Data Science, 2022
Tut 6: Machine Learning 1

1. [Probability] Assume that the probability of obtaining heads when tossing a coin is λ.
   a. What is the probability of obtaining the first head at the (k + 1)-th toss?
   b. What is the expected number of tosses needed to get the first head?

Tutorial 6

1.a. P(k tails in first k tosses, then 1 head) = $(1-\lambda)^k$

b. Let n be number of tosses.

$$X = E[n]$$

If the tosses are independent

$$X = \lambda \times 1 + (1-\lambda)(X+1)$$

$$X = \frac{1}{\lambda}.$$

2. [Probability] Assume X is a random variable.
   a. We define the variance of X as: $Var(X) = E[(X - E[X])^2]$. Prove that $Var(X) = E[X^2] - E[X]^2$.
   b. If $E[X] = 0$ and $E[X^2] = 1$, what is the variance of X? If $Y = a + bX$, what is the variance of Y ?

2 $X \to$ random variable

a. $\text{var}[X] = E[(X - E[X])^2]$

to prove, $\text{var}(X) = E[X^2] - E[X]^2$

$$\begin{aligned}
\text{var}(X) &= E[X^2 - 2X E[X] + E[X]^2] \\
&= E[X^2] - 2E[X E[X]] + E[X]^2 \\
&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2
\end{aligned}$$

HP

b. $E[X] = 0 \qquad E[X^2] = 1$

$y = a + bX$

$$\begin{aligned}
\text{var}(X) &= E[X^2] - E[X]^2 \\
&= 1 - 0^2 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\text{var}(y) &= \text{var}(a + bX) \\
&= E[y^2] - E[y]^2
\end{aligned}$$

$$\begin{aligned}
E[y^2] &= E[(a+bX)^2] \\
&= E[a^2 + 2abX + b^2X^2] \\
&= a^2 + 2ab E[X] + b^2 E[X^2] \\
&= a^2 + b^2
\end{aligned}$$

$$\begin{aligned}
E[y] &= E[a + bX] \\
&= a + bE[X] \\
&= a.
\end{aligned}$$

$\text{var}(y) = a^2 + b^2 - (a)^2 = b^2$

3. [Probability] Your friend Aku is a great predictor about winning horse race. Assume that we know three facts: 1) If Aku tells you that a horse name black beauty will win, it will win with probability 0.99. 2) If Aku tells you that a black beauty will not win, it will not win with probability 0.99999. 3) With probability 10^{-5}, Aku predicts that a black beauty is a winning horse. This also means that with probability $1 - 10^{-5}$, Aku predicts that a black beauty will not win.

a. Given a horse, what is the probability that it wins?

b. What is the probability that Aku correctly predicts a black beauty is winning ?

Make Assumptions about values when it is necessary in consistent manner. Refer necessary table from following link when necessary.

https://www.sheffield.ac.uk/polopoly_fs/1.43999!/file/tutorial-10-reading-tables.pdf

Qn 3. Let A be the event "Alex predicts that the horse is a winning horse."

Let $\neg A$ be the event "Alex predicts that the horse is not a winning horse."

Let W be the event that the horse is a winning horse.

Let $\neg W$ be the event that horse is not a winning horse

Given $P(W|A) = 0.99$
$P(\neg W|\neg A) : 0.99999$

$P(A) = 10^{-5}$

a) $P(W) : P(W, A) + P(W, \neg A)$

Probability of winning

$: P(W|A)\,(P A) + P(W|\neg A)\,P(\neg A)$
$= 0.99 \times 10^{-5} + (1 - 0.99999)(1 - 10^{-5})$
$\approx 1.99 \times 10^{-5}$

b) Prob that Alex predicts winning correctly.

$$P(A|W) = \frac{P(A, W)}{P(W)}$$

$$: \frac{P(W|A)\; P(A)}{P(W)}$$

$$: \frac{0.99 \times 10^{-5}}{1.99 \times 10^{-5}}$$

≈ 0.4975