REPORT

Date
November 29, 2022

Reference number


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# Navier–Stokes in Cylindrical formulation

## Weak form and Nek implementation

## 1 Formulation of Navier–Stokes

### 1.1 Mapping from Cartesian to Cylindrical

$$\begin{aligned} x_c &= x && \text{(1a)} \\ y_c &= r\cos(\theta) && \text{(1b)} \\ z_c &= r\sin(\theta) && \text{(1c)} \end{aligned}$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r\sin(\theta) \\ 0 & \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{2}$$

$$\implies \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \tag{3}$$

$$\begin{aligned} \hat{x} &= \hat{x}_c && \text{(4a)} \\ \hat{r} &= \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c && \text{(4b)} \\ \hat{\theta} &= -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c && \text{(4c)} \end{aligned}$$

$$\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{5}$$

## 1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$
\begin{aligned}
\nabla(\psi) = &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_c}\right)\hat{x_c} + \\
&\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y_c}\right)\hat{y_c} + \\
&\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z_c}\right)\hat{z_c}
\end{aligned}
$$

$$
\therefore \nabla(\psi) = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} \tag{6}
$$

Laplacian of a scalar field

$$
\nabla\cdot\nabla(\psi) = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right]\cdot\left[\hat{x}\frac{\partial\psi}{\partial x} + \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]
$$

$$
= \begin{cases}
\left[\hat{x}\cdot\hat{x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{r}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\cdot\hat{x}\right] \\
+\left[\hat{r}\cdot\hat{x}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{r}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\cdot\hat{r}\right] \\
+\left[\hat{\theta}\cdot\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial x} + \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial\theta}\cdot\hat{\theta}\right] + \left[\hat{\theta}\cdot\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial\theta}\cdot\hat{\theta}\right] \\
+\left[\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\theta}\cdot\hat{\theta}\right]
\end{cases}
$$

$$
\therefore \nabla^2(\psi) = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta} \tag{7}
$$

Divergence of a Vector field

$$
\nabla\cdot(\boldsymbol{u}) = \left(\hat{x}\cdot\frac{\partial}{\partial x} + \hat{r}\cdot\frac{\partial}{\partial r} + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(u_x\hat{x} + u_r\hat{r} + u_\theta\hat{\theta}\right)
$$

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{u}) = \quad &\left(\hat{x}\cdot\hat{x}\frac{\partial u_x}{\partial x} + \hat{x}\cdot\frac{\partial\hat{x}}{\partial x}u_x + \hat{x}\cdot\hat{r}\frac{\partial u_r}{\partial x} + \hat{x}\cdot\frac{\partial\hat{r}}{\partial x}u_r + \hat{x}\cdot\hat{\theta}\frac{\partial u_\theta}{\partial x} + \hat{x}\cdot\frac{\partial\hat{\theta}}{\partial x}u_\theta\right) \\
+ &\left(\hat{r}\cdot\hat{x}\frac{\partial u_x}{\partial r} + \hat{r}\cdot\frac{\partial\hat{x}}{\partial r}u_x + \hat{r}\cdot\hat{r}\frac{\partial u_r}{\partial r} + \hat{r}\cdot\frac{\partial\hat{r}}{\partial r}u_r + \hat{r}\cdot\hat{\theta}\frac{\partial u_\theta}{\partial r} + \hat{r}\cdot\frac{\partial\hat{\theta}}{\partial r}u_\theta\right) \\
+ &\left(\hat{\theta}\cdot\hat{x}\frac{1}{r}\frac{\partial u_x}{\partial\theta} + \hat{\theta}\cdot\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}u_x + \hat{\theta}\cdot\hat{r}\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \hat{\theta}\cdot\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}u_r + \hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \hat{\theta}\cdot\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}u_\theta\right)
\end{aligned}
$$

$$\nabla \cdot (\boldsymbol{u}) = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \right) \tag{8}$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\nabla(\boldsymbol{u}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right]\left[u_x\hat{x} + u_r\hat{r} + u_\theta\hat{\theta}\right]$$

