REPORT

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Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$
\begin{aligned}
x_c &= x && \text{(1a)}\\
y_c &= r\cos(\theta) && \text{(1b)}\\
z_c &= r\sin(\theta) && \text{(1c)}
\end{aligned}
$$

$$
\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r\sin(\theta) \\ 0 & \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \tag{2}
$$

$$
\implies \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \tag{3}
$$

$$
\begin{aligned}
\hat{x} &= \hat{x}_c && \text{(4a)}\\
\hat{r} &= \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c && \text{(4b)}\\
\hat{\theta} &= -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c && \text{(4c)}
\end{aligned}
$$

$$
\begin{aligned}
\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \\
\implies \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} && \text{(5)}
\end{aligned}
$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) = &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_c}\right)\hat{x_c} + \\ &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y_c}\right)\hat{y_c} + \\ &\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z_c}\right)\hat{z_c}\end{aligned}$$

$$\therefore \nabla(\psi) = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} \tag{6}$$

Laplacian of a scalar field

$$\nabla\cdot\nabla(\psi) = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right]\cdot\left[\hat{x}\frac{\partial\psi}{\partial x} + \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]$$

$$= \begin{cases} \left[\hat{x}\cdot\hat{x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{r}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial x}\cdot\hat{x}\right] + \left[\hat{x}\cdot\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\cdot\hat{x}\right] \\ + \left[\hat{r}\cdot\hat{x}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{r}\frac{\partial}{\partial r}\frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial r}\cdot\hat{r}\right] + \left[\hat{r}\cdot\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\cdot\hat{r}\right] \\ + \left[\hat{\theta}\cdot\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial x} + \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\hat{x}}{\partial\theta}\cdot\hat{\theta}\right] + \left[\hat{\theta}\cdot\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\hat{r}}{\partial\theta}\cdot\hat{\theta}\right] \\ + \left[\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\frac{\partial\hat{\theta}}{\partial\theta}\cdot\hat{\theta}\right] \end{cases}$$

$$\therefore \nabla^2(\psi) = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta} \tag{7}$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\nabla(\boldsymbol{u}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right]\left[\hat{x}u_x + \hat{r}u_r + \hat{\theta}u_\theta\right]$$

$$= \begin{cases} \left[\hat{x}\frac{\partial u_x}{\partial x}\hat{x} + \hat{x}u_x\frac{\partial\hat{x}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_r}{\partial x}\hat{r} + \hat{x}u_r\frac{\partial\hat{r}}{\partial x}\right] + \left[\hat{x}\frac{\partial u_\theta}{\partial x}\hat{\theta} + \hat{x}u_\theta\frac{\partial\hat{\theta}}{\partial x}\right] \\ + \left[\hat{r}\frac{\partial u_x}{\partial r}\hat{x} + \hat{r}u_x\frac{\partial\hat{x}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_r}{\partial r}\hat{r} + \hat{r}u_r\frac{\partial\hat{r}}{\partial r}\right] + \left[\hat{r}\frac{\partial u_\theta}{\partial r}\hat{\theta} + \hat{r}u_\theta\frac{\partial\hat{\theta}}{\partial r}\right] \\ + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\hat{x} + \hat{\theta}\frac{u_x}{r}\frac{\partial\hat{x}}{\partial\theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\hat{r} + \hat{\theta}\frac{u_r}{r}\frac{\partial\hat{r}}{\partial\theta}\right] + \left[\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\hat{\theta} + \hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{\theta}}{\partial\theta}\right] \end{cases} \tag{8}$$

$$\therefore \nabla(\boldsymbol{u}) = \begin{bmatrix} \hat{x}\dfrac{\partial u_x}{\partial x}\hat{x} & \hat{x}\dfrac{\partial u_r}{\partial x}\hat{r} & \hat{x}\dfrac{\partial u_\theta}{\partial x}\hat{\theta} \\ \hat{r}\dfrac{\partial u_x}{\partial r}\hat{x} & \hat{r}\dfrac{\partial u_r}{\partial r}\hat{r} & \hat{r}\dfrac{\partial u_\theta}{\partial r}\hat{\theta} \\ \hat{\theta}\dfrac{1}{r}\dfrac{\partial u_x}{\partial \theta}\hat{x} & \hat{\theta}\left(\color{blue}{-\dfrac{u_\theta}{r}}\color{black} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right)\hat{r} & \hat{\theta}\left(\color{red}{\dfrac{u_r}{r}}\color{black} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{bmatrix} \tag{9}$$

