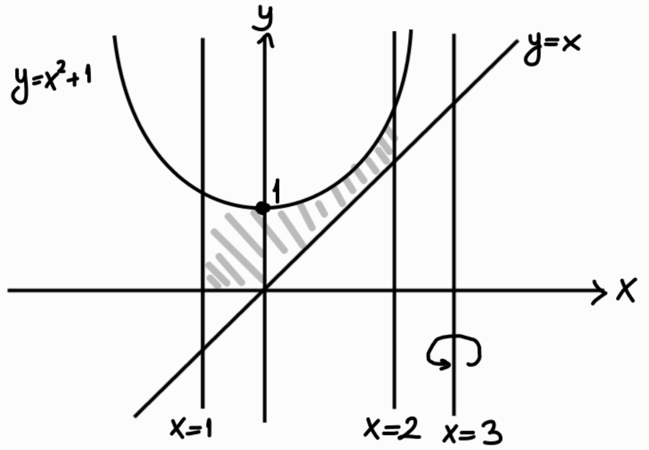
AREA AND VOLUME

The question is "Write the integral to find the area of the region bounded by the given curves-lines-axes. Write the integral to evaluate the volume of the solid obtained by rotating the region about the specified axis-line."

Remark: No calculation will be made.

1) Region: $y=x^2+1$, $y=x$, $x=-1$, $x=2$

Rotation: about $x=3$.

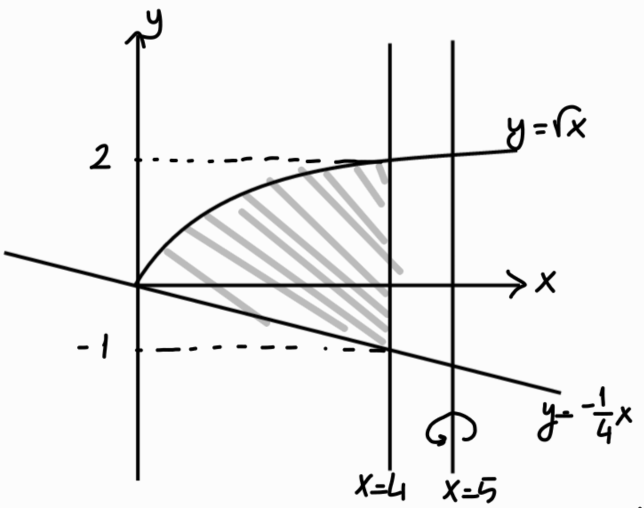


$$A=\int_{-1}^{2}\left[(x^2+1)-x\right]dx$$

$$V=2\pi\int_{-1}^{2}\underbrace{(3-x)}_{r}\cdot\underbrace{(x^2+1-x)}_{h}\,dx$$
(Shell)

2) Region: $y=\sqrt{x}$, $y=-\frac{1}{4}x$, $x=4$

Rotation: a) about y-axis b) about $x=5$.



$$A=\int_{0}^{4}\left[\sqrt{x}-\left(-\frac{1}{4}x\right)\right]dx$$
w.r.t x

OR

$$A=\int_{-1}^{0}\left[4-(-4y)\right]dy+\int_{0}^{2}(4-y^2)\,dy$$
w.r.t y

a) $$V_a=\pi\int_{-1}^{0}\left[4^2-(-4y)^2\right]dy+\pi\int_{0}^{2}(4^2-y^2)\,dy$$
Washer

OR

$y=-\frac{1}{4}x \Rightarrow x=-4y$

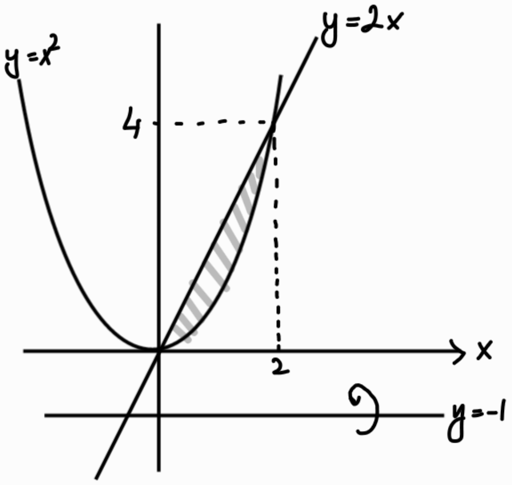
$y=\sqrt{x} \Rightarrow x=y^2$

$$V_a=2\pi\int_{0}^{4}\underbrace{\left[\sqrt{x}-\left(-\frac{x}{4}\right)\right]}_{h}\cdot\underbrace{x}_{r}\,dx$$
Shell

b) $$V_b=2\pi\int_{0}^{4}\underbrace{\left[\sqrt{x}-\left(-\frac{x}{4}\right)\right]}_{h}\underbrace{(5-x)}_{r}\,dx$$
Shell

