

# **Applied Algorithms**

## **CSCI-B505 / INFO-I500**

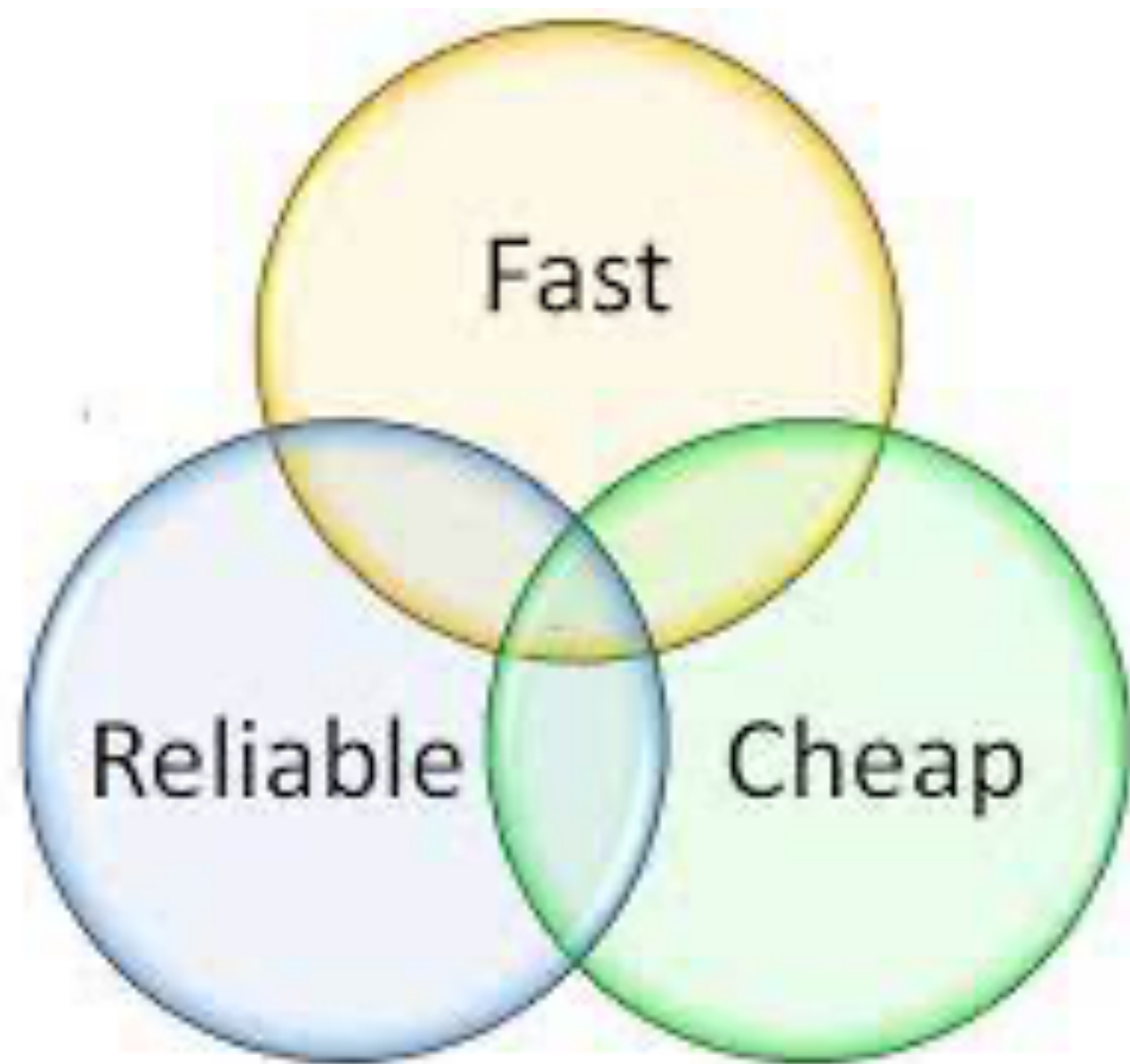
### **Lecture 1.**

### **Algorithm Analysis and Asymptotic Notation**

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# How can we measure the performance of an algorithm?

