



Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$x_c = x \quad (1a)$$

$$y_c = r \cos(\theta) \quad (1b)$$

$$z_c = r \sin(\theta) \quad (1c)$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r \sin(\theta) \\ 0 & \sin(\theta) & r \cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (2)$$

$$\Rightarrow \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \quad (3)$$

$$\hat{x} = \hat{x}_c \quad (4a)$$

$$\hat{r} = \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c \quad (4b)$$

$$\hat{\theta} = -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \quad (4c)$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (5)$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) &= \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial x_c} \right) \hat{x}_c + \\ &\quad \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial y_c} \right) \hat{y}_c + \\ &\quad \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial z_c} \right) \hat{z}_c \\ \therefore \nabla(\psi) &= \frac{\partial\psi}{\partial x} \hat{x} + \frac{\partial\psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \hat{\theta}\end{aligned}\tag{6}$$

Laplacian of a scalar field

$$\begin{aligned}\nabla \cdot \nabla(\psi) &= \left[\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right] \cdot \left[\hat{x} \frac{\partial\psi}{\partial x} + \hat{r} \frac{\partial\psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right] \\ &= \left\{ \begin{aligned} &\left[\hat{x} \cdot \hat{x} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial \hat{x}}{\partial x} \cdot \hat{x} \right] + \left[\hat{x} \cdot \hat{r} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial \hat{x}}{\partial x} \cdot \hat{r} \right] + \left[\hat{x} \cdot \hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{x}}{\partial x} \cdot \hat{\theta} \right] \\ &+ \left[\hat{r} \cdot \hat{x} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial \hat{r}}{\partial r} \cdot \hat{x} \right] + \left[\hat{r} \cdot \hat{r} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial \hat{r}}{\partial r} \cdot \hat{r} \right] + \left[\hat{r} \cdot \hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \cdot \hat{\theta} \right] \\ &+ \left[\hat{\theta} \cdot \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial x} + \frac{1}{r} \frac{\partial\psi}{\partial x} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{x} \right] + \left[\hat{\theta} \cdot \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial r} + \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{r} \right] \\ &+ \left[\hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{\theta} \right] \end{aligned} \right\} \\ \therefore \nabla^2(\psi) &= \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\psi}{\partial \theta^2}\end{aligned}\tag{7}$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\begin{aligned}\nabla(\mathbf{u}) &= \left[\hat{x} \frac{\partial}{\partial x} + \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right] \left[\hat{x} u_x + \hat{r} u_r + \hat{\theta} u_\theta \right] \\ &= \left\{ \begin{aligned} &\left[\hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} u_x \frac{\partial \hat{x}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} u_r \frac{\partial \hat{r}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{x} u_\theta \frac{\partial \hat{\theta}}{\partial x} \right] \\ &+ \left[\hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} u_x \frac{\partial \hat{x}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} u_r \frac{\partial \hat{r}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{r} u_\theta \frac{\partial \hat{\theta}}{\partial r} \right] \\ &+ \left[\hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{u_x}{r} \frac{\partial \hat{x}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{r}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} + \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{\theta}}{\partial \theta} \right] \end{aligned} \right\}\end{aligned}\tag{8}$$

$$\therefore \nabla(\mathbf{u}) = \begin{bmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{bmatrix} \quad (9)$$

Vector Laplacian:

$$\nabla \cdot \nabla(\mathbf{u}) = \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \begin{pmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_x}{\partial x} + \hat{r} \frac{\partial u_x}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} \\
= & \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{x} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{x}}{\partial x} \\
& + \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{x} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{x}}{\partial r} \\
& + \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{x}}{\partial x} \\
& + \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{r} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_x}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{r} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{\theta} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{\theta} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& = \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r} \frac{\partial u_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} \tag{10}
\end{aligned}$$

Second column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_r}{\partial x} + \hat{r} \frac{\partial u_r}{\partial r} - \hat{\theta} \frac{u_\theta}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_r}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{x} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{x} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(-\hat{\theta} \frac{\partial}{\partial x} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial x} \frac{u_\theta}{r} \right) \hat{r} - \hat{x} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{r} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{r} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(-\hat{\theta} \frac{\partial}{\partial r} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial r} \frac{u_\theta}{r} \right) \hat{r} - \hat{r} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{\theta} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{\theta} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(-\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_\theta}{r} \right) \hat{r} - \hat{\theta} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} \right] \hat{r} + \left[-\frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right] \hat{\theta} \quad (11)
\end{aligned}$$