$$= \begin{cases} \left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}u_x\frac{\partial \hat{x}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}u_r\frac{\partial \hat{r}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{x}u_\theta\frac{\partial \hat{\theta}}{\partial x}\right] \\ + \left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}u_x\frac{\partial \hat{x}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}u_r\frac{\partial \hat{r}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{r}u_\theta\frac{\partial \hat{\theta}}{\partial r}\right] \\ + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} + \hat{\theta}\frac{u_x}{r}\frac{\partial \hat{x}}{\partial \theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\frac{\partial \hat{r}}{\partial \theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} + \hat{\theta}\frac{u_\theta}{r}\frac{\partial \hat{\theta}}{\partial \theta}\right] \end{cases} \tag{9}$$

$$= \begin{cases} \left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta}\right] \\ + \left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta}\right] \\ + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r}\right] \end{cases} \tag{10}$$

Which, I rearrange as gradients of individual velocity components (and their respective unit vectors)

$$\nabla(u_x\hat{x}) = \hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} \tag{11a}$$

$$\nabla(u_r\hat{r}) = \hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta} \tag{11b}$$

$$\nabla(u_\theta\hat{\theta}) = \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r} \tag{11c}$$

$$\therefore \nabla(\boldsymbol{u}) = \begin{bmatrix} \hat{x}\frac{\partial u_x}{\partial x}\hat{x} & \hat{x}\frac{\partial u_r}{\partial x}\hat{r} & \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} \\ \hat{r}\frac{\partial u_x}{\partial r}\hat{x} & \hat{r}\frac{\partial u_r}{\partial r}\hat{r} & \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} \\ \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x} & \hat{\theta}\left(-\frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{r} & \hat{\theta}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{bmatrix} \tag{12}$$

Vector Laplacian:

$$\nabla \cdot \nabla(\boldsymbol{u}) = \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\right) \begin{pmatrix} \hat{x}\frac{\partial u_x}{\partial x}\hat{x}+ & \hat{x}\frac{\partial u_r}{\partial x}\hat{r}+ & \hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta}+ \\ \hat{r}\frac{\partial u_x}{\partial r}\hat{x}+ & \hat{r}\frac{\partial u_r}{\partial r}\hat{r}+ & \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta}+ \\ \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\hat{x}+ & \hat{\theta}\left(-\frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{r}+ & \hat{\theta}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned}
&\left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(\hat{x}\frac{\partial u_x}{\partial x} + \hat{r}\frac{\partial u_x}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right)\hat{x} \\
= \quad & \hat{x} \cdot \left(\hat{x}\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial \hat{x}}{\partial x}\frac{\partial u_x}{\partial x}\right)\hat{x} + \hat{x} \cdot \hat{x}\frac{\partial u_x}{\partial x}\frac{\partial \hat{x}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial x}\frac{\partial u_x}{\partial r}\right)\hat{x} + \hat{x} \cdot \hat{r}\frac{\partial u_x}{\partial r}\frac{\partial \hat{x}}{\partial r} \\
&+ \hat{x} \cdot \left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right) + \frac{\partial \hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right)\hat{x} + \hat{x} \cdot \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\frac{\partial \hat{x}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial r}\frac{\partial u_x}{\partial x}\right)\hat{x} + \hat{r} \cdot \hat{x}\frac{\partial u_x}{\partial x}\frac{\partial \hat{x}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r}\frac{\partial^2 u_x}{\partial r^2} + \frac{\partial \hat{r}}{\partial r}\frac{\partial u_x}{\partial r}\right)\hat{x} + \hat{r} \cdot \hat{r}\frac{\partial u_x}{\partial r}\frac{\partial \hat{x}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right) + \frac{\partial \hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right)\hat{x} + \hat{r} \cdot \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\frac{\partial \hat{x}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial \theta}\frac{1}{r}\frac{\partial u_x}{\partial x}\right)\hat{x} + \hat{\theta} \cdot \hat{x}\frac{\partial u_x}{\partial x}\frac{1}{r}\frac{\partial \hat{x}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial \theta}\frac{1}{r}\frac{\partial u_x}{\partial r}\right)\hat{x} + \hat{\theta} \cdot \hat{r}\frac{\partial u_x}{\partial r}\frac{1}{r}\frac{\partial \hat{x}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial \theta}\right)\hat{x} + \frac{\partial \hat{\theta}}{\partial \theta}\frac{1}{r^2}\frac{\partial u_x}{\partial \theta}\right)\hat{x} + \hat{\theta} \cdot \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial \theta}\frac{1}{r}\frac{\partial \hat{x}}{\partial \theta} \\
&\qquad = \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_x}{\partial \theta^2}\right]\hat{x}
\end{aligned} \tag{13}$$

