Assignment 14

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment14

1 Problem

(hoffman/page208/1b) :
Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P^{-1}AP}$ and $\mathbf{P^{-1}BP}$ are both diagonal where $\mathbf{A}$ and $\mathbf{B}$ are real matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.0.1}$$

$$\mathbf{B} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \tag{1.0.2}$$

2 Explanation

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\|x\mathbf{I} - \mathbf{A}\right\|$$
Theorem	According to theorem 8, if a 2×2 matrix has two characteristics values then the $\mathbf{P}$ that diagonalize $\mathbf{A}$ will necessarily also diagonalize any $\mathbf{B}$ that commutes with $\mathbf{A}$.
Basis	Let there exist a $\mathbf{P}$ in basis $\beta = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathbb{V}$ consisting of eigen vector which are common to both $\mathbf{A}$ and $\mathbf{B}$ such that $$\mathbf{A}\mathbf{b}_i = \lambda_i \mathbf{b}_i \qquad \mathbf{B}\mathbf{b}_i = \mu_i \mathbf{b}_i$$ $$\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$ $$\Lambda_A = \mathbf{P^{-1}AP} \qquad \Lambda_B = \mathbf{P^{-1}BP}$$

TABLE 1: Definitions and theorem used

3 Solution

Operations	Matrix A	Matrix B
Characteristic Polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$ $= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix}$ $= (x-1)(x-1) - 1$	$p(x) = \lvert x\mathbf{I} - \mathbf{B} \rvert$ $= \begin{vmatrix} x-1 & -a \\ -a & x-1 \end{vmatrix}$ $= (x-1)(x-1) - a^2$
Characteristic values	$p(x) = 0$ $x(x-1) = 0$ $\lambda_1 = 0\ ,\lambda_2 = 2$	$p(x) = 0$ $(x-1)^2 - a^2 = 0$ $\mu_1 = (1-a)\ ,\ \mu_2 = (1+a)$
Basis for Characteristics Values	$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{X} = 0$ $\implies$ For $\lambda_1 = 0$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\implies$ For $\lambda_2 = 2$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$(\mathbf{B} - \mu_i\mathbf{I})\mathbf{X} = 0$ $\implies$ For $\mu_1 = (1-a)$ $\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\implies$ For $\mu_2 = (1+a)$ $\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Invertible matrix	Let $\mathbf{P} = \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} \end{pmatrix}$ Then, $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	Let $\mathbf{P} = \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} \end{pmatrix}$ Then, $\implies \mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

Verification	$\Lambda_A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $\Lambda_A = \mathbf{P^{-1}AP}$ $\Lambda_A = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$	$\Lambda_B = \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix}$ $\Lambda_B = \mathbf{P^{-1}BP}$ $\Lambda_B = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix} = \Lambda_B$
Conclusion	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

TABLE 2: Finding an Invertible matrix