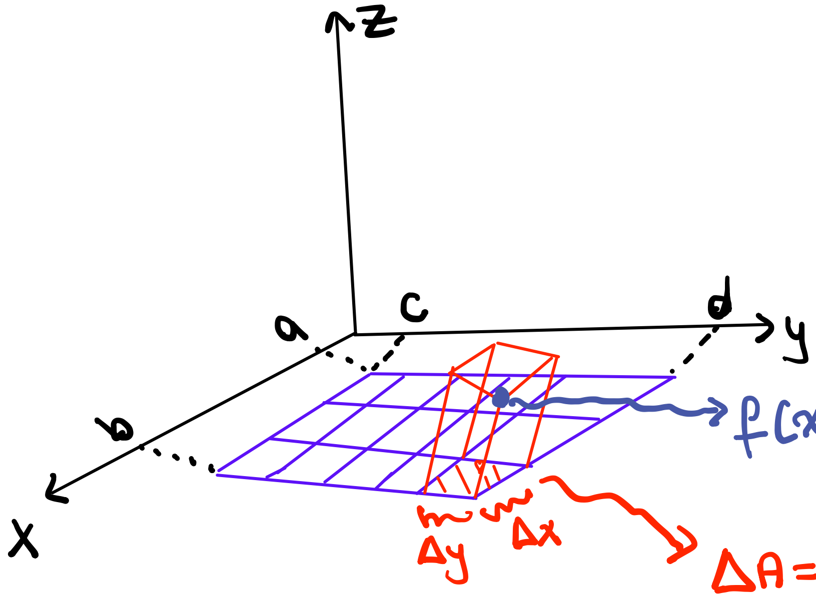
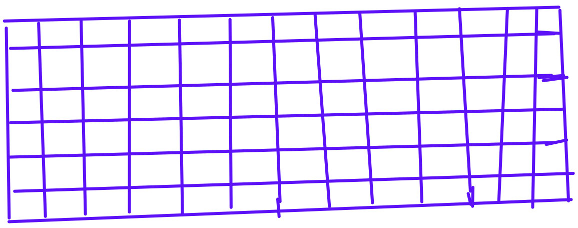
Multiple Integrals

Double and Iterated Integrals over Rectangles



Let $f(x_k, y_k) = z_k$

$\Delta A = \Delta x . \Delta y$ (Area of the base rectangle)



- Let n be the number of small rectangles
- If the number of rectangles is increased too much ($n \to \infty$), then the area of these rectangles approaches zero.

(Volume) $\Delta V =$ Area of the base . height

$$\Delta V = \Delta A \cdot f(x_k, y_k)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i) . \Delta A$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta x_i \Delta y_i$$

$$= \iint_D f(x,y)\, dA$$ where $D = \{(x,y) : a \le x \le b \text{ and } c \le y \le d\}$

D: Region of the integration

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example: Let $D=\{(x,y): 1 \leq x \leq 2 \text{ and } 2 \leq y \leq 3\}$.

$$\iint_D (3x^2+2y)\,dA = ?$$

Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*
$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

   (a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

   (b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

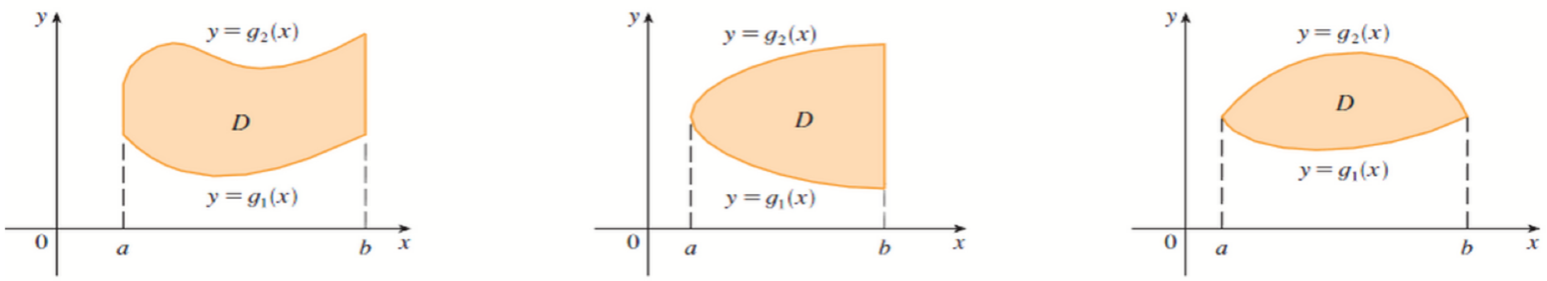
   if R is the union of two nonoverlapping regions R_1 and R_2