Third column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_\theta}{\partial x} + \hat{r} \frac{\partial u_\theta}{\partial r} + \hat{\theta} \frac{u_r}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{x} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{x} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial x} \frac{u_r}{r} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{r} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{r} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial r} \frac{u_r}{r} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{\theta} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_r}{r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{u_r}{r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \right] \hat{\theta} + \left[-\frac{u_r}{r^2} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} \quad (12)
\end{aligned}$$

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \begin{cases} \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} + \\ \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} + \\ \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \hat{\theta} \end{cases} \quad (13)$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\theta} \quad (14)$$

2 Nek Implementations:

2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$\begin{aligned} & - \int_{\Omega} v \frac{\partial}{\partial x_j} (p \delta_{ij}) d\Omega \\ & - \int_{\Omega} \frac{\partial}{\partial x_j} (v p \delta_{ij}) d\Omega + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \\ & - \int_{\partial\Omega} (v p \delta_{ij}) n_j dA + \int_{\Omega} p \delta_{ij} \frac{\partial}{\partial x_j} (v) d\Omega \end{aligned}$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as \mathcal{J} . In the continuous formulation, this becomes:

$$\begin{aligned} w_i &= \int_{\Omega} p \frac{\partial v}{\partial x_i} d\Omega \\ &= \int_{\Omega} p \frac{\partial v}{\partial x_i} \left(\frac{\partial \Omega}{\partial \hat{\Omega}} \right) d\hat{\Omega} \\ &= \int_{\Omega} p \left(\frac{1}{\mathcal{J}} \frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}} \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}} \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) \mathcal{J} d\hat{\Omega} \\ &= \int_{\Omega} p \left(\frac{\partial r}{\partial x_i} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \\ \implies w_i &= \int_{\Omega} p \left(\frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j} \right) d\hat{\Omega}, \end{aligned} \quad (15)$$

where, r_j represents the reference element coordinate directions r, s, t . After discretization, this becomes:

$$\implies w_i = \sum_k W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial v}{\partial r_j}(x_k) \right) p$$

When cross derivative terms like $\partial s/\partial x$ etc. are non-zero, we have the following expression

$$\begin{aligned} w_i = & (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x_i} \cdot * W \cdot * p \right) + \\ & (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial x_i} \cdot * W \cdot * p \right) + \\ & (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial x_i} \cdot * W \cdot * p \right) \end{aligned} \quad (16)$$

In the case of no cross derivative terms, the above expression simplifies to:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x} \cdot * W \cdot * p \right) \quad (17a)$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial y} \cdot * W \cdot * p \right) \quad (17b)$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial z} \cdot * W \cdot * p \right) \quad (17c)$$

which can be further expressed purely as kronecker products, taking $\partial r/\partial x$, $\partial s/\partial y$, $\partial t/\partial z$ etc. as diagonal matrices for the respective one dimensional problems.

$$w_1 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} W_{r2}) p \quad (18a)$$

$$w_2 = (\mathcal{I}_{t12}^T \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial y} W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \quad (18b)$$

$$w_3 = (\mathcal{D}_{t12}^T \frac{\partial t}{\partial z} W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 W_{r2}) p \quad (18c)$$

For the cylindrical case we encounter some differences. There is an additional factor of R in front of the integral, *i.e.* $\partial \Omega_{x,y,z} \rightarrow R \partial \Omega_{x,R,\theta} \rightarrow R \mathcal{J} \partial \hat{\Omega}$. Also, for the θ term we have an additional division by R .

$$\begin{aligned} w_x &= \int_{\Omega} p \left(\frac{\partial r}{\partial x} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial x} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial x} \frac{\partial v}{\partial t} \right) R d\hat{\Omega} \\ w_R &= \int_{\Omega} p \left(\frac{\partial r}{\partial R} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial R} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial R} \frac{\partial v}{\partial t} \right) R d\hat{\Omega} \\ w_{\theta} &= \int_{\Omega} p \left(\frac{\partial r}{\partial \theta} \frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta} \frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta} \frac{\partial v}{\partial t} \right) d\hat{\Omega} \end{aligned}$$

Which will lead to the similar (but not identical) expressions for opgradt. If we assume the grid is Cartesian in the $x - R - \theta$ space, *i.e.* there are no cross derivatives.

