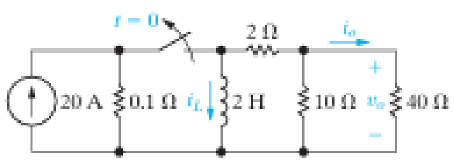


Figure 7.6 ▲ The circuit for Example 7.1.

Solution

Use Analysis Method 7.1.

a) **Step 1:** To determine the initial current in the inductor, draw the circuit in Fig. 7.6 for $t < 0$. The switch has been closed for a long time prior to $t = 0$, so we know the inductor voltage is zero at $t = 0^-$ and the inductor can be replaced by a short circuit. The result is shown in Fig. 7.7. The short circuit shunts all of the resistors, so it has all of the current from the source. Therefore, the current in the inductor at $t = 0^-$ is 20 A and

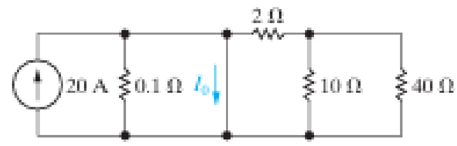


Figure 7.7 ▲ The circuit for Example 7.1 when $t < 0$.

$$I_0 = i_L(0^-) = i_L(0^+) = 20 \text{ A}.$$

Step 2: To calculate the time constant, τ, we need to find the equivalent resistance attached to the inductor when $t \geq 0$. To do this, draw the circuit in Fig. 7.6 for $t \geq 0$. Since the switch is now open, the current source and its parallel resistor are removed from the circuit, as shown in Fig. 7.8.

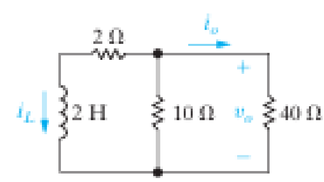


Figure 7.8 ▲ The circuit for Example 7.1 when $t \geq 0$.

From this circuit you can see that the equivalent resistance attached to the inductor is

$$R_{eq} = 2 + (40 \| 10) = 10 \ \Omega$$

and the time constant of the circuit is

$$\tau = \frac{L}{R_{eq}} = \frac{2}{10} = 0.2 \text{ s}.$$

Step 3: Write the equation for the inductor current by substituting the values for the initial current and the time constant into Eq. 7.4 to give

$$i_L(t) = I_0 e^{-t/\tau} = 20e^{-t/0.2} = 20e^{-5t} \text{ A}, \quad t \geq 0.$$

Step 4: We use resistive circuit analysis in the remaining parts of this problem to find additional currents and voltages.

b) We find the current in the 40 Ω resistor in Fig. 7.8 using current division; that is,

$$i_o = -i_L \frac{10}{10 + 40}.$$

Note that this expression is valid for $t \geq 0^+$ because $i_o = 0$ at $t = 0^-$, so the resistor current i_o changes instantaneously. Thus,

$$i_o(t) = -0.25 i_L(t) = -4e^{-5t} \text{ A}, t \geq 0^+.$$

c) We find the voltage $_o$ in Fig. 7.8 by applying Ohm's law:

$$v_o(t) = 40 i_o = -160e^{-5t} \text{ V}, \quad t \geq 0^+.$$

d) The power dissipated in the 10 Ω resistor in Fig. 7.8 is

$$p_{10\Omega}(t) = \frac{v_o^2}{10} = 2560e^{-10t} \text{ W}, \quad t \geq 0^+.$$

The total energy dissipated in the 10 Ω resistor is

$$w_{10\Omega}(t) = \int_0^\infty 2560e^{-10t}\, dt = 256 \text{ J}.$$

The initial energy stored in the 2 H inductor is

$$w(0) = \frac{1}{2} L i_L^2(0) = \frac{1}{2}(2)(20)^2 = 400 \text{ J}.$$

Therefore, the percentage of energy dissipated in the 10 Ω resistor is

$$\frac{256}{400}(100) = 64\%.$$

EXAMPLE 7.2 Determining the Natural Response of an *RL* Circuit with Parallel Inductors

In the circuit shown in Fig. 7.10, the initial currents in inductors L_1 and L_2 have been established by sources not shown. The switch is opened at $t = 0$.

a) Find i_1, i_2, and i_3 for $t \geq 0$.

b) Calculate the initial energy stored in the parallel inductors.

c) Determine how much energy is stored in the inductors as $t \to \infty$.

d) Show that the total energy delivered to the resistive network equals the difference between the results obtained in (b) and (c).

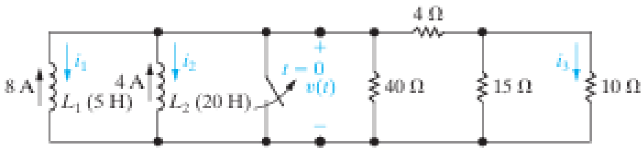


Figure 7.10 ▲ The circuit for Example 7.2.

Solution

a) The key to finding currents i_1, i_2, and i_3 lies in knowing the voltage $v(t)$. We can easily find $v(t)$ if we simplify the circuit shown in Fig. 7.10 to the equivalent form shown in Fig. 7.11. The parallel inductors combine to give an equivalent inductance of 4 H, carrying an initial current of 12 A. The resistive network reduces to a single resistance of $40 \parallel [4 + (15 \parallel 10)] = 8\ \Omega$. We can now use Analysis Method 7.1.

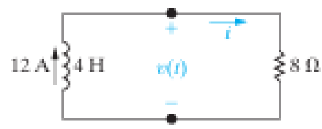


Figure 7.11 ▲ A simplification of the circuit shown in Fig. 7.10.

Step 1: The initial current in the inductor in Fig. 7.11 is $I_0 = 12$ A.

Step 2: The equivalent resistance attached to the inductor in Fig. 7.11 is 8 Ω. Therefore, the time constant is

$$\tau = \frac{L}{R} = \frac{4}{8} = 0.5 \text{ s.}$$

Step 3: The inductor current in Fig. 7.11 is

$$i(t) = I_0 e^{-t/\tau} = 12e^{-t/0.5} = 12e^{-2t}\text{ A}, \quad t \geq 0.$$

Step 4: We will use additional circuit analysis techniques to find the currents i_1, i_2, and i_3. To begin, note that in Fig. 7.11, $v(t) = 8i(t)$, so

$$v(t) = 96e^{-2t}\text{ V}, \quad t \geq 0^+.$$

From the circuit in Fig. 7.10, we see that that $v(t) = 0$ at $t = 0^-$, so the expression for $v(t)$ is valid for $t \geq 0^+$. After obtaining $v(t)$, we can calculate i_1 and i_2 using the relationship between current and voltage in inductors:

$$\begin{aligned} i_1 &= \frac{1}{5}\int_0^t 96e^{-2x}\,dx - 8 \\ &= 1.6 - 9.6e^{-2t}\text{ A}, \quad t \geq 0, \\ i_2 &= \frac{1}{20}\int_0^t 96e^{-2x}\,dx - 4 \\ &= -1.6 - 2.4e^{-2t}\text{ A}, \quad t \geq 0. \end{aligned}$$

We will use two steps to find i_3; in the first step, calculate the voltage across the parallel 15 Ω and 10 Ω resistors using voltage division. Calling that voltage $v_{15\|10}$, positive at the top of the circuit, we get

$$v_{15\|10} = \frac{15\|10}{4 + 15\|10}v = \frac{6}{10}(96e^{-2t}) = 57.6e^{-2t}\text{ V}, t \geq 0^+.$$

Now use Ohm's law to calculate i_3, giving

$$i_3 = \frac{v_{15\|10}}{10} = 5.76e^{-2t}\text{ A}, \quad t \geq 0^+.$$

Note that the expressions for the inductor currents i_1 and i_2 are valid for $t \geq 0$, whereas the expression for the resistor current i_3 is valid for $t \geq 0^+$.

b) The initial energy stored in the inductors is

$$w = \frac{1}{2}(5)(8)^2 + \frac{1}{2}(20)(4)^2 = 320 \text{ J.}$$

c) As $t \to \infty$, $i_1 \to 1.6$ A and $i_2 \to -1.6$ A. Therefore, a long time after the switch opens, the energy stored in the two inductors is

$$w = \frac{1}{2}(5)(1.6)^2 + \frac{1}{2}(20)(-1.6)^2 = 32 \text{ J.}$$

d) We obtain the total energy delivered to the resistive network by integrating the expression for the instantaneous power from zero to infinity:

$$w = \int_0^\infty p\,dt = \int_0^\infty (96e^{-2t})(12e^{-2t})\,dt$$

$$= 1152\frac{e^{-4t}}{-4}\bigg|_0^\infty = 288 \text{ J.}$$

This result is the difference between the initially stored energy (320 J) and the energy trapped in the parallel inductors (32 J). Also, note that the equivalent inductor for the parallel inductors (which predicts the terminal behavior of the parallel combination) has an initial energy of $\frac{1}{2}(4)(12)^2 = 288$ J; that is, the energy stored in the equivalent inductor represents the amount of energy that will be delivered to the resistive network at the terminals of the original inductors.

