Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$\begin{aligned} x_c &= x & \text{(1a)} \\ y_c &= r\cos(\theta) & \text{(1b)} \\ z_c &= r\sin(\theta) & \text{(1c)} \end{aligned}$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r\sin(\theta) \\ 0 & \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{2}$$

$$\implies \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \tag{3}$$

$$\begin{aligned} \hat{x} &= \hat{x}_c & \text{(4a)} \\ \hat{r} &= \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c & \text{(4b)} \\ \hat{\theta} &= -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c & \text{(4c)} \end{aligned}$$

$$\begin{aligned} \implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \\ \implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} & \text{(5)} \end{aligned}$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$
\begin{aligned}
\nabla(\psi) = &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_c}\right)\hat{x}_c + \\
&\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y_c}\right)\hat{y}_c + \\
&\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z_c}\right)\hat{z}_c
\end{aligned}
$$

$$
\therefore \nabla(\psi) = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} \tag{6}
$$

Laplacian of a scalar field

$$
\nabla\cdot\nabla(\psi) = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right]\cdot\left[\hat{x}\frac{\partial\psi}{\partial x} + \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]
$$

$$
= \begin{cases}
\left[\hat{x}\cdot\hat{x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{r}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\cdot\hat{x}\right] \\
+\left[\hat{r}\cdot\hat{x}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{r}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\cdot\hat{r}\right] \\
+\left[\hat{\theta}\cdot\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial x} + \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial\theta}\cdot\hat{\theta}\right] + \left[\hat{\theta}\cdot\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial\theta}\cdot\hat{\theta}\right] \\
+\left[\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\theta}\cdot\hat{\theta}\right]
\end{cases}
$$

$$
\therefore \nabla^2(\psi) = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta} \tag{7}
$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$
\nabla(\boldsymbol{u}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right]\left[\hat{x}u_x + \hat{r}u_r + \hat{\theta}u_\theta\right]
$$

$$
= \begin{cases}
\left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}u_x\frac{\partial\hat{x}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}u_r\frac{\partial\hat{r}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{x}u_\theta\frac{\partial\hat{\theta}}{\partial x}\right] \\
+\left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}u_x\frac{\partial\hat{x}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}u_r\frac{\partial\hat{r}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{r}u_\theta\frac{\partial\hat{\theta}}{\partial r}\right] \\
+\left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\hat{x} + \hat{\theta}\frac{u_x}{r}\frac{\partial\hat{x}}{\partial\theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\hat{r} + \hat{\theta}\textcolor{red}{\frac{u_r}{r}\frac{\partial\hat{r}}{\partial\theta}}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\theta} + \hat{\theta}\textcolor{blue}{\frac{u_\theta}{r}\frac{\partial\hat{\theta}}{\partial\theta}}\right]
\end{cases} \tag{8}
$$

$$\therefore \nabla(\boldsymbol{u}) = \begin{bmatrix} \hat{x}\dfrac{\partial u_x}{\partial x}\hat{x} & \hat{x}\dfrac{\partial u_r}{\partial x}\hat{r} & \hat{x}\dfrac{\partial u_\theta}{\partial x}\hat{\theta} \\ \hat{r}\dfrac{\partial u_x}{\partial r}\hat{x} & \hat{r}\dfrac{\partial u_r}{\partial r}\hat{r} & \hat{r}\dfrac{\partial u_\theta}{\partial r}\hat{\theta} \\ \hat{\theta}\dfrac{1}{r}\dfrac{\partial u_x}{\partial \theta}\hat{x} & \hat{\theta}\left(-\dfrac{u_\theta}{r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right)\hat{r} & \hat{\theta}\left(\dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{bmatrix} \tag{9}$$

Vector Laplacian:

$$\nabla \cdot \nabla(\boldsymbol{u}) = \left(\hat{x}\cdot\frac{\partial}{\partial x} + \hat{r}\cdot\frac{\partial}{\partial r} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial \theta}\right)\begin{pmatrix} \hat{x}\dfrac{\partial u_x}{\partial x}\hat{x}+ & \hat{x}\dfrac{\partial u_r}{\partial x}\hat{r}+ & \hat{x}\dfrac{\partial u_\theta}{\partial x}\hat{\theta}+ \\ \hat{r}\dfrac{\partial u_x}{\partial r}\hat{x}+ & \hat{r}\dfrac{\partial u_r}{\partial r}\hat{r}+ & \hat{r}\dfrac{\partial u_\theta}{\partial r}\hat{\theta}+ \\ \hat{\theta}\dfrac{1}{r}\dfrac{\partial u_x}{\partial \theta}\hat{x}+ & \hat{\theta}\left(-\dfrac{u_\theta}{r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right)\hat{r}+ & \hat{\theta}\left(\dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{pmatrix}$$

First Column

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_x}{\partial x}+\hat{r}\frac{\partial u_x}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}\\
= \quad & \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{x}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{x}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{r}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_x}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{r}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{\theta}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{\theta}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial r^2}+\frac{1}{r}\frac{\partial u_x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}
\end{aligned}
\tag{10}
$$