## The parameters



**Correctness:** Works on all cases, general

**Elegance:** Easy to communicate and code

**Efficiency :**

- Time
- Space

### ON THE COMPUTATIONAL COMPLEXITY OF ALGORITHMS

BY  
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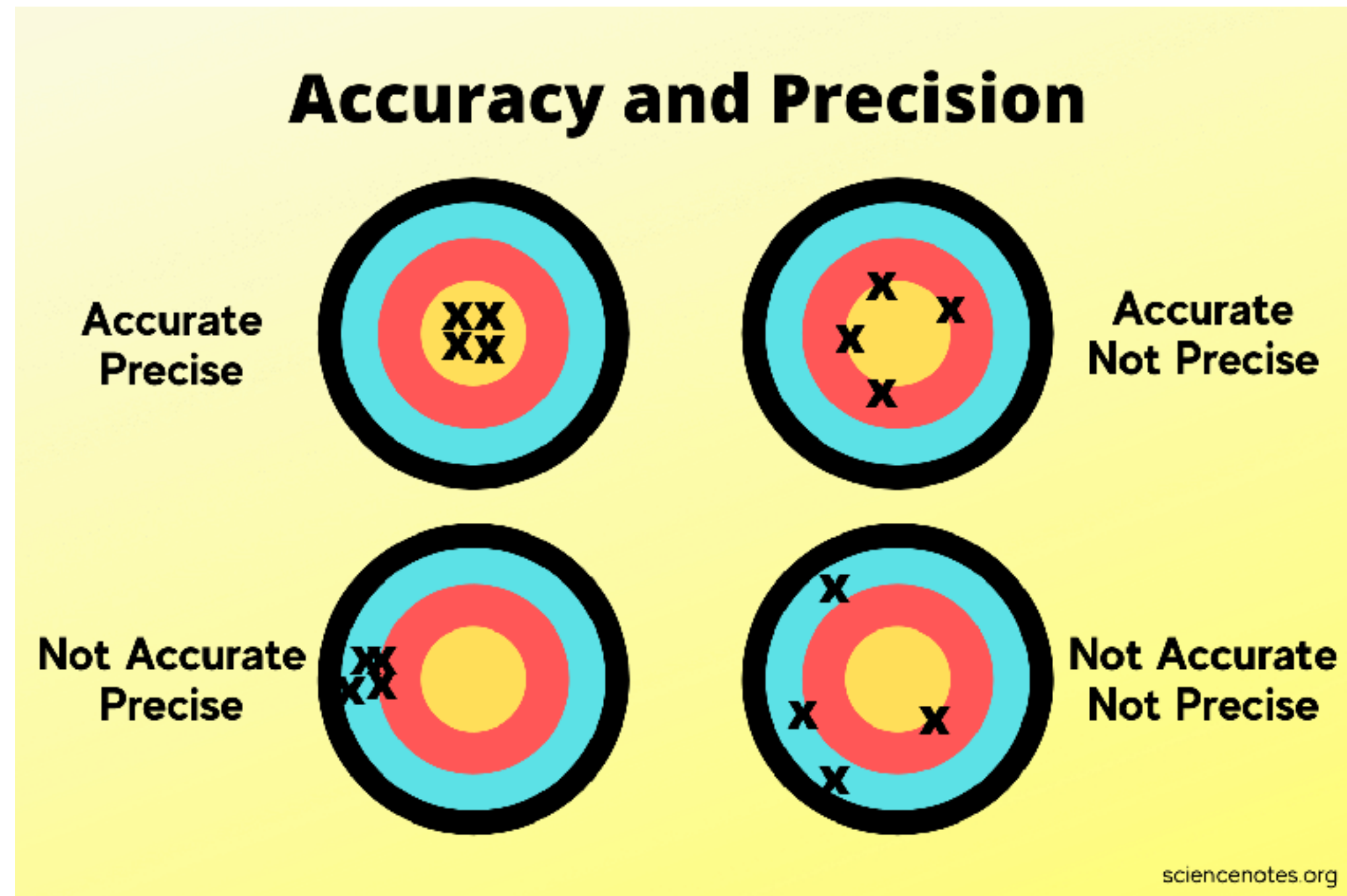
I. Introduction. In his celebrated paper [1], A. M. Turing investigated the computability of sequences (functions) by mechanical procedures and showed that the set of sequences can be partitioned into computable and noncomputable sequences. One finds, however, that some computable sequences are very easy to compute whereas other computable sequences seem to have an inherent complexity that makes them difficult to compute. In this paper, we investigate a scheme of classifying sequences according to how hard they are to compute. This scheme puts a rich structure on the computable sequences and a variety of theorems are established. Furthermore, this scheme can be generalized to classify numbers, functions, or recognition problems according to their computational complexity.

Experimental versus theoretical *evaluation* ?

**Hartmanis & Stearn'1965:**  
**Time and space complexity**  
**analysis of algorithms**

# Experimental vs. Theoretical Measurements?

## Experimental vs. Theoretical Analysis



<https://sciencenotes.org/what-is-the-difference-between-accuracy-and-precision/>

Experimental: PRECISE but SPECIFIC

Theoretical: ACCURATE and GENERAL

To measure the general time/space performance of an algorithm, we follow the theoretical complexity analysis by using the mathematical notion of ASYMPTOTIC ANALYSIS

**ACCURACY:** How close your measurements to the truth

**PRECISION:** How close your measurements to each other

# Experimental vs. Theoretical Analysis

## The experimental analysis

- Implement the algorithm and execute on some inputs !
- Many specifications such as the programming language, programmer, compiler, the machine (CPU, memory, I/O channels, etc ...) are required.

**Very PRECISE on the SPECIFIC case, but ...**

### Some further questions:

- How long will it take on another input size ?
- Best, average, and worst case scenarios ?
- Is it optimal ?
- ...

If we have an **abstract** way of analyzing the performance, such questions would be easier to answer **without need of explicit experiments.**



# Experimental vs. Theoretical Measurements?

## The theoretical analysis

- Assume an abstract machine (*actually, this is the **word-RAM** model*) such that :
  - All instructions (arithmetic, memory access, control, etc...) are *atomic*, which take equal time
  - We have enough memory, and can read/write in blocks of  $\log n$  bits, for an input size of  $n$ .

