1) Let D be the region enclosed by the lines $x-y=2$, $x-y=4$, $x+y=1$ and $x+y=2$. Which of the following integrals will we obtain when we rewrite the integral $\iint_D (x^2-y^2)\,dx\,dy$ by using the transformation $u=x-y$, $v=x+y$?

$= (x-y)(x+y) = uv$

A) $\int_1^2\int_2^4 \frac{uv}{2}\,du\,dv$ B) $\int_2^4\int_1^2 uv\,du\,dv$ C) $\int_1^2\int_2^4 uv\,du\,dv$ D) $\int_2^4\int_1^2 \frac{uv}{2}\,du\,dv$ E) $\int_2^4\int_1^2 u^2v^2\,du\,dv$

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\left.\begin{array}{l} u = x-y \\ v = x+y \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{1}{2}(u+v) \\ y = -\frac{1}{2}(u-v) \end{array}$$

D	D^*
$x-y=2$	$u=2$
$x-y=4$	$u=4$
$x+y=1$	$v=1$
$x+y=2$	$v=2$

$$J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix}$$

$$= \frac{1}{2}\cdot\frac{1}{2} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

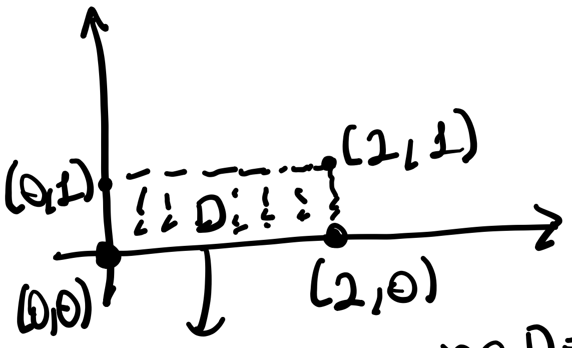
$$\Rightarrow \int_1^2\int_2^4 u\cdot v\cdot \frac{1}{2}\,du\,dv = \int_1^2\int_2^4 \frac{uv}{2}\,du\,dv$$

21) Which of the following is the average value of the function $f(x,y)=xy+2x$ on the rectangular region with vertices $(0,0),(0,1),(2,0)$ and $(2,1)$?

A) 3 B) 2 C) $\frac{5}{2}$ D) $\frac{7}{2}$ E) $\frac{9}{2}$

$$\text{Average value of } f = \frac{1}{\text{Area on the region } D} \cdot \iint_D f(x,y)\,dA$$

$$= \frac{1}{2} \cdot \int_0^1 \int_0^2 (xy+2x)\,dx\,dy$$



Area of region $D = 1 \cdot 2 = 2$

$$= \frac{1}{2} \int_0^1 \left[\frac{x^2 y}{2} + x^2 \Big|_0^2 \right] dy$$

$$= \frac{1}{2} \int_0^1 (2y+4)\,dy$$

$$= \frac{1}{2} \left[y^2 + 4y \Big|_0^1 \right]$$

$$= \frac{5}{2}$$

$\overline{\ 7\ }$

3) What is the arc length of the curve $r(t) = (2t)\vec{i} + \left(\frac{1}{2}t^2\right)\vec{j} + (\ln(t^2))\vec{k}$ in the interval $\underset{a}{1} \le t \le \underset{b}{e}$?

A) $\frac{e^9+11}{2}$ B) $\frac{e^2+3}{2}$ C) $\frac{e^4+7}{2}$ D) $\frac{e^6+11}{2}$ E) $\frac{e+1}{2}$

$$L = \int_a^b |r'(t)|\,dt$$

$$r'(t) = 2\vec{i} + t\cdot\vec{j} + \frac{2}{t}\vec{k} \Rightarrow$$

$$|r'(t)| = \sqrt{2^2 + t^2 + \left(\frac{2}{t}\right)^2} = \sqrt{\frac{t^4+4t^2+4}{t^2}}$$

$$\Rightarrow |r'(t)| = \sqrt{\frac{(t^2+2)^2}{t^2}} \underset{1\le t\le e}{=} \frac{t^2+2}{t} = t + 2\cdot\frac{1}{t}$$

