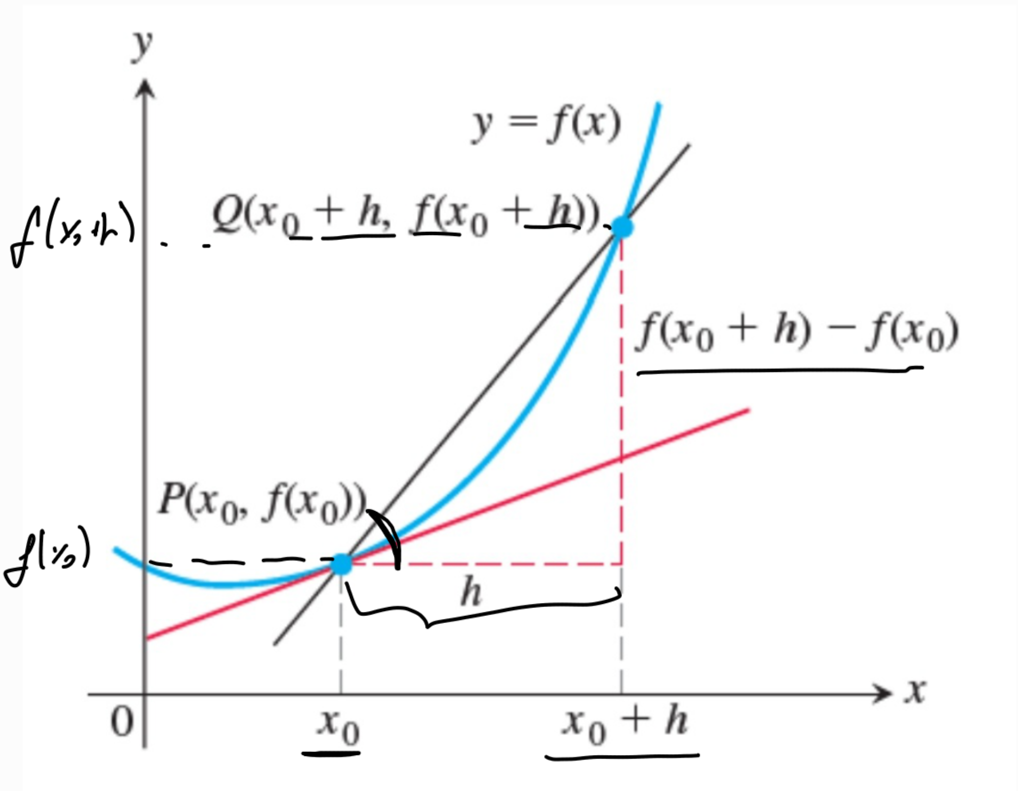
DERIVATIVES

Tangent Lines and the Derivative at a Point

Definition: The slope of the curve $y=f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad \text{(provided the limit exists)}$$

The tangent line to the curve at P is the line through P with this slope.



The slope of the tangent line at P is

$$\boxed{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}.$$

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \quad h\neq 0$$

is called the difference quotient of f at x_0 with increment h. If the difference quotient has a limit as h appoaches zero, that limit is given a special name and notation.

<u>Definition</u>: The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided the limit exists.

