DOMAIN

The question is "Find the domain of the function $f(x)$" for all in this file.

① $f(x) = \underbrace{\sin(3^x)}_{D_1} \cdot \underbrace{\sqrt{|x^2|-4}}_{D_2}$

For D_1

$D_1 = \mathbb{R}$. Because, sine and exponential function is defined for all $x \in \mathbb{R}$.

For D_2

$\sqrt{|x^2|-4} \Rightarrow |x^2| - 4 \geq 0 \Rightarrow \sqrt{|x^2|} \geq \sqrt{4} \Rightarrow |x| \geq 2$

(inside of $\sqrt{\ }$ must be ≥ 0) (|.| is always nonnegative)

$\Rightarrow x \leq -2$ or $x \geq 2 \Rightarrow D_2 = (-\infty, -2] \cup [2, \infty)$ (or $\mathbb{R} \setminus (-2,2)$)

(inside of |.| can be both positive or negative)

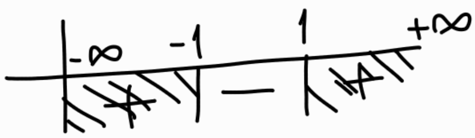
$\Rightarrow D(f) = D_1 \cap D_2 = (-\infty, -2] \cup [2, \infty) = \mathbb{R} \setminus (-2, 2)$

② $f(x) = \underbrace{\cos\left(\frac{1}{\sqrt{x^2-1}}\right)}_{D_1} - e^{\overbrace{3x + \frac{2}{x}}^{D_2}}$

For D_1

$\sqrt{x^2-1} \neq 0$ and $x^2 - 1 \geq 0 \Rightarrow x^2 - 1 > 0 \Rightarrow \underbrace{(x-1)}_{x=1} \cdot \underbrace{(x+1)}_{x=-1} > 0$

(Can't divide by 0) (inside $\sqrt{\ }$ can't be negative)



$D_1 = (-\infty, -1) \cup (1, \infty)$ (or $\mathbb{R} \setminus [-1, 1]$)

For D_2

Exponential function is continuous on $\mathbb{R}$. But can't make a division with 0.

$$\Rightarrow x \neq 0 \Rightarrow D_2 = \mathbb{R} \setminus \{0\}$$

$$D(f) = D_1 \cap D_2 = (-\infty, -1) \cup (1, \infty) = \mathbb{R} \setminus [-1, 1]$$

($x = 0$ is already not in D_1)

(3) $$f(x) = \underbrace{\frac{1}{6+x-x^2}}_{D_1} + \underbrace{\ln\left(\sqrt{1-x}\right)}_{D_2}$$

For D_1

Polynomials are continuous on $\mathbb{R}$. But can't make a division with 0.

$$6 + x - x^2 \neq 0 \Rightarrow (3-x) \cdot (2+x) \neq 0 \Rightarrow x \neq -2, 3$$

$$x = 3 \qquad x = -2$$

$$D_1 = \mathbb{R} \setminus \{-2, 3\}$$

For D_2

Inside of a logarithm function must be positive.

$$\sqrt{1-x} > 0 \Rightarrow 1 - x > 0 \Rightarrow 1 > x \qquad D_2 = (-\infty, 1)$$

(The result of $\sqrt{\ }$ can be 0 or positive)

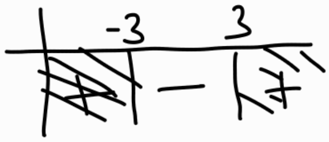
$$D(f) = D_1 \cap D_2 = (-\infty, 1) \setminus \{-2\} = (-\infty, -2) \cup (-2, 1)$$

$$④\quad f(x)=\frac{\overbrace{7-\sqrt{x^2-9}}^{D_1}}{\underbrace{\sqrt{25-x^2}}_{D_2}}+\underbrace{e^{\ln x}}_{D_3}$$

For D_1

Inside of $\sqrt{\ }$ can't be negative. (Can be zero)

$$x^2-9\geq 0 \Rightarrow \underset{\substack{\Downarrow\\ x=3}}{(x-3)}\cdot\underset{\substack{\Downarrow\\ x=-3}}{(x+3)}\geq 0$$