Second column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_r}{\partial x}+\hat{r}\frac{\partial u_r}{\partial r}-\hat{\theta}\frac{u_\theta}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}\\
&\quad=\quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{x}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{x}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(-\hat{\theta}\frac{\partial}{\partial x}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial x}\frac{u_\theta}{r}\right)\hat{r}-\hat{x}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{r}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{r}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(-\hat{\theta}\frac{\partial}{\partial r}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial r}\frac{u_\theta}{r}\right)\hat{r}-\hat{r}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{\theta}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{\theta}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(-\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_\theta}{r}\right)\hat{r}-\hat{\theta}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\right]\hat{r}+\left[-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{\theta}
\end{aligned}
\tag{11}
$$

Third column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_\theta}{\partial x}+\hat{r}\frac{\partial u_\theta}{\partial r}+\hat{\theta}\frac{u_r}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}\\
&\quad= \quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{x}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{x}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial x}\frac{u_r}{r}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\\
&\qquad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{r}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{r}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial r}\frac{u_r}{r}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\\
&\qquad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{\theta}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_r}{r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{u_r}{r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}\right]\hat{\theta}+\left[-\frac{u_r}{r^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r}
\end{aligned}
\tag{12}
$$

$$
\therefore \nabla\cdot\nabla(\boldsymbol{u})=\begin{cases}
\left[\dfrac{\partial^2 u_x}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_x}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}+\\
\left[\dfrac{\partial^2 u_r}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_r}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial\theta^2}-\dfrac{u_r}{r^2}-\dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial\theta}\right]\hat{r}+\\
\left[\dfrac{\partial^2 u_\theta}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_\theta}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial\theta^2}+\dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r^2}\right]\hat{\theta}
\end{cases}
\tag{13}
$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla\cdot\nabla(\boldsymbol{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\hat{\theta} \quad (14)$$

2 Nek Implementations:

2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$-\int_\Omega v\frac{\partial}{\partial x_j}(p\delta_{ij})d\Omega$$

$$-\int_\Omega \frac{\partial}{\partial x_j}(vp\delta_{ij})d\Omega + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega$$

$$-\int_{\partial\Omega}(vp\delta_{ij})n_j dA + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as $\mathcal{J}$. In the continuous formulation, this becomes:

$$\begin{aligned}
w_i &= \int_\Omega p\frac{\partial v}{\partial x_i}d\Omega \\
&= \int_\Omega p\frac{\partial v}{\partial x_i}\left(\frac{\partial\Omega}{\partial\hat{\Omega}}\right)d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{1}{\mathcal{J}}\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}}\frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}}\frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)d\hat{\Omega} \\
\implies w_i &= \int_\Omega p\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}\right)d\hat{\Omega},
\end{aligned} \quad (15)$$

where, r_j represents the reference element coordinate directions r, s, t. After discretization, this becomes:

$$\implies w_i = \sum_k W(x_k)\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}(x_k)\right)p$$

When cross derivative terms like $\partial s/\partial x$ *etc.* are non-zero, we have the following expression

$$
\begin{aligned}
w_i =& (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x_i}.*W.*p)+ \\
& (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial x_i}.*W.*p)+ \\
& (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial x_i}.*W.*p)
\end{aligned} \tag{16}
$$

In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x}.*W.*p) \tag{17a}$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial y}.*W.*p) \tag{17b}$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial z}.*W.*p) \tag{17c}$$

which can be further expressed purely as kronecker products, taking $\partial r/\partial x$, $\partial s/\partial y$, $\partial t/\partial z$ *etc.* as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} W_{r2})p \tag{18a}$$

$$w_2 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial y} W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2})p \tag{18b}$$

$$w_3 = (\mathcal{D}_{t12}^T \frac{\partial t}{\partial z} W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2})p \tag{18c}$$

For the cylindrical case we encounter some differences. There is an additional factor of R in front of the integral, *i.e.* $\partial\Omega_{x,y,z} \rightarrow R\partial\Omega_{x,R,\theta} \rightarrow R\mathcal{J}\partial\hat{\Omega}$. Also, for the θ term we have an additional division by R.

$$
\begin{aligned}
w_x =& \int_\Omega p\left(\frac{\partial r}{\partial x}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial v}{\partial t}\right) R d\hat{\Omega} \\
w_R =& \int_\Omega p\left(\frac{\partial r}{\partial R}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial R}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial R}\frac{\partial v}{\partial t}\right) R d\hat{\Omega} \\
w_\theta =& \int_\Omega p\left(\frac{\partial r}{\partial \theta}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta}\frac{\partial v}{\partial t}\right) d\hat{\Omega}
\end{aligned}
$$

Which will lead to the similar (but not identical) expressions for opgradt. If we assume the grid is Cartesian in the $x - R - \theta$ space, *i.e.* there are no cross derivatives.

