MAT1320 LINEAR ALGEBRA EXERCISES I

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1. Let A be a 3×3 matrix such that the sum of its columns is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$. Then, which of the followings is equal to the matrix AB?

   a) $\begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ e) None of them

1) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then, see that

$a_{11}+a_{12}+a_{13}=0$, $a_{21}+a_{22}+a_{23}=0$, $a_{31}+a_{32}+a_{33}=0$. Since the columns of the matrix B are $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, the elements of AB are of the form

$k.(a_{11}+a_{12}+a_{13})$, $k.(a_{21}+a_{22}+a_{23})$, $k.(a_{31}+a_{32}+a_{33})$ where $k \in \{-1,1,2\}$.

$\Rightarrow$ Then, $A.B = 0$ matrix.

2) $X_{2\times 3} \Rightarrow X^T_{3\times 2} \Rightarrow \underbrace{X^T_{3\times 2} . Y_{2\times 3}}_{3\times 3} = C_{3\times 3}$

$\Rightarrow A_{4\times 3} . C_{3\times 3} . A^T_{3\times 4} = B_{4\times 4}$

$\Rightarrow B^T_{4\times 4}$

2. For the matrices $X_{2\times 3}, Y_{2\times 3}$ and $A_{4\times 3}$, which of the followings is equal to the dimension of the matrix $\left[A\left(X^TY\right)^{-1}A^T\right]^T$?

   a) (4×4) b) (4×3) c) (3×4) d) (2×2) e) (4×2)

3. If $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$, then which of the followings is equal to x?

   a) -2 b) 2 c) $-\frac{1}{7}$ d) $\frac{7}{8}$ e) $\frac{8}{3}$

3) $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4+4x & 6+x \end{bmatrix}}_{1\times 3}$

$\Rightarrow \begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 4+4x & 6+x \end{bmatrix} \begin{bmatrix} x \\ -1 \\ 2 \end{bmatrix}$

$= x - 4 - 6x + 12 + 2x = 0$

$\Rightarrow 8 = 3x \Rightarrow \boxed{x = \frac{8}{3}}$

4) $A.C.B.D = I_n$ } multiply by A^{-1} from left side

$\Rightarrow CBD = A^{-1}$ } multiply by D^{-1} from right side

$\Rightarrow CB = A^{-1}D^{-1}$ } multiply by B^{-1} from right side

$\Rightarrow C = A^{-1}D^{-1}B^{-1}$

$\Rightarrow C^{-1} = \left(A^{-1}D^{-1}B^{-1}\right)^{-1}$

$= BDA$

4. Let A, B, C and D be $n \times n$ matrices having the inverse. If $ACBD = I_n$, then which of the followings is equal to C^{-1} ?

   a) BAD b) BDA c) DBA d) $A^{-1}D^{-1}B^{-}$

   e) C^{-1} may not be exist.

5. If $A = [a_{ij}]_{p\times q}$, $B = [b_{jk}]_{q\times n}$ and $C = [c_{kl}]_{n\times m}$, then which of the followings is the dimension of the matrix ABC?

a) $q\times m$ b) $q\times n$ c) $p\times m$

d) $m\times p$ e) $p\times q$

5) $$\underbrace{\underbrace{A_{p\times q}\cdot B_{q\times n}}_{p\times n}\cdot \underbrace{C_{n\times m}}_{n\times m}}_{p\times m}$$

$p\times m$ matrix.

6) $A^2 = A.A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 2A^2 = 2 \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow 2A^2 + A - 5.I_2 = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} -2 & -5 \\ 0 & -2 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, then which of the followings is equal to the matrix $2A^2 + A - 5I_2$?

a) $\begin{bmatrix} -2 & -5 \\ 0 & -2 \end{bmatrix}$ b) $\begin{bmatrix} -2 & -5 \\ 0 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 2 & -5 \\ 0 & -2 \end{bmatrix}$

d) $\begin{bmatrix} -2 & 5 \\ 0 & -2 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 5 \\ 0 & -2 \end{bmatrix}$

7. Let the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and the polynomial $f(x) = 2x^3 - x^2 + 3x + 7$ be given. Then, which of the followings is the matrix $f(A)$?

a) $\begin{bmatrix} 16 & 16 \\ 6 & 9 \end{bmatrix}$ b) $\begin{bmatrix} 17 & 12 \\ 6 & 9 \end{bmatrix}$ c) $\begin{bmatrix} 16 & 16 \\ 9 & 6 \end{bmatrix}$

d) $\begin{bmatrix} 14 & 16 \\ 6 & 10 \end{bmatrix}$ e) $\begin{bmatrix} 8 & 6 \\ 4 & 12 \end{bmatrix}$