Summary

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

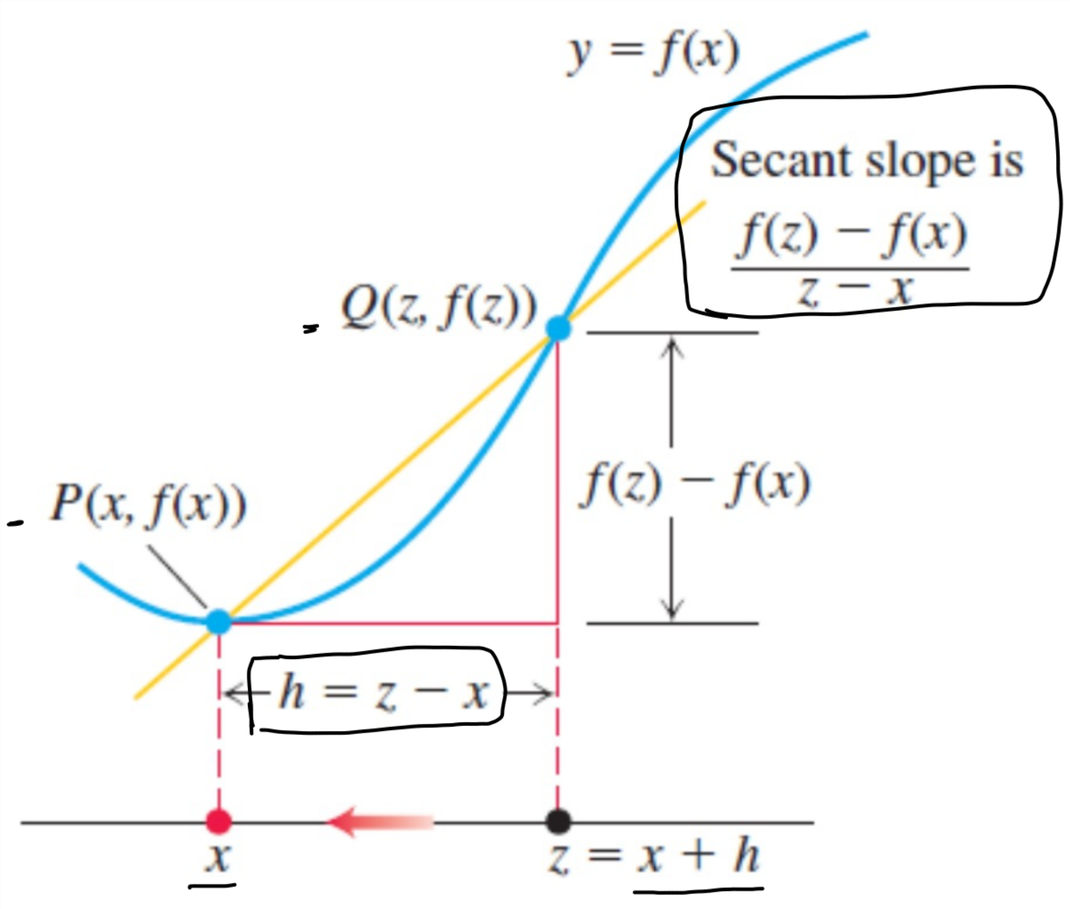
1) The slope of the graph of $y=f(x)$ at $x=x_0$
2) The slope of the tangent line to the curve $y=f(x)$ at $x=x_0$
3) The rate of change of $f(x)$ w.r.t. x at the $x=x_0$
4) The derivative $f'(x_0)$ at $x=x_0$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \qquad (h = z - x)$$

Equivalent notation

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$



Example: Differentiate from first principles $f(x) = \dfrac{x}{x-1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h).(x-1) - x.(x+h-1)}{(x+h-1).(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{\cancel{x^2} \cancel{-x} + \cancel{hx} - h - \cancel{x^2} - \cancel{hx} \cancel{+x}}{(x+h-1).(x-1)} = \lim_{h\to 0} \frac{1}{\cancel{h}} \cdot \frac{-\cancel{h}}{(x+h-1).(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1).(x-1)} = \frac{-1}{(x-1).(x-1)} = \frac{-1}{(x-1)^2}.$$

Example: Differentiate $f(x)=\sqrt{x}$ by using the alternative formula for derivatives.

$$f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x}=\lim_{z\to x}\frac{\cancel{\sqrt{z}-\sqrt{x}}}{\cancel{(\sqrt{z}-\sqrt{x})}\cdot(\sqrt{z}+\sqrt{x})}$$

$$\lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Note: For $f'(x)$ at $x=4$, one sometimes writes

$$f'(4)=\frac{d}{dx}\sqrt{x}\Big|_{x=4}=\frac{1}{2\sqrt{x}}\Big|_{x=4}$$

Differentiable on an Interval; One Sided Derivatives

A function $y=f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the limits

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \quad \text{(Right-hand derivative at a)}$$

$$\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h} \quad \text{(Left-hand derivative at b)}$$

exist at the endpoints.

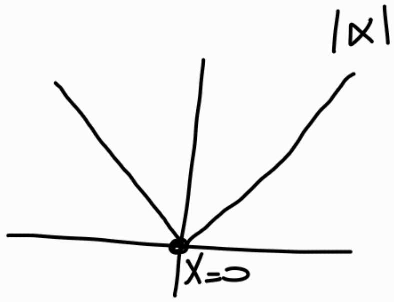
$f'(x_0)$ exists $\Longleftrightarrow$ $f'_+(x_0)=f'_-(x_0)$ and both are exist.

