MAT1320 LINEAR ALGEBRA EXERCISES II

Name Surname:	**Group No:**
Student No:	**Duration:**
Department:	**Date:** 10 October 2022
Lecturer: Dr. Mustafa SARI	**Signature:**

1. Let $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ be an involut matrix ($A^2 = I$), let $B = \begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix}$ be a nonzero idempotent ($A^2 = A$) matrix and let $C = \begin{pmatrix} 1 & -3 \\ z & 3 \end{pmatrix}$ be a singular matrix (not invertible). Then, which of the followings is the value $x - y - z$?

$z = -1,\ 1 = -1.$

a) -1 b) 0 c) 1 d) -2 e) 2

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to x = 0.$

$\begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} y^2 & 0 \\ y^2 & 0 \end{pmatrix}$ $\quad y^2 = y$ $\quad y = 1,\ \cancel{0}$

$z = -1$

$-1 + 1 + 0 = 0.$

$A^T = \begin{pmatrix} 0 & a & 4 \\ a+b & 2 & a+b \\ c+2 & c & 4 \end{pmatrix} = \begin{pmatrix} 0 & a+b & c+2 \\ a & 2 & c \\ 4 & a+b & 4 \end{pmatrix}$

$a + b = a \Rightarrow \boxed{b = 0}$ $\quad a + b = c = 2 \Rightarrow \boxed{a = 2}$

$c + 2 = 4 \Rightarrow \boxed{c = 2}$

$A^T = A$

2. If $A = \begin{bmatrix} 0 & a+b & c+2 \\ a & 2 & c \\ 4 & a+b & 4 \end{bmatrix}$ is a symmetric matrix, then which of the followings is true for the matrix $B = \begin{bmatrix} b & a & -2 \\ b-a & 0 & 1 \\ c & -1 & b \end{bmatrix}$? $\to \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} = B$

I. B is a skew-symmetric matrix.

II. B^2 is a symmetric matrix.

III. $\mathrm{Tr}(B) = \mathrm{Tr}(A)$. $\quad 0 \neq 6$

$B^T = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$

a) Only I b) I and II c) II and III

d) I and III e) All of them

3. Let A be $n \times n$ skew-symmetric matrix and let x be a vector with n components. Which of the followings is equal to $x^T A x$ for all $x \in \mathbb{R}^n$?

a) $\mathbf{0} \in \mathbb{R}^n$ b) x^T c) $0 \in \mathbb{R}$

d) x e) $-x^T$

Taking $n = 2$, let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Then, $x^T A x = [2\ \ 3]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= (-3\ \ 2)\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= 0 \in \mathbb{R}.$

It can be seen that the result is 0 always.

$B^2 = \begin{pmatrix} -3 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 43 & -21 & -10 \\ -21 & 29 & 9 \\ -10 & 9 & 22 \end{pmatrix}$

$\Rightarrow B^2$ is symmetric.

$-1 = -2 + c \Rightarrow \boxed{c = 1}$ $\quad \boxed{b = 0},\ a + 3 = 0 \Rightarrow \boxed{a = -3}$

$-d = a - 2 \Rightarrow \boxed{d = 5}$ $\quad \Rightarrow a + b = -e \Rightarrow \boxed{e = 3}$

4. If $A = \begin{bmatrix} 0 & a-2 & 2-c \\ d & b & a+b \\ -1 & e & a+3 \end{bmatrix}$ is a skew-symmetric matrix, then which of the followings is true for the matrix $B = \begin{bmatrix} a & 5 & e \\ d & b & b-2 \\ 3 & a+c & c+2 \end{bmatrix}$? $\to \begin{bmatrix} -3 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 3 \end{bmatrix} = B$

I. B is a symmetric matrix. $\quad \mathrm{Tr}(B) = 0 = \mathrm{Tr}(A)$

II. B^2 is a symmetric matrix.

III. $\mathrm{Tr}(B) = \mathrm{Tr}(A)$.

a) Only I b) I and II c) II and III

d) I and III e) All of them

5. Let $A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ be an involut matrix, let $B = \begin{bmatrix} y & 0 \\ y & 0 \end{bmatrix}$ be a nonzero idempotent matrix and let $C = \begin{bmatrix} 1 & -2 \\ z & 2 \end{bmatrix}$ be a noninvertible matrix. Then, which of the followings is equal to the value $x + y + z$?

a) -1 b) 1 c) 0 d) -2 e) 2

$A^2 = I \Rightarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow x = 0.$

$B^2 = B \Rightarrow \begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} y^2 & 0 \\ y^2 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix}$

$\Rightarrow y^2 = y \Rightarrow y^2 - y = y(y-1) = 0 \Rightarrow y = 0, 1$

Since B is nonzero, $y = 1$.

