Module 2

Probability

Random experiment:

If an experiment is conducted, any number of times, under essentially identical conditions, there is a set of all possible outcomes associated with it. If the result is not certain and is anyone of the several possible outcomes, the experiment is called a random trail or a random experiment. The outcomes are known as elementary events and a set of outcomes is an event. Thus, an elementary event is also an event.

Equally likely events:

Events are said to be equally likely when there is no reason to expect anyone of them rather than anyone of the others.

Example:

When a card is drawn from a pack, any card may be obtained. In this trail, all the 52 elementary events are equally likely.

Exhaustive Events:

All possible events in any trial are known as exhaustive events.

Example:

1. In tossing a coin, there are two exhaustive elementary events, like head and tail.
2. In throwing a die, there are six exhaustive elementary events i.e., getting 1 or 2 or 3 or 4 or 5 or 6.

Mutually exclusive events:

Events are said to be mutually exclusive, if the happening of anyone of the events in a trial excludes the happening of any one of the others i.e., if no two or more of the events can happen simultaneously in the same trial.

Probability:

If a random experiment or a trial results 'n' exhaustive, mutually exclusive and equally likely outcomes, out of which 'm' are favorable to the to the occurrence of event E, then the probability 'p' of occurrence or happening of E, usually denoted by P(E), is given by

$$p = P(E) = \frac{No.\ of\ favourable\ cases}{total\ no.\ of\ exhaustive\ cases} = \frac{m}{n}$$

Note:

1. Since $m \geq 0,\ n > 0\ and\ m \leq n$, we get $P(E) \geq 0$ and $P(E) \leq 1$, then $0 \leq P(E) \leq 1$ or $0 \leq P(\bar{E}) \leq 1$.
2. The non-happening of the event E is called the complementary event of E and is denoted by $\bar{E}\ or\ E^c$. The number of cases favorable to E i.e., non-happening of E is $n - m$. Then the probability q that E will not happen is given by:

   $q = P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p.$

   Then, $p + q = 1$

$$q = P(\bar{E}) = 1 - P(E)$$
$$P(E) = 1 - P(\bar{E})$$
$$P(E) + P(\bar{E}) = 1$$

3. If $P(E) = 1,\ E\ is\ called$ certain event and if $P(E) = 0,\ E$ is called impossible event.

Example

1. What is the chance of getting 4 on rolling a die.

Solution:

There are six possible ways in which die can roll.

There is one way of getting 4.

i.e., the required choice of getting 4 is $\frac{1}{6}$.

2. What is the chance of that a leap year selected at random will contain 53 Sundays.

Solution:

Number of days in a leap year is 366.

Number of full weeks, in a leap year 52+ 2 days.

These two days can be any one of the following 7 ways.

Out of these 7 cases the last two are favorable.

Hence the required probability is $\frac{2}{7}$.

Simple Event:

An event in a trial that cannot be further split is called a simple event or an elementary event.

Sample space:

The set of all possible simple events in a trial is called a sample space for the trial. Each element of a sample space is called a sample point.

Example:

Two coins are tossed, then the possible simple events of the trial are HH, HT, TH, TT.

i.e., the sample space is $S = \{\text{HH, HT, TH, TT}\}$

Axioms of Probability:

Let E be the random experiment whose sample space is S. If C is a subset of sample space. We define the following three axioms:

1. $P(C) \geq 0$, for every $C \subseteq S$

2. $P(S) = 1$
3. $P(C_1 \cup C_2 \cup C_3 \cup \dots) = P(C_1) + P(C_2) + P(C_3) + \cdots$, where $C_1, C_2, C_3, \dots$ are subsets of S and they are mutually disjoint. i.e., $C_1 \cap C_2 = \emptyset,\ for\ i \neq j$.

Properties of the probability function:

1. For each $C \subseteq S$, $P(C) = 1 - P(C^*)$, where C^* is the compliment of C in S.

2. The Probability of null set is zero, i.e., $P(\emptyset) = 0$,

3. If C_1 and C_2 are subsets of S such that then $P(C_1) \leq P(C_2)$.

4. $C \subset S,\ 0 \leq P(C) \leq 1$

5. If C_1 and C_2 are subsets of S such then $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$.

Example:

If the sample space is , $S = C_1 \cup C_2$, if $P(C_1) = 0.8,\ P(C_2) = 0.5$ then find $P(C_1 \cap C_2)$.

Solution:

Given that $S = C_1 \cup C_2$,

$$P(C_1) = 0.8,\ P(C_2) = 0.5$$

By the addition law of probability

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

$$P(C_1 \cap C_2) = P(C_1) + P(C_2) - P(S)$$

$$P(C_1 \cap C_2) = 0.8 + 0.5 - 1$$

$= 1.3 - 1$

$= 0.3.$

Conditional Event:

If C_1, C_2 are events of a sample space S and if C_2 occurs after the occurrence of C_1, then the event of occurrence of C_2 after the event C_1 called conditional event of C_2 given C_1. It is denoted by $\frac{C_2}{C_1}$. Similarly, we define $\frac{C_1}{C_2}$.

