Tangent Lines, and Normal Lines

Example: Show that the point $(2,4)$ lies on the curve $x^3+y^3-9xy=0$. Then find the tangent line and normal line to the curve there.

Put $(2,4)$ into the equation of the curve.

$$2^3+4^3-9\cdot 2\cdot 4 = 8+64-72=0$$

$$3x^2+3y^2\cdot y' - 9\left(y+x\cdot y'\right) = 0$$

$$\cancel{3}\cdot 4+\cancel{3}\cdot 16\cdot y'\big|_{(2,4)} - \underset{3}{\cancel{9}}\left(4+2\cdot y'\big|_{(2,4)}\right) = 0$$

$$4+16y'\big|_P - 12 - 6y'\big|_P = 0 \qquad (P=(2,4))$$

$$10y'\big|_P - 8 = 0 \Rightarrow y'\big|_P = \frac{8}{10} = \frac{4}{5} = m_T$$

__Tangent Line:__

$$y-y_0 = m(x-x_0)$$

$$y-4=\frac{4}{5}(x-2)$$

$$y = 4+\frac{4x}{5}-\frac{8}{5} = \underline{\frac{4x}{5}+\frac{12}{5}}$$

$$m_T\cdot m_N = -1 \Rightarrow \frac{4}{5}\cdot m_N = -1 \Rightarrow m_N = -\frac{5}{4}$$

__Normal Line:__ $y-4 = -\frac{5}{4}(x-2) = -\frac{5}{4}x+\frac{13}{2}$.

Example: Find equation of the tangent line to the curve $f(x) = \sin(\sin x)$ at the point with x-coordinate is $x = \pi$.

$$(x, y) = (\pi, f(\pi)) = (\pi, \underbrace{\sin(\underbrace{\sin\pi}_{0})}_{0}) = (\pi, 0)$$

$y = f(x)$

$$f'(x) = \cos(\sin x) \cdot \cos x$$

$$f'(\pi) = \underbrace{\cos(\underbrace{\sin\pi}_{0})}_{1} \cdot \underbrace{\cos\pi}_{-1} = -1 = m_T$$

Tangent Line: $y - 0 = -1 \cdot (x - \pi) \Rightarrow y = -x + \pi$

Normal Line: $y - 0 = 1(x - \pi) \Rightarrow y = x - \pi$

$(m_T \cdot m_N = -1 \Rightarrow m_N = 1)$

Linearization and Differentials

Definition: If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

is the linearization of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the standard linear approximation of f at a. The point $x = a$ is the center of the approximation.

Example: Find the linearization of $f(x)=\cos x$ at $x=\frac{\pi}{2}$.

$f\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$, $f'(x)=-\sin x \Rightarrow f'\left(\frac{\pi}{2}\right)=-1$

$L(x)=0+(-1)\cdot\left(x-\frac{\pi}{2}\right)=\underline{-x+\frac{\pi}{2}}$.

Example: Find the linearization of $f(x)=\sqrt{1+x}$ at $x=3$.

$a=3$ $f(3)=2$, $f'(x)=\frac{1}{2\sqrt{1+x}}$ $f'(3)=\frac{1}{4}$

$L(x)=2+\frac{1}{4}(x-3)=\frac{x+5}{4}$

At $x=3.2$

$L(3.2)=\frac{3.2+5}{4}=\frac{8.2}{4}=\underline{2.05}\approx f(3.2)$

$f(3.2)=\sqrt{1+3.2}=\sqrt{4.2}\approx\underline{2.04939}$

Definition: Let $y=f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is $dy=\underline{f'(x)}\cdot dx$.

Example:

(a) $d(\tan 2x) = 2 \cdot \sec^2(2x)\,dx$

(b) $d\left(\frac{x}{x+1}\right) = \frac{x+1-x}{(x+1)^2}dx = \frac{1}{(x+1)^2}dx$

$$f(a+dx) \approx f(a) + dy \qquad \Delta x = dx \;, \quad \Delta y \approx dy$$

The approximation $\Delta y \cong dy$ can be used to estimate $f(a+dx)$ when $f(a)$ is known, dx is small, and

