Basics of QCD Perturbation Theory

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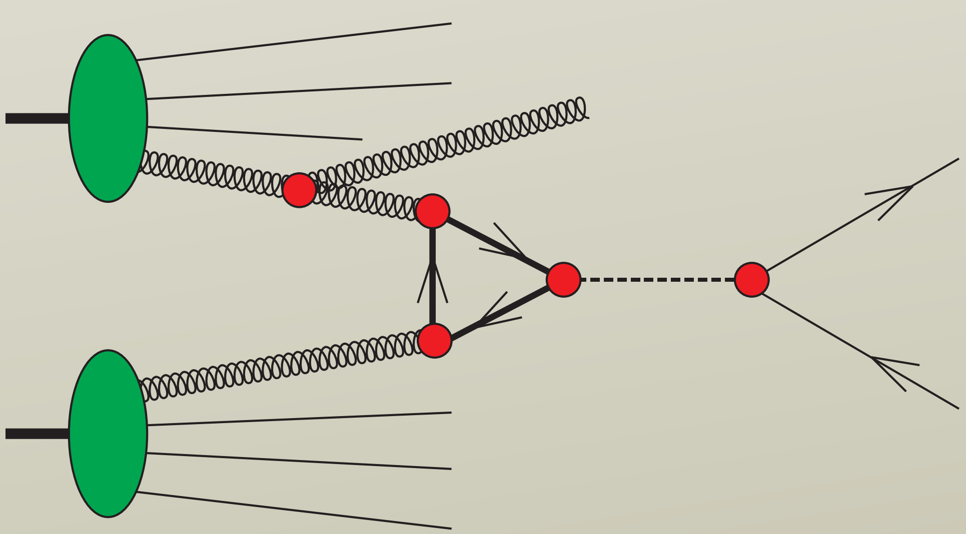
Why study QCD?

- You may want to discover something new at the LHC: squarks, dark matter particles, ...

- For example, maybe you want to discover anomalous couplings of the Higgs boson.

- Then QCD is part of the process and you need to understand how it works.



Abstract

- Applications of QCD to experiment combine
  - a particular calculation of Feynman diagrams;
  - general features of the theory that enable the calculation to apply to the experiment.
- We will study the general features.

Some general features of QCD

- Jet structure.
- Renormalization group and running coupling.
- Existence of infrared safe observables.
- Ability to isolate soft initial state physics in parton distribution functions.

Along the way...

- We will study the three basic processes:
  - electron-positron annihilation,
  - deeply inelastic scattering,
  - hard processes in hadron-hadron collisions.
- And we will learn some kinematics that "everybody knows."

Electron-positron annihilation and jets

Exploring the QCD final state

Topics

- Structure of the cross section.
- General nature of the singularities.
- Null-plane coordinates.
- Space-time picture.
- Infrared-safe observables.

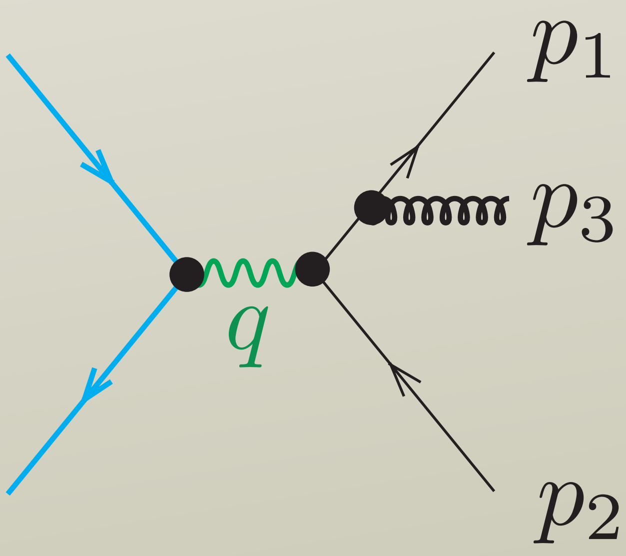
Cross section for e⁺e⁻⟶3 partons

$$\frac{1}{\sigma_0}\,\frac{d\sigma}{dE_3\,d\cos\theta_{13}} = \frac{\alpha_s}{2\pi}\,C_F\,\frac{f(E_3,\theta_{13})}{E_3(1-\cos\theta_{13})}$$

where $f(E_3,\theta_{13})$ is straightforward but slightly messy. It is finite for $E_3 \to 0$ or $\theta_{13} \to 0$.

- Collinear singularity:

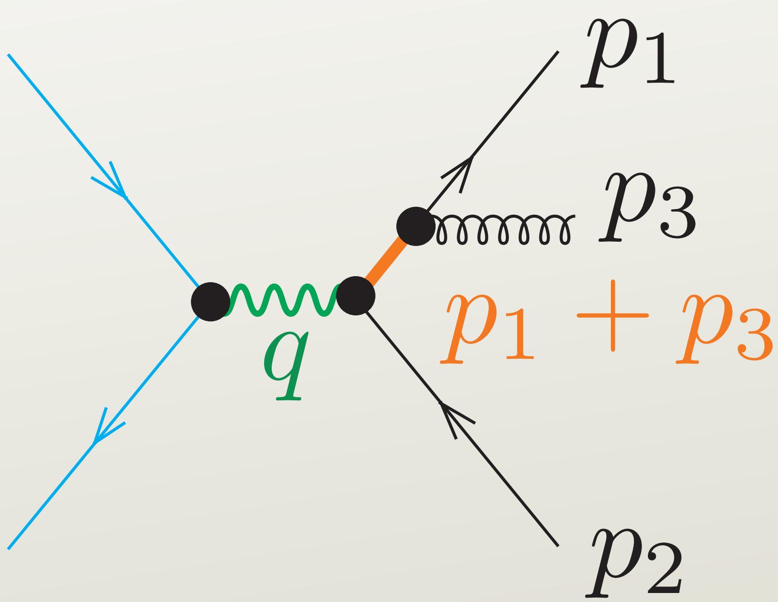
$$\int_0^1 d\cos\theta_{13}\;\frac{d\sigma}{dE_3\,d\cos\theta_{13}} = \log(\infty)$$



- Soft singularity:

$$\int_0^a dE_3\;\frac{d\sigma}{dE_3\,d\cos\theta_{13}} = \log(\infty)$$

General nature of the singularities



- $\mathcal{M}$ contains a factor $1/(p_1+p_3)^2$.

  $(p_1+p_3)^2 = 2E_1E_3(1-\cos\theta_{13})$

- This is singular for $\theta_{13} \to 0$ and for $E_3 \to 0$.

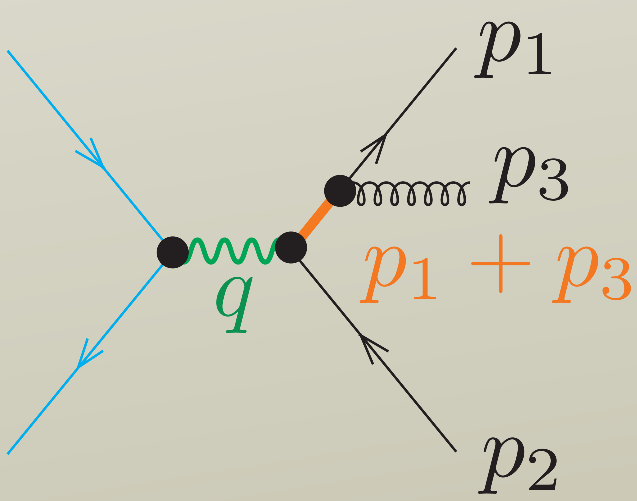
- The numerator has a factor θ_{13} for small θ_{13}.

- So

$$|\mathcal{M}|^2 \propto \frac{1}{E_3^2\theta_{13}^2}, \qquad \theta_{13} \to 0 \text{ or } E_3 \to 0$$

- This gives logarithmically divergent integrals

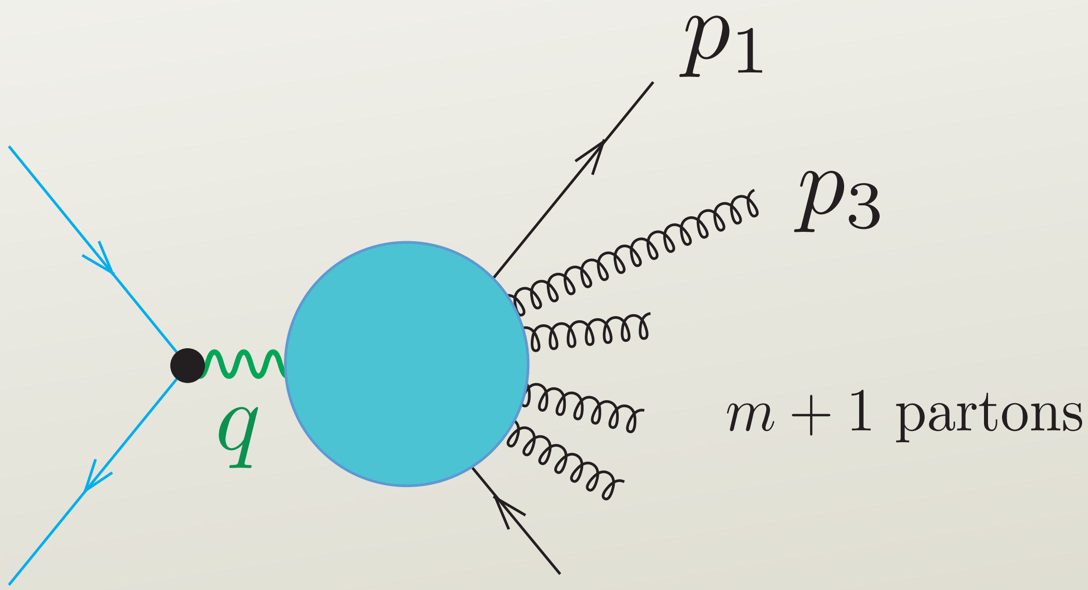
$$
\begin{aligned}
\int d\sigma \quad &\sim \quad \int \frac{d^3\vec{p}_3}{E_3}\,\frac{1}{E_3^2\theta_{13}^2} \\
&= \quad \int \frac{E_3^2 dE_3\, d\cos\theta_{13}\, d\phi}{E_3}\;\frac{1}{E_3^2\theta_{13}^2} \\
&\sim \quad \int \frac{dE_3}{E_3}\,\frac{d\theta_{13}^2}{\theta_{13}^2}\; d\phi
\end{aligned}
$$



This structure is general for tree graphs.

- Suppose that partons 1 and 3 become collinear.

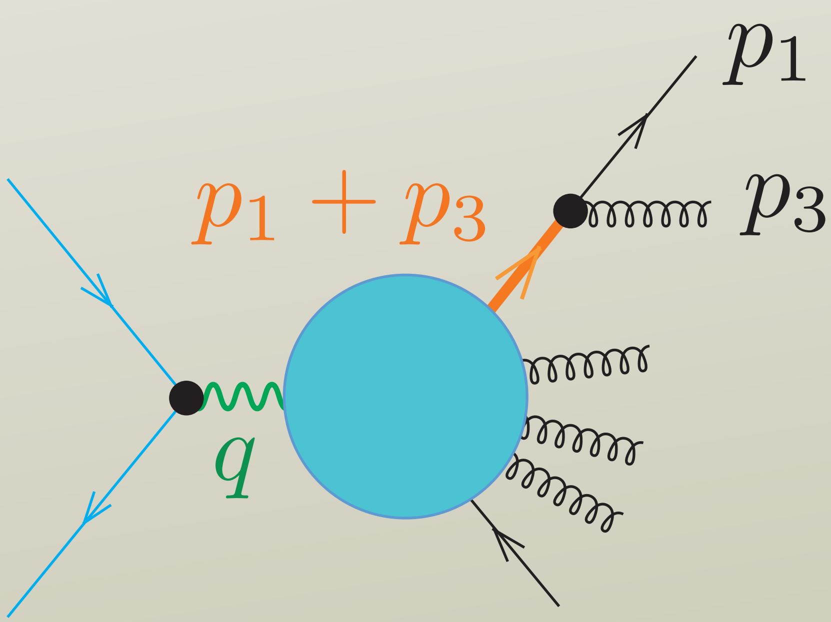
$$p_3 \to z(p_1 + p_3)$$



- Then

$$\mathcal{M}_{m+1} \sim \left[\mathcal{M}_m\right]_{\{1,3\}\ \text{on-shell}} \frac{\text{spinors}}{(p_1+p_3)^2}$$

