Q1)

Let g be a differentiable function where $g'(4)=2$ and z be defined implicitly as a function of x and y by the following equation: $g(xz)=\frac{xy}{z}$. Then which of the following is the value of the derivative $\frac{\partial z}{\partial x}$ at the point $(x,y,z)=(2,2,2)$?

A) 0 B) $-\frac{7}{5}$ C) $-\frac{1}{7}$ D) $-\frac{2}{7}$ E) $-\frac{3}{5}$

$$g(xz)-\frac{xy}{z}=0 \quad \text{and} \quad z=f(x,y) \Rightarrow$$

$$\left(1\cdot z+x\cdot\frac{\partial z}{\partial x}\right)\cdot g'(\underbrace{xz}_{2\cdot 2 = 4})-\frac{y}{z}+\frac{xy}{z^2}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow$$

$$\left(2+2\cdot\frac{\partial z}{\partial x}\right)\cdot \underbrace{g'(4)}_{=2}-\frac{2}{2}+\frac{4}{4}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow$$

$$4+4\cdot\frac{\partial z}{\partial x}-1+\frac{\partial z}{\partial x}=0 \Rightarrow$$

$$3+5\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,2,2)}=-\frac{3}{5}$$

Question 2: Let f be a differentiable function and $u = x^3y^2$, $v = \sin(\pi x)$ and $z = f(u,v)$. If $f_u(2,0) = \frac{1}{3}$ and $f_v(2,0) = \frac{1}{\pi}$, then what is the value of $\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(2,\frac{1}{2})} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(2,\frac{1}{2})}$?

A) $\frac{7}{2}$ B) $\frac{8}{3}$ C) $\frac{14}{3}$ D) $\frac{17}{3}$ E) $\frac{13}{6}$

$x=2$ and $y=\frac{1}{2} \Rightarrow u = 2^3 \cdot \left(\frac{1}{2}\right)^2 = 8 \cdot \frac{1}{4} = 2$

$v = \sin(2\pi) = 0$

$$\frac{\partial z}{\partial x} = f_u(2,0) \cdot \frac{\partial u}{\partial x} + f_v(2,0) \cdot \frac{\partial v}{\partial x}$$

$$= \frac{1}{3} \cdot (3x^2y^2) + \frac{1}{\pi} \cdot (\pi \cdot \cos(\pi x))$$

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1/2)} = \frac{1}{3} \cdot \left(3 \cdot 4 \cdot \frac{1}{4}\right) + \frac{1}{\pi} \cdot \pi \cdot \underbrace{\cos(2\pi)}_{=1}$$

$$= 1 + 1 = 2$$

$$\frac{\partial z}{\partial y} = f_u(2,0) \cdot \frac{\partial u}{\partial y} + f_v(2,0) \cdot \frac{\partial v}{\partial y}$$

$$= \frac{1}{3} \cdot (2x^3y) + \frac{1}{\pi} \cdot (0 \cdot \cos(\pi x)) \Rightarrow$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,1/2)} = \frac{1}{3} \cdot 2 \cdot 8 \cdot \frac{1}{2} + \cancel{\frac{1}{\pi} \cdot 0 \cdot \cos(2\pi)}^{0} = \frac{8}{3}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1/2)} + \left.\frac{\partial z}{\partial y}\right|_{(2,1/2)} = 2 + \frac{8}{3} = \frac{14}{3}$$

Question 3: Let $f(x,y)=\begin{cases} \dfrac{\cos x - e^{2y^2}}{\sin x}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

What is the value of $f_x(0,0)$?

A) 0 B) $-\frac{1}{2}$ C) $-\frac{1}{4}$ D) $-\frac{1}{8}$ E) $-\frac{1}{6}$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$f(h,0)=\frac{\cos h - e^0}{\sin h}=\frac{\cos h-1}{\sin h}$$

$$f(0,0)=0$$

$$f_x(0,0)=\lim_{h\to 0}\frac{\frac{\cos h-1}{\sin h}-0}{h}=\lim_{h\to 0}\frac{\cos h-1}{h\cdot\sin h}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{h\to 0}\frac{\cos h-1}{h\cdot \sin h}\cdot\frac{\cos h+1}{\cos h+1}=\lim_{h\to 0}\frac{\cos^2 h-1}{(h\cdot\sin h)\cdot(\cos h+1)}$$

$$=\lim_{h\to 0}\frac{-\sin^2 h}{h\cdot\sin h\cdot[\cos h+1]}=\lim_{h\to 0}\frac{-\sin h}{h\cdot[\cos h+1]}$$

$$=\Big[-\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{=1}\Big]\cdot\Big[\lim_{h\to 0}\frac{1}{\cos h+1}\Big]$$

$$=-1\cdot\frac{1}{\underbrace{\cos 0}_{=1}+1}=-\frac{1}{2}$$

Question 4: Let $f(x,y)=\sqrt[3]{x^4+y^2}$.