C is noninvertible, then second row must be a multiple of the first row.

$\Rightarrow \begin{pmatrix} 1 & 2 \\ z & -2 \end{pmatrix} \Rightarrow (z \;\; -2) = -1 (1 \;\; 2)$

$\Rightarrow z = -1$

$\Rightarrow x + y + z = 0 + 1 - 1 = 0.$

$\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & 3 \end{pmatrix} = A^2$

$A^3 = A^2 A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Since $A^k = 0$ for $k = 3$, A is nilpotent.

6. Which of the followings is true for the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$?

a) Idempotent b) Involut c) Orthogonal

d) Nilpotent e) None of them

7. Which of the followings is true for the matrix $B = \begin{bmatrix} ab & -a^2 \\ b^2 & -ab \end{bmatrix}$?

a) Involut ($B^2 = I$) b) Orthogonal ($B^T = B^{-1}$) c) Idempotent ($B^2 = B$)

d) Hermitian ($B^H = B$) e) Nilpotent ($B^k = 0$.)

$B^2 = \begin{bmatrix} ab & -a^2 \\ b^2 & -ab \end{bmatrix}\begin{bmatrix} ab & -a^2 \\ b^2 & -ab \end{bmatrix}$

$= \begin{bmatrix} a^2b^2 + -a^2b^2 & -a^3b + a^3b \\ ab^3 - ab^3 & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\bar{X} = \begin{pmatrix} -i & 1+i & 2 \\ a_1 - a_2 i & -3i & b_1 - b_2 i \\ c_1 - c_2 i & -i & 0 \end{pmatrix} \Rightarrow (\bar{X})^T = \begin{pmatrix} -i & a_1 - a_2 i & c_1 - c_1 i \\ 1+i & -3i & -i \\ 2 & b_1 - b_2 i & 0 \end{pmatrix} = -X$

$a_1 - a_2 i = -1 + i \Rightarrow a_1 = -1, \; a_2 = -1$ $\quad \boxed{a = -1 - i}$

$c_1 - c_2 i = -2 \Rightarrow c_1 = -2, \; c_2 = 0 \Rightarrow \boxed{c = -2}$

$a_1 + a_2 i$

$(\bar{X})^T = X^H = -X$

$a + b + c = -1 - i + i - 2 = -3$

8. If $X = \begin{bmatrix} i & 1-i & 2 \\ a & 3i & b \\ c & i & 0 \end{bmatrix}$ is a skew-Hermitian matrix, then which of the followings is the value $a + b + c$?

$b_1 - b_2 i = -i \Rightarrow b_1 = 0, \; b_2 = 1$

a) $3 - 2i$ b) -3 c) $1 - i$

d) 2 e) $-3 + i$

$\boxed{b = 0 + 1 \cdot i = i}$

9. Let A and B be 3×3 real nonzero matrices and $(AB)^T + B^{-1}A = 0$. If B is an orthogonal matrix, then which of the followings is true for the matrix A?

a) Symmetric b) Idempotent c) Nilpotent d) Orthogonal e) Skew-symmetric

$(AB)^T + B^{-1}A = 0$, $B^T = B^{-1}$

$\Downarrow$

$B^T A^T + B^{-1}A = 0$ Since $B^T = B^{-1}$

$\Rightarrow B^{-1}A^T + B^{-1}A = 0$

$\Rightarrow B^{-1}(A^T + A) = 0$ Since B is invertible

$\Rightarrow A^T + A = 0$

$\Rightarrow A^T = -A \Rightarrow A$ is skew-symmetric.