

Assignment 17

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Download the latex-tikz codes from

<https://github.com/neharani289/MatrixTheory/Assignment17>

1 PROBLEM

(ugcjune/2018/28) :

If \mathbf{A} is a 2×2 matrix over \mathbb{R} with $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$, then we can conclude that

- 1) $\det(\mathbf{A}) = 0$
- 2) $\mathbf{A} = 0$
- 3) $\text{tr}(\mathbf{A}) = 0$
- 4) \mathbf{A} is non singular.

2 SOLUTION

Given	<p>\mathbf{A} be a 2×2 matrix over \mathbb{R} with</p> $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$
Explanation	<p>If \mathbf{X} is an eigen vector of matrix \mathbf{A} corresponding to the eigen value λ i.e</p> $\mathbf{AX} = \lambda\mathbf{X}$ <p>then, $(\mathbf{I} + \mathbf{A})\mathbf{X} = (1 + \lambda)\mathbf{X}$</p> <p>Thus, \mathbf{X} is an eigen vector of $(\mathbf{A} + \mathbf{I})$ corresponding to the eigen value $(1 + \lambda)$.</p> <p>Let λ_1, λ_2 be two eigen values of \mathbf{A} and $(1 + \lambda_1), (1 + \lambda_2)$ be the eigen values of $(\mathbf{A} + \mathbf{I})$.</p> <p>$\Rightarrow$ Eigen value of $\mathbf{A} = \lambda_1, \lambda_2$</p> <p>$\Rightarrow$ Eigen value of $(\mathbf{A} + \mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1$</p>

Since,

$$\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

Trace of any matrix is sum of its eigen values.

Determinant of matrix is product of its eigen values

$$\Rightarrow (\lambda_1 + 1)(\lambda_2 + 1) = 1 + (\lambda_1 \lambda_2)$$

$$\Rightarrow \boxed{\lambda_1 + \lambda_2 = 0}$$

$$\Rightarrow \boxed{\text{tr}(\mathbf{A}) = 0}$$

Option 1 : $\det \mathbf{A} = 0$

Consider an example where $\det \mathbf{A} = 0$

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\det(\mathbf{A} + \mathbf{I}) = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

Conclusion:

- 1) $\text{tr}(\mathbf{A}) = 0$
- 2) $\det \mathbf{A} = 0$
- 3) $\mathbf{A} \neq \mathbf{0}$
- 4) \mathbf{A} is singular.

Consider an example where $\det \mathbf{A} \neq 0$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\det \mathbf{A} = 1$$

$$\det \mathbf{A} + \mathbf{I} = 2$$

$$\Rightarrow \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

	<p>Conclusion:</p> <ol style="list-style-type: none"> 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} \neq 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) \mathbf{A} is singular.
Option 2 : $\mathbf{A} = \mathbf{0}$	<p>Consider an example where $\mathbf{A} = \mathbf{0}$;</p> $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\det \mathbf{A} = 0$ $\det (\mathbf{A} + \mathbf{I}) = 1$ $\implies \det (\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ <p>Conclusion:</p> <ol style="list-style-type: none"> 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} = \mathbf{0}$ 4) \mathbf{A} is singular. <p>Consider an example where $\mathbf{A} \neq \mathbf{0}$;</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\det \mathbf{A} = 0$ $\det (\mathbf{A} + \mathbf{I}) = 1$ $\implies \det (\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ <p>Conclusion:</p> <ol style="list-style-type: none"> 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) \mathbf{A} is singular.
Option 4: \mathbf{A} is non singular	<p>Non Singular Matrix: A non-singular matrix is a square one whose determinant is not zero. non-singular matrix is also a full rank matrix.</p> <p>Consider an example where \mathbf{A} is non singular.</p> $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\det \mathbf{A} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

$$(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\det (\mathbf{A} + \mathbf{I}) = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$\implies \det (\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

Conclusion:

- 1) $tr(\mathbf{A}) = 0$
- 2) $\det \mathbf{A} \neq 0$
- 3) $\mathbf{A} \neq \mathbf{0}$
- 4) \mathbf{A} is non singular.

Consider an example where \mathbf{A} is singular.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\det \mathbf{A} = 0$$

$$\det(\mathbf{A} + \mathbf{I}) = 1$$

$$\implies \det (\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

Conclusion:

- 1) $tr(\mathbf{A}) = 0$
- 2) $\det \mathbf{A} \neq 0$
- 3) $\mathbf{A} \neq \mathbf{0}$
- 4) \mathbf{A} is non singular.

Option 3 : $tr(\mathbf{A}) = 0$

Consider an example where $tr(\mathbf{A}) = 0$;

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\det \mathbf{A} = -1$$

$$\det (\mathbf{A} + \mathbf{I}) = 2$$

$$\implies \det (\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

Conclusion:

- 1) $tr(\mathbf{A}) = 0$
- 2) $\det \mathbf{A} \neq 0$
- 3) $\mathbf{A} \neq \mathbf{0}$
- 4) \mathbf{A} is non singular.

	<p>Consider an example where $tr(\mathbf{A}) \neq \mathbf{0}$;</p> $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\det \mathbf{A} = 0$ $\det (\mathbf{A} + \mathbf{I}) = 2$ $\implies \det (\mathbf{A} + \mathbf{I}) \neq 1 + \det(\mathbf{A})$ <p>Thus, the given condition not satisfied in this case.</p>
Conclusion	<p>In all options, $tr(\mathbf{A}) = 0$ satisfied.</p> <p>Thus, we can conclude Option 3 is correct.</p>

TABLE 1: Solution Summary