Vector Laplacian:

$$\nabla \cdot \nabla(\boldsymbol{u}) = \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\right) \begin{pmatrix} \hat{x}\dfrac{\partial u_x}{\partial x}\hat{x}+ & \hat{x}\dfrac{\partial u_r}{\partial x}\hat{r}+ & \hat{x}\dfrac{\partial u_\theta}{\partial x}\hat{\theta}+ \\ \hat{r}\dfrac{\partial u_x}{\partial r}\hat{x}+ & \hat{r}\dfrac{\partial u_r}{\partial r}\hat{r}+ & \hat{r}\dfrac{\partial u_\theta}{\partial r}\hat{\theta}+ \\ \hat{\theta}\dfrac{1}{r}\dfrac{\partial u_x}{\partial \theta}\hat{x}+ & \hat{\theta}\left(-\dfrac{u_\theta}{r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right)\hat{r}+ & \hat{\theta}\left(\dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)\hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_x}{\partial x}+\hat{r}\frac{\partial u_x}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}\\
= \quad &\hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{x}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{x}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial x}\\
&+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{r}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_x}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{r}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{\partial\hat{x}}{\partial r}\\
&+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial x}\right)\hat{x}+\hat{\theta}\cdot\hat{x}\frac{\partial u_x}{\partial x}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_x}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_x}{\partial r}\right)\hat{x}+\hat{\theta}\cdot\hat{r}\frac{\partial u_x}{\partial r}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_x}{\partial\theta}\right)\hat{x}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_x}{\partial\theta}\frac{1}{r}\frac{\partial\hat{x}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial r^2}+\frac{1}{r}\frac{\partial u_x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}
\end{aligned}\tag{10}$$

Second column:

$$
\begin{aligned}
&\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_r}{\partial x}+\hat{r}\frac{\partial u_r}{\partial r}-\hat{\theta}\frac{u_\theta}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}\\
&= \quad \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{x}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial x}\\
&\quad +\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{x}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial x}\\
&\quad +\hat{x}\cdot\left(-\hat{\theta}\frac{\partial}{\partial x}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial x}\frac{u_\theta}{r}\right)\hat{r}-\hat{x}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial x}\\
&\quad +\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial x}\\
&\quad +\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{r}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{\partial\hat{r}}{\partial r}\\
&\quad +\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{r}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{\partial\hat{r}}{\partial r}\\
&\quad +\hat{r}\cdot\left(-\hat{\theta}\frac{\partial}{\partial r}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial r}\frac{u_\theta}{r}\right)\hat{r}-\hat{r}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{\partial\hat{r}}{\partial r}\\
&\quad +\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{\partial\hat{r}}{\partial r}\\
&\quad +\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial x}\right)\hat{r}+\hat{\theta}\cdot\hat{x}\frac{\partial u_r}{\partial x}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\quad +\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_r}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_r}{\partial r}\right)\hat{r}+\hat{\theta}\cdot\hat{r}\frac{\partial u_r}{\partial r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\quad +\hat{\theta}\cdot\left(-\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_\theta}{r}-\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_\theta}{r}\right)\hat{r}-\hat{\theta}\cdot\hat{\theta}\frac{u_\theta}{r}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&\quad +\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right)\hat{r}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_r}{\partial\theta}\frac{1}{r}\frac{\partial\hat{r}}{\partial\theta}\\
&=\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\right]\hat{r}+\left[-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{\theta}
\end{aligned}
\tag{11}
$$

Third column:

$$\left(\hat{x}\cdot\frac{\partial}{\partial x}+\hat{r}\cdot\frac{\partial}{\partial r}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\hat{x}\frac{\partial u_\theta}{\partial x}+\hat{r}\frac{\partial u_\theta}{\partial r}+\hat{\theta}\frac{u_r}{r}+\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}$$

$$\begin{aligned}
= \quad & \hat{x}\cdot\left(\hat{x}\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial\hat{x}}{\partial x}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{x}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{r}\frac{\partial}{\partial x}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial x}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{x}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial x}\frac{u_r}{r}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial x}\\
&+\hat{x}\cdot\left(\hat{\theta}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial x}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{x}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial x}\\
&+\hat{r}\cdot\left(\hat{x}\frac{\partial}{\partial r}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{r}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{\partial\hat{\theta}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{r}\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial\hat{r}}{\partial r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{r}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{\partial\hat{\theta}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial r}\frac{u_r}{r}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{u_r}{r}\frac{\partial\hat{\theta}}{\partial r}\\
&+\hat{r}\cdot\left(\hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{r}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{\partial\hat{\theta}}{\partial r}\\
&+\hat{\theta}\cdot\left(\hat{x}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial x}+\frac{\partial\hat{x}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial x}\right)\hat{\theta}+\hat{\theta}\cdot\hat{x}\frac{\partial u_\theta}{\partial x}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{r}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial u_\theta}{\partial r}+\frac{\partial\hat{r}}{\partial\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{r}\frac{\partial u_\theta}{\partial r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u_r}{r}+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r}\frac{u_r}{r}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{u_r}{r}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\\
&+\hat{\theta}\cdot\left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\frac{\partial\hat{\theta}}{\partial\theta}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)\hat{\theta}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}
\end{aligned}$$