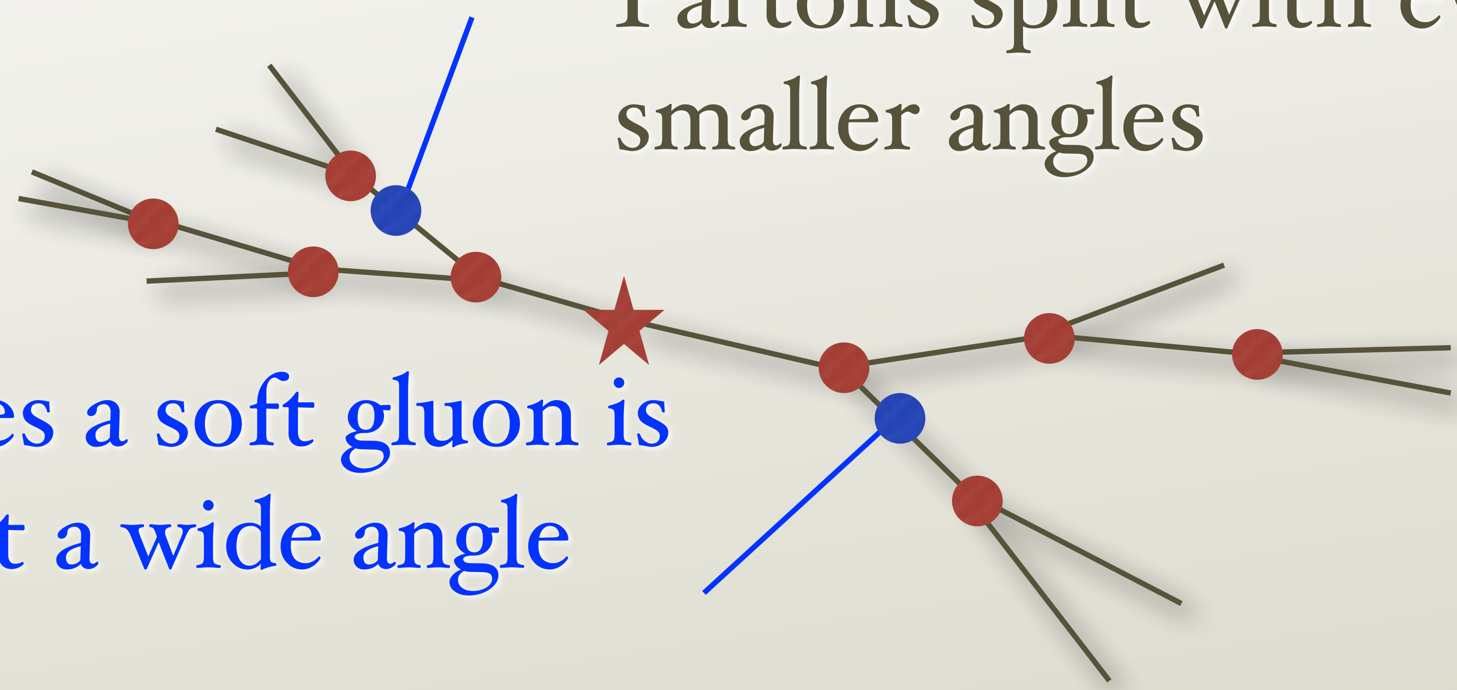
splitting amplitude



- This is how one starts to define a parton shower.

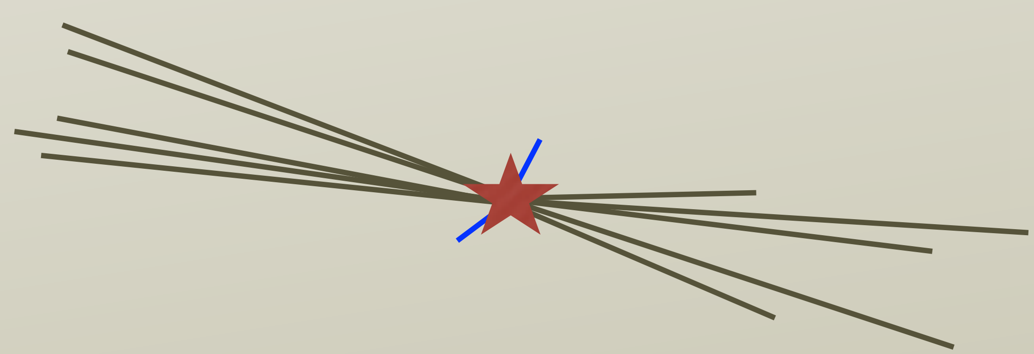
This suggests the following structure of events:

Partons split with ever smaller angles



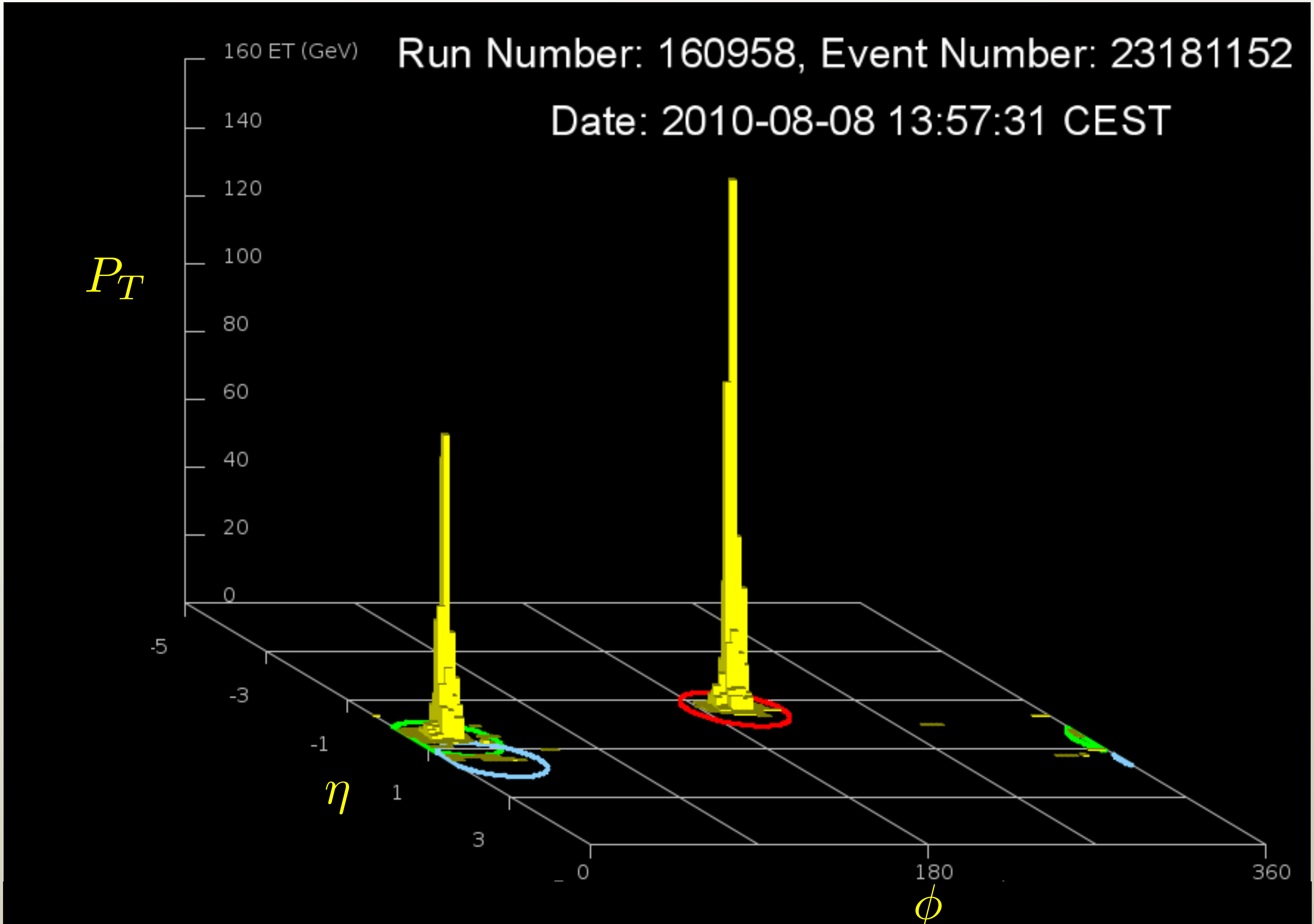
Sometimes a soft gluon is emitted at a wide angle

- The corresponding picture for the final state particles is



- These sprays of particles are called jets.

Jets exist in nature...



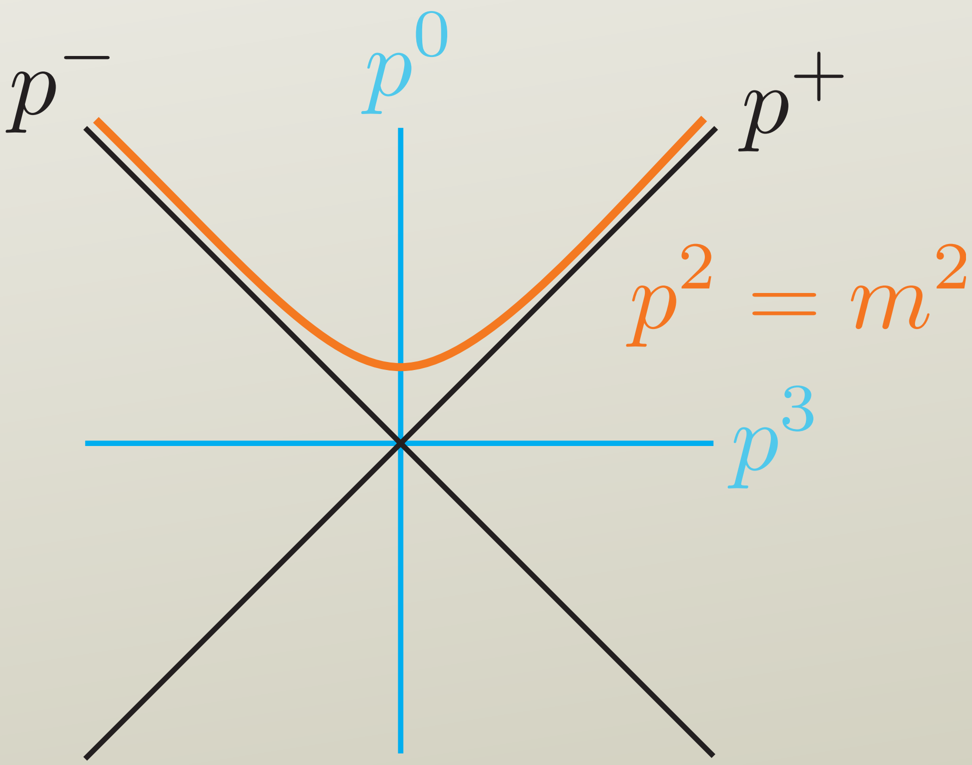
An Atlas event with two jets

Null-plane coordinates

Use $p^\mu = (p^+, p^-, p^1, p^2)$ where

$$p^\pm = (p^0 \pm p^3)/\sqrt{2}$$

- Often one chooses the axes so that a particle or group of particles of interest have large p^+ and small p^- and $\boldsymbol{p}_T$.

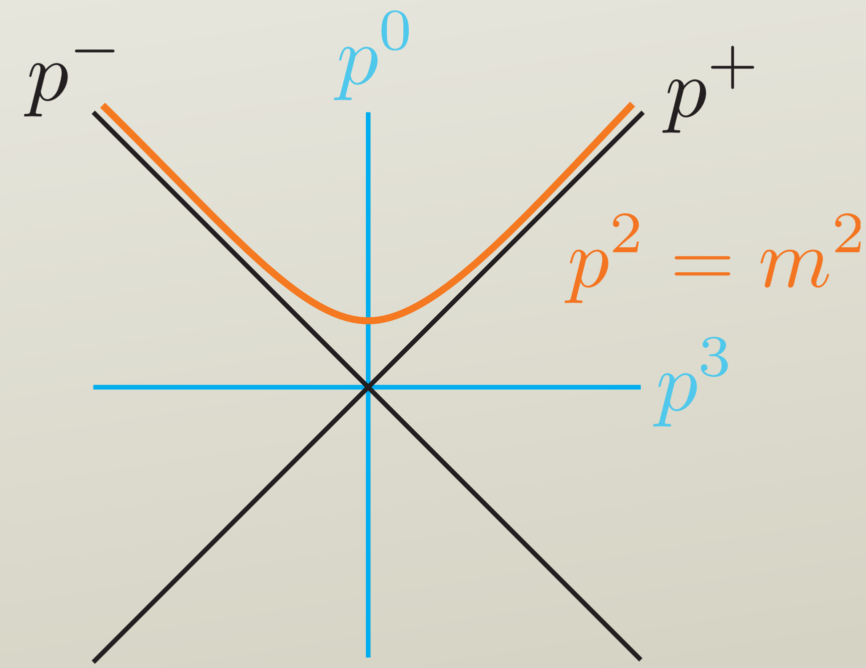


Some properties of null-plane coordinates

- Recall $p^{\pm} = (p^0 \pm p^3)/\sqrt{2}$

- Covariant square

$$p^2 = 2p^+ p^- - \boldsymbol{p}_T^2$$



- For a particle on its mass shell

$$p^+ > 0\,, \qquad p^- > 0$$

$$p^- = \frac{\boldsymbol{p}_T^2 + m^2}{2p^+}$$

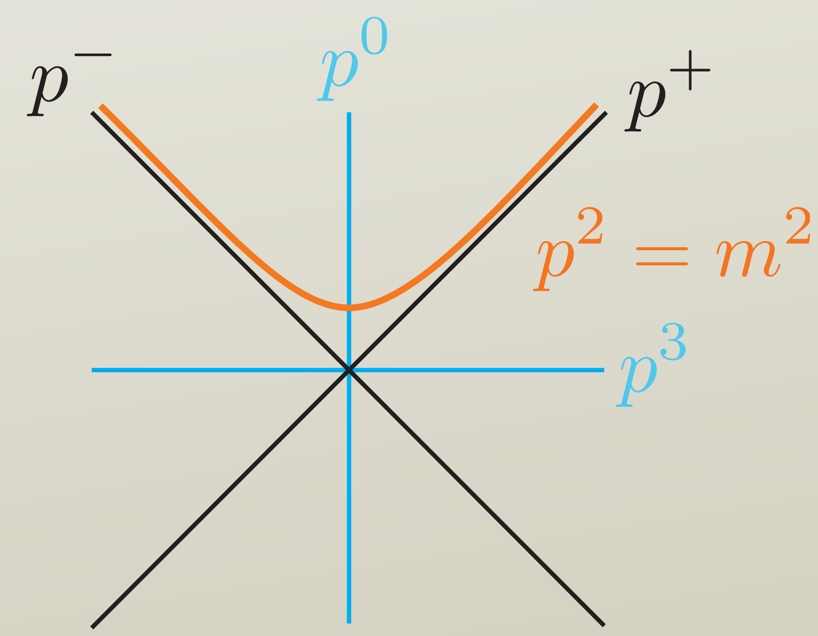
- A particle with limited $\boldsymbol{p}_T$ and large p^+ has small p^-.

