



Navier–Stokes in Cylindrical formulation

Weak form and Nek implementation

1 Formulation of Navier–Stokes

1.1 Mapping from Cartesian to Cylindrical

$$x_c = x \quad (1a)$$

$$y_c = r \cos(\theta) \quad (1b)$$

$$z_c = r \sin(\theta) \quad (1c)$$

$$\begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r \sin(\theta) \\ 0 & \sin(\theta) & r \cos(\theta) \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (2)$$

$$\Rightarrow \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{Bmatrix} dx_c \\ dy_c \\ dz_c \end{Bmatrix} \quad (3)$$

$$\hat{x} = \hat{x}_c \quad (4a)$$

$$\hat{r} = \cos(\theta)\hat{y}_c + \sin(\theta)\hat{z}_c \quad (4b)$$

$$\hat{\theta} = -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \quad (4c)$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\theta)\hat{y}_c + \cos(\theta)\hat{z}_c \\ 0 & 0 & -\cos(\theta)\hat{y}_c - \sin(\theta)\hat{z}_c \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} d\hat{x} \\ d\hat{r} \\ d\hat{\theta} \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\theta} \\ 0 & 0 & -\hat{r} \end{pmatrix} \begin{Bmatrix} dx \\ dr \\ d\theta \end{Bmatrix} \quad (5)$$

1.2 Gradient/Divergence/Laplace Operators

For scalar fields

$$\begin{aligned}\nabla(\psi) &= \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial x_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial x_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial x_c} \right) \hat{x}_c + \\ &\quad \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial y_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial y_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial y_c} \right) \hat{y}_c + \\ &\quad \left(\frac{\partial\psi}{\partial x} \frac{\partial x}{\partial z_c} + \frac{\partial\psi}{\partial r} \frac{\partial r}{\partial z_c} + \frac{\partial\psi}{\partial \theta} \frac{\partial \theta}{\partial z_c} \right) \hat{z}_c \\ \therefore \nabla(\psi) &= \frac{\partial\psi}{\partial x} \hat{x} + \frac{\partial\psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \hat{\theta}\end{aligned}\tag{6}$$

Laplacian of a scalar field

$$\begin{aligned}\nabla \cdot \nabla(\psi) &= \left[\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right] \cdot \left[\hat{x} \frac{\partial\psi}{\partial x} + \hat{r} \frac{\partial\psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial\psi}{\partial \theta} \right] \\ &= \left\{ \begin{aligned} &\left[\hat{x} \cdot \hat{x} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial \hat{x}}{\partial x} \cdot \hat{x} \right] + \left[\hat{x} \cdot \hat{r} \frac{\partial}{\partial x} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial \hat{x}}{\partial x} \cdot \hat{r} \right] + \left[\hat{x} \cdot \hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{x}}{\partial x} \cdot \hat{\theta} \right] \\ &+ \left[\hat{r} \cdot \hat{x} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial \hat{r}}{\partial r} \cdot \hat{x} \right] + \left[\hat{r} \cdot \hat{r} \frac{\partial}{\partial r} \frac{\partial\psi}{\partial r} + \frac{\partial\psi}{\partial r} \frac{\partial \hat{r}}{\partial r} \cdot \hat{r} \right] + \left[\hat{r} \cdot \hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \cdot \hat{\theta} \right] \\ &+ \left[\hat{\theta} \cdot \hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial x} + \frac{1}{r} \frac{\partial\psi}{\partial x} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{x} \right] + \left[\hat{\theta} \cdot \hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial\psi}{\partial r} + \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{r} \right] \\ &+ \left[\hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial\psi}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial\psi}{\partial \theta} \frac{\partial \hat{\theta}}{\partial \theta} \cdot \hat{\theta} \right] \end{aligned} \right\} \\ \therefore \nabla^2(\psi) &= \frac{\partial^2\psi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\psi}{\partial \theta^2}\end{aligned}\tag{7}$$

Vector gradient. Highlighted in color are the only terms which are non-zero differentials of unit vectors.

$$\begin{aligned}\nabla(\mathbf{u}) &= \left[\hat{x} \frac{\partial}{\partial x} + \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right] \left[\hat{x} u_x + \hat{r} u_r + \hat{\theta} u_\theta \right] \\ &= \left\{ \begin{aligned} &\left[\hat{x} \frac{\partial u_x}{\partial x} \hat{x} + \hat{x} u_x \frac{\partial \hat{x}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_r}{\partial x} \hat{r} + \hat{x} u_r \frac{\partial \hat{r}}{\partial x} \right] + \left[\hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \hat{x} u_\theta \frac{\partial \hat{\theta}}{\partial x} \right] \\ &+ \left[\hat{r} \frac{\partial u_x}{\partial r} \hat{x} + \hat{r} u_x \frac{\partial \hat{x}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_r}{\partial r} \hat{r} + \hat{r} u_r \frac{\partial \hat{r}}{\partial r} \right] + \left[\hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \hat{r} u_\theta \frac{\partial \hat{\theta}}{\partial r} \right] \\ &+ \left[\hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + \hat{\theta} \frac{u_x}{r} \frac{\partial \hat{x}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \hat{r} + \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{r}}{\partial \theta} \right] + \left[\hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \hat{\theta} + \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{\theta}}{\partial \theta} \right] \end{aligned} \right\}\end{aligned}\tag{8}$$

$$\therefore \nabla(\mathbf{u}) = \begin{bmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{bmatrix} \quad (9)$$

