INTEGRATION TECHNIQUES 1

1) $\int_0^1 5x\sqrt{x^2+3}\,dx = ?$

$x^2+3=t^2 \qquad x=0 \Rightarrow t=\sqrt{3}$

$\cancel{2}x\,dx=\cancel{2}t\,dt \qquad x=1 \Rightarrow t=2$

$I=\int_{\sqrt{3}}^{2} 5t\sqrt{t^2}\,dt=\int_{\sqrt{3}}^{2}5t^2\,dt=\frac{5t^3}{3}\Big|_{\sqrt{3}}^{2}=\frac{5}{3}\left(8-3\sqrt{3}\right).$

2) $\int \frac{e^x}{4+e^x}\,dx=?$

$4+e^x=t$

$e^x dx=dt$

$I=\int\frac{dt}{t}=\ln|t|+c=\ln(\underbrace{4+e^x}_{>0})+c$ (no need for abs. value)

3) $\int \frac{\ln x^2\,dx}{x}=? \qquad \int\frac{2\ln x}{x}\,dx$

$\ln x=t$

$\frac{dx}{x}=dt$

$I=\int 2t\,dt=t^2+c=(\ln x)^2+c=\ln^2 x+c$

power of ln

4) $\int\frac{(\ln x)^2}{x}\,dx=?$

$\ln x=t$

$\frac{dx}{x}=dt$

$I=\int t^2\,dt=\frac{t^3}{3}+c=\frac{\ln^3 x}{3}+c$

5) $\int\frac{dx}{\ln x^x}=? \qquad \int\frac{dx}{x.\ln x}$

$\ln x=t$

$\frac{dx}{x}=dt$

$I=\int\frac{dt}{t}=\ln|t|+c=\ln|\ln x|+c$

⑥ $\int \frac{x+3}{x-2}\,dx = ?$

$x-2=t$
$dx=dt$

$$\int \frac{t+5}{t}\,dt = \int\left(1+\frac{5}{t}\right)dt = t+5\ln|t|+c = x\overset{\text{constant}}{-2}+\ln|x-2|^5+c$$
$$= x+\ln|x-2|^5+c$$

⑦ $\int \frac{x\,dx}{\sqrt{x-2}} = ?$

$x-2=t^2$
$dx=2t\,dt$
$x=t^2+2$

$$I=\int \frac{(t^2+2).2\cancel{t}\,dt}{\cancel{t}} = 2\left(\frac{t^3}{3}+2t\right)+c = \frac{2}{3}(x-2)^{3/2}+4\sqrt{x-2}+c$$

⑧ $\int \frac{e^{1/x}}{x^2}\,dx = ?$

$\frac{1}{x}=x^{-1}=t$
$-\frac{1}{x^2}dx=dt$

$$I=\int -e^t\,dt = -e^t+c = -e^{1/x}+c$$

⑨ $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx = ?$

$e^x+e^{-x}=t$
$(e^x-e^{-x})dx=dt$

$$I=\int\frac{dt}{t} = \ln|t|+c = \ln\left(e^x+e^{-x}\right)+c$$

⑩ $\int \frac{x^2+2x}{(x+1)^2}\,dx = ?$ $\qquad \frac{x^2+2x}{(x+1)^2} = \frac{x^2+2x+1-1}{(x+1)^2} = \frac{(x+1)^2-1}{(x+1)^2} = 1-\frac{1}{(x+1)^2}$

$x+1=t$
$dx=dt$

$$I=\int\left[1-\frac{1}{(x+1)^2}\right]dx = \int dx+\int\frac{-dt}{t^2} = x+\frac{1}{t}+c = x+\frac{1}{x+1}+c$$
$$= \frac{x^2+x+1}{x+1}+c.$$

(11) $\int \frac{dx}{x.\ln x.[\ln(\ln x)]} = ?$

1st way: $\ln x = t$ $\quad I = \int \frac{dt}{t\ln t}$ $\quad \ln t = u$ $\quad I = \int \frac{du}{u} = \ln|u| + c$

$\frac{1}{x}dx = dt$ $\quad \frac{dt}{t} = du$

$= \ln|\ln t| + c = \ln|\ln(\ln x)| + c$

2nd way: $\ln(\ln x) = t$ $\quad I = \int \frac{dt}{t} = \ln|t| + c = \ln|\ln(\ln x)| + c$

