CSCE 5150 – Analysis of Computer Algorithms

Homework No. 1 - Introduction & Growth of Functions

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Q1. [10 points] Give asymptotic upper and lower bounds for T(n) in each of the following recurrences. Assume that T(n) is constant for $n \leq 3$. Make your bounds as tight as possible, and justify your answers. (Hint: You can use the best matching case of Master Method to determine the tight bounds of Θ notation)

(a) T(n) = 2T(n/3) + n lg n

(b) T(n) = 3T(n/5) + lg2 n

(c) T(n) = 7T(n/2) + n 3

(d) T(n) = T($\sqrt{n}$) + Θ(lg lg n)

(e) T(n) = 10T(n/3) + 17n 1.2

$$\boxed{T(n) = a\,T\left(\frac{n}{b}\right) + \theta\left(n^k \log^p n\right)}$$

Master's Theorem

Here, a >= 1, b > 1, k >= 0 and p is a real number.

Case-01:

If $a > b^k$, then $T(n) = \theta(n^{\log_b a})$

Case-02:

If $a = b^k$ and

- If p < -1, then $T(n) = \theta(n^{\log_b a})$
- If p = -1, then $T(n) = \theta(n^{\log_b a}.\log^2 n)$
- If p > -1, then $T(n) = \theta(n^{\log_b a}.\log^{p+1} n)$

Case-03:

If $a < b^k$ and

- If p < 0, then $T(n) = O(n^k)$
- If p >= 0, then $T(n) = \theta(n^k \log^p n)$

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a) $T(n) = 2T(n/3) + n \lg n$ $\left\{\begin{matrix} a < b^k \\ 2 < 3 \end{matrix}\right\}$ case 3

$a = 2, b = 3, k = 1, p = 1$

As $p \geq 0$, $\Rightarrow \Theta(n^k \log^p n) \Rightarrow \Theta(n \lg n)$

It is case 3 of the master theorem.

b) $T(n) = 3T\left(\frac{n}{5}\right) + \lg^2 n$ case 1 $\left\{\begin{matrix} a > b^k \\ 3 > 5^0 \end{matrix}\right.$

$a = 3, b = 5, k = 0, p = 2$

$T(n) = \Theta(n^{\log_5 3})$

It is case 1 of master theorem

c) $T(n) = 7T(n/2) + n^3$

$a = 7, b = 2, k = 3, p = 0$ case 3 $\left\{\begin{matrix} 7 < 2^3 \\ a < b^k \end{matrix}\right.$

$T(n) = \Theta(n^3 (\log n)^0) = \Theta(n^3)$

It is case 3 of masters theorem

d) $T(n) = T(\sqrt{n}) + \Theta(\lg \lg n)$

$a = 1, b = 1,$ $k = \lg n$ $p = 1$ $\left\{\begin{matrix} 1 = 1^1 \\ a = b^k \end{matrix}\right.$ case 2

$T(n) = \Theta((\lg \lg n)^2)$

It is case 2 of masters theorem

e) $T(n) = 10T(n/3) + 17n^{1.2}$

$a = 10, b = 3, k = 1.2$ case 1 $\left\{\begin{matrix} a > b^k \\ 10 > 3^{1.2} \end{matrix}\right.$

$T(n) = n^{\log_3 10}$

It is case 1 of masters theorem

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Q2. [5 points] By using the substitution method, show that the solution of the recurrence T(n) = T(n-1) + n is O(n 2). (Exercise 4.3-1)

2. Assume $T(1) = \Theta(1)$

Guess $O(n^2)$, Assume $T(n) \leq cn^2, k < n$

Prove $T(n) < cn^2$

$T(n) = T(n-1) + n \leq cn^2$

$\leq c(n-1)^2 + n = cn^2 - 2cn + c + n$

$\leq cn^2 + (1-2c)n + c$

$cn^2 \geq cn^2 + (1-2c)n + c$

It holds for $c > \frac{1}{2}$.