③ Region : $y=x^2$, $y=2x$

Rotation : a) about x-axis b) about y-axis c) about $y=-1$



$$A=\int_0^2 \underbrace{(2x-x^2)\,dx}_{\text{w.r.t } x}=\int_0^4 \underbrace{\left(\sqrt{y}-\frac{y}{2}\right)dy}_{\text{w.r.t } y}$$

a) (Washer) (Shell)
$$V_a=\pi\int_0^2\left[(2x)^2-(x^2)^2\right]dx=2\pi\int_0^4\underbrace{\left[\sqrt{y}-\frac{y}{2}\right]}_{h}\cdot\underbrace{y}_{r}\,dy$$

b) (Washer) (Shell)
$$V_b=\pi\int_0^4\left[(\sqrt{y})^2-\left(\frac{y}{2}\right)^2\right]dy=2\pi\int_0^2\underbrace{(2x-x^2)}_{h}\cdot\underbrace{x}_{r}\,dx$$

$x^2=2x \Rightarrow x=0,\ x=2$

$y=x^2 \Rightarrow x=\sqrt{y}$

$y=2x \Rightarrow x=\frac{y}{2}$

c) (Washer) (Shell)
$$V_c=\pi\int_0^2\left[(2x+1)^2-(x^2+1)^2\right]dx=2\pi\int_0^4\underbrace{\left[\sqrt{y}-\frac{y}{2}\right]}_{h}\underbrace{\left[y-(-1)\right]}_{r}dy$$

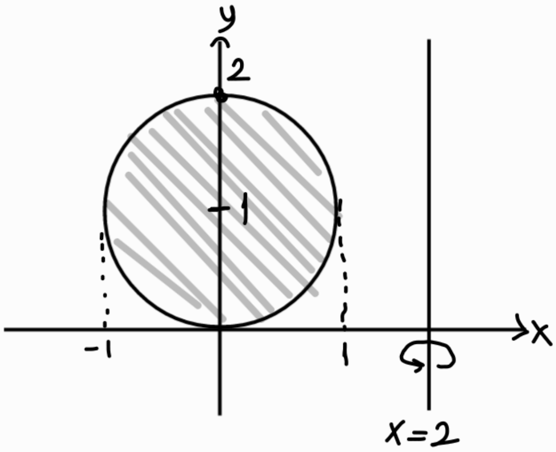
④ Region : $x^2+y^2-2y=0$, x-axis ($y=0$)

Rotation : a) about $y=0$ b) about $x=2$

$x^2+(y^2-2y+1)=1$

$x^2+(y-1)^2=1$

$$A=\int_0^2\left[\sqrt{2y-y^2}-\left(-\sqrt{2y-y^2}\right)\right]dy=2\int_0^2\sqrt{2y-y^2}\,dy$$



a)
$$V_a=2\pi\int_0^2\underbrace{\left(\sqrt{2y-y^2}-\left(-\sqrt{2y-y^2}\right)\right)}_{h}\cdot\underbrace{y}_{r}\,dy \quad \text{(Shell)}$$

$$V_a=\pi\int_{-1}^1\left[\left(1+\sqrt{1-x^2}\right)^2-\left(1-\sqrt{1-x^2}\right)^2\right]dx \quad \text{(Washer)}$$