$dy = f'(a)\,dx$

Example: Use differentials to estimate

(a) $7.97^{1/3}$ (b) $= \sin\left(\frac{\pi}{6} + 0.01\right)$

(a) $f(x) = x^{1/3}$ $\quad dy = \frac{1}{3} \cdot x^{-2/3}\,dx$

Set $a=8$. $\quad a+dx = 7.97 \Rightarrow dx = -0.03$

$$f(7.97) = f(\underline{a+dx}) \approx f(a) + dy$$
$$= 8^{1/3} + \frac{1}{3} \cdot 8^{-2/3} \cdot (-0.03)$$
$$= 2 + \frac{1}{3} \cdot \frac{1}{4} \cdot (-0.03) = 1.9975.$$

$\Rightarrow \underline{f(7.97) \approx 1.9975}$

$7.97^{1/3} \cong 1.997497$

(b) $y = \sin x \quad dy = \cos x . dx \quad \left[\frac{dy}{dx} = \cos x \Rightarrow dy = \cos x dx \right]$

$a = \frac{\pi}{6} \quad dx = 0.01$

$$f\left(\frac{\pi}{6} + 0.01\right) \approx f\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right).(0.01)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(0.01) \approx 0.5087$$

$$\sin\left(\frac{\pi}{6} + 0.01\right) \approx 0.508635$$

Indeterminate Forms and L'Hôpital's Rule

Indeterminate Form $\frac{0}{0}, \frac{\infty}{\infty}$

If the functions $f(x)$ and $g(x)$ are both zero or both $\mp\infty$ at $x=a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x=a$. The substitution produces $\frac{0}{0}$ or $\frac{\infty}{\infty}$, a meaningless expressions (indeterminate forms), that we cannot evaluate.

The indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

Theorem: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\mp\infty}{\mp\infty}$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or $\mp\infty$

Remark: Do not make the mistake of the derivative of $\frac{f}{g}$. The quotient to use is $\frac{f'}{g'}$, not $\left(\frac{f}{g}\right)'$

Ex.: $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \overset{L}{=} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2}$$

Ex.: $\displaystyle\lim_{x\to 0} \frac{3x-\sin x}{x} = \left(\frac{0}{0}\right) \overset{L}{=} \lim_{x\to 0} \frac{3-\cos x}{1} = 2$

$\frac{1}{2}(x+1)^{-1/2}$

Ex.: $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \left(\frac{0}{0}\right) \overset{L}{=} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+1}}-\frac{1}{2}}{2x} = \left(\frac{0}{0}\right)$

$$\overset{L}{=} \lim_{x\to 0} \frac{-\frac{1}{4}\cdot(x+1)^{-3/2}}{2} = -\frac{1}{8}$$

Ex.: $\displaystyle\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\sin\left(\frac{1}{x}\right)} = \left(\frac{0}{0}\right) \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cos\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}} = \frac{1}{1} = 1$

$\frac{1}{x} = t$, $x \to \infty \Rightarrow t = 0$ $\quad \lim_{t \to 0} \frac{\ln(1+t)}{\sin t} \overset{L}{=} \lim_{t \to 0} \frac{\frac{1}{1+t}}{\cos t} = 1$

Ex.: $$\lim_{x \to (\frac{\pi}{2})^-} \frac{\sec x}{1+\tan x} = \left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\cancel{\sec x} \cdot \tan x}{\sec^{\cancel{2}} x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}} = 1$$

Ex.: $$\lim_{x \to 1} \frac{\ln x}{x-1} = \left(\frac{0}{0}\right) \overset{L}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

Ex.: $$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Indeterminate Products $0 \cdot \mp\infty$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \mp\infty$, then it is not clear what the value of $\lim_{x \to a} f(x) \cdot g(x)$, if any.

This type is called an indeterminate form of type $0 \cdot \infty$.

We can deal with it by writing the product as a quotient:

$$f \cdot g = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

Ex.: $$\lim_{x \to \infty} \left(x \cdot \sin \frac{1}{x}\right) = (\infty \cdot 0) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \left(= 1\right)$$

$$\overset{L}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = 1$$

Ex.: $\lim_{x\to 0^+} \sqrt{x}\cdot \ln x = (0\cdot\infty) = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}\cdot x^{-3/2}}$

$= \lim_{x\to 0^+} -2\cdot x^{-1}\cdot x^{+3/2} = \lim_{x\to 0^+} -2\cdot x^{1/2} = 0$

Indeterminate Differences $\infty-\infty$

If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then the limit

$$\lim_{x\to a}[f(x)-g(x)]$$

is called an indeterminate form of type $\infty-\infty$.