Examples:

1. Two coins are tossed. The event of getting two tails given that there is at least one tail is a conditional event.
2. A die is thrown three times. The event of getting the sum of the numbers thrown is 15 when it is known that the first throw was a 5 is a conditional event.

Conditional Probability:

Let S be the sample space of a random experiment. Let $C_1 \subset S$, further let $C_2 \subset C_1$, then the conditional event C_1has already occurred, denoted by $P(C_2/C_1)$ is defined as

$$P(C_2/C_1) = \frac{P(C_2 \cap C_1)}{P(C_1)},\ if\ P(C_1) \neq 0$$

Or

$$P(C_2 \cap C_1) = P(C_1)P(C_2/C_1)$$

Note:

1. IfC_1, C_2, C_3 are any three events, then
$$P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2/C_1)P(C_3/C_1 \cap C_2)$$
2. For any events $C_1, C_2, C_3, \dots, C_n$, then

$$P(C_1 \cap C_2 \cap C_3 \cap \dots \cap C_n) = P(C_1)P(C_2/C_1)P(C_3/C_1 \cap C_2) \dots P(C_n/C_1 \cap C_2 \cap \dots \cap C_n)$$

Examples:

1. A box contains 12 items of which 4 are defective the items are drawn at random from the box one after the other. Find the probability that all three are non-defective.

Solution:

There are 8 non-defective items

Total no. of items=12

Let $C_1,\ C_2,\ C_3$ be the events getting non-defective items on first, second and third drawn.

$$P(C_1) = \frac{8}{12}$$

$$P(C_2/C_1) = \frac{7}{11}$$

$$P(C_3/C_1 \cap C_2) = \frac{6}{10}$$

The probability of three are non-defective

$$P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2/C_1)P(C_3/C_1 \cap C_2)$$
$$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} = \frac{42}{55}$$

2. A box contains 20 balls of which 5 are red, 15 are white. If 3 balls are selected at random and are drawn in succession without replacement. Find the probability that all three balls selected are red.

Solution:

Total balls=20

Where 5 red and 15 white

$$P(C_1) = \frac{5}{20}$$

$$P(C_2/C_1) = \frac{4}{19}$$

$$P(C_3/C_1 \cap C_2) = \frac{3}{18}$$

$$P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2/C_1)P(C_3/C_1 \cap C_2)$$
$$= \frac{5}{20} \times \frac{4}{19} \times \frac{3}{18} = \frac{1}{114}$$

3. The probability that A hits the target is ¼ and the probability B hits is 2/5. What is the probability the target will be hit if A and B each shoot at the target?

Solution:

Given that

$$P(A) = \frac{1}{4}$$

$$P(B) = \frac{2}{5}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B)$$

$$P(A \cup B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{4} \times \frac{2}{5} = \frac{11}{20}$$

Bayes theorem:

Let $C_1, C_2, C_3, \dots, C_n$ be a partition of sample space and let C be any event which is a subset of $\bigcup_{i=1}^{n} C_i$ such that

$$\mathrm{P}(C) > 0,\ then\ P(C_\mathrm{i}/\mathrm{C}) = \frac{P(C_i)P(C/C_i)}{\sum_{i=1}^{n} P(C_i)P(C/C_i)}.$$

Proof:

Let S be the sample space.

Let $C_1,\ C_2,\ C_3, \dots,\ C_n$ be a mutually disjoint event.

Let $C \subset \bigcup_{i=1}^{n} C_i$ such that $P(C) > 0$

$$C \subset \bigcup_{i=1}^{n} C_i$$

$$C = C \cap \left(\bigcup_{i=1}^{n} C_i\right)$$

$$C = \bigcup_{i=1}^{n} (C \cap C_i)$$

$P(C) = \sum_{i=1}^{n} P(C \cap C_i)$

$P(C) = \sum_{i=1}^{n} P(C_i) P(C/C_i)$ (1)

Now, $P(C_i/C) = \frac{P(C_i \cap C)}{P(C)}$

$$P(C_i/C) = \frac{P(C_i)P(C/C_i)}{P(C)}$$

$P(C_i/C) = \frac{P(C_i)P(C/C_i)}{\sum_{i=1}^{n} P(C_i)P(C/C_i)}$ (using eq. (1))

Example 1:

A box contains 3 blue, 2 red marbles while another box contains 2 blue, 5 red. A marble drawn at random from one of the boxes terms out to be blue. What is the probability that it come from the first box?

Solution:

Let $A_1,\ A_2$ be boxes, then $P(A_1) = \frac{1}{2}$ and $P(A_2) = \frac{1}{2}$

Let C_2 be the event of drawing blue marble $P(C_2/A_1) = \frac{3}{5}$

Let C_2 be the event of drawing blue marble $P(C_2/A_2) = \frac{2}{7}$

We have to require

$$P(A_1/C_2) = \frac{P(A_1)P(C_2/A_1)}{P(A_1)P(C_2/A_1) + P(A_2)P(C_2/A_2)}$$

$$P(A_1/C_2) = \frac{\frac{1}{2}\times\frac{3}{5}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{2}{7}}$$

$$= \frac{\frac{3}{10}}{\frac{3}{10}+\frac{2}{14}} = \frac{21}{31}$$

Example 2:

A bowl one contains 6 red chips and 4 blue chips, 5 of these are selected at random and put in bowl 2 which was originally empty. One chip is drawn at random from bowl 2 relative to the hypothesis that this chip is blue. Find the conditional probability that 2 red chips and 3 blue chips are transferred from bowl 1 to bowl 2.