DOUBLE INTEGRALS OVER GENERAL REGIONS

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.



3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

(•Calculating double integrals using perpendicular cross-sections)

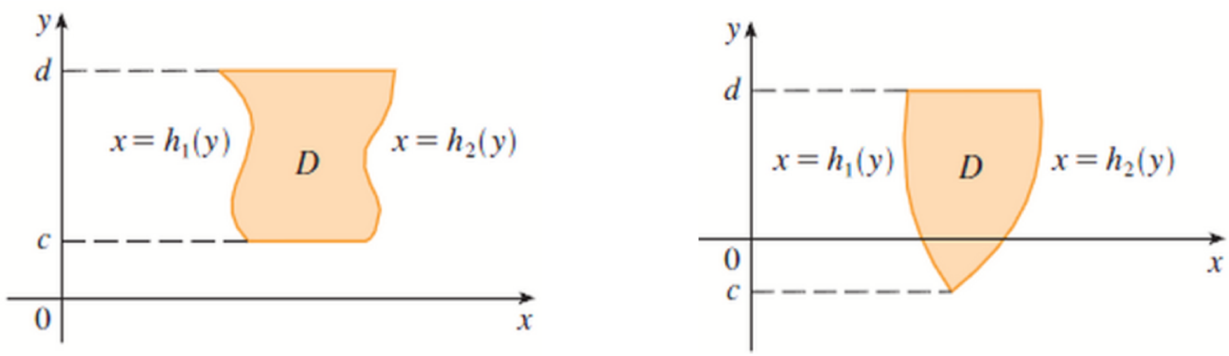
Example : $D=\{(x,y): 1\le x\le 3,\ x\le y\le 2\sqrt{x}\}$

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.



Using the same methods that were used in establishing (3), we can show that

5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

(1. Calculating double integrals using horizontal cross-sections)

where D is a type II region given by Equation 4.

V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example 2: Let D: $x=0$, $x=2$ and
and
$y=0$, $y=x^2$

$$I=\iint_D y\,dx\,dy.$$

a) What is the value of I according to type 1?
b) What is the value of I according to type 2?

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

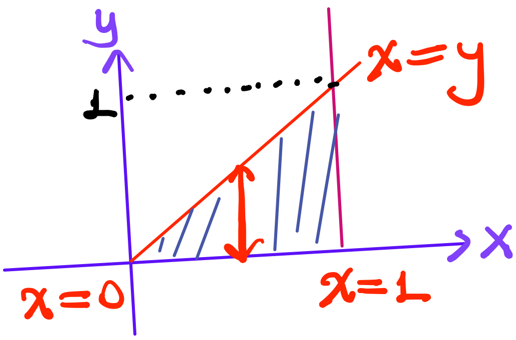
$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example: Let $f(x,y) = 1 - 6x^2y$ and $D: 0 \leq x \leq 2$, $-1 \leq y \leq 1$. Calculate the integral $\iint_D f(x,y)\, dA$.

* Changing the order of integration in double integrals *

• Example: $I = \int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$

- Here, it is difficult to integrate with respect to x. Therefore, since the function does not include the variable y, it becomes easier to integrate when you change the order of integration.