Second column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_r}{\partial x}+\hat{r}\frac{\partial u_r}{\partial r}-\hat{\theta}\frac{u_\theta}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}\\
&\quad=\quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{x}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{x}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(-\hat{\theta}\frac{\partial}{\partial x}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial x}\frac{u_\theta}{r}\right)\hat{r}-\hat{x}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial x}\\
&\qquad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{r}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{r}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(-\hat{\theta}\frac{\partial}{\partial r}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial r}\frac{u_\theta}{r}\right)\hat{r}-\hat{r}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial r}\\
&\qquad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{\theta}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{\theta}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(-\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_\theta}{r}\right)\hat{r}-\hat{\theta}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\qquad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\right]\hat{r}+\left[-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{\theta}
\end{aligned}
\tag{14}
$$

Third column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_\theta}{\partial x}+\hat{r}\frac{\partial u_\theta}{\partial r}+\hat{\theta}\frac{u_r}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}\\
&= \quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{x}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{x}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial x}\frac{u_r}{r}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\\
&\quad+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{r}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{r}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial r}\frac{u_r}{r}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\\
&\quad+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{\theta}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_r}{r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{u_r}{r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&\quad+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}\right]\hat{\theta}+\left[-\frac{u_r}{r^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r}
\end{aligned}
\tag{15}
$$

$$
\therefore \nabla\cdot\nabla(\boldsymbol{u})=\begin{cases}
\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}+\\
\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r}+\\
\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right]\hat{\theta}
\end{cases}
\tag{16}
$$

Which can be written using the Laplacian of individual scalar fields, $u_x, u_r, u_\theta$ as,

$$\therefore \nabla\cdot\nabla(\boldsymbol{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\hat{\theta} \quad (17)$$

For the weak Laplacian we need the gradient of the test functions:

$$\nabla(v_x\hat{x})\cdot = \left(\hat{x}\frac{\partial v_x}{\partial x}\hat{x}\cdot + \hat{x}\frac{\partial v_x}{\partial r}\hat{r}\cdot + \hat{x}\frac{1}{r}\frac{\partial v_x}{\partial\theta}\hat{\theta}\cdot\right) \quad (18a)$$

$$\nabla(v_r\hat{r})\cdot = \left(\hat{r}\frac{\partial v_r}{\partial x}\hat{x}\cdot + \hat{r}\frac{\partial v_r}{\partial r}\hat{r}\cdot + \hat{r}\frac{1}{r}\frac{\partial v_r}{\partial\theta}\hat{\theta}\cdot + \hat{\theta}\frac{v_r}{r}\hat{\theta}\cdot\right) \quad (18b)$$

$$\nabla(v_\theta\hat{\theta})\cdot = \left(\hat{\theta}\frac{\partial v_\theta}{\partial x}\hat{x}\cdot + \hat{\theta}\frac{\partial v_\theta}{\partial r}\hat{r}\cdot + \hat{\theta}\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\hat{\theta}\cdot - \hat{r}\frac{v_\theta}{r}\hat{\theta}\cdot\right) \quad (18c)$$