Ex.: $\lim_{x\to \frac{\pi}{2}^-} (\sec x - \tan x) = (\infty-\infty) = \lim_{x\to \frac{\pi}{2}^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$= \lim_{x\to \frac{\pi}{2}^-} \frac{(1-\sin x)}{\cos x} \overset{L}{=} \lim_{x\to \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = 0$

Ex.: $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = (\infty-\infty) = \lim_{x\to 0}\frac{x-\sin x}{x.\sin x} = \left(\frac{0}{0}\right)$

(x) $(\sin x)$

$\overset{L}{=} \lim_{x\to 0}\frac{1-\cos x}{\sin x + x.\cos x} \overset{L}{=} \lim_{x\to 0}\frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$ //

Indeterminate Powers $(0^0, \infty^0, 1^\infty)$

These several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

Each of these three cases can be treated by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\cdot \ln[f(x)]}$$

and then

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)\cdot \ln[f(x)]}$$

where the indeterminate product $g(x)\ln[f(x)]$ is of type $0.\infty$.

<u>Ex.</u>: $\lim_{x\to\infty} x^{\frac{1}{x}} = (\infty^0)$ $\qquad y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\cdot \ln x$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x}\cdot \ln x = \lim_{x\to\infty} \frac{\ln x}{x} = \left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to\infty} \ln y = 0 \Rightarrow \ln\left(\lim_{x\to\infty} y\right) = 0 \Rightarrow e^{\ln\left(\lim_{x\to\infty} y\right)} = e^0$$

$$\Rightarrow \lim_{x\to\infty} y = 1$$

<u>Ex.</u>: $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = (1^\infty)$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \underbrace{\cot x}_{(\infty}\cdot \underbrace{\ln(1+\sin 4x)}_{0)} = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \left(\frac{0}{0}\right)$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = \frac{\frac{4}{1+0}}{1} = \underline{4} \Rightarrow \lim_{x\to 0^+} y = \underline{e^4}$$

Ex.: $\lim_{x\to 0^+} (1+x)^{1/x} = (1^\infty)$

$$\underbrace{\lim_{x\to 0^+} \ln y} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \left(\frac{0}{0}\right) \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

$$\lim_{x\to 0^+} y = e^1 = e.$$

APPLICATIONS OF DERIVATIVES

Extreme Values of Functions on Closed Intervals

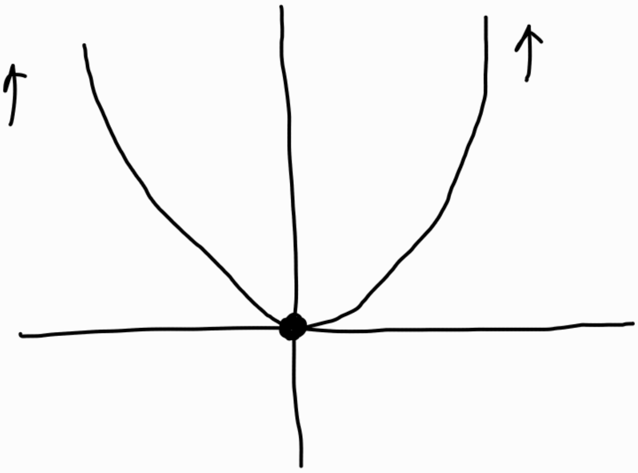
Definition: Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c), \quad \forall x \in D$$

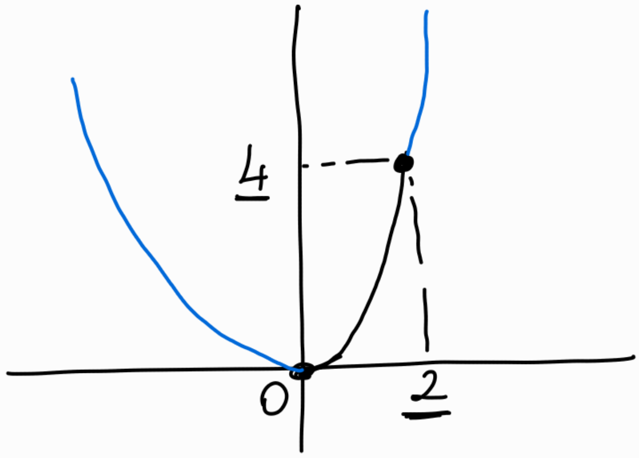
and an absolute minimum value on D at c if

