



# Navier–Stokes in Cylindrical formulation

## Weak form and Nek implementation

### 1 Formulation of Navier–Stokes

#### 1.1 Mapping from Cartesian to Cylindrical

$$x_c = x \quad (1a)$$

$$y_c = r \cos(\theta) \quad (1b)$$

$$z_c = r \sin(\theta) \quad (1c)$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r \sin(\theta) \\ 0 & \sin(\theta) & r \cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (2)$$

$$\Rightarrow \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \quad (3)$$

$$\hat{x} = \hat{x}_c \quad (4a)$$

$$\hat{r} = \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c \quad (4b)$$

$$\hat{\theta} = -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \quad (4c)$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (5)$$

## 1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) &= \left( \frac{\partial\psi}{\partial x} \frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial x_c} \right) \hat{x}_c + \\ &\quad \left( \frac{\partial\psi}{\partial x} \frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial y_c} \right) \hat{y}_c + \\ &\quad \left( \frac{\partial\psi}{\partial x} \frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial z_c} \right) \hat{z}_c \\ \therefore \nabla(\psi) &= \frac{\partial\psi}{\partial x} \hat{x} + \frac{\partial\psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \hat{\theta}\end{aligned}\tag{6}$$

Laplacian of a scalar field

$$\begin{aligned}\nabla \cdot \nabla(\psi) &= \left[ \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right] \cdot \left[ \hat{x} \frac{\partial\psi}{\partial x} + \hat{r} \frac{\partial\psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right] \\ &= \left\{ \begin{aligned} &\left[ \hat{x} \cdot \hat{x} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial\hat{x}}{\partial x} \cdot \hat{x} \right] + \left[ \hat{x} \cdot \hat{r} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial\hat{x}}{\partial x} \cdot \hat{r} \right] + \left[ \hat{x} \cdot \hat{\theta} \frac{\partial}{\partial x} \left( \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{x}}{\partial x} \cdot \hat{\theta} \right] \\ &+ \left[ \hat{r} \cdot \hat{x} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial\hat{r}}{\partial r} \cdot \hat{x} \right] + \left[ \hat{r} \cdot \hat{r} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial\hat{r}}{\partial r} \cdot \hat{r} \right] + \left[ \hat{r} \cdot \hat{\theta} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{r}}{\partial r} \cdot \hat{\theta} \right] \\ &+ \left[ \hat{\theta} \cdot \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial x} + \frac{1}{r} \frac{\partial\psi}{\partial x} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{x} \right] + \left[ \hat{\theta} \cdot \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial r} + \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{r} \right] \\ &+ \left[ \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{\theta} \right] \end{aligned} \right\} \\ \therefore \nabla^2(\psi) &= \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\psi}{\partial \theta^2}\end{aligned}\tag{7}$$

Divergence of a Vector field

$$\begin{aligned}\nabla \cdot (\mathbf{u}) &= \left( \hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) (u_x \hat{x} + u_r \hat{r} + u_\theta \hat{\theta}) \\ \nabla \cdot (\mathbf{u}) &= \left( \hat{x} \cdot \hat{x} \frac{\partial u_x}{\partial x} + \hat{x} \cdot \frac{\partial\hat{x}}{\partial x} u_x + \hat{x} \cdot \hat{r} \frac{\partial u_r}{\partial x} + \hat{x} \cdot \frac{\partial\hat{r}}{\partial x} u_r + \hat{x} \cdot \hat{\theta} \frac{\partial u_\theta}{\partial x} + \hat{x} \cdot \frac{\partial\hat{\theta}}{\partial x} u_\theta \right) \\ &+ \left( \hat{r} \cdot \hat{x} \frac{\partial u_x}{\partial r} + \hat{r} \cdot \frac{\partial\hat{x}}{\partial r} u_x + \hat{r} \cdot \hat{r} \frac{\partial u_r}{\partial r} + \hat{r} \cdot \frac{\partial\hat{r}}{\partial r} u_r + \hat{r} \cdot \hat{\theta} \frac{\partial u_\theta}{\partial r} + \hat{r} \cdot \frac{\partial\hat{\theta}}{\partial r} u_\theta \right) \\ &+ \left( \hat{\theta} \cdot \hat{x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} + \hat{\theta} \cdot \frac{\partial\hat{x}}{\partial \theta} \frac{1}{r} u_x + \hat{\theta} \cdot \hat{r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \hat{\theta} \cdot \frac{\partial\hat{r}}{\partial \theta} \frac{1}{r} u_r + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \hat{\theta} \cdot \frac{\partial\hat{\theta}}{\partial \theta} \frac{1}{r} u_\theta \right)\end{aligned}$$