- Integration over the mass shell:

$$(2\pi)^{-3}\int \frac{d^3\vec{p}}{2\sqrt{\vec{p}^{\,2}+m^2}}\cdots = (2\pi)^{-3}\int d^2\mathbf{p}_T \int_0^\infty \frac{dp^+}{2p^+}\cdots .$$

- Fourier transform:

$$p\cdot x = p^+x^- + p^-x^+ - \boldsymbol{p}_T\cdot\boldsymbol{x}_T$$



- so x^+ is Fourier conjugate to p^-
  and x^- is Fourier conjugate to p^+
  (Sorry.)

Boosts

$$
\begin{aligned}
v^{+}_{\text{new}} &= e^{\omega} v^{+}_{\text{old}} \\
v^{-}_{\text{new}} &= e^{-\omega} v^{-}_{\text{old}} \\
\boldsymbol{v}_{T,\text{new}} &= \boldsymbol{v}_{T,\text{old}}
\end{aligned}
$$

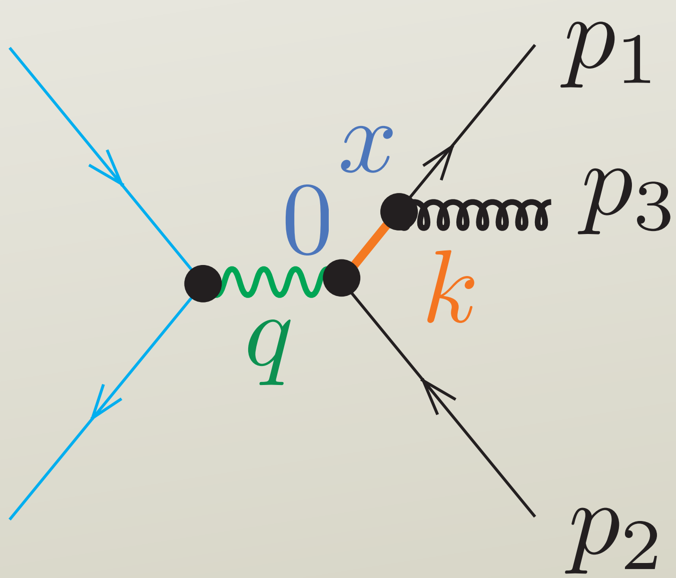
- So

$$
p \cdot x = p^{+} x^{-} + p^{-} x^{+} - \boldsymbol{p}_T \cdot \boldsymbol{x}_T
$$

is invariant.

Space-time picture of the singularities

- Write the "amputated" diagram in coordinate space.



- Define $p_1^\mu + p_3^\mu = k^\mu$.
- Use coordinates with k^+ large and $\boldsymbol{k}_T = 0$.
- Then $k^2 = 2k^+k^-$ becomes small when k^- becomes small. (Collinear or soft limit, $m = 0$.)

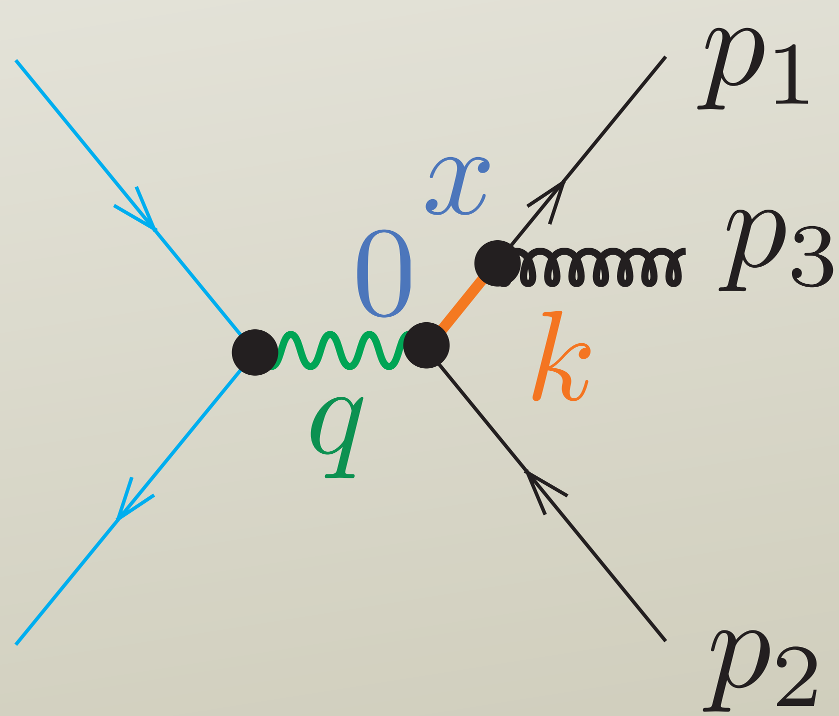
$$k^- = \frac{\boldsymbol{p}_3^2}{2p_1^+} + \frac{\boldsymbol{p}_3^2}{2p_3^+}$$

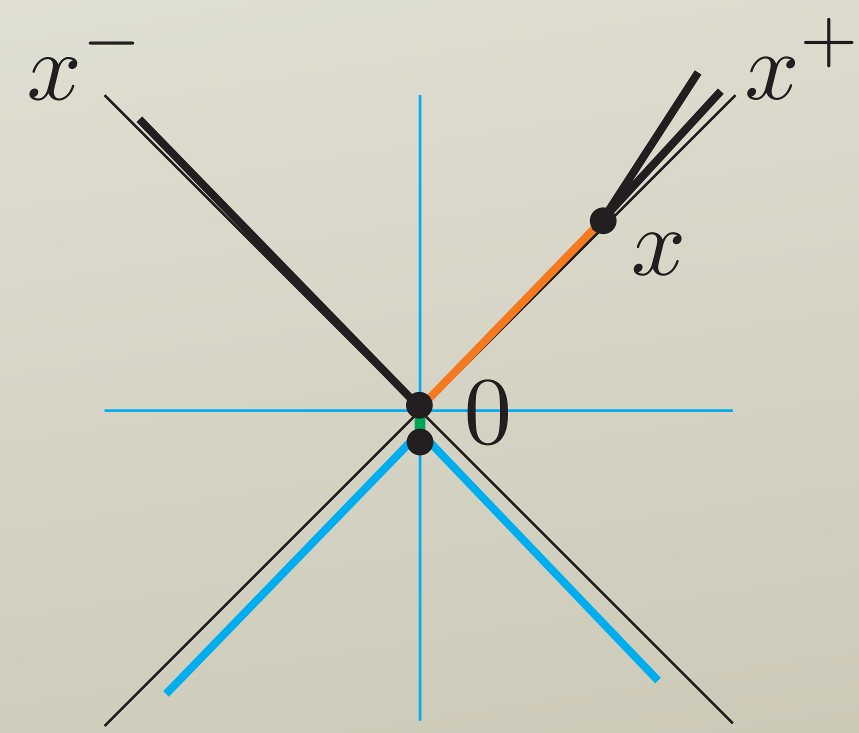
Consider the Fourier transform.

- The singularity corresponds to large k^+ and small k^-.

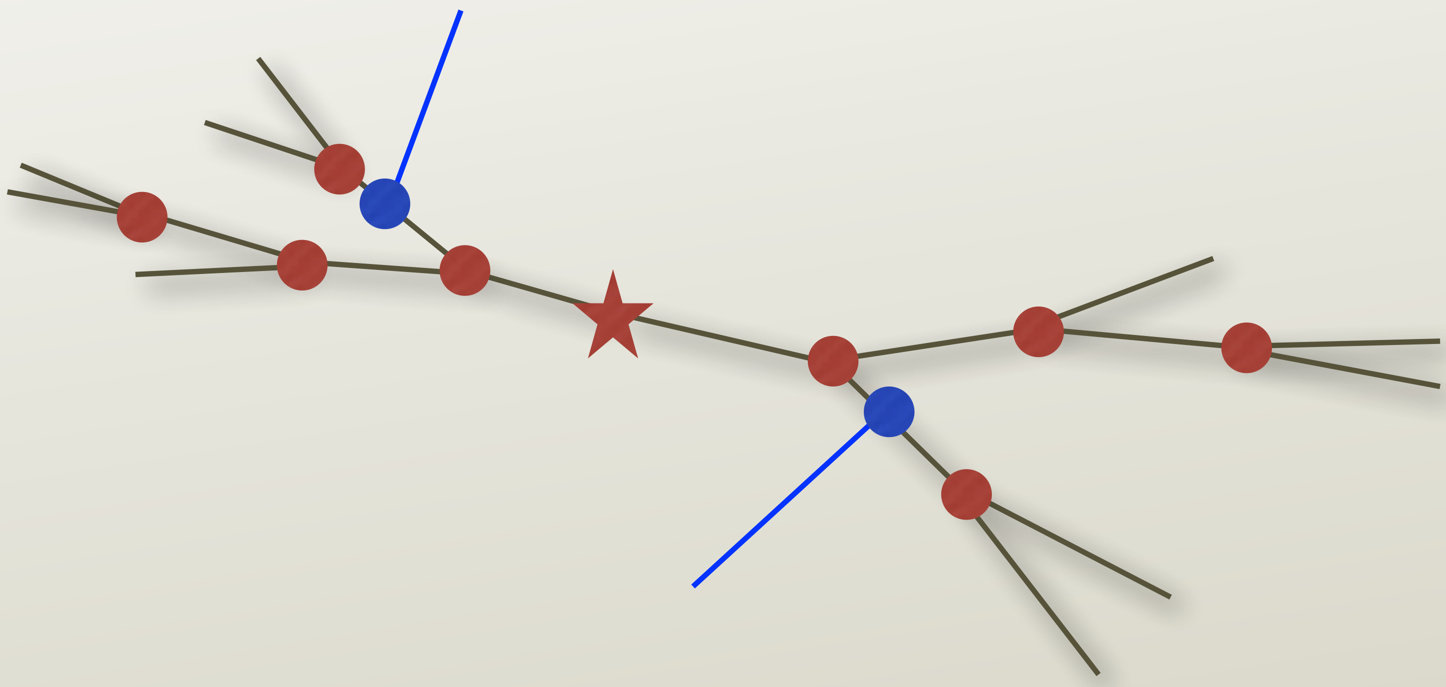
$$S_F(k) = \int dx^+ dx^- d\boldsymbol{x}\ \exp(i[k^+ x^- + k^- x^+ - \boldsymbol{k}\cdot\boldsymbol{x}])\ S_F(x).$$

- Contributing positions have large x^+ and small x^-.





- Thus in the picture



the first splittings happen relatively early, the next ones are much later.

- For example, 0.002 fm, 0.02 fm, 0.2 fm ...

- Beware...
- We will find that perturbative QCD cannot predict long time physics very well.
- But the detector is a long distance away.
- How can we have any sound predictions?