SELF-CHECK: Also try Chapter Problem 7.19.

EXAMPLE 7.3 Determining the Natural Response of an *RC* Circuit

The switch in the circuit shown in Fig. 7.15 has been in position x for a long time. At $t = 0$, the switch moves instantaneously to position y. Find

a) $v_C(t)$ for $t \geq 0$,

b) $v_o(t)$ for $t \geq 0^+$,

c) $i_o(t)$ for $t \geq 0^+$, and

d) the total energy dissipated in the 60 kΩ resistor.

Solution

Use Analysis Method 7.2.

a) **Step 1:** Determine the initial capacitor voltage V_0 by drawing the circuit in Fig. 7.15 for $t < 0$. The result is shown in Fig. 7.16, and since the capacitor behaves like an open circuit, its initial voltage equals the source voltage:

$$V_0 = 100 \text{ V}.$$

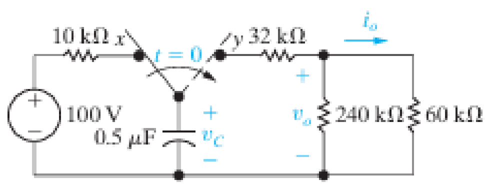


Figure 7.15 ▲ The circuit for Example 7.3.

Step 2: Calculate the time constant. To do this, draw the circuit in Fig. 7.15 for $t \geq 0$, as shown in Fig. 7.17, and find the equivalent resistance attached to the capacitor:

$$R_{eq} = 32 \times 10^3 + (240 \times 10^3 \| 60 \times 10^3)$$
$$= 80 \text{ k}\Omega,$$
$$\tau = R_{eq}C = (80 \times 10^3)(0.5 \times 10^{-6})$$
$$= 40 \text{ ms}.$$

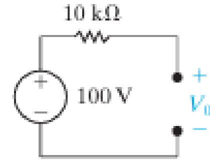


Figure 7.16 ▲ The circuit in Fig. 7.15 for $t < 0$.

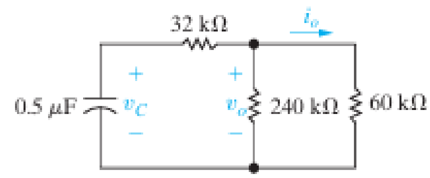


Figure 7.17 ▲ The circuit in Fig. 7.15 for $t \geq 0$.

Step 3: Write the equation for the capacitor voltage by substituting the values for V_0 and τ into Eq. 7.9:

$$v_C(t) = 100e^{-t/0.04} = 100e^{-25t} \text{ V}, \quad t \geq 0.$$

Step 4: Determine the remaining quantities using resistive circuit analysis techniques for the circuit in Fig. 7.17.

b) To find $v_o(t)$ in Fig. 7.17, note that the resistive circuit forms a voltage divider across the terminals of the capacitor. Thus

$$v_o(t) = \frac{240 \times 10^3 \| 60 \times 10^3}{32 \times 10^3 + (240 \times 10^3 \| 60 \times 10^3)} v_C(t)$$
$$= 0.6(100e^{-25t}) = 60e^{-25t} \text{ V}, \quad t \geq 0^+.$$

This expression for $v_o(t)$ is valid for $t \geq 0^+$ because $v_o(0^-)$ is zero. Thus, we have an instantaneous change in the voltage across the 240 kΩ resistor.

c) We find the current $i_o(t)$ from Ohm's law:

$$i_o(t) = \frac{v_o(t)}{60 \times 10^3} = e^{-25t} \text{ mA}, \quad t \geq 0^+.$$

d) The power dissipated in the 60 kΩ resistor is

$$p_{60\text{k}\Omega}(t) = i_o^2(t)(60 \times 10^3) = 60e^{-50t} \text{ mW}, \quad t \geq 0^+.$$

The total energy dissipated is

$$w_{60\text{k}\Omega} = \int_0^{\infty} i_o^2(t)(60 \times 10^3)dt = 1.2 \text{ mJ}.$$

EXAMPLE 7.4 Determining the Natural Response of an *RC* Circuit with Series Capacitors

The initial voltages on capacitors C_1 and C_2 in the circuit shown in Fig. 7.18 have been established by sources not shown. The switch is closed at $t = 0$.

a) Find $v_1(t)$, $v_2(t)$, and $v(t)$ for $t \geq 0$ and $i(t)$ for $t \geq 0^+$.
b) Calculate the initial energy stored in the capacitors C_1 and C_2.
c) Determine how much energy is stored in the capacitors as $t \rightarrow \infty$.
d) Show that the total energy delivered to the 250 kΩ resistor is the difference between the results obtained in (b) and (c).

Solution

a) Once we know $v(t)$, we can obtain the current $i(t)$ from Ohm's law. After determining $i(t)$, we can calculate $v_1(t)$ and $v_2(t)$ because the voltage across a capacitor is a function of the capacitor current. To find $v(t)$, we replace the series-connected capacitors with an equivalent capacitor. It has a capacitance of 4μF and is charged to a voltage of 20 V. Therefore, the circuit shown in Fig. 7.18 reduces to the one shown in Fig. 7.19. We can now use Analysis Method 7.2 to determine $v(t)$.

Step 1: The initial voltage across the capacitor in Fig. 7.19 is $V_0 = 20$ V.

Step 2: The resistance attached to the capacitor in Fig. 7.19 is 250 kΩ. Therefore, the time constant is

$$\tau = (250 \times 10^3)(4 \times 10^{-6}) = 1 \text{ s}.$$

Step 3: Write the equation for the capacitor voltage by substituting the values for V_0 and τ in Eq. 7.9 to give

$$v(t) = 20e^{-t}\text{ V}, \quad t \geq 0.$$

Step 4: Determine the currents and voltages requested using the techniques described at the start of the Solution.

For the circuit in Fig. 7.19, use Ohm's law to find the current $i(t)$:

$$i(t) = \frac{v(t)}{250{,}000} = 80e^{-t}\,\mu\text{A}, \quad t \geq 0^+.$$

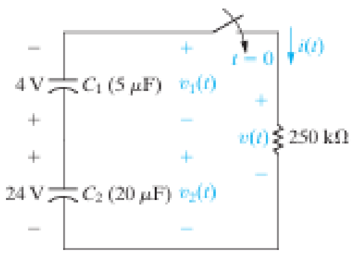


Figure 7.18 ▲ The circuit for Example 7.4.

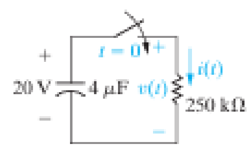


Figure 7.19 ▲ A simplification of the circuit shown in Fig. 7.18.

