MAT1320-Linear Algebra
Lecture Notes

Echelon Form of a Matrix

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d) If a column contains a leading one, then all other entries in that column are zero.

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- An, $m \times n$ matrix satisfying properties $a)$, $b)$, and $c)$ is said to be in row echelon form (REF). There may be no zero rows.
- A similar definition can be formulated in the obvious manner for reduced column echelon form and column echelon form.

Example

The following are matrices in reduced row echelon form, since they satisfy properties $a)$, $b)$, and $d)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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and

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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The following are matrices in row echelon form.

$$H = \begin{pmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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Note: We shall now show that every matrix can be put into row (column) echelon form, or into reduced row (column) echelon form, by means of certain row (column) operations.

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- Replace row (column) j by k times row (column) $i+$ row (column) j, Type III: $k\mathbf{r}_i + \mathbf{r}_j \rightarrow \mathbf{r}_j$ $(k\mathbf{c}_i + \mathbf{c}_j \rightarrow \mathbf{c}_j)$

Example

Let $A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{pmatrix}$.

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Interchanging rows 1 and 3 of A, we obtain

$$B = A_{r_1 \leftrightarrow r_3} = \begin{pmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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Multiplying the third row of A by $\frac{1}{3}$, we obtain

$$C = A_{\frac{1}{3} r_3 \to r_3} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{pmatrix}$$

Echelon Form of a Matrix

Adding (-2) times row 2 of A to row 3 of A, we obtain

$$D = A_{-2r_2+r_3\to r_3} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{pmatrix}$$

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Note: An $m \times n$ matrix B is said to be row (column) equivalent to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

Example

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Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1 + r_2 \to r_2} = \underbrace{\begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}}_{\text{REF}}$$

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Example (cont.)

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \begin{pmatrix} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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Adding (-1) times row 2 of E to row 1 of E, we obtain

$$F = E_{-r_2+r_1\to r_1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{pmatrix}}_{\text{RREF}}$$

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Note: The number of nonzero rows in the REF or RREF of a matrix $\mathbf{A}$ is called the rank of $\mathbf{A}$ and denoted by $rank\,(\mathbf{A})$.

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Note: The number of nonzero rows in the REF or RREF of a matrix $\mathbf{A}$ is called the rank of $\mathbf{A}$ and denoted by $rank(\mathbf{A})$. The rank of matrix A given in the previous example is 3.

Finding an Inverse using Elementary Row Operations

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$$\mathbf{E}_n \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}$$

which means that $\mathbf{E}_n \dots \mathbf{E}_2 \mathbf{E}_1 = \mathbf{A}^{-1}$.

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  which means that $\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 = \mathbf{A}^{-1}$.
- Furthermore, because

$$\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{I} = \mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1$$

  we can use the following technique:

Finding an Inverse using Elementary Row Operations

- Write **A** and **I** side-by-side.

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- Write **A** and **I** side-by-side.
- Then apply the same row operations to both **A** and **I** until **A** is reduced to the identity matrix.

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- Then apply the same row operations to both $\mathbf{A}$ and $\mathbf{I}$ until $\mathbf{A}$ is reduced to the identity matrix.
- At the same time, the identity matrix will be "reduced" to the inverse matrix.

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Here is the fully worked out example:

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$$B = [A|I_3] = \left(\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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Multiplying the first row of B by $\frac{1}{3}$, we obtain

$$C = B_{\frac{1}{3}r_1 \to r_1} = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2 \to r_2} \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-1) times row 2 of E to row 1 of E, we obtain

$$\underset{-r_2+r_1\to r_1}{E} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-1) times row 2 of E to row 1 of E, we obtain

$$\underset{-r_2+r_1\to r_1}{E} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right), \ A^{-1} = \begin{pmatrix} 1 & -1 & -6 \\ \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

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