Infrared safe observables

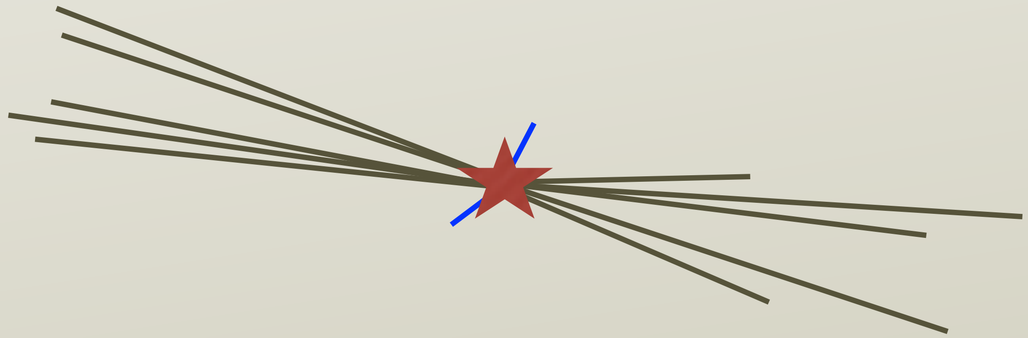
- Definition of an observable

$$
\begin{aligned}
\sigma[F] \quad = \quad & \frac{1}{2!}\int d\Omega_2 \; \frac{d\sigma[2]}{d\Omega_2} \; F_2(p_1^\mu, p_2^\mu) \\
& + \frac{1}{3!}\int d\Omega_2 dE_3 d\Omega_3 \; \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} \; F_3(p_1^\mu, p_2^\mu, p_3^\mu) \\
& + \frac{1}{4!}\int d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4 \\
& \qquad \times \frac{d\sigma[4]}{d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4} \; F_4(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) \\
& + \cdots \, .
\end{aligned}
$$

- The observable F is "infrared safe" if

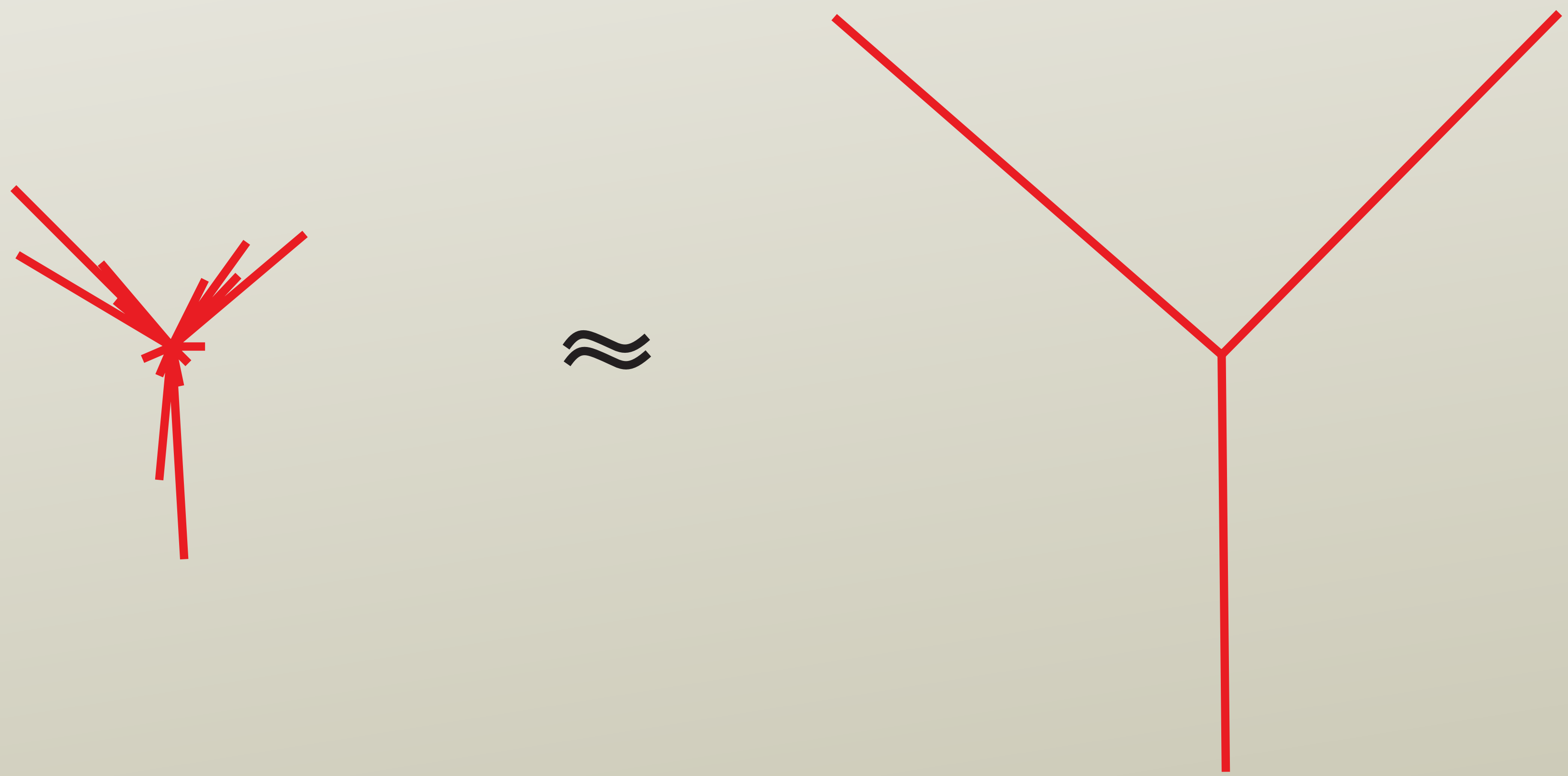
  for $z = 0$ or $0 < z < 1$

$$F_{m+1}(p_1^\mu, \dots, (1-z)p_m^\mu, z p_m^\mu) = F_m(p_1^\mu, \dots, p_m^\mu).$$

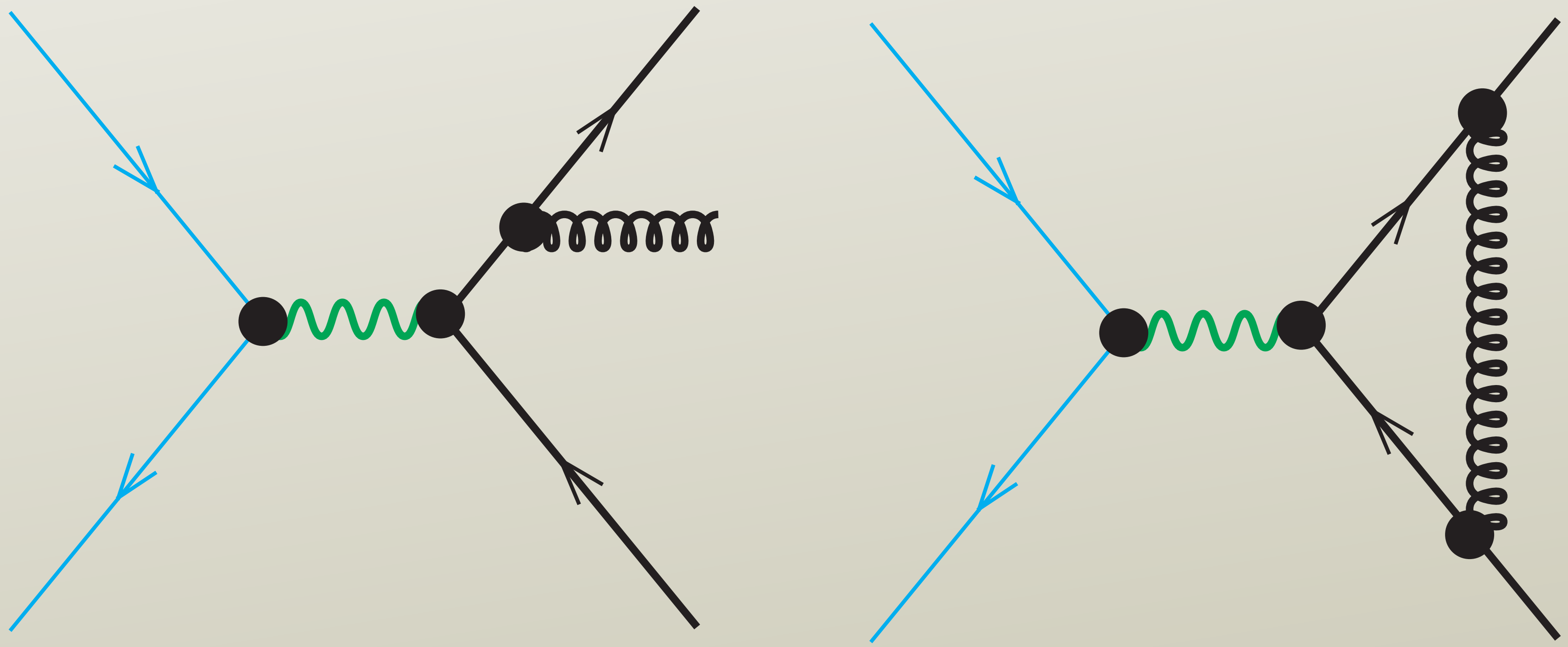


- *Caveat:* $F_m(p_1^\mu, \dots, p_m^\mu)$ should be smooth functions.

- For a physical event, infrared safety means that the actual event gives approximately the same result as when the hadrons in a jet are combined to make a few “parton jets”.



- For a calculated cross section, the infrared infinities cancel.



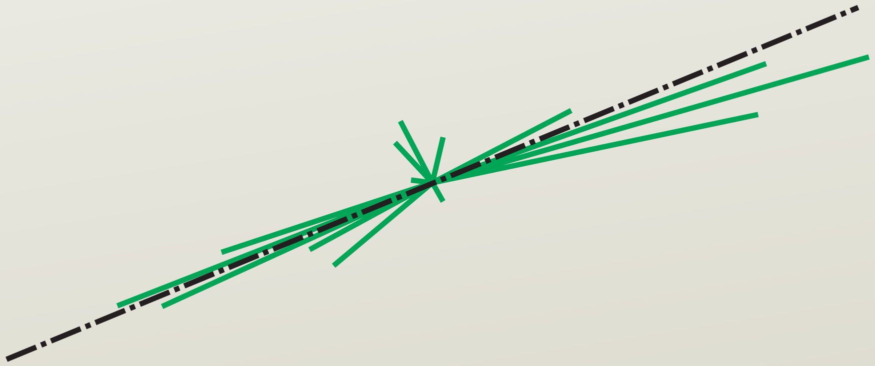
- The simplest example of an infrared safe observable in electron positron annihilation is the total cross section.

$$F_m(p_1^\mu, \ldots, p_m^\mu) = 1.$$

A more interesting example is the thrust distribution $d\sigma/dT$.

$$F_m(p_1^\mu, \ldots, p_m^\mu) = \delta\left(T - \mathcal{T}_m(p_1^\mu, \ldots, p_m^\mu)\right)$$

$$\mathcal{T}_m(p_1^\mu, \ldots, p_m^\mu) = \max_{\vec{u}} \frac{\sum_{i=1}^m |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^m |\vec{p}_i|}$$

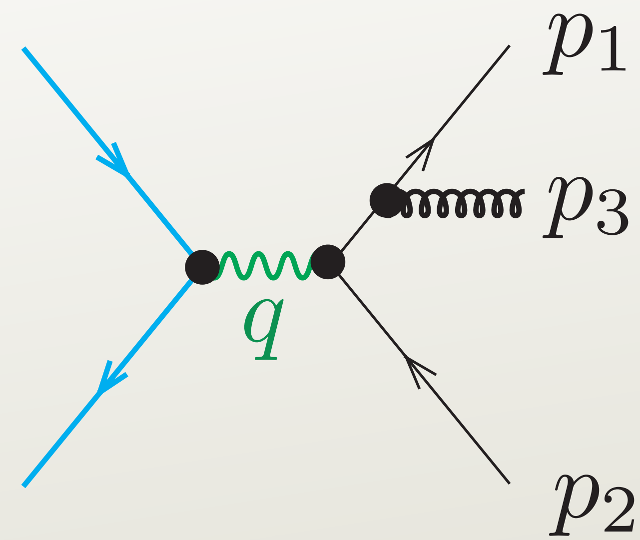


- Contribution from a particle with $\vec{p} = 0$ drops out.
- Replacing one parton by two collinear partons does not change T.

$$|(1-z)\,\vec{p}_m \cdot \vec{u}| + |z\,\vec{p}_m \cdot \vec{u}| = |\vec{p}_m \cdot \vec{u}|$$

$$|(1-z)\,\vec{p}_m| + |z\,\vec{p}_m| = |\vec{p}_m|$$

- Since the thrust distribution is infrared safe, we can calculate it in perturbation theory.



- At first order, one finds

$$\frac{d\sigma}{dT} = \left(\frac{4\pi\alpha^2}{Q^2}\sum e_f^2\right)\frac{C_{\rm F}\alpha_{\rm s}}{2\pi}$$
$$\times\left[\frac{3(2-T)(2-3T)}{1-T} + \frac{4-6T(1-T)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right)\right]$$

- $Q^2 = q^2 = (p_1+p_2+p_3)^2$.
- $C_{\rm F} = 4/3$.
- $T \to 1$ corresponds to two narrow jets.
- $d\sigma/dT$ is singular in this limit.