Knowing $i(t)$, we calculate the expressions for $v_1(t)$ and $v_2(t)$ for the circuit in Fig. 7.18:

$$v_1(t) = -\frac{1}{5 \times 10^{-6}}\int_0^t 80 \times 10^{-6}e^{-x}dx - 4$$
$$= (16e^{-t} - 20)\text{ V}, \quad t \geq 0,$$

$$v_2(t) = -\frac{1}{20 \times 10^{-6}}\int_0^t 80 \times 10^{-6}e^{-x}dx + 24$$
$$= (4e^{-t} + 20)\text{ V}, \quad t \geq 0.$$

b) The initial energy stored in C_1 is

$$w_1 = \frac{1}{2}(5 \times 10^{-6})(4)^2 = 40\,\mu\text{J}.$$

The initial energy stored in C_2 is

$$w_2 = \frac{1}{2}(20 \times 10^{-6})(24)^2 = 5760\,\mu\text{J}.$$

The total energy stored in the two capacitors is

$$w_o = 40 + 5760 = 5800\,\mu\text{J}.$$

c) As $t \rightarrow \infty$,

$$v_1 \rightarrow -20\text{ V and } v_2 \rightarrow +20\text{ V}.$$

Therefore, the energy stored in the two capacitors is

$$w_\infty = \frac{1}{2}(5 \times 10^{-6})(-20)^2 + \frac{1}{2}(20 \times 10^{-6})(20)^2$$
$$= 5000\,\mu\text{J}.$$

d) The total energy delivered to the 250 kΩ resistor is

$$= \int_0^\infty p\,dt = \int_0^\infty (20e^{-t})(80 \times 10^{-6}e^{-t})dt = 800\mu\text{J}.$$

Comparing the results obtained in (b) and (c) shows that

$$800\,\mu\text{J} = (5800 - 5000)\,\mu\text{J}.$$

The energy stored in the equivalent capacitor in Fig. 7.19 is $\frac{1}{2}(4 \times 10^{-6})(20)^2 = 800\,\mu$J. Because this capacitor predicts the terminal behavior of the original series-connected capacitors, the energy stored in the equivalent capacitor is the energy delivered to the 250 kΩ resistor.

EXAMPLE 7.5 Determining the Step Response of an *RL* Circuit

The switch in the circuit shown in Fig. 7.21 has been in position a for a long time. At $t = 0$, the switch moves from position a to position b. The switch is a make-before-break type; that is, the connection at position b is established before the connection at position a is broken, so the inductor current is continuous.

a) Find the expression for $i(t)$ for $t \geq 0$.

b) What is the initial voltage across the inductor just after the switch has been moved to position b?

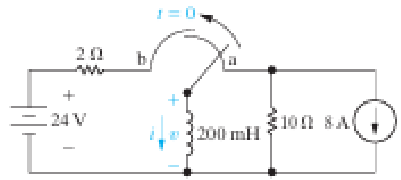


Figure 7.21 ▲ The circuit for Example 7.5.

c) Does this initial voltage make sense in terms of circuit behavior?

d) How many milliseconds after the switch has been moved does the inductor voltage equal 24 V?

e) Plot both $i(t)$ and $v(t)$ versus t.

Solution

a) Use Analysis Method 7.3 to find the inductor current.

Step 1: Determine the initial current in the inductor. To do this, draw the circuit in Fig. 7.21 when $t < 0$ and the switch is in position a, as shown in Fig. 7.22. Note that since the switch has been in position a for a long time, the inductor behaves like a short circuit that carries all of the current from the 8 A current source. Therefore, $I_0 = -8$ A because the inductor current and the source current are in opposite directions.

Step 2: Calculate the time constant for the circuit. Start by drawing the circuit in Fig. 7.21 when $t \geq 0$ and the switch is in position b, as shown in Fig. 7.23. Then determine the Thévenin equivalent resistance for the circuit attached to the inductor. Since the circuit attached to the inductor is already a Thévenin equivalent circuit, the Thévenin equivalent resistance is 2 Ω and $\tau = 0.2/2 = 0.1$ s.

Step 3: Calculate the final value for the inductor current. To do this, draw the circuit in Fig. 7.21 as $t \to \infty$, when the switch is in position b, as shown in Fig. 7.24. Since the switch has been in position b for a long time, the inductor behaves like a short circuit, as seen in Fig. 7.24, and the current can be found from Ohm's law. Therefore, $I_f = 24/2 = 12$ A.

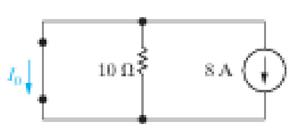


Figure 7.22 ▲ The circuit in Fig. 7.21 for $t < 0$.

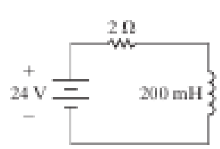


Figure 7.23 ▲ The circuit in Fig. 7.21 for $t \geq 0$.

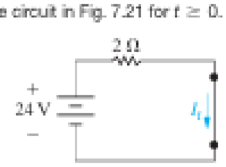


Figure 7.24 ▲ The circuit in Fig. 7.21 as $t \to \infty$.

Step 4: Write the equation for the inductor current when $t \geq 0$ by substituting the values for the initial current, the time constant, and the final current into Eq. 7.12 to give

$$
\begin{aligned}
i &= I_f + (I_0 - I_f)e^{-t/\tau} \\
&= 12 + (-8 - 12)e^{-t/0.1} \\
&= 12 - 20e^{-10t}\ \text{A},\ t \geq 0.
\end{aligned}
$$

Step 5: Calculate any other quantities of interest, which we do in the remainder of this example.

b) The voltage across the inductor is

$$
\begin{aligned}
v &= L\frac{di}{dt} \\
&= 0.2(200e^{-10t}) \\
&= 40e^{-10t}\ \text{V}, \quad t \geq 0^+.
\end{aligned}
$$

The initial inductor voltage is

$$v(0^+) = 40\ \text{V}.$$

c) Yes. In the instant after the switch has been moved to position b, the inductor current is 8 A counterclockwise around the newly formed closed path. This current causes a 16 V drop across the 2 Ω resistor. This voltage drop adds to the 24 V drop across the source, producing a 40 V drop across the inductor.

d) We find the time at which the inductor voltage equals 24 V by solving the expression

$$24 = 40e^{-10t}$$

for t:

$$
\begin{aligned}
t &= \frac{1}{10}\ln\frac{40}{24} \\
&= 51.08\ \text{ms}.
\end{aligned}
$$

e) Figure 7.25 shows the graphs of $i(t)$ and $v(t)$ versus t. Note that at the instant of time when the current equals zero, the inductor voltage equals the source voltage of 24 V, as predicted by Kirchhoff's voltage law.

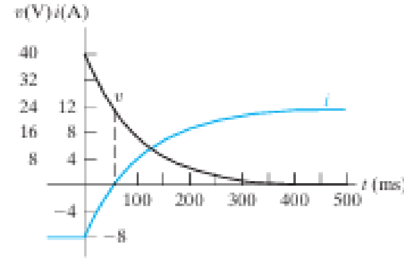


Figure 7.25 ▲ The current and voltage waveforms for Example 7.5.

EXAMPLE 7.6 Determining the Step Response of an *RC* Circuit

The switch in the circuit shown in Fig. 7.29 has been in position 1 for a long time. At $t = 0$, the switch moves to position 2. Find

a) $v_o(t)$ for $t \geq 0$ and

b) $i_o(t)$ for $t \geq 0^+$.

Solution

Use Analysis Method 7.4.

a) **Step 1:** Determine the initial voltage across the capacitor by analyzing the circuit in Fig. 7.29 for $t < 0$. Do this by redrawing the circuit with the switch in position 1, as shown in Fig. 7.30. Note that the capacitor behaves like an open circuit because the switch has been in position 1 for a long time. The capacitor's initial voltage is the same as the voltage across the 60 kΩ resistor, which we can find using voltage division:

$$V_0 = \frac{60{,}000}{60{,}000 + 20{,}000}(40) = 30 \text{ V}.$$

Step 2: Calculate the time constant by finding the equivalent resistance attached to the capacitor for $t \geq 0$ in the circuit of Fig. 7.29. Begin by drawing the circuit in Fig. 7.29 with the switch in position 2, as shown in Fig. 7.31(a). Then find the Norton equivalent with respect to the terminals of the capacitor. Begin by computing the open-circuit voltage, which is given by the −75 V

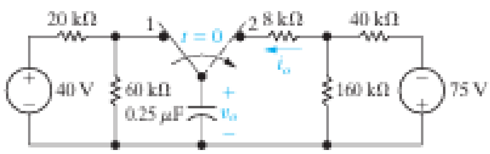


Figure 7.29 ▲ The circuit for Example 7.6.