Which of the following is true about the $f_x(0,0)$ and $f_y(0,0)$?

A) $f_x(0,0)=0$ and there is no $f_y(0,0)$

B) There are no $f_x(0,0)$ and $f_y(0,0)$

C) $f_x(0,0)=0=f_y(0,0)$

D) There is no $f_x(0,0)$ and $f_y(0,0)=0$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt[3]{h^4}-0}{h}=\lim_{h\to 0}h^{1/3}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt[3]{h^2}-0}{h}=\lim_{h\to 0}h^{-1/3}$$

there is no limit so there is no $f_y(0,0)$

Hence, $f_x(0,0)=0$ and there is no $f_y(0,0)$

Question 5: Let $f(x,y) = y\ln x + xe^y$ $(x>0, y>0)$.
Which of the following is the function $h(x,y)$ that satisfies the equation
$h(x,y) \cdot f_{xx} + x f_{xy} - f_{yy} = 0$?

A) $h(x,y) = -\frac{y}{x}$ B) $h(x,y) = \frac{x^2}{y}$ C) $h(x,y) = x^2 y$

D) $h(x,y) = -x^2 y$ E) $h(x,y) = \frac{1}{xy}$

$f_x = 0 . \ln x + y \cdot \frac{1}{x} + 1 \cdot e^y + x \cdot 0 = \frac{y}{x} + e^y$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x} + e^y\right) = -\frac{y}{x^2} + 0 = -\frac{y}{x^2}$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x} + e^y\right) = \frac{1}{x} + e^y$

$f_y = 1 \cdot \ln x + y \cdot 0 + 0 . e^y + x . e^y . 1 = \ln x + xe^y$

$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\ln x + xe^y\right) = 0 + 0 . e^y + x . 1 . e^y$
$= xe^y$

$h \cdot \left(-\frac{y}{x^2}\right) + x\left[\frac{1}{x} + e^y\right] - xe^y = 0 \Rightarrow$

$h \cdot \left(-\frac{y}{x^2}\right) + 1 + \underbrace{xe^y - xe^y}_{=0} = 0 \Rightarrow$

$h\left(-\frac{y}{x^2}\right) = -1 \Rightarrow h = \frac{-1}{-y/x^2} = \frac{x^2}{y}$

Question 6: Let $u=x-y$, $v=y-z$ and $w=z-x$ transformations be made in the function $F=F(x-y,y-z,z-x)$. Which of the following is the sum of $\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}$?

A) $2F_u$ B) $F_u+F_v+F_w$ C) 0 D) F_v-F_w E) $2F_u-F_w$

$F=F(u,v,w)$

$$\frac{\partial F}{\partial x}=F_u\cdot u_x+F_v\cdot v_x+F_w\cdot w_x$$
$$=F_u\cdot 1+F_v\cdot 0+F_w\cdot(-1)=F_u-F_w$$

$$\frac{\partial F}{\partial y}=F_u\cdot u_y+F_v\cdot v_y+F_w\cdot w_y$$
$$=F_u\cdot(-1)+F_v\cdot 1+F_w\cdot 0=-F_u+F_v$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}=F_u-F_w-F_u+F_v=\underline{F_v-F_w}$$

Question 7: Let $f(x,y)=\begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Which of the following is true about the partial derivatives $f_x(0,0)$, $f_y(0,0)$ and the continuity of the $f(x,y)$ at the point $(0,0)$?

A) There are no $f_x(0,0)$ and $f_y(0,0)$. $f(x,y)$ is discontinuous at the $(0,0)$.

B) $f_x(0,0)=f_y(0,0)=0$ and $f(x,y)$ is continuous at $(0,0)$.

C) $f_x(0,0)=f_y(0,0)=0$ and f is discontinuous at the $(0,0)$.

D) $f_x(0,0)=f_y(0,0)=0$ and f is continuous.

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{h\cdot 0^2}{h^2+0^4}-0}{h}=\lim_{h\to 0}\left(\frac{0}{h}\right)=\lim_{h\to 0}0=0$$

$$\Rightarrow f_x(0,0)=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{0.h^2}{0^2+h^4}-0}{h}=\lim_{h\to 0}\left(\frac{0}{h}\right)$$

$$=\lim_{h\to 0} 0=0 \Rightarrow$$

$f_y(0,0)=0$

If we take $x=k.y^2$ for $k\in\mathbb{R}$, then

$$\lim_{y\to 0}\frac{ky^2.y^2}{(ky)^2+y^4}=\lim_{y\to 0}\frac{ky^4}{k^2y^4+y^4}$$

$$=\lim_{y\to 0}\frac{k\cancel{y^4}}{(k^2+1)\cancel{y^4}}=\frac{k}{k^2+1}$$

so limit is depending on k. Thus, there is no limit. Hence, f is discontinuous at the $(0,0)$.