Without executing the code itself,  
just investigate how many instructions would be  
executed on an input size of  $n$ ,  
and return it as a function of the input size.

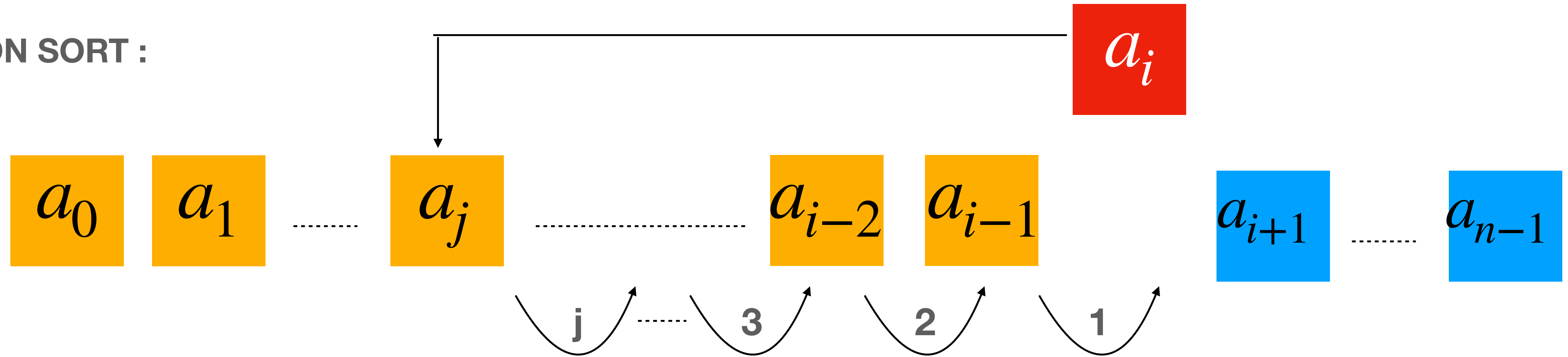
**Very ACCURATE on the GENERAL case!**

Let's pass over an example case...

# Experimental vs. Theoretical Measurements?

## Time Complexity of Insertion Sort

INSERTION SORT :



- Assume  $a_0$  to  $a_{i-1}$  is sorted
- Save  $a_i$  in a temporary variable
- Scan elements towards left, and shift right by one if it is smaller than  $a_i$
- When the check fails, or the beginning of the list is reached, insert  $a_0$  to the vacant position

**Iterate through  $i = 1$  to  $i = n - 1$**

# Investigating the Time Complexity of Insertion Sort

`a[0..n-1]` contains elements to be sorted

		EXECUTION COUNT	COST
		<hr/>	<hr/>
1	<code>for (i = 1; i &lt; n; i++ ) {</code>	$n$	$c_1$
2	<code>    // Invariant: a[0..i-1] is sorted</code>		
	<code>    // Invariant: a[i..n-1] not yet sorted</code>		
3	<code>    int tmp = a[i];</code>	$n - 1$	$c_2$
4	<code>    for (j=i; (j&gt;0 &amp;&amp; a[j]&gt;tmp); j--) {</code>	$\sum_{j=1}^{n-1} t_j$	$c_3$
	<code>        // Invariant: hole is at a[j]</code>		
5	<code>        a[j] = a[j-1];</code>	$\sum_{j=1}^{n-1} (t_j - 1)$	$c_4$
6	<code>    }</code>		
7	<code>    a[j] = tmp;</code>	$n - 1$	$c_5$
	<code>}</code>		

$t_j$  is the number times  $a_j$  is compared with the previous elements.

Notice that  $t_j$  is not related with the input size  $n$ , but the values in the array.

# Experimental vs. Theoretical Measurements?

## Investigating the Time Complexity of Insertion Sort

Total running time  
of insertion sort is

$$C(n) = c_1 \cdot n + (c_2 + c_5) \cdot (n - 1) + c_3 \cdot \sum_{j=1}^{j=n} t_j + c_4 \cdot \sum_{j=1}^{j=n} (t_j - 1)$$

**Worst case:** Input sequence is in decreasing order, thus  $t_j = j$  and

quadratic  
function

$$C(n) = c_1 \cdot n + (c_2 + c_5) \cdot (n - 1) + c_3 \cdot \frac{n(n+1)}{2} + c_4 \cdot \frac{n(n-1)}{2} = A \cdot n^2 + B \cdot n + C$$

**Best case:** Input sequence is in increasing order, thus  $t_j = 1$  and

$$C(n) = c_1 \cdot n + (c_2 + c_5) \cdot (n - 1) + c_3 \cdot n + c_4 \cdot 0 = D \cdot n + E$$