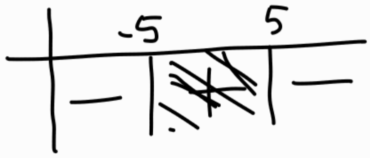


$D_1=(-\infty,-3]\cup[3,\infty)$

For D_2

Can't divide by 0 and inside of $\sqrt{\ }$ can't be negative.

$$25-x^2>0 \Rightarrow \underset{\substack{\Downarrow\\ x=5}}{(5-x)}\cdot\underset{\substack{\Downarrow\\ x=-5}}{(5+x)}>0$$



$D_2=(-5,5)$

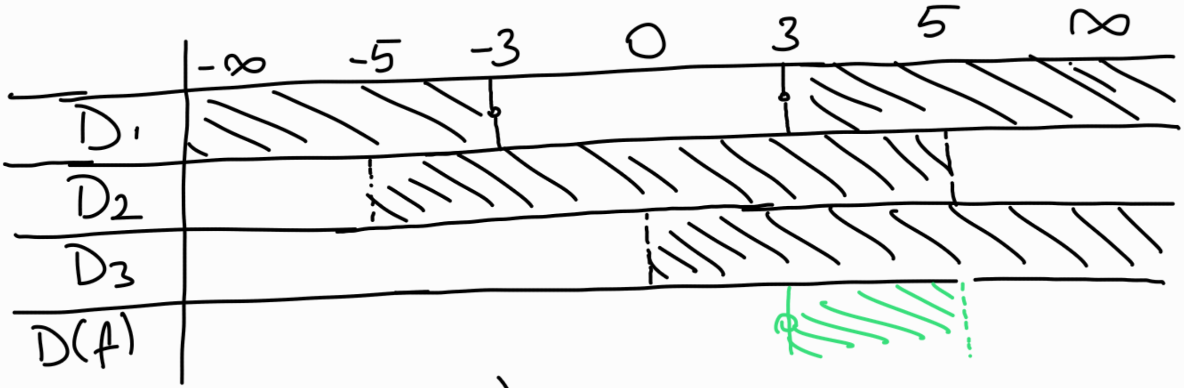
For D_3

e is continuous on $\mathbb{R}$. But inside of $\ln$ must be positive.

$x>0 \Rightarrow D_3=(0,\infty)$

($e^{\ln x}=x$ from the properties. But we <u>must</u> consider its original form $e^{\ln x}$ not just x!)

For the intersection, we can use the sign table.



$\Rightarrow D(f)=[-3,5)$

$$⑤\ f(x)=\underbrace{\arctan\left(2^{3x+5}\right)}_{D_1}-\underbrace{\frac{1}{\ln(2-5x)}}_{D_2}+\underbrace{\frac{1}{x^2+5}}_{D_3}$$

For D_1

arctan is defined for all x in $\mathbb{R}$. Also, $3x+5$ is defined for all $x\in\mathbb{R}$

(Remember the range of tan, that is $\mathbb{R}$)

$D_1=\mathbb{R}$

For D_2

i) Can't divide by $0 \Rightarrow \ln(2-5x)\neq 0 \Rightarrow 2-5x\neq 1 \Rightarrow x\neq \frac{1}{5}$

(ln or logarithm has the value 0, if the function inside is 1)

ii) Inside of log must be positive.

$2-5x>0 \Rightarrow 2>5x \Rightarrow \frac{2}{5}>x$

$\Rightarrow D_2=(-\infty,\frac{2}{5})\setminus\{\frac{1}{5}\}$ (or $(-\infty,\frac{1}{5})\cup(\frac{1}{5},\frac{2}{5})$)

For D_3

Can't divide by 0. But x^2+5 doesn't have a root in $\mathbb{R}$.

($\Delta=b^2-4ac \Rightarrow 0-4.1.5=-20<0 \Rightarrow$ no real root)

$D_3=\mathbb{R}$

$\Rightarrow D(f)=D_1\cap D_2\cap D_3=(-\infty,\frac{2}{5})\setminus\{\frac{1}{5}\}=(-\infty,\frac{1}{5})\cup(\frac{1}{5},\frac{2}{5})$.