$$L = \int_1^e \left(t + 2\cdot\frac{1}{t}\right)dt = \left[\frac{t^2}{2} + 2\ln t\right]\Bigg|_1^e$$

$$= \left(\frac{e^2}{2} + 2\underbrace{\ln e}_{1}\right) - \left(\frac{1}{2} + 2\underbrace{\ln 1}_{0}\right)$$

$$= \frac{e^2}{2} + 2 - \frac{1}{2} = \frac{e^2+3}{2}$$

4) Which of the following is the form that the $I_1 = \iint_{D_1} xy^3 dA$ integral will take with the transformation x=v/6u, y=2u, where the region D_1 is the region bounded by xy=1, xy=3, y=2 and y=6?

A) $I_1 = \frac{4}{9}\int_3^9 \int_1^3 uv\,du\,dv$ B) $I_1 = \frac{2}{9}\int_6^9 \int_1^2 uv\,du\,dv$ C) $I_1 = \frac{2}{3}\int_3^{12} \int_1^2 uv\,du\,dv$

D) $I_1 = \frac{2}{9}\int_3^9 \int_1^3 uv\,du\,dv$ E) $I_1 = \frac{4}{9}\int_2^4 \int_1^3 uv\,du\,dv$

$$\iint_{D_1} f(x,y)\,dA = \iint_{D_1^*} f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{vmatrix} = -\frac{1}{3u}$$

$$\Rightarrow |J(u,v)| = \frac{1}{3u}$$

$$xy = \frac{v}{6u}\cdot 2u = \frac{v}{3}, \quad y = 2u \Rightarrow$$

D_1		D_1^*
$xy=1$	$\longrightarrow$	$v=3$ $\left(\frac{v}{3}=1\right)$
$xy=3$	$\longrightarrow$	$v=9$ $\left(\frac{v}{3}=3\right)$
$y=2$	$\longrightarrow$	$u=1$ $(2u=2)$
$y=6$	$\longrightarrow$	$u=3$ $(2u=6)$

and $xy^3 = \frac{v}{6u}\cdot(2u)^3$

$= \frac{v}{6u}\cdot 8u^3$

$= \frac{4}{3}u^2 v$

$$\Rightarrow I_1 = \int_3^9 \int_1^3 \frac{4}{3} u^2 v \cdot \frac{1}{3u} \, du \, dv$$

$$= \frac{4}{9} \int_3^9 \int_1^3 u v \, du \, dv$$

5) Which of the following is the integral $\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \left(x^2+y^2\right) dy\,dx$ written in polar coordinates ?

A) $\int_{\frac{3\pi}{2}}^{2\pi} \int_0^1 r^3\,dr\,d\theta$ B) $\int_{\pi}^{\frac{3\pi}{2}} \int_0^1 r^3\,dr\,d\theta$ C) $\int_{\frac{\pi}{2}}^{\pi} \int_0^1 r^3\,dr\,d\theta$ D) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3\,dr\,d\theta$ E) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^3\,dr\,d\theta$

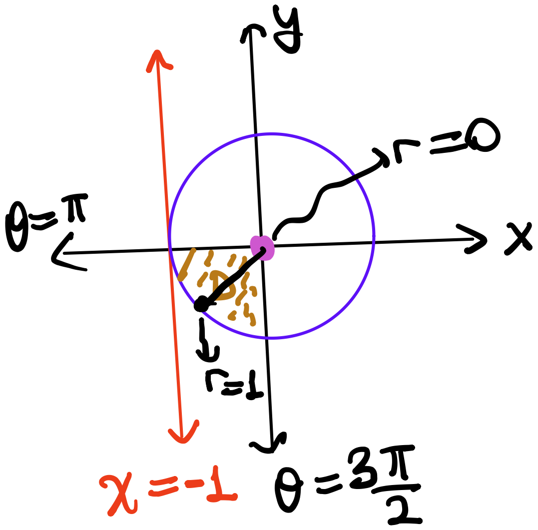
$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$x^2+y^2=r^2$$

D

$x=0 \quad y=0$

$x=-1 \quad y=-\sqrt{1-x^2} \Rightarrow y^2=1-x^2 \Rightarrow x^2+y^2=1$



$$\rightarrow \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 r^2 \cdot r.\,dr.\,d\theta$$

$$= \int_{\pi}^{3\pi/2} \int_0^1 r^3\,dr\,d\theta$$

6) Let region $D=\{(x,y)\in\mathbb{R}^2 : 1\leq x^2+y^2\leq e, y\leq 0\}$.
Which of the following integral gives the volume of the solid bounded by the $z=\dfrac{\ln(x^2+y^2)}{x^2+y^2}$ -surface above and the region D in the xy-plane below?