- D : $y=0$ $\quad x=y$
  $y=1$ $\quad x=1$

- If we look from bottom to top and left to right in this shaded area, then:

  $x=0$ to $x=1$ $\qquad y=0$ to $y=x$

- The integral borders are determined by looking from bottom to top for y and from left to right for x!

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sin(x^2)\,dy\,dx = \int_0^1 \left(y\sin(x^2)\right)\Big|_0^x dx$$

$$= \int_0^1 x\sin(x^2)\,dx$$

If we say $x^2 = u$ and $x=1 \Rightarrow u = 1^2 = 1$, then

$x = 0 \Rightarrow u = 0^2 = 0$

$2x\,dx = du$

$x\,dx = \frac{1}{2}du$

$$I = \int_0^1 \sin u \cdot \frac{1}{2}du = \frac{1}{2}\int_0^1 \sin u\,du$$

$$= \left[-\frac{1}{2}\cos u\right]\Big|_0^1$$

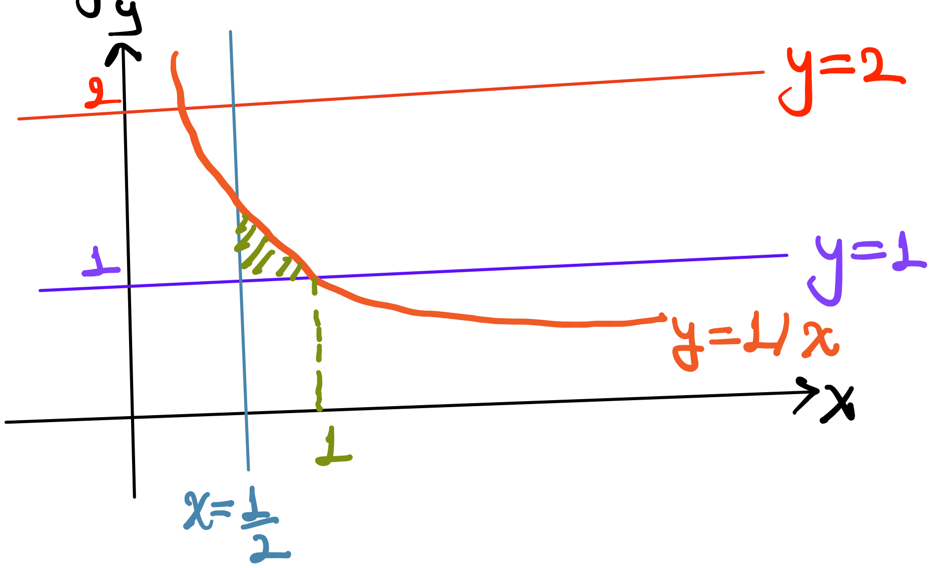
$$= -\frac{1}{2}\left[\cos 1 - \underbrace{\cos 0}_{=1}\right]$$

$$= \frac{1}{2}\left[1 - \cos 1\right]$$

Example: $I = \int_{1}^{2} \int_{1/2}^{1/y} \left(e^{\ln x - x}\right) dx\, dy = ?$

- Because of the difficulty of integration here, we must change the order of integration.

D: $y = 1$ $\quad x = 1/2$
$y = 2$ $\quad x = 1/y$



$\uparrow \begin{matrix} y = 1/x \\ y = 1 \end{matrix}$ $\qquad x = \frac{1}{2} \longrightarrow x = 1$

$$I = \int_{1/2}^{1} \int_{1}^{1/x} \left(e^{\ln x - x}\right) dy\, dx = \int_{1/2}^{1} \left(y . e^{\ln x - x}\right) \Big|_{1}^{1/x} dx$$