Example: Show that the function $f(x)=|x|$ is not differentiable at $x=0$.

$$f'_+(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{|0+h|-|0|}{h}$$

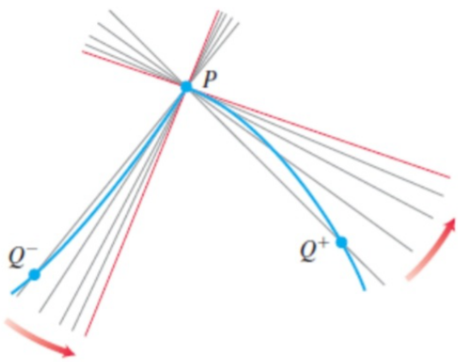
$$=\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{\cancel{h}}{\cancel{h}}=1$$

$$f'_-(0)=\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-\cancel{h}}{\cancel{h}}=-1$$

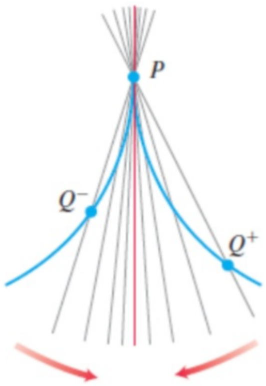


$1\neq -1 \Rightarrow f(x)$ is not diff.ble $x=0$.

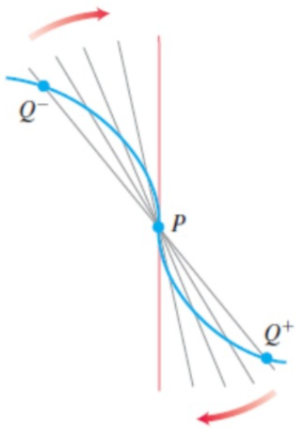
When Does a Function Not Have a Derivative at a Point



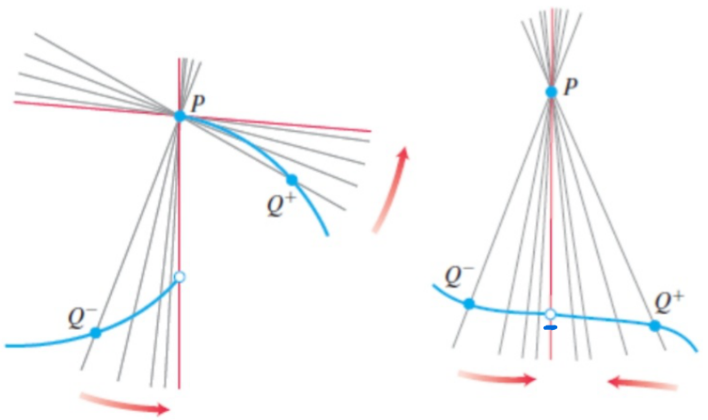
1. a *corner*, where the one-sided derivatives differ



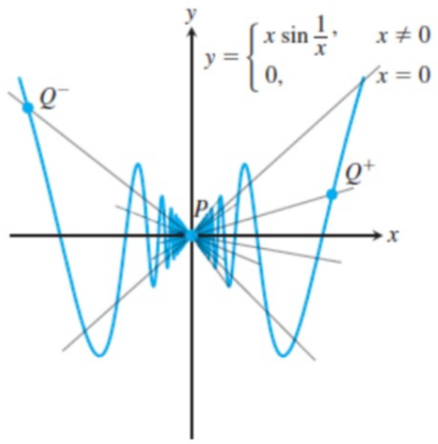
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, it approaches $-\infty$)



4. a *discontinuity* (two examples shown)