A) $\int_{\pi}^{2\pi}\int_{0}^{e}\frac{2\ln r}{r}\,dr\,d\theta$

B) $\int_{\pi}^{2\pi}\int_{1}^{e}\frac{2\ln r}{r^2}\,dr\,d\theta$

C) $\int_{\pi}^{2\pi}\int_{1}^{e}\frac{2\ln r}{r}\,dr\,d\theta$

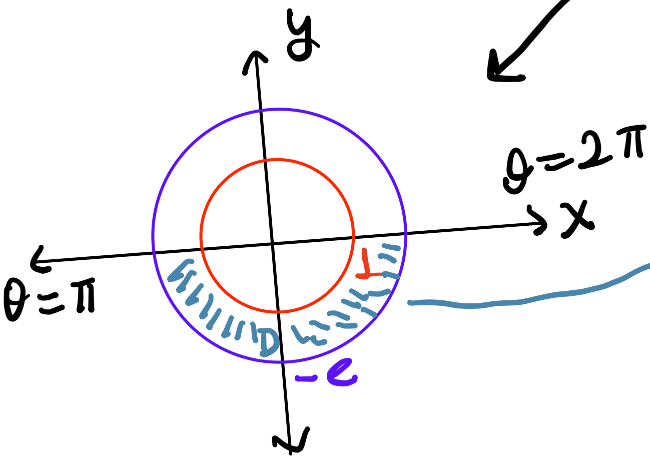
D) $\int_{\pi}^{2\pi}\int_{1}^{e}\frac{2\ln r}{r}\,d\theta\,dr$

E) $\int_{1}^{e}\int_{\pi}^{2\pi}\frac{2\ln r}{r}\,dr\,d\theta$

$x^2+y^2=r^2$

volume $=\iint_D z\,dA$

D
$1\leq x^2+y^2$
$x^2+y^2\leq e$
$y\leq 0$



$$\text{volume}=\int_{\pi}^{2\pi}\int_{1}^{e}\frac{\ln(r^2)}{r^2}\,r\,dr\,d\theta$$

$$=\int_{\pi}^{2\pi}\int_{1}^{e}\frac{2\ln r}{r}\,dr\,d\theta$$

7) Which of the following is equal to the integral $\int_0^2 \int_0^{\sqrt{2y-y^2}} \frac{xy}{x^2+y^2}\,dx\,dy$?

A) $\int_0^{\frac{\pi}{2}} \int_0^{2\sin\theta} \sin\theta\cos\theta\, r\,dr\,d\theta$ B) $\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sin\theta\cos\theta\, r\,dr\,d\theta$ C) $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} \tan\theta\, r\,dr\,d\theta$

D) $\int_0^{\frac{\pi}{2}} \int_0^{2\sin\theta} \sin\theta\cos\theta\,dr\,d\theta$ E) $\int_0^{2\pi} \int_{\cos\theta}^{\sin\theta} \sin\theta\cos\theta\, r\,dr\,d\theta$

