Example 2 : Let D: $x=0$, $x=2$ and $y=0$, $y=x^2$ and

$$I=\iint_D y\,dx\,dy .$$

a) What is the value of I according to type 1?

b) What is the value of I according to type 2?

a) Type 1 (orthogonal cross-section)

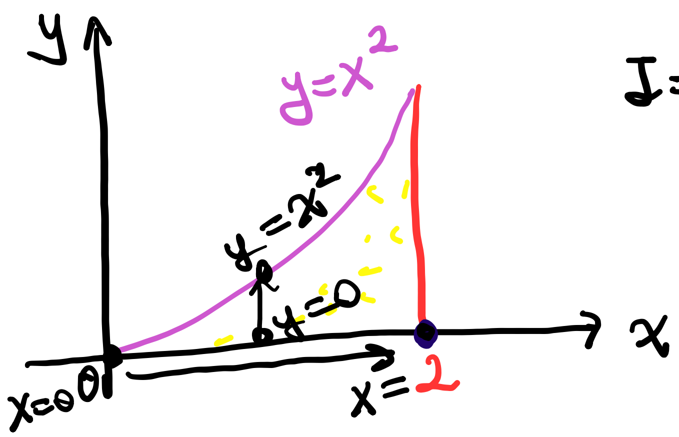
$D=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Region D

$x=0$ $y=0$

$x=2$ $y=x^2$



$$I=\int_0^2\int_0^{x^2} y\,dy\,dx$$

$$=\int_0^2\left(\frac{y^2}{2}\Big|_0^{x^2}\right)dx$$

x

$x=0 = a$ $\longrightarrow$ $x=2 = b$

y

$g_2(x)=x^2$

$g_1(x)=0$

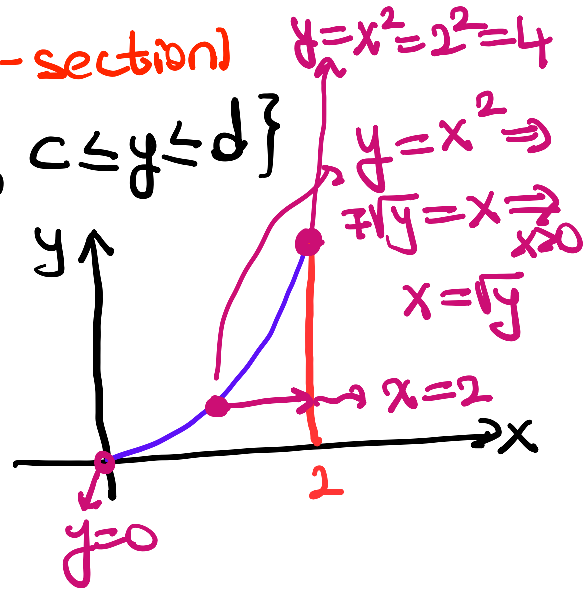
$$=\int_0^2 \frac{x^4}{2}\,dx$$

$$=\frac{1}{2}\cdot\frac{x^5}{5}\Big|_0^2$$

$$=\frac{16}{5}-0=\frac{16}{5}$$

b) <u>Type 2</u> (Horizontal cross-section)

$$D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$\underline{x}$

$h_1(y) = \sqrt{y}$ $\qquad$ $h_2(y) = 2$

$\left(y = x^2 \underset{x>0}{\Rightarrow} x = \sqrt{y} \right)$

$\underline{y}$

$y = 4$ $\left(x = 2 \Rightarrow y = 2^2 = 4\right)$

$y = x^2$

$y = 0$

$c = 0$ $\qquad$ $d = 4$

$$I = \int_0^4 \int_{\sqrt{y}}^2 y\,dx\,dy = \int_0^4 \left(xy \Big|_{\sqrt{y}}^{2} \right) dy$$

$$= \int_0^4 \left(2y - y^{3/2} \right) dy$$

$$= \left[y^2 - \frac{2}{5} y^{5/2} \right] \Big|_0^4$$

$$= \left[16 - \frac{2}{5} \cdot (2^2)^{5/2} \right] - 0$$

$$= 16 - \frac{64}{5} = \frac{16}{5}$$

7

• **Question:** Which of the following integrals is obtained when the integral of $\int_0^3 \int_{x^2}^{3x} f(x,y)\,dy\,dx$ is rewritten by changing the order of integration?

