



Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$x_c = x \quad (1a)$$

$$y_c = r \cos(\theta) \quad (1b)$$

$$z_c = r \sin(\theta) \quad (1c)$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r \sin(\theta) \\ 0 & \sin(\theta) & r \cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (2)$$

$$\Rightarrow \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \quad (3)$$

$$\hat{x} = \hat{x}_c \quad (4a)$$

$$\hat{r} = \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c \quad (4b)$$

$$\hat{\theta} = -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \quad (4c)$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (5)$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) = & \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial x_c} \right) \hat{x}_c + \\ & \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial y_c} \right) \hat{y}_c + \\ & \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial z_c} \right) \hat{z}_c\end{aligned}$$

$$\therefore \nabla(\psi) = \frac{\partial\psi}{\partial x} \hat{x} + \frac{\partial\psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \hat{\theta} \quad (6)$$

Laplacian of a scalar field

$$\begin{aligned}\nabla \cdot \nabla(\psi) &= \left[\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right] \cdot \left[\hat{x} \frac{\partial\psi}{\partial x} + \hat{r} \frac{\partial\psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right] \\ &= \left\{ \begin{aligned} & \left[\hat{x} \cdot \hat{x} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial\hat{x}}{\partial x} \cdot \hat{x} \right] + \left[\hat{x} \cdot \hat{r} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial\hat{x}}{\partial x} \cdot \hat{r} \right] + \left[\hat{x} \cdot \hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{x}}{\partial x} \cdot \hat{\theta} \right] \\ & + \left[\hat{r} \cdot \hat{x} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial\hat{r}}{\partial r} \cdot \hat{x} \right] + \left[\hat{r} \cdot \hat{r} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial\hat{r}}{\partial r} \cdot \hat{r} \right] + \left[\hat{r} \cdot \hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{r}}{\partial r} \cdot \hat{\theta} \right] \\ & + \left[\hat{\theta} \cdot \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial x} + \frac{1}{r} \frac{\partial\psi}{\partial x} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{x} \right] + \left[\hat{\theta} \cdot \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial r} + \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{r} \right] \\ & + \left[\hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial\psi}{\partial \theta} \frac{\partial\hat{\theta}}{\partial \theta} \cdot \hat{\theta} \right] \end{aligned} \right\}\end{aligned}$$

$$\therefore \nabla^2(\psi) = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\psi}{\partial \theta^2} \quad (7)$$

Vector gradient. Highlighted in color are the only terms which are non-zero dif-

ferentials of unit vectors.

$$\begin{aligned} \nabla(\mathbf{u}) &= \left[\hat{x} \frac{\partial}{\partial x} + \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right] \left[u_x \hat{x} + u_r \hat{r} + u_\theta \hat{\theta} \right] \\ &= \left\{ \begin{aligned} &\left[\hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} u_x \frac{\partial \hat{x}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} u_r \frac{\partial \hat{r}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{x} u_\theta \frac{\partial \hat{\theta}}{\partial x} \right] \\ &+ \left[\hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} u_x \frac{\partial \hat{x}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} u_r \frac{\partial \hat{r}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{r} u_\theta \frac{\partial \hat{\theta}}{\partial r} \right] \\ &+ \left[\hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{u_x}{r} \frac{\partial \hat{x}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{r}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} + \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{\theta}}{\partial \theta} \right] \end{aligned} \right. \quad (8) \end{aligned}$$

$$= \left\{ \begin{aligned} &\left[\hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \right] \\ &+ \left[\hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \right] \\ &+ \left[\hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \right] \end{aligned} \right. \quad (9)$$

Which, I rearrange as gradients of individual velocity components (and their respective unit vectors)

$$\nabla(u_x \hat{x}) = \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} \quad (10a)$$

$$\nabla(u_r \hat{r}) = \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} \quad (10b)$$

$$\nabla(u_\theta \hat{\theta}) = \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \quad (10c)$$

$$\therefore \nabla(\mathbf{u}) = \begin{bmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{bmatrix} \quad (11)$$

Vector Laplacian:

$$\nabla \cdot \nabla(\mathbf{u}) = \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \begin{pmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_x}{\partial x} + \hat{r} \frac{\partial u_x}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} \\
= & \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{x} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{x}}{\partial x} \\
& + \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{x} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{x}}{\partial r} \\
& + \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{x}}{\partial x} \\
& + \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{r} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_x}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{r} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{\theta} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{\theta} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& = \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r} \frac{\partial u_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} \tag{12}
\end{aligned}$$

