Assignment 5

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***Abstract*—This document explains the the concept of finding the angle between the two straight lines from given second degree equation**

Download all latex-tikz codes from

https://github.com/neharani289/MatrixTheory/tree/master/Assignment5

1 Problem

Prove that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents two straight lines and find the angle between them

2 Solution

The general second order equation is given by ,

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{2.0.1}$$

Given,

$$12x^2+7xy-10y^2+13x+45y-35=0 \tag{2.0.2}$$

The above equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.0.3}$$

where

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u}=\begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \tag{2.0.5}$$

$$f=-35 \tag{2.0.6}$$

(2.0.3) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}=0 \tag{2.0.7}$$

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}=\begin{vmatrix}12 & \frac{7}{2} & \frac{13}{2}\\ \frac{7}{2} & -10 & \frac{45}{2}\\ \frac{13}{2} & \frac{45}{2} & -35\end{vmatrix} \tag{2.0.8}$$

$$\implies 12\begin{vmatrix}-10 & \frac{45}{2}\\ \frac{45}{2} & -35\end{vmatrix}-\frac{7}{2}\begin{vmatrix}\frac{7}{2} & \frac{45}{2}\\ \frac{13}{2} & -35\end{vmatrix}+\frac{13}{2}\begin{vmatrix}\frac{7}{2} & -10\\ \frac{13}{2} & \frac{45}{2}\end{vmatrix}=0 \tag{2.0.9}$$

The lines intercept if

$$\left|\mathbf{V}\right|<0 \tag{2.0.10}$$

$$\left|\mathbf{V}\right|=-\frac{529}{4}<0 \tag{2.0.11}$$

From (2.0.9) and (2.0.11) it can be concluded that the given equation represents a pair of intersecting lines.
Let (α,β) be their point of intersection, then

$$\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\frac{-13}{2}\\ -\frac{45}{2}\end{pmatrix} \tag{2.0.12}$$

$$\implies \begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}-1\\ 2\end{pmatrix} \tag{2.0.13}$$

$$\text{From } \textit{Spectral theorem, } \mathbf{V}=\mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.14}$$

$$\mathbf{V}=\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \tag{2.0.15}$$

$$\mathbf{P}=\begin{pmatrix}\frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2}\\ 1 & 1\end{pmatrix} \tag{2.0.16}$$

$$\mathbf{D}=\begin{pmatrix}1+\frac{\sqrt{533}}{2} & 0\\ 0 & 1-\frac{\sqrt{533}}{2}\end{pmatrix} \tag{2.0.17}$$

Using *Spectral decomposition* of matrix we can express equation as

$$u_1(x-\alpha)+u_2(y-\beta)=\pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha)+v_2(y-\beta)) \tag{2.0.18}$$

Substituting values in above equation we get;

$$\frac{\sqrt{533}-22}{2}(x+1)+(y-2) = \pm\sqrt{-\frac{1-\frac{\sqrt{533}}{2}}{1+\frac{\sqrt{533}}{2}}}\left(\frac{-22-\sqrt{533}}{2}(x+1)+(y-2)\right) \tag{2.0.19}$$

Simplifying (2.0.19),

$$3x-2y+7=0 \text{ and } 4x+5y-5=0 \tag{2.0.20}$$

$$\implies (3x-2y+7)(4x+5y-5)=0 \tag{2.0.21}$$

Thus the equation of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} = 5 \tag{2.0.22}$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -7 \tag{2.0.23}$$

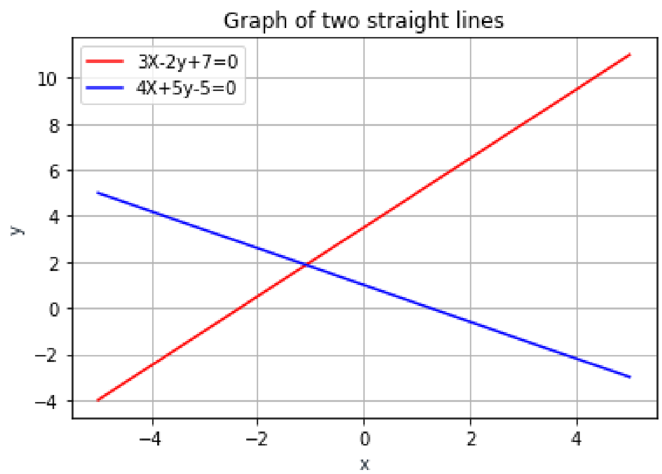

Fig. 1: Pair of straight lines

3 Angle between the straight lines

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.0.1}$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{||\mathbf{n_1}|| \, ||\mathbf{n_2}||} \tag{3.0.2}$$

$$\implies \quad \theta = \cos^{-1}(\frac{2}{\sqrt{533}}) = \tan^{-1}(\frac{23}{2}) \tag{3.0.3}$$