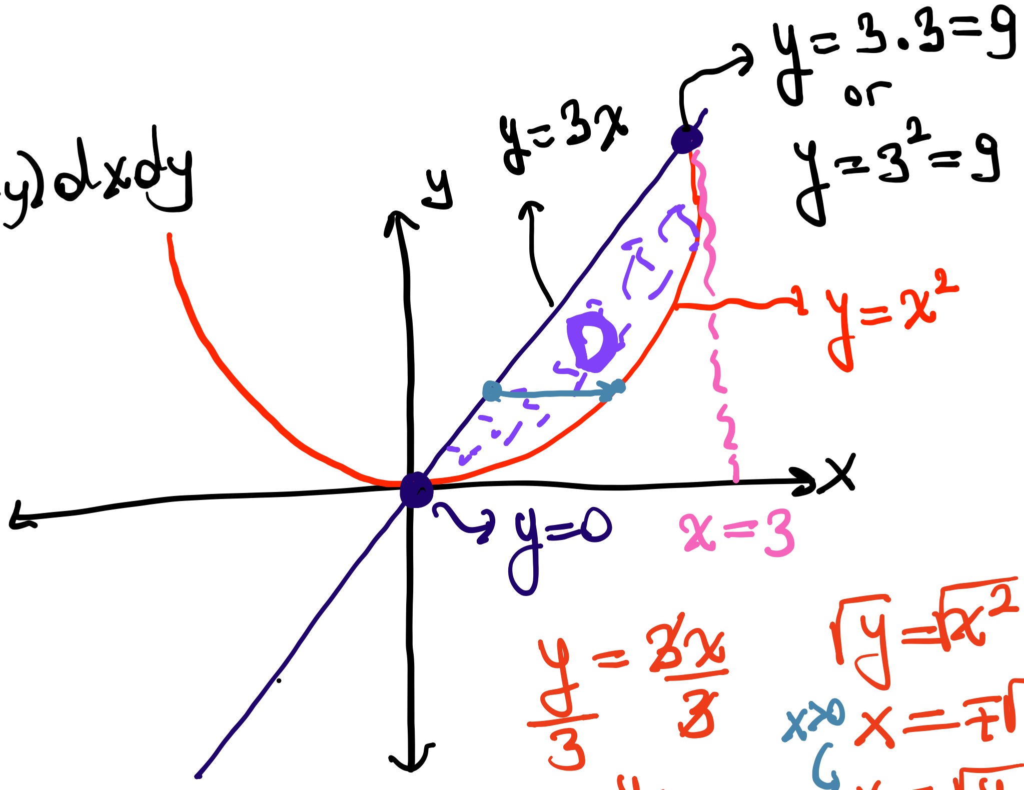
A) $\int_0^3 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy$

B) $\int_0^{16} \int_{y/4}^{\sqrt{y}} f(x,y)\,dx\,dy$

C) $\int_0^{35} \int_{y/5}^{\sqrt{y}} f(x,y)\,dx\,dy$

D) $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

E) $\int_0^9 \int_{y/3}^{\sqrt{y}} f(x,y)\,dx\,dy$



$\frac{y}{3} = \frac{\cancel{3}x}{\cancel{3}}$

$x = y/3$

$\overset{\shortparallel}{h_1(y)}$

$\sqrt{y} = \sqrt{x^2}$

$x = \mp\sqrt{y}$

$x>0 \;\hookrightarrow\; x = \sqrt{y}$

$\overset{\shortparallel}{h_2(y)}$

$$\int_0^9 \int_{y/3}^{\sqrt{y}} f(x,y)\,\underbrace{dx\,dy}$$

↳ type 2

$D = \{(x,y) : h_1(y) \le x \le h_2(y),\; c \le y \le d\}$

• Question : Let f be an integrable and <u>even</u> function satisfying the condition $\int_0^1 f(y)dy = \frac{2}{7}$. Accordingly, which of the following is the result of the integral of $\int_{-1}^{1}\int_{0}^{1} f(y)\,x\,e^{\frac{x^2}{2}}\,dx\,dy$?