$$= \int_{1/2}^{1} \left(\frac{1}{x} - 1\right) e^{\ln x - x} dx$$

- If we say $\ln x - x = u$, then $\left(\frac{1}{x} - 1\right) dx = du$

$$I = \int e^u \, du = e^u = e^{\ln x - x} \Big|_{1/2}^{1}$$

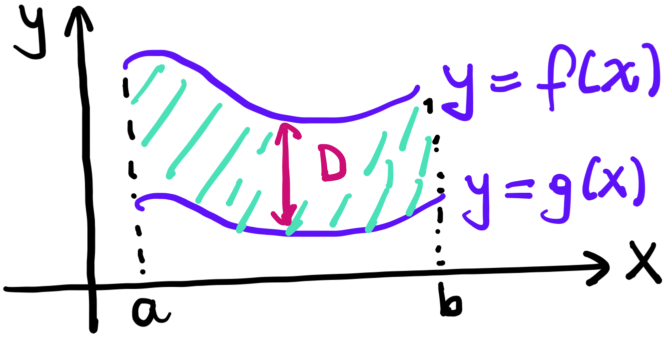
$$= e^{\ln 1 - 1} - e^{\ln\left(\frac{1}{2}\right) - \frac{1}{2}}$$

$$= e^{-1} - \frac{e^{\ln\left(\frac{1}{2}\right)}}{e^{1/2}}$$

$$= \frac{1}{e} - \frac{1/2}{\sqrt{e}}$$

$$= \frac{1}{e} - \frac{1}{2\sqrt{e}}$$

Calculating Area with Double Integral

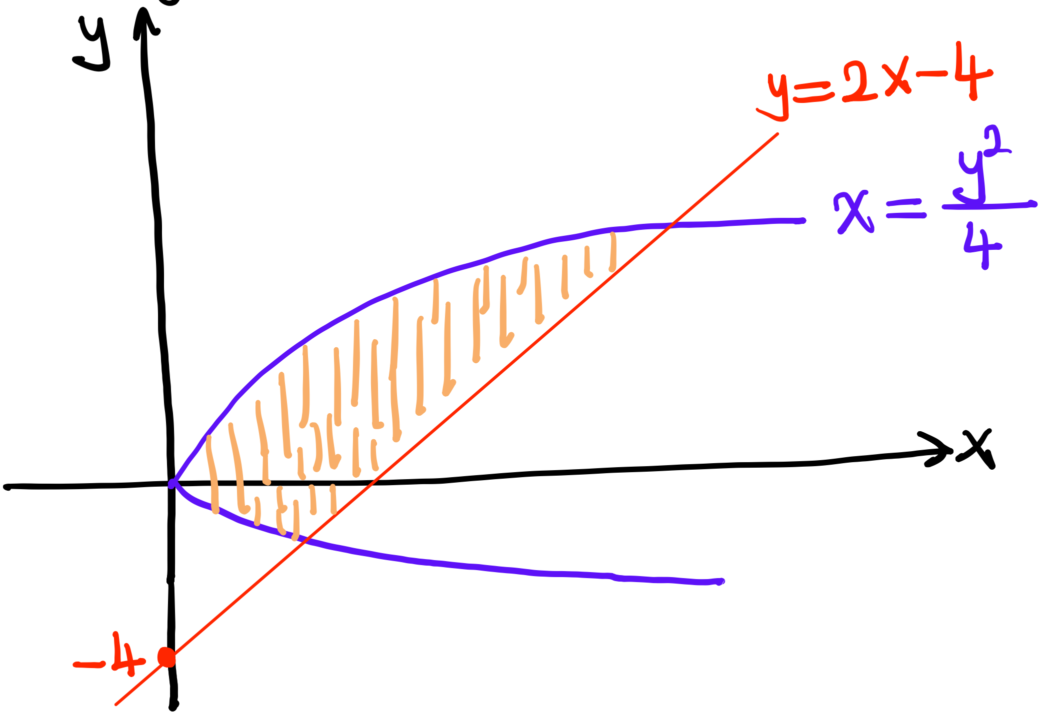


$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx = \int_a^b \int_{g(x)}^{f(x)} 1 \cdot dy\,dx$$

- If $f(x,y) = 1$ in the $\iint_D f(x,y)\,dx\,dy$, then

$$\boxed{\text{Area} = \iint_D dx\,dy}$$

Example: Calculate the area of the region between the curve $y^2 = 4x$ and the line $y = 2x - 4$ using the double integral.



