Assignment 13

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Download the latex-tikz codes from

https://github.com/neharani289/MatrixTheory/Assignment13

1 Problem

(hoffman/page198/9) :
Let A be an nxn matrix with characteristics polynomial

$$f = (x - c_1)^{d_1} ....(x - c_k)^{d_k}$$

Show that

$$c_1 d_1 + ... + c_k d_k = trace(A)$$

2 Solution

Given	Let A be an nxn $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$ and Characteristics polynomial $$f = (x - c_1)^{d_1} ....(x - c_k)^{d_k}$$
To proof	$c_1 d_1 + ... + c_k d_k = trace(A)$

Claim	$det(xI - A) = x^2 - trace(A)x^{n-1} + ... + (-1)^n det(A)$
To prove that the coefficient of x^{n-1} is $trace(A)$	Proof by method of induction for n=2 Let, A_2 be $2x2$matrix. $\mathbf{A_2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\det(xI - A) = \begin{vmatrix} (x - a_{11}) & -a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\implies det(xI - A) = x^2 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})$ $\boxed{trace(A_2) = a_{11} + a_{12}}$ and the coefficient of $\boxed{x^{n-1} = x}$. Therefore , claim is true for $n = 2$. Assume that it is true for upto $n - 1$. Then, Coefficient of x^{n-2} will be $s = a_{22} + a_{33} + a_{44} + ... + a_{nn}$ for the matrix, $A_{n-1Xn-1}$, $A_{n-1} = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$

TABLE 1: Proof for Coefficients of x^{n-1}

To prove that the claim is true for n	$\det(xI-A) = \begin{vmatrix} (x-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (x-a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (x-a_{nn}) \end{vmatrix}$ $det(xI-A) = (x-a_{11})cofactor(a_{11}) + a_{21}cofactor(-a_{21})$ $+ ... - (-1)^{n+1}cofactor(-a_{n1})$ $det(xI-A) = (x-a_{11})(x^{n-1} - sx^{n-2} + ..) + a_{21}$(polynomial of degree $n-2$) $+ ..\text{-}(-1)^{n+1}a_{n1}$(polynomial of degree $n-2$) $det(xI-A) = x^n - (a_{11}+s)x^{n-1}+$ polynomial of degree at most $n-2$ $det(xI-A) = x^n - (a_{11}+a_{22}+..+a_{nn})x^{n-1}+$polynomial of degree at most $n-2$ $\implies det(xI-A) = x^n - trace(A)x^{n-1}$ + polynomial of degree atmost $n-2$ Given characteristic polynomial of A is, $f = (x-c_1)^{d_1}....(x-c_k)^{d_k}$ where, $d_1 + d_2 + ... + d_k = n$ Since,expansion of $(x-r)^t = x^t - rtx^{t-1} + .. + (-1)^t r^t$ Therefore, $f = (x-c_1)^{d_1}....(x-c_k)^{d_k}$ $f = (x^{d_1} - c_1d_1x^{d_1-1} + ..)..(x^{d_k} - c_kd_kx^{d_k-1} + ..)$ $\implies \boxed{f = x^n - (c_1d_1 + c_2d_2 + ... + c_kd_k)x^{n-1} + ..}$ Since, $f = det(xI-A)$ $\implies x^n - (c_1d_1 + c_2d_2 + ... + c_kd_k)x^{n-1} + .. = x^n - trace(A)x^{n-1}+$ polynomial of degree atmost $n-2$ By comparing corresponding terms in above equation ; $\boxed{c_1d_1 + c_2d_2 + ... + c_kd_k = trace(A)}$ Hence, Proved.

TABLE 2: Solution Summary