$$\nabla \cdot (\mathbf{u}) = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \quad (8)$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\begin{aligned} \nabla(\mathbf{u}) &= \left[ \hat{x} \frac{\partial}{\partial x} + \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right] \left[ u_x \hat{x} + u_r \hat{r} + u_\theta \hat{\theta} \right] \\ &= \left\{ \begin{aligned} &\left[ \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} u_x \frac{\partial \hat{x}}{\partial x} \right] + \left[ \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} u_r \frac{\partial \hat{r}}{\partial x} \right] + \left[ \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{x} u_\theta \frac{\partial \hat{\theta}}{\partial x} \right] \\ &+ \left[ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} u_x \frac{\partial \hat{x}}{\partial r} \right] + \left[ \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} u_r \frac{\partial \hat{r}}{\partial r} \right] + \left[ \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{r} u_\theta \frac{\partial \hat{\theta}}{\partial r} \right] \\ &+ \left[ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{u_x}{r} \frac{\partial \hat{x}}{\partial \theta} \right] + \left[ \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{r}}{\partial \theta} \right] + \left[ \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} + \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{\theta}}{\partial \theta} \right] \end{aligned} \right. \quad (9) \end{aligned}$$

$$= \left\{ \begin{aligned} &\left[ \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \right] \\ &+ \left[ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \right] \\ &+ \left[ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \right] \end{aligned} \right. \quad (10)$$

Which, I rearrange as gradients of individual velocity components (and their respective unit vectors)

$$\nabla(u_x \hat{x}) = \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} \quad (11a)$$

$$\nabla(u_r \hat{r}) = \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} \quad (11b)$$

$$\nabla(u_\theta \hat{\theta}) = \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \quad (11c)$$

$$\therefore \nabla(\mathbf{u}) = \begin{bmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} & \hat{\theta} \left( -\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} & \hat{\theta} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{bmatrix} \quad (12)$$

Vector Laplacian:

$$\nabla \cdot \nabla(\mathbf{u}) = \left( \hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \begin{pmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \left( -\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{\theta} \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned} & \left( \hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left( \hat{x} \frac{\partial u_x}{\partial x} + \hat{r} \frac{\partial u_x}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} \\ = & \hat{x} \cdot \left( \hat{x} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{x} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{x}}{\partial x} \\ & + \hat{x} \cdot \left( \hat{r} \frac{\partial}{\partial x} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{x} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{x}}{\partial r} \\ & + \hat{x} \cdot \left( \hat{\theta} \frac{\partial}{\partial x} \left( \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{x}}{\partial \theta} \\ & + \hat{r} \cdot \left( \hat{x} \frac{\partial}{\partial r} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{r} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{r}}{\partial r} \\ & + \hat{r} \cdot \left( \hat{r} \frac{\partial^2 u_x}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{r} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{r}}{\partial r} \\ & + \hat{r} \cdot \left( \hat{\theta} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{r}}{\partial \theta} \\ & + \hat{\theta} \cdot \left( \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{\theta} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\ & + \hat{\theta} \cdot \left( \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{\theta} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\ & + \hat{\theta} \cdot \left( \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\ & = \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r} \frac{\partial u_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} \end{aligned} \tag{13}$$

