

Assignment 13

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Download the latex-tikz codes from

<https://github.com/neharani289/MatrixTheory/Assignment13>

1 PROBLEM

(hoffman/page198/9) :

Let A be an $n \times n$ matrix with characteristics polynomial

$$f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$$

Show that

$$c_1 d_1 + \dots + c_k d_k = \text{trace}(A)$$

2 SOLUTION

Given	<p>Let A be an $n \times n$</p> $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ <p>and Characteristics polynomial</p> $f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$
To proof	$c_1 d_1 + \dots + c_k d_k = \text{trace}(A)$
Claim	$\det(xI - A) = x^n - \text{trace}(A)x^{n-1} + \dots + (-1)^n \det(A)$
To prove that the coefficient	Consider the method of induction to proof for $n=2$

of x^{n-1} is $\text{trace}(A)$

Let, A_2 be 2×2 matrix.

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(xI - A) = \begin{vmatrix} (x - a_{11}) & -a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\implies \det(xI - A) = x^2 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})$$

$$\boxed{\text{trace}(A_2) = a_{11} + a_{12}} \text{ and the coefficient of } \boxed{x^{n-1} = x}.$$

Therefore, claim is true for $n = 2$.

Assume that it is true for upto $n - 1$.

Then, Coefficient of x^{n-2} will be

$$\boxed{s = a_{22} + a_{33} + a_{44} + \dots + a_{nn}} \text{ for the matrix, } A_{n-1 \times n-1},$$

$$A_{n-1} = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

To prove that the claim is true for n

$$\det(xI - A) = \begin{vmatrix} (x - a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (x - a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (x - a_{nn}) \end{vmatrix}$$

$$\det(xI - A) = (x - a_{11})\text{cofactor}(a_{11}) + a_{21}\text{cofactor}(-a_{21}) + \dots - (-1)^{n+1}\text{cofactor}(-a_{n1})$$

$$\det(xI - A) = (x - a_{11})(x^{n-1} - sx^{n-2} + \dots) + a_{21}(\text{polynomial of degree } n-2) + \dots - (-1)^{n+1}a_{n1}(\text{polynomial of degree } n-2)$$

$$\det(xI - A) = x^n - (a_{11} + s)x^{n-1} + \text{polynomial of degree at most } n-2$$

$$\det(xI - A) = x^n - (a_{11} + a_{22} + \dots + a_{nn})x^{n-1} + \text{polynomial of degree at most } n-2$$

$$\implies \det(xI - A) = x^n - \text{trace}(A)x^{n-1} + \text{polynomial of degree at most } n-2$$

Given characteristic polynomial of A is,

$$f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$$

where,

$$d_1 + d_2 + \dots + d_k = n$$

$$\text{Since, expansion of } (x - r)^t = x^t - tx^{t-1} + \dots + (-1)^t r^t$$

Therefore,

$$f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}$$

$$f = (x^{d_1} - c_1 d_1 x^{d_1-1} + \dots)(x^{d_k} - c_k d_k x^{d_k-1} + \dots)$$

$$\Rightarrow \boxed{f = x^n - (c_1 d_1 + c_2 d_2 + \dots + c_k d_k) x^{n-1} + \dots}$$

Since,

$$f = \det(xI - A)$$

$$\Rightarrow x^n - (c_1 d_1 + c_2 d_2 + \dots + c_k d_k) x^{n-1} + \dots = x^n - \text{trace}(A) x^{n-1} + \text{polynomial of degree at most } n-2$$

By comparing corresponding terms in above equation ;

$$\boxed{c_1 d_1 + c_2 d_2 + \dots + c_k d_k = \text{trace}(A)}$$

Hence, Proved.

TABLE 1: Solution Summary