$$f(x) \geq f(c), \quad \forall x \in D.$$

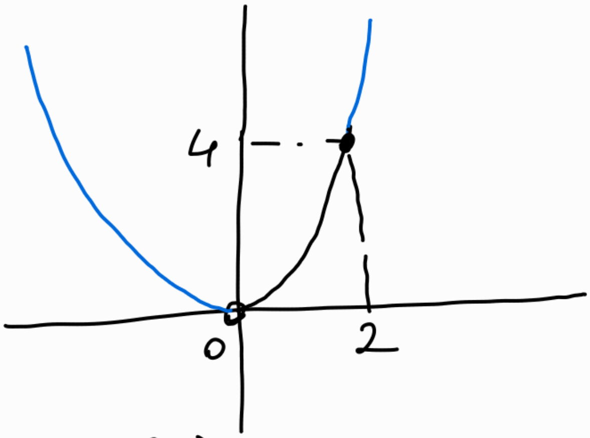
Function rule	Domain	Absolute Extrema on D
(a) $y=x^2$	$(-\infty,\infty)$	No absolute max. Abs. min of 0 at $x=0$
(b) $y=x^2$	$[0,2]$	Abs. max. of 4 at $x=2$ Abs. min of 0 at $x=0$
(c) $y=x^2$	$(0,2]$	Abs. max. of 4 at $x=2$ No abs. min.
(d) $y=x^2$	$(0,2)$	No absolute extrema.



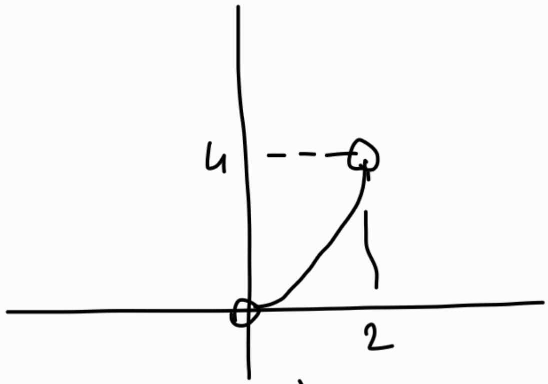
(a)



(b)

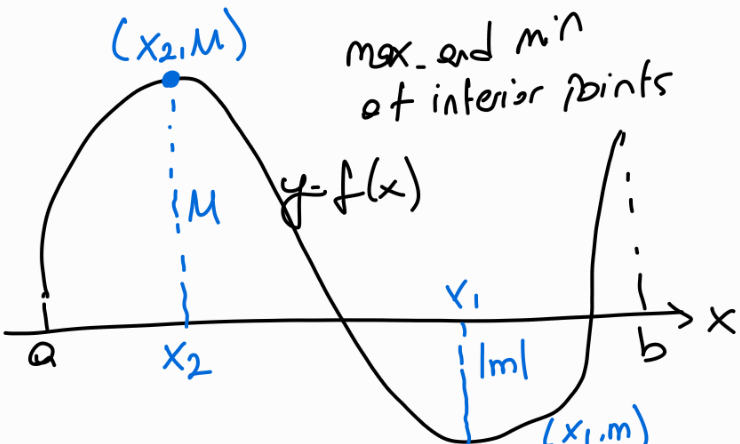


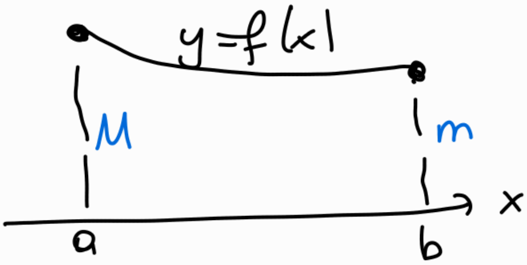
(c)



