

Assignment 13

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Download the latex-tikz codes from

<https://github.com/neharani289/MatrixTheory/Assignment13>

1 PROBLEM

(hoffman/page198/9) :

Let A be an $n \times n$ matrix with characteristics polynomial

$$f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$$

Show that

$$c_1 d_1 + \dots + c_k d_k = \text{trace}(A)$$

2 SOLUTION

Given	<p>Let A be an $n \times n$</p> $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ <p>and Characteristics polynomial</p> $f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$
To proof	$c_1 d_1 + \dots + c_k d_k = \text{trace}(A)$

Claim	$\det(xI - A) = x^n - \text{trace}(A)x^{n-1} + \dots + (-1)^n \det(A)$
<p>To prove that the coefficient of x^{n-1} is $\text{trace}(A)$</p>	<p>Proof by method of induction for $n=2$</p> <p>Let, A_2 be 2×2 matrix.</p> $A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\det(xI - A) = \begin{vmatrix} (x - a_{11}) & -a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\Rightarrow \det(xI - A) = x^2 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})$ <p>$\boxed{\text{trace}(A_2) = a_{11} + a_{12}}$ and the coefficient of $\boxed{x^{n-1} = x}$.</p> <p>Therefore, claim is true for $n = 2$.</p> <p>Assume that it is true for upto $n - 1$.</p> <p>Then, Coefficient of x^{n-2} will be $\boxed{s = a_{22} + a_{33} + a_{44} + \dots + a_{nn}}$ for the matrix, $A_{n-1 \times n-1}$,</p> $A_{n-1} = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$

TABLE 1: Proof for Coefficients of x^{n-1}

To prove that the claim is true for n	$\det(xI - A) = \begin{vmatrix} (x - a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (x - a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (x - a_{nn}) \end{vmatrix}$ $\det(xI - A) = (x - a_{11})\text{cofactor}(a_{11}) + a_{21}\text{cofactor}(-a_{21}) + \dots - (-1)^{n+1}\text{cofactor}(-a_{n1})$ $\det(xI - A) = (x - a_{11})(x^{n-1} - sx^{n-2} + \dots) + a_{21}(\text{polynomial of degree } n - 2) + \dots - (-1)^{n+1}a_{n1}(\text{polynomial of degree } n - 2)$ $\det(xI - A) = x^n - (a_{11} + s)x^{n-1} + \text{polynomial of degree at most } n - 2$ $\det(xI - A) = x^n - (a_{11} + a_{22} + \dots + a_{nn})x^{n-1} + \text{polynomial of degree at most } n - 2$ $\implies \det(xI - A) = x^n - \text{trace}(A)x^{n-1} + \text{polynomial of degree at most } n - 2$ <p>Given characteristic polynomial of A is, $f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$ where, $d_1 + d_2 + \dots + d_k = n$</p> <p>Since, expansion of $(x - r)^t = x^t - rt x^{t-1} + \dots + (-1)^t r^t$</p> <p>Therefore, $f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$ $f = (x^{d_1} - c_1 d_1 x^{d_1-1} + \dots) \dots (x^{d_k} - c_k d_k x^{d_k-1} + \dots)$</p> $\implies \boxed{f = x^n - (c_1 d_1 + c_2 d_2 + \dots + c_k d_k) x^{n-1} + \dots}$ <p>Since, $f = \det(xI - A)$ $\implies x^n - (c_1 d_1 + c_2 d_2 + \dots + c_k d_k) x^{n-1} + \dots = x^n - \text{trace}(A) x^{n-1} + \text{polynomial of degree at most } n - 2$</p> <p>By comparing corresponding terms in above equation ;</p> $\boxed{c_1 d_1 + c_2 d_2 + \dots + c_k d_k = \text{trace}(A)}$ <p>Hence, Proved.</p>
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TABLE 2: Solution Summary