⑥ $f(x)=\underbrace{\sqrt{\frac{x^2-6x+9}{x^3-4x^2+6x}}}_{D_1} + \underbrace{\cos\left(\ln\left(\frac{1}{x}\right)\right)}_{D_2}$

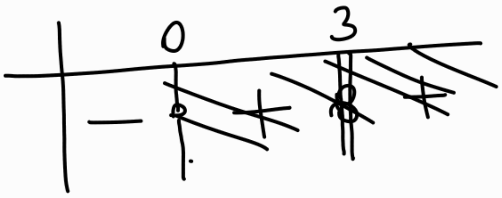
For D_1

$$\frac{x^2-6x+9}{x^3-4x^2+6x}\geq 0 \Rightarrow \frac{\overbrace{(x-3)^2}^{(i)}}{\underbrace{x}_{(ii)}\cdot\underbrace{(x^2-4x+6)}_{(iii)}} \geq 0$$

(i) $x=3$ is double root (won't change the sign)

(ii) $x=0$ is root and $x\neq 0$ since it means division by 0.

(iii) For x^2-4x+6, $\Delta=b^2-4ac=16-24=-8<0$, no real root.



$D_1=(0,\infty)$

(If the function is given as $\sqrt{\frac{f(x)}{g(x)}}$ it can't be considered as $\frac{\sqrt{f(x)}}{\sqrt{g(x)}}$ for the domain!)

For D_2

cos is defined for all $x\in\mathbb{R}$, but ln is not. Inside of ln must be positive.

$\frac{1}{x}>0 \Rightarrow x>0$ ($x\neq 0$, because it is a divisor)

$D_2=(0,\infty)$

$D(f)=D_1\cap D_2=(0,\infty)$

7 $f(x) = \underbrace{\ln\left(\dfrac{x^2-4x+3}{x.(x+1)}\right)}_{D_1} + \underbrace{|x^2|}_{D_2}$

For D_1

Inside of $\ln$ must be positive.

$$\frac{x^2-4x+3}{x.(x+1)} > 0 \Rightarrow \frac{\overbrace{(x-1)}^{x=1}.\overbrace{(x-3)}^{x=3}}{\underbrace{x}_{x=0}.\underbrace{(x+1)}_{x=-1}} > 0$$

Divisors can't be $0 \Rightarrow x \neq 0,\ x \neq -1$

Roots can't be $0 \Rightarrow x \neq 1,\ x \neq 3$ (>0 not ≥ 0!)

$D_1 = (-\infty, -1) \cup (0, 1) \cup (3, \infty)$

(If the function is in the form $\ln\left(\dfrac{f(x)}{g(x)}\right)$ it <u>can't</u> be considered as $\ln(f(x)) - \ln(g(x))$ for the domain!)

For D_2

$D_2 = \mathbb{R}$ (absolute value is defined for all $x \in \mathbb{R}$, so is x^2)

$D(f) = D_1 \cap D_2 = (-\infty, -1) \cup (0, 1) \cup (3, \infty)$

8 $f(x) = \underbrace{\dfrac{x^2+x}{x\sqrt{x+2}}}_{D_1} \cdot \underbrace{\arcsin(x^2)}_{D_2}$

For D_1

(i) Can't divide by 0. (ii) Inside of $\sqrt{\ }$ can't be negative

$x \neq 0$ and $x+2 > 0 \Rightarrow x > -2 \Rightarrow D_1 = (-2, 0) \cup (0, \infty)$
(or $(-2, \infty) \setminus \{0\}$)

($x^2 + x = x(x+1)$ and can be cancelled with denominator x.
For domain, <u>don't</u> make any cancellation, consider $x \neq 0$!)

For D_2

Domain of arcsine is $[-1,1]$ (Remember the range of sine!)
and x^2 is defined for this interval (for $\mathbb{R}$, in general)

$D_2 = [-1,1]$

$D(f) = D_1 \cap D_2 = [-1,0) \cup (0,1] = [-1,1] \setminus \{0\}$

(9) $f(x) = \dfrac{\overbrace{\ln(\ln(\sqrt{x+3}))}^{D_1}}{\underbrace{\sqrt{|x|-x}}_{D_2}}$

For D_1

Inside of ln must be positive.

