

Assignment 7

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Abstract—This problem demonstrate a method to find the foot perpendicular from a given point to a given plane using Singular Value Decomposition.

All the codes for the figure in this document can be found at

<https://github.com/neharani289/Assignment7/tree/master/Assignment7>

1 PROBLEM

Set up the equation of a plane through the point A (-2, -3, 4) and perpendicular to the line

$$\frac{x}{4} = \frac{y-3}{6} = \frac{z+2}{-12} \quad (1.0.1)$$

2 SOLUTION

Let the equation of plane is

$$ax + by + cz + d = 0 \quad (2.0.1)$$

Direction ratio of the line (1.0.1) is given as

$$\mathbf{D} = \begin{pmatrix} 4 \\ 6 \\ -12 \end{pmatrix} \quad (2.0.2)$$

Now let consider

$$\mathbf{A} = (-2 \quad -3 \quad 4) \quad (2.0.3)$$

Since plane is passing through the point A (-2, -3, 4) and perpendicular to the line (1.0.1), hence

$$\mathbf{AD} + d = 0 \quad (2.0.4)$$

$$\Rightarrow d = 37 \quad (2.0.5)$$

Hence equation of the plane is

$$2x + 3y + 6z + 37 = 0 \quad (2.0.6)$$

$$\Rightarrow 2x + 3y - 6z = -37 \quad (2.0.7)$$

equation (2.0.7) can written as :

$$(2 \quad 3 \quad -6)\mathbf{x} = -37 \quad (2.0.8)$$

For foot perpendicular we need to find the distance between the plane and point P (0, 3, -2).

First we find orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (2.0.9)$$

$$\Rightarrow (a \quad b \quad c) \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 0 \quad (2.0.10)$$

$$\Rightarrow 2a + 3b - 6c = 0 \quad (2.0.11)$$

Putting a=1 and b=0 we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix} \quad (2.0.12)$$

Putting a=0 and b=1 we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.13)$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \quad (2.0.14)$$

Putting values in (2.0.14),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \quad (2.0.15)$$

Now, to solve (2.0.15), we perform Singular Value Decomposition on \mathbf{M} as follows,

$$\mathbf{M} = \mathbf{USV}^T \quad (2.0.16)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of \mathbf{MM}^T and \mathbf{S} is diagonal matrix of singular value of

eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix} \quad (2.0.17)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36} \end{pmatrix} \quad (2.0.18)$$

From (2.0.14) putting (2.0.16) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (2.0.19)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (2.0.20)$$

Where \mathbf{S}_+ is Moore-Penrose Pseudo-Inverse of \mathbf{S} . Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (2.0.21)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & \frac{1}{3} \\ 0 & 1-\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36}-\lambda \end{pmatrix} = 0 \quad (2.0.22)$$

$$\Rightarrow \lambda(\lambda-1)(\lambda-\frac{49}{36}) = 0 \quad (2.0.23)$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad (2.0.24)$$

$$\lambda_2 = 1 \quad (2.0.25)$$

$$\lambda_3 = 0 \quad (2.0.26)$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{12}{13} \\ \frac{18}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} \quad (2.0.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{18}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-2}{7} \\ \frac{-3}{7} \\ \frac{6}{7} \end{pmatrix} \quad (2.0.29)$$

Hence we obtain \mathbf{U} of (2.0.16) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \quad (2.0.30)$$

After computing the singular values from eigen

values $\lambda_1, \lambda_2, \lambda_3$ we get \mathbf{S} of (2.0.16) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.31)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.32)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \quad (2.0.33)$$

$$\Rightarrow \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \quad (2.0.34)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (2.0.35)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (2.0.36)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (2.0.37)$$

Hence we obtain \mathbf{V} of (2.0.16) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (2.0.38)$$

Finally from (2.0.16) we get the Singular Value Decomposition of \mathbf{M} as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \quad (2.0.39)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.40)$$

From (2.0.20) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{6}{\sqrt{13}} \\ -3 \end{pmatrix} \quad (2.0.41)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{24}{7\sqrt{13}} \\ \frac{6}{\sqrt{13}} \end{pmatrix} \quad (2.0.42)$$

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \quad (2.0.43)$$

Verifying the solution of (2.0.43) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (2.0.44)$$

Evaluating the R.H.S in (2.0.44) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \quad (2.0.45)$$

$$\Rightarrow \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \quad (2.0.46)$$

Solving the augmented matrix of (2.0.46) we get,

$$\left(\begin{array}{cc|c} \frac{10}{9} & \frac{1}{6} & \frac{-2}{3} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{array} \right) \xleftrightarrow{R_1 = \frac{9R_1}{10}} \left(\begin{array}{cc|c} 1 & \frac{3}{20} & \frac{-3}{5} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{array} \right) \quad (2.0.47)$$

$$\xleftrightarrow{R_2 = R_2 - \frac{R_1}{6}} \left(\begin{array}{cc|c} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & \frac{49}{40} & \frac{21}{10} \end{array} \right) \quad (2.0.48)$$

$$\xleftrightarrow{R_2 = \frac{40}{49} R_2} \left(\begin{array}{cc|c} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & 1 & \frac{5}{7} \end{array} \right) \quad (2.0.49)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3R_2}{20}} \left(\begin{array}{cc|c} 1 & 0 & \frac{-6}{7} \\ 0 & 1 & \frac{5}{7} \end{array} \right) \quad (2.0.50)$$

Hence, Solution of (2.0.44) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \quad (2.0.51)$$

Comparing results of \mathbf{x} from (2.0.43) and (2.0.51) we conclude that the solution is verified.