

# Assignment 5

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**Abstract**—This document explains the the concept of finding the angle between the two straight lines from given second degree equation

Download all latex-tikz codes from

<https://github.com/neharani289/ee14014/tree/master/Assignment5>

## 1 PROBLEM

Prove that the equation  $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$  represents two straight lines and find the angle between them

## 2 SOLUTION

The general second order equation is given by ,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (2.0.2)$$

The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (2.0.5)$$

$$f = -35 \quad (2.0.6)$$

(2.0.3) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.7)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (2.0.8)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{45}{2} & \end{vmatrix} = 0 \quad (2.0.9)$$

The lines intercept if

$$|\mathbf{V}| < 0 \quad (2.0.10)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (2.0.11)$$

From (2.0.9) and (2.0.11) it can be concluded that the given equation represents a pair of intersecting lines.

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (2.0.12)$$

Substituting in (2.0.12)

$$\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{13}{2} \\ -\frac{45}{2} \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.14)$$

$$\text{From Spectral theorem, } \mathbf{V} = \mathbf{PDP}^T \quad (2.0.15)$$

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2} \\ 1 & 1 \end{pmatrix} \quad (2.0.17)$$

$$\mathbf{D} = \begin{pmatrix} 1 + \frac{\sqrt{533}}{2} & 0 \\ 0 & 1 - \frac{\sqrt{533}}{2} \end{pmatrix} \quad (2.0.18)$$

Under Affine transformation,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + \mathbf{c} \quad (2.0.19)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (2.0.20)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \quad (2.0.21)$$

under transformation (2.0.21) will become,

$$\Rightarrow \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \quad (2.0.22)$$

(2.0.22) is called *Spectral decomposition* of matrix

$$\Rightarrow X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \quad (2.0.23)$$

$$u_1 X + u_2 Y = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1 X + v_2 Y) \quad (2.0.24)$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \quad (2.0.25)$$

Substituting (2.0.14), (2.0.17) and (2.0.18) in (2.0.25),

$$\begin{aligned} & \frac{\sqrt{533} - 22}{2}(x + 1) + (y - 2) \\ &= \pm \sqrt{-\frac{1 - \frac{\sqrt{533}}{2}}{1 + \frac{\sqrt{533}}{2}}} \left( \frac{-22 - \sqrt{533}}{2}(x + 1) + (y - 2) \right) \end{aligned} \quad (2.0.26)$$

Simplifying (2.0.26),

$$3x - 2y + 7 = 0 \text{ and } 4x + 5y - 5 = 0 \quad (2.0.27)$$

$$\Rightarrow (3x - 2y + 7)(4x + 5y - 5) = 0 \quad (2.0.28)$$

Thus the equation of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} = 5 \quad (2.0.29)$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -7 \quad (2.0.30)$$

### 3 ANGLE BETWEEN THE STRAIGHT LINES

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (3.0.1)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (3.0.2)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (3.0.3)$$

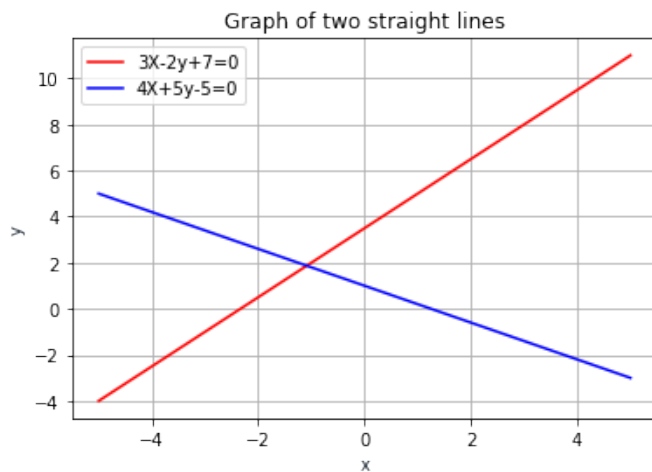


Fig. 1: Pair of straight lines