Second column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_r}{\partial x} + \hat{r} \frac{\partial u_r}{\partial r} - \hat{\theta} \frac{u_\theta}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_r}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{x} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{x} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(-\hat{\theta} \frac{\partial}{\partial x} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial x} \frac{u_\theta}{r} \right) \hat{r} - \hat{x} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{r} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{r} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(-\hat{\theta} \frac{\partial}{\partial r} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial r} \frac{u_\theta}{r} \right) \hat{r} - \hat{r} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{\theta} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{\theta} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(-\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_\theta}{r} \right) \hat{r} - \hat{\theta} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} \right] \hat{r} + \left[-\frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right] \hat{\theta} \quad (13)
\end{aligned}$$

Third column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_\theta}{\partial x} + \hat{r} \frac{\partial u_\theta}{\partial r} + \hat{\theta} \frac{u_r}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{x} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{x} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial x} \frac{u_r}{r} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{r} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{r} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial r} \frac{u_r}{r} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{\theta} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_r}{r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{u_r}{r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \right] \hat{\theta} + \left[-\frac{u_r}{r^2} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} \quad (14)
\end{aligned}$$

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \begin{cases} \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} + \\ \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} + \\ \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \hat{\theta} \end{cases} \quad (15)$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\theta} \quad (16)$$

Weak Laplacian:

$$\begin{aligned} \nabla(v_x \hat{x}) \cdot \nabla(u_x \hat{x}) &= \left(\hat{x} \frac{\partial v_x}{\partial x} \hat{x} + \hat{r} \frac{\partial v_x}{\partial r} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial v_x}{\partial \theta} \hat{x} \right) \cdot \left(\hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} \right) \\ &= \hat{x} \frac{\partial v_x}{\partial x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} \frac{\partial v_x}{\partial r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{x} \frac{1}{r^2} \frac{\partial v_x}{\partial \theta} \frac{\partial u_x}{\partial \theta} \hat{x} \end{aligned} \quad (17)$$

$$\begin{aligned} \nabla(v_r \hat{r}) \cdot \nabla(u_r \hat{r}) &= \left(\hat{x} \frac{\partial v_r}{\partial x} \hat{r} + \hat{r} \frac{\partial v_r}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial v_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{v_r}{r} \hat{\theta} \right) \cdot \left(\hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \hat{\theta} \right) \\ &= \begin{cases} \hat{r} \frac{\partial v_r}{\partial x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{r} \frac{\partial v_r}{\partial r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} \frac{1}{r^2} \frac{\partial v_r}{\partial \theta} \frac{\partial u_r}{\partial \theta} \hat{r} + \\ \hat{\theta} \frac{v_r}{r} \frac{u_r}{r} \hat{\theta} + \hat{r} \frac{1}{r} \frac{\partial v_r}{\partial \theta} \frac{u_r}{r} \hat{\theta} + \hat{\theta} \frac{v_r}{r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} \end{cases} \end{aligned} \quad (18)$$

$$\begin{aligned} \nabla(v_\theta \hat{\theta}) \cdot \nabla(u_\theta \hat{\theta}) &= \left(\hat{x} \frac{\partial v_\theta}{\partial x} \hat{\theta} + \hat{r} \frac{\partial v_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{v_\theta}{r} \hat{r} \right) \cdot \left(\hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} - \hat{\theta} \frac{u_\theta}{r} \hat{r} \right) \\ &= \begin{cases} \hat{\theta} \frac{\partial v_\theta}{\partial x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{\theta} \frac{\partial v_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{\theta} \frac{1}{r^2} \frac{\partial v_\theta}{\partial \theta} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} \\ + \hat{r} \frac{v_\theta}{r} \frac{u_\theta}{r} \hat{r} - \hat{\theta} \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \frac{u_\theta}{r} \hat{r} - \hat{r} \frac{v_\theta}{r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} \end{cases} \end{aligned} \quad (19)$$