Solution:

Bowl-1: 6 red and 4 blue

Bowl-2: 2 red and 3 blue

Let E be the event 2 red and 3 blue chips are transferred from bowl 1 to bowl 2.

Then $P(E) = \frac{6_{C_2}\times 4_{C_3}}{10_{C_5}}$

Let B be the event that a blue chip is drawn from bowl 2.

Then $P(B/E) = \frac{3}{5}$

$$P(E/B) = \frac{P(E)P(B/E)}{P(B)}$$

$$P(E/B) = \frac{\frac{6_{C_2} \times 4_{C_3}}{10_{C_5}} \times \frac{3}{5}}{1} = \frac{1}{7}$$

Random Variable:

A random variable is a function that associates a real number with each element in the sample space.

Example:

Suppose that a coin is tossed twice so that the sample space is S={HH, TH, HT, TT}.

Let X represents the number of heads which can come up with each sample point. We can associate a number for X as:

Sample point:	HH	TH	HT	TT
X :	2	1	1	0

There are two types of random variables

(i) Discrete Random Variable

(ii) Continuous Random Variable

(i) Discrete Random Variable: A Random Variable which takes on a finite (or) countably infinite

number of values is called a Discrete Random Variable**.**

(ii) Continuous Random Variable: A Random Variable which takes on non-countable infinite

number of values is called as non- Discrete (or) Continuous Random Variable.

Probability Mass Function (P.M.F):

The set of ordered pairs $\left(x,\ f(x)\right)$ is a probability function of Probability Mass Function of a Discrete Random Variable x.

If for each possible outcome $x,\ f(x)$ must be

(i) $f(x) \geq 0$
(ii) $\sum f(x) = 1$

(iii) $P(X = x) = \ f(x)$

The Probability Mass Function is also denoted by $P_X(x) = P(X\ = x)$.

Probability Density Function (P.D.F):

The function $f(x)$ is a Probability Density Function for the Continuous Random Variable x defined over the set of real numbers $R,\ if$

(i) $f(x) \geq 0,\ \forall\ x \in R$

(ii) $\int_{-\infty}^{+\infty} f(x)\ dx = 1$

(iii) $P(a < X < b) = \int_a^b f(x)\ dx$

Cumulative Distribution Function $F(x)$:

The Cumulative density distribution function of a discrete random variable X with probability distribution function $f(x)$ as

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

Example:

The Probability function of a random variable X is

X	1	2	3
f(x)	½	1/3	1/6

Find the cumulative distribution of X.

Sol:

The cumulative distribution of X is

X	1	2	3
f(x)	½	5/6	1

Problem:

A shipment of 8 similar computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 computers. Find the probability distribution are the number of defective.

Sol:

Let X be a random variable whose values x at the possible number of defective computers purchased by the school then x maybe 0,1,2

Now, $f(X = x = 0) = P(X = 0)$

$$= \frac{3_{C_0} \times 5_{C_2}}{8_{C_2}} = \frac{10}{28} = \frac{5}{14}$$

$$f(X = x = 1) = P(X = 1)$$

$$= \frac{3_{C_1} \times 5_{C_1}}{8_{C_2}} = \frac{15}{28}$$

$$f(X = x = 2) = P(X = 2)$$

$$= \frac{3_{C_2} \times 5_{C_0}}{8_{C_2}} = \frac{3}{28}$$

The probability distribution function of X is

X	0	1	2
f(x)	10/28	15/28	3/28

Problem:

A random variable X has density function

$$f(x) = \begin{cases} Ce^{-3x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find

(i) The constant C

(ii) $P(1 < x < 2)$

(iii) $P(X \geq 3)$

(iv) $P(X < 1)$

Sol:

Given that

$$f(x) = \begin{cases} Ce^{-3x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

(i) $\int_0^\infty f(x)\,dx = 1$

$$\left\{ since \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{+\infty} f(x)\,dx \right\}$$

$$\int_0^\infty Ce^{-3x}\,dx = 1$$

$$C\left[0 + \frac{1}{3}\right] = 1$$

$$C = 3.$$

(ii) $\int_1^2 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_1^2$$

$$= \left[\frac{e^{-3x}}{-1}\right]_1^2$$

$$= -[e^{-6} - e^{-3}]$$

$$= e^{-3} - e^{-6}$$

(iii) $\int_3^\infty f(x)\,dx = \int_3^\infty 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_3^\infty$$

$$= 0 + e^{-9}$$

$$= e^{-9}$$

(iv) $\int_{-\infty}^1 f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^1 f(x)\,dx$

$$= 0 + \int_0^1 3e^{-3x}\, dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_0^1$$

$$= 1 - e^{-3}$$

Problem:

Let X be a random variable of discrete type having

$$f(x) = \frac{4!}{x!\ (4-x)!}\left(\frac{1}{2}\right)^4,\ x = 0,1,2,3,4.$$

Check whether $f(x)$ is actual a probability density function, if so find $P(A_1)$, where $A_1 = \{0,1\}$.