Second column:

$$\begin{aligned}
& \left( \hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left( \hat{x} \frac{\partial u_r}{\partial x} + \hat{r} \frac{\partial u_r}{\partial r} - \hat{\theta} \frac{u_\theta}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} \\
&= \hat{x} \cdot \left( \hat{x} \frac{\partial^2 u_r}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{x} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left( \hat{r} \frac{\partial}{\partial x} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{x} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left( -\hat{\theta} \frac{\partial}{\partial x} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial x} \frac{u_\theta}{r} \right) \hat{r} - \hat{x} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left( \hat{\theta} \frac{\partial}{\partial x} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{r} \cdot \left( \hat{x} \frac{\partial}{\partial r} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{r} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left( \hat{r} \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{r} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left( -\hat{\theta} \frac{\partial}{\partial r} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial r} \frac{u_\theta}{r} \right) \hat{r} - \hat{r} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left( \hat{\theta} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{\theta} \cdot \left( \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{\theta} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{\theta} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( -\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_\theta}{r} \right) \hat{r} - \hat{\theta} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&= \left[ \frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} \right] \hat{r} + \left[ -\frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right] \hat{\theta} \tag{14}
\end{aligned}$$

Third column:

$$\begin{aligned}
& \left( \hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left( \hat{x} \frac{\partial u_\theta}{\partial x} + \hat{r} \frac{\partial u_\theta}{\partial r} + \hat{\theta} \frac{u_r}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \\
&= \hat{x} \cdot \left( \hat{x} \frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{x} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left( \hat{r} \frac{\partial}{\partial x} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{x} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left( \hat{\theta} \frac{\partial}{\partial x} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial x} \frac{u_r}{r} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left( \hat{\theta} \frac{\partial}{\partial x} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{r} \cdot \left( \hat{x} \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{r} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left( \hat{r} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{r} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left( \hat{\theta} \frac{\partial}{\partial r} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial r} \frac{u_r}{r} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left( \hat{\theta} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{\theta} \cdot \left( \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{\theta} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_r}{r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{u_r}{r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left( \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&= \left[ \frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \right] \hat{\theta} + \left[ -\frac{u_r}{r^2} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} \quad (15)
\end{aligned}$$

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \begin{cases} \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} + \\ \left[ \frac{\partial^2 u_r}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} + \\ \left[ \frac{\partial^2 u_\theta}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \hat{\theta} \end{cases} \quad (16)$$

Which can be written using the Laplacian of individual scalar fields,  $u_x, u_r, u_\theta$  as,

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \nabla^2(u_x)\hat{x} + \left( \nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{r} + \left( \nabla^2(u_\theta) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\theta} \quad (17)$$

For the weak Laplacian we need the gradient of the test functions:

$$\nabla(v_x \hat{x}) \cdot = \left( \hat{x} \frac{\partial v_x}{\partial x} \hat{x} \cdot + \hat{x} \frac{\partial v_x}{\partial r} \hat{r} \cdot + \hat{x} \frac{1}{r} \frac{\partial v_x}{\partial \theta} \hat{\theta} \cdot \right) \quad (18a)$$

$$\nabla(v_r \hat{r}) \cdot = \left( \hat{r} \frac{\partial v_r}{\partial x} \hat{x} \cdot + \hat{r} \frac{\partial v_r}{\partial r} \hat{r} \cdot + \hat{r} \frac{1}{r} \frac{\partial v_r}{\partial \theta} \hat{\theta} \cdot + \hat{\theta} \frac{v_r}{r} \hat{\theta} \cdot \right) \quad (18b)$$

$$\nabla(v_\theta \hat{\theta}) \cdot = \left( \hat{\theta} \frac{\partial v_\theta}{\partial x} \hat{x} \cdot + \hat{\theta} \frac{\partial v_\theta}{\partial r} \hat{r} \cdot + \hat{\theta} \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \hat{\theta} \cdot - \hat{r} \frac{v_\theta}{r} \hat{\theta} \cdot \right) \quad (18c)$$