A) $2\sqrt{e}-2$ B) $2e-1$ C) $4\sqrt{e}-4$ D) $4e-1$ E) 0

• Since f is an even function, $2=\int_0^1 f(y)dy = \int_{-1}^{0} f(y)dy$ so

$$\int_{-1}^{1} f(y)dy = \int_{-1}^{0} f(y)dy + \int_{0}^{1} f(y)dy = 2+2=4$$

$$\int_{-1}^{1}\int_{0}^{1} f(y)\,x e^{\frac{x^2}{2}}\,dx\,dy = \int_{0}^{1}\int_{-1}^{1} f(y)\,x e^{\frac{x^2}{2}}\,dy\,dx$$

$$= \int_0^1 4xe^{\frac{x^2}{2}}dx$$

$$=4\int_0^1 \underbrace{xe^{x^2/2}dx}_{} \rightarrow e^{x^2/2}x\,dx$$

if $\frac{x^2}{2}=u \Rightarrow$

$x\,dx = du$

and

$x=1 \Rightarrow u=1^2/2=1/2$

$x=0 \Rightarrow u=0^2/2=0$

$$=4\int_0^{1/2} e^u du = 4e^u\Big|_0^{1/2}$$

$$=4e^{1/2}-4e^0$$

$$=4\sqrt{e^1}-4$$

Question: Which of the following is equal to sum

$$\underbrace{\int_{-3}^{1-e}\int_{1-x}^{4} f(x,y)\,dydx}_{D_1} + \underbrace{\int_{1-e}^{1}\int_{e}^{4} f(x,y)\,dydx}_{D_2} + \underbrace{\int_{1}^{\ln 4}\int_{e^x}^{4} f(x,y)\,dydx}_{D_3}$$

$D = D_1 + D_2 + D_3$

A) $\int_{e}^{4}\int_{1-y}^{\ln y} f(x,y)\,dxdy$ B) $\int_{e}^{4}\int_{1+y}^{\ln y} f(x,y)\,dxdy$ C) $\int_{0}^{1}\int_{e}^{e^y} f(x,y)\,dxdy$ D) $\int_{-3}^{1}\int_{1+y}^{e^y} f(x,y)\,dxdy$ E) $\int_{-3}^{1}\int_{e^y}^{4} f(x,y)\,dxdy$

1

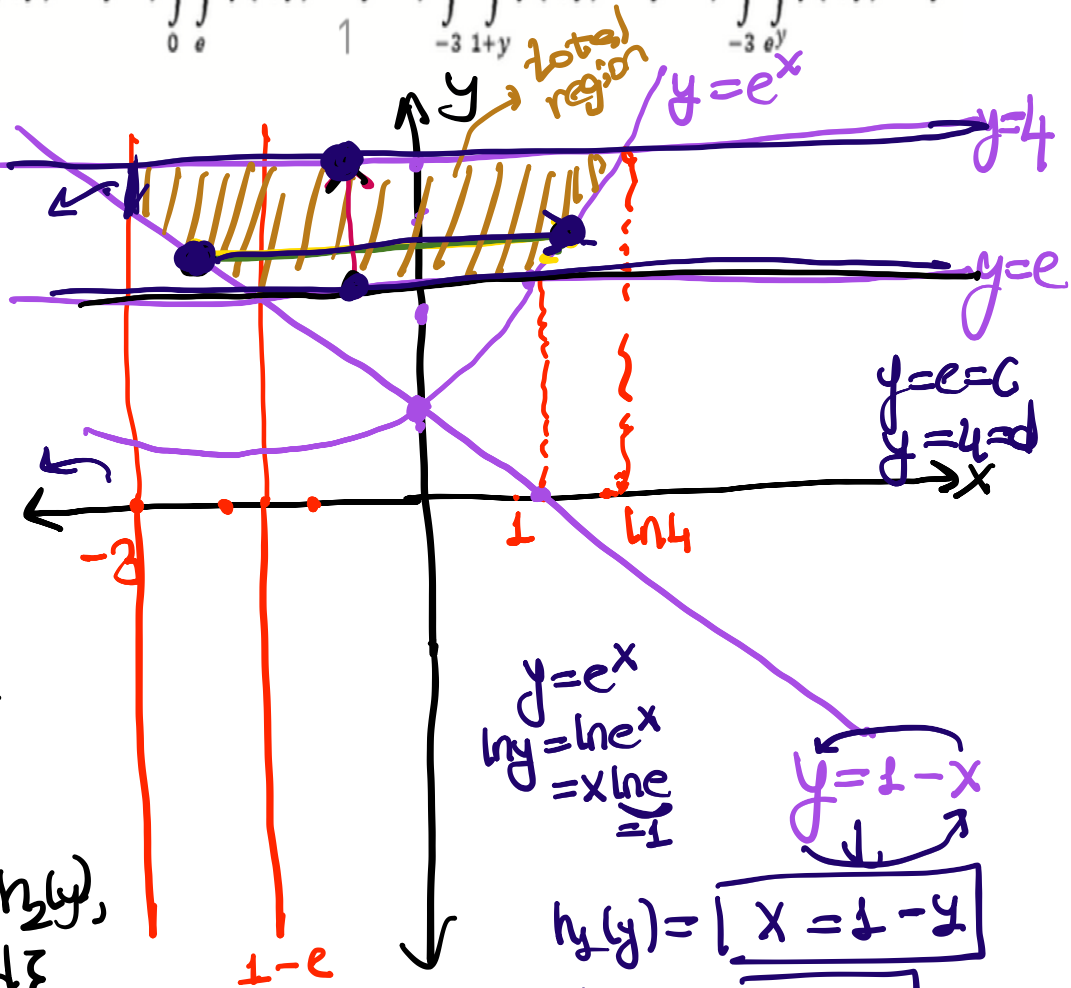
Region R:

$x=-3 \qquad y=1-x$

$x=1-e \qquad y=4$

$x=1 \qquad y=e$

$x=\ln 4 \qquad y=e^x$



$y=e^x$

$\ln y = \ln e^x$

$= x \underbrace{\ln e}_{=1}$

$h_1(y) = \boxed{x = 1-y}$

$h_2(y)\ \boxed{x = \ln y}$

Region type II:

$\iint_D f(x,y)\,dxdy$

$R = \{(x,y): h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

x

$y=1-x \Rightarrow x = 1-y$ $= h_1(y)$

$y=e^x \Rightarrow x = \ln y$ $= h_2(y)$

y

$y=4=d$

$y=e=c$

$$\int_{e}^{4}\int_{1-y}^{\ln y} f(x,y)\,dxdy$$

EXAMPLE 2 If $f(x, y) = \frac{xy}{x^2 + y^2}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

- $\lim_{y\to 0}\Big(\underbrace{\lim_{x\to 0}\frac{xy}{x^2+y^2}}_{=\frac{0}{y^2}=0}\Big) = \lim_{y\to 0} 0 = 0$
- $\lim_{x\to 0}\Big(\underbrace{\lim_{y\to 0}\frac{xy}{x^2+y^2}}_{=\frac{0}{x^2}=0}\Big) = \lim_{x\to 0} 0 = 0$
- Double limit is equal to 0.

Note: The existence of the double limit does not guarantee that the function f(x,y) has a limit at the point (a,b).

- If we write $y = kx$, $k \in \mathbb{R}$, then

$\lim_{x\to 0}\frac{k\cancel{x^2}}{\cancel{x^2}(1+k^2)} = \lim_{x\to 0}\frac{k}{1+k^2}$ so limit is depending on k. Thus, there is no limit at the (0,0).

EXAMPLE 3 If $f(x, y) = \frac{xy^2}{x^2 + y^4}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

- $\lim_{y\to 0}\Big(\underbrace{\lim_{x\to 0}\frac{xy^2}{x^2+y^4}}_{\frac{0}{y^4}=0}\Big) = \lim_{y\to 0} 0 = 0$
- $\lim_{x\to 0}\Big(\underbrace{\lim_{y\to 0}\frac{xy^2}{x^2+y^4}}_{=\frac{0}{x^2}=0}\Big) = \lim_{x\to 0} 0 = 0$

Double limit is equal to 0.

Note: The existence of the double limit does not guarantee that the function f(x,y) has a limit at the point (a,b).

• If we write $x = ky^2$ $[k \in \mathbb{R}]$, then

$$\lim_{y \to 0} \frac{k \cancel{y^4}}{\cancel{y^4}(k^2+1)} = \lim_{y \to 0} \frac{k}{k^2+1}$$

so limit is depending on k. Hence, there is no limit at the (0,0).