linear  
function

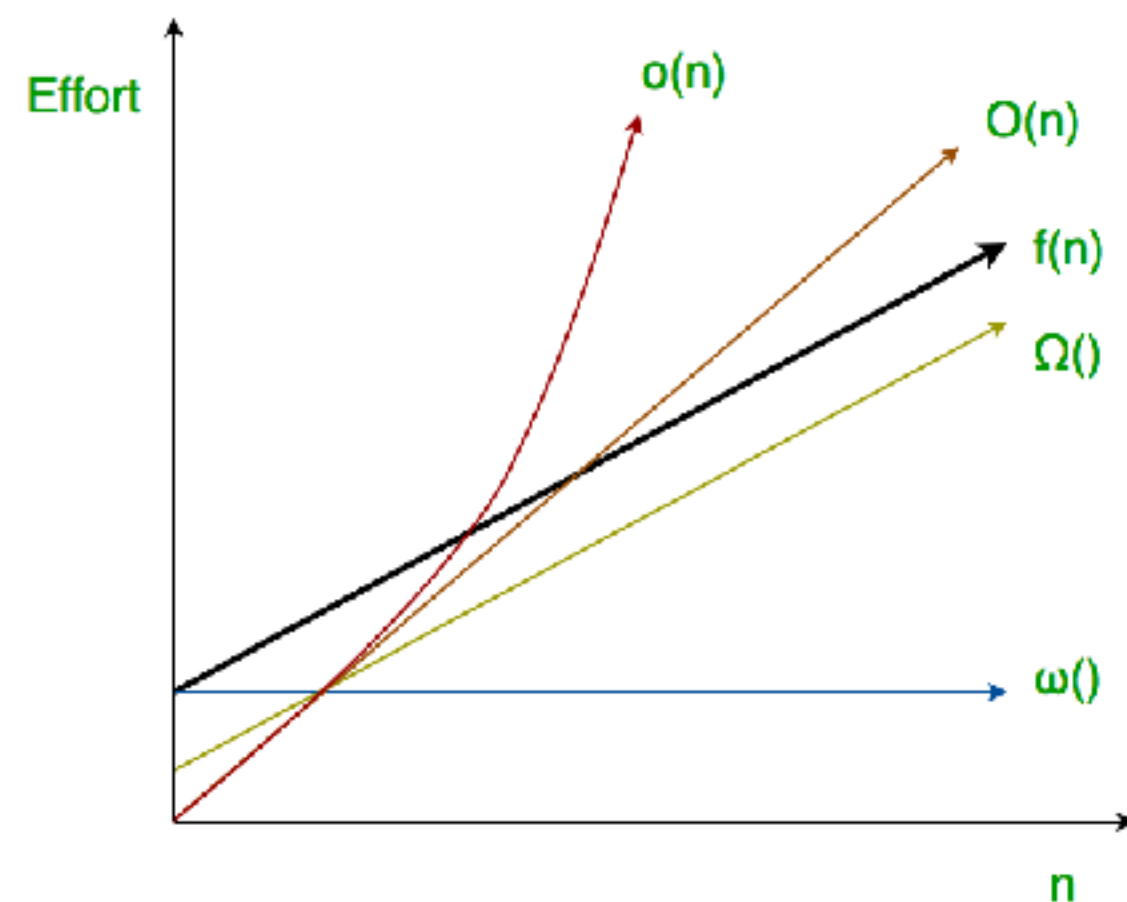
We don't know the values of the constants  $A, B, C, D, E$  exactly,  
but we know how  $C(n)$  will change according to input size  $n$  !



# Asymptotic Notation

## Big-Oh, Big-Omega, Big-Theta, Little-Oh, Little-Omega

How do the running time and the memory consumption of an algorithm change according to the input size ?