Strain rate tensor  $\mathcal{S} = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ :

$$2\mathcal{S} = \begin{cases} \left( \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} \right) + \left( \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} \frac{\partial u_x}{\partial r} \hat{r} + \hat{x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{\theta} \right) \\ \left( \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} \right) + \left( \hat{r} \frac{\partial u_r}{\partial x} \hat{x} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{\theta} + \hat{\theta} \frac{u_r}{r} \hat{\theta} \right) \\ \left( \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \right) + \left( \hat{\theta} \frac{\partial u_\theta}{\partial x} \hat{x} + \hat{\theta} \frac{\partial u_\theta}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{r} \frac{u_\theta}{r} \hat{\theta} \right) \end{cases} \quad (19)$$

Rearranging terms for convenience:

$$2\mathcal{S} = \begin{cases} \hat{x} \left( 2 \frac{\partial u_x}{\partial x} \right) \hat{x} & + \hat{x} \left( \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) \hat{r} & + \hat{x} \left( \frac{\partial u_\theta}{\partial x} + \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{\theta} \\ + \hat{r} \left( \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) \hat{x} & + \hat{r} \left( 2 \frac{\partial u_r}{\partial r} \right) \hat{r} & + \hat{r} \left( \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) \hat{\theta} \\ + \hat{\theta} \left( \frac{1}{r} \frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x} \right) \hat{x} & + \hat{\theta} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \hat{r} & + \hat{\theta} \left( \frac{2}{r} \frac{\partial u_\theta}{\partial \theta} + 2 \frac{u_r}{r} \right) \hat{\theta} \end{cases} \quad (20)$$

Now taking the dot product with the gradient of the test function,  $\nabla(v) \cdot$ .

$$\nabla(v_x \hat{x}) \cdot \mathcal{S} = \begin{cases} \frac{\partial v_x}{\partial x} \frac{\partial u_x}{\partial x} + \frac{1}{2} \frac{\partial v_x}{\partial r} \left( \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) \\ + \frac{1}{2r} \frac{\partial v_x}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x} \right) \end{cases} \quad (21a)$$

$$\nabla(v_r \hat{r}) \cdot \mathcal{S} = \begin{cases} \frac{1}{2} \frac{\partial v_r}{\partial x} \left( \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) + \frac{\partial v_r}{\partial r} \frac{\partial u_r}{\partial r} \\ + \frac{1}{2r} \frac{\partial v_r}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \frac{v_r}{r} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \end{cases} \quad (21b)$$

$$\nabla(v_\theta \hat{\theta}) \cdot \mathcal{S} = \begin{cases} \frac{1}{2} \frac{\partial v_\theta}{\partial x} \left( \frac{\partial u_\theta}{\partial x} + \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{1}{2} \frac{\partial v_\theta}{\partial r} \left( \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) \\ + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{v_\theta}{2r} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \end{cases} \quad (21c)$$

## 2 Nek Implementations:

### 2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$\begin{aligned} & - \int_{\Omega} v \frac{\partial}{\partial x_j} (p \delta_{ij}) d\Omega \\ & - \int_{\Omega} \frac{\partial}{\partial x_j} (v p \delta_{ij}) d\Omega + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \\ & - \int_{\partial\Omega} (v p \delta_{ij}) n_j dA + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \end{aligned}$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as  $\mathcal{J}$ . In the continuous formulation, this becomes:

$$\begin{aligned} w_i &= \int_{\Omega} p \frac{\partial v}{\partial x_i} d\Omega \\ &= \int_{\Omega} p \frac{\partial v}{\partial x_i} \left( \frac{\partial \Omega}{\partial \hat{\Omega}} \right) d\hat{\Omega} \\ &= \int_{\Omega} p \left( \frac{1}{\mathcal{J}} \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}} \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}} \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) \mathcal{J} d\hat{\Omega} \\ &= \int_{\Omega} p \left( \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \\ \Rightarrow w_i &= \int_{\Omega} p \left( \frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j} \right) d\hat{\Omega}, \end{aligned} \tag{22}$$

where,  $r_j$  represents the reference element coordinate directions  $r, s, t$ . After discretization, this becomes:

$$\Rightarrow w_i = \sum_k W(x_k) \left( \frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j}(x_k) \right) p$$