Question: Surface $x^2+y^2+z^2 = 4y+2z+5$ and surface $3x^2+2y^2-2z=3$ at the point $P(1,2,4)$. What is the angle of the intersection?

A) $\frac{\pi}{4}$ B) $\frac{\pi}{6}$ C) $\frac{\pi}{2}$ D) $\frac{\pi}{3}$ E) $\frac{2\pi}{3}$

$f(x,y,z) = x^2+y^2+z^2-4y-2z-5 \Rightarrow \vec{\nabla f} = f_x\vec{i}+f_y\vec{j}+f_z\vec{k} \Rightarrow$

$\vec{\nabla f} = (2x)\vec{i}+(2y-4)\vec{j}+(2z-2)\vec{k} \Rightarrow \vec{\nabla f}|_P = 2\vec{i}+0\vec{j}+6\vec{k}$
$= \langle 2,0,6 \rangle$

$g(x,y,z) = 3x^2+2y^2-2z-3 \Rightarrow \vec{\nabla g} = (6x)\vec{i}+(4y)\vec{j}+(-2)\vec{k}$

$\vec{\nabla g}|_P = 6\vec{i}+8\vec{j}+(-2)\vec{k} = \langle 6,8,-2 \rangle$

$$\cos\theta = \frac{\vec{\nabla f}|_P \cdot \vec{\nabla g}|_P}{|\vec{\nabla f}|_P||\vec{\nabla g}|_P|} = \frac{0}{|\vec{\nabla f}|_P||\vec{\nabla g}|_P|} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$$

$\vec{\nabla f}|_P \cdot \vec{\nabla g}|_P = \langle 2,0,6 \rangle \cdot \langle 6,8,-2 \rangle = 2.6+0.8+6.(-2)$
$= 12+0+(-12) = 0$

• Question : If the local minimum point of the function $f(x,y)=\frac{1}{3}y^3+x^2-2xy-3y$ is (K,L), then what is the value of the $K+L$?

A) −10 B) −4 C) −2 D) 2 E) 6

$f_x = 2x-2y=0 \Rightarrow x=y$

$f_y = y^2-2x-3 = y^2-2y-3=0$

$\rightarrow f_{yx}=-2$

$(y-3)(y+1)=0$

$y=3$ or $y=-1$

$x=y$, $x=3$; $x=-1$

Critical Points:

$(-1,-1)$

$(3,3)$

$f_{xx}=2=A>0$

$f_{xy}=-2=B$

$f_{yy}=2y=C$

	A	B	C ($2y$)	B^2-AC
saddle point $(-1,-1)$	$2>0$	-2	-2	$(-2)^2-2.(-2)$ $4+4=8>0$
$(3,3)$ (K, L)	$2>0$	-2	6	$(-2)^2-2.6$ $4-12=-8$ <0

local min.

$A>0$

$B^2-AC<0$

$K+L=3+3=6$

7

14) Which of the following is the value of f_x at the point P , if $\vec{i}-\vec{j}-\vec{k}$ is the direction in which the function $f(x,y,z)$ increases most rapidly and the value of this directional derivative is equal to $2\sqrt{3}$ at P ?

A) $\sqrt{3}$ B) 4 C) 3 D) 2 E) 1

• If f increases most rapidly, then

direction: $\vec{\nabla} f \parallel \vec{i}-\vec{j}-\vec{k} = \langle 1,-1,-1 \rangle \Rightarrow$

$$\vec{\nabla} f = a\langle 1,-1,-1 \rangle = \langle \underset{f_x|_P}{\underset{\shortparallel}{a}}, \underset{f_y|_P}{\underset{\shortparallel}{-a}}, \underset{f_z|_P}{\underset{\shortparallel}{-a}} \rangle, \ a>0$$

• If f increases most rapidly, then

$$(D_u f)|_P = |\vec{\nabla} f|_P| = \sqrt{a^2+(-a)^2+(-a)^2}$$

$$= \sqrt{3a^2} = |a|\sqrt{3}$$

$$\underset{a>0}{=} a\sqrt{3} = 2\sqrt{3} \Rightarrow$$

$$a=2$$

so

$$f_x|_P = a = 2$$

7