D

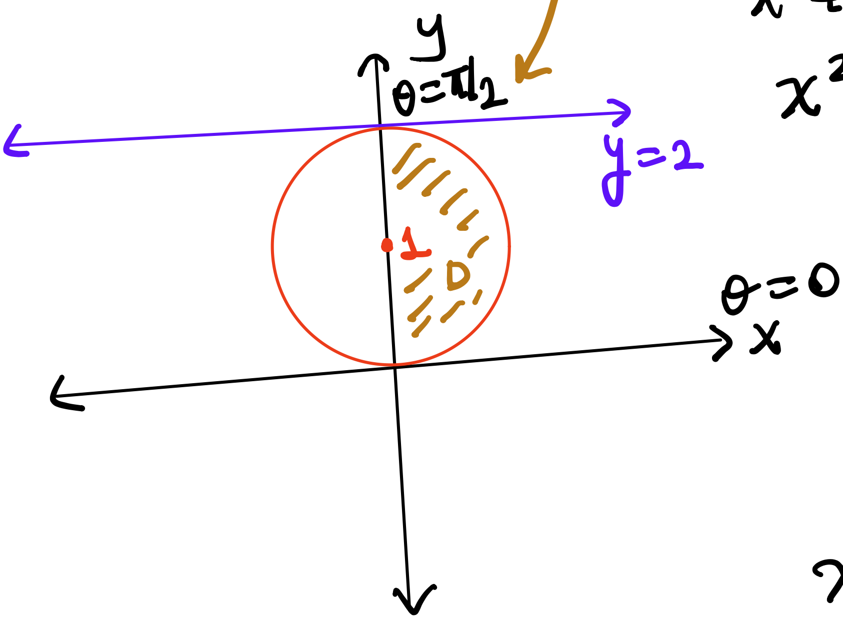
$y=0 \qquad x=0$

$y=2 \qquad x=\sqrt{2y-y^2} \Rightarrow x^2=2y-y^2 \Rightarrow x^2+(y-1)^2=1$

$x^2+y^2-2y=0$

$x^2+y^2-2y+1-1=0$

$x^2+y^2-2y+1=1$



$x=r\cos\theta$

$y=r\sin\theta$

$x^2+y^2=r^2$

$$\frac{xy}{x^2+y^2}=\frac{r^2\cos\theta\sin\theta}{r^2}$$

$$=\cos\theta\sin\theta$$

$$x = \sqrt{2y - y^2} \Rightarrow x^2 = 2y - y^2$$

$$\Rightarrow r^2\cos^2\theta = 2r\sin\theta - r^2\sin^2\theta$$

$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0$$

$$\Rightarrow r^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) - 2r\sin\theta = 0$$

$$\Rightarrow r(r - 2\sin\theta) = 0$$

$$\Rightarrow r = 0 \quad \text{or} \quad r = 2\sin\theta$$

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\sin\theta} \cos\theta \sin\theta \, r\,dr\,d\theta$$

8) In which direction is the derivative of the function $f(x,y)=xy$ at the point $(0,3)$ equal to -1?

A) $\vec{u}=\frac{2\sqrt{2}}{3}\vec{i}\pm\frac{1}{3}\vec{j}$ B) $\vec{u}=-\frac{1}{3}\vec{i}\pm\frac{1}{3}\vec{j}$ C) $\vec{u}=\frac{1}{3}\vec{i}\pm\frac{2\sqrt{2}}{3}\vec{j}$ D) $\vec{u}=-\frac{1}{3}\vec{i}\pm\frac{2\sqrt{2}}{3}\vec{j}$ E) $\vec{u}=\pm\frac{2\sqrt{2}}{3}\vec{i}+\frac{1}{3}\vec{j}$

$$\vec{\nabla} f\big|_{(0,3)} = f_x\big|_{(0,3)}\vec{i} + f_y\big|_{(0,3)}\vec{j}$$

$$= (y)\big|_{(0,3)}\vec{i} + x\big|_{(0,3)}\vec{j}$$

$$= 3\vec{i} + 0\cdot\vec{j} = \langle 3,0\rangle$$

$$\boxed{(D_u f)_p = \vec{\nabla} f\big|_{(0,3)}\cdot\vec{u}}$$

Let $\vec{u}=u_1\vec{i}+u_2\vec{j}$
$=\langle u_1,u_2\rangle$

unit

$|u|=\sqrt{u_1^2+u_2^2}=1$

$\Rightarrow u_1^2+u_2^2=1$

$$-1=\langle 3,0\rangle\langle u_1,u_2\rangle \Rightarrow$$

$$3u_1=-1 \Rightarrow u_1=-\frac{1}{3}$$

$$\left(-\frac{1}{3}\right)^2+u_2^2=1 \Rightarrow$$

$$u_2^2=\frac{8}{9} \Rightarrow u_2=\pm\frac{2\sqrt{2}}{3}$$

$$\Rightarrow u=-\frac{1}{3}\vec{i}\pm\frac{2\sqrt{2}}{3}\vec{j}$$

9) Let $\vec{F}(t) = \left(\frac{t^2+3t-10}{t^2-t-2}, \sin\left(\frac{\pi}{2}t\right), \frac{\sin(t-2)}{t-2} \right)$.