When cross derivative terms like  $\partial s / \partial x$  etc. are non-zero, we have the following expression

$$\begin{aligned} w_i &= (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left( \frac{\partial r}{\partial x_i} \cdot * W \cdot p \right) + \\ & (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left( \frac{\partial s}{\partial x_i} \cdot * W \cdot p \right) + \\ & (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left( \frac{\partial t}{\partial x_i} \cdot * W \cdot p \right) \end{aligned} \tag{23}$$



In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left( \frac{\partial r}{\partial x} \cdot * W \cdot * p \right) \quad (24a)$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left( \frac{\partial s}{\partial y} \cdot * W \cdot * p \right) \quad (24b)$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left( \frac{\partial t}{\partial z} \cdot * W \cdot * p \right) \quad (24c)$$

which can be further expressed purely as kronecker products, taking  $\partial r/\partial x$ ,  $\partial s/\partial y$ ,  $\partial t/\partial z$  etc. as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} W_{r2}) p \quad (25a)$$

$$w_2 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial y} W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \quad (25b)$$

$$w_3 = (\mathcal{D}_{t12}^T \frac{\partial t}{\partial z} W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \quad (25c)$$

For the cylindrical case we encounter some differences. There is an additional factor of  $R$  in front of the integral, *i.e.*  $\partial \Omega_{x,y,z} \rightarrow R \partial \Omega_{x,R,\theta} \rightarrow R \mathcal{J} \partial \hat{\Omega}$ . Also, for the  $\theta$  term we have an additional division by  $R$ . Finally, there is an additional term for  $R$  component for the derivative of the unit vector.

$$\begin{aligned} w_x &= \int_{\Omega} p \left( \frac{\partial r}{\partial x} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x} \frac{\partial v}{\partial t} \right) R d\hat{\Omega} \\ w_R &= \int_{\Omega} p \left( \frac{\partial r}{\partial R} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial R} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial R} \frac{\partial v}{\partial t} + \frac{v \mathcal{J}}{R} \right) R d\hat{\Omega} \\ w_{\theta} &= \int_{\Omega} p \left( \frac{\partial r}{\partial \theta} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \end{aligned}$$

Which will lead to the similar (but not identical) expressions for opgradt. Substituting  $\tilde{\mathcal{D}}_{s12}^T = (\partial s/\partial y) \mathcal{D}_{s12}^T + \mathcal{I}_{s12}^T/R$ , and if we assume the grid is Cartesian in the  $x - R - \theta$  space, *i.e.* there are no cross derivatives.

$$w_x = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left( \frac{\partial r}{\partial x} \cdot * R \cdot * W \cdot * p \right) \quad (26a)$$

$$w_R = \left( \mathcal{I}_{t12}^T \otimes \left( \frac{1}{\mathcal{J}} \frac{\partial s}{\partial y} \mathcal{D}_{s12}^T + \frac{\mathcal{I}_{s12}^T}{R} \right) \otimes \mathcal{I}_{r12}^T \right) (R \cdot * W \cdot * \mathcal{J} \cdot * p) \quad (26b)$$

$$w_{\theta} = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left( \frac{\partial t}{\partial z} \cdot * W \cdot * p \right) \quad (26c)$$

## 2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field  $u$ , defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can

represent  $\partial r/\partial x, \partial s/\partial x, \dots$  as matrices  $D_x r, D_x s, \dots$ . For a one dimensional case,  $D_x r$  is a diagonal matrix. Therefore, in one dimension,  $D_x = D_x r D_r$  and  $D_x^T = D_r^T (D_x r)^T$  etc.