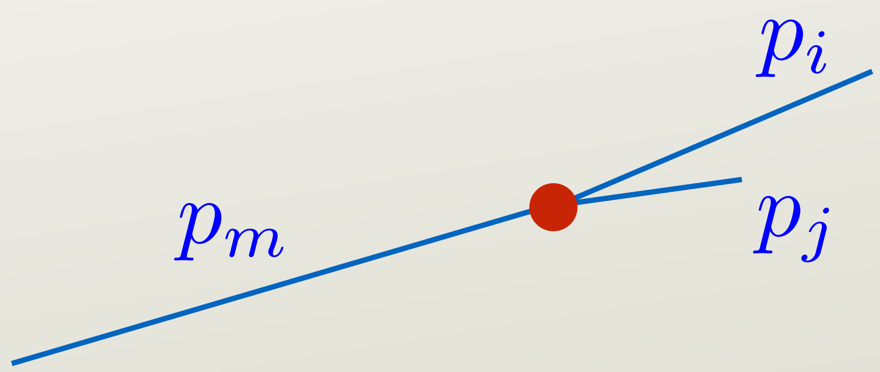
Quantitative measure of infrared safety

- Usually, infrared safety of an observable F is taken to be a "yes"-"no" question.
- But I find it helpful for an infrared safe observable to ask "how safe."
- We can say "safe at scale $Q^2(F)$."

See section III of Z. Nagy and D.E. Soper, Phys. Rev. D **98**, 014034 (2018).

- For partons i and j becoming collinear or one becoming soft,

$$p_i \to z p_m$$
$$p_j \to (1-z) p_m$$



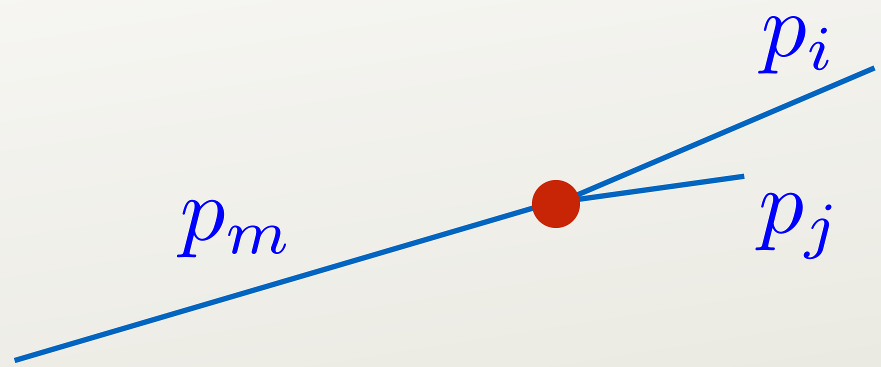
define a splitting scale

$$\mu_{ij}^2 = \frac{Q}{E_i + E_j}\,(p_i + p_j)^2$$

- Then the soft or collinear limit is $\mu_{ij}^2 \to 0$.

- There is a corresponding change δF in the observable F in the limit $\mu_{ij}^2 \to 0$.

$$\mu_{ij}^2 = \frac{Q}{E_i + E_j}\,(p_i + p_j)^2$$



- The change δF in F induces a change in the cross section, $\delta\sigma[F]$.

- We can define the infrared-safety scale $Q^2(F)$ by

$$\frac{1}{Q^2(F)} = \max\left[\frac{\delta\sigma[F]/\,\sigma[F]}{\mu_{ij}^2}\right]$$

- For thrust, one finds

$$\delta T \approx T \; \frac{\mu_{ij}^2}{Q^2}$$

- Then, since $\sigma[F] = d\sigma/dT$ is approximately $\sigma[T] \propto 1/(1-T)$, we have

$$\frac{\delta\sigma[F]}{\sigma[F]} \approx \frac{\delta T}{1-T} \approx \frac{T}{1-T} \; \frac{\mu_{ij}^2}{Q^2}$$

so

$$Q^2(F) \equiv \mu_{ij}^2 \; \frac{\sigma[F]}{\delta\sigma[F]} = \mu_{ij}^2 \; \frac{1-T}{T} \frac{Q}{\mu_{ij}^2} \approx \frac{1-T}{T} \; Q^2$$

- $Q^2(F)$ is of order Q^2 for $(1-T)$ not small but becomes small when $(1-T) \to 0$.

Review

- QCD Feynman diagrams are singular when any two partons become collinear or a gluon becomes soft.
- This property, with “singular” modified to “big” is the basis for parton shower Monte Carlos.
- Physically, it means that jets appear.
- The small virtuality splittings happen late.
- We can look at the small time physics by using infrared safe observables.
- We can define a scale for an infrared safe observable.

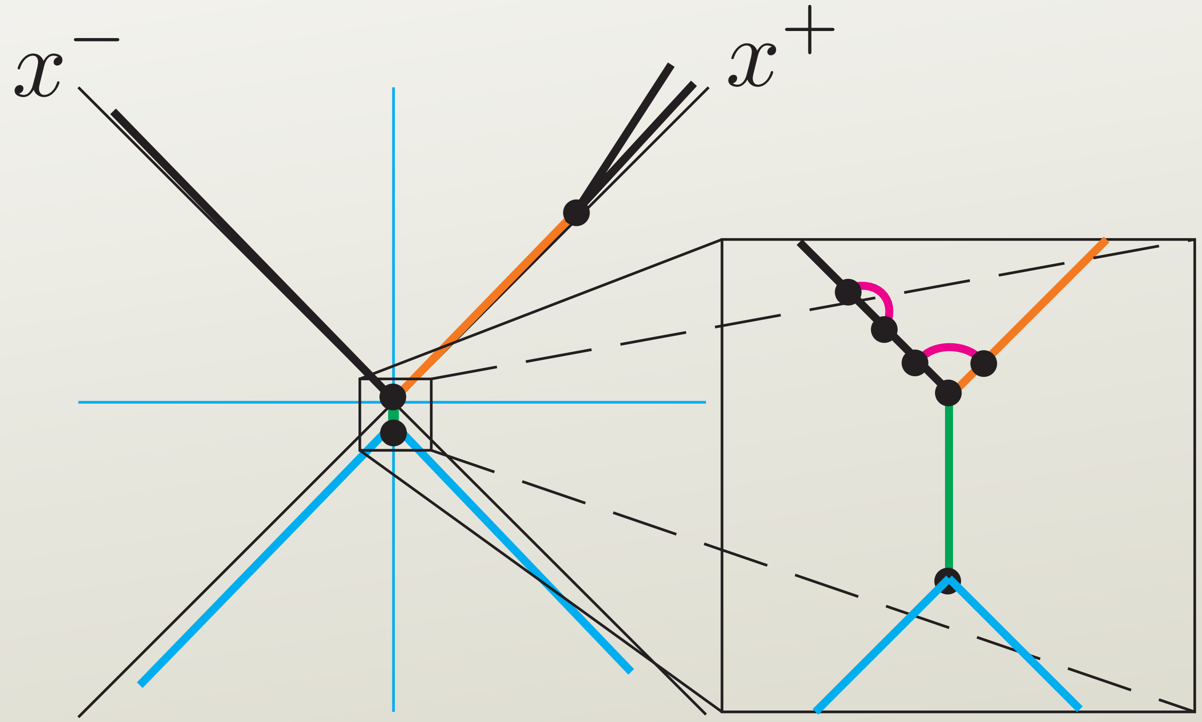
Renormalization and the running coupling

How quantum field theory hides the truth

Topics

- What renormalization does.
- The running coupling.
- The choice of scale.
- Beyond the Standard Model.

What renormalization does



- There are quantum fluctuations at very small distance scales.

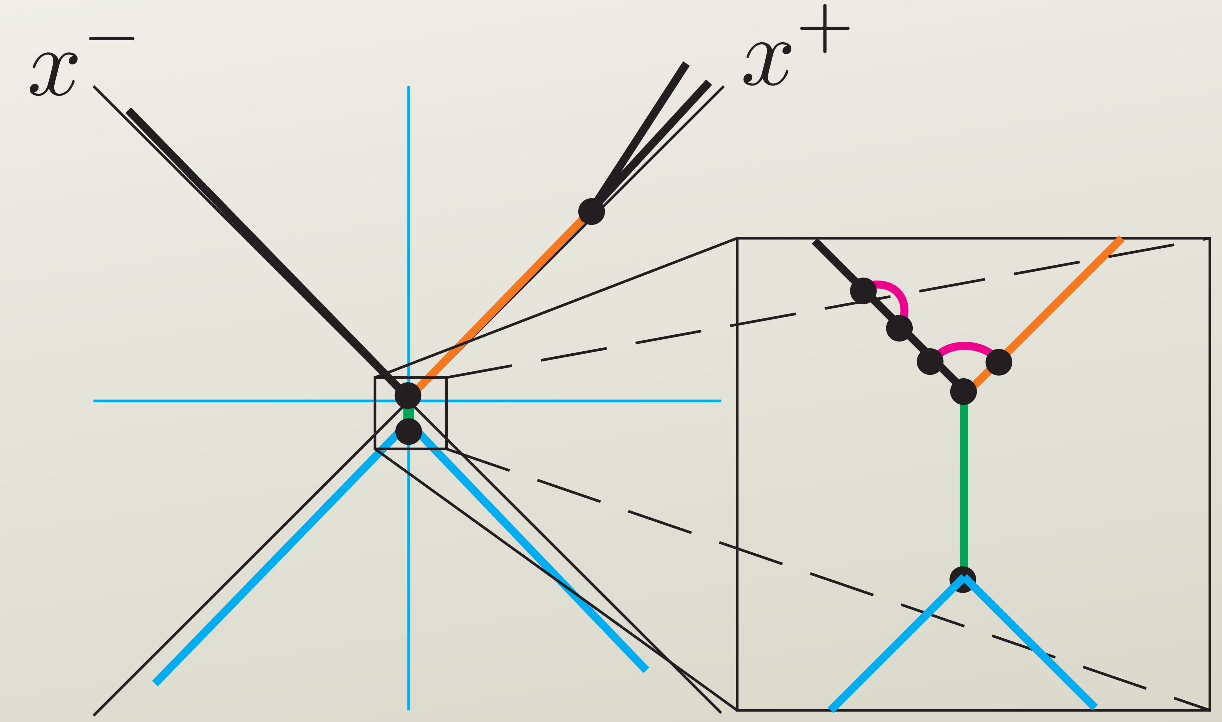
- They have a big effect.

- Renormalization accounts for their effect while eliminating the details below some scale.

Use $\overline{\text{MS}}$ renormalization with renormalization scale μ:

- Physics of time scales $|t| \ll 1/\mu$ removed from perturbative calculation.

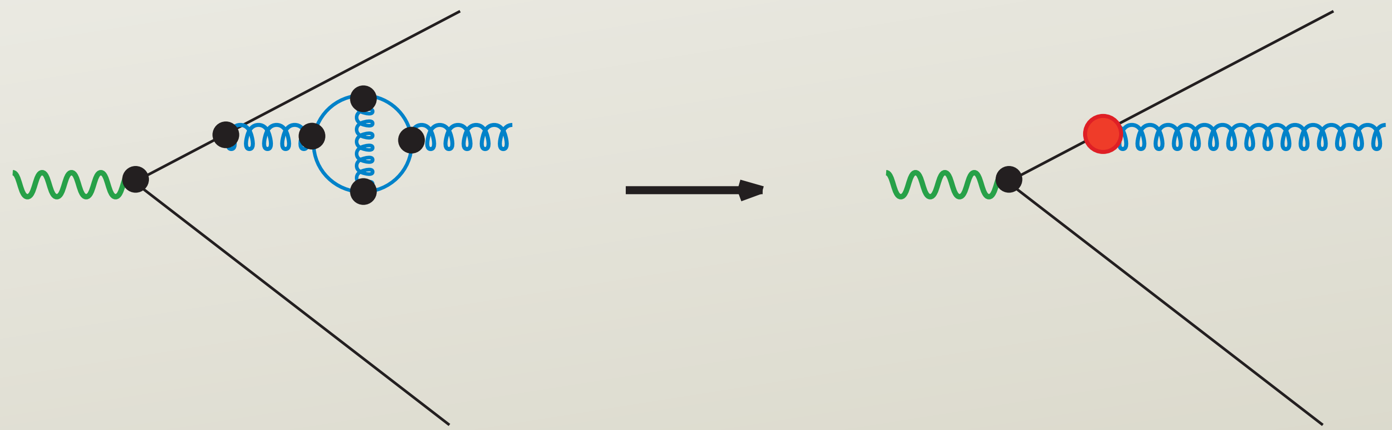
- Effect of small time physics accounted for by adjusting value of the coupling*: $\alpha_s = \alpha_s(\mu)$.