Question 8: Let $f(x,y)=\begin{cases}(x^2+y)\cdot\sin\left(\frac{1}{x+y}\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Which of the following is true about the $f_x(0,0)$ and $f_y(0,0)$?

A) $f_x(0,0)=0=f_y(0,0)$

B) $f_x(0,0)=1$ and there is no $f_y(0,0)$

C) $f_x(0,0)=1$ and $f_y(0,0)=0$

D) $f_x(0,0)=0$ and there is no $f_y(0,0)$

E) There are no $f_x(0,0)$ and $f_y(0,0)$

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to0}\frac{h^2\cdot\sin\left(\frac{1}{h}\right)-0}{h}$$

$$=\lim_{h\to0}h\cdot\sin\left(\frac{1}{h}\right)$$

$\left(h\to0 \begin{smallmatrix}\to0^+\\ \to0^-\end{smallmatrix}\right)$ For $h\neq0$, $-1\leq\sin\left(\frac{1}{h}\right)\leq1 \Rightarrow$

$$-h\leq h\sin\left(\frac{1}{h}\right)\leq h$$

$$\lim_{h\to0}(-h)=\lim_{h\to0}h=0$$

$$\underbrace{\lim_{h\to 0}(-h)}_{=0} \le \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) \le \underbrace{\lim_{h\to 0} h}_{=0}$$

so $\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$ by the Sandwich Theorem.

Hence, $f_x(0,0) = 0$.

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\cdot\sin\left(\frac{1}{h}\right) - 0}{\cancel{h}} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

there is no limit

so there is no $f_y(0,0)$

Question 9: Let $z=f(x,y)$ be a differentiable function with $yz_x - xz_y = 0$. What is z_θ when we switch to polar coordinates?

- **Polar Coordinates**: $x = r\cos\theta$
  $y = r\sin\theta$

$$z_\theta = z_x \cdot x_\theta + z_y \cdot y_\theta$$
$$= z_x \cdot (\underbrace{-r\sin\theta}_{=-y}) + z_y \cdot (\underbrace{r\cos\theta}_{=x})$$
$$= -yz_x + xz_y$$
$$= -[\underbrace{yz_x - xz_y}_{=0}]$$

$$\Rightarrow z_\theta = 0$$

Question 1: Let the function $f(x,y)=\sin(xy-y^3x)+\ln\left(\frac{1}{x+y}\right)$ be given. Then, which of the following is the value of $\frac{f_y(1,1)}{f_{yy}(1,1)}$?

A) $\frac{16}{23}$ B) $\frac{10}{23}$ C) $\frac{13}{12}$ D) $\frac{17}{12}$ E) 2

$$f_y=[x-3y^2x].\cos(xy-y^3x)-\frac{1/(x+y)^2}{1/(x+y)}$$

$$=[x-3y^2x].\cos(xy-y^3x)-\frac{1}{x+y}\Rightarrow$$

$$f_y(1,1)=-2.\underbrace{\cos 0}_{=1}-\frac{1}{2}=-2-\frac{1}{2}=-\frac{5}{2}$$

$$f_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left[(x-3y^2x).\cos(xy-y^3x)-\frac{1}{x+y}\right]$$

$$=(-6yx).\cos(xy-y^3x)+(x-3y^2x)^2.(-\sin(xy-y^3x))+\frac{1}{(x+y)^2}\Rightarrow$$

$$f_{yy}(1,1)=-6.\underbrace{\cos 0}_{=1}+\cancelto{0}{(-2)^2.(-\underbrace{\sin 0}_{=0})}+\frac{1}{4}$$

$$=-6+\frac{1}{4}=-\frac{23}{4}$$

$$\frac{f_y(1,1)}{f_{yy}(1,1)}=\frac{-5/2}{-23/4}=\frac{10}{23}$$

Question 2: Let $z = \sin(x^2y^2) + \tan(xy)$.

If $\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0)} = A$ and $\left.\frac{\partial^2 z}{\partial^2 y}\right|_{(0,0)} = B$, then

what is the value of $A+B$?