5. wild oscillation

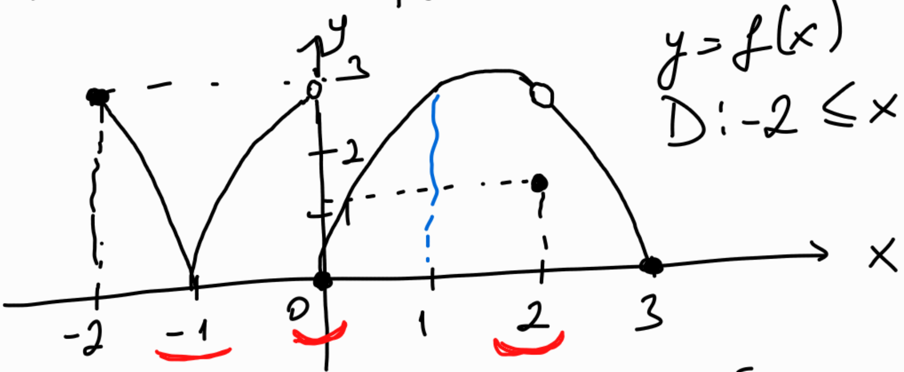
Theorem: If f has a derivative at $x=c$, then f is continuous at $x=c$.

$$\text{Differentiable} \underset{\not\Leftarrow}{\Rightarrow} \text{Continuous} \underset{\not\Leftarrow}{\Rightarrow} \text{Limit}$$

Caution! A function need not have a derivative at a point where it is continuous.

Example: The figure below shows the graph of a function over a closed interval D. At what domain points does the function appear to be

1) differentiable?
2) continuous but not diff-ble?
3) neither cont. nor diff-ble?



$y=f(x)$
$D: -2 \leq x \leq 3$

1) f is diff-ble on $[-2,3] \setminus \{-1,0,2\}$

2) f is continuous but not diff-ble at $x=-1$ because $f(-1)=\lim_{x\to -1} f(x)=0$ but there is a corner at $x=-1$

3) f is neither cont. nor diff-ble at $x=0$ because $\underbrace{\lim_{x\to 0} f(x) \text{ doesn't exist}}_{\text{jump disc.}}$, and $x=2$ because $\underbrace{f(2) \neq \lim_{x\to 2} f(x)}_{\text{removable disc.}}$

Differentiation Rules

1) Constant Rule: $f(x) = c \Rightarrow f'(x) = 0$
2) Constant Multiple Rule: $f(x) = c \cdot g(x) \Rightarrow f'(x) = c \cdot g'(x)$
3) Power Rule: $f(x) = x^n \Rightarrow f'(x) = n \cdot x^{n-1}$
4) Sum Rule: $f(x) + g(x) \Rightarrow f'(x) + g'(x)$
5) Difference: $f(x) - g(x) \Rightarrow f'(x) - g'(x)$
6) Product Rule: $f(x) \cdot g(x) \Rightarrow f'(x) \cdot g(x) + f(x) \cdot g'(x)$
7) Quotient Rule: $\dfrac{f(x)}{g(x)} \Rightarrow \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
8) Chain Rule: $f(g(x)) \Rightarrow f'(g(x)) \cdot g'(x)$
9) Exponent Rule: $f(x) = e^x \Rightarrow f'(x) = e^x$ (e: Euler's Number)
10) Constant Exp-Rule: $f(x) = a^x \Rightarrow f'(x) = a^x \cdot \ln a$
11) Natural Log Rule: $f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x}$
12) Logarithm Rule: $f(x) = \log_a x \Rightarrow f'(x) = \dfrac{1}{x \cdot \ln a} = \dfrac{1}{x} \cdot \dfrac{1}{\ln a}$
13) Sine Rule: $f(x) = \sin x \Rightarrow f'(x) = \cos x$
14) Cosine Rule: $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
15) Tangent Rule: $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x = 1 + \tan^2 x$
16) Cotangent Rule: $f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x = -(1 + \cot^2 x)$
17) Secant Rule: $f(x) = \sec x \Rightarrow f'(x) = \sec x \cdot \tan x$
18) Cosecant Rule: $f(x) = \csc x \Rightarrow f'(x) = -\csc x \cdot \cot x$

Ex.: Find the derivative of $f(x) = \dfrac{x^2-1 \to a(x)}{x^2+1 \to b(x)}$

$$\boxed{x^2 \Rightarrow 2.x^{2-1}}$$

$$f'(x) = \frac{a'(x).b(x) - a(x).b'(x)}{[b(x)]^2}$$

$$= \frac{2x.(x^2+1) - (x^2-1).2x}{[x^2+1]^2}$$

$$= \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{[x^2+1]^2} = \frac{4x}{(x^2+1)^2}$$