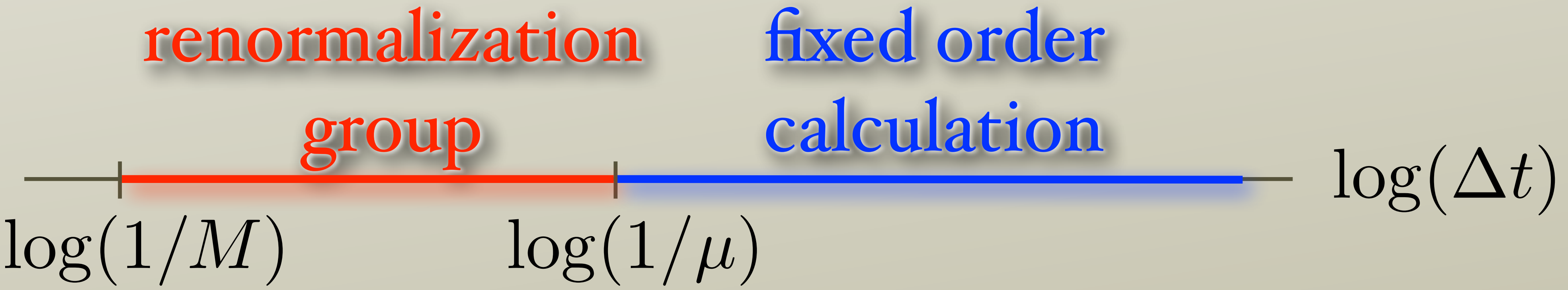
* This is not exactly the truth. There are also running masses $m(\mu)$ and there are μ dependent adjustments to the normalizations of the field operators. In addition, renormalization by dimensional regularization and minimal subtraction is not exactly the same as imposing a cutoff $|\Delta x| > 1/\mu$.

The running coupling

- We account for time scales much smaller than $1/\mu$ (but bigger than a cutoff M at the "GUT scale") by using the running coupling.



- This sums the effects of short time fluctuations of the fields.



Result of the one loop renormalization group equation:

$$\begin{aligned}\alpha_s(\mu) &\sim \alpha_s(M) - (\beta_0/4\pi)\log(\mu^2/M^2)\ \alpha_s^2(M) \\ &\qquad +(\beta_0/4\pi)^2\log^2(\mu^2/M^2)\ \alpha_s^3(M) + \cdots \\ &= \frac{\alpha_s(M)}{1+(\beta_0/4\pi)\alpha_s(M)\log(\mu^2/M^2)} \\ &= \frac{4\pi}{\beta_0\log(\mu^2/\Lambda^2)} \\ &= \frac{\alpha_s(M_Z)}{1+(\beta_0/4\pi)\alpha_s(M_Z)\log(\mu^2/M_Z^2)}\end{aligned}$$

The choice of scale

- Our example: $e^+e^- \to \gamma^* \to \text{hadrons}$

$$\sigma_{\text{tot}} = \frac{4\pi\alpha^2}{Q^2} \left(\sum e_f^2\right) [1 + \Delta]$$

$$\Delta(\mu) =$$

$$\frac{\alpha_s(\mu)}{\pi} + \left[1.4092 + 1.9167\ \log\left(\mu^2/Q^2\right)\right] \left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$

$$+ \left[-12.805 + 7.8179\ \log\left(\mu^2/Q^2\right) + 3.674\ \log^2\left(\mu^2/Q^2\right)\right]$$

$$\times \left(\frac{\alpha_s(\mu)}{\pi}\right)^3$$

$$+ \cdots$$

$$
\begin{aligned}
\Delta(\mu) \quad = \quad & \frac{\alpha_s(\mu)}{\pi} + \left[1.4092 + 1.9167\ \log\left(\mu^2/Q^2\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \\
& + \left[-12.805 + 7.8179\ \log\left(\mu^2/Q^2\right) + 3.674\ \log^2\left(\mu^2/Q^2\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \\
& + \cdots
\end{aligned}
$$

- The coefficients depend on μ.
- $\alpha_s(\mu)$ depends on μ.
- The "exact" Δ does not depend on μ.
- The more terms we have, the less μ dependence there is.

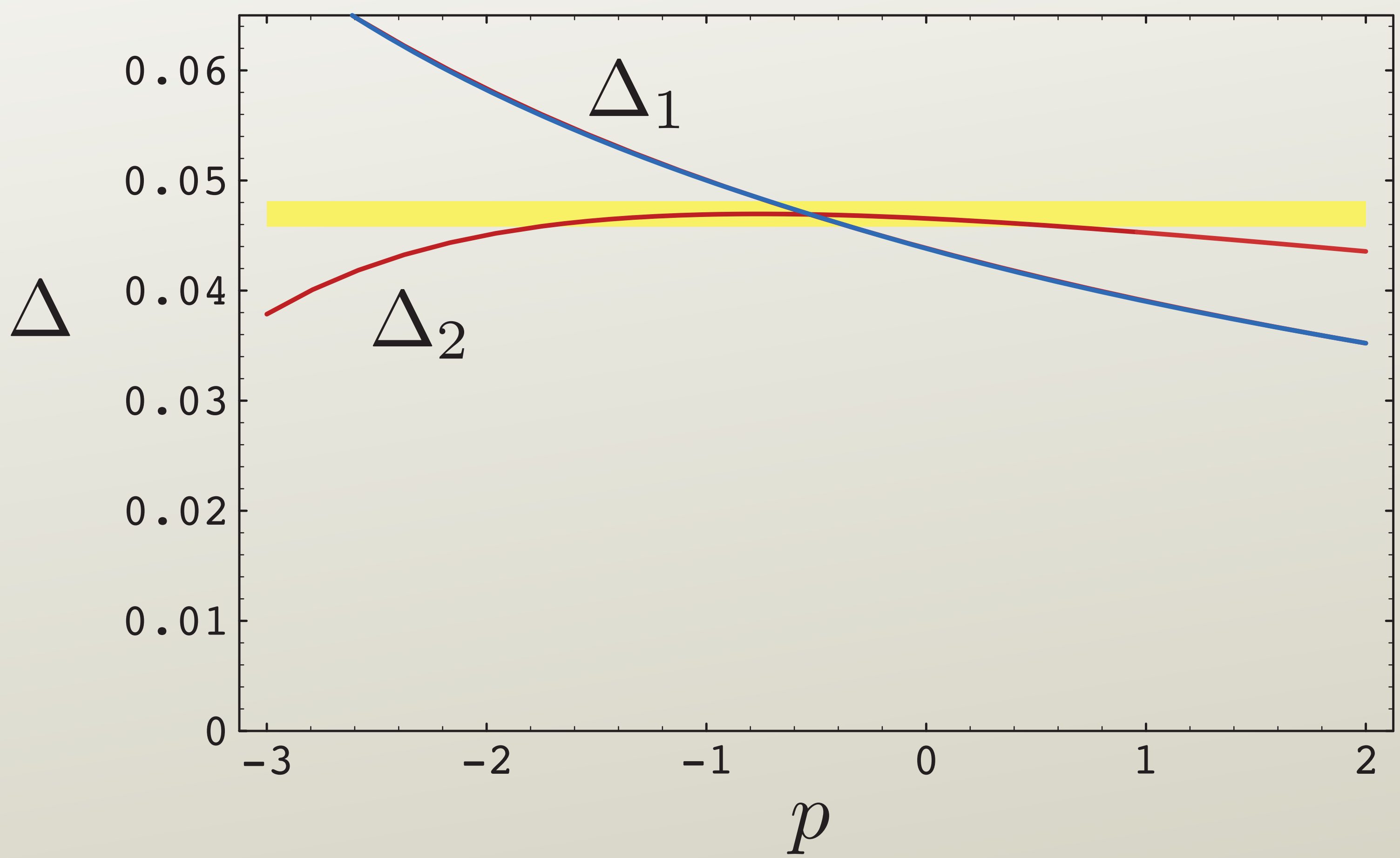
$$
\frac{d}{d\log\mu}\sum_{n=1}^{N} c_n(\mu)\ \alpha_s(\mu)^n \sim \mathcal{O}(\alpha_s(\mu)^{N+1})
$$

$$
\begin{aligned}
\Delta(\mu) \quad = \quad & \frac{\alpha_s(\mu)}{\pi} + \left[1.4092 + 1.9167\ \log\left(\mu^2/Q^2\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \\
& + \left[-12.805 + 7.8179\ \log\left(\mu^2/Q^2\right) + 3.674\ \log^2\left(\mu^2/Q^2\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \\
& + \cdots
\end{aligned}
$$

- What scale should you choose?
- Clearly a scale μ much different from Q is not good.
- How could you estimate the error if you know only the α_s^2 terms?
- Let's look at these questions numerically.

I take $\alpha_s(M_Z) = 0.117$, $Q = 34$ GeV, 5 flavors of quarks.

I plot $\Delta(\mu)$ versus p defined by $\mu = 2^p Q$.

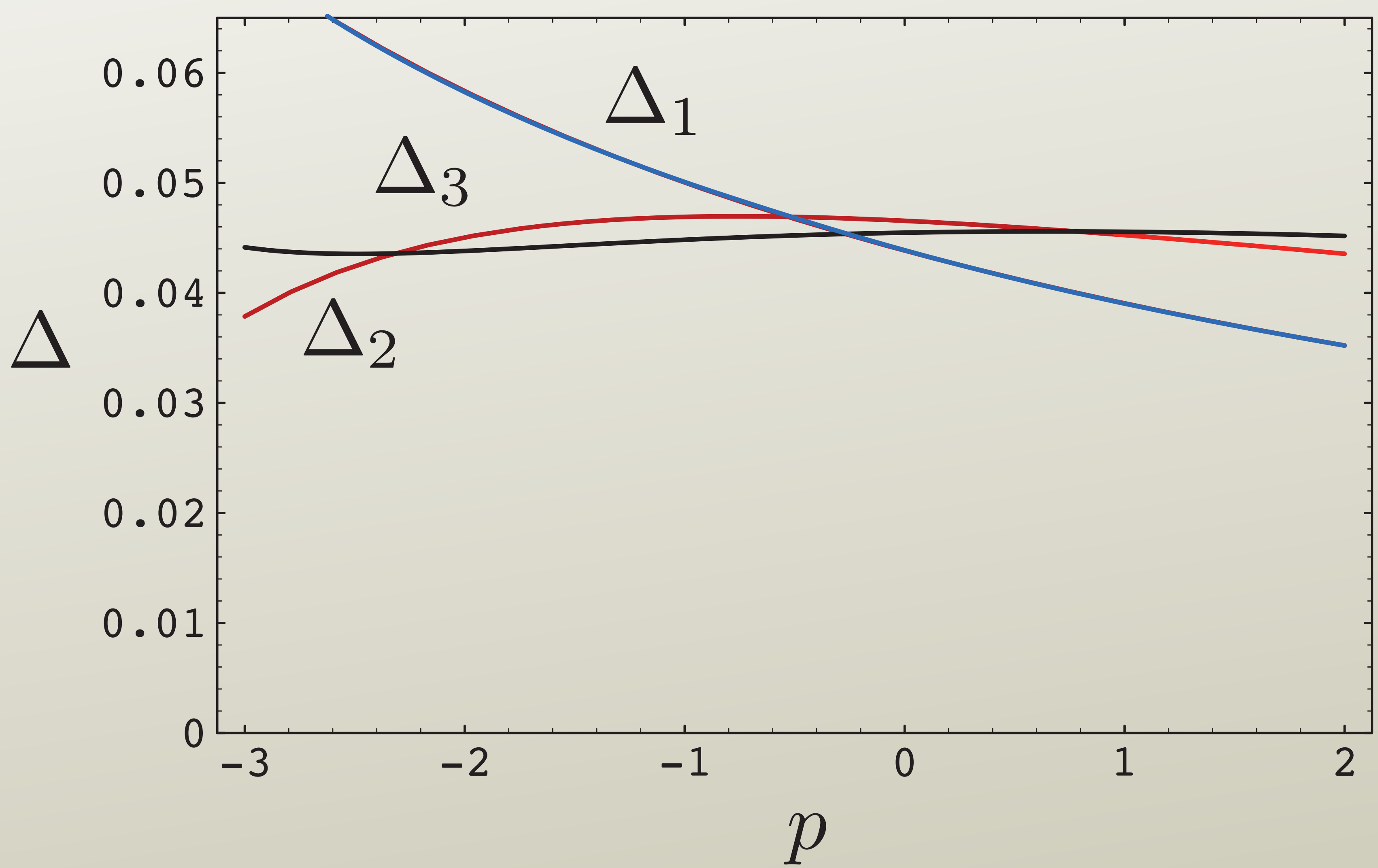


- Δ_1 includes one term.
- Δ_2 includes two terms.