Solution:

Given the sample space of random variables is $S = \{0,1,2,3,4\}$.

$$P(S) = \sum_{x=0}^{4} f(x)$$

$$= f(0) + f(1) + f(2) + f(3) + f(4)$$

$$= \frac{4!}{4!}\left(\frac{1}{2}\right)^4 + \frac{4!}{1!\,3!}\left(\frac{1}{2}\right)^4 + \frac{4!}{2!\,2!}\left(\frac{1}{2}\right)^4 + \frac{4!}{3!\,1!}\left(\frac{1}{2}\right)^4 + \frac{4!}{4!}\left(\frac{1}{2}\right)^4$$

$$= \left(\frac{1}{2}\right)^4 (1 + 4 + 6 + 4 + 1) = 1$$

$$P(S) = 1$$

We observe that $f(x)$ is probability density function of given random variable of discrete type.

If $A_1 = \{0,1\}$

Then $P(A_1) = f(0) + f(1) = \frac{4!}{4!}\left(\frac{1}{2}\right)^4 + \frac{4!}{1!3!}\left(\frac{1}{2}\right)^4 = \frac{5}{16}$

Problem:

Let x be a random variable of continuous type with probability density function

$$f(x) = \begin{cases} e^{-x}, \; for \; 0 < x < \infty \\ 0 \; otherwise \end{cases}$$

Find $(x \in A_1)$, $A_1 : 0 < x < 1$.

.**Solution:**

$$f(x) = \begin{cases} e^{-x}, \; for \; 0 < x < \infty \\ 0 \; otherwise \end{cases}$$

We observe that given $f(x)$ is probability density function

$$\int_0^{\infty} f(x) \, dx = 1$$

$$\int_0^{\infty} e^{-x} \, dx = 1 \left(since \text{ since } \int_{-\infty}^{+\infty} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_0^{+\infty} f(x) \, dx \right)$$

$$1 = 1$$

$$P(A_1) = P(0 < x < 1) = \int_0^1 e^{-x} \, dx = 1 - \frac{1}{e}$$

Problem:

Determine the value of the constant k and the distribution function of the continuous type of random variables x, whose p.d.f. is

$$f(x) = \begin{cases} 0 & for \quad x < 0 \\ kx & for \quad 0 \le x \le 1 \\ k & for \quad 1 \le x \le 2 \\ -kx + 3k & for \quad 2 \le x \le 3 \\ 0 & for \quad x > 3 \end{cases}$$

Solution:

We know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} kx\,dx + \int_{1}^{2} k\,dx + \int_{2}^{3} (-kx + 3k)\,dx + 0 = 1$$

$$\left[k\frac{x^2}{2}\right]_1^0 + [kx]_1^2 + \left[-k\frac{x^2}{2}\right]_2^3 + [3kx]_2^3 = 1$$

$$\frac{k}{2} + 2k - k + \left(-\frac{9k}{2} + 2k\right) + 9k - 6k = 1$$

$$-4k + 13k - 7k = 1$$

$$-11k + 13k = 1$$

$$k = \frac{1}{2}$$

The cumulative distribution function $F(x)$

$$\int_{0}^{x} kt\,dt = \frac{1}{2}\int_{0}^{x} t\,dt = \frac{1}{2}\frac{x^2}{2}\ for \quad 0 \le x \le 1$$

$$\int_{0}^{1} kt\,dt + \int_{1}^{x} k\,dt = \int_{0}^{1} \frac{t}{2}\,dt + \int_{1}^{x} \frac{t}{2}\,dt = \frac{x}{2} - \frac{1}{4},\ for \quad 1 \le x \le 2$$

$$\int_{0}^{1} kt\,dt + \int_{1}^{2} k\,dt + \int_{2}^{x} (-kt + 3k)\,dt$$

$$= \frac{1}{4} + \frac{1}{2} + \left[-\frac{1}{2}\frac{t^2}{2} + \frac{3}{2}t \right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{x^2}{2} + \frac{3}{2}x + 1 - 3$$

$$= -\frac{x^2}{2} + \frac{3}{2}x - \frac{5}{4};\ 2 \leq x \leq 3$$

$$1,\ for\ x > 3$$

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ \frac{x^2}{4} & for \quad 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{4} & for \quad 1 \leq x \leq 2 \\ -\frac{x^2}{2} + \frac{3}{2}x - \frac{5}{4} & for \quad 2 \leq x \leq 3 \\ 1,\ for \quad x > 3 & \end{cases}$$

Mathematical Expectation:

Let X be a random variable with probability distribution $f(x)$, then the mean or mathematical expectation of X is denoted by $E(X)$ and it is denoted by

$E(X) = \sum x\, f(x)$, where $X$is a discrete random variable

$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$,where $X$is a continuous random variable

X be a random variable with pdf $f(x)$ and the mean μ, then the variance of X is

$V(x) = \sigma^2 = \mathrm{E}[(X-\mu)^2] = \sum (X-\mu)^2 f(x)$, where $X$is a discrete random variable

$V(x) = \sigma^2 = \int_{-\infty}^{+\infty} (X-\mu)^2 f(x)$, where $X$is a continuous random variable

The positive square root of variance is a standard deviation of X. It is denoted by $\sigma(S.D)$.