Strain rate tensor $\mathcal{S} = 1/2(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$:

$$2\mathcal{S} = \begin{cases} \left(\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\hat{x}\right) + \left(\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}\frac{\partial u_x}{\partial r}\hat{r} + \hat{x}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\hat{\theta}\right) \\ \left(\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\hat{\theta}\right) + \left(\hat{r}\frac{\partial u_r}{\partial x}\hat{x} + \hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\hat{\theta} + \hat{\theta}\frac{u_r}{r}\hat{\theta}\right) \\ \left(\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\theta} - \hat{\theta}\frac{u_\theta}{r}\hat{r}\right) + \left(\hat{\theta}\frac{\partial u_\theta}{\partial x}\hat{x} + \hat{\theta}\frac{\partial u_\theta}{\partial r}\hat{r} + \hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\theta} - \hat{r}\frac{u_\theta}{r}\hat{\theta}\right) \end{cases} \quad (19)$$

Rearranging terms for convenience:

$$2\mathcal{S} = \begin{cases} \hat{x}\left(2\frac{\partial u_x}{\partial x}\right)\hat{x} & +\hat{x}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right)\hat{r} & +\hat{x}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{\theta} \\ +\hat{r}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right)\hat{x} & +\hat{r}\left(2\frac{\partial u_r}{\partial r}\right)\hat{r} & +\hat{r}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\hat{\theta} \\ +\hat{\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta} + \frac{\partial u_\theta}{\partial x}\right)\hat{x} & +\hat{\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\hat{r} & +\hat{\theta}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta} + 2\frac{u_r}{r}\right)\hat{\theta} \end{cases} \quad (20)$$

Now taking the dot product with the gradient of the test function, $\nabla(v)\cdot(2\mu\mathcal{S})$

$$\nabla(v_x\hat{x})\cdot(2\mu\mathcal{S}) = \begin{cases} 2\frac{\partial v_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial v_x}{\partial r}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right) \\ +\frac{1}{r}\frac{\partial v_x}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta} + \frac{\partial u_\theta}{\partial x}\right) \end{cases} \quad (21a)$$

$$\nabla(v_r\hat{r})\cdot(2\mu\mathcal{S}) = \begin{cases} \frac{\partial v_r}{\partial x}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right) + 2\frac{\partial v_r}{\partial r}\frac{\partial u_r}{\partial r} \\ +\frac{1}{r}\frac{\partial v_r}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{v_r}{r}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta} + 2\frac{u_r}{r}\right) \end{cases} \quad (21b)$$

$$\nabla(v_\theta\hat{\theta})\cdot(2\mu\mathcal{S}) = \begin{cases} \frac{\partial v_\theta}{\partial x}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial\theta}\right) + \frac{\partial v_\theta}{\partial r}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right) \\ +\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta} + 2\frac{u_r}{r}\right) - \frac{v_\theta}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \end{cases} \quad (21c)$$

# 2 Nek Implementations:

## 2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$-\int_\Omega v \frac{\partial}{\partial x_j}(p\delta_{ij}) d\Omega$$
$$-\int_\Omega \frac{\partial}{\partial x_j}(vp\delta_{ij}) d\Omega + \int_\Omega p\delta_{ij} \frac{\partial}{\partial x_j}(v) d\Omega$$
$$-\int_{\partial\Omega} (vp\delta_{ij}) n_j dA + \int_\Omega p\delta_{ij} \frac{\partial}{\partial x_j}(v) d\Omega$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as $\mathcal{J}$. In the continuous formulation, this becomes:

$$\begin{aligned}
w_i &= \int_\Omega p \frac{\partial v}{\partial x_i} d\Omega \\
&= \int_\Omega p \frac{\partial v}{\partial x_i} \left( \frac{\partial \Omega}{\partial \hat{\Omega}} \right) d\hat{\Omega} \\
&= \int_\Omega p \left( \frac{1}{\mathcal{J}} \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}} \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}} \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) \mathcal{J} d\hat{\Omega} \\
&= \int_\Omega p \left( \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \\
\implies w_i &= \int_\Omega p \left( \frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j} \right) d\hat{\Omega},
\end{aligned} \tag{22}$$

where, $r_j$ represents the reference element coordinate directions $r, s, t$. After discretization, this becomes:

$$\implies w_i = \sum_k W(x_k) \left( \frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j}(x_k) \right) p$$