A) 0 B) 1 C) 2 D) 3 E) 4

$$\frac{\partial z}{\partial x} = (2xy^2)\cdot\cos(x^2y^2) + y\cdot[1+\tan^2(xy)]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[(2xy^2)\cdot\cos(x^2y^2) + y\cdot(1+\tan^2(xy))\right]$$

$$= (4xy)\cdot\cos(x^2y^2) + 4x^2y^4\cdot(-\sin(x^2y^2)) + y^2\cdot 2\tan(xy)\cdot[1+\tan^2(xy)]$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0)} = 0\cdot\underbrace{\cos 0}_{=1} + 0\cdot(-\cancel{\sin 0}^{\,0}) + 0^2\cdot 2\cdot\cancel{\underbrace{\tan 0}_{=0}}^{\,0}[1+\underbrace{\tan^2 0}_{\neq 0}]$$

$$= 0 = A$$

$$\frac{\partial z}{\partial y} = (2x^2y)\cos(x^2y^2) + x[1+\tan^2(xy)]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[(2x^2y)\cos(x^2y^2) + x[1+\tan^2(xy)]\right]$$

$$= (2x^2)\cos(x^2y^2) + 4x^4y^2(-\sin(x^2y^2))$$

$$+ x^2 \cdot 2 \cdot \tan(xy) \cdot [1 + \tan^2(xy)]$$

$$\Rightarrow \left.\frac{\partial^2 z}{\partial y^2}\right|_{(0,0)} = 0 \cdot \cancel{\cos 0}^{\,0=1} + 4 \cdot 0 \cdot \cancel{(-\sin 0)}^{\,0}_{=0}$$

$$+ 0^2 \cdot 2 \cdot \underbrace{\cancel{\tan 0}}_{=0} \cdot [1 + \underbrace{\tan^2 0}_{=0}]$$

$$= 0 = B$$

$$\Rightarrow A + B = 0 + 0 = 0$$

Question 3: Let $f(x,y) = (x^2+y^2).\tan(\ln(xy)) + e^{(x-y)}$.
Then, what is the value of $2f_x(1,1) - f_y(1,1)$?

A) 3 B) 4 C) 5 D) 6 E) 7

$$f_x = (2x).\tan(\ln(xy)) + (x^2+y^2).\frac{y}{xy}.[1+\tan^2(\ln(xy))] + e^{(x-y)} \Rightarrow$$

$$f_x(1,1) = 2.\underbrace{\tan(\underbrace{\ln 1}_{=0})}_{=0} + 2.1.[1+\underbrace{\tan^2(\underbrace{\ln 1}_{=0})}_{=0}] + \underbrace{e^0}_{=1}$$

$$= 0 + 2 + 1 = 3$$

$$2f_x(1,1) = 2.3 = 6$$

$$f_y = (2y).\tan(\ln(xy)) + (x^2+y^2).\frac{x}{xy}.[1+\tan^2(\ln(xy))] - e^{(x-y)} \Rightarrow$$

$$f_y(1,1) = 2.\underbrace{\tan(\underbrace{\ln 1}_{=0})}_{=0} + 2.1.[1+\underbrace{\tan^2(\underbrace{\ln 1}_{=0})}_{=0}] - \underbrace{e^0}_{=1}$$

$$= 2.0 + 2.1.[1+0] - 1$$

$$= 2 - 1 = 1$$

$$\Rightarrow 2f_x(1,1) - f_y(1,1) = 6 - 1 = 5$$

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Question 4: Let $f(x,y)=\begin{cases} \dfrac{x^3-xy^3}{2x^2+y^2} &, (x,y)\neq(0,0) \\ 0 &, (x,y)=(0,0) \end{cases}$

What is the value of $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)}$?

A) -1 B) $-\frac{1}{2}$ C) 0 D) $\frac{1}{2}$ E) 1

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)}=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h}{2}-0}{h}=$$

$$f(h,0)=\frac{h^3-h\cdot 0^3}{2h^2+0^2}=\frac{h^3}{2h^2}=\frac{h}{2}$$

$$f(0,0)=0$$

$$\lim_{h\to 0}\frac{\cancel{h}}{2\cancel{h}}=\frac{1}{2}$$

Question 5: Let $w=(x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$ and $z=\sin(r+s)$. What is the value of $\left.\frac{\partial w}{\partial r}\right|_{\substack{r=1\\ s=-1}}$

A) 6 B) 8 C) 10 D) 12 E) 14

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$=[2(x+y+z).1].1+[2(x+y+z).1].(1.(-\sin(r+s)))$$
$$+[2.(x+y+z).1].(1.\cos(r+s))$$

$r=1$ and $s=-1 \Rightarrow x=1-(-1)=2$
$$y=\cos(1+(-1))=\cos 0=1$$
$$+\underline{\;z=\sin(1+(-1))=\sin 0=0\;}$$
$$=3$$

$$\Rightarrow 2.3+\cancel{2.3}(\underbrace{-\sin 0}_{=0})+2.3.\underbrace{\cos 0}_{=1}=$$

$$6+6=\underline{12}$$