Calculating Volume with Double Integral

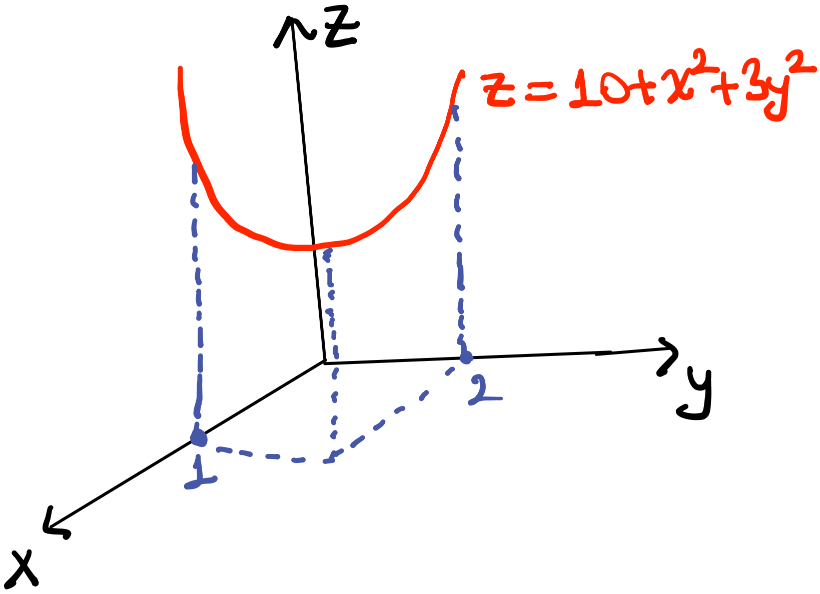
① Let $f(x,y)$ be a positive function on a region D. In this case, the volume of the solid formed by bounded by $z=f(x,y)$ from above and $z=0$ from below on the region D is as follows:

$$V=\iint_D f(x,y)\,dx\,dy$$

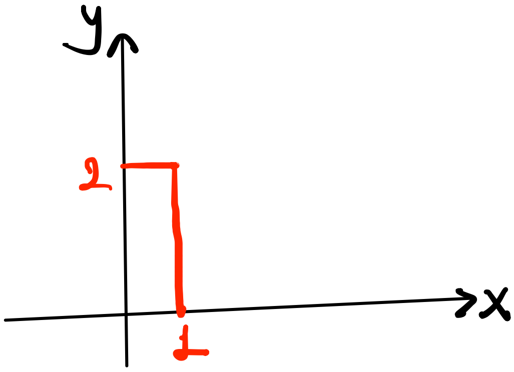
② If the solid is bounded by the surfaces $z=f_1(x,y)$ from above and $z=f_2(x,y)$ from below; if the projection of these surfaces on the xy-plane is the region D, the resulting volume is as follows:

$$V=\iint_D [f_1(x,y)-f_2(x,y)]\,dx\,dy$$

<u>Example</u>: Find the volume of the region bounded by the paraboloid $z=10+x^2+3y^2$ above and the rectangle $D:\begin{cases}0\le x\le 1\\ 0\le y\le 2\end{cases}$ below.