$O()$ ,  $\Omega()$ ,  $\Theta()$ ,  $o()$ ,  $w()$

**Bachman (1894),**

**Landau (1909):**

**Hardy&Littlewood(1914);**

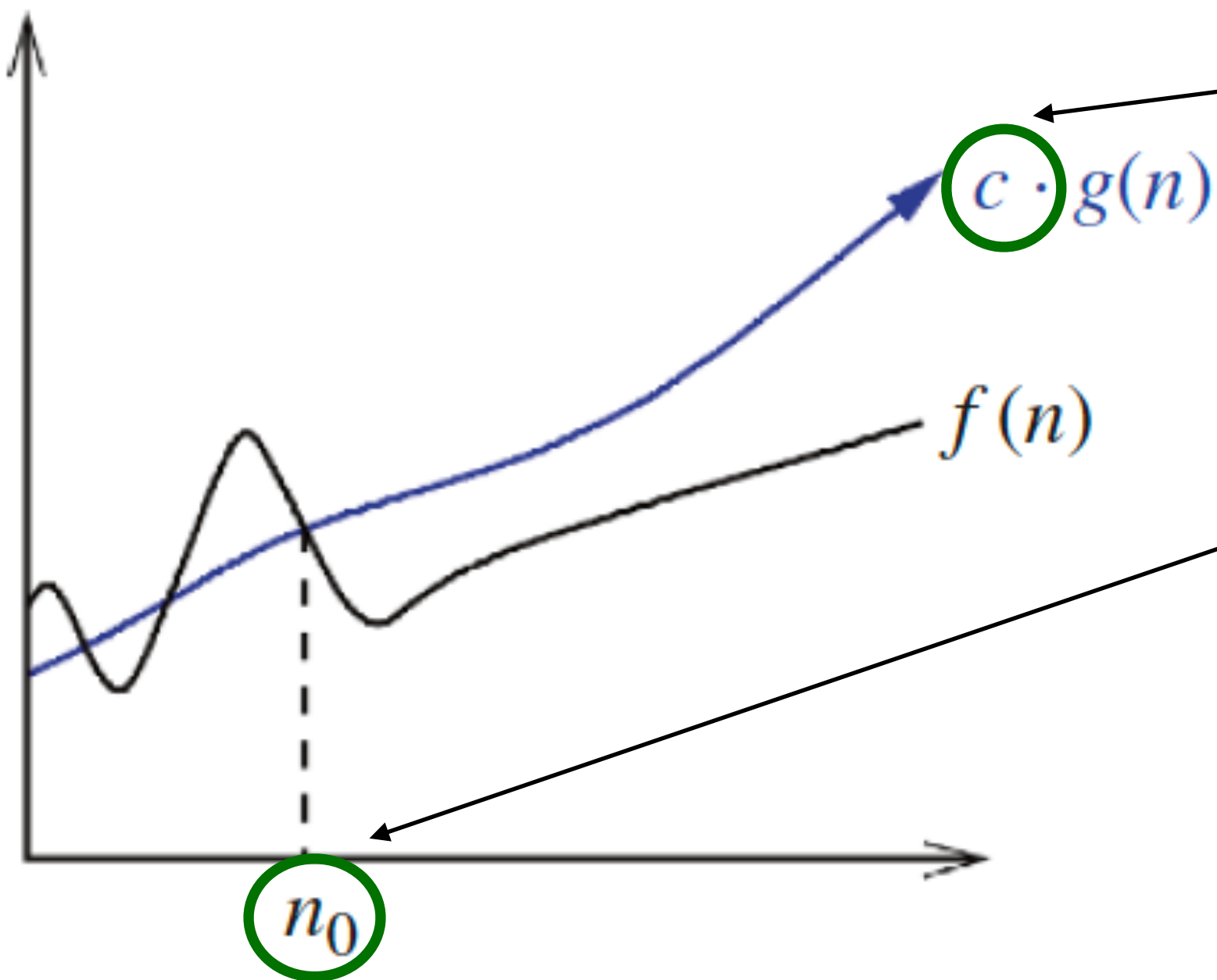
**Knuth (1970s)**

- Represent the number of instructions and the memory allocation of an algorithm as a function of the input size.
- Such functions may include many terms of different orders
- **Lower order terms will not make much difference when input size gets large**
- $O()$ ,  $\Omega()$ ,  $\Theta()$ ,  $o()$ ,  $w()$  notations are used to observe the growth of functions

# Big-Oh

$O(g(n))$  defines a **SET** of functions such that there exist positive constants  $c > 0$  and  $n_0 > 0$ , where  $f(n)$  is always less than or equal to  $c \cdot g(n)$  for all  $n > n_0$ .

$$O(g(n)) = \{f(n) \mid 0 \leq f(n) \leq c \cdot g(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$



All  $f(n)$  functions are  
UPPER BOUNDED  
by  $g(n)$

# Big-Oh

$$O(g(n)) = \{f(n) \mid 0 \leq f(n) \leq c \cdot g(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$

To prove  $f(n) \in O(g(n))$  we need to find explicit values of  $c$  and  $n_0$ , where  $f(n)$  is always less than or equal to  $c \cdot g(n)$  for all  $n > n_0$ .

Example:

If  $f(n) = 4n^2 + 10000000n + 10^{34}$ , then  $f(n) \in O(n^2)$ .

Proof:

For  $c = 5$ ,  $n_0 = 10^{18}$ ;  $4n^2 + 10^6n + 10^{34} \leq 5 \cdot n^2, \forall n > 10^{18}$

*After a threshold value,  $f(n)$  is **upper bounded** by  $g(n) = n^2$ .*

# Little-Oh

$o(g(n))$  defines a **SET** of functions such that **for any**  $c > 0$ , there is an  $n_0 > 0$ , where  $f(n)$  is always **less than**  $c \cdot g(n)$  for all  $n \geq n_0$ .

$$o(g(n)) = \{f(n) \mid 0 \leq f(n) < c \cdot g(n), \forall n \geq n_0, \forall c > 0, \exists n_0 > 0\}$$

$$O(g(n)) = \{f(n) \mid 0 \leq f(n) \leq c \cdot g(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$

Examples:

- $5n + 17 \in o(n^2)$
- $n \log n \in o(n^{1.0001})$

Alternative definition:

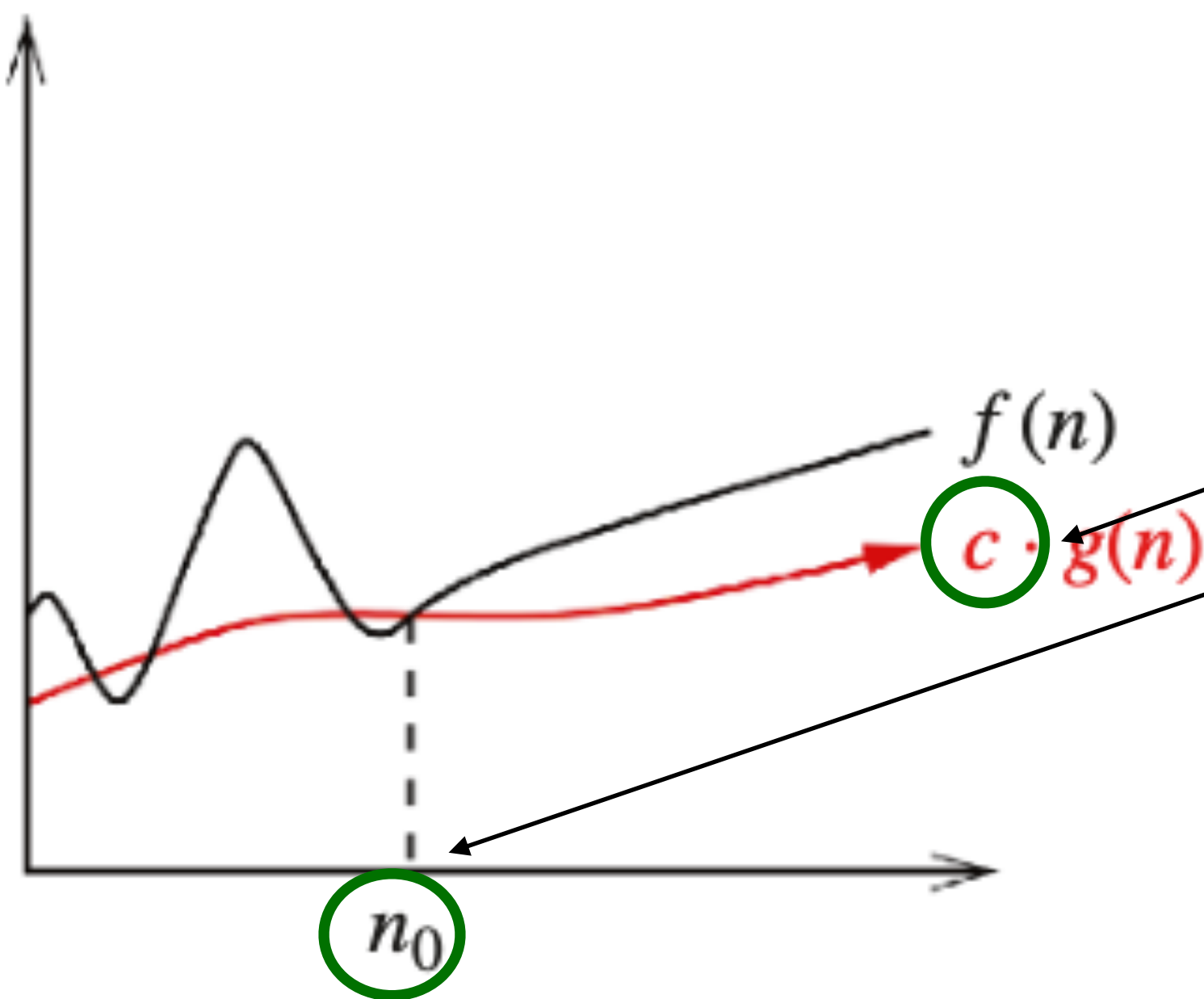
$$\{f(n) \mid \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0\}$$



# Big-Omega

$\Omega(g(n))$  defines a **SET** of functions such that there exist positive constants  $c > 0$  and  $n_0 > 0$ , where  $f(n)$  is always greater than or equal to  $c \cdot g(n)$  for all  $n > n_0$ .

$$\Omega(g(n)) = \{f(n) \mid 0 \leq c \cdot g(n) \leq f(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$



All  $f(n)$  functions are  
**LOWER BOUNDED**  
by  $g(n)$

# Big-Omega

$$\Omega(g(n)) = \{f(n) \mid 0 \leq c \cdot g(n) \leq f(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$

To prove  $f(n) \in \Omega(g(n))$  we need to find explicit values of  $c$  and  $n_0$ , where  $f(n)$  is always **larger than or equal** to  $c \cdot g(n)$  for all  $n > n_0$ .

Example:

Given  $f(n) = 0.01n^2 + 0.1n + 3$ ,  $f(n) \in \Omega(n)$ .

Proof:

For  $c = 1$ ,  $n_0 = 90$ ;  $0.01n^2 + 4n + 3 \geq 1 \cdot n, \forall n > 90$

*After a threshold value,  $f(n)$  is **lower bounded** by  $g(n) = n$ .*

# Little-Omega

$\omega(g(n))$  defines a **SET** of functions such that **for any**  $c > 0$ , there is an  $n_0 > 0$ , where  $f(n)$  is always **greater than**  $c \cdot g(n)$  for all  $n \geq n_0$ .

$$\omega(g(n)) = \{f(n) \mid 0 \leq c \cdot g(n) < f(n), \forall n \geq n_0, \forall c > 0, \exists n_0 > 0\}$$



$$\Omega(g(n)) = \{f(n) \mid 0 \leq c \cdot g(n) \leq f(n), \forall n \geq n_0, \exists c > 0, \exists n_0 > 0\}$$

Examples:

- $f(n) = 5n^3 + 6n - 17 \in \omega(g(n) = n^2)$
- $f(n) = n^{1.0001} \in \omega(g(n) = \log n)$