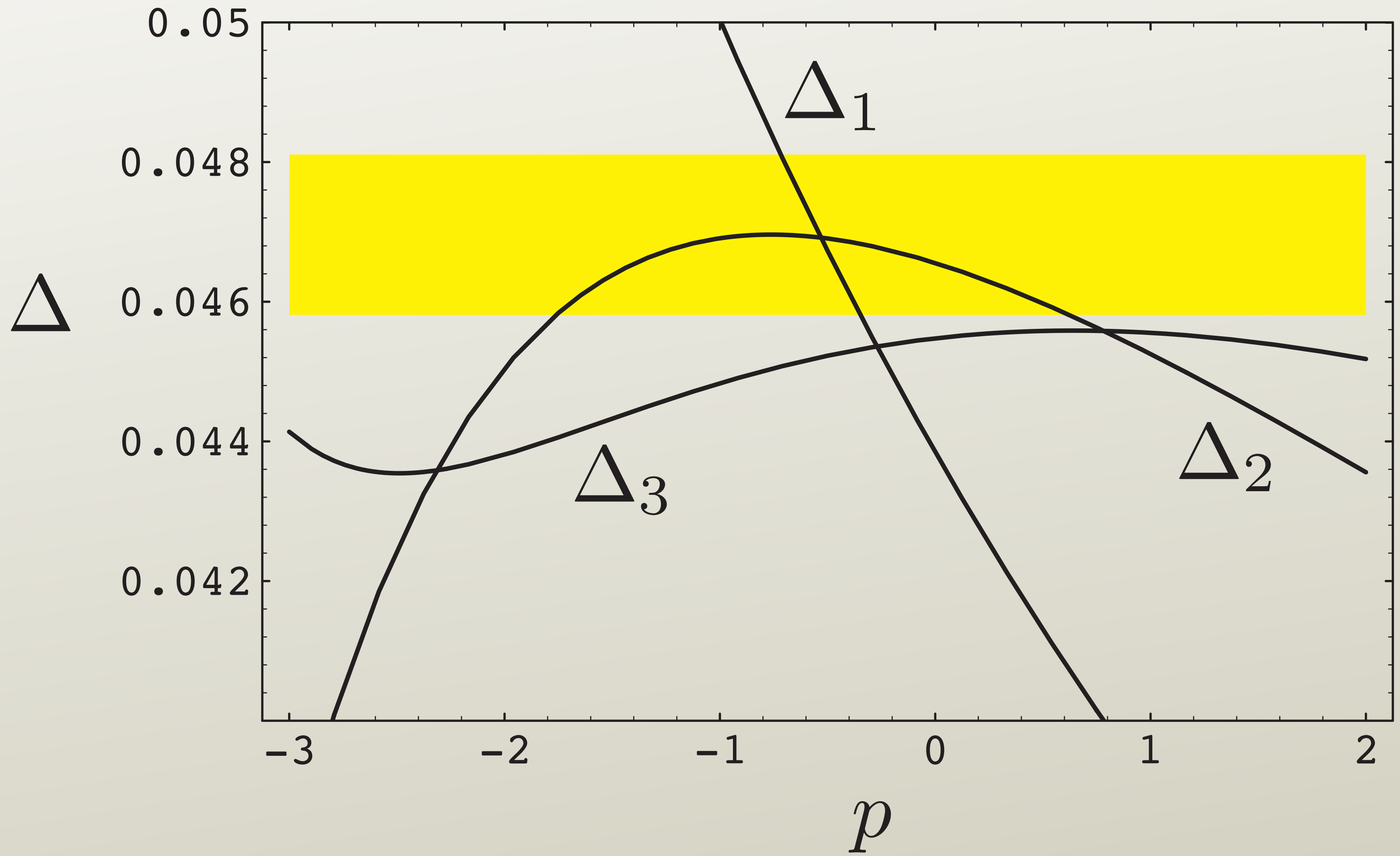
- Possible choice: "principle of minimal sensitivity" point $\hat{\mu}$ where Δ_2 is flat.

- Error band estimated using $\mu = 2\hat{\mu}$ or $\mu = \hat{\mu}/2$.

- One more order.

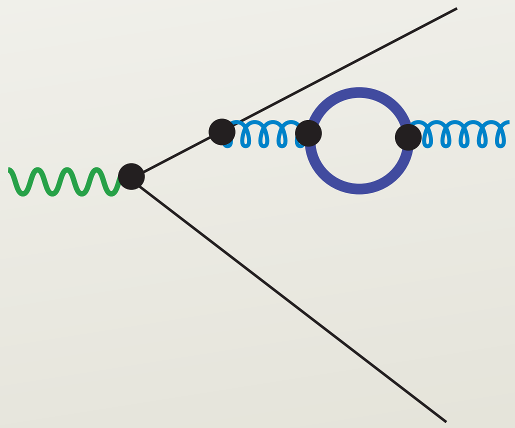


- Magnified view.



- Were the value and the error estimate about right?

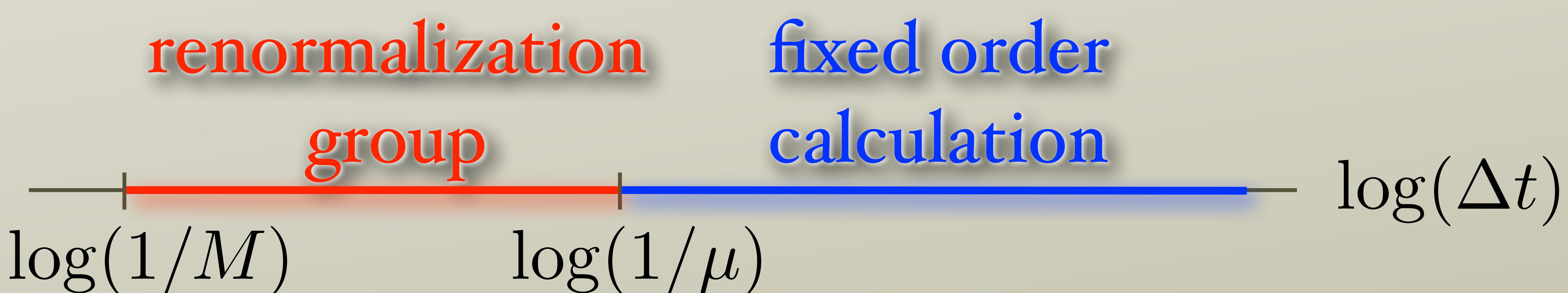
Beyond the Standard Model



- If we knew about new very heavy particles and interactions, we should put that into loop graphs.
- The new physics then affects predictions for lower energies.

- Unfortunately, the effect of the new physics is to change the value of the couplings and masses of the Standard Model.

- That is, the new physics provides the initial conditions for the renormalization group equations.



- A really good theory would predict the couplings and masses of the standard model.
- So far, that hasn't happened.
- Except for that possibility, and one more, the secrets of very short distance physics are pretty well hidden from us until we have enough energy to directly probe the small times.

- The exception: new short time physics can introduce small new terms in the "effective lagrangian" such as

$$\Delta\mathcal{L} = \frac{\tilde{g}^2}{M^2}\ \bar{\psi}\gamma^\mu\psi\ \psi\gamma_\mu\psi$$

- Such a term can lead to an effect that is small, but visible with a precision experiment.

- An especially good place to look is in signals that are forbidden by symmetries of the standard model.

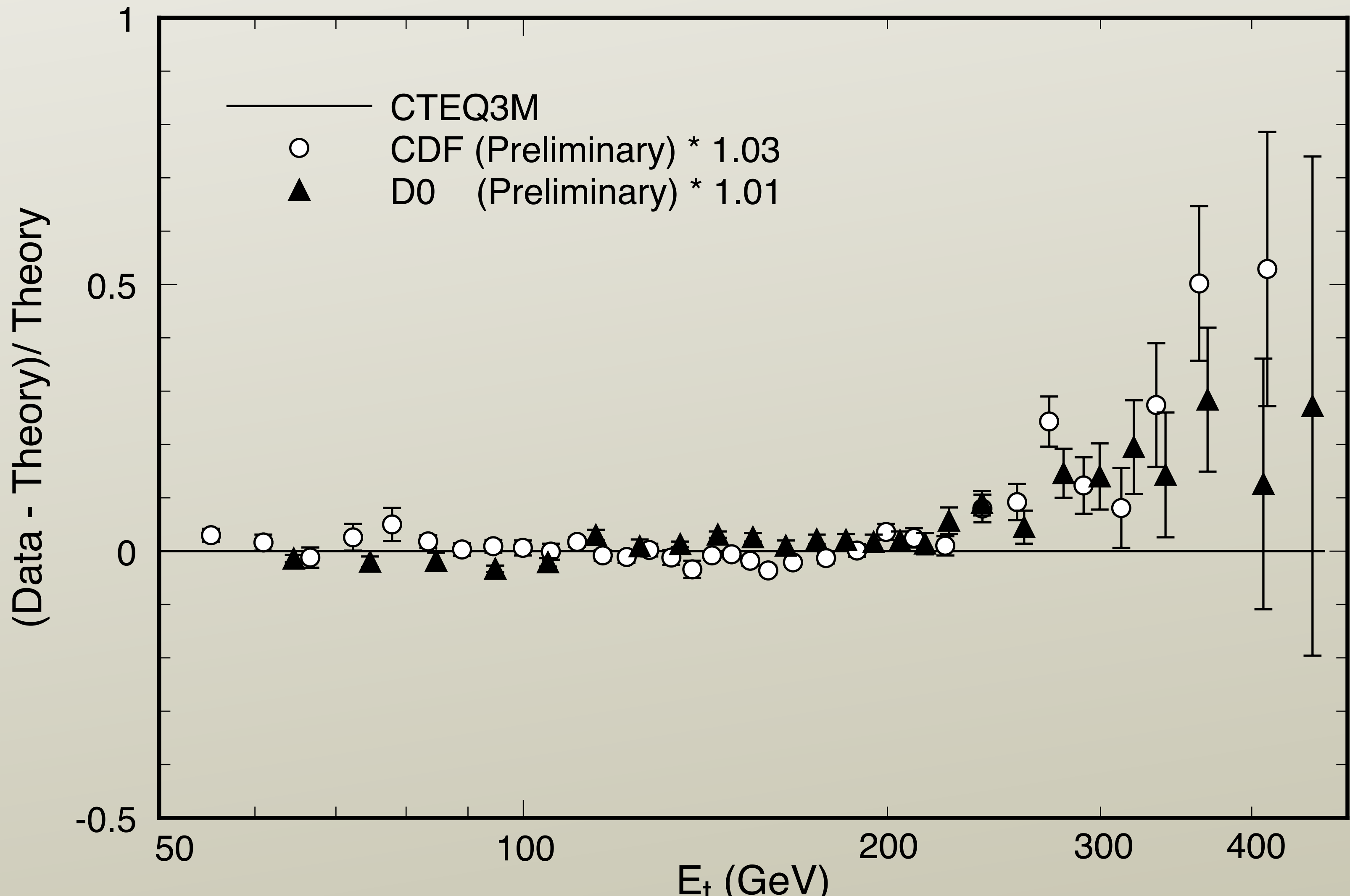
- Additionally, the effect can grow with energy.

- A systematic examination of this is the domain of *effective field theory.*

- An example is the cross section for jet production at a hadron collider:

$$\frac{\text{data} - \text{theory}}{\text{theory}} \propto \frac{\tilde{g}^2 \; E_T^2}{M^2}$$

- These data were eventually explained by something else, but illustrate what to look for.



Review

- Loop graphs "know" about physics at very small time scales.

- We remove these effects below a time scale $1/\mu$ from perturbative graphs and incorporate them into the couplings, eg. $\alpha_\mathrm{s}(\mu)$.

- One chooses μ to be on the order of the physical scale of the problem.

- The effects of very small time scale physics are mostly hidden.

Deeply inelastic scattering

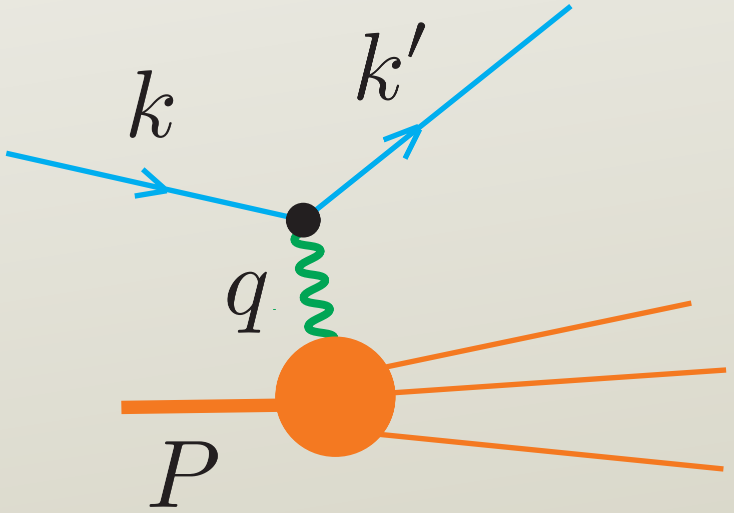
The effect of the initial state

Topics

- Kinematics of deeply inelastic scattering (DIS).
- The space-time picture.
    - Small x.
- Partons.
- The factored cross section.

