

Assignment 14

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Download the latex-tikz codes from

<https://github.com/neharani289/MatrixTheory/Assignment14>

1 PROBLEM

(hoffman/page208/1b) :

Find an invertible matrix \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ are both diagonal where \mathbf{A} and \mathbf{B} are real matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (1.0.1)$$

$$\mathbf{B} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad (1.0.2)$$

2 EXPLANATION

Characteristic Polynomial	For an $n \times n$ matrix \mathbf{A} , characteristic polynomial is defined by, $p(x) = x\mathbf{I} - \mathbf{A} $
Theorem	According to theorem 8, if a 2×2 matrix has two characteristics values then the \mathbf{P} that diagonalize \mathbf{A} will necessarily also diagonalize any \mathbf{B} that commutes with \mathbf{A} .
Basis	Let there exist a \mathbf{P} in basis $\beta = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of \mathbb{V} consisting of eigen vector which are common to both \mathbf{A} and \mathbf{B} such that $\mathbf{A}\mathbf{b}_i = \lambda_i\mathbf{b}_i \quad \mathbf{B}\mathbf{b}_i = \mu_i\mathbf{b}_i$ $\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \quad \Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$

TABLE 1: Definitions and theorem used

3 SOLUTION

Operations	Matrix A	Matrix B
Characteristic Polynomial	$p(x) = x\mathbf{I} - \mathbf{A} $ $= \begin{vmatrix} x-1 & 1 \\ 1 & x-1 \end{vmatrix}$ $= (x-1)(x-1) - 1$	$p(x) = x\mathbf{I} - \mathbf{B} $ $= \begin{vmatrix} x-1 & a \\ a & x-1 \end{vmatrix}$ $= (x-1)(x-1) - a^2$
Characteristic values	$p(x) = 0$ $x(x-1) = 0$ $\lambda_1 = 0, \lambda_2 = 2$	$p(x) = 0$ $(x-1)^2 - a^2 = 0$ $\mu_1 = (1-a), \mu_2 = (1+a)$
Basis for Characteristics Values	$(\mathbf{A} - \lambda\mathbf{I})\mathbf{b} = 0$ <p>For $\lambda_1 = 0$</p> $\mathbf{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ <p>For $\lambda_2 = 2$</p> $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$(\mathbf{B} - \mu\mathbf{I})\mathbf{b} = 0$ <p>For $\mu_1 = (1-a)$</p> $\mathbf{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ <p>For $\mu_2 = (1+a)$</p> $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Invertible matrix	$\Rightarrow \mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	$\Rightarrow \mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$
Verification	$\Lambda_A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ $\Lambda_A = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$	$\Lambda_B = \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix}$ $\Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ $\Lambda_B = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix} = \Lambda_B$
Conclusion	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

TABLE 2: Solution Summary