Assignment 17

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment17

1 Problem

(ugcjune/2018/28) :

If $\mathbf{A}$ is a 2×2 matrix over $\mathbb{R}$ with $\det(\mathbf{A}+\mathbf{I}) = 1 + \det(\mathbf{A})$, then we can conclude that

1) $\det(\mathbf{A}) = 0$
2) $\mathbf{A} = 0$
3) $tr(\mathbf{A}) = 0$
4) $\mathbf{A}$ is non singular.

2 Solution

<table>
<tr>
<td>Given</td>
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Let $\mathbf{A}$ be a 2×2 matrix over $\mathbb{R}$.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\boxed{\det \mathbf{A} = ad - bc}$$

$$\boxed{tr(\mathbf{A}) = (a + d)}$$

Now,

$$(\mathbf{A}+\mathbf{I}) = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}$$

$$\det(\mathbf{A}+\mathbf{I}) = \begin{vmatrix} a+1 & b \\ c & d+1 \end{vmatrix} = (a+1)(d+1) - (bc) \tag{2.0.1}$$

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	$$\boxed{1 + \det \mathbf{A} = 1 + (ad - bc)} \tag{2.0.2}$$ Since, $\det(\mathbf{A} + \mathbf{I}) = \det(\mathbf{A})$ Equating (2.0.1) and (2.0.2); $$(a+1)(d+1) - (bc) = 1 + (ad - bc)$$ $$\implies \boxed{(a+d) = 0 = tr(\mathbf{A})}$$
Option 1	Let, $$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$ $$\det \mathbf{A} = \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0$$ $$(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$ $$\det(\mathbf{A} + \mathbf{I}) = \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 1$$ $$\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is non singular.
Option 2	Let, $$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ $$\det \mathbf{A} = 0$$ $$\det(\mathbf{A} + \mathbf{I}) = 1$$ $$\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$

	Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} = 0$ 3) $\mathbf{A} = \mathbf{0}$ 4) $\mathbf{A}$ is singular.
Option 3	Let , $$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$ Eigen value of $\mathbf{A} = \lambda_1, \lambda_2$ Eigen value of $(\mathbf{A} + \mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1$ Since,$\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ $$\implies (\lambda_1 + 1)(\lambda_2 + 1) = 1 + (\lambda_1 \lambda_2)$$ $$\implies \lambda_1 + \lambda_2 = 0$$ trace of any matrix is sum of its eigen values. $$\implies tr(\mathbf{A}) = 0$$
Option 4	Let, $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ $$\det \mathbf{A} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$ $$(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$ $$\det(\mathbf{A} + \mathbf{I}) = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$$ $$\implies \det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$ Conclusion: 1) $tr(\mathbf{A}) = 0$ 2) $\det \mathbf{A} \neq 0$ 3) $\mathbf{A} \neq \mathbf{0}$ 4) $\mathbf{A}$ is non singular.

Conclusion	In all options, $tr(\mathbf{A}) = 0$ satisfied. Hence, Option 3 is correct.

TABLE 1: Solution Summary