Assignment 17

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment17

1 Problem

(ugcjune/2018/28) :
If $\mathbf{A}$ is a 2×2 matrix over $\mathbb{R}$ with $\det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$, then we can conclude that

1) $\det(\mathbf{A}) = 0$
2) $\mathbf{A} = 0$
3) $tr(\mathbf{A}) = 0$
4) $\mathbf{A}$ is non singular.

2 Solution

Given	$\mathbf{A}$ be a 2×2 matrix over $\mathbb{R}$ with $$\det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$$
Explanation	If $\mathbf{X}$ is an eigen vector of matrix $\mathbf{A}$ corresponding to the eigen value λ i.e $$\mathbf{AX} = \lambda\mathbf{X}$$ then, $(\mathbf{I}+\mathbf{A})\mathbf{X} = (1+\lambda)\mathbf{X}$ Thus, $\mathbf{X}$ is an eigen vector of $(\mathbf{A}+\mathbf{I})$ corresponding to the eigen value $(1+\lambda)$. Let λ_1, λ_2 be two eigen values of $\mathbf{A}$ and $(1+\lambda_1), (1+\lambda_2)$ be the eigen values of $(\mathbf{A}+\mathbf{I})$. $\implies$ Eigen value of $\mathbf{A} = \lambda_1, \lambda_2$ $\implies$ Eigen value of $(\mathbf{A}+\mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1$

	Since, $$\det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$$ Trace of any matrix is sum of its eigen values. Determinant of matrix is product of its eigen values $$\implies (\lambda_1+1)(\lambda_2+1) = 1 + (\lambda_1\lambda_2)$$ $$\implies \boxed{\lambda_1+\lambda_2=0}$$ $$\implies \boxed{tr(\mathbf{A})=0}$$
Option 1 : $\det\mathbf{A} = 0$	Consider an example where $\det\mathbf{A} = 0$ $$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$ $$\det\mathbf{A} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$ $$(\mathbf{A}+\mathbf{I}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$ $$\det(\mathbf{A}+\mathbf{I}) = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$ $$\implies \det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det\mathbf{A} = 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is singular. Consider an example where $\det\mathbf{A} \neq 0$ $$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ $\det\mathbf{A} = 1$ $\det\mathbf{A}+\mathbf{I} = 2$ $\implies \det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$

	Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} \neq 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is non singular.
Option 2 : $\mathbf{A} = \mathbf{0}$	Consider an example where $\mathbf{A} = 0$; $$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ $$\det \mathbf{A} = 0$$ $$\det(\mathbf{A} + \mathbf{I}) = 1$$ $$\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} = \mathbf{0}$ 4) $\mathbf{A}$ is singular. Consider an example where $\mathbf{A} \neq \mathbf{0}$; $$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ $$\det \mathbf{A} = 0$$ $$\det(\mathbf{A} + \mathbf{I}) = 1$$ $$\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is singular.
Option 4: $\mathbf{A}$ is non singular	Non Singular Matrix: A non-singular matrix is a square one whose determinant is not zero.non-singular matrix is also a full rank matrix. Consider an example where $\mathbf{A}$ is non singular. $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

	$\det \mathbf{A} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$ $(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\det(\mathbf{A} + \mathbf{I}) = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$ $\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} \neq 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is non singular. Consider an example where $\mathbf{A}$ is singular. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\det \mathbf{A} = 0$ $\det(\mathbf{A} + \mathbf{I}) = 1$ $\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is singular.
Option 3 : $tr(\mathbf{A}) = \mathbf{0}$	Consider an example where $tr(\mathbf{A}) = 0$; $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\det \mathbf{A} = -1$ $\det(\mathbf{A} + \mathbf{I}) = 2$ $\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} \neq 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is non singular.

	Consider an example where $tr(\mathbf{A}) \neq \mathbf{0}$; $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ $$\det \mathbf{A} = 0$$ $$\det(\mathbf{A} + \mathbf{I}) = 2$$ $$\implies \det(\mathbf{A} + \mathbf{I}) \neq 1 + \det(\mathbf{A})$$ Thus, the given condition not satisfied in this case.
Conclusion	In all options, $tr(\mathbf{A}) = 0$ satisfied. Thus, we can conclude Option 3 is correct.

TABLE 1: Solution Summary