Ex.: $f(x) = \dfrac{\cos x}{1-\sin x}$

$$f(x) = \cos x \cdot \frac{1}{1-\sin x}$$

$$\boxed{= \underbrace{\cos x}_{a(x)} \cdot \underbrace{(1-\sin x)^{-1}}_{b(x)}}$$

$$f'(x) = a'(x).b(x) + a(x).b'(x)$$

$$= -\sin x.(1-\sin x)^{-1} + \cos x.(-1).(1-\sin x)^{-2}.(-\cos x)$$

$$= \frac{-\sin x}{1-\sin x} + \frac{\cos^2 x}{(1-\sin x)^2} = \frac{-\sin x + \underline{\sin^2 x + \cos^2 x}}{(1-\sin x)^2}$$

$(1-\sin x)$

$$= \frac{\cancel{1-\sin x}}{(1-\sin x)^{\cancel{2}}} = \frac{1}{1-\sin x}$$

Ex.: $f(x) = \sin(x^2+e^x)$

$$\frac{d}{dx}\sin(\underbrace{x^2+e^x}_{\text{inside}}) = \cos(\underbrace{x^2+e^x}_{\text{inside left alone}}).(\underbrace{2x+e^x}_{\text{derivative of the inside}})$$

Ex: $f(t) = \dfrac{\tan t \to a(t)}{10^t \to b(t)}$ $\qquad a^x \Rightarrow a^x . \ln a$, $a \to 10$

$$D(f)(t) = \frac{a'(t) . b(t) - a(t) . b'(t)}{[b(t)]^2}$$

$$= \frac{\sec^2 t . \cancel{10^t} - \tan t . \cancel{10^t} . \ln 10}{(10^t)^{\cancel{2}}} = \frac{\sec^2 t - \ln 10 . \tan t}{10^t}$$

Second and Higher Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the second derivative of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means that the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30 . x^4$$

Thus $D^2(x^6) = 30 . x^4$.

If y'' is differentiable, its derivative $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$.
is the third derivative of y w.r.t x.

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

denoting the n-th derivative of y w.r.t x for any positive integer n.

Ex.: If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

$$f(x) = x^3 - x$$
$$f'(x) = \frac{d}{dx}(x^3 - x) = 3x^2 - 1$$
$$f''(x) = \frac{d}{dx}(3x^2 - 1) = 6x$$
$$f'''(x) = \frac{d}{dx}(6x) = 6 = 3!$$
$$f^{(4)}(x) = \frac{d}{dx}(6) = 0$$

Ex.: Find the 27th derivative of $\cos x$.

$$27 \equiv 3 \pmod 4$$

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x = f^{(27)}(x)$$
$$f^{(4)}(x) = \cos x = f(x)$$
$$f^{(5)}(x) = -\sin x = f'(x)$$

Implicit Differentiation

We want to compute y' but do not have an explicit relation $y=f(x)$ available. Rather, we have an implicit relation $F(x,y)=0$ between x and y.

Example: Find $\frac{dy}{dx}$ or circle $x^2+y^2=25$ at the point $(3,4)$.

$$\rightarrow \quad x^2+y^2=25$$

$$\frac{d}{dx}(x^2+y^2)=\frac{d(25)}{dx}$$

$$\boxed{2x+2y\cdot\frac{dy}{dx}=0} \xrightarrow{y=f(x)} \underbrace{2x\cdot\frac{dx}{dx}}_{1}+2y\frac{dy}{dx}$$

$$2x+2y\cdot\frac{dy}{dx}$$

$$\frac{\cancel{2}y}{y}\cdot\frac{dy}{dx}=-\frac{\cancel{2}x}{y}$$

$$\frac{dy}{dx}=\frac{-x}{y} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,4)}=-\frac{3}{4}$$

$$x \quad y$$

Formula: $\boxed{F(x,y)=0} \Rightarrow F'(x,y)=-\frac{F_x}{F_y}$

$F_x \rightarrow$ anything other than x is constant

$F_y \rightarrow$ anything other than y is constant

$$x^2+y^2=25 \Rightarrow F(x,y)=x^2+y^2-25=0$$

$$F'(x,y)=-\frac{2x}{2y}=-\frac{x}{y} \Rightarrow F'(3,4)=-\frac{3}{4}$$

Ex.: Find y' if $y^2 = x^2 + \sin(xy)$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}\left(x^2 + \sin(xy)\right) \qquad [y = f(x)]$$