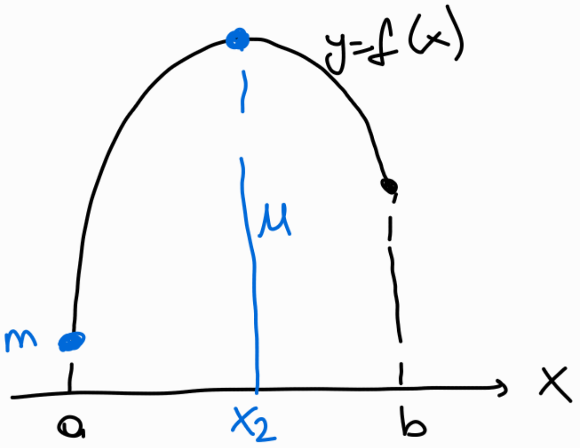
(d)

Theorem (The Extreme Value Theorem): If f is continuous on a closed interval $[a,b]$, then f attains both an absolute max. value M and abs. min. value m in $[a,b]$. That is there are numbers x_1 and x_2 in $[a,b]$ with $f(x_1)=m$, $f(x_2)=M$, and $m \le f(x) \le M$ for all x in $[a,b]$.

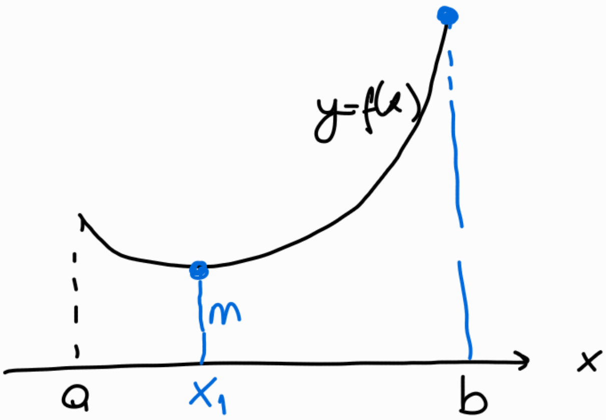




max. and min. at endpoints.