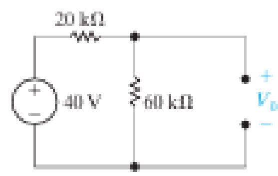


Figure 7.30 ▲ The circuit in Fig. 7.29 when $t < 0$.

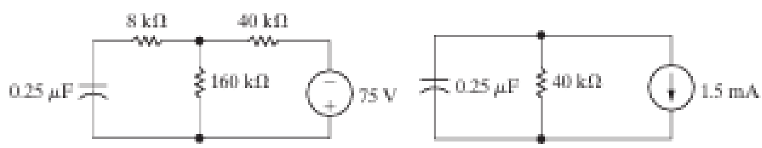


Figure 7.31 ▲ (a) The circuit in Fig. 7.29 when $t \geq 0$; (b) replacing the circuit to the right of the capacitor in part (a) with its Norton equivalent.

source divided across the 40 kΩ and 160 kΩ resistors:

$$V_{oc} = \frac{160 \times 10^3}{(40 + 160) \times 10^3}(-75) = -60 \text{ V}.$$

Next, calculate the Thévenin resistance, as seen to the right of the capacitor, by shorting the 75 V source and making series and parallel combinations of the resistors:

$$R_{Th} = 8000 + 40{,}000 \| 160{,}000 = 40 \text{ k}\Omega.$$

The value of the Norton current source is the ratio of the open-circuit voltage to the Thévenin resistance, or $-60/(40 \times 10^3) = -1.5$ mA. The resulting Norton equivalent circuit is shown in Fig. 7.31(b). From Fig. 7.31(b) we see that the equivalent resistance attached to the capacitor is 40 kΩ, so the time constant is

$$\tau = RC = (40 \times 10^3)(0.25 \times 10^{-6}) = 10 \text{ ms}.$$

Step 3: Calculate the final value of the capacitor voltage by analyzing the circuit in Fig. 7.29 as $t \to \infty$. The circuit is shown in Fig. 7.32, and since the switch has been in position 2 for a long time, the capacitor behaves like an open circuit. The final capacitor voltage equals the voltage across the 40 kΩ resistor, so

$$V_f = -(40 \times 10^3)(1.5 \times 10^{-3}) = -60 \text{ V}.$$

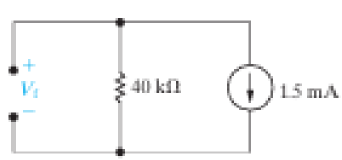


Figure 7.32 ▲ The circuit in Fig. 7.29 as $t \to \infty$.

Step 4: Write the equation for capacitor voltage by substituting the values for initial capacitor voltage, time constant, and final capacitor voltage into Eq. 7.20 to give

$$\begin{aligned} v_o &= V_f + (V_0 - V_f)e^{-t/\tau} \\ &= -60 + [30 - (-60)]e^{-t/0.01} \\ &= -60 + 90e^{-100t} \text{ V}, \quad t \geq 0. \end{aligned}$$

Step 5: We calculate the other quantity of interest, i_o, in part (b).

b) Write the solution for i_o using the relationship between current and voltage in a capacitor to give

$$\begin{aligned} i_o &= C\frac{dv_o}{dt} = (0.25 \times 10^{-6})(-9000e^{-100t}) \\ &= -2.25e^{-100t} \text{ mA}. \end{aligned}$$

Because $dv_o(0^-)/dt = 0$, the expression for i_o clearly is valid only for $t \geq 0^+$.

EXAMPLE 7.7 Using the General Solution Method to Find an *RL* Circuit's Natural Response

The switch in the circuit shown in Fig. 7.33 has been closed for a long time. At $t = 0$ the switch opens and remains open.

a) What is the initial value of i_o?

b) What is the time constant of the circuit when the switch is open?

c) What is the final value of i_o?

d) What is the expression for $i_o(t)$ when $t \geq 0$?

e) What is the expression for $v_o(t)$ when $t \geq 0$?

f) Find $v_o(0^-)$ and $v_o(0^+)$.

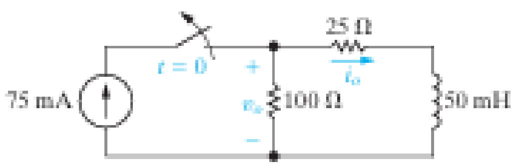


Figure 7.33 ▲ The circuit for Example 7.7.

Solution

Use Analysis Method 7.5.

a) **Step 1**: Identify the inductor current, i_o, as the variable of interest, because this is an *RL* circuit. **Step 2**: Calculate the initial value of i_o. The switch has been closed for a long time, so the inductor behaves like a short circuit. Therefore, the current through the inductor is the current in the 25 Ω resistor. Using current division, the current in the 25 Ω resistor is $[(100 \| 25)/25](0.075) = 60$ mA, so $I_0 = i_o(0) = 60$ mA.

b) **Step 3**: Calculate the time constant $\tau = L/R$. When $t \geq 0$, the equivalent resistance attached to the inductor is the series combination of the 100 Ω and 25 Ω resistors, or 125 Ω and. Therefore,

$$\tau = \frac{0.05}{125} = 0.4 \text{ ms}.$$

c) **Step 4**: Calculate the final value for the inductor current, I_f. This is a natural-response problem because for $t \geq 0$ there is no source in the circuit. Eventually, all of the energy stored in the inductor before the switch opens is dissipated by the resistors and the inductor current is zero, so $I_f = 0$.

d) **Step 5:** Write the equation for the inductor current by substituting the values for I_o, τ, and I_f into Eq. 7.23 to give

$$i_o(t) = I_f + (I_0 - I_f)e^{-t/\tau} = 0 + (0.06 - 0)e^{-t/0.4\times10^{-3}}$$

$$= 60e^{-2500t} \text{ mA}, \quad t \geq 0.$$

e) **Step 6:** Use the inductor current to find the voltage across the 100 Ω, using Ohm's law. The result is

$$v_o(t) = -100i_o = -6e^{-2500t} \text{ V}, t \geq 0^+.$$

f) From part (a), when $t < 0$ the switch is closed, and the current divides between the 100 Ω and 25 Ω resistors. We know that the current in the 25 Ω is 60 mA, so the current in the 100 Ω must be $75 - 60 = 15$ mA. Using Ohm's law,

$$v_o(0^-) = 100(0.015) = 1.5 \text{ V}.$$

From part (e)

$$v_o(0^+) = -6e^{-2500(0^+)} = -6 \text{ V}.$$

There is a discontinuity in the voltage across the 100 Ω resistor at $t = 0$.

EXAMPLE 7.8 Using the General Solution Method to Find an *RC* Circuit's Step Response

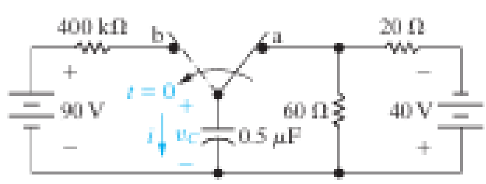
The switch in the circuit shown in Fig. 7.34 has been in position a for a long time. At $t = 0$ the switch is moved to position b.

a) What is the expression for $v_C(t)$ when $t \geq 0$?

b) What is the expression for $i(t)$ when $t \geq 0^+$?

c) How long after the switch is in position b does the capacitor voltage equal zero?

d) Plot $v_C(t)$ and $i(t)$ versus t.



Solution

Use Analysis Method 7.5.

a) **Step 1:** Identify the capacitor voltage, v_C, as the variable of interest, because this is an *RC* circuit.

Step 2: Calculate the initial value of v_C. The switch has been in position a for a long time, so the capacitor looks like an open circuit. Therefore, the voltage across the capacitor is the voltage across the 60 Ω resistor. Using voltage division, the voltage across the 60 Ω resistor is $[60/(60 + 20)](40) = 30$ V, positive at the lower terminal of the resistor. But v_C is positive at the upper terminal of the capacitor, so $V_0 = v_C(0) = -30$ V.