z
$z = 10 + x^2 + 3y^2$
y
x
1
2



y
x
2
1

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1. \end{aligned}$$

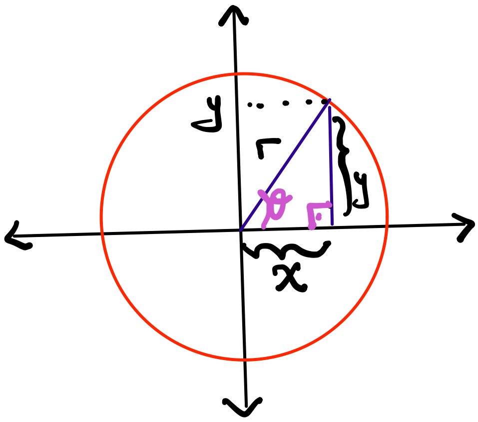
When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$\begin{aligned} V &= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\ &= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\ &= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1. \end{aligned}$$

The two integrals are equal, as they should be. ■

Double Integrals in Polar Form

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$



- $\sin\theta = \frac{y}{r} \Rightarrow$

  $y = r\sin\theta$

- $\cos\theta = \frac{x}{r} \Rightarrow$

  $x = r\cos\theta$

- $r^2 = x^2 + y^2$

2 **CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL** If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_D dx\,dy = A = \iint_R r\,dr\,d\theta = \int_{\alpha}^{\beta}\int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

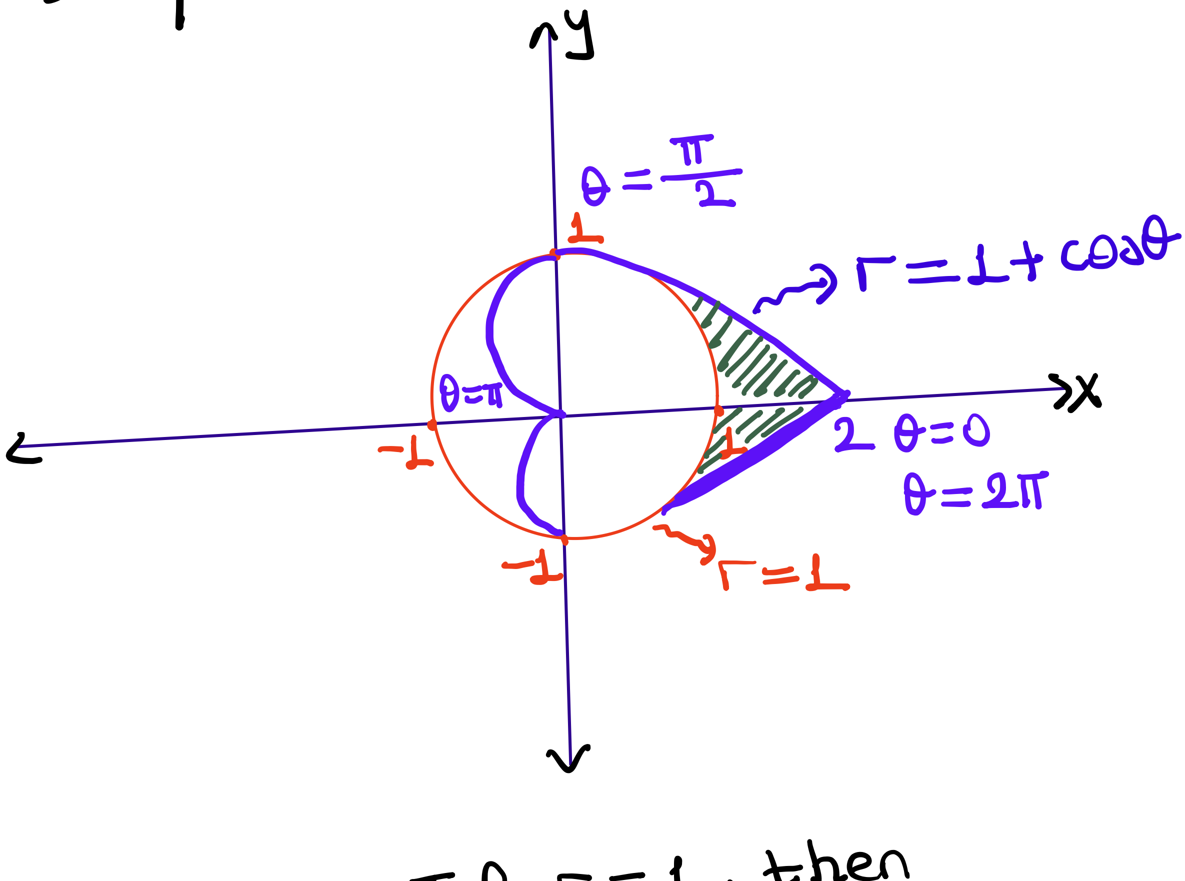
Example: $I = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = ?$