$x=\mp\sqrt{1-(y-1)^2}=\mp\sqrt{2y-y^2}$

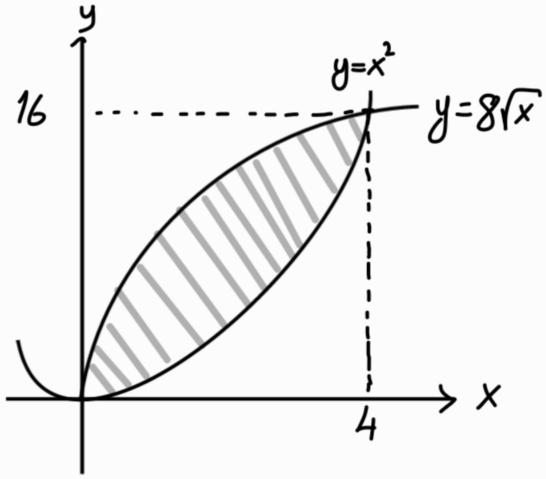
$y=\mp\sqrt{1-x^2}+1$

b) (Washer)
$$V_b=\pi\int_0^2\left[\left(2-\left(-\sqrt{2y-y^2}\right)\right)^2-\left(2-\sqrt{2y-y^2}\right)^2\right]dy$$

$$V_b=2\pi\int_{-1}^1\underbrace{\left[\left(1+\sqrt{1-x^2}\right)-\left(1-\sqrt{1-x^2}\right)\right]}_{h}\cdot\underbrace{(2-x)}_{r}\,dx \quad \text{(Shell)}$$

5) Region: $y = 8\sqrt{x}$, $y = x^2$

Rotation: a) about x-axis b) about y-axis



$y = x^2 \Longleftrightarrow x = \mp\sqrt{y}$

$y = 8\sqrt{x} \Longleftrightarrow x = y^2/64$

$8\sqrt{x} = x^2 \Longleftrightarrow x=0,\ x=4$

w.r.t x w.r.t. y

$$A = \int_0^4 (8\sqrt{x} - x^2)\,dx = \int_0^{16} \left(\sqrt{y} - \frac{y^2}{64}\right) dy$$

(Washer) (Shell)

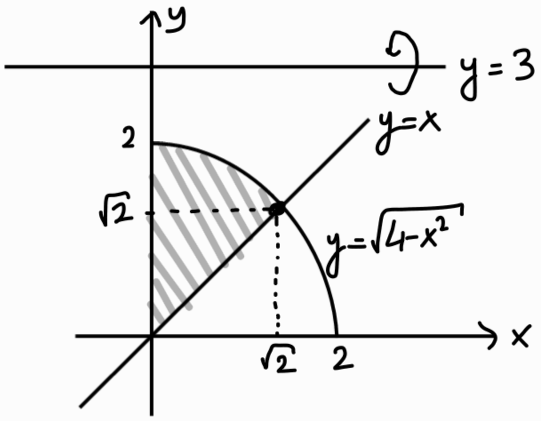
$$\text{a) } V_a = \pi\int_0^4 \left[(8\sqrt{x})^2 - (x^2)^2\right] dx = 2\pi \int_0^{16} \left(\sqrt{y} - \frac{y^2}{64}\right) \cdot y\,dy$$

$$\text{b) } V_b = \pi\int_0^{16} \left[(\sqrt{y})^2 - \left(\frac{y^2}{64}\right)^2\right] dy \quad \text{(Washer)}$$

$$= 2\pi \int_0^4 \underbrace{(8\sqrt{x} - x^2)}_{h}\,\underbrace{x}_{r}\,dx \quad \text{(Shell)}$$

6) Region: $y = \sqrt{4-x^2}$, $y = x$, $x = 0$

Rotation: a) about x-axis b) about y-axis c) about $y=3$



w.r.t. x w.r.t. y

$$A = \int_0^{\sqrt{2}} \left(\sqrt{4-x^2} - x\right) dx = \int_0^{\sqrt{2}} y\,dy + \int_{\sqrt{2}}^{2} \sqrt{4-y^2}\,dy$$

$$\text{a) } V_a = \pi\int_0^{\sqrt{2}} \left[(\sqrt{4-x^2})^2 - x^2\right] dx \quad \text{(Washer)}$$

$$= 2\pi\int_0^{\sqrt{2}} \underbrace{y}_{h}\cdot\underbrace{y}_{r}\,dy + 2\pi\int_{\sqrt{2}}^{2} \underbrace{\sqrt{4-y^2}}_{h}\,\underbrace{y}_{r}\,dy \quad \text{(Shell)}$$

