REPORT

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Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$x_c = x \tag{1a}$$
$$y_c = r\cos(\theta) \tag{1b}$$
$$z_c = r\sin(\theta) \tag{1c}$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r\sin(\theta) \\ 0 & \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{2}$$

$$\implies \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \tag{3}$$

$$\hat{x} = \hat{x}_c \tag{4a}$$
$$\hat{r} = \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c \tag{4b}$$
$$\hat{\theta} = -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \tag{4c}$$

$$\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{5}$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) = &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_c}\right)\hat{x_c} + \\ &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y_c}\right)\hat{y_c} + \\ &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z_c}\right)\hat{z_c}\end{aligned}$$

$$\therefore \nabla(\psi) = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} \tag{6}$$

Laplacian of a scalar field

$$\nabla\cdot\nabla(\psi) = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right]\cdot\left[\hat{x}\frac{\partial\psi}{\partial x} + \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]$$

$$= \begin{cases} \left[\hat{x}\cdot\hat{x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{r}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\cdot\hat{x}\right] \\ + \left[\hat{r}\cdot\hat{x}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{r}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\cdot\hat{r}\right] \\ + \left[\hat{\theta}\cdot\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial x} + \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial\theta}\cdot\hat{\theta}\right] + \left[\hat{\theta}\cdot\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial\theta}\cdot\hat{\theta}\right] \\ + \left[\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\theta}\cdot\hat{\theta}\right] \end{cases}$$

$$\therefore \nabla^2(\psi) = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta} \tag{7}$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\nabla(\boldsymbol{u}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right]\left[u_x\hat{x} + u_r\hat{r} + u_\theta\hat{\theta}\right]$$

$$= \begin{cases} \left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}u_x\frac{\partial \hat{x}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}u_r\frac{\partial \hat{r}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{x}u_\theta\frac{\partial \hat{\theta}}{\partial x}\right] \\ + \left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}u_x\frac{\partial \hat{x}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}u_r\frac{\partial \hat{r}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{r}u_\theta\frac{\partial \hat{\theta}}{\partial r}\right] \\ + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} + \hat{\theta}\frac{u_x}{r}\frac{\partial \hat{x}}{\partial \theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\frac{\partial \hat{r}}{\partial \theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} + \hat{\theta}\frac{u_\theta}{r}\frac{\partial \hat{\theta}}{\partial \theta}\right] \end{cases} \tag{8}$$

$$= \begin{cases} \left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta}\right] \\ + \left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta}\right] \\ + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r}\right] \end{cases} \tag{9}$$

Which, I rearrange as gradients of individual velocity components (and their respective unit vectors)

$$\nabla(u_x\hat{x}) = \hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} \tag{10a}$$

$$\nabla(u_r\hat{r}) = \hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta} \tag{10b}$$

$$\nabla(u_\theta\hat{\theta}) = \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r} \tag{10c}$$

$$\therefore \nabla(\boldsymbol{u}) = \begin{bmatrix} \hat{x}\frac{\partial u_x}{\partial x}\hat{x} & \hat{x}\frac{\partial u_r}{\partial x}\hat{r} & \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} \\ \hat{r}\frac{\partial u_x}{\partial r}\hat{x} & \hat{r}\frac{\partial u_r}{\partial r}\hat{r} & \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} \\ \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} & \hat{\theta}\left(-\frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{r} & \hat{\theta}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{bmatrix} \tag{11}$$

Vector Laplacian:

$$\nabla\cdot\nabla(\boldsymbol{u}) = \left(\hat{x}\cdot\frac{\partial}{\partial x} + \hat{r}\cdot\frac{\partial}{\partial r} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial \theta}\right)\begin{pmatrix} \hat{x}\frac{\partial u_x}{\partial x}\hat{x}+ & \hat{x}\frac{\partial u_r}{\partial x}\hat{r}+ & \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta}+ \\ \hat{r}\frac{\partial u_x}{\partial r}\hat{x}+ & \hat{r}\frac{\partial u_r}{\partial r}\hat{r}+ & \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta}+ \\ \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x}+ & \hat{\theta}\left(-\frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{r}+ & \hat{\theta}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{pmatrix}$$