$$=\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}\right]\hat{\theta}+\left[-\frac{u_r}{r^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{r} \tag{12}$$

$$\therefore \nabla\cdot\nabla(\boldsymbol{u})=\begin{cases}\left[\dfrac{\partial^2 u_x}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_x}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_x}{\partial\theta^2}\right]\hat{x}+\\ \left[\dfrac{\partial^2 u_r}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_r}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial\theta^2}-\dfrac{u_r}{r^2}-\dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial\theta}\right]\hat{r}+\\ \left[\dfrac{\partial^2 u_\theta}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_\theta}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial\theta^2}+\dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta}-\dfrac{u_\theta}{r^2}\right]\hat{\theta}\end{cases} \tag{13}$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla \cdot \nabla(\boldsymbol{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)\hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)\hat{\theta} \quad (14)$$

2 Nek Implementations:

2.1 Opgradt

The opgradt subroutine implements the gradient of a field defined on Mesh 2 (pressure mesh), integrated w.r.t the Mesh 1 test functions. This is done with integration by parts which creates a boundary term.

Pressure term in the momentum equations:

$$-\int_\Omega v \frac{\partial}{\partial x_j}(p\delta_{ij})d\Omega$$
$$-\int_\Omega \frac{\partial}{\partial x_j}(vp\delta_{ij})d\Omega + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega$$
$$-\int_{\partial\Omega} (vp\delta_{ij})n_j dA + \int_\Omega p\delta_{ij}\frac{\partial}{\partial x_j}(v)d\Omega$$

The first term is the boundary condition. The second term is what opgradt evaluates. I represent the determinant of the mapping between reference element and physical coordinates, the Jacobian as $\mathcal{J}$. In the continuous formulation, this becomes:

$$\begin{aligned}
w_i &= \int_\Omega p\frac{\partial v}{\partial x_i}d\Omega \\
&= \int_\Omega p\frac{\partial v}{\partial x_i}\left(\frac{\partial \Omega}{\partial \hat{\Omega}}\right)d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{1}{\mathcal{J}}\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{1}{\mathcal{J}}\frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{1}{\mathcal{J}}\frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)\mathcal{J}d\hat{\Omega} \\
&= \int_\Omega p\left(\frac{\partial r}{\partial x_i}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x_i}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x_i}\frac{\partial v}{\partial t}\right)d\hat{\Omega} \\
\implies w_i &= \int_\Omega p\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}\right)d\hat{\Omega}, \qquad (15)
\end{aligned}$$

where, r_i represents the reference element coordinate directions r, s, t After discretization, this becomes:

$$\implies w_i = \sum_k W(x_k)\left(\frac{\partial r_j}{\partial x_i}\frac{\partial v}{\partial r_j}(x_k)\right)p$$

In the case of no cross derivative terms like $\partial s/\partial x$ etc., this has a simple tensor product evaluation using:

$$w_1 = (\mathcal{I}^T_{z12} \otimes \mathcal{I}^T_{y12} \otimes \mathcal{D}^T_{x12})(W. * \frac{\partial r}{\partial x}. * p) \tag{16a}$$

$$w_2 = (\mathcal{I}^T_{z12} \otimes \mathcal{D}^T_{y12} \otimes \mathcal{I}^T_{x12})(W. * \frac{\partial s}{\partial y}. * p) \tag{16b}$$

$$w_3 = (\mathcal{D}^T_{z12} \otimes \mathcal{I}^T_{y12} \otimes \mathcal{I}^T_{x12})(W. * \frac{\partial t}{\partial z}. * p) \tag{16c}$$

For the cylindrical case we have

$$\begin{aligned}
w_x = &\quad \int_\Omega p \left(\frac{\partial r}{\partial x}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial v}{\partial t} \right) d\hat{\Omega} \\
w_R = &\quad \int_\Omega p \left(\frac{\partial r}{\partial R}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial R}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial R}\frac{\partial v}{\partial t} \right) d\hat{\Omega} \\
w_\theta = &\quad \int_\Omega \frac{p}{R} \left(\frac{\partial r}{\partial \theta}\frac{\partial v}{\partial r} + \frac{\partial s}{\partial \theta}\frac{\partial v}{\partial s} + \frac{\partial t}{\partial \theta}\frac{\partial v}{\partial t} \right) d\hat{\Omega}
\end{aligned}$$

References