MAT1320 LINEAR ALGEBRA EXERCISES IX-X

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1. (A points) Let $\vec{u}$ and $\vec{v}$ be two unit vectors. If $\vec{u}+2\vec{v}$ is orthogonal to $5\vec{u}-4\vec{v}$, then which of the followings is the angle between the vectors $\vec{u}$ and $\vec{v}$?

   a) $60°$ b) $90°$ c) $30°$ d) $\arccos\left(\frac{1}{3}\right)$ e) $\arccos\left(\frac{2}{7}\right)$

$(\vec u + 2\vec v)\cdot(5\vec u - 4\vec v) = 0$

$\Rightarrow 5\vec u\cdot\vec u - 4\vec u\cdot\vec v + 10\underbrace{\vec v\cdot\vec u}_{\vec u\cdot\vec v} - 8\vec v\cdot\vec v = 0$

Since $\vec u\cdot\vec u = |\vec u|^2$ and $\vec v\cdot\vec v = |\vec v|^2$, and $|\vec u| = |\vec v| = 1$,

$5 + 6\vec u\cdot\vec v - 8 = 0 \Rightarrow \vec u\cdot\vec v = \frac{1}{2}$

$\Rightarrow \underbrace{\vec u\cdot\vec v}_{\frac12} = \underbrace{|\vec u|}_{1}\underbrace{|\vec v|}_{1}\cos\alpha \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = 60°.$

Let $\vec u = (u_1, u_2, u_3)$.

$\vec u\cdot\vec a = 0 \Rightarrow 2u_1 + 0.u_2 + 1.u_3 = 0 \Rightarrow 2u_1 + u_3 = 0$.

$\vec c\times\vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = (-10, -3, 7)$

$\vec u\times\vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ u_1 & u_2 & u_3 \\ 1 & 1 & 1 \end{vmatrix} = (u_2 - u_3,\ u_1 - u_3,\ u_1 - u_2)$

$u_1 - u_3 = -3$ and $2u_1 = -u_3 \Rightarrow u_1 + 2u_1 = -3 \Rightarrow \boxed{u_1 = -1} \Rightarrow \boxed{u_3 = 2}$

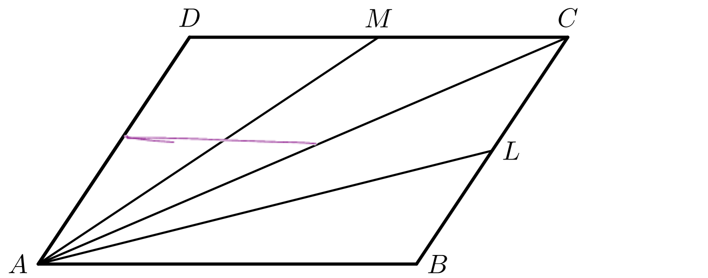
$u_1 - u_2 = 7 \Rightarrow \boxed{u_2 = -8} \Rightarrow \vec u = (-1, -8, 2)$

2. (D points) Let $\vec{a} = 2\mathbf{i} + \mathbf{k}$, $\vec{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\vec{c} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. If $\vec{u}\times\vec{b} = \vec{c}\times\vec{b}$ and $\vec{u}\cdot\vec{a} = 0$, then which of the followings is the vector $\vec{u}$?

   a) $\mathbf{i} + 8\mathbf{j} + \mathbf{k}$ b) $\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ c) $2\mathbf{i} + \mathbf{j} - 8\mathbf{k}$

   d) $-\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$ e) $\mathbf{i} - 8\mathbf{j} + \mathbf{k}$

3. (C points) For the following parallelogram $ABCD$, the points L and M are the middle points of the sides BC and CD, respectively. Then, which of the followings is the vector $AL + AM$?



   a) $\frac{1}{2}AC$ b) AC c) $\frac{3}{2}AC$ d) $2AC$

   e) None of them

$\vec{AL} = \vec{AB} + \vec{BL}$

$\vec{AM} = \vec{AD} + \vec{DM}$

$\Rightarrow \vec{AL} + \vec{AM} = \underbrace{\vec{AB} + \vec{AD}}_{\vec{AC}} + \underbrace{\vec{BL} + \vec{DM}}_{\frac12 \vec{AC}} = \frac{3}{2}\vec{AC}$

T: $3t^2 \in T$, $3t \in T$ but $3t^2 + 3t \notin T$.

$\Rightarrow T$ is not subspace of P_2.

A: Let $a_1t^2 + b_1t + c_1$, $a_2t^2 + b_2t + c_2 \in A$. Then,

$b_1 = 3c_1$, $b_2 = 3c_2$.

$\Rightarrow (a_1t^2 + b_1t + c_1) + (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + c_1 + c_2 \in A$ ✓

See that $(b_1 + b_2) = 3(c_1 + c_2)$.

$\forall k \in \mathbb{R}$,

$\Rightarrow k.(a_1t^2 + b_1t + c_1) = ka_1t^2 + kb_1t + kc_1 \in A$ since $kb_1 = 3kc_1$. ✓

4. (A points) Let P_2 be the set of all polynomials over real numbers whose degrees are at most 2. Recall that P_2 is a vector space with usual addition and multiplication by a scalar on polynomials. Then, which of the following subsets is a subspace of P_2?

$$\mathcal{M} = \{at^2 + bt + c \mid c = 0\}$$
$$\mathcal{A} = \{at^2 + bt + c \mid b = 3c\}$$
$$\mathcal{T} = \{at^2 + bt + c \mid a + b + c = 3\}$$

   a) $\mathcal{M}$ and $\mathcal{A}$ ~~b) $\mathcal{M}$ and $\mathcal{T}$~~ ~~c) $\mathcal{A}$ and $\mathcal{T}$~~

   ~~d) Only $\mathcal{M}$~~ ~~e) All of them~~

5. Which of the following subsets are subspaces of the given vector spaces ?

$$\mathcal{Y} = \left\{ \begin{bmatrix} x \\ x^2 \end{bmatrix} \mid x \in \mathbb{R} \right\} \subset \mathbb{R}^2$$

$$\mathcal{T} = \left\{ \begin{bmatrix} x \\ x+1 \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

$$\mathcal{U} = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

a) ~~Only $\mathcal{Y}$~~ b) ~~Only $\mathcal{T}$~~ c) Only $\mathcal{U}$

d) ~~$\mathcal{Y}$ and $\mathcal{T}$~~ e) ~~$\mathcal{T}$ and $\mathcal{U}$~~

Y: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \in Y$ but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \notin Y$ because $3^2 \neq 5$.

T: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \in T$ but $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \notin T$ since $5 \neq 3+1$.

$\Rightarrow$ Then, see that $\mathcal{U}$ is closed under addition and multiplication by scalars.

$\Rightarrow \mathcal{U}$ is a subspace