What is the value of $\lim_{t\to 2} \vec{F}(t)$?

A) $\left(\frac{7}{3}, 0, 1\right)$ B) $\left(\frac{7}{4}, 0, 1\right)$ C) $\left(\frac{7}{2}, 1, 1\right)$

D) $\left(-\frac{7}{3}, 0, 1\right)$ E) $\left(\frac{7}{3}, 1, 1\right)$

$$\lim_{t\to 2} \frac{(t+5)\cancel{(t-2)}}{(t+1)\cancel{(t-2)}} = \frac{2+5}{2+1} = \frac{7}{3}$$

$$\lim_{t\to 2} \sin\left(\frac{\pi}{2}t\right) = \sin\left(\frac{\pi}{2}\cdot 2\right) = \sin(\pi) = 0$$

$$\lim_{t\to 2} \frac{\sin(t-2)}{t-2} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$(t-2=\theta \Rightarrow$
$t\to 2,\ \theta\to 0)$

$$\Rightarrow \lim_{t\to 2} \vec{F}(t) = \left(\frac{7}{3}, 0, 1\right)$$

10) Let $f(x,y)=\frac{1}{x}$. What is the average value of the f on the region $0\leq x\leq 1$, $x^2\leq y\leq \sqrt{x}$?

A) $\frac{9}{4}$ B) $\frac{9}{2}$ C) $\frac{9}{5}$ D) $\frac{3}{4}$ E) $\frac{3}{2}$

Average value of f $= \dfrac{1}{\text{Area of the region } D} \cdot \iint\limits_D f(x,y)\,dA$

Type 1 region

dydx

$0\leq x\leq 1$

$x^2\leq y\leq \sqrt{x}$

Area of the region D $=\iint\limits_D 1\cdot dy\,dx$

$$=\int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx$$

$$=\int_0^1 y\Big|_{x^2}^{\sqrt{x}} dx$$

$$=\int_0^1 (x^{1/2}-x^2)\,dx$$

$$=\frac{2}{3}x^{3/2}-\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$$

$$A.V = \frac{1}{1/3} \int_0^1 \int_{x^2}^{\sqrt{x}} \frac{1}{x}\, dy\, dx$$

$$= 3 \cdot \int_0^1 \left(\frac{y}{x} \Big|_{x^2}^{\sqrt{x}} \right) dx$$

$$= 3 \cdot \int_0^1 (x^{-1/2} - x)\, dx$$

$$= 3 \left[2\sqrt{x} - \frac{x^2}{2} \Big|_0^1 \right] = 3 \cdot \left[2 - \frac{1}{2} \right]$$

$$= 3 \cdot \frac{3}{2}$$

$$= \frac{\frac{9}{2}}{7}$$

11) Which of the following is the equation of the tangent line to the $\vec{r}(t) = (\sin t, \cos t, t)$ at the point $(0,1,0)$?

A) $\vec{l}(t) = (t, 1, t)$ B) $\vec{l}(t) = (t, -1, t)$

C) $\vec{l}(t) = (2t, 1, t)$ D) $\vec{l}(t) = (1, t, t)$

E) $\vec{l}(t) = (t, t, 1)$

$$\vec{l}(t) = (x_0, y_0, t_0) + t \cdot \text{slope}\big|_{(x_0, y_0, t_0)}$$

$(x_0, y_0, t_0) = (0, 1, 0) \Rightarrow \sin t = 0, \cos t = 1, t = 0$

$\text{slope} = r'(t) = (\cos t, -\sin t, 1) \Rightarrow$

$r'(0) = (\cos 0, -\sin 0, 1) = (1, 0, 1)$

$$\vec{l}(t) = (0, 1, 0) + t(1, 0, 1)$$
$$= (0, 1, 0) + (t, 0, t)$$
$$= (0 + t, 1 + 0, 0 + t)$$
$$= (t, 1, t)$$