$y = \sqrt{4-x^2} \Leftrightarrow x = \sqrt{4-y^2}$

$\sqrt{4-x^2} = x \Rightarrow 4 - x^2 = x^2 \Rightarrow x^2 = 2$

$\Rightarrow x = \mp\sqrt{2}$

(Disk) (Shell)

$$\text{b) } V_b = \pi\int_0^{\sqrt{2}} \left(\sqrt{4-y^2}\right)^2 dy + \pi\int_{\sqrt{2}}^{2} y^2\,dy = 2\pi\int_0^{\sqrt{2}} \underbrace{\left(\sqrt{4-x^2} - x\right)}_{h}\,\underbrace{x}_{r}\,dx$$

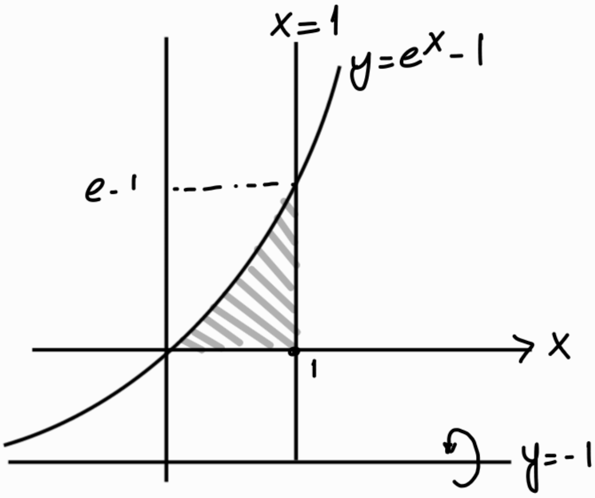
(Washer)

$$\text{c) } V_c = \pi\int_0^{\sqrt{2}} \left[(3-x)^2 - \left(3 - \sqrt{4-x^2}\right)^2\right] dx$$

(Horizontally the region needs to be separated into two pieces.)

7) Region: $y=e^x-1$, $x=1$, x-axis

Rotation: a) about $x=0$ b) about $y=-1$



$y=e^x-1 \iff x=\ln(y+1)$

$$A=\int_0^1 (e^x-1)\,dx = \int_0^{e-1}[1-\ln(y+1)]\,dy$$
w.r.t. x w.r.t. y

a) $V_a=\pi\int_0^{e-1}[1-\ln(y+1)]^2\,dy$ (Disk)

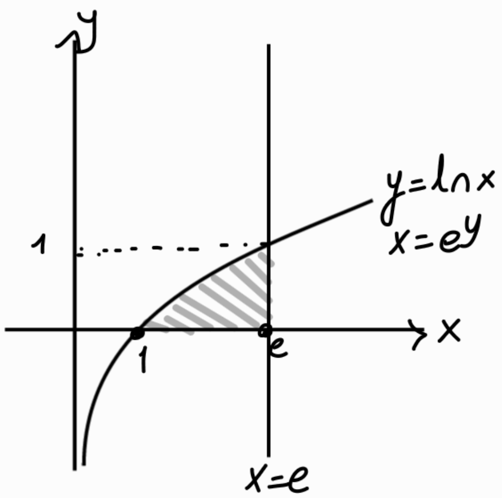
$=2\pi\int_0^1 \underbrace{(e^x-1)}_{h}\cdot\underbrace{x}_{r}\,dx$ (Shell)

b) $V_b=\pi\int_0^1[(e^x-1-(-1)]^2\,dx$ (Disk)

$=2\pi\int_0^{e-1}\underbrace{[1-\ln(y+1)]}_{h}\cdot\underbrace{(y+1)}_{r}\,dy$ (Shell)

8) Region: $y=\ln x$, $x=e$, x-axis

Rotation: a) about $x=0$ b) about $y=0$



$$A=\int_1^e \ln x\,dx = \int_0^1 (e-e^y)\,dy$$
w.r.t x w.r.t. y

a) $V_a=\pi\int_0^1(e^2-e^{2y})\,dy = 2\pi\int_1^e \underbrace{\ln x}_{h}\cdot\underbrace{x}_{r}\,dx$
(Washer) (Shell)

b) $V_b=\pi\int_1^e(\ln x)^2\,dx = 2\pi\int_0^1\underbrace{(e-e^y)}_{h}\underbrace{y}_{r}\,dy$
(Disk) (Shell)

$\begin{pmatrix} x=0 \iff y\text{-axis} \\ y=0 \iff x\text{-axis}\end{pmatrix}$