Kinematics of DIS

- The process is $l + h \to l' + X$.



$$Q^2 = -q^2$$
$$x_{bj} = \frac{Q^2}{2P \cdot q}$$

$$\text{“deeply inelastic”} \implies Q^2 \to \infty,\ x \text{ fixed}$$

$$W^2 = (P+q)^2 = m_h^2 + \frac{1-x}{x} Q^2$$
$$y = \frac{P \cdot q}{P \cdot k}.$$

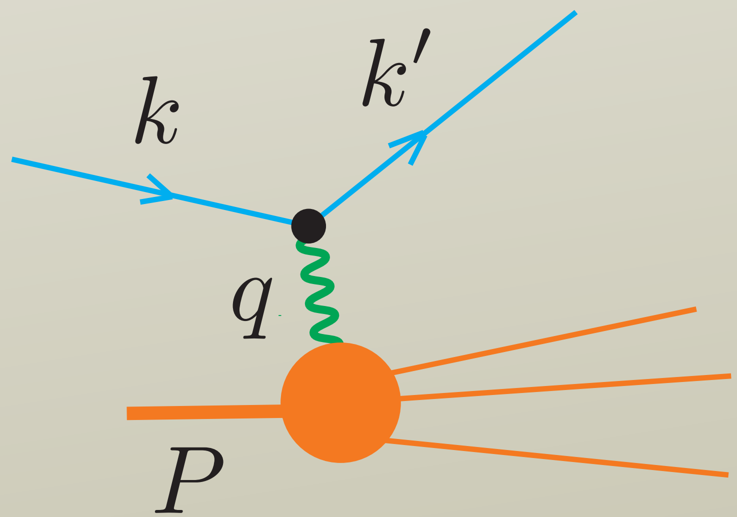
Structure functions

- This standard analysis requires only electroweak physics and symmetries, not QCD.

- I simplify a little by including just Z exchange.

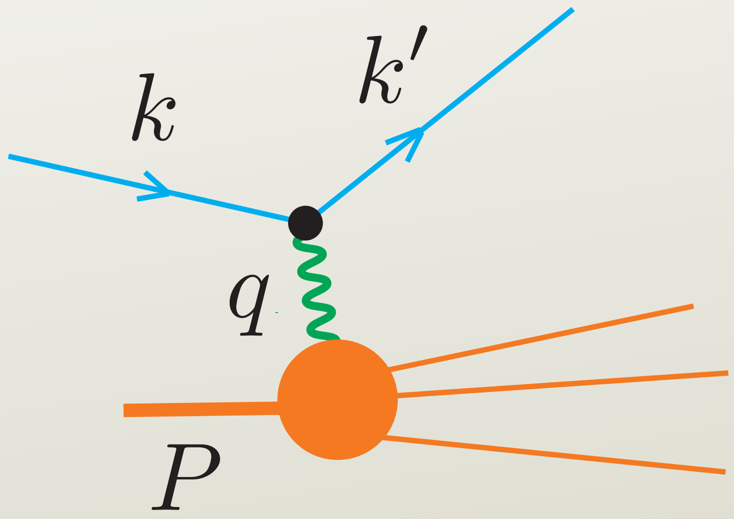
$$d\sigma = \frac{4\alpha^2}{s}\frac{d^3\vec{k}'}{2|\vec{k}'|}\frac{1}{(q^2-M^2)^2}\, L^{\mu\nu}(k,q)\, W_{\mu\nu}(P,q).$$

$$L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left(k\cdot\gamma\,\Gamma^\mu k'\cdot\gamma\,\Gamma^\nu\right).$$



$$d\sigma = \frac{4\alpha^2}{s}\,\frac{d^3\vec{k}'}{2|\vec{k}'|}\,\frac{1}{(q^2 - M^2)^2}\,L^{\mu\nu}(k,q)\,W_{\mu\nu}(P,q).$$

The structure of $W_{\mu\nu}$:



$$\begin{aligned} W_{\mu\nu} = &-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) \\ &+\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right)\frac{1}{P\cdot q}\, F_2(x,Q^2) \\ &-i\epsilon_{\mu\nu\lambda\sigma}P^\lambda q^\sigma \frac{1}{P\cdot q}\, F_3(x,Q^2). \end{aligned}$$

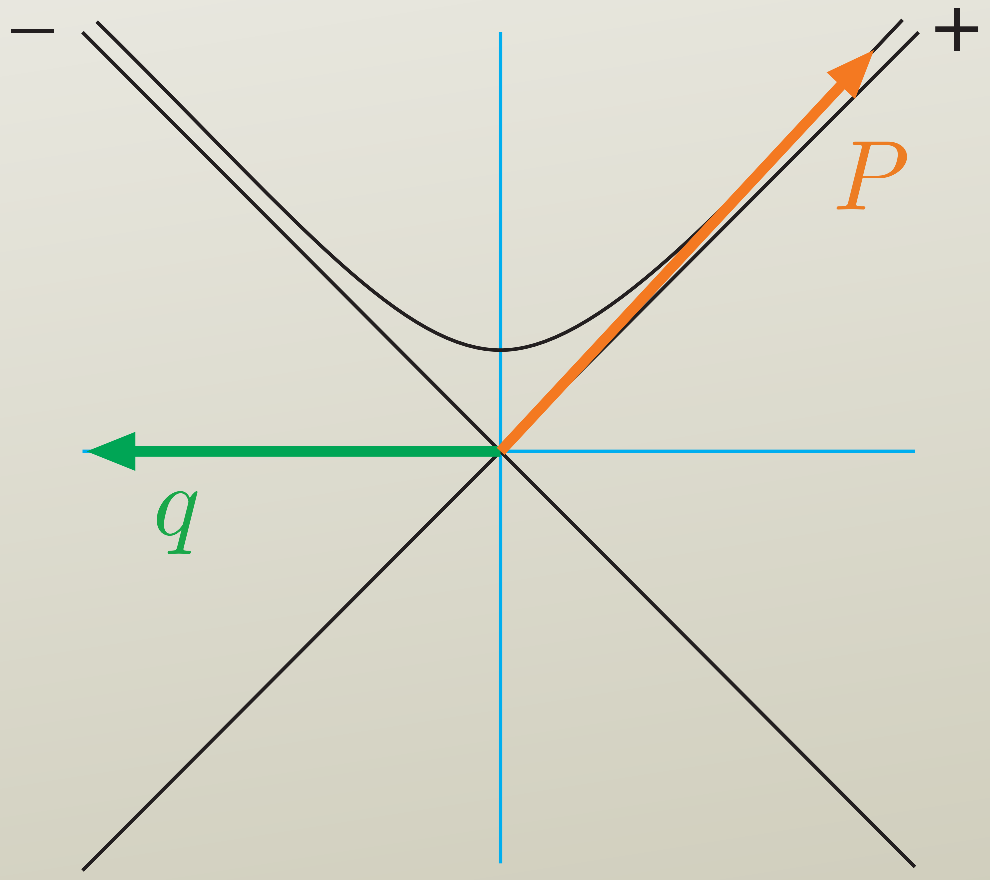
The functions F_1, F_2 and F_3 are the structure functions.

Space-time picture of DIS

- A convenient reference frame is

$$(q^+, q^-, \boldsymbol{q}) = \frac{1}{\sqrt{2}} \left(-Q, Q, \mathbf{0}\right)$$

$$(P^+, P^-, \boldsymbol{P}) \approx \frac{1}{\sqrt{2}} \left(\frac{Q}{x}, \frac{x m_h^2}{Q}, \mathbf{0}\right)$$



- Hadron momentum is big; momentum transfer is big.

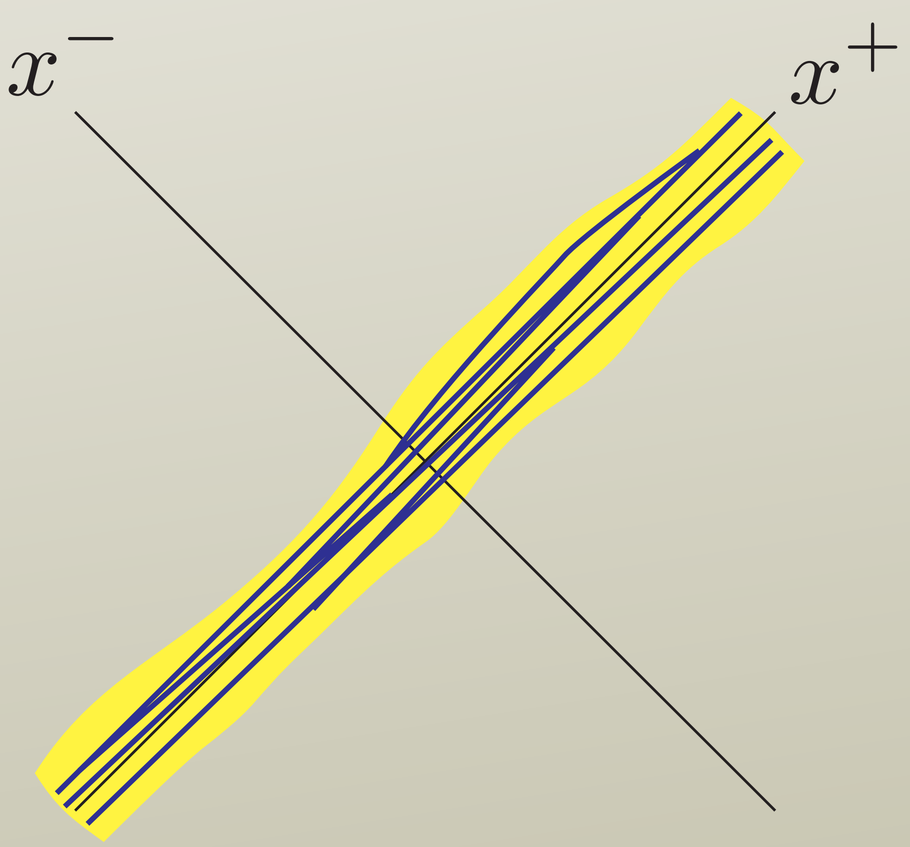
- Picture for a fast moving hadron is Lorentz transformation from rest frame picture.

$$e^{\omega} = \frac{P^+}{P^+_{\rm rest}} = \frac{Q}{mx}$$

- Separations Δx^μ between interactions:

$$\Delta x^+ \sim \frac{1}{m} \times \frac{Q}{mx} = \frac{Q}{m^2 x}$$

$$\Delta x^- \sim \frac{1}{m} \times \frac{mx}{Q} = \frac{x}{Q}$$



- The hadron meets the virtual photon.

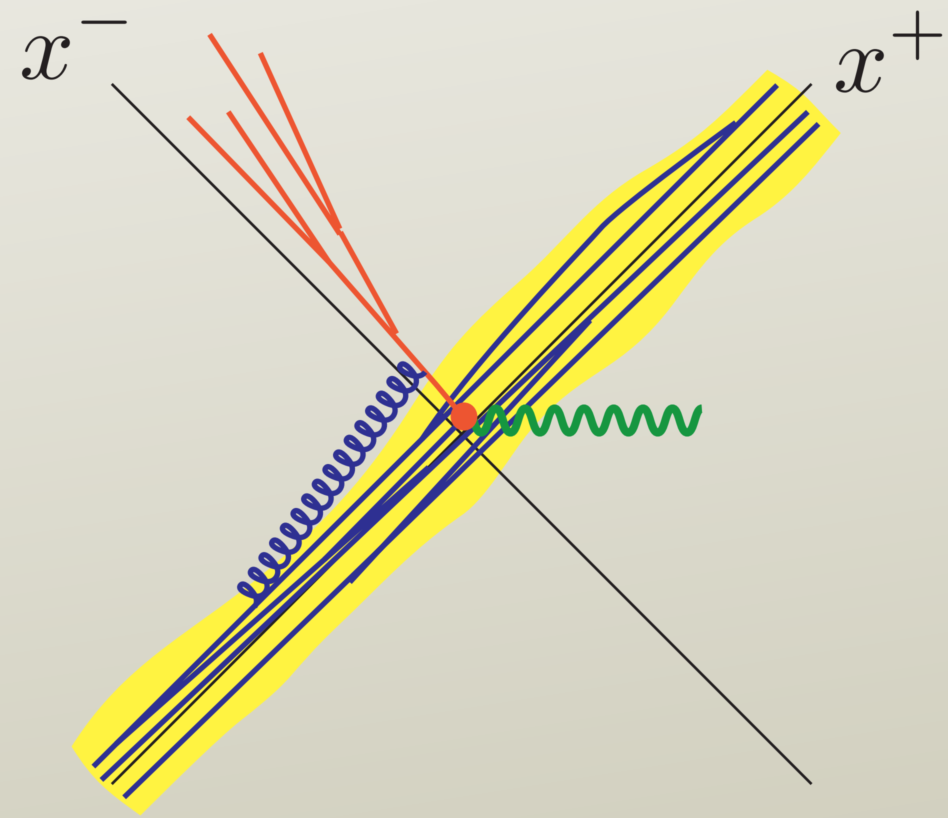
- Virtual photon has $q^- \sim Q$ so its interaction takes place over

$$\Delta x^+ \sim 1/Q$$

- But interactions in the proton happen at a scale