First Column

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_x}{\partial x}+\hat{r}\frac{\partial u_x}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}\\
= \quad & \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{x}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{x}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{r}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_x}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{r}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{\theta}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{\theta}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&\qquad\qquad=\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial r^2}+\frac{1}{r}\frac{\partial u_x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}
\end{aligned}
\tag{12}
$$

Second column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_r}{\partial x}+\hat{r}\frac{\partial u_r}{\partial r}-\hat{\theta}\frac{u_\theta}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}\\
&\quad= \quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{x}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{x}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(-\hat{\theta}\frac{\partial}{\partial x}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial x}\frac{u_\theta}{r}\right)\hat{r}-\hat{x}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{r}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{r}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(-\hat{\theta}\frac{\partial}{\partial r}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial r}\frac{u_\theta}{r}\right)\hat{r}-\hat{r}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{\theta}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{\theta}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(-\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_\theta}{r}\right)\hat{r}-\hat{\theta}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\right]\hat{r}+\left[-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{\theta}
\end{aligned}
\tag{13}
$$

Third column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_\theta}{\partial x}+\hat{r}\frac{\partial u_\theta}{\partial r}+\hat{\theta}\frac{u_r}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}\\
&=\quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{x}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{x}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial x}\frac{u_r}{r}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{r}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{r}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial r}\frac{u_r}{r}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{\theta}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_r}{r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{u_r}{r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}\right]\hat{\theta}+\left[-\frac{u_r}{r^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r}
\end{aligned}
\tag{14}
$$

$$
\therefore\ \nabla\cdot\nabla(\boldsymbol{u})=\begin{cases}
\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}+\\
\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r}+\\
\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right]\hat{\theta}
\end{cases}
\tag{15}
$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla \cdot \nabla(\boldsymbol{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)\hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)\hat{\theta} \quad (16)$$

For the weak Laplacian we need the gradient of the test functions:

$$\nabla(v_x\hat{x})\cdot = \left(\hat{x}\frac{\partial v_x}{\partial x}\hat{x}\cdot + \hat{x}\frac{\partial v_x}{\partial r}\hat{r}\cdot + \hat{x}\frac{1}{r}\frac{\partial v_x}{\partial \theta}\hat{\theta}\cdot\right) \quad (17a)$$

$$\nabla(v_r\hat{r})\cdot = \left(\hat{r}\frac{\partial v_r}{\partial x}\hat{x}\cdot + \hat{r}\frac{\partial v_r}{\partial r}\hat{r}\cdot + \hat{r}\frac{1}{r}\frac{\partial v_r}{\partial \theta}\hat{\theta}\cdot + \hat{\theta}\frac{v_r}{r}\hat{\theta}\cdot\right) \quad (17b)$$

$$\nabla(v_\theta\hat{\theta})\cdot = \left(\hat{\theta}\frac{\partial v_\theta}{\partial x}\hat{x}\cdot + \hat{\theta}\frac{\partial v_\theta}{\partial r}\hat{r}\cdot + \hat{\theta}\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\hat{\theta}\cdot - \hat{r}\frac{v_\theta}{r}\hat{\theta}\cdot\right) \quad (17c)$$

Strain rate tensor $\mathcal{S} = 1/2(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$:

$$2\mathcal{S} = \begin{cases} \left(\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x}\right) + \left(\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}\frac{\partial u_x}{\partial r}\hat{r} + \hat{x}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{\theta}\right) \\ \left(\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta}\right) + \left(\hat{r}\frac{\partial u_r}{\partial x}\hat{x} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{\theta} + \hat{\theta}\frac{u_r}{r}\hat{\theta}\right) \\ \left(\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r}\right) + \left(\hat{\theta}\frac{\partial u_\theta}{\partial x}\hat{x} + \hat{\theta}\frac{\partial u_\theta}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{r}\frac{u_\theta}{r}\hat{\theta}\right) \end{cases} \quad (18)$$

Rearranging terms for convenience:

$$2\mathcal{S} = \begin{cases} \hat{x}\left(2\frac{\partial u_x}{\partial x}\right)\hat{x} & +\hat{x}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right)\hat{r} & +\hat{x}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial \theta}\right)\hat{\theta} \\ +\hat{r}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right)\hat{x} & +\hat{r}\left(2\frac{\partial u_r}{\partial r}\right)\hat{r} & +\hat{r}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\hat{\theta} \\ +\hat{\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x}\right)\hat{x} & +\hat{\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\hat{r} & +\hat{\theta}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial \theta} + 2\frac{u_r}{r}\right)\hat{\theta} \end{cases} \quad (19)$$