Note:

$E(x^2) = \sum x^2 \, f(x)$ (discrete)

$E(x^2) = \int_{-\infty}^{+\infty} x^2 \, f(x)$ (continuous)

Problem:

If X is a random variable whose pdf is

$f(x) = \begin{cases} \frac{x}{3} & x = 1,2 \\ 0 & otherwise \end{cases}$ find the mathematical expectation of

(i) x (ii) x^2 (iii) $15 - 6x$

Sol:

Given $f(x) = \begin{cases} \frac{x}{3}, & x = 1,2 \\ 0, & otherwise \end{cases}$

We know that $E(X) = \int x \, f(x) \, dx$ (continuous)

(i) $E(x) = \sum_{x=1}^{2} x \, f(x)$ (discrete)

$$= 1 \, f(1) + 2 \, f\ (2)$$

$$= 1\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = \frac{5}{3}$$

(ii) $E(x^2) = \sum_{x=1}^{2} x^2 \, f(x)$ (discrete)

$$= 1 \, f(1) + 4 \, f\ (2)$$

$$= 1\left(\frac{1}{3}\right) + 4\left(\frac{2}{3}\right) = \frac{9}{3} = 3$$

(iii) $E(15 - 6x) = \sum_{x=1}^{2} 15 - 6x \, f(x)$

$$= (15 - 6(1))\, f(1) + (15 - 6(2))\, f\,(2) \qquad \text{(since E(c) = c)}$$

$$= 9\left(\frac{1}{3}\right) + 3\left(\frac{2}{3}\right) = \frac{15}{3} = 5$$

Problem:

If X is a random variable whose pdf is

$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & otherwise \end{cases}$ find the mathematical expectation of

(i) x (ii) x^2 (iii)$6x - 3x^2$

Sol:

Given $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & otherwise \end{cases}$

(i) $E(x) = \int_0^1 2x\,(1-x)dx$

$$= \int_0^1 2x\,dx - \int_0^1 2x^2 dx$$

$$= 2\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

(ii) $E(x^2) = \int_0^1 2x^2\,(1-x)dx$

$$= \int_0^1 2x^2\,dx - \int_0^1 2x^3 dx$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1\right]$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{6}$$

(iii) $E(6x) - E(3x^2) = 6\int_0^1 2x\,(1-x)dx - 3\int_0^1 2x^2\,(1-x)dx$

$$= 12\int_0^1 (x - x^2)dx - 6\int_0^1 (x^2 - x^3)dx$$

$$= 12\left[\frac{1}{2} - \frac{1}{3}\right] - 6\left[\frac{1}{12}\right] = \frac{3}{2}$$

Problem:

If X is a random variable whose pdf is

$f(x) = \begin{cases} \frac{1}{\pi}\ \frac{1}{(1+x)^2} & -\infty < x < \infty \\ 0 & otherwise \end{cases}$, then show that $E(x)$ does not exist.

Sol:

$$E(x) = \int_{-\infty}^{+\infty} x\,f(x)\ dx$$

$$= \int_{-\infty}^{+\infty} \frac{x}{\pi}\ \frac{1}{(1+x)^2}\ \ dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2x}{(1+x)^2}\ \ dx$$

$$= \frac{1}{2\pi}(\log(1+x)^2)_{-\infty}^{+\infty}$$

Therefore $E(x)$ does not exist.

Result 1:

If k is a constant $E(k) = k$ itself.

Proof:

We know that

$$E(x) = \int_{-\infty}^{+\infty} x\,f(x)\ dx$$

$$E(k) = \int_{-\infty}^{+\infty} k\, f(k)\ dk$$

$$= k \int_{-\infty}^{+\infty} f(k)\ dk$$

$$= k.1 = k.$$

$$E(k) = k$$

Result 2:

If $a\ and\ b$ are constants and X is a random variable with pdf $f(x)$ then $(ax + b) = aE(x) + b$.

Proof:

We have

$$E(ax + b) = \int_{-\infty}^{+\infty} (ax + b)\, f(x)\ dx$$

$$= \int_{-\infty}^{+\infty} a\, x\, f(x)\ dx + \int_{-\infty}^{+\infty} b\, f(x)\ dx$$

$$= aE(x) + b.1$$

$= aE(x) + b$

Result 3:

The variance of a random variable X is $\sigma^2 = E(x^2) - \mu^2$.