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T) \left(\frac{\partial r}{\partial x} \cdot * R \cdot * W \cdot * p \right) \quad (19a)$$

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial s}{\partial y} \cdot * R \cdot * W \cdot * p \right) \quad (19b)$$

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T) \left(\frac{\partial t}{\partial z} \cdot * W \cdot * p \right) \quad (19c)$$

The factors of R , W and the geometric factors can also be expressed in kronecker product form:

$$w_1 = (\mathcal{I}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{D}_{r12}^T)(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \frac{\partial r}{\partial x})(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (20a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (20b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 R_2 W_{s2} \otimes \mathcal{D}_{r12}^T \frac{\partial r}{\partial x} \mathcal{I}_2 W_{r2})p \quad (20c)$$

R -direction:

$$w_2 = (\mathcal{I}_{t12}^T \otimes \mathcal{D}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\mathcal{I}_2 \otimes \frac{\partial s}{\partial R} \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes R_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (21a)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (21b)$$

$$= (\mathcal{I}_{t12}^T \mathcal{I}_2 \mathcal{I}_2 W_{t2} \otimes \mathcal{D}_{s12}^T \frac{\partial s}{\partial R} R_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p \quad (21c)$$

θ -direction:

$$w_3 = (\mathcal{D}_{t12}^T \otimes \mathcal{I}_{s12}^T \otimes \mathcal{I}_{r12}^T)(\frac{\partial t}{\partial \theta} \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(\mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (22a)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2)(W_{t2} \otimes W_{s2} \otimes W_{r2})p \quad (22b)$$

$$= (\mathcal{D}_{t12}^T \frac{\partial t}{\partial \theta} \mathcal{I}_2 W_{t2} \otimes \mathcal{I}_{s12}^T \mathcal{I}_2 \mathcal{I}_2 W_{s2} \otimes \mathcal{I}_{r12}^T \mathcal{I}_2 \mathcal{I}_2 W_{r2})p \quad (22c)$$

2.2 Opdiv

The opdiv subroutine implements the divergence operation for a vector field u , defined on Mesh 1 (velocity mesh), integrated w.r.t the Mesh 2 test functions. We can represent $\partial r/\partial x, \partial s/\partial x, \dots$ as matrices $D_x r, D_x s, \dots$. For a one dimensional case, $D_x r$ is a diagonal matrix. Therefore, in one dimension, $D_x = D_x r D_r$ and $D_x^T = D_r^T (D_x r)^T$ etc.

$$\begin{aligned} \int_{\Omega} q \nabla \cdot u &= \int_{\Omega} q \frac{\partial}{\partial x_i} (u_i) d\Omega \\ &= \int_{\Omega} q \frac{1}{\mathcal{J}} \frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} \mathcal{J} d\hat{\Omega} \\ &= \sum_k q(x_k) W(x_k) \left(\frac{\partial r_j}{\partial x_i} \frac{\partial u_i}{\partial r_j} (x_k) \right) \\ &= \sum_k q(x_k) W(x_k) (D_{x_i} r_j) (D_{r_j} u_i) \\ &= q_k W_k (D_{x_i} r_j) (D_{r_j} u_i) \end{aligned}$$

In the absence of cross geometric factors this becomes

$$\begin{aligned} \int_{\Omega} q \nabla \cdot u &= q_k W_k (D_x r D_r u + D_y s D_s v + D_z t D_t w) \\ \int_{\Omega} q \nabla \cdot u &= \begin{cases} qW(\mathcal{I}_{t12} \otimes \mathcal{I}_{s12} \otimes D_x r D_r)u \\ +qW(\mathcal{I}_{t12} \otimes D_y s D_s \otimes \mathcal{I}_{r12})u \\ +qW(D_z t D_t \otimes \mathcal{I}_{s12} \otimes \mathcal{I}_{r12})u \end{cases} \end{aligned} \quad (23)$$

Here D_r, D_s, D_t is essential $D_{r12}, D_{s12}, D_{t12}$, since we are evaluating the derivative of a Mesh 1 field on Mesh 2 points.

2.3 Pressure Pseudo-Laplacian

$$S_{\Delta t} = DQD^T \quad (24)$$

where, if $Q = H^{-1}$ there is no decoupling error and we have the Uzawa algorithm. Alternately, $Q = B^{-1}$ in which case we incur a decoupling error but avoid the nested iterations since B being diagonal can be trivially inverted.

References