Assignment 16

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment16

1 Problem

(hoffman/page226/9) :

Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial(they necessarily have the same characteristic polynomial) but which are not similar.

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$

TABLE 1: Definitions

3 Solution

Given	Let **A** and **B** be two nilpotent matrix. $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
Characteristic polynomial	For Matrix **A**; $$p_A(x) = \left\|x\mathbf{I} - \mathbf{A}\right\|$$ $$p_A(x) = \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$ For Matrix **B**; $$p_B(x) = \left\|x\mathbf{I} - \mathbf{B}\right\|$$ $$p_B(x) = \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$ Therefore, Characteristics polynomial are same for both matrix **A** and **B**.
Minimal Polynomial	For Matrix **A**; $p_A(x) = x^4$ Let, minimal polynomial of **A** is $m_A(x)$ $m_A(x)$ always divide $p_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix.

	$$m_A(A) = \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$ $$m_A(\mathbf{A}) = \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$ $$\implies m_A(x) = x^2$$ $\implies$ x^2 is a minimal polynomial of Matrix **A** For Matrix **B**; $$m_B(B) = \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$ $$m_B(\mathbf{B}) = \mathbf{B}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$ $$\implies m_B(x) = x^2$$ $\implies$ x^2 is a minimal polynomial of Matrix **B** Therefore, minimial polynomial for both Matrix **A** and **B** are same.
Checking whether Matrix **A** and **B** are same .	$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ rank of matrix = no of linearly independent row or column vectors in the matrix $rank(\mathbf{A}) = 2$ $$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $rank(\mathbf{B}) = 1$

	$\implies rank\left(\mathbf{A}\right) \neq rank\left(\mathbf{B}\right)$ Therefore, Matrix **A** and **B** are not same.
Conclusion	1) Characteristics polynomial for both Matrix **A** and **B** are same. 2) Minimal polynomial for both matrix are same i.e $m_A\left(x\right) = m_B\left(x\right) = x^2$. 3) Matrix **A** and **B** are not same .

TABLE 2: Solution Summary