Proof:

Let X be a random variable then variance $\sigma^2 = \mathrm{E}[(X - \mu)^2] = \sum (X - \mu)^2 f(x)$

$$\sigma^2 = \sum (x^2 + \mu^2 - 2x\mu) f(x)$$

$$= \sum x^2 f(x) + \sum \mu^2 f(x) - \sum 2x\mu f(x)$$

$$= E(x^2) + \mu^2 - 2\mu E(x)$$

$$= E(x^2) + \mu^2 - 2\mu^2$$

$$= E(x^2) - \mu^2$$

$$i.e., \ \sigma^2 = E(x^2) - \mu^2$$

Try your self

Find the mean and variance of a random variable whose pdf is

$$f(x) = \begin{cases} \frac{x}{15} & for\ x = 1,2,3,4,5. \\ 0 & otherwise \end{cases}$$

Discrete random variables $X\ and\ Y$:

Joint Probability Distribution function of (X, Y)

The set of triples $(X_i, Y_j, P_{ij}), i = 1,2,3, \ldots, n\ ;\ j = 1,2,3, \ldots, m$ is called the joint probability distribution function of (X, Y) and it can be represented in the form of table as follows:

Y \ X	Y_1	Y_2	Y_3	$\cdots$	Y_m	$P_X(X_i)$
X_1	P_{11}	P_{12}	P_{13}	$\cdots$	P_{1m}	P_{1*}
X_2	P_{21}	P_{22}	P_{23}	$\cdots$	P_{2m}	P_{2*}
X_3	P_{31}	P_{32}	P_{33}		P_{3m}	P_{3*}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_n	P_{n1}	P_{n2}	P_{n3}	$\cdots$	P_{nm}	P_{n*}
$P_Y(Y_j)$	P_{*1}	P_{*2}	P_{*3}	$\cdots$	P_{*m}	1

Marginal Probability Distribution

Let (X, Y) be a two-dimensional discrete random variable. Then the marginal probability function of the random variable X is defined as

$$P(X = x_i) = \sum_{j=1}^{m} P_{ij} = P_{i*}$$

The marginal probability function of the random variable Y is defined as

$$P(Y = y_j) = \sum_{i=1}^{n} P_{ij} = P_{*j}$$

The marginal distribution of X is the coefficient of pairs (x_i, P_{i*}) and of Y is (y_j, P_{*j}).

Conditional Probability Distribution

Let (X, Y) be two-dimensional discrete random variable, then

$$P\left(X = x_i/Y = y_j\right) = \frac{P\left(X = x_i, Y = y_j\right)}{P\left(Y = y_j\right)} = \frac{P_{ij}}{P_{*j}}$$

is called the conditional probability function of X given $Y = y_j$.

and

$$P\left(Y = y_j/X = x_i\right) = \frac{P\left(X = x_i, Y = y_j\right)}{P(X = x_i)} = \frac{P_{ij}}{P_{i*}}$$

is called the conditional probability function of X given $X = x_i$.

Problem 1:

For the bivariate probability distribution of (X, Y) given below, find

$P(X \leq 1), P(Y \leq 3), P(X \leq 1, Y \leq 3\,), P(X \leq 1/Y \leq 3\,)$ *and* $P(Y \leq 3/X \leq 1\,)$.

Y \ X	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Problem 2:

A random observation on a bivariate population (X, Y) can yield one of the following pairs of values with probabilities noted against them:

For each observation pair	Probability
(1,1); (2,1); (3,3); (4,3)	1/20
(3,1); (4,1); (1,2); (2,2); (3,2); (4,2); (1,3); (2,3)	1/10

Find the probability that $Y = 2$ given that $X = 4$. Also find the probability that $Y = 2$. Examine if the two events $X = 4$ and $Y = 2$ are independent.

Problem 3:

The joint probability distribution of two random variables X *and* Y is given by: $P(X = 0, Y = 1) = \frac{1}{3}, P(X = 1, Y = -1) = \frac{1}{3}, and\ P(X = 1, Y = 1) = \frac{1}{3}$.

Find

(i) Marginal distributions of X *and* Y

(ii) the conditional probability distribution of X given $Y = 1$.

Try Yourself:

(a) A two-dimensional random variable (X, Y) have a bivariate distribution given by:

$P(X = x, Y = y) = \frac{x^2+y}{32}, for\ x = 0,1,2,3\ and\ y = 0,1.$

Find the marginal distribution of $X\ and\ Y$

(b) a two-dimensional random variable (X, Y) have a joint probability mass function: $P(x, y) = \frac{1}{27}(2x + y)$, where $x\ and\ y$ can assume only the integer values 0,1 and 2.

Find the conditional distribution of Y for $X = x$.

Continuous random variables $X\ and\ Y$:

Joint Probability Density function of (X, Y)

Let (X, Y) be a two-dimensional continuous random variable such that

$$P\left(X - \frac{dX}{2} \le X \le X + \frac{dX}{2}, \qquad Y - \frac{dY}{2} \le Y \le Y + \frac{dY}{2}\right) = \iint f(X,Y)\ dXdY$$

Then $f(X, Y)$ is called the joint density function of (X, Y), if it satisfies the following conditions:

(i) $f(X, Y) \ge 0, for\ all\ (X, Y) \in R$, where R is the range space.

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(X, Y)\ dXdY = 1$

Moreover, if $(a, b), (c, d) \in R$, then

(iii) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(X, Y)\ dXdY = 1$

Marginal Probability Distribution:

When (X, Y) be a two-dimensional continuous random variable, then the marginal density function of the random variable X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

The marginal density function of the random variable Y is defined as

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Conditional Probability Distribution

Let (X, Y) be two-dimensional continuous random variable, then

$$f(x/y) = \frac{f(x, y)}{f_Y(y)}$$

is called the conditional probability function of X given Y.

and, $f(y/x) = \frac{f(x,y)}{f_X(x)}$

is called the conditional probability function of Y given X.