Example : Calculate the area of the region D inside $r = 1 + \cos\theta$ and outside $r = 1$ using the double integral.

-

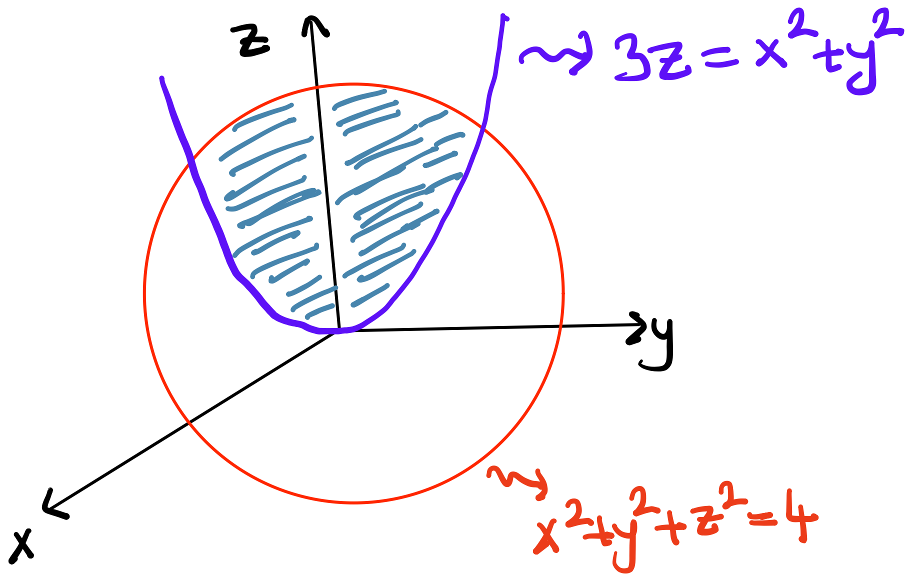
θ	$r = 1 + \cos\theta$
0	2
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	1
2π	2

- $r = 1 \Rightarrow$
  $r^2 = 1 \Rightarrow$
  $r^2 = x^2 + y^2$
  $x^2 + y^2 = 1$



- **Intersection :** If $r = 1$, then
  $1 = 1 + \cos\theta \Rightarrow \cos\theta = 0$
  $\Rightarrow \theta_1 = \pi/2$ and
  $\theta_2 = -\pi/2$

Example: Find the volume of the region bounded by the $3z = x^2+y^2$ above and the $x^2+y^2+z^2=4$ below.