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot \mathbf{u} &= \int_{\Omega} q \frac{\partial}{\partial x_i} (u_i) d\Omega \\
&= \int_{\Omega} q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\
&= \sum_k q(x_k) W(x_k) \left( \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} (x_k) \right) \\
&= \sum_k q(x_k) W(x_k) (D_{x_i} r_j) (D_{r_j} u_i) \\
&= q_k W_k (D_{x_i} r_j) (D_{r_j} u_i)
\end{aligned}$$

In the absence of cross geometric factors this becomes

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot \mathbf{u} &= q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w) \\
\int_{\Omega} q \nabla \cdot \mathbf{u} &= \begin{cases} q W (\mathcal{I}_{t12} \otimes \mathcal{I}_{s12} \otimes D_x r D_r) u \\ + q W (\mathcal{I}_{t12} \otimes D_y s D_s \otimes \mathcal{I}_{r12}) v \\ + q W (D_z t D_t \otimes \mathcal{I}_{s12} \otimes \mathcal{I}_{r12}) w \end{cases} \quad (27)
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot \mathbf{u} &= \begin{cases} q (W_t \mathcal{I}_{t12} \otimes W_s \mathcal{I}_{s12} \otimes W_r D_x r D_r) u \\ + q (W_t \mathcal{I}_{t12} \otimes W_s D_y s D_s \otimes W_r \mathcal{I}_{r12}) v \\ + q (W_t D_z t D_t \otimes W_s \mathcal{I}_{s12} \otimes W_r \mathcal{I}_{r12}) w \end{cases} \quad (28)
\end{aligned}$$

Here  $D_r, D_s, D_t$  is essentially  $D_{r12}, D_{s12}, D_{t12}$ , since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

For the cylindrical case we have

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot \mathbf{u} &= \int_{\Omega} q \frac{\partial}{\partial x_i} (u_i) d\Omega \\
&= \int_{\Omega} q \frac{\partial u_i}{\partial x_i} R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left( \frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial (Rv)}{\partial R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial R} + \frac{v}{R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left( \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{v}{R} + \frac{1}{R} \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left( \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{\mathcal{J} v}{R} + \frac{1}{R} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R d\hat{\Omega} \\
&= W_k q_k \left( R \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + R \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{R \mathcal{J} v}{R} + \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) \quad (29)
\end{aligned}$$

For undeformed elements,  $\mathcal{J}$  is a constant throughout the element. Therefore in kronecker notation we have

$$\int_{\Omega} q \nabla \cdot \mathbf{u} = \begin{cases} Wq(\mathcal{I}_{12} \otimes R \otimes D_{x12}rD_r)u \\ +Wq(\mathcal{I}_{12} \otimes (RD_{R12}sD_s + \mathcal{J}\mathcal{I}_{12}) \otimes \mathcal{I}_{12})v \\ +Wq(D_{\theta12}tD_t \otimes \mathcal{I}_{12} \otimes \mathcal{I}_{12})w \end{cases} \quad (30)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u} = \begin{cases} q(W_t\mathcal{I}_{12} \otimes W_sR \otimes W_rD_{x12}rD_r)u \\ +q(W_t\mathcal{I}_{12} \otimes W_s(RD_{R12}sD_s + \mathcal{J}\mathcal{I}_{12}) \otimes W_r\mathcal{I}_{12})v \\ +q(W_tD_{\theta12}tD_t \otimes W_s\mathcal{I}_{12} \otimes W_r\mathcal{I}_{12})w \end{cases} \quad (31)$$

### 2.3 Pressure Pseudo-Laplacian

$$S_{\Delta t} = DQD^T \quad (32)$$

where, if  $Q = H^{-1}$  there is no decoupling error and we have the Uzawa algorithm. Alternately,  $Q = B^{-1}$  in which case we incur a decoupling error but avoid the nested iterations since  $B$ , being diagonal, can be trivially inverted. In Nek, the operator  $D^T = \int_{\Omega} p \nabla \cdot v d\Omega$ , and  $D = \int_{\Omega} q \nabla \cdot u d\Omega$