Assignment 5

Neha Rani

***Abstract*—This document explains the the concept of finding the angle between the two straight lines from given second degree equation**

Download all latex-tikz codes from

https://github.com/neharani289/ee14014/tree/master/Assignment5

1 Problem

Prove that the equation $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$ represents two straight lines and find the angle between them

1.1 Pair of staraight lines

The general second order equation is given by ,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.1.1}$$

the above equation (1.1.1) can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{1.1.4}$$

the above equation (1.1.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.1.5}$$

2 Solution

Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \tag{2.0.1}$$

The above equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \tag{2.0.4}$$

$$f = -35 \tag{2.0.5}$$

(2.0.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.6}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \tag{2.0.7}$$

$$\implies 12\begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2}\begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2}\begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \tag{2.0.8}$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \tag{2.0.9}$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \tag{2.0.10}$$

From (2.0.8) and (2.0.10) it can be concluded that the given equation represents a pair of intersecting lines.

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{2.0.11}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{2.0.12}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2.0.13}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{2.0.14}$$

under transformation (2.0.14) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{2.0.15}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.16}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.17}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{2.0.18}$$

$$\text{where } X' = Xu_1 + Yu_2 \text{ and } Y' = Xv_1 + Yv_2 \tag{2.0.19}$$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \tag{2.0.20}$$

(2.0.20) is called *Spectral decomposition* of matrix

$$\implies X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{2.0.21}$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \tag{2.0.22}$$

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{2.0.23}$$

Substituting in (2.0.11)

$$\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{-13}{2} \\ -\frac{45}{2} \end{pmatrix} \tag{2.0.24}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.25}$$

$$\text{From } \textit{Spectral theorem, } \mathbf{V} = \mathbf{PDP}^T \tag{2.0.26}$$

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{2.0.27}$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2} \\ 1 & 1 \end{pmatrix} \tag{2.0.28}$$

$$\mathbf{D} = \begin{pmatrix} 1 + \frac{\sqrt{533}}{2} & 0 \\ 0 & 1 - \frac{\sqrt{533}}{2} \end{pmatrix} \tag{2.0.29}$$

Substituting (2.0.25), (2.0.28) and (2.0.29) in (2.0.23),

$$\frac{\sqrt{533}-22}{2}(x+1) + (y-2) = \pm\sqrt{-\frac{1-\frac{\sqrt{533}}{2}}{1+\frac{\sqrt{533}}{2}}}\left(\frac{-22-\sqrt{533}}{2}(x+1) + (y-2)\right) \tag{2.0.30}$$

Simplifying (2.0.30),

$$3x - 2y + 7 = 0 \text{ and } 4x + 5y - 5 = 0 \tag{2.0.31}$$

$$\implies (3x - 2y + 7)(4x + 5y - 5) = 0 \tag{2.0.32}$$

Thus the equation of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} = 5 \tag{2.0.33}$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix}\mathbf{x} = -7 \tag{2.0.34}$$

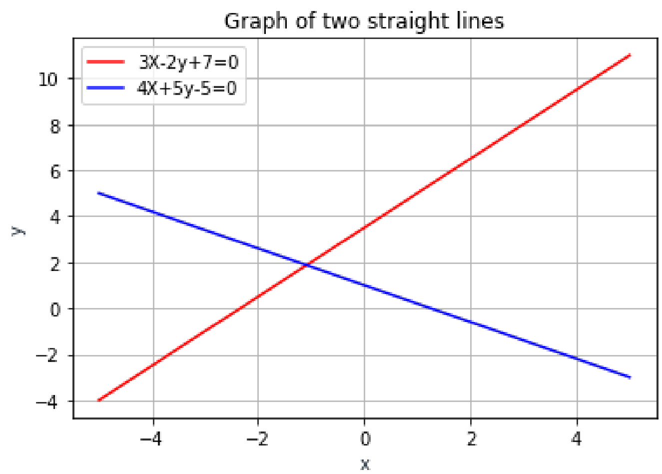

Fig. 1: Pair of straight lines

3 Angle between the straight lines

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.0.1}$$

as

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{||\mathbf{n_1}||\,||\mathbf{n_2}||} \tag{3.0.2}$$

$$\implies \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \tag{3.0.3}$$