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Q3. [5 points] Use a recursion tree to determine a good asymptotic upper bound on the recurrence T(n) = T(n/2) + n 2 . You can use the substitution method to verify your answer. (Exercise 4.4-2)

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3. $T(n) = T\left(\frac{n}{2}\right) + n^2$

<u>Recursion tree:</u>

T(n)	node		Sum
$T(n)$	0	n^2	n^2
$T\left(\frac{n}{2}\right)$	1	$\left(\frac{n}{2}\right)^2 \quad \left(\frac{n}{2}\right)^2$	$2\left(\frac{n}{2}\right)^2$
$T\left(\frac{n}{4}\right)$	2	$\left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2$	$4\left(\frac{n}{4}\right)^2$
⋮	⋮	$\Theta(1)$	⋮
$T\left(\frac{n}{2^i}\right)$	i		

Total $= n^2 + 2\left(\frac{n}{2}\right)^2 + 4\left(\frac{n}{4}\right)^2 + \cdots$

$$\text{Total} = n^2\left(\left(\frac{1}{2^0}\right)^2 + 2\left(\frac{1}{2^1}\right)^2 + 4\left(\frac{1}{2^2}\right)^2 + \cdots\right)$$

$$= n^2 \sum_{i=0}^{\log n} \left(\frac{1}{4}\right)^i$$

$T(n) = \Theta(n^2)$

Substitution Method:

Assume $T(1) = \Theta(1)$

Guess $\Theta(n^2)$.

Assume $T(n) \le cn^2$
$k < n$

Prove $T(n) < cn^2$

~~Upper bound~~: $T(n) \le T\left(\frac{n}{2}\right) + n^2 \le cn^2$

$$\le c\left(\frac{n}{2}\right)^2 + n^2$$

$$\le \frac{cn^2}{4} + n^2$$

$$\le n^2\left(\frac{c+4}{4}\right)$$

We have assumed that $cn^2 \ge n^2\left(\frac{c+4}{4}\right)$

$$4c \ge c + 4$$

$$c \ge 4/3$$

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Q4. [5 points] Use Strassen's algorithm to compute the matrix product and show your work. (Exercise 4.2-1)

(1 3   (6 8

7 5)   4 2)

Output from the PA1 code:

```
Enter matrix size for matrix A and matrix B : 2
Enter numbers in the first matrix A, one after the other: 1
Enter numbers in the first matrix A, one after the other: 3
Enter numbers in the first matrix A, one after the other: 7
Enter numbers in the first matrix A, one after the other: 5
Enter numbers in the second matrix B, one after the other: 6
Enter numbers in the second matrix B, one after the other: 8
Enter numbers in the second matrix B, one after the other: 4
Enter numbers in the second matrix B, one after the other: 2
Below is the matrix_A [[1. 3.]
 [7. 5.]]
Below is the matrix_B [[6. 8.]
 [4. 2.]]
Strassens Matrix multiplication result for above matrix_A and matrix_B is :
[[18 14]
 [62 66]]
```

Validation:

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$A = \begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 8 \\ 4 & 2 \end{bmatrix}$

$A11 = 1$ $B11 = 6$
$A12 = 3$ $B12 = 8$
$A21 = 7$ $B21 = 4$
$A22 = 5$ $B22 = 2$

Sum:

$S_1 = B12 - B22 = 8 - 2 = 6$
$S_2 = A11 - A12 = 1+3 = 4$
$S_3 = A21 + A22 = 7+5 = 12$
$S_4 = B21 - B11 = 4 - 6 = -2$
$S_5 = A11 + A22 = 1+5 = 6$
$S_6 = B11 + B22 = 6+2 = 8$
$S_7 = A12 - A22 = 3-5 = -2$
$S_8 = B21 + B22 = 4+2 = 6$
$S_9 = A11 - A21 = 1-7 = -6$
$S_{10} = B11 + B12 = 6+8 = 14$

Product:

$P_1 = A11 * S_1 = 1*6 = 6$
$P_2 = S_2 * B22 = 4*2 = 8$
$P_3 = S_3 * B11 = 12*6 = 72$
$P_4 = A22 * S_4 = 5*-2 = -10$
$P_5 = S_5 * S_6 = 6*8 = 48$
$P_6 = S_7 * S_8 = -2*6 = -12$
$P_7 = S_9 * S_{10} = -6*14 = -84$

Result matrix: $C_{11} = P_5 + P_4 - P_2 + P_6 = 18$
$C_{12} = P_1 + P_2 = 6+8 = 14$
$C_{21} = P_3 + P_4 = 72-10 = 62$
$C_{22} = P_5 + P_1 - P_3 - P_7 = 48+6-72+84 = 66$

$C = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$