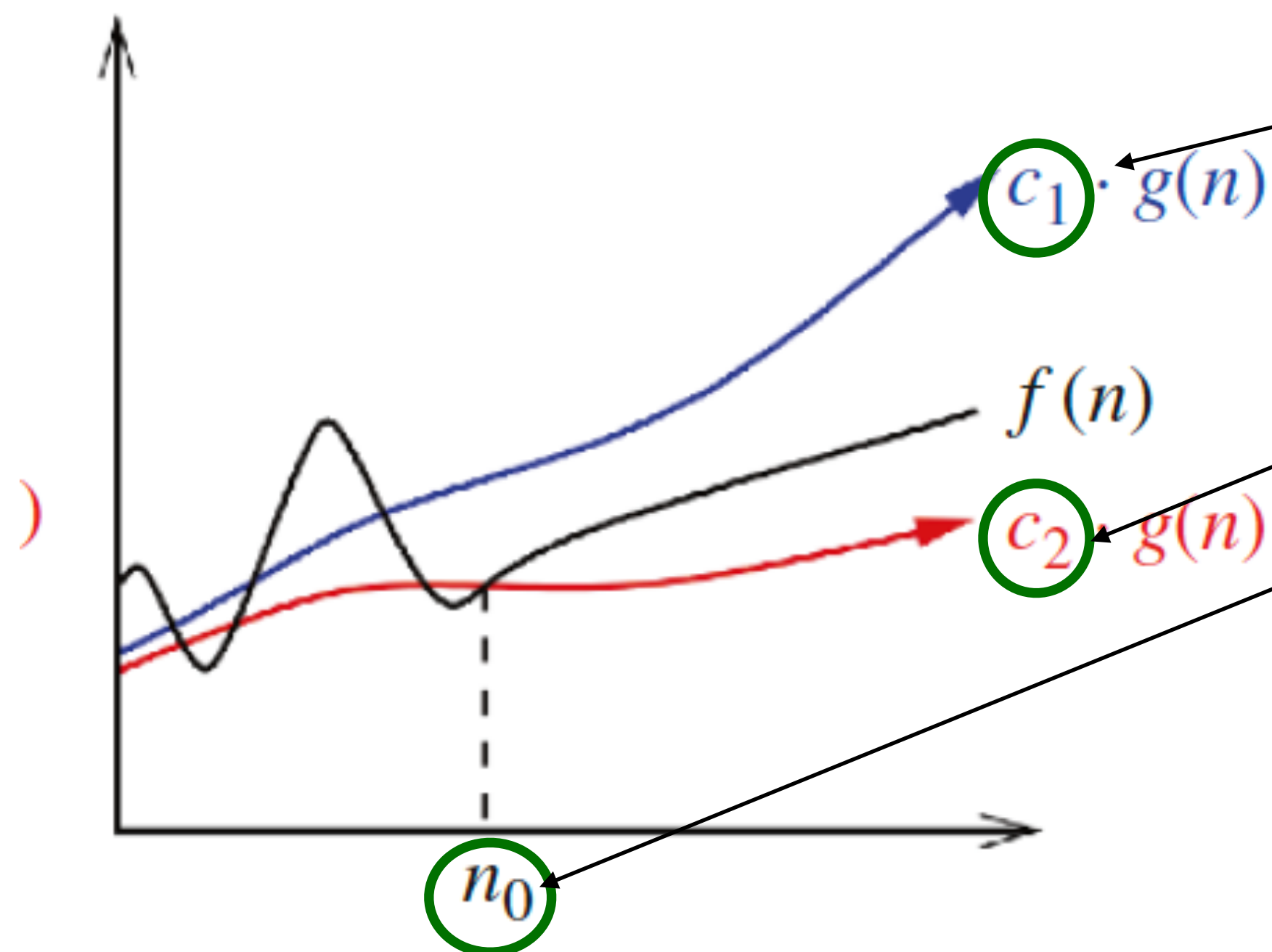
**Alternative definition:**

$$\{f(n) \mid \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0\}$$

# Big-Theta $\Theta()$

$\Theta(g(n))$  defines a **SET** of functions such that there exist positive constants  $c_1 > 0$  and  $c_2 > 0$  and  $n_0 > 0$ , where  $f(n)$  is always less than or equal to  $c_1 \cdot g(n)$  and greater than or equal to  $c_2 \cdot g(n)$  for all  $n > n_0$ .

$$\Theta(g(n)) = \{f(n) \mid 0 \leq c_2 \cdot g(n) \leq f(n) \leq c_1 \cdot g(n), \forall n \geq n_0, \exists(c_1 > 0, c_2 > 0, n_0 > 0)\}$$



$\Theta(n) = O(n) \cap \Omega(n)$   
All  $f(n)$  functions that are upper and lower bounded by the SAME  $g(n)$ .



# Big-Theta

$$\Theta(g(n)) = \{f(n) \mid 0 \leq c_2 \cdot g(n) \leq f(n) \leq c_1 \cdot g(n), \forall n \geq n_0, \exists(c_1 > 0, c_2 > 0, n_0 > 0)\}$$

To prove  $f(n) \in \Theta(g(n))$  we need to find explicit values of  $c_1, c_2$  and  $n_0$ , where  $f(n)$  is always less than or equal to  $c_1 \cdot g(n)$  and greater than  $c_2 \cdot g(n)$  for all  $n > n_0$ .