Problem 1:

Joint distribution of $X\ and\ Y$ is given by $f(x, y) = 4xye^{-(x^2+y^2)}; x \geq 0, y \geq 0$, test whether $X\ and\ Y$ are independent. For the above joint distribution, find the conditional density of $X\ given\ Y = y$.

Solution:

Joint probability distribution function of $X\ and\ Y$ is $f(x, y) = 4xye^{-(x^2+y^2)}; x \geq 0, y \geq 0$

The marginal density of X is given by

$$f_X(x) = \int_0^\infty f(x,y)\, dy$$

$$= \int_0^\infty 4xye^{-(x^2+y^2)} dy = 4x\, e^{-x^2} \int_0^\infty y\, e^{-y^2} dy$$

$$= 4x\, e^{-x^2} \int_0^\infty e^{-t} \frac{dt}{2}$$

$$= 2x\, e^{-x^2}; x \geq 0$$

Similarly,

$$f_Y(y) = \int_0^\infty f(x,y)\, dx$$

$$= 2y\, e^{-y^2}; y \geq 0$$

Since $f_{XY}(x,y) = f_X(x).f_Y(y), X \text{ and } Y$ are independently distributed.

The conditional distribution of X is given by $Y = y$

$$f_{X/Y}(X = x, Y = y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{4xye^{-(x^2+y^2)}}{2y\, e^{-y^2}} = 2x\, e^{-x^2};\ x \geq 0$$

Problem 2:

Suppose that two-dimensional continuous random variable (X, Y) has joint probability density function given by

$$f(x,y) = \begin{cases} 6x^2y, & 0 < x < 1, 0 < y < 1 \\ 0, & elsewhere \end{cases}$$

(i) Verify that $\int_0^1 \int_0^1 f(x,y)dxdy = 1$

(ii) Find $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right),\ P(X + Y < 1), P(X > Y) and\ P(X < 1/Y < 2)$

Solution:

(i) $\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \int_0^1 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^1 dx = \int_0^1 3x^2\, dx = 1$

(ii) $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^1 6x^2y\, dxdy + \int_0^{3/4} \int_1^2 0\, dxdy$

$$= \int_0^{3/4} 6x^2 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{8}{9}\int_0^{3/4} 3x^2\, dx = \frac{3}{8}$$

$P(X + Y < 1)$

$$= \int_0^1 \int_0^{1-x} 6x^2 y\, dxdy = \int_0^1 6x^2 \left(\frac{y^2}{2}\right)_0^{1-x} dx = \int_0^1 3x^2(1-x)^2 dx$$

$$= \frac{1}{10}$$

$$P(X > Y) = \int_0^1 \int_0^x 6x^2 y\, dxdy = \int_0^1 3x^2 (y^2)_0^x\, dx = \int_0^1 3x^4 dx = \frac{3}{5}$$

$$P(X < 1/Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)}$$

$$P(X < 1 \cap Y < 2) = \int_0^1 \int_0^1 6x^2 y\, dxdy + \int_0^1 \int_1^2 0.\ dxdy = 1$$

$$P(Y < 2) = \int_0^1 \int_0^2 f(x,y)\, dxdy = \int_0^1 \int_0^1 6x^2 y\, dxdy + \int_0^1 \int_1^2 0. dxdy = 1$$

$$P(X < 1/Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)} = 1$$

Try yourself:

1. If X *and* Y are two random variables having joint density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0 \le x < 2, \quad 2 \le y < 4 \\ 0, & elsewhere \end{cases}$$

Find

(i) $P(X < 1 \cap Y < 3)$
(ii) $P(X + Y < 3)$
(iii) $P(X < 1/Y < 3)$

2. If $f(x_1, x_2) = \begin{cases} 4x_1x_2, & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & elsewhere \end{cases}$ is a joint p.d.f. of x_1 and x_2.

   Then find $P\left(0 < x_1 < \frac{1}{2},\ \frac{1}{4} < x_2 < 1\right)$.

Moments:

The r^{th} moment about the origin of a random variable X denoted by μ_r is $E(X^r)$, i.e.,

$$\mu_0 = E(X^0) = E(1) = 1$$

$$\mu_1 = E(X^1) = E(X) = \mu$$

$$\mu_2 = E(X^2) - \big(E(X)\big)^2 = E(X^2) - \mu^2$$

$$E(X^2) = \sigma^2 + \mu^2$$

Moment Generating function (MGF):

The MGF of the distribution of a random variable completely describes the nature of the distribution.

Let having PDF $f(X)$, then the MGF of the distribution of X is denoted by $M(t)$ and is defined as $M(t) = E(e^{tx})$.

Thus, the MGF $M(t) = \begin{cases} \sum e^{tx} f(x), & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, & if\ x\ is\ contuous \end{cases}$

We know that $M(t) = E(e^{tx})$

$$M(t) = E\left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right)$$

$$= E(1) + E(tx) + E\left(\frac{t^2x^2}{2!}\right) + E\left(\frac{t^3x^3}{3!}\right) + \cdots$$

$$= 1 + t.E(x) + \frac{t^2}{2!}.E(x^2) + \frac{t^3}{3!}.E(x^3) + \cdots$$

$$= 1 + t.\mu_1 + \frac{t^2}{2!}.\mu_2 + \cdots$$

$$M(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r{}'$$

The coefficient of $\frac{t^r}{r!}$ is about the origin is $\mu_r{}'$.