2 Nek Implementations:

2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$\begin{aligned} & - \int_{\Omega} v \frac{\partial}{\partial x_j} (p \delta_{ij}) d\Omega \\ & - \int_{\Omega} \frac{\partial}{\partial x_j} (v p \delta_{ij}) d\Omega + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \\ & - \int_{\partial\Omega} (v p \delta_{ij}) n_j dA + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \end{aligned}$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as \mathcal{J} . In the continuous formulation, this becomes:

$$\begin{aligned}
w_i &= \int_{\Omega} p \frac{\partial v}{\partial x_i} d\Omega \\
&= \int_{\Omega} p \frac{\partial v}{\partial x_i} \left(\frac{\partial \Omega}{\partial \hat{\Omega}} \right) d\hat{\Omega} \\
&= \int_{\Omega} p \left(\frac{1}{\mathcal{J}} \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}} \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}} \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} p \left(\frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \\
\Rightarrow w_i &= \int_{\Omega} p \left(\frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j} \right) d\hat{\Omega}, \tag{20}
\end{aligned}$$

where, r_j represents the reference element coordinate directions r, s, t . After discretization, this becomes:

$$\Rightarrow w_i = \sum_k W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j}(x_k) \right) p$$

When cross derivative terms like $\partial s / \partial x$ etc. are non-zero, we have the following expression

$$\begin{aligned}
w_i &= (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x_i} \cdot * W \cdot * p \right) + \\
&\quad (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial x_i} \cdot * W \cdot * p \right) + \\
&\quad (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial x_i} \cdot * W \cdot * p \right)
\end{aligned} \tag{21}$$

In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x} \cdot * W \cdot * p \right) \tag{22a}$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial y} \cdot * W \cdot * p \right) \tag{22b}$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial z} \cdot * W \cdot * p \right) \tag{22c}$$

which can be further expressed purely as kronecker products, taking $\partial r / \partial x$, $\partial s / \partial y$, $\partial t / \partial z$ etc. as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} W_{r2}) p \tag{23a}$$

$$w_2 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial y} W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \tag{23b}$$

$$w_3 = (\mathcal{D}_{t12}^T \frac{\partial t}{\partial z} W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \tag{23c}$$

For the cylindrical case we encounter some differences. There is an additional factor of R in front of the integral, *i.e.* $\partial\Omega_{x,y,z} \rightarrow R\partial\Omega_{x,R,\theta} \rightarrow R\mathcal{J}\partial\hat{\Omega}$. Also, for the θ term we have an additional division by R .

$$\begin{aligned} w_x &= \int_{\Omega} p \left(\frac{\partial r}{\partial x} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x} \frac{\partial v}{\partial t} \right) R d\hat{\Omega} \\ w_R &= \int_{\Omega} p \left(\frac{\partial r}{\partial R} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial R} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial R} \frac{\partial v}{\partial t} \right) R d\hat{\Omega} \\ w_{\theta} &= \int_{\Omega} p \left(\frac{\partial r}{\partial \theta} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \end{aligned}$$

Which will lead to the similar (but not identical) expressions for opgradt. If we assume the grid is Cartesian in the $x - R - \theta$ space, *i.e.* there are no cross derivatives.

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x} \cdot * R. * W. * p \right) \quad (24a)$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial y} \cdot * R. * W. * p \right) \quad (24b)$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial z} \cdot * W. * p \right) \quad (24c)$$

The factors of R , W and the geometric factors can also be expressed in kronecker product form:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) (\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \frac{\partial r}{\partial x}) (\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (25a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (25b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2 W_{r2}) p \quad (25c)$$

R -direction:

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) (\mathcal{I}_2 \otimes \frac{\partial s}{\partial R} \otimes \mathcal{I}_2) (\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (26a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (26b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2}) p \quad (26c)$$

θ -direction:

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial \theta} \otimes \mathcal{I}_2 \otimes \mathcal{I}_2 \right) (\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (27a)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2) (W_{t2} \otimes W_{s2} \otimes W_{r2}) p \quad (27b)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2}) p \quad (27c)$$

2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field u , defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can represent $\partial r/\partial x, \partial s/\partial x, \dots$ as matrices $D_x r, D_x s, \dots$. For a one dimensional case, $D_x r$ is a diagonal matrix. Therefore, in one dimension, $D_x = D_x r D_r$ and $D_x^T = D_r^T (D_x r)^T$ etc.