Example:

If  $f(n) = 4n^2 + 10^6n + 10^{34}$ , then  $f(n) \in \Theta(n^2)$ .

Proof:

For  $c_1 = 5$  and  $c_2 = 1$  and  $n_0 = 10^{18}$ ;

$$1 \cdot n^2 \leq 4n^2 + 10^6n + 10^{34} \leq 5 \cdot n^2, \forall n > n_0$$

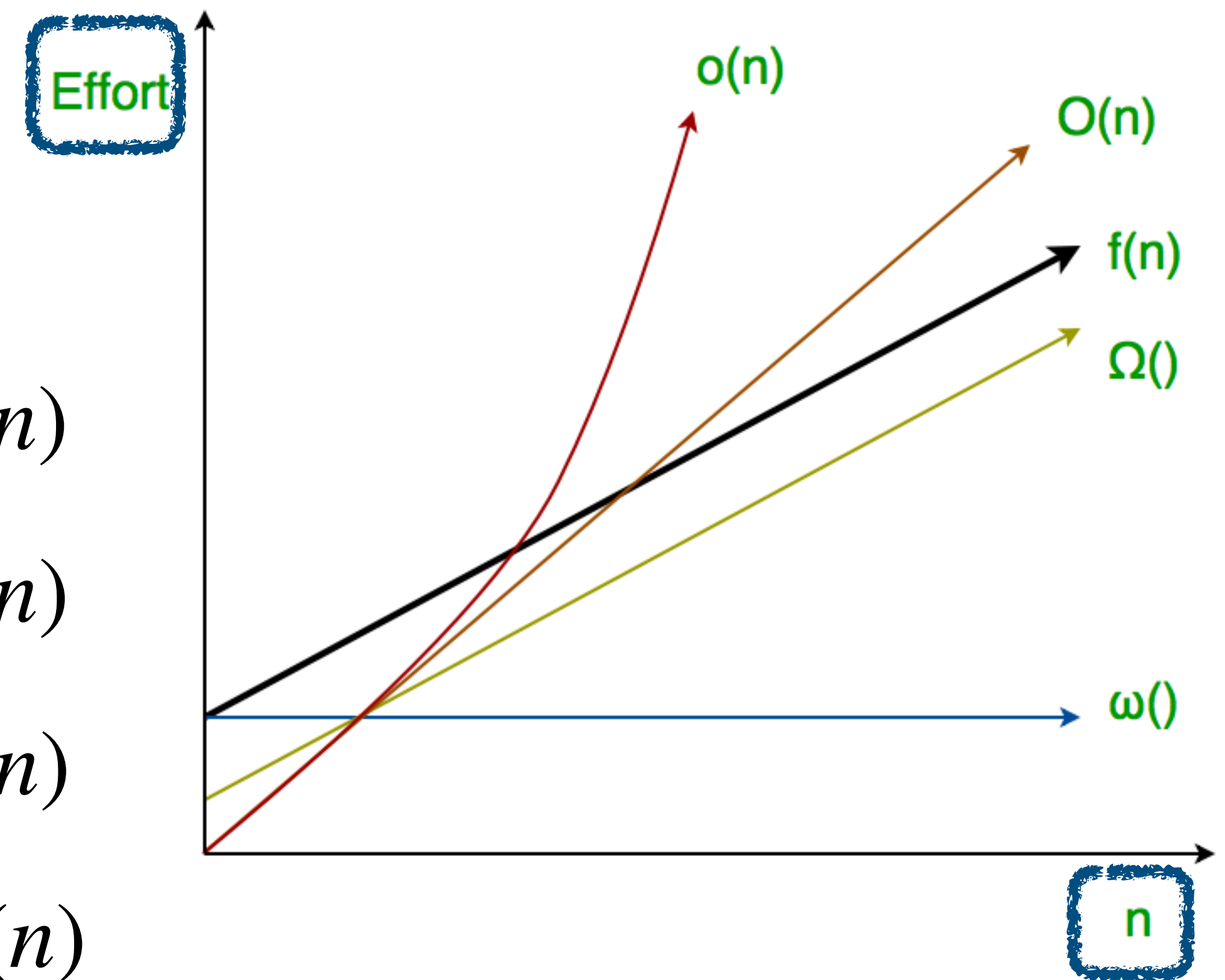
*After a threshold value,  $f(n)$  is **upper and lower bounded** by  $g(n) = n^2$ .*

# Some Relationships in Asymptotic Notation

- $f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$
- $f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \text{ AND } f(n) \in \Omega(g(n))$
- Can you prove them ?

# Some Analogies in Asymptotic Notation

- $f(n) \in O(g(n))$  is like  $\leq$  operator:  $f(n) \leq cg(n)$
- $f(n) \in o(g(n))$  is like  $<$  operator:  $f(n) < cg(n)$
- $f(n) \in \Omega(g(n))$  is like  $\geq$  operator:  $f(n) \geq cg(n)$
- $f(n) \in \omega(g(n))$  is like  $>$  operator:  $f(n) > cg(n)$
- $f(n) \in \Theta(g(n))$  is like  $=$  operator:  $f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$



# Questions or comments ?

- Please make sure you understand the meaning and the math behind.

*Next lecture we will be studying analysis of algorithms with asymptotic notation.*