Modeling Foul Trouble in Basketball

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1 Terminology

- **(Excessive Fouls)** When I say that a player has excessive fouls, I mean that they if they continue accumulating fouls at the same pace, they will foul out before the end of the game.

2 Model 1: Constant Value of Minutes

For the basic model, we'll say our objective is to maximize the number of minutes a fixed star player is on the floor. Note two important properties of this objective:

- We don't care which minutes a player plays - clutch time minutes are just as valuable to us as first quarter points
- We don't care whether these minutes are consecutive or split up. To us, an extreme minute-on-minute-off rule is indistinguishable from putting the player on the floor for an entire half.

Let's say that our star player has k fouls at time n, where $n < 48 \cdot \frac{k}{6}$ (i.e. the player is on pace to foul out before the game clock elapses). At this moment, the coach has two choices: pull the player and put them back in when $\frac{k}{6}$th of the game has elapsed, or keep them in until they foul out. Since our objective is to maximize the number of minutes played, we need to compare the expected minutes resulting from each of these choices.

For simplicity, assume that there is probability p that the player commits a foul for every minute they are on the floor. Let E(k) represent the expected number of minutes before the player accumulates k fouls.

2.1 Expected minutes when player remains in game

The expected minutes if the coach keeps the player on the floor can be broken down as follows:

$$\text{[Minutes played]} + \text{[Expected minutes before accumulating (6-k) fouls]}$$
$$= [n] + [E(6-k)]$$

2.2 Expected minutes when player is benched

If the player is benched until the $48 \cdot \frac{k}{6}$ mark, then we can breakdown total minutes as follows:

[Minutes already played] + min([Expected minutes before 6th foul], [Remaining minutes in game])

Rewritten, this sum is equivalent to:

$$[n] + \min(E(6-k), 48 \cdot [1-\tfrac{k}{6}])$$

2.3 Determining the function E

Let X be our random variable representing the number of minutes before a player commits k fouls. Given probability p that a player commits a foul over the span of a minute, we have the following probability mass function corresponding to a geometric distribution:

$$P(X = x) = p \cdot (1 - p)^{x-1}$$

with expected value over a single minute given by $\frac{1}{p}$.

Let X_1 be the number of minutes until the first foul, X_2 the number of additional minutes until the second foul, and so on. Each X_i is then a geometric variable with success factor p and therefore expected value $\frac{1}{p}$.

Let $Y := \sum_{i=1}^{x} X_i$, i.e. the number of minutes until the player accumulates x fouls. By linearity of expectation, we have:

$$E(Y) = E(\sum_{i=1}^{x} X_i) = \sum_{i=1}^{x} E(X_i) = \frac{x}{p}$$

So, the expected number of minutes before a given player accumulates x fouls, assuming a probability p of collecting a foul over the course of a minute, is equal to $\frac{x}{p}$. We now have an expression for the function E:

$$E(k) = \frac{k}{p}$$

2.4 Computing Expected Minutes

Beginning with the first strategy - keeping the player in the game - we can now compute our expression for expected minutes as:

$$[n] + [E(6-k)] = n + \frac{6-k}{p}$$

Next, consider the second strategy of benching the player until the $48 \cdot \frac{k}{6}$ mark. We have the following expression:

$$[n] + \min(E(6-k), 48 \cdot [1-\tfrac{k}{6}]) = n + \min(\tfrac{6-k}{p}, 48 \cdot [1-\tfrac{k}{6}])$$

To determine the maximum of the expected minutes for both of these strategies, we have to find the maximum of $\frac{6-k}{p}$ and min($\frac{6-k}{p}$, 48·[1-$\frac{k}{6}$]).

This computation is trivial. If 48·[1-$\frac{k}{6}$] $<$ $\frac{6-k}{p}$ (i.e. if the number of minutes left in the game after reinserting the player is less than the expected number of minutes before fouling out), then keeping the player on the floor maximizes their expected minutes over the course of the game. If 48·[1-$\frac{k}{6}$] $>$ $\frac{6-k}{p}$, we still get the same result, since the minimum function in the expression evaluates to $\frac{6-k}{p}$.

Concluding, with this basic model, the optimal strategy when a player is in foul trouble is to keep them in the game. Intuitively, taking the player out and reinserting them only shifts their remaining minutes to later in the game. At best (of course, using the assumption that minutes later have the same value as minutes now), such a policy has no advantage over keeping the player on the court. At its worst, if those expected remaining minutes span a longer period of time than remains in the game after reinsertion, removing the player could come at the opportunity cost of more playing time.

Note that the key characteristic of this model is not its objective function; It is the assumption of constant value of minutes. For instance, if we used maximizing point differential as our objective, while assuming that the player increases differential by a constant σ points per minute, the optimal strategy would still be to keep the player in the game. Why? Our new objective function is simply σ times expected minutes, which is just a monotonic shift of the old objective function.

The models become substantively different when we reject the two assumptions implicit in the objective function:

- **First,** that points earlier in the game are equally valuable as "clutch" points.
- **Second,** that player minutes are just as productive with excessive fouls as without.

2.5 Evaluating Assumptions

2.5.1 Does the value of points change by quarter?

I performed a logistic regression, determining the relationship between a point increase in score differential in each quarter on the probability of winning a game. I then computed the odds ratio for a point increase in differential for each quarter, and found the ratio of these odds ratios.

- **Quarter 1 odds ratio**: 1.358
- **Quarter 2 odds ratio**: 1.419

- **Quarter 3 odds ratio**: 1.397
- **Quarter 4 odds ratio**: 1.376

These are all fairly similar, indicating that changes in point differential in the fourth quarter are no more valuable to winning than identical changes in the first, second or third quarters.

2.5.2 Does the value of player minutes change by quarter?

Critical to the above model is the assumption that a player will be just as productive in minutes following the accumulation of excessive fouls as they would be without those fouls. In practice, however, players may be fearful of collecting further fouls, leading them to play less aggressively. To confirm, I examined the difference in player efficiency in possessions where they have excessive fouls and possessions where they don't. This season, players shoot 47.76 percent from the field while in foul trouble, similar to a 47.47 percent when not in foul trouble.

Measuring the change in defensive aggressiveness due to foul trouble is more difficult, because