$\ln(\sqrt{x+3}) > 0 \Rightarrow (\sqrt{x+3})^2 > 1^2 \Rightarrow x+3 > 1 \Rightarrow x > -2$

$D_1 = (-2, \infty)$

(Value of ln is positive if the function inside is > 1.
For values between 0 and 1, ln gets negative velue.)

For D_2

Divisor can't be 0 and inside of $\sqrt{\ }$ must be nonnegative.

$|x| - x > 0 \Rightarrow |x| > x \Rightarrow x < 0 \qquad D_2 = (-\infty, 0)$

($|\cdot|$ of x is greater then x iff x is negative)

$D(f) = D_1 \cap D_2 = (-2,0)$

(10) $f(x) = \dfrac{\overbrace{\sin(e^x \cdot \pi)}^{D_1}}{\underbrace{\sqrt{\ln x} + e^x}_{D_2}}$

For D_1

The function sine is defined for all $x \in \mathbb{R}$ and so does exponential function.

$\Rightarrow D_1 = \mathbb{R}$

For D_2

(i) Denominator can't be 0.

$$\sqrt{\ln x} + e^x \neq 0.$$

We know that the value of $\sqrt{\ }$ is nonnegative and so does exponential function. Also, their sum can't be 0, because e^x is always positive. No problem.

(ii) Inside of $\sqrt{\ }$ must be nonnegative and $\ln$ is nonnegative if the function inside is $\geqslant 1$.

$$\ln x \geqslant 0 \Rightarrow x \geqslant 1 \Rightarrow D_2 = [1, \infty)$$

$$D(f) = D_1 \cap D_2 = [1, \infty)$$

11) $f(x) = \underbrace{\sqrt{\ln(2x-x^2)}}_{D_1} + \underbrace{\arccos\left(\frac{1}{x}\right)}_{D_2}$

For D_1

Inside of $\sqrt{\ }$ must be nonnegative and $\ln$ is nonnegative for inputs $\geqslant 1$.

$$\ln(2x-x^2) \geqslant 0 \Rightarrow 2x - x^2 \geqslant 1 \Rightarrow x^2 - 2x + 1 \leqslant 0 \Rightarrow (x-1)^2 \leqslant 0$$

$\Rightarrow x = 1$ (square of something can't be negative)

$$D_1 = \{1\}$$

For D_2

The domain of arccos is between $[-1, 1]$ and $x \neq 0$.

(For the domain of arccos, remember the range of cos)

$$-1 \leqslant \frac{1}{x} \leqslant 1 \Rightarrow \text{(i) } -1 \leqslant \frac{1}{x} \quad \underline{\text{and}} \quad \text{(ii) } \frac{1}{x} \leqslant 1$$

We have two cases: $x > 0$ <u>or</u> $x < 0$

(If $x < 0$, multiplication by x will reverse inequalities)

Case 1 (If $x>0$) Multiply both sides with x

(i) $-x \leq 1 \Rightarrow 1 \leq x$ ⎫ $x \geq 1 \Rightarrow x \in [1, \infty)$
(ii) $1 \leq x$ ⎭

Case 2 (If $x<0$) Multiply both sides with x

(i) $-x \geq 1 \Rightarrow -1 \geq x$ ⎫ $x \leq -1 \Rightarrow x \in (-\infty, -1]$
(ii) $1 \geq x$ ⎭

(and means "both must be satisfied"
or means "at least one of them must be satisfied")

$D_2 = (-\infty, -1] \cup [1, \infty) = \mathbb{R} \setminus (-1, 1)$

$D(f) = D_1 \cap D_2 = \{1\}$

(12) $f(x) = \dfrac{\overbrace{\arccos(\ln x^2)}^{D_1}}{\underbrace{x^2+1}_{D_2}}$

For D_1

Domain of arccos is $[-1,1]$. Put $\ln x^2$ into this interval.

