

Assignment 18

Neha Rani
EE20MTECH14014

Download the latex-tikz codes from

<https://github.com/neharani289/MatrixTheory/Assignment18>

1 PROBLEM

(ugcdec/2015/72) :

Let \mathbf{V} be the vector space of polynomials over \mathbb{R} of degree less than or equal to n . For $p(x) = a_0 + a_{n-1}x + \dots + a_n x^n$ in \mathbf{V} , define a linear transformation $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ by $(\mathbf{T}p)(x) = a_n + a_{n-1}x + \dots + a_0 x^n$. Then

- 1) \mathbf{T} is one to one.
- 2) \mathbf{T} is onto.
- 3) \mathbf{T} is invertible.
- 4) $\det \mathbf{T} = \pm 1$.

2 DEFINITION AND THEOREM USED

Theorem	<p>Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the linear transformation $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where \mathbf{A} is an $m \times n$ matrix.</p> <ol style="list-style-type: none"> 1) T is one to one if the columns of \mathbf{A} are linearly independent, which happens precisely when \mathbf{A} has a pivot position in every column. 2) T is onto if an over \mathbb{R} only if the span of the columns of \mathbf{A} is \mathbb{R}^n, which happens precisely when \mathbf{A} has a pivot position in every row.
$Range(\mathbf{T})$	<p>It is column-space of linear operator \mathbf{T}.</p> $\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v}$ <p>where $\mathbf{x}, \mathbf{v} \in \mathbf{V}$ and We can also say that</p> $Range(\mathbf{T}) = C(\mathbf{A})$ <p>where $C(\mathbf{A})$ is column space of \mathbf{A}.</p>
$rank(\mathbf{T})$	$rank(\mathbf{T}) = rank(\mathbf{A})$

TABLE 1: Definitions and Theorem

3 SOLUTION

Given	<p>\mathbf{V} be a vector space of polynomials over \mathbb{R} of degree less than n</p> $p(x) = a_0 + a_{n-1}x + \dots + a_n x^n$ <p>$\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$</p> $(\mathbf{T}p)(x) = a_n + a_{n-1}x + \dots + a_0 x^n$
Explanation	<p>We know that Basis for a polynomial vector space $P = (p_1, p_2, \dots, p_n)$ is a set of vectors that spans the space, and is linearly independent .</p> $\text{Basis} = (1, x, x^2, \dots, x^n)$ $\mathbf{T}(1) = x^n = 0.1 + 0.x + \dots + 0.x^{n-1} + 1.x^n$ $\mathbf{T}(x) = x^{n-1} = 0.1 + 0.x + \dots + 1.x^{n-1} + 0.x^n$ \vdots $\mathbf{T}(x^n) = 1 = 1.1 + 0.x + \dots + 0.x^{n-1} + 0.x^n$ <p>Expressing \mathbf{T} in matrix form</p> $\mathbf{T} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$
Example	<p>For Simplicity , Let $n = 3$</p> $\Rightarrow p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ $\Rightarrow (\mathbf{T})p(x) = a_3 + a_2x + a_1x^2 + a_0x^3$ <p>Basis = $(1, x, x^2, x^3)$</p> $\mathbf{T}(1) = 0.0 + 0.x + 0.x^2 + 1.x^3$ $\mathbf{T}(x) = 0.0 + 0.x + 1.x^2 + 0.x^3$ $\mathbf{T}(x^2) = 0.0 + 1.x + 0.x^2 + 0.x^3$ $\mathbf{T}(x^3) = 1.1 + 0.x + 0.x^2 + 0.x^3$ <p>Expressing \mathbf{T} in matrix form;</p>

	$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
Statement 1: \mathbf{T} is one to one	True
	<p>$\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation</p> <p>\mathbf{T} is one-to-one if and only if the nullity of \mathbf{T} is zero.</p> <p>According to rank-nullity theorem. $\dim(\mathbf{V}) = \text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T})$</p> $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ <p>Here, $\dim(\mathbf{V}) = 4$</p> <p>$\text{rank}(\mathbf{T}) = \text{no. of linearly independent column or row} = 4$</p> <p>$\implies \text{nullity}(\mathbf{T}) = 0$</p> <p>Thus, we can conclude \mathbf{T} is one to one .</p>
Statement 2: \mathbf{T} is onto	True
	<p>A matrix transformation is onto if and only if the matrix has a pivot position in each row, if the number of pivots is equal to the number of rows.</p> $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ <p>$\implies \text{rank}(\mathbf{T}) = 4$ which is equal to no of rows.</p> <p>Thus, we can conclude \mathbf{T} is onto.</p>
Statement 3: \mathbf{T} is invertible	True
	<p>Theorem : A linear transformation $T : V \rightarrow W$ is invertible if there exists another linear transformation $U : W \rightarrow V$ such that UT is the <i>identity</i> transformation on V and TU is the <i>identity</i> transformation on W , where U is called Inverse of \mathbf{T}.</p> <p>\mathbf{T} is invertible if and only if \mathbf{T} is <i>one – one</i> and <i>onto</i></p>

	$\Rightarrow \mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\mathbf{T}^{-1} = \mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{T}$ $\mathbf{UT} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ <p>Thus, we can conclude \mathbf{T} is invertible.</p>
Statement 4: $\det \mathbf{T} = \pm 1$	True
	$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ where } \mathbf{T} \text{ is a permutation matrix .}$ <p>A permutation matrix is nonsingular matrix, and determinant is ± 1. Permutation matrix \mathbf{A} satisfies $\mathbf{AA}^T = \mathbf{I}$</p> <p>Here,</p> $\mathbf{TT}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\mathbf{TT}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}, \text{ also an Involutory matrix .}$ <p>Involutory matrix: an involutory matrix is a matrix that is its own inverse. That is, multiplication by matrix \mathbf{A} is an involution if and only if $\mathbf{A}^2 = \mathbf{I}$ and Determinant of an involutory matrix over any field is ± 1</p> <p>Since, $\mathbf{T}^{-1} = \mathbf{T}$ and $\mathbf{T}^2 = \mathbf{I}$</p> <p>We can say \mathbf{T} is also an Involutory matrix. Thus, we can conclude $\det \mathbf{T} = \pm 1$</p>

TABLE 2: Solution Summary