$\frac{\frac{1}{x}}{\ln x}dx = \frac{dx}{x\ln x} = dt$

(12) $\int \frac{dx}{\sqrt{x}(3-e^{-\sqrt{x}})} = ?$

$\sqrt{x} = t$ $\quad I = \int \frac{2dt}{3-e^{-t}} \overset{*\left(\frac{e^t}{e^t}\right)}{=} \int \frac{2e^t dt}{3e^t - 1}$ $\quad 3e^t - 1 = u$

$\frac{1}{2\sqrt{x}}dx = dt$ $\quad 3e^t dt = du$

$$I = \int \frac{2}{3}\frac{du}{u} = \frac{2}{3}\ln|u| + c = \frac{2}{3}\ln|3e^t - 1| + c = \frac{2}{3}\ln|3e^{\sqrt{x}} - 1| + c$$

(13) $\int \frac{dx}{(1+\sqrt{x})^3} = ?$ $\quad 1+\sqrt{x} = t \Rightarrow \sqrt{x} = t-1$

$\frac{1}{2\sqrt{x}}dx = dt \Rightarrow dx = 2(t-1)dt$

$$I = \int \frac{2(t-1)dt}{t^3} = 2\int (t-1).t^{-3}dt = 2\int (t^{-2} - t^{-3})dt = 2\left[\frac{t^{-1}}{-1} - \frac{t^{-2}}{-2}\right] + c$$

$$= 2\left[-\frac{1}{t} + \frac{1}{2t^2}\right] + c = \frac{-2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + c = \frac{-1-2\sqrt{x}}{(1+\sqrt{x})^2} + c$$

(14) $\int_1^e \frac{\ln x^2}{x(1+\ln^2 x)}dx = ?$ $\quad \ln x^2 = 2\ln x$

$\ln x = t$ $\quad x = 1 \Rightarrow t = 0$ $\quad I = \int_0^1 \frac{2t\,dt}{1+t^2} = \ln|1+t^2| \Big|_0^1 = \ln 2 - \underbrace{\ln 1}_{0} = \ln 2$

$\frac{dx}{x} = dt$ $\quad x = e \Rightarrow t = 1$

15) $\displaystyle\int_1^{e^2} \frac{\sqrt{1+4\ln x}}{3x}\,dx = ?$

$1+4\ln x = t^2 \qquad x=1 \Rightarrow t=1$

$\frac{4}{x}dx = 2t\,dt \qquad x=e^2 \Rightarrow t=3$

$$I=\int_1^3 \frac{t^2\,dt}{6} = \frac{t^3}{18}\Big|_1^3 = \frac{1}{18}(27-1) = \frac{26}{18} = \frac{13}{9}.$$

16) $\displaystyle\int_0^9 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = ?$

$1+\sqrt{x} = t \quad \Rightarrow \sqrt{x} = t-1 \qquad x=0 \Rightarrow t=1$

$\frac{dx}{2\sqrt{x}} = dt \quad \Rightarrow dx = 2(t-1)dt \qquad x=9 \Rightarrow t=4$

$$I=\int_1^4 \frac{(t-1)2(t-1)dt}{t}$$

$$=2\int_1^4 \frac{(t^2-2t+1)dt}{t} = 2\int_1^4 \left(t-2+\frac{1}{t}\right)dt = 2\left[\frac{t^2}{2}-2t+\ln t\right]_1^4 = 3+4\ln 2$$

17) $\displaystyle\int_1^{16} \frac{x-1}{x+\sqrt{x}}\,dx = ?$

$(x-\sqrt{x})$

$$\frac{(x-1)(x-\sqrt{x})}{x^2-x} = \frac{\cancel{(x-1)}(x-\sqrt{x})}{x\cancel{(x-1)}} = \frac{x-\sqrt{x}}{x} = 1-\frac{1}{\sqrt{x}}$$

$$I=\int_1^{16}\left(1-\frac{1}{\sqrt{x}}\right)dx = x\Big|_1^{16} - \int_1^{16}\frac{dx}{\sqrt{x}}$$

$\sqrt{x}=t \qquad x=1 \Rightarrow t=1$

$\frac{1}{2\sqrt{x}}dx = dt \qquad x=16 \Rightarrow t=4$

$$15-\int_1^4 2dt = 15-2t\Big|_1^4 = 15-2(\underbrace{4-1}_{3}) = 9.$$

18) $\displaystyle\int_1^9 \frac{dx}{(x+1)\sqrt{x}+2x} = ?$

$$\frac{1}{(x+1)\sqrt{x}+2\sqrt{x}\sqrt{x}} = \frac{1}{\sqrt{x}(x+2\sqrt{x}+1)} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$$