$-1 \leq \ln x^2 \leq 1 \underset{\text{use exponential function}}{\Rightarrow} e^{-1} \leq e^{\ln x^2} \leq e^1 \underset{(*)}{\Rightarrow} \sqrt{\frac{1}{e}} \leq \sqrt{x^2} \leq \sqrt{e}$

$\Rightarrow \sqrt{\frac{1}{e}} \leq |x| \leq \sqrt{e} \Rightarrow$ (i) $\frac{1}{\sqrt{e}} \leq x \leq \sqrt{e}$ or $-\sqrt{e} \leq x \leq -\frac{1}{\sqrt{e}}$

$\Rightarrow D_1 = \left[-\sqrt{e}, -\frac{1}{\sqrt{e}}\right] \cup \left[\frac{1}{\sqrt{e}}, \sqrt{e}\right]$

(*) Remember the property $e^{\ln a} = a^{\ln e} = a^1 = a$

For D_2

x^2+1 is defined for all $x \in \mathbb{R}$ and $x^2+1 \neq 0 \Rightarrow D_2 = \mathbb{R}$

$D(f) = D_1 \cap D_2 = \left[-\sqrt{e}, -\frac{1}{\sqrt{e}}\right] \cup \left[\frac{1}{\sqrt{e}}, \sqrt{e}\right]$

13) $f(x) = \underbrace{\ln\left(\dfrac{9-x^2}{x^2+x}\right)}_{D_1} \cdot \underbrace{\operatorname{arccot}(\sqrt{x})}_{D_2}$

For D_1

Inside of ln must be positive.

$\dfrac{9-x^2}{x^2+x} > 0 \Rightarrow \dfrac{(3-x).(3+x)}{x(x+1)} > 0$ $\quad$ $x=3,\ x=-3$, $x=0,\ x=-1$ } roots (will change sign)

		-3		-1		0		3	
$9-x^2$	—	o						o	—
x^2+x				o	—	o			
$\dfrac{9-x^2}{x^2+x}$	—	o		o	—	o		o	—

$D_1 = (-3,-1) \cup (0,3)$

(Consider inside of ln all together, don't break into several ln's)

For D_2

Domain of arccot is $\mathbb{R}$ (remember the range of cot)

But inside $\sqrt{\ }$ must be ≥ 0

$x \geq 0 \Rightarrow D_2 = [0,\infty)$

$D(f) = D_1 \cap D_2 = (0,3)$

14) $f(x) = \dfrac{\overbrace{\ln(18-2x^2)}^{D_1}}{\underbrace{|2x-5|}_{D_2}} + \underbrace{\arcsin(x-3)}_{D_3}$

For D_1

Inside of ln must be positive

$18-2x^2 > 0 \Rightarrow 18 > 2x^2 \Rightarrow \sqrt{9} > \sqrt{x^2} \Rightarrow |x| < 3$

$\Rightarrow -3 < x < 3$

$D_1 = (-3,3)$

For D_2

Divisor can't be zero. (No problem with absolute value.)

$2x-5 \neq 0 \Rightarrow x \neq \frac{5}{2} \quad (x \neq 2.5)$

$D_2 = \mathbb{R} \setminus \{\frac{5}{2}\}$

For D_3

Domain of arcsin is $[-1,1]$.

$-1 \leq x-3 \leq 1 \Rightarrow 2 \leq x \leq 4$

$D_3 = [2,4]$

$D(f) = D_1 \cap D_2 \cap D_3 = [2, \frac{5}{2}) \cup (\frac{5}{2}, 3) = [2,3) \setminus \{\frac{5}{2}\}$

15) $f(x) = \dfrac{\overbrace{\cos(\arccos(x+1))}^{D_1}}{\underbrace{\log_5(x+3)}_{D_2}}$

For D_1

Domain of arccos is $[-1,1]$ and cos is $\mathbb{R}$.

$-1 \leq x+1 \leq 1 \Rightarrow -2 \leq x \leq 0$

$D_1 = [-2,0]$

(cos and arccos are inverse of each other. Thus, the value of it is same as $x+1$ $(f(f^{-1}(x+1)) = x+1)$. But domain must be considered with its given form!)

For D_2

Divisor can't be 0 and inside of ln must be positive

$x+3 \neq 1 \Rightarrow x \neq -2$ $(\log_5 1 = 0)$

$x+3 > 0 \Rightarrow x > -3$

$D_2 = (-3, \infty) \setminus \{-2\}$

$D(f) = D_1 \cap D_2 = (-2, 0]$