max. at interior point,
min. at endpoint

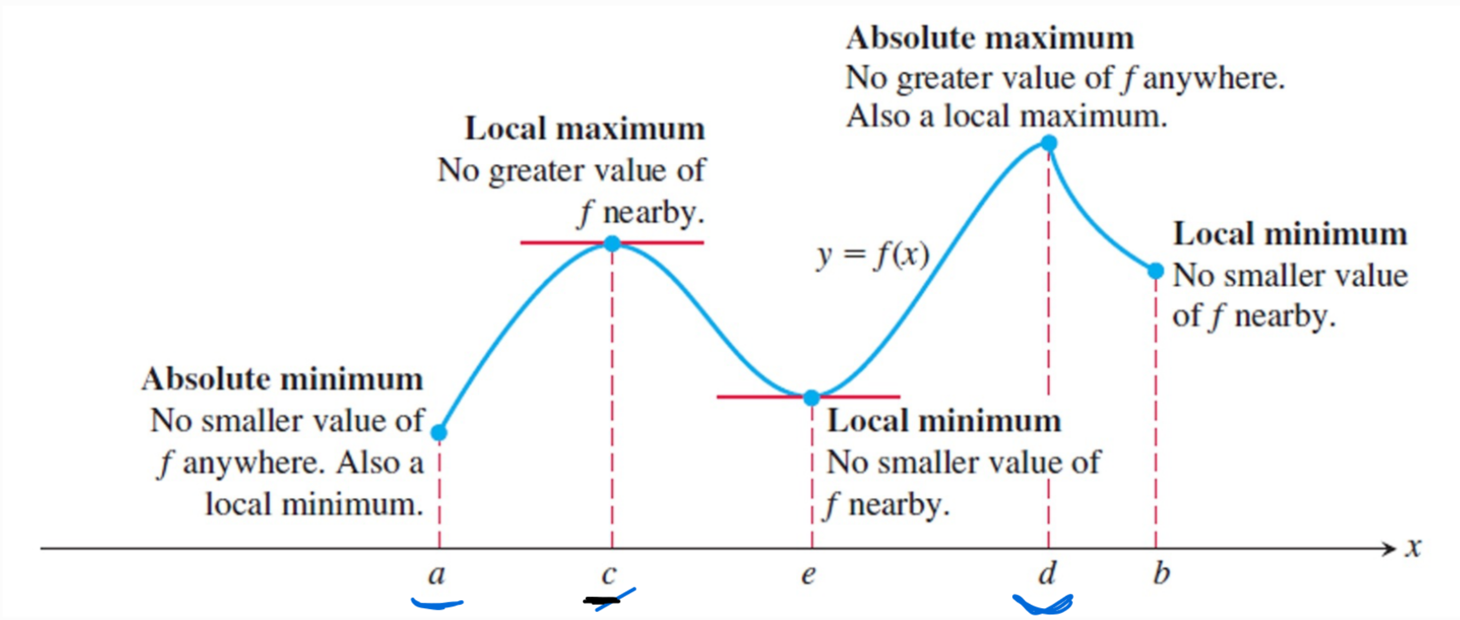


mini. at interior point.
max. at endpoint.

Local (Relative) Extreme Values

<u>Definition:</u> A function f has a local max. value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a local min. value at c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.



Finding Extrema

Theorem (The 1st Derivative Theorem for Local Extreme Values): If f has a local max. or min. value at an interior point c of its domain, and if f' is defined at c, then $f'(c)=0$.

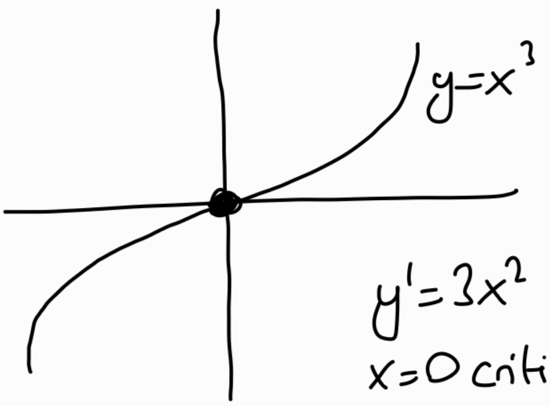
The only places where a function f can possibly have an extreme value are

1) interior points where $f'=0$,
2) interior points where f' is undefined,
3) endpoints of an interval in the domain of f.

Definition: An interior point of the domain of a function f where f' is zero or undefined is a critical point of.

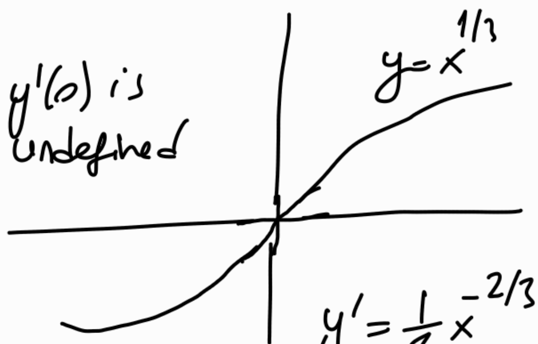
Thus the only domain points where a function can assume extreme values are critical points and endpoints.

Remark: A function might not have a local max. or min. at a critical point.



$y'=3x^2$ $(y'(0)=0)$

$x=0$ critical (but no local ext.)



$y'=\frac{1}{3}x^{-2/3}$

$x=0$ is critical

Finding the Abs. Ext. of a Cont. Funct. f on a Finite Closed Interval

1) Find all critical points of f on the interval
2) Evaluate f at all critical points and endpoints
3) Take the largest and smallest of those values.

Ex.1 Find the abs. max. and min. values of

$f(x) = 10x(2-\ln x)$ on the interval $[1, e^2]$

$$f'(x) = 10(2-\ln x) - 10x \cdot \frac{1}{x} = 10(1-\ln x)$$

Critical points: $f'(x)=0 \Rightarrow 10(1-\ln x)=0$

$$1-\ln x = 0$$

$$\ln x = 1 \Rightarrow x = e$$

$f(e) = 10e \rightarrow$ abs. max.

$f(1) = 20 \rightarrow$ local ext. (don't know max or min)

$f(e^2) = 0 \rightarrow$ abs. min.

Ex.: $f(x) = x^{2/3}$ on $[-2, 3]$

$f'(x) = \frac{2}{3} x^{-1/3} \rightarrow \frac{2}{3\sqrt[3]{x}}$ $\quad x=0 \rightarrow$ critical point (undefined)

$f(0) = 0 \rightarrow$ abs. min

$f(-2) = 4^{1/3} \rightarrow$ local ext.

$f(3) = 9^{1/3} \rightarrow$ abs. max.