$\sqrt{x}+1 = t \qquad x=1 \Rightarrow t=2$

$\frac{1}{2\sqrt{x}}dx = dt \qquad x=9 \Rightarrow t=4$

$$I=\int_2^4 \frac{2dt}{t^2} = -\frac{2}{t}\Big|_2^4 = -\Big(\underbrace{\frac{1}{2}-1}_{-\frac{1}{2}}\Big) = \frac{1}{2}$$

19 $\int \frac{dx}{2+\sqrt{x}} = ?$

$2+\sqrt{x} = t \Rightarrow \sqrt{x} = t-2$

$\frac{1}{2\sqrt{x}}dx = dt \Rightarrow dx = 2(t-2)dt$

constant

$I = \int \frac{2(t-2)dt}{t} = \int \left(2-\frac{4}{t}\right)dt = 2t - 4\ln t + c = 4 + \sqrt{x} - 4\ln(2+\sqrt{x}) + c$

OR $= \sqrt{x} - \ln(2+\sqrt{x})^4 + c$

20 $\int_0^1 (x+1)\sqrt{1-x}\,dx = ?$

$1-x = t^2 \Rightarrow x = 1-t^2$ $\qquad x=0 \Rightarrow t=1$

$-dx = 2t\,dt$ $\qquad x=1 \Rightarrow t=0$

$I = \int_1^0 (2-t^2)\underbrace{\sqrt{t^2}}_{t}\cdot(-2)\cdot t\,dt = \int_0^1 (4t^2-2t^4)dt = \frac{4}{3}t^3 - \frac{2}{5}t^5 \Big|_0^1 = \frac{4}{3} - \frac{2}{5} = \frac{14}{15}$

change boundaries

21 $\int_1^e \frac{2(\ln x^x)+1}{x^2}dx = ?$

$I = \int_1^e \frac{2x\ln x + 1}{x^2}dx = \int_1^e \frac{2\ln x}{x}dx + \int_1^e \frac{dx}{x^2}$ $\qquad \ln x = t \quad x=1 \Rightarrow t=0$

$\frac{dx}{x} = dt \quad x=e \Rightarrow t=1$

$I = 2\int_0^1 t\,dt + \int_1^e \frac{dx}{x^2} = t^2\Big|_0^1 - \frac{1}{x}\Big|_1^e = 1 - \frac{1}{e} + 1 = 2 - \frac{1}{e}$

22 $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx = ?$

$\sin x = t$

$\cos x\,dx = dt$

$I = \int \frac{dt}{\sqrt{4-t^2}} = \arcsin\left(\frac{t}{2}\right) + c = \arcsin\left(\frac{\sin x}{2}\right) + c.$

23) $\int_0^{\pi/4} e^{(\tan x - 2\ln(\cos x))} dx = ?$

$$e^{\tan x - 2\ln(\cos x)} = e^{\tan x} \cdot e^{-2\ln(\cos x)} = e^{\tan x} \cdot e^{\ln(\cos x)^{-2}} = e^{\tan x} \cdot \frac{1}{\cos^2 x}$$

$$I = \int_0^{\pi/4} e^{\tan x} \cdot \sec^2 x dx \qquad \begin{array}{l} \tan x = t \\ \sec^2 x dx = dt \end{array} \qquad \begin{array}{l} x=0 \Rightarrow t=0 \\ x=\frac{\pi}{4} \Rightarrow t=1 \end{array}$$

$$I = \int_0^1 e^t dt = e^t \Big|_0^1 = e^1 - e^0 = e - 1.$$

24) $\int \sin x \cdot \cos x \cdot \sqrt{2+\sin^2 x}\, dx = ?$

$$\begin{array}{l} 2+\sin^2 x = t^2 \\ \cancel{2} \sin x \cos x dx = \cancel{2} t dt \end{array} \qquad I = \int \underbrace{\sqrt{t^2}}_{t} \cdot t dt = \frac{t^3}{3} + c = \frac{(2+\sin^2 x)^{3/2}}{3} + c$$