7) $f(A) = 2A^3 - A^2 + 3A + 7.I_2$

$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$

$\Rightarrow f(A) = 2A^3 - A^2 + 3A + 7.I_2$

$= \begin{bmatrix} 10 & 12 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 17 & 16 \\ 8 & 9 \end{bmatrix}$

8. Let $B = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$. If $(AB)^{-1} = (CB)^T$, then which of the following is equal to the matrix A^{-1}?

a) $\begin{bmatrix} -44 & 17 \\ 31 & -12 \end{bmatrix}$ b) $\begin{bmatrix} 44 & -17 \\ 31 & 12 \end{bmatrix}$ c) $\begin{bmatrix} 44 & -17 \\ 31 & -12 \end{bmatrix}$

d) $\begin{bmatrix} -44 & 17 \\ -31 & -12 \end{bmatrix}$ e) $\begin{bmatrix} -44 & -17 \\ 31 & 12 \end{bmatrix}$

$(AB)^{-1} = (CB)^T \Rightarrow B^{-1}A^{-1} = (CB)^T$

$\Rightarrow A^{-1} = B(CB)^T$

$CB = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 2 & -5 \end{bmatrix}$

$(CB)^T = \begin{bmatrix} -5 & 2 \\ 13 & -5 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 \\ 13 & -5 \end{bmatrix} = \begin{bmatrix} 44 & -17 \\ 31 & -12 \end{bmatrix}$

9. Let $A = \begin{bmatrix} -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Which of the following is the $(3,3)$-entry of the inverse matrix A^{-1} ? $\Rightarrow AA^{-1} = I_4$

a) 0 b) -2 c) 1 d) -1 e) 2

$$\begin{bmatrix} -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

?

Multiplying the second row of A and the third column of A^{-1}, we get

$$0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} + 0 \cdot a_{43} = 0$$

$$\Rightarrow \boxed{a_{33} = 0}$$

10) To find $(3,3)$-entry of the matrix AB, we need to calculate the product of third row of A and third column of B.

$$\begin{bmatrix} & & & \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} & & b_{13} & \\ & & b_{23} & \\ & & b_{33} & \\ & & b_{43} & \end{bmatrix} = \begin{bmatrix} & & \vdots & \\ & & c_{33} & \end{bmatrix}$$

$a_{31} = 2$, $a_{32} = 1$, $a_{33} = 6$, $a_{34} = 7$

$b_{13} = -1$, $b_{23} = 1$, $b_{33} = 4$, $b_{43} = 4$

$\Rightarrow c_{33} = 2 \cdot -1 + 1 \cdot 1 + 6 \cdot 4 + 7 \cdot 4 = 51$

10. Let $A = [a_{ij}]_{n\times m}$ and $B = [b_{ij}]_{m\times r}$ be two matrices where $a_{ij} = \begin{cases} i+j, & i \le j \\ i-j & i > j \end{cases}$ and $b_{ij} = \begin{cases} 2i-j, & i < j \\ j+1, & i \ge j \end{cases}$. Then, which of the following is the $(3,3)$-entry of the matrix AB? for $m=4$?

a) [redacted] 51 b) 54 c) 55 d) 56 e) 57

11. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 4 & -1 & 0 & 3 \end{bmatrix}$, then which of the following is the entry b_{24}?

a) -2 b) 3 c) 8 d) -4 e) 11

$$A_{2\times 2} \cdot B_{m\times n} = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 4 & -1 & 0 & 3 \end{bmatrix}_{2\times 4} \Rightarrow m=2,\ n=4.$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 4 & -1 & 0 & 3 \end{bmatrix}$$

?

$\Rightarrow 3 \cdot b_{14} + 4 \cdot b_{24} = 5$

$-3 /\ 1 \cdot b_{14} + 1 \cdot b_{24} = 3$

$+$

$b_{24} = 5 - 3 \cdot 3 = -4$

12. (D points) Let $B = \begin{bmatrix} -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Which of the followings is the $(4,4)$-entry of the matrix B^{-1}? $\Rightarrow BB^{-1} = I_4$

a) 0 b) -2 c) 1 d) -1 e) 2

$$\begin{bmatrix} -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Second row, fourth column: $\Rightarrow b_{34} = 0$.

→ third row, fourth column: $b_{24} + b_{34} = 0$

$\Rightarrow b_{24} = 0$.

→ first row, fourth column:

$-2 \cdot b_{14} + 1 \cdot b_{24} + 0 \cdot b_{34} + 2 b_{44} = 0$

$\Rightarrow b_{14} + b_{44} = 0 \Rightarrow b_{14} = -b_{44}$

→ fourth row fourth column:

$3 \cdot b_{14} + 2 \cdot b_{44} = 1 \Rightarrow -b_{44} = 1 \Rightarrow \boxed{b_{44} = -1}$