Step 3: Calculate the time constant $\tau = RC$. When $t \geq 0$, the equivalent resistance attached to the capacitor has the value 400 kΩ. Therefore,

$$\tau = (400 \times 10^3)(0.5 \times 10^{-6}) = 0.2 \text{ s}.$$

Step 4: Calculate the final value for the capacitor voltage, V_f. As $t \to \infty$, the switch has been in position b for a long time, and the capacitor behaves like an open circuit in the presence of the 90 V source. Because of the open circuit, there is no current in the 400 kΩ resistor, so $V_f = 90$ V.

Step 5: Write the equation for capacitor voltage by substituting the values for V_o, τ, and V_f into Eq. 7.23 to give

$$v_C(t) = V_f + (V_0 - V_f)e^{-t/\tau} = 90 + (-30 - 90)e^{-t/0.2}$$

$$= 90 - 120e^{-5t} \text{ V}, \quad t \geq 0.$$

b) **Step 6:** Use the relationship between voltage and current for capacitors to find the capacitor voltage. The result is

$$i(t) = C\frac{dv_C}{dt} = (0.5 \times 10^{-6})[-5(-120e^{-5t})]$$

$$= 300e^{-5t} \ \mu\text{A}, \quad t \geq 0^+.$$

c) To find how long the switch must be in position b before the capacitor voltage becomes zero, we solve the equation derived in (a) for the time when $v_C(t) = 0$:

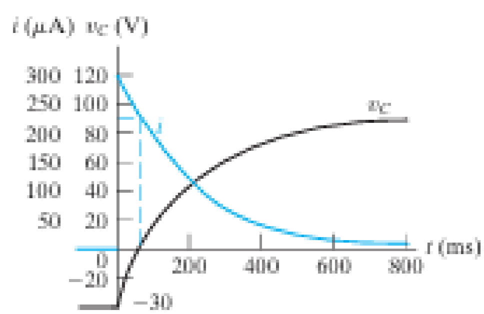
$$120e^{-5t} = 90 \text{ or } e^{5t} = \frac{120}{90},$$

so

$$t = \frac{1}{5}\ln\left(\frac{4}{3}\right) = 57.54 \text{ ms}.$$

Note that when $v_C = 0$, the voltage drop across the 400 kΩ resistor is 90 V so $i = 225\ \mu$A.

d) Figure 7.35 shows the graphs of $v_C(t)$ and $i(t)$ versus t.



EXAMPLE 7.9 Using the General Solution Method to Find an *RL* Circuit's Step Response

The switch in the circuit shown in Fig. 7.36 has been open for a long time. At $t = 0$ the switch is closed. Find the expression for

a) $i(t)$ when $t \geq 0$ and

b) $v(t)$ when $t \geq 0^+$.

Solution

Use Analysis Method 7.5.

a) **Step 1:** Identify the inductor current, i, as the variable of interest, because this is an *RL* circuit.

Step 2: Calculate the initial value of i. The switch has been open for a long time, so from Ohm's law, the initial current in the inductor is $20/(1 + 3) = 5$ A. Thus, $I_0 = i(0) = 5$ A.

Step 3: Calculate the time constant $\tau = L/R$. When $t \geq 0$, the switch is closed, shunting the 3 Ω resistor. The remaining resistance attached to the inductor has the value 1 Ω. Therefore,

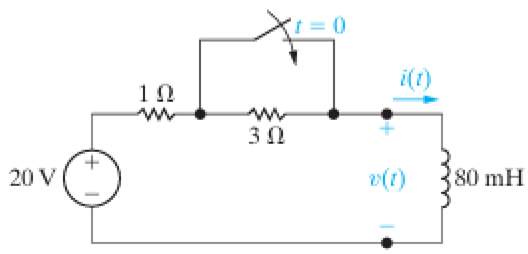


Figure 7.36 ▲ The circuit for Example 7.9.

$$\tau = \frac{80 \times 10^{-3}}{1} = 80 \text{ ms}.$$

Step 4: Calculate the final value for the inductor current. As $t \to \infty$, the switch has been closed for a long time, and the inductor behaves like a short circuit in the presence of the 20 V source. Using Ohm's law, the current in the inductor is $20/1 = 20$ A, so $I_f = 20$ A.

Step 5: Write the equation for inductor current by substituting the values for I_o, τ, and I_f into Eq. 7.23 to give

$$i(t) = I_f + (I_0 - I_f)e^{-t/\tau} = 20 + (5 - 20)e^{-t/0.08}$$
$$= 20 - 15e^{-12.5t} \text{ A}, \quad t \geq 0.$$

b) **Step 6:** Use the relationship between voltage and current for inductors to find the inductor voltage. The result is

$$v(t) = L\frac{di}{dt} = (80 \times 10^{-3})[-12.5(-15e^{-12.5t})]$$
$$= 15e^{-12.5t} \text{ V}, \quad t \geq 0^+.$$

EXAMPLE 7.10 Determining the Step Response of a Circuit with Magnetically Coupled Coils

There is no energy stored in the circuit in Fig. 7.37 at the time the switch is closed.

a) Find the solutions for i_o, $_o$, i_1, and i_2.

b) Show that the solutions obtained in (a) make sense in terms of known circuit behavior.

Solution

a) For the circuit in Fig. 7.37, the magnetically coupled coils can be replaced by a single inductor having an inductance of

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M} = \frac{45 - 36}{18 - 12} = 1.5 \text{ H}.$$

(See Problem 6.41.) It follows that the circuit in Fig. 7.37 can be simplified, as shown in Fig. 7.38. We can apply Analysis Method 7.5 to the circuit in Fig. 7.38.

Step 1: Identify the inductor current, i_o, as the variable of interest, because this is an RL circuit.

Step 2: Calculate the initial value of i. By hypothesis, there is no initial energy stored in the coils, so there is no initial current in the equivalent 1.5 H inductor. Thus, $I_0 = i(0) = 0$.

Step 3: Calculate the time constant $\tau = L/R$. When $t \geq 0$, the switch is closed and the resistance attached to the inductor has the value 7.5 Ω. Therefore,

$$\tau = \frac{1.5}{7.5} = 0.2 \text{ s}.$$

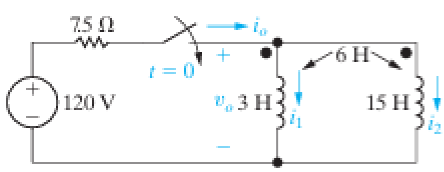


Figure 7.37 ▲ The circuit for Example 7.10.

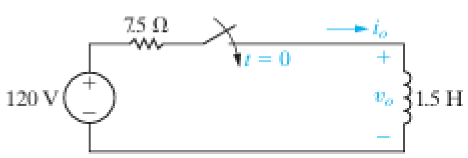


Figure 7.38 ▲ The circuit in Fig. 7.37 with the magnetically coupled coils replaced by an equivalent coil.

Step 4: Calculate the final value for the inductor current. As $t \to \infty$, the switch has been closed for a long time, and the inductor behaves like a short circuit in the presence of the 120 V source. Using Ohm's law, the current in the inductor is $120/7.5 = 16$ A, so $I_f = 16$ A.