Now taking the dot product with the gradient of the test function, $\nabla(v)\cdot$

$$\nabla(v_x\hat{x})\cdot\mathcal{S} = \begin{cases} \frac{\partial v_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{1}{2}\frac{\partial v_x}{\partial r}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right) \\ +\frac{1}{2r}\frac{\partial v_x}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x}\right) \end{cases} \quad (20a)$$

$$\nabla(v_r\hat{r})\cdot\mathcal{S} = \begin{cases} \frac{1}{2}\frac{\partial v_r}{\partial x}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right) + \frac{\partial v_r}{\partial r}\frac{\partial u_r}{\partial r} \\ +\frac{1}{2r}\frac{\partial v_r}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{v_r}{r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \end{cases} \quad (20b)$$

$$\nabla(v_\theta\hat{\theta})\cdot\mathcal{S} = \begin{cases} \frac{1}{2}\frac{\partial v_\theta}{\partial x}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial \theta}\right) + \frac{1}{2}\frac{\partial v_\theta}{\partial r}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) \\ +\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - \frac{v_\theta}{2r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \end{cases} \quad (20c)$$

2 Nek Implementations:

2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$
\begin{gathered}
-\int_\Omega v \frac{\partial}{\partial x_j}(p\delta_{ij})d\Omega \\
-\int_\Omega \frac{\partial}{\partial x_j}(vp\delta_{ij})d\Omega + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega \\
-\int_{\partial\Omega} (vp\delta_{ij})n_j dA + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega
\end{gathered}
$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as $\mathcal{J}$. In the continuous formulation, this becomes:

$$
\begin{aligned}
w_i &= \int_\Omega p\frac{\partial v}{\partial x_i}d\Omega \\
&= \int_\Omega p\frac{\partial v}{\partial x_i}\left(\frac{\partial \Omega}{\partial \hat{\Omega}}\right)d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{1}{\mathcal{J}}\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}}\frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}}\frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)d\hat{\Omega} \\
\implies w_i &= \int_\Omega p\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}\right)d\hat{\Omega},
\end{aligned} \tag{21}
$$

where, r_j represents the reference element coordinate directions r, s, t. After discretization, this becomes:

$$
\implies w_i = \sum_k W(x_k)\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}(x_k)\right)p
$$

When cross derivative terms like $\partial s/\partial x$ *etc.* are non-zero, we have the following expression

$$
\begin{aligned}
w_i =& (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x_i}.*W.*p)+ \\
& (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial x_i}.*W.*p)+ \\
& (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial x_i}.*W.*p)
\end{aligned} \tag{22}
$$

In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}^T_{t12} \otimes \mathcal{I}^T_{s12} \otimes \mathcal{D}^T_{r12})(\frac{\partial r}{\partial x}. * W. * p) \tag{23a}$$

$$w_2 = (\mathcal{I}^T_{t12} \otimes \mathcal{D}^T_{s12} \otimes \mathcal{I}^T_{r12})(\frac{\partial s}{\partial y}. * W. * p) \tag{23b}$$

$$w_3 = (\mathcal{D}^T_{t12} \otimes \mathcal{I}^T_{s12} \otimes \mathcal{I}^T_{r12})(\frac{\partial t}{\partial z}. * W. * p) \tag{23c}$$

which can be further expressed purely as kronecker products, taking $\partial r/\partial x$, $\partial s/\partial y$, $\partial t/\partial z$ *etc.* as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}^T_{t12}\mathcal{I}_2 W_{t2} \otimes \mathcal{I}^T_{s12}\mathcal{I}_2 W_{s2} \otimes \mathcal{D}^T_{r12}\frac{\partial r}{\partial x} W_{r2})p \tag{24a}$$

$$w_2 = (\mathcal{I}^T_{t12}\mathcal{I}_2 W_{t2} \otimes \mathcal{D}^T_{s12}\frac{\partial s}{\partial y} W_{s2} \otimes \mathcal{I}^T_{r12}\mathcal{I}_2 W_{r2})p \tag{24b}$$

$$w_3 = (\mathcal{D}^T_{t12}\frac{\partial t}{\partial z} W_{t2} \otimes \mathcal{I}^T_{s12}\mathcal{I}_2 W_{s2} \otimes \mathcal{I}^T_{r12}\mathcal{I}_2 W_{r2})p \tag{24c}$$

