Assignment 18

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment18

1 Problem

(ugcdec/2015/72) :
Let $\mathbf{V}$ be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to n. For $p(x) = a_0 + a_{n-1}x + .. + a_n x^n$ in $\mathbf{V}$, define a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ by $(\mathbf{T}p)(x) = a_n + a_{n-1}x + .. + a_0 x^n$. Then

1) $\mathbf{T}$ is one to one.
2) $\mathbf{T}$ is onto.
3) $\mathbf{T}$ is invertible.
4) $\det \mathbf{T} = \pm 1$.

2 Definition and theorem used

Theorem	Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ s an $m \times n$ matrix. 1) T is **one to one** if the columns of $\mathbf{A}$ are linearly independent , which happens precisely when $\mathbf{A}$ has a pivot position in every column. 2) T is **onto** if an over $\mathbb{R}$ only if the span of the columns of $\mathbf{A}$ is $\mathbb{R}^n$, which happens precisely when $\mathbf{A}$ has a pivot position in every row.
$Range(\mathbf{T})$	It is column-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v}$$ where $\mathbf{x}, \mathbf{v} \in \mathbf{V}$ and We can also say that $$Range(\mathbf{T}) = C(\mathbf{A})$$ where $C(\mathbf{A})$ is column space of $\mathbf{A}$.
$rank(\mathbf{T})$	$rank(\mathbf{T}) = rank(\mathbf{A})$

TABLE 1: Definitions and Theorem

3 Solution

Given	$\mathbf{V}$ be a vector space of polynomials over $\mathbb{R}$ of degree less then n $$p(x) = a_0 + a_{n-1}x + .. + a_n x^n$$ $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ $$(\mathbf{T}p)(x) = a_n + a_{n-1}x + .. + a_0 x^n$$
Explanation	We know that Basis for a polynomial vector space $P = (p_1, p_2, .., p_n)$ is a set of vectors that spans the space, and is linearly independent . $$\text{Basis} = \left(1, x, x^2, ..., x^n\right)$$ $$\begin{aligned} &\mathbf{T}(1) = x^n = 0.1 + 0.x + .. + 0.x^{n-1} + 1.x^n \\ &\mathbf{T}(x) = x^{n-1} = 0.1 + 0.x + .. + 1.x^{n-1} + 0.x^n \\ &\vdots \\ &\mathbf{T}(x^n) = 1 = 1.1 + 0.x + .. + 0.x^{n-1} + 0.x \end{aligned}$$ Expressing $\mathbf{T}$ in matrix form $$\mathbf{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
Example	For Simplicity , Let $n = 3$ $$\implies p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$ $$\implies (\mathbf{T})p(x) = a_3 + a_2 x + a_1 x^2 + a_0 x^3$$ Basis $= \left(1, x, x^2, x^3\right)$ $$\mathbf{T}(1) = 0.0 + 0.x + 0.x^2 + 1.x^3$$ $$\mathbf{T}(x) = 0.0 + 0.x + 1.x^2 + 0.x^3$$ $$\mathbf{T}(x^2) = 0.0 + 1.x + 0.x^2 + 0.x^3$$ $$\mathbf{T}(x^3) = 1.1 + 0.x + 0.x^2 + 0.x^3$$ Expressing $\mathbf{T}$ in matrix form;

	$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
Statement 1:**T** is one to one	True
.	$\mathbf{T} : \mathbf{V} \to \mathbf{V}$ be a linear transformation **T** is one-to-one if and only if the nullity of **T** is zero. According to rank-nullity theorem. $dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T})$ $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ Here, $dim(\mathbf{V}) = 4$ $rank(\mathbf{T})$ = no. of linearly independent column or row = 4 $\implies nullity(\mathbf{T}) = 0$ Thus, we can conclude **T** is one to one .
Statement 2:**T** is onto	True
	A matrix transformation is onto if and only if the matrix has a pivot position in each row, if the number of pivots is equal to the number of rows. $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\implies rank(\mathbf{T}) = 4$ which is equal to no of rows. Thus, we can conclude **T** is onto.
Statement 3:**T** is invertible	True
	Theorem : A linear transformation $T : V \to W$ is **invertible** if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the *identity* transformation on W , where U is called Inverse of **T**. **T** is **invertible** if and only if **T** is $one - one$ and *onto*

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$$\implies \mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{T}$$

$$\mathbf{UT} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Thus, we can conclude **T** is invertible.

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Statement 4: $\det \mathbf{T} = \pm 1$

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True

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$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ where } \mathbf{T} \text{ is a permutation matrix }.$$

A permutation matrix is nonsingular matrix, and determinant is ± 1. Permutation matrix **A** satisfies $\mathbf{A}\mathbf{A}^T = \mathbf{I}$

Here, $$\mathbf{T}\mathbf{T}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{T}\mathbf{T}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \text{ , also an Involutory matrix }.$$

Involutory matrix: an involutory matrix is a matrix that is its own inverse. That is, multiplication by matrix **A** is an involution if and only if $\mathbf{A}^2 = \mathbf{I}$.

Since, $\mathbf{T}^{-1} = \mathbf{T}$ and $\mathbf{T}^2 = \mathbf{I}$

We can say **T** is also an Involutory matrix.
Thus, we can conclude $\det \mathbf{T} = \pm 1$

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TABLE 2: Solution Summary