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot u &= \int_{\Omega} q \frac{\partial}{\partial x_i} (u_i) d\Omega \\
&= \int_{\Omega} q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\
&= \sum_k q(x_k) W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} (x_k) \right) \\
&= \sum_k q(x_k) W(x_k) (D_{x_i} r_j) (D_{r_j} u_i) \\
&= q_k W_k (D_{x_i} r_j) (D_{r_j} u_i)
\end{aligned}$$

In the absence of cross geometric factors this becomes

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot u &= q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w) \\
\int_{\Omega} q \nabla \cdot u &= \begin{cases} q W (\mathcal{I}_{t12} \otimes \mathcal{I}_{s12} \otimes D_x r D_r) u \\ + q W (\mathcal{I}_{t12} \otimes D_y s D_s \otimes \mathcal{I}_{r12}) v \\ + q W (D_z t D_t \otimes \mathcal{I}_{s12} \otimes \mathcal{I}_{r12}) w \end{cases} \quad (28)
\end{aligned}$$

$$\int_{\Omega} q \nabla \cdot u = \begin{cases} q (W_t \mathcal{I}_{t12} \otimes W_s \mathcal{I}_{s12} \otimes W_r D_x r D_r) u \\ + q (W_t \mathcal{I}_{t12} \otimes W_s D_y s D_s \otimes W_r \mathcal{I}_{r12}) v \\ + q (W_t D_z t D_t \otimes W_s \mathcal{I}_{s12} \otimes W_r \mathcal{I}_{r12}) w \end{cases} \quad (29)$$

Here D_r, D_s, D_t is essential $D_{r12}, D_{s12}, D_{t12}$, since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

For the cylindrical case we have

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot u &= \int_{\Omega} q \frac{\partial}{\partial x_i} (u_i) d\Omega \\
&= \int_{\Omega} q \frac{\partial u_i}{\partial x_i} R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left(\frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial(Rv)}{\partial R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial R} + \frac{v}{R} + \frac{1}{R} \frac{\partial w}{\partial \theta} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left(\frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{v}{R} + \frac{1}{R} \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R \mathcal{J} d\hat{\Omega} \\
&= \int_{\Omega} q \left(\frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{\mathcal{J}v}{R} + \frac{1}{R} \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) R d\hat{\Omega} \\
&= W_k q_k \left(R \frac{\partial r_j}{\partial x} \frac{\partial u}{\partial r_j} + R \frac{\partial r_j}{\partial R} \frac{\partial v}{\partial r_j} + \frac{R \mathcal{J}v}{R} + \frac{\partial r_j}{\partial \theta} \frac{\partial w}{\partial r_j} \right) \quad (30)
\end{aligned}$$

For undeformed elements, \mathcal{J} is a constant throughout the element. Therefore in kronecker notation we have

$$\int_{\Omega} q \nabla \cdot u = \begin{cases} W q (\mathcal{I}_{12} \otimes R \otimes D_{x12} r D_r) u \\ + W q (\mathcal{I}_{12} \otimes (R D_{R12} s D_s + \mathcal{J} \mathcal{I}_{12}) \otimes \mathcal{I}_{12}) v \\ + W q (D_{\theta12} t D_t \otimes \mathcal{I}_{12} \otimes \mathcal{I}_{12}) w \end{cases} \quad (31)$$

$$\int_{\Omega} q \nabla \cdot u = \begin{cases} q (W_t \mathcal{I}_{12} \otimes W_s R \otimes W_r D_{x12} r D_r) u \\ + q (W_t \mathcal{I}_{12} \otimes W_s (R D_{R12} s D_s + \mathcal{J} \mathcal{I}_{12}) \otimes W_r \mathcal{I}_{12}) v \\ + q (W_t D_{\theta12} t D_t \otimes W_s \mathcal{I}_{12} \otimes W_r \mathcal{I}_{12}) w \end{cases} \quad (32)$$

2.3 Pressure Pseudo-Laplacian

$$S_{\Delta t} = D Q D^T \quad (33)$$

where, if $Q = H^{-1}$ there is no decoupling error and we have the Uzawa algorithm. Alternately, $Q = B^{-1}$ in which case we incur a decoupling error but avoid the nested iterations since B , being diagonal, can be trivially inverted.