$$2y \cdot y' = 2x + \cos(xy)\left(\frac{dx}{dx} \cdot y + x \cdot \frac{dy}{dx}\right)$$

$$2y \cdot y' = 2x + \cos(xy)(y + xy')$$

$$2y \cdot y' = 2x + y \cdot \cos(xy) + \underline{xy' \cdot \cos(xy)}$$

$$2y \cdot y' - x\cos(xy) \cdot y' = 2x + y \cdot \cos(xy)$$

$$y' \cdot \left(2y - x \cdot \cos(xy)\right) = 2x + y \cdot \cos(xy)$$

$$y' = \frac{2x + y \cdot \cos(xy)}{2y - x \cdot \cos(xy)}$$

Derivatives of Inverse Functions and Logarithms

Theorem (The Derivative Rule for Inverses)

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$ $[f(a) = b]$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \text{or}$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Example: $f(x) = x^2$, $x \geq 0$ continued.

$f^{-1}(x) = \sqrt{x}$ and $\boxed{f'(x) = 2x}$ so that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2 \cdot f^{-1}(x)} = \frac{1}{2\sqrt{x}}$$

Logarithmic Differentiation

Example: Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)\cdot(x+3)^{1/2}}{x-1}$, $x > 1$.

$$\ln y = \ln\left[\frac{(x^2+1)\cdot(x+3)^{1/2}}{x-1}\right]$$

$$= \ln\left[(x^2+1)\cdot(x+3)^{1/2}\right] - \ln(x-1)$$

$$= \ln(x^2+1) + \ln\left((x+3)^{1/2}\right) - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \frac{1}{2}\cdot\ln(x+3) - \ln(x-1) \quad \text{Take derivative.}$$

$$\Rightarrow \cancel{y}\cdot\frac{1}{\cancel{y}}\cdot\frac{dy}{dx} = \left(\frac{2x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}\right)\cdot y$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)\cdot\frac{(x^2+1)\cdot(x+3)^{1/2}}{x-1}$$

Example: Differentiate $y = x^{\sqrt{x}}$.

$$\ln y = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x} \cdot \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\sqrt{x} \cdot \ln x\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}$$

$$y' = \left[\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right] \cdot y = \left[\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right] \cdot x^{\sqrt{x}}$$

$$= \left(\frac{\ln x + 2}{2\sqrt{x}}\right) \cdot x^{\sqrt{x}}$$

Derivatives of the inverse trigonometric functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}} \quad (|x| < 1)$$

$$\frac{d}{dy}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x| \cdot \sqrt{x^2-1}} \quad (|x| > 1)$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x| \cdot \sqrt{x^2-1}} \quad (|x| > 1)$$

Ex.: $\frac{d}{dx}\left(\arcsin \underline{x^2}\right) = \frac{1}{\sqrt{1-x^4}} \cdot 2x$

Ex.: $\frac{d}{dx}\left(\frac{1}{\arctan x}\right) = \frac{d}{dx}\left(\arctan x\right)^{-1}$

$$= -1 \cdot (\arctan x)^{-2} \cdot \frac{1}{1+x^2}$$

$$= \frac{-1}{(1+x^2)\cdot(\arctan x)^2}$$

Ex.: $\frac{d}{dx}\left(\operatorname{arcsec}(5x^4)\right) = \frac{1}{|\cancel{5}\cancel{x^4}^{\,x}| \cdot \sqrt{25x^8-1}} \cdot \overset{4}{\cancel{20}} \cdot \cancel{x^3}$ $\quad (5x^4 > 1)$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

$$\frac{d}{dx}\left(\frac{1}{\arctan(2x)}\right) = \frac{d}{dx}\left(\arctan(2x)\right)^{-1}$$

$$-1 \cdot \left(\arctan(\underline{2x})\right)^{-2} \cdot \frac{1}{1+4x^2} \cdot 2$$

$$f\left(g\underbrace{(x)}_{h(x)}\right) \Rightarrow f'(g(x)) \cdot g'(x)$$

$$f\left(\operatorname{arcsec}(\underline{5x^4})\right)$$