Step 5: Write the equation for inductor current by substituting the values for I_o, τ, and I_f into Eq. 7.23 to give

$$i_o(t) = I_f + (I_0 - I_f)e^{-t/\tau} = 16 + (0 - 16)e^{-t/0.2}$$
$$= 16 - 16e^{-5t} \text{ A}, t \geq 0.$$

Step 6: Use the inductor current and the relationship between voltage and current for inductors to find the inductor voltage. The result is

$$v_o(t) = L\frac{di}{dt} = (1.5)[-5(-16e^{-5t})]$$
$$= 120e^{-5t} \text{ V}, t \geq 0^+.$$

To find i_1 and i_2 we use KVL for the mesh containing the coupled coils in Fig. 7.37 to see that

$$3\frac{di_1}{dt} + 6\frac{di_2}{dt} = 6\frac{di_1}{dt} + 15\frac{di_2}{dt}$$

or

$$\frac{di_1}{dt} = -3\frac{di_2}{dt}.$$

It also follows from Fig. 7.37 and KCL that $i_o = i_1 + i_2$, so

$$\frac{di_o}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = -3\frac{di_2}{dt} + \frac{di_2}{dt} = -2\frac{di_2}{dt}.$$

Therefore

$$80e^{-5t} = -2\frac{di_2}{dt}.$$

Because $i_2(0)$ is zero, we have

$$i_2 = \int_0^t -40e^{-5x}\,dx$$
$$= -8 + 8e^{-5t} \text{ A}, \quad t \geq 0.$$

Using $i_o = i_1 + i_2$, we get

$$i_1 = (16 - 16e^{-5t}) - (-8 + 8e^{-5t})$$
$$= 24 - 24e^{-5t} \text{ A}, \quad t \geq 0.$$

b) First, we observe that $i_o(0)$, $i_1(0)$, and $i_2(0)$ are all zero, which is consistent with the statement that no energy is stored in the circuit at the instant the switch is closed. Then we observe that $v_o(0^+) = 120$ V, which is consistent with the fact that $i_o(0) = 0$. Now we see that the solutions for i_1 and i_2 are consistent with the solution for $_o$ by observing

$$\begin{aligned} v_o &= 3\frac{di_1}{dt} + 6\frac{di_2}{dt} \\ &= 360e^{-5t} - 240e^{-5t} \\ &= 120e^{-5t}\text{ V}, \quad t \geq 0^+, \end{aligned}$$

or

$$\begin{aligned} v_o &= 6\frac{di_1}{dt} + 15\frac{di_2}{dt} \\ &= 720e^{-5t} - 600e^{-5t} \\ &= 120e^{-5t}\text{ V}, \quad t \geq 0^+. \end{aligned}$$

The final values of i_1 and i_2 can be checked using flux linkages. The flux linking the 3 H coil (λ_1) must be equal to the flux linking the 15 H coil (λ_2) because

$$v_o = \frac{d\lambda_1}{dt} = \frac{d\lambda_2}{dt}.$$

Now

$$\lambda_1 = 3i_1 + 6i_2 \text{ Wb-turns}$$

and

$$\lambda_2 = 6i_1 + 15i_2 \text{ Wb-turns.}$$

Regardless of which expression we use, we obtain

$$\lambda_1 = \lambda_2 = 24 - 24e^{-5t} \text{ Wb-turns.}$$

Note the solution for λ_1 or λ_2 is consistent with the solution for v_o.

The final value of the flux linking either coil 1 or coil 2 is 24 Wb-turns; that is,

$$\lambda_1(\infty) = \lambda_2(\infty) = 24 \text{ Wb-turns.}$$

The final value of i_1 is

$$i_1(\infty) = 24 \text{ A}$$

and the final value of i_2 is

$$i_2(\infty) = -8 \text{ A.}$$

The consistency between these final values for i_1 and i_2 and the final value of the flux linkage can be seen from the expressions:

$$\begin{aligned} \lambda_1(\infty) &= 3i_1(\infty) + 6i_2(\infty) \\ &= 3(24) + 6(-8) = 24 \text{ Wb-turns,} \\ \lambda_2(\infty) &= 6i_1(\infty) + 15i_2(\infty) \\ &= 6(24) + 15(-8) = 24 \text{ Wb-turns.} \end{aligned}$$

The final values of i_1 and i_2 can only be checked via flux linkage because at $t = \infty$ the two coils are ideal short circuits. We cannot use current division when the two branches have no resistance

EXAMPLE 7.11 Analyzing an *RL* Circuit that has Sequential Switching

The two switches in the circuit shown in Fig. 7.39 have been closed for a long time. At $t = 0$, switch 1 is opened. Then, 35 ms later, switch 2 is opened.

a) Find $i_L(t)$ for $0 \leq t \leq 35$ ms.

b) Find i_L for $t \geq 35$ ms.

c) What percentage of the initial energy stored in the 150 mH inductor is dissipated in the 18 Ω resistor?

d) Repeat (c) for the 3 Ω resistor.

e) Repeat (c) for the 6 Ω resistor.

Solution

We use Analysis Method 7.5 to solve this problem.

a) **Step 1:** Identify the inductor current, i_L, as the variable of interest, because this is an *RL* circuit.

Step 2: Calculate the initial value of i. For $t < 0$ both switches are closed, causing the 150 mH inductor to short-circuit the 18 Ω resistor. The equivalent circuit is shown in Fig. 7.40. We determine the initial current in the inductor by solving for $i_L(0^-)$ in the circuit shown in Fig. 7.40. After making several source transformations, we find $i_L(0^-)$ to be 6 A, so $I_0 = 6$ A.

Step 3: Calculate the time constant $\tau = L/R$. For $0 \leq t \leq 35$ ms, switch 1 is open (switch 2 is closed), which disconnects the 60 V voltage

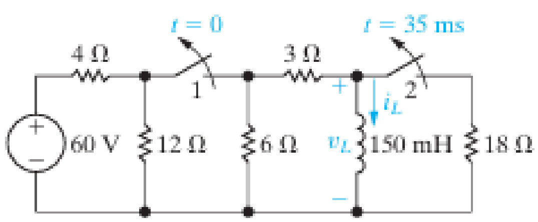


Figure 7.39 ▲ The circuit for Example 7.11.

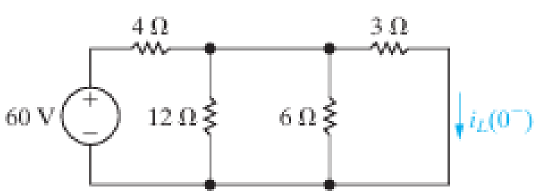


Figure 7.40 ▲ The circuit shown in Fig. 7.39, for $t < 0$.

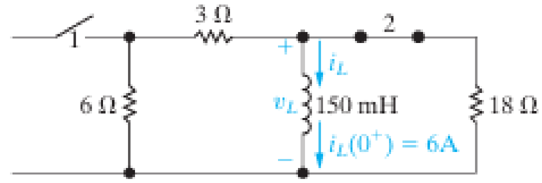


Figure 7.41 ▲ The circuit shown in Fig. 7.39, for $0 \leq t \leq 35$ ms.

source and the 4 Ω and 12 Ω resistors from the circuit. The inductor is no longer behaving as a short circuit (because the dc source is no longer in the circuit), so the 18 Ω resistor is no longer short-circuited. The equivalent circuit is shown in Fig. 7.41. Note that the equivalent resistance across the terminals of the inductor is the parallel combination of 9 Ω and 18 Ω, or 6 Ω. Therefore,

$$\tau = \frac{0.15}{6} = 25 \text{ ms}.$$

Step 4: Calculate the final value for the inductor current. For $0 \leq t \leq 35$ ms there is no source in the circuit, so during this time period we have a natural-response problem and the final value of the inductor current is zero. Thus, $I_f = 0$.

Step 5: Write the equation for the inductor current for $0 \leq t \leq 35$ ms by substituting the values for I_o, τ, and I_f into Eq. 7.23 to give

$$i_L(t) = I_f + (I_0 - I_f)e^{-t/\tau} = 0 + (6 - 0)e^{-t/0.025}$$
$$= 6e^{-40t} \text{ A}, 0 \leq t \leq 35 \text{ ms}.$$

b) Now we repeat Steps 2–5 for $t \geq 35$ ms.

Step 2: Calculate the initial value of the inductor current for this time segment. When $t = 35$ ms, the value of the inductor current is determined from the inductor current equation for the previous time segment because the inductor current must be continuous for all time. So,

$$i_L(35 \times 10^{-3}) = 6e^{-40(35\times10^{-3})} = 6e^{-1.4} = 1.48 \text{ A}.$$

Thus, for $t \geq 35$ ms, $I_0 = 1.48$ A.

Step 3: Calculate the time constant $\tau = L/R$. For $t \geq 35$ ms, both switches are open, and the circuit reduces to the one shown in Fig. 7.42. Note that the

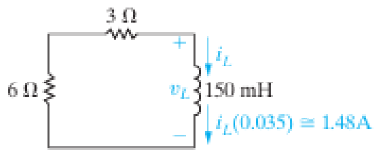


Figure 7.42 ▲ The circuit shown in Fig. 7.39, for $t \geq 35$ ms.

equivalent resistance across the terminals of the inductor is the series combination of 3 Ω and 6 Ω or 9 Ω. Therefore,

$$\tau = \frac{0.15}{9} = 16.67 \text{ ms.}$$

Step 4: Calculate the final value for the inductor current. For $t \geq 35$ ms there is no source in the circuit, so during this time period we have a natural-response problem, and the final value of the inductor current is zero. Thus, $I_f = 0$.