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x}.*R.*W.*p) \tag{19a}$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial y}.*R.*W.*p) \tag{19b}$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial z}.*W.*p) \tag{19c}$$

The factors of R, W and the geometric factors can also be expressed in kronecker product form:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \frac{\partial r}{\partial x})(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (20a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (20b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2 W_{r2})p \quad (20c)$$

R-direction:

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\mathcal{I}_2 \otimes \frac{\partial s}{\partial R} \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (21a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (21b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p \quad (21c)$$

θ-direction:

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial \theta} \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (22a)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (22b)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p \quad (22c)$$

2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field u, defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can represent $\partial r/\partial x, \partial s/\partial x, \ldots$ as matrices $D_x r, D_x s, \ldots$. For a one dimensional case, $D_x r$ is a diagonal matrix. Therefore, in one dimension, $D_x = D_x r D_r$ and $D_x^T = D_r^T (D_x r)^T$ *etc.*

$$\begin{aligned}
\int_\Omega q \nabla \cdot u = & \int_\Omega q \frac{\partial}{\partial x_i}(u_i) d\Omega \\
= & \int_\Omega q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\
= & \sum_k q(x_k) W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j}(x_k) \right) \\
= & \sum_k q(x_k) W(x_k) (D_{x_i} r_j)(D_{r_j} u_i) \\
= & q_k W_k (D_{x_i} r_j)(D_{r_j} u_i)
\end{aligned}$$

In the absence of cross geometric factors this becomes

$$\int_\Omega q\nabla\cdot u = \quad q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w)$$

$$\int_\Omega q\nabla\cdot u = \quad \begin{cases} qW(\mathcal{I}_{t12}\otimes\mathcal{I}_{s12}\otimes D_x r D_r)u \\ +qW(\mathcal{I}_{t12}\otimes D_y s D_s\otimes\mathcal{I}_{r12})v \\ +qW(D_z t D_t\otimes\mathcal{I}_{s12}\otimes\mathcal{I}_{r12})w \end{cases} \tag{23}$$

$$\int_\Omega q\nabla\cdot u = \begin{cases} q(W_t\mathcal{I}_{t12}\otimes W_s\mathcal{I}_{s12}\otimes W_r D_x r D_r)u \\ +q(W_t\mathcal{I}_{t12}\otimes W_s D_y s D_s\otimes W_r\mathcal{I}_{r12})v \\ +q(W_t D_z t D_t\otimes W_s\mathcal{I}_{s12}\otimes W_r\mathcal{I}_{r12})w \end{cases} \tag{24}$$

Here D_r, D_s, D_t is essential $D_{r12}, D_{s12}, D_{t12}$, since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

For the cylindrical case we have

$$\begin{aligned}
\int_\Omega q\nabla\cdot u &= \int_\Omega q\frac{\partial}{\partial x_i}(u_i)d\Omega \\
&= \int_\Omega q\frac{\partial u_i}{\partial x_i}R\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega q\left(\frac{\partial u}{\partial x}+\frac{1}{R}\frac{\partial(Rv)}{\partial R}+\frac{1}{R}\frac{\partial w}{\partial\theta}\right)R\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega q\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial R}+\frac{v}{R}+\frac{1}{R}\frac{\partial w}{\partial\theta}\right)R\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega q\left(\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{v}{R}+\frac{1}{R}\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)R\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega q\left(\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{\mathcal{J}v}{R}+\frac{1}{R}\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)Rd\hat{\Omega} \\
&= W_k q_k\left(R\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+R\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{R\mathcal{J}v}{R}+\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)
\end{aligned} \tag{25}$$

For undeformed elements, $\mathcal{J}$ is a constant throughout the element. Therefore in kronecker notation we have

$$\int_\Omega q\nabla\cdot u = \begin{cases} Wq(\mathcal{I}_{12}\otimes R\otimes D_{x12} r D_r)u \\ +Wq(\mathcal{I}_{12}\otimes(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes\mathcal{I}_{12})v \\ +Wq(D_{\theta12}tD_t\otimes\mathcal{I}_{12}\otimes\mathcal{I}_{12})w \end{cases} \tag{26}$$

$$\int_\Omega q\nabla\cdot u = \begin{cases} q(W_t\mathcal{I}_{12}\otimes W_s R\otimes W_r D_{x12} r D_r)u \\ +q(W_t\mathcal{I}_{12}\otimes W_s(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes W_r\mathcal{I}_{12})v \\ +q(W_t D_{\theta12}tD_t\otimes W_s\mathcal{I}_{12}\otimes W_r\mathcal{I}_{12})w \end{cases} \tag{27}$$

2.3 Pressure Pseudo-Laplacian

$$S_{\Delta t} = DQD^T \tag{28}$$

where, if $Q = H^{-1}$ there is no decoupling error and we have the Uzawa algorithm. Alternately, $Q = B^{-1}$ in which case we incur a decoupling error but avoid the nested iterations since B, being diagonal, can be trivially inverted.