• **Intersection**: $\underbrace{x^2+y^2}_{=3z}+z^2=4 \Rightarrow z^2+3z-4=0$

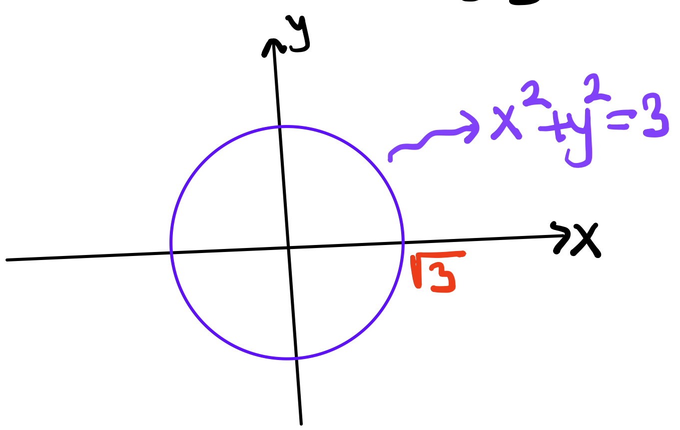
$\Rightarrow (z+4)(z-1)=0$

$\Rightarrow z=1 \ (z>0)$

$\Rightarrow x^2+y^2+1=4$

$\Rightarrow \boxed{x^2+y^2=3}$

Projection region D



DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{1}$$

The Jacobian can also be denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

• Note : $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$

THEOREM 3—Substitution for Double Integrals Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \tag{2}$$

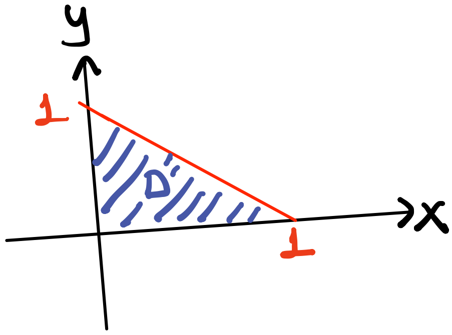
• $\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} \Rightarrow \begin{aligned} &D \longrightarrow D^* \\ &dxdy \longrightarrow |J(u,v)|\,du\,dv \end{aligned}$

EXAMPLE 3 Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

Let $\left.\begin{aligned} u &= x+y \\ v &= y-2x \end{aligned}\right\} \Rightarrow x = \dfrac{u-v}{3}$ and $y = \dfrac{2u}{3} + \dfrac{v}{3}$

$D:\; x=0 \quad x=1$
$\quad\;\; y=0 \quad y=1-x$

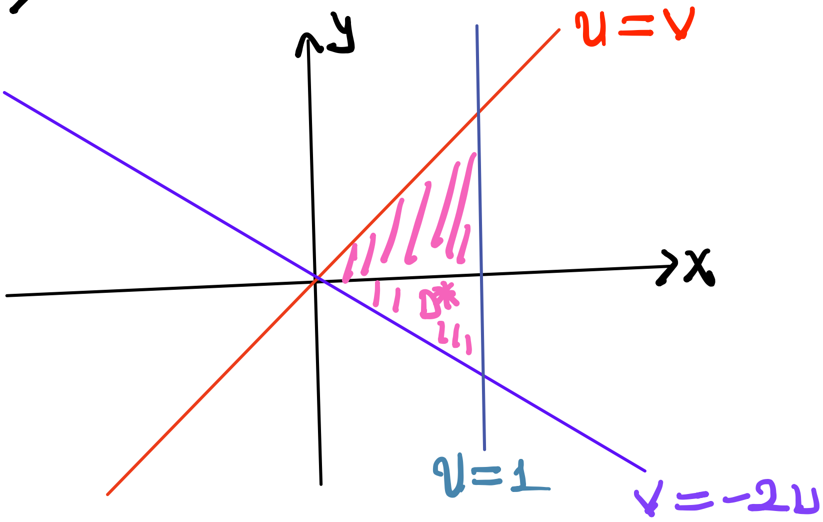


$\underline{D}$ $\qquad\qquad\qquad\qquad$ $\underline{D^*}$

$y=1-x \Rightarrow x+y=1 \Rightarrow \boxed{u=1}$

$x=0 \Rightarrow u-v=0 \Rightarrow \boxed{u=v}$

$y=0 \Rightarrow 2u+v=0 \Rightarrow \boxed{v=-2u}$



$$\textbf{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \qquad (3)$$

Example: Find the average value of the function $f(x,y) = x^2 + y^2$ in a right triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$.