Step 5: Write the equation for inductor current when $t \geq 35$ ms by substituting the values for I_o, τ, and I_f into Eq. 7.23 to give

$$\begin{aligned} i_L(t) &= I_f + (I_0 - I_f)e^{-(t-t_0)/\tau} \\ &= 0 + (1.48 - 0)e^{-(t-0.035)/0.01667} \\ &= 1.48e^{-60(t-0.035)} \text{ A, t} \geq 35 \text{ ms.} \end{aligned}$$

Note that when switch 2 is opened, the time constant changes and the exponential function is shifted in time by 35 ms.

Step 6: Use the inductor current to solve the remaining parts of this problem.

c) The 18 Ω resistor is in the circuit only during the first 35 ms of the switching sequence. During this interval, the voltage across the resistor is

$$\begin{aligned} v_L &= 0.15\frac{d}{dt}(6e^{-40t}) \\ &= -36e^{-40t} \text{ V}, \quad 0 < t < 35 \text{ ms.} \end{aligned}$$

The power dissipated in the 18 Ω resistor is

$$p = \frac{v_L^2}{18} = 72e^{-80t} \text{ W}, \quad 0 < t < 35 \text{ ms.}$$

Hence, the energy dissipated is

$$\begin{aligned} w &= \int_0^{0.035} 72e^{-80t}\, dt \\ &= \frac{72}{-80} e^{-80t}\Big|_0^{0.035} \\ &= 0.9(1 - e^{-2.8}) = 845.27 \text{ mJ.} \end{aligned}$$

The initial energy stored in the 150 mH inductor is

$$w_0 = \frac{1}{2}(0.15)(6)^2 = 2.7 \text{ J} = 2700 \text{ mJ.}$$

Therefore, $(845.27/2700) \times 100$, or 31.31% of the initial energy stored in the 150 mH inductor is dissipated in the 18 Ω resistor.

d) For $0 < t < 35$ ms, the voltage across the 3 Ω resistor is

$$\begin{aligned} v_{3\Omega} &= \left(\frac{v_L}{9}\right)(3) \\ &= \frac{1}{3} v_L \\ &= -12e^{-40t} \text{ V.} \end{aligned}$$

Therefore, the energy dissipated in the 3 Ω resistor in the first 35 ms is

$$\begin{aligned} w_{3\Omega} &= \int_0^{0.035} \frac{(-12e^{-40t})^2}{3}\, dt \\ &= 0.6(1 - e^{-2.8}) \\ &= 563.51 \text{ mJ.} \end{aligned}$$

For $t > 35$ ms, the current in the 3 Ω resistor is

$$i_{3\Omega} = i_L = (6e^{-1.4})e^{-60(t-0.035)} \text{ A.}$$

Hence, the energy dissipated in the 3 Ω resistor for $t > 35$ ms is

$$\begin{aligned} w_{3\Omega} &= \int_{0.035}^{\infty} 3i_{3\Omega}^2 dt \\ &= \int_{0.035}^{\infty} 3(6e^{-1.4})^2(e^{-60(t-0.035)})^2 dt \\ &= 108e^{-2.8} \times \frac{e^{-120(t-0.035)}}{-120}\Big|_{0.035}^{\infty} \\ &= \frac{108}{120}e^{-2.8} = 54.73 \text{ mJ.} \end{aligned}$$

The total energy dissipated in the 3 Ω resistor is

$$\begin{aligned} w_{3\Omega}(\text{total}) &= 563.51 + 54.73 \\ &= 618.24 \text{ mJ.} \end{aligned}$$

The percentage of the initial energy stored is

$$\frac{618.24}{2700} \times 100 = 22.90\%.$$

e) Because the 6 Ω resistor is in series with the 3 Ω resistor, the energy dissipated and the percentage of the initial energy stored will be twice that of the 3 Ω resistor:

$$w_{6\Omega}(\text{total}) = 1236.48 \text{ mJ,}$$

and the percentage of the initial energy stored is 45.80%. We check these calculations by observing that

$$1236.48 + 618.24 + 845.27 = 2699.99 \text{ mJ}$$

and

$$31.31 + 22.90 + 45.80 = 100.01\%.$$

The small discrepancies in the summations are the result of roundoff errors.

EXAMPLE 7.12 Analyzing an *RC* Circuit that Has Sequential Switching

The uncharged capacitor in the circuit shown in Fig. 7.43 is initially switched to terminal a of the three-position switch. At $t = 0$, the switch is moved to position b, where it remains for 15 ms. After the 15 ms delay, the switch is moved to position c, where it remains indefinitely.

a) Derive the numerical expression for the voltage across the capacitor.

b) Plot the capacitor voltage versus time.

c) When will the voltage on the capacitor equal 200 V?

Solution

We use Analysis Method 7.5 to solve this problem.

a) **Step 1:** Identify the capacitor voltage, v, as the variable of interest, because this is an *RC* circuit.

Step 2: Calculate the initial value of v. For $t < 0$, the capacitor is initially uncharged, so $V_0 = v(0) = 0$ V.

Step 3: Calculate the time constant $\tau = RC$. When $0 \leq t \leq 15$ ms, the equivalent resistance attached to the capacitor has the value 100 kΩ. Therefore,

$$\tau = (100 \times 10^3)(0.1 \times 10^{-6}) = 10 \text{ ms}.$$

Step 4: Calculate the final value for the capacitor voltage, V_f. If the switch were to remain in position b for a long time, the capacitor would eventually behave like an open circuit in the presence of the 400 V source. Because of the open circuit, there would be no current in the 100 k Ω resistor, so $V_f = 400$ V.

Step 5: Write the equation for capacitor voltage by substituting the values for V_o, τ, and V_f into Eq. 7.23 to give

$$v(t) = V_f + (V_0 - V_f)e^{-t/\tau} = 400 + (0 - 400)e^{-t/0.01}$$

$$= 400 - 400e^{-100t} \text{ V}, \quad 0 \leq t \leq 15 \text{ ms}.$$

Now we repeat Steps 2–5 for the next time interval, $t \geq 15$ ms.

Step 2: Calculate the initial value of v. At $t = 15$ ms, the capacitor voltage is determined by the equation we derived for the previous time interval. So,

$$v(0.015) = 400 - 400e^{-100(0.015)} = 400 - 400e^{-1.5}$$

$$= 310.75 \text{ V}.$$

Thus, $V_0 = v(0.015) = 310.75$ V.

Step 3: Calculate the time constant $\tau = RC$. When $t \geq 15$ ms, the equivalent resistance attached to the capacitor has the value 50 kΩ. Therefore,

$$\tau = (50 \times 10^3)(0.1 \times 10^{-6}) = 5 \text{ ms}.$$

Step 4: Calculate the final value for the capacitor voltage, V_f. For $t \geq 15$ ms, the switch remains in position c for a long time, and there is no source in the circuit. During this time interval, the circuit exhibits a natural response, so $V_f = 0$.

Step 5: Write the equation for capacitor voltage by substituting the values for V_o, τ, and V_f into Eq. 7.23 to give

$$v(t) = V_f + (V_0 - V_f)e^{-(t-t_0)/\tau}$$
$$= 0 + (310.75 - 0)e^{-(t-0.015)/0.005}$$
$$= 310.75e^{-200(t-0.015)} \text{ V}, \quad t \geq 15 \text{ ms}.$$

Step 6: Use the capacitor voltage to solve the remaining parts of this problem.

b) Figure 7.44 shows the plot of v versus t.