When cross derivative terms like $\partial s/\partial x$ *etc.* are non-zero, we have the following expression

$$\begin{aligned}
w_i =& (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x_i}.*W.*p) + \\
& (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial x_i}.*W.*p) + \\
& (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial x_i}.*W.*p)
\end{aligned} \tag{23}$$

In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x}.*W.*p) \tag{24a}$$
$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial s}{\partial y}.*W.*p) \tag{24b}$$
$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial z}.*W.*p) \tag{24c}$$

which can be further expressed purely as kronecker products, taking $\partial r/\partial x,\ \partial s/\partial y,\ \partial t/\partial z$ *etc.* as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}_{t12}^T\mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T\mathcal{I}_2 W_{s2} \otimes \mathcal{D}_{r12}^T\frac{\partial r}{\partial x}W_{r2})p \tag{25a}$$
$$w_2 = (\mathcal{I}_{t12}^T\mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T\frac{\partial s}{\partial y}W_{s2} \otimes \mathcal{I}_{r12}^T\mathcal{I}_2 W_{r2})p \tag{25b}$$
$$w_3 = (\mathcal{D}_{t12}^T\frac{\partial t}{\partial z}W_{t2} \otimes \mathcal{I}_{s12}^T\mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T\mathcal{I}_2 W_{r2})p \tag{25c}$$

For the cylindrical case we encounter some differences. There is an additional factor of $R$ in front of the integral, *i.e.* $\partial\Omega_{x,y,z} \rightarrow R\partial\Omega_{x,R,\theta} \rightarrow R\mathcal{J}\partial\hat{\Omega}$. Also, for the $\theta$ term we have an additional division by $R$. Finally, there is an additional term for $R$ component for the derivative of the unit vector.

$$\begin{aligned}
w_x = & \int_\Omega p\left(\frac{\partial r}{\partial x}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial v}{\partial t}\right) R d\hat{\Omega} \\
w_R = & \int_\Omega p\left(\frac{\partial r}{\partial R}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial R}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial R}\frac{\partial v}{\partial t} + \frac{v\mathcal{J}}{R}\right) R d\hat{\Omega} \\
w_\theta = & \int_\Omega p\left(\frac{\partial r}{\partial \theta}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta}\frac{\partial v}{\partial t}\right) d\hat{\Omega}
\end{aligned}$$

Which will lead to the similar (but not identical) expressions for opgradt. Substituting $\widetilde{\mathcal{D}}_{s12}^T = (\partial s/\partial y)\mathcal{D}_{s12}^T + \mathcal{I}_{s12}^T/R$, and if we assume the grid is Cartesian in the $x - R - \theta$ space, *i.e.* there are no cross derivatives.

$$w_x = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\frac{\partial r}{\partial x}.*R.*W.*p) \tag{26a}$$
$$w_R = \left(\mathcal{I}_{t12}^T \otimes (\frac{1}{\mathcal{J}}\frac{\partial s}{\partial y}\mathcal{D}_{s12}^T + \frac{\mathcal{I}_{s12}^T}{R}) \otimes \mathcal{I}_{r12}^T\right)(R.*W.*\mathcal{J}.*p) \tag{26b}$$
$$w_\theta = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial z}.*W.*p) \tag{26c}$$

## 2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field $u$, defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can represent $\partial r/\partial x, \partial s/\partial x, \ldots$ as matrices $D_x r, D_x s, \ldots$. For a one dimensional case, $D_x r$ is a diagonal matrix. Therefore, in one dimension, $D_x = D_x r D_r$ and $D_x^T = D_r^T (D_x r)^T$ *etc.*

$$\begin{aligned}
\int_\Omega q \nabla \cdot \boldsymbol{u} = & \int_\Omega q \frac{\partial}{\partial x_i}(u_i) d\Omega \\
= & \int_\Omega q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\
= & \sum_k q(x_k) W(x_k) \left( \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j}(x_k) \right) \\
= & \sum_k q(x_k) W(x_k) (D_{x_i} r_j)(D_{r_j} u_i) \\
= & q_k W_k (D_{x_i} r_j)(D_{r_j} u_i)
\end{aligned}$$