For the cylindrical case we encounter some differences. There is an additional factor of R in front of the integral, *i.e.* $\partial\Omega_{x,y,z} \to R\partial\Omega_{x,R,\theta} \to R\mathcal{J}\partial\hat{\Omega}$. Also, for the θ term we have an additional division by R. Finally, there is an additional term for R, θ components for the derivative of the unit vectors.

$$\begin{aligned} w_x = & \int_\Omega p\left(\frac{\partial r}{\partial x}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial v}{\partial t}\right) R d\hat{\Omega} \\ w_R = & \int_\Omega p\left(\frac{\partial r}{\partial R}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial R}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial R}\frac{\partial v}{\partial t}\right) R d\hat{\Omega} \\ w_\theta = & \int_\Omega p\left(\frac{\partial r}{\partial \theta}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta}\frac{\partial v}{\partial t}\right) d\hat{\Omega} \end{aligned}$$

Which will lead to the similar (but not identical) expressions for opgradt. If we assume the grid is Cartesian in the $x - R - \theta$ space, *i.e.* there are no cross derivatives.

$$w_x = (\mathcal{I}^T_{t12} \otimes \mathcal{I}^T_{s12} \otimes \mathcal{D}^T_{r12})(\frac{\partial r}{\partial x}. * R. * W. * p) \tag{25a}$$

$$w_R = (\mathcal{I}^T_{t12} \otimes \mathcal{D}^T_{s12} \otimes \mathcal{I}^T_{r12})(\frac{\partial s}{\partial y}. * R. * W. * p) \tag{25b}$$

$$w_\theta = (\mathcal{D}^T_{t12} \otimes \mathcal{I}^T_{s12} \otimes \mathcal{I}^T_{r12})(\frac{\partial t}{\partial z}. * W. * p) \tag{25c}$$

The factors of R, W and the geometric factors can also be expressed in kronecker product form:

$$w_x = (\mathcal{I}^T_{t12} \otimes \mathcal{I}^T_{s12} \otimes \mathcal{D}^T_{r12})(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \frac{\partial r}{\partial x})(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \tag{26a}$$

$$= (\mathcal{I}^T_{t12}\mathcal{I}_2\mathcal{I}_2 \otimes \mathcal{I}^T_{s12}\mathcal{I}_2 R_2 \otimes \mathcal{D}^T_{r12}\frac{\partial r}{\partial x}\mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \tag{26b}$$

$$= (\mathcal{I}^T_{t12}\mathcal{I}_2\mathcal{I}_2 W_{t2} \otimes \mathcal{I}^T_{s12}\mathcal{I}_2 R_2 W_{s2} \otimes \mathcal{D}^T_{r12}\frac{\partial r}{\partial x}\mathcal{I}_2 W_{r2})p \tag{26c}$$

R-direction:

$$
\begin{aligned}
w_R = \quad & (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\mathcal{I}_2 \otimes \frac{\partial s}{\partial R} \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p & \text{(27a)} \\
= \quad & (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p & \text{(27b)} \\
= \quad & (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p & \text{(27c)}
\end{aligned}
$$

θ-direction:

$$
\begin{aligned}
w_\theta = \quad & (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial \theta} \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p & \text{(28a)} \\
= \quad & (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p & \text{(28b)} \\
= \quad & (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p & \text{(28c)}
\end{aligned}
$$

2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field u, defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can represent $\partial r/\partial x, \partial s/\partial x, \ldots$ as matrices $D_x r, D_x s, \ldots$. For a one dimensional case, $D_x r$ is a diagonal matrix. Therefore, in one dimension, $D_x = D_x r D_r$ and $D_x^T = D_r^T (D_x r)^T$ *etc.*

$$
\begin{aligned}
\int_\Omega q \nabla \cdot u = \quad & \int_\Omega q \frac{\partial}{\partial x_i}(u_i) d\Omega \\
= \quad & \int_\Omega q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\
= \quad & \sum_k q(x_k) W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j}(x_k) \right) \\
= \quad & \sum_k q(x_k) W(x_k) (D_{x_i} r_j)(D_{r_j} u_i) \\
= \quad & q_k W_k (D_{x_i} r_j)(D_{r_j} u_i)
\end{aligned}
$$