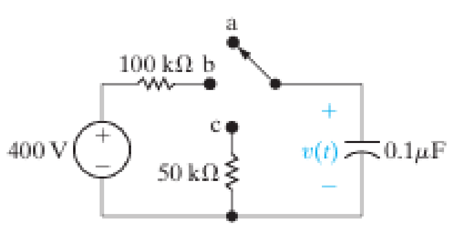


Figure 7.43 ▲ The circuit for Example 7.12.

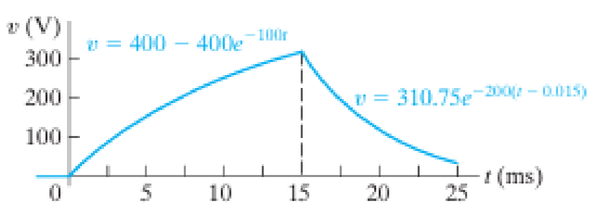


Figure 7.44 ▲ The capacitor voltage for Example 7.12.

c) The plot in Fig. 7.44 reveals that the capacitor voltage will equal 200 V at two different times: once in the interval between 0 and 15 ms and once after 15 ms. We find the first time by solving the expression

$$200 = 400 - 400e^{-100t_1},$$

which yields $t_1 = 6.93$ ms. We find the second time by solving the expression

$$200 = 310.75e^{-200(t_2 - 0.015)}.$$

In this case, $t_2 = 17.20$ ms.

EXAMPLE 7.13 Finding the Unbounded Response in an *RC* Circuit

a) When the switch is closed in the circuit shown in Fig. 7.45, the voltage on the capacitor is 10 V. Find the expression for v_o for $t \geq 0$.

b) Assume that the capacitor short-circuits when its terminal voltage reaches 150 V. How many milliseconds elapse before the capacitor short-circuits?

Solution

a) We need to write the differential equation that describes the capacitor voltage, v_o. To make this task easier, let's simplify the circuit attached to the capacitor by replacing it with its Thévenin equivalent. This subcircuit is shown in Fig. 7.46, and as you can see, it does not contain an independent source. Thus, the Thévenin equivalent consists of a single resistor. To find the Thévenin equivalent resistance for the circuit in Fig. 7.46, we use the test-source method (see Example 4.18, p. 152), where v_T is the test voltage and i_T is the test current. Writing a KCL equation at the top node, we get

$$i_T = \frac{v_T}{10 \times 10^3} - 7\left(\frac{v_T}{20 \times 10^3}\right) + \frac{v_T}{20 \times 10^3}$$

Solving for the ratio v_T/i_T yields the Thévenin resistance:

$$R_{\text{Th}} = \frac{v_T}{i_T} = -5\text{ k}\Omega.$$

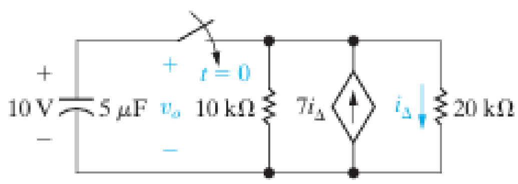


Figure 7.45 ▲ The circuit for Example 7.13.

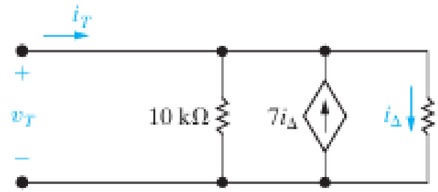


Figure 7.46 ▲ The test-source method used to find R_{Th}.

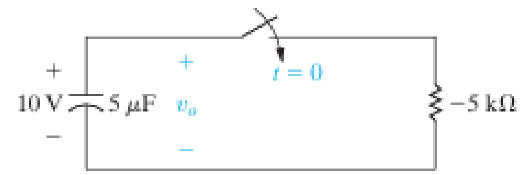


Figure 7.47 ▲ A simplification of the circuit shown in Fig. 7.45.

We replace the two resistors and the dependent source in Fig. 7.45 with R_{Th} to get the circuit shown in Fig. 7.47. For $t \geq 0$, write a KCL equation at the top node to construct the differential equation describing this circuit:

$$(5 \times 10^{-6})\frac{dv_o}{dt} + \frac{v_o}{-5000} = 0$$

Dividing by the coefficient of the first derivative yields

$$\frac{dv_o}{dt} - 40v_o = 0.$$

This equation has the same form as Eq. 7.1, so we can find $v_o(t)$ using the same separation of variables technique applied to Eq. 7.1 (see p. 250). Thus, the capacitor voltage is

$$v_o(t) = 10e^{40t}\text{ V}, \quad t \geq 0.$$

b) $v_o = 150$ V when $e^{40t} = 15$.
Therefore, $40t = \ln 15$, and $t = 67.70$ ms.

EXAMPLE 7.14 Analyzing an Integrating Amplifier

Assume that the numerical values for the voltage shown in Fig. 7.49 are $V_m = 50$ mV and $t_1 = 1$ s. We apply this voltage to the integrating-amplifier circuit shown in Fig. 7.48. The circuit parameters of the amplifier are $R_s = 100\ \text{k}\Omega$, $C_f = 0.1\ \mu\text{F}$, and $V_{CC} = 6$ V. The capacitor's initial voltage is zero.

a) Calculate $v_o(t)$.

b) Plot $v_o(t)$ versus t.

Solution

a) For $0 \leq t \leq 1$ s,

$$v_o = \frac{-1}{(100 \times 10^3)(0.1 \times 10^{-6})} 50 \times 10^{-3} t + 0$$

$$= -5t \text{ V}, \quad 0 \leq t \leq 1 \text{ s}.$$

For $1 \leq t \leq 2$ s,

$$v_o = (5t - 10) \text{ V}.$$

b) Figure 7.51 shows a plot of $v_o(t)$ versus t.

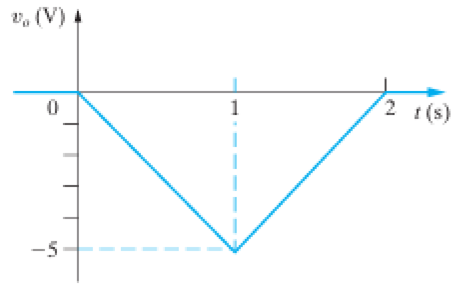


Figure 7.51 ▲ The output voltage for Example 7.14.

EXAMPLE 7.15 Analyzing an Integrating Amplifier that Has Sequential Switching

At the instant the switch makes contact with terminal a in the circuit shown in Fig. 7.52, the voltage on the 0.1 μF capacitor is 5 V. The switch remains at terminal a for 9 ms and then moves instantaneously to terminal b. How many milliseconds after making contact with terminal b does the op amp saturate?

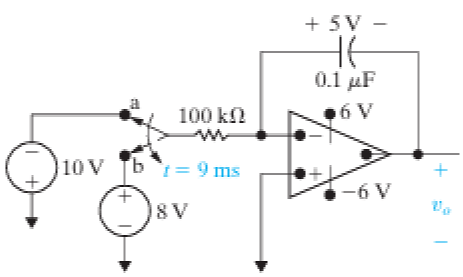


Figure 7.52 ▲ The circuit for Example 7.15.

Solution

The expression for the output voltage during the time the switch is at terminal a is

$$v_o = \frac{-1}{(100 \times 10^3)(0.1 \times 10^{-6})} \int_0^t (-10) dy + (-5)$$

$$= (1000t - 5) \text{ V}.$$

Thus, 9 ms after the switch makes contact with terminal a, the output voltage is $1000(9 \times 10^{-3}) - 5 = 4$ V. Note that the op amp does not saturate during its first 9 ms of operation.

The expression for the output voltage after the switch moves to terminal b is

$$v_o = \frac{-1}{(100 \times 10^3)(0.1 \times 10^{-6})} \int_{9\times10^{-3}}^t 8dy + 4$$

$$= -800(t - 9 \times 10^{-3}) + 4$$

$$= (11.2 - 800t) \text{ V}.$$

When the switch is at terminal b, the voltage is decreasing, and the op amp eventually saturates at −6 V. Therefore, we set the expression for v_o equal to −6 V to obtain the saturation time t_s:

$$11.2 - 800t_s = -6,$$

or

$$t_s = 21.5 \text{ ms}.$$

Thus, the integrating amplifier saturates 21.5 ms after making contact with terminal b.