If X be a continuous random variable, then MGF is

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$M'(t) = \int_{-\infty}^{\infty} x . e^{tx} f(x) dx$$

$$M''(t) = \int_{-\infty}^{\infty} x^2 . e^{tx} f(x) dx, \ldots$$

Now at $t = 0$

$$M(0) = E(1) = 1$$

$$M'(0) = E(x) = \mu$$

$$M''(0) = E(x^2) = \sigma^2 + \mu^2$$

Mean is $\mu = M'(0)$

Variance is $M''(0) = M''(0) - \left(M'(0)\right)^2$

$$\mu_r{}' = \frac{\partial^r}{\partial t^r} \left(M(t)\right) \;; \quad r = 0,1,2,\ldots$$

Example 1:

Obtain the moment generating function of the probability density function is

$f(x) = \begin{cases} xe^{-x}, & 0 < x < \infty \\ 0, & otherwise \end{cases}$.

Solution:

The MGF of the distribution is

$$M(t) = E(e^{tx})$$

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx}\, xe^{-x} dx$$

$$= \int_0^{\infty} xe^{(t-1)x} dx$$

$$= \int_0^{\infty} xe^{-(1-t)x} dx$$

$$= \frac{1}{(1-t)^2}, \qquad for\ t < 1$$

Example 2:

Find the moment generating function of the probability distribution function is

$$f(x) = \begin{cases} \frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3, & x = 0,1,2,3 \\ 0, & otherwise \end{cases}.$$

Solution:

The MGF of the distribution is

$$M(t) = \sum_{x=0}^{3} e^{tx} f(x)$$

$$= \sum_{x=0}^{3} e^{tx} \frac{3!}{x!\,(3-x)!}\left(\frac{1}{2}\right)^3$$

$$= \left(\frac{1}{2}\right)^3 \left[\frac{3!}{3!} + e^t \frac{3!}{1!\,2!} + e^{2t}\frac{3!}{2!\,1!} + e^{3t}\frac{3!}{3!\,0!}\right]$$

$$M(t) = \left(\frac{1}{2}\right)^3 \left[1 + 3e^t + 2e^{2t} + e^{3t}\right]$$

Characteristic function:

The characteristic function is defined as

$$\emptyset_X(t) = E\left(e^{itX}\right) = \begin{cases} \sum_x e^{itX} f(x), & for\ discrete\ probability\ distribution \\ \int e^{itX} f(x)dx, for\ continuous\ probability\ distribution \end{cases}$$

If $F_X(x)$ is the distribution function of a continuous random variable X, then

$$\emptyset_X(t) = \int_{-\infty}^{\infty} e^{itX}\, dF(x)$$

Where, $dF(x) = C\ \frac{1}{(1+x^2)^m}; m > 1, -\infty < x < \infty$

For discrete case, we have

$$\emptyset_X(t) = E\left(e^{itX}\right)$$

$$= E\left(1 + itX + \frac{(it)^2X^2}{2!} + \frac{(it)^3x^3}{3!} + \cdots\right)$$

$$= E(1) + E(itX) + E\left(\frac{(it)^2X^2}{2!}\right) + E\left(\frac{(it)^3X^3}{3!}\right) + \cdots$$

$$= 1 + it.E(X) + \frac{(it)^2}{2!}.E(X^2) + \frac{(it)^3}{3!}.E(X^3) + \cdots$$

$$= 1 + it.\mu_1 + \frac{(it)^2}{2!}.\mu_2 + \cdots$$

$$M(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r{}'$$

The coefficient of $\frac{(it)^r}{r!}$ is about the origin is $\mu_r{}'$.

Properties of characteristic function:

1. If the distribution function of a random variable X is symmetrical about zero, i.e., if $f(-x) = f(x)$, then $\emptyset_X(t)$ is real valued function of t.
2. $\emptyset_{cX}(t) = \emptyset_X(ct)$, c being a constant.
3. If X_1 and X_2 are independent random variables, then $\emptyset_{X_1+X_2}(t) = \emptyset_{X_1}(t).\emptyset_{X_2}(t)$
4. $\emptyset_X(-t)$ and $\emptyset_X(t)$ are conjugate functions, i.e., $\emptyset_X(-t) = \overline{\emptyset_X(t)}$.

Covariance:

If X and Y are two random variables, then the Covariance between them is defined as

$Cov(X,Y) = E(XY) - E(X)E(Y)$

Properties:

1. If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$
2. $Cov(X + a, Y + b) = Cov(X,Y)$

3. $Cov(aX + b, cY + d) = ac\, Cov(X, Y)$

Problem:

Two random variables X and Y have the following joint pdf

$$f(x, y) = \begin{cases} 2 - x - y, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

Find the

i. Variance of X

ii. Variance of Y

iii. Covariance of X and Y.