In the absence of cross geometric factors this becomes

$$\int_\Omega q \nabla \cdot \boldsymbol{u} = q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w)$$

$$\int_\Omega q \nabla \cdot \boldsymbol{u} = \begin{cases} qW(\mathcal{I}_{t12} \otimes \mathcal{I}_{s12} \otimes D_x r D_r) u \\ +qW(\mathcal{I}_{t12} \otimes D_y s D_s \otimes \mathcal{I}_{r12}) v \\ +qW(D_z t D_t \otimes \mathcal{I}_{s12} \otimes \mathcal{I}_{r12}) w \end{cases} \tag{27}$$

$$\int_\Omega q \nabla \cdot \boldsymbol{u} = \begin{cases} q(W_t \mathcal{I}_{t12} \otimes W_s \mathcal{I}_{s12} \otimes W_r D_x r D_r) u \\ +q(W_t \mathcal{I}_{t12} \otimes W_s D_y s D_s \otimes W_r \mathcal{I}_{r12}) v \\ +q(W_t D_z t D_t \otimes W_s \mathcal{I}_{s12} \otimes W_r \mathcal{I}_{r12}) w \end{cases} \tag{28}$$

Here $D_r, D_s, D_t$ is essentially $D_{r12}, D_{s12}, D_{t12}$, since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

For the cylindrical case we have

$$\begin{aligned}
\int_\Omega q \nabla \cdot \boldsymbol{u} = & \int_\Omega q \frac{\partial}{\partial x_i}(u_i) d\Omega \\
= & \int_\Omega q \frac{\partial u_i}{\partial x_i} R \mathcal{J} d\hat{\Omega} \\
= & \int_\Omega q \left( \frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial (Rv)}{\partial R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
= & \int_\Omega q \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial R} + \frac{v}{R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
= & \int_\Omega q \left( \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{v}{R} + \frac{1}{R} \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R \mathcal{J} d\hat{\Omega} \\
= & \int_\Omega q \left( \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{\mathcal{J} v}{R} + \frac{1}{R} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R d\hat{\Omega} \\
= & W_k q_k \left( R \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + R \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{R \mathcal{J} v}{R} + \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right)
\end{aligned} \tag{29}$$

For undeformed elements, $\mathcal{J}$ is a constant throughout the element. Therefore in kronecker notation we have

$$\int_\Omega q\nabla\cdot\boldsymbol{u} = \begin{cases} Wq(\mathcal{I}_{12}\otimes R\otimes D_{x12}rD_r)u \\ +Wq(\mathcal{I}_{12}\otimes(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes\mathcal{I}_{12})v \\ +Wq(D_{\theta 12}tD_t\otimes\mathcal{I}_{12}\otimes\mathcal{I}_{12})w \end{cases} \tag{30}$$

$$\int_\Omega q\nabla\cdot\boldsymbol{u} = \begin{cases} q(W_t\mathcal{I}_{12}\otimes W_sR\otimes W_rD_{x12}rD_r)u \\ +q(W_t\mathcal{I}_{12}\otimes W_s(RD_{R12}sD_s+\mathcal{J}\mathcal{I}_{12})\otimes W_r\mathcal{I}_{12})v \\ +q(W_tD_{\theta 12}tD_t\otimes W_s\mathcal{I}_{12}\otimes W_r\mathcal{I}_{12})w \end{cases} \tag{31}$$

## 2.3 Pressure Pseudo-Laplacian

$$S_{\Delta t} = DQD^T \tag{32}$$

where, if $Q = H^{-1}$ there is no decoupling error and we have the Uzawa algorithm. Alternately, $Q = B^{-1}$ in which case we incur a decoupling error but avoid the nested iterations since $B$, being diagonal, can be trivially inverted. In Nek, the operator $D^T = \int_\Omega p\nabla\cdot v d\Omega$, and $D = \int_\Omega q\nabla\cdot u d\Omega$