Assignment 16

Neha Rani
EE20MTECH14014

Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment16

1 Problem

(hoffman/page226/9) :
Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial(they necessarily have the same characteristic polynomial) but which are not similar.

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Similar matrices	Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be similar if 1) $det(\mathbf{A}) = det(\mathbf{B})$ 2) $tr(\mathbf{A}) = tr(\mathbf{B})$ 3) $rank(\mathbf{A}) = rank(\mathbf{B})$
Nilpotent Matrix	A square matrix $\mathbf{A}$ is called a Nilpotent matrix if there exist a positive integer $'m'$ such that $\mathbf{A}^m = \mathbf{0}$ and $'m'$ is called Index of nilpotent matrix $\mathbf{A}$. The determinant and trace of a nilpotent matrix are always zero.

TABLE 1: Definitions

3 Solution

Given	Let **A** and **B** be two nilpotent matrix. $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
Characteristic polynomial	For Matrix **A**; $$p_A(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$ $$p_A(x) = \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$ For Matrix **B**; $$p_B(x) = \left\lvert x\mathbf{I} - \mathbf{B} \right\rvert$$ $$p_B(x) = \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$ Therefore, Characteristics polynomial are same for both matrix **A** and **B**.
Minimal Polynomial	For Matrix **A**; $p_A(x) = x^4$ Let, minimal polynomial of **A** is $m_A(x)$ $m_A(x)$ always divide $p_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix.

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$$m_A(\mathbf{A}) = \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$

$$m_A(\mathbf{A}) = \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

$$\implies m_A(x) = x^2$$

$\implies$ x^2 is a minimal polynomial of Matrix **A**

For Matrix **B**;

$$m_B(\mathbf{B}) = \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$

$$m_B(\mathbf{B}) = \mathbf{B}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

$$\implies m_B(x) = x^2$$

$\implies$ x^2 is a minimal polynomial of Matrix **B**

Therefore, minimial polynomial for both Matrix **A** and **B** are same.

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Checking whether Matrix **A**

and **B** are similar .

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$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank of matrix = no of linearly independent row or column vectors in the matrix

$rank(\mathbf{A}) = 2$

$tr(\mathbf{A}) = 0$

$det(\mathbf{A}) = 0$

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	$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $rank\,(\mathbf{B}) = 1$ $tr(\mathbf{B}) = 0$ $det(\mathbf{B}) = 0$ $\implies rank\,(\mathbf{A}) \neq rank\,(\mathbf{B})$ Therefore, Matrix **A** and **B** are not similar.
Conclusion	1) Characteristics polynomial for both Matrix **A** and **B** are same. 2) Minimal polynomial for both matrix are same i.e $m_A\,(x) = m_B\,(x) = x^2$. 3) Matrix **A** and **B** are not similar .

TABLE 2: Solution Summary