In the absence of cross geometric factors this becomes

$$
\int_\Omega q \nabla \cdot u = \quad q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w)
$$

$$
\int_\Omega q \nabla \cdot u = \quad \begin{cases} qW(\mathcal{I}_{t12} \otimes \mathcal{I}_{s12} \otimes D_x r D_r)u \\ +qW(\mathcal{I}_{t12} \otimes D_y s D_s \otimes \mathcal{I}_{r12})v \\ +qW(D_z t D_t \otimes \mathcal{I}_{s12} \otimes \mathcal{I}_{r12})w \end{cases} \tag{29}
$$

$$
\int_\Omega q \nabla \cdot u = \quad \begin{cases} q(W_t \mathcal{I}_{t12} \otimes W_s \mathcal{I}_{s12} \otimes W_r D_x r D_r)u \\ +q(W_t \mathcal{I}_{t12} \otimes W_s D_y s D_s \otimes W_r \mathcal{I}_{r12})v \\ +q(W_t D_z t D_t \otimes W_s \mathcal{I}_{s12} \otimes W_r \mathcal{I}_{r12})w \end{cases} \tag{30}
$$

Here D_r, D_s, D_t is essential $D_{r12}, D_{s12}, D_{t12}$, since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

For the cylindrical case we have

$$
\begin{aligned}
\int_\Omega q\nabla\cdot u = & \int_\Omega q\frac{\partial}{\partial x_i}(u_i)d\Omega \\
= & \int_\Omega q\frac{\partial u_i}{\partial x_i}R\mathcal{J}d\hat{\Omega} \\
= & \int_\Omega q\left(\frac{\partial u}{\partial x}+\frac{1}{R}\frac{\partial(Rv)}{\partial R}+\frac{1}{R}\frac{\partial w}{\partial\theta}\right)R\mathcal{J}d\hat{\Omega} \\
= & \int_\Omega q\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial R}+\frac{v}{R}+\frac{1}{R}\frac{\partial w}{\partial\theta}\right)R\mathcal{J}d\hat{\Omega} \\
= & \int_\Omega q\left(\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{v}{R}+\frac{1}{R}\frac{1}{\mathcal{J}}\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)R\mathcal{J}d\hat{\Omega} \\
= & \int_\Omega q\left(\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{\mathcal{J}v}{R}+\frac{1}{R}\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)Rd\hat{\Omega} \\
= & W_k q_k\left(R\frac{\partial r_j}{\partial x}\frac{\partial u}{\partial r_j}+R\frac{\partial r_j}{\partial R}\frac{\partial v}{\partial r_j}+\frac{R\mathcal{J}v}{R}+\frac{\partial r_j}{\partial\theta}\frac{\partial w}{\partial r_j}\right)
\end{aligned}
\tag{31}
$$

For undeformed elements, $\mathcal{J}$ is a constant throughout the element. Therefore in kronecker notation we have

$$
\int_\Omega q\nabla\cdot u = \begin{cases} Wq(\mathcal{I}_{12}\otimes R\otimes D_{x12}rD_r)u \\ +Wq(\mathcal{I}_{12}\otimes(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes\mathcal{I}_{12})v \\ +Wq(D_{\theta 12}tD_t\otimes\mathcal{I}_{12}\otimes\mathcal{I}_{12})w \end{cases} \tag{32}
$$

$$
\int_\Omega q\nabla\cdot u = \begin{cases} q(W_t\mathcal{I}_{12}\otimes W_sR\otimes W_rD_{x12}rD_r)u \\ +q(W_t\mathcal{I}_{12}\otimes W_s(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes W_r\mathcal{I}_{12})v \\ +q(W_tD_{\theta 12}tD_t\otimes W_s\mathcal{I}_{12}\otimes W_r\mathcal{I}_{12})w \end{cases} \tag{33}
$$

2.3 Pressure Pseudo-Laplacian

$$
S_{\Delta t} = DQD^T \tag{34}
$$

where, if $Q = H^{-1}$ there is no decoupling error and we have the Uzawa algorithm. Alternately, $Q = B^{-1}$ in which case we incur a decoupling error but avoid the nested iterations since B, being diagonal, can be trivially inverted.