Vector Laplacian:

$$\nabla \cdot \nabla(\mathbf{u}) = \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \begin{pmatrix} \hat{x} \frac{\partial u_x}{\partial x} \hat{x} + & \hat{x} \frac{\partial u_r}{\partial x} \hat{r} + & \hat{x} \frac{\partial u_\theta}{\partial x} \hat{\theta} + \\ \hat{r} \frac{\partial u_x}{\partial r} \hat{x} + & \hat{r} \frac{\partial u_r}{\partial r} \hat{r} + & \hat{r} \frac{\partial u_\theta}{\partial r} \hat{\theta} + \\ \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \hat{x} + & \hat{\theta} \left(-\frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + & \hat{\theta} \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \end{pmatrix}$$

First Column

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_x}{\partial x} + \hat{r} \frac{\partial u_x}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} \\
= & \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{x} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{x}}{\partial x} \\
& + \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{x} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{x}}{\partial r} \\
& + \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{x}}{\partial \theta} \\
& + \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{r} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_x}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{r} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
& + \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{\partial \hat{r}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial x} \right) \hat{x} + \hat{\theta} \cdot \hat{x} \frac{\partial u_x}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_x}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_x}{\partial r} \right) \hat{x} + \hat{\theta} \cdot \hat{r} \frac{\partial u_x}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& + \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_x}{\partial \theta} \right) \hat{x} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_x}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
& = \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r} \frac{\partial u_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} \tag{10}
\end{aligned}$$

Second column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_r}{\partial x} + \hat{r} \frac{\partial u_r}{\partial r} - \hat{\theta} \frac{u_\theta}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_r}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{x} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{x} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(-\hat{\theta} \frac{\partial}{\partial x} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial x} \frac{u_\theta}{r} \right) \hat{r} - \hat{x} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{r} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{r} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(-\hat{\theta} \frac{\partial}{\partial r} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial r} \frac{u_\theta}{r} \right) \hat{r} - \hat{r} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial \hat{r}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial x} \right) \hat{r} + \hat{\theta} \cdot \hat{x} \frac{\partial u_r}{\partial x} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_r}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_r}{\partial r} \right) \hat{r} + \hat{\theta} \cdot \hat{r} \frac{\partial u_r}{\partial r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(-\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_\theta}{r} - \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_\theta}{r} \right) \hat{r} - \hat{\theta} \cdot \hat{\theta} \frac{u_\theta}{r} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{r} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{1}{r} \frac{\partial \hat{r}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} \right] \hat{r} + \left[-\frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right] \hat{\theta} \quad (11)
\end{aligned}$$

Third column:

$$\begin{aligned}
& \left(\hat{x} \cdot \frac{\partial}{\partial x} + \hat{r} \cdot \frac{\partial}{\partial r} + \hat{\theta} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \right) \left(\hat{x} \frac{\partial u_\theta}{\partial x} + \hat{r} \frac{\partial u_\theta}{\partial r} + \hat{\theta} \frac{u_r}{r} + \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} \\
&= \hat{x} \cdot \left(\hat{x} \frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial \hat{x}}{\partial x} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{x} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{r} \frac{\partial}{\partial x} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial x} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{x} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial x} \frac{u_r}{r} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{x} \cdot \left(\hat{\theta} \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial x} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{x} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial x} \\
&+ \hat{r} \cdot \left(\hat{x} \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{r} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{r} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial \hat{r}}{\partial r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{r} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial r} \frac{u_r}{r} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{u_r}{r} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{r} \cdot \left(\hat{\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{r} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{\partial \hat{\theta}}{\partial r} \\
&+ \hat{\theta} \cdot \left(\hat{x} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial x} + \frac{\partial \hat{x}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial x} \right) \hat{\theta} + \hat{\theta} \cdot \hat{x} \frac{\partial u_\theta}{\partial x} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{r} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial u_\theta}{\partial r} + \frac{\partial \hat{r}}{\partial \theta} \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{r} \frac{\partial u_\theta}{\partial r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{u_r}{r} + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r} \frac{u_r}{r} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{u_r}{r} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&+ \hat{\theta} \cdot \left(\hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial \hat{\theta}}{\partial \theta} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{\theta} + \hat{\theta} \cdot \hat{\theta} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{1}{r} \frac{\partial \hat{\theta}}{\partial \theta} \\
&= \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \right] \hat{\theta} + \left[-\frac{u_r}{r^2} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} \quad (12)
\end{aligned}$$

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \begin{cases} \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} \right] \hat{x} + \\ \left[\frac{\partial^2 u_r}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] \hat{r} + \\ \left[\frac{\partial^2 u_\theta}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \hat{\theta} \end{cases} \quad (13)$$

Which can be written using the Laplacian of individual scalar fields, u_x, u_r, u_θ as,

$$\therefore \nabla \cdot \nabla(\mathbf{u}) = \nabla^2(u_x)\hat{x} + \left(\nabla^2(u_r) - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{r} + \left(\nabla^2(u_\theta) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\theta} \quad (14)$$

References