25) $\int \frac{\sec^2 x dx}{\tan^3 x} = ?$

$$\begin{array}{l} \tan x = t \\ \sec^2 x dx = dt \end{array} \qquad I = \int \frac{dt}{\underset{\downarrow}{t^3}} = \frac{t^{-2}}{-2} + c = \frac{-2}{\tan^2 x} + c \quad \underline{OR} \quad -2\cot^2 x + c$$

$$t^{-3}$$

26) $\int \sin^7 x \cdot \cos^5 x\, dx = ?$

For 1st way　　For 2nd way

$$\int \sin^7 x \cdot \underbrace{\cos^4 x}_{(1-\sin^2 x)^2} \cdot \cos x dx = \int \underbrace{\sin^6 x}_{(1-\cos^2 x)^3} \cdot \cos^5 x \cdot \sin x dx$$

$\underline{1.st\ way}$: $\begin{array}{l} \sin x = t \\ \cos x dx = dt \end{array}$　$I = \int t^7 \underbrace{(1-t^2)^2}_{(1-2t^2+t^4)} dt = \int (t^7 - 2t^9 + t^{11}) dt$

$$\Rightarrow I = \frac{t^8}{8} - \frac{t^{10}}{5} + \frac{t^{12}}{12} + c = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{5} + \frac{\sin^{12} x}{12} + c$$

$\underline{2nd\ way}$. $\begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array}$　$I = -\int \underbrace{(1-t^2)^3}_{(1-3t^2+3t^4-t^6)} \cdot t^5 dt = -\int (t^5 - 3t^7 + 3t^9 - t^{11}) dt$

$$\Rightarrow I = -\frac{t^6}{6} + \frac{3}{8}t^8 - \frac{3}{10}t^{10} + \frac{t^{12}}{12} + c = -\frac{\cos^6 x}{6} + \frac{3}{8}\cos^8 x - \frac{3}{10}\cos^{10} x + \frac{\cos^{12} x}{12} + c.$$

27) $\int \sin^3 x \cos^4 x dx = ?$ $\qquad \int \underbrace{\sin^2 x}_{(1-\cos^2 x)} . \cos^4 x . \sin x \, dx$

$\cos x = t$
$-\sin x dx = dt$ $\qquad I = \int -(1-t^2).\, t^4 dt = \int (t^6 - t^4) dt = \frac{t^7}{7} - \frac{t^5}{5} + c$

$\Rightarrow I = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$

28) $\int \sin^4 x . \cos^3 x dx = ?$ $\qquad \int \sin^4 x . \underbrace{\cos^2 x}_{(1-\sin^2 x)} . \cos x \, dx$

$\sin x = t$
$\cos x dx = dt$ $\qquad I = \int t^4 (1-t^2) dt = \int (t^4 - t^6) dt = \frac{t^5}{5} - \frac{t^7}{7} + c$

$\Rightarrow I = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

29) $\int \sin^2 x . \cos^4 x dx = ?$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$I = \int \left(\frac{1-\cos 2x}{2}\right) . \underbrace{\left(\frac{1+\cos 2x}{2}\right)^2}_{\frac{1+2\cos 2x + \cos^2 2x}{4}} dx$$

$$I = \frac{1}{8} \int (1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}} - \cos^3 2x) dx$$

$$= \frac{1}{8}\left(x + \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8}\right) - \frac{1}{8} \underbrace{\int \cos^3 2x}_{J}$$

$J = \int \underbrace{\cos^2 2x}_{(1-\sin^2 2x)} . \cos 2x dx$ $\qquad \sin 2x = t$, $2\cos 2x dx = dt$ $\qquad J = \frac{1}{2} \int (1-t^2) dt$

$$\Rightarrow J = \frac{1}{2}\left(t - \frac{t^3}{3}\right) + c = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + c$$

$$\Rightarrow I = \frac{1}{16}\left(x + \sin 2x - \frac{\sin 4x}{4}\right) - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + c$$

$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4} + \frac{\sin^3 2x}{3}\right) + c$$

(30) $\displaystyle\int_0^{\pi/4} \frac{24\tan x\sec^2 x}{(1+2\tan x)^2}\,dx = ?$

$1+2\tan x = t \qquad x=0 \Rightarrow t=1$

$2\sec^2 x\,dx = dt \qquad x=\frac{\pi}{4} \Rightarrow t=3$

$$I=\int_1^3 \frac{24\left(\frac{t-1}{2}\right)\cdot\frac{dt}{2}}{t^2} = 6\int_1^3 \frac{t-1}{t^2}\,dt = 6\int_1^3\left[\frac{1}{t}-\frac{1}{t^2}\right]dt = 6\left[\ln|t|+\frac{1}{t}\right]_1^3$$

$$= 6\left[\ln 3+\frac{1}{3}-\cancel{\ln 1}-1\right] = 6\ln 3-\frac{2}{3}.$$

(31) $\displaystyle\int \frac{x^{2/3}-x^{5/2}}{x^{1/2}}\,dx = ?$

$x=t^6$

$dx = 6t^5$

$$I=\int \frac{t^2-t^{15}}{t^3}\,dt = \int\left[\frac{1}{t}-t^{12}\right]dt = \ln|t| - \frac{t^{13}}{13}+c$$

$$I=\ln\left(x^{1/6}\right)-\frac{x^{13/6}}{13}+c.$$

(32) $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx = ?$

$$I=\underbrace{\int\frac{x\,dx}{\sqrt{1-x^2}}}_{-\sqrt{1-x^2}} + \underbrace{\int\frac{\arcsin x\,dx}{\sqrt{1-x^2}}}_{J}$$

$\arcsin x = t$

$\frac{dx}{\sqrt{1-x^2}} = dt$

$$\Rightarrow J=\int t\,dt = \frac{t^2}{2}+c$$

$$\Rightarrow I=-\sqrt{1-x^2}+\frac{\arcsin^2 x}{2}+c.$$

(33) $\displaystyle\int_0^{\pi/3} \frac{\tan x\sqrt{\sec x}+\sec x\sqrt{\tan x}}{\cos x}\,dx = ?$

$$I=\underbrace{\int_0^{\pi/3}\tan x.\sec x.\sqrt{\sec x}\,dx}_{J} + \underbrace{\int_0^{\pi/3}\sec^2 x\sqrt{\tan x}\,dx}_{K} = J+K$$

For J: $\sec x = t^2$ $\quad x=0 \Rightarrow t=1$

$\sec x.\tan x\,dx = 2t\,dt$ $\quad x=\frac{\pi}{3} \Rightarrow t=\sqrt{2}$

$$J = \int_1^{\sqrt{2}} 2t\underbrace{\sqrt{t^2}}_{t}\,dt = \int_1^{\sqrt{2}} 2t^2\,dt = \frac{2}{3}t^3 \Big|_1^{\sqrt{2}} = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

For K: $\tan x = t^2$ $\quad x=0 \Rightarrow t=0$

$\sec^2 x\,dx = 2t\,dt$ $\quad x=\frac{\pi}{3} \Rightarrow t=\sqrt[4]{3}$

$$K = \int_0^{\sqrt[4]{3}} 2t\sqrt{t^2}\,dt = \frac{2}{3}t^3 \Big|_0^{\sqrt[4]{3}} = \frac{2}{3}3^{3/4} = \frac{2}{3}\sqrt[4]{27}$$

$$\Rightarrow I = \frac{2}{3}\left(2\sqrt{2}-1+\sqrt[4]{27}\right)$$

34) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = ?$

$\ln x = t$ $\quad x=1 \Rightarrow t=0$

$\frac{dx}{x} = dt$ $\quad x=e \Rightarrow t=1$

$$I = \int_0^1 \frac{dt}{\sqrt{1+t^2}}$$

$t=\tan u$ $\quad t=0 \Rightarrow u=0$

$dt = \sec^2 u\,du$ $\quad t=1 \Rightarrow u=\frac{\pi}{4}$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\sec^{\cancel{2}} u\,du}{\cancel{\underbrace{\sqrt{1+\tan^2 u}}_{\sqrt{\sec^2 u}=\sec u}}} = \ln|\sec u + \tan u| \Big|_0^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln 1 = \ln\left(\sqrt{2}+1\right).$$

35) $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin x}\,.\cos^3 x\,dx = ?$

$\cos^3 x = \cos^2 x.\cos x = (1-\sin^2 x).\cos x$ $\quad 2+2\sin x = 2(1+\sin x)$

$1+\sin x = t^2$ $\quad x=-\frac{\pi}{2} \Rightarrow t=0$

$\cos x\,dx = 2t\,dt$ $\quad x=\frac{\pi}{2} \Rightarrow t=\sqrt{2}$

$$I = \int_0^{\sqrt{2}} \sqrt{2}.\underbrace{\sqrt{t^2}}_{t}.t^2.(2-t^2)2t\,dt$$

$$\Rightarrow I = 2\sqrt{2}\int_0^{\sqrt{2}} (2t^4 - t^6)\,dt = 2\sqrt{2}\left[\frac{2}{5}t^5 - \frac{t^7}{7}\right]_0^{\sqrt{2}} = \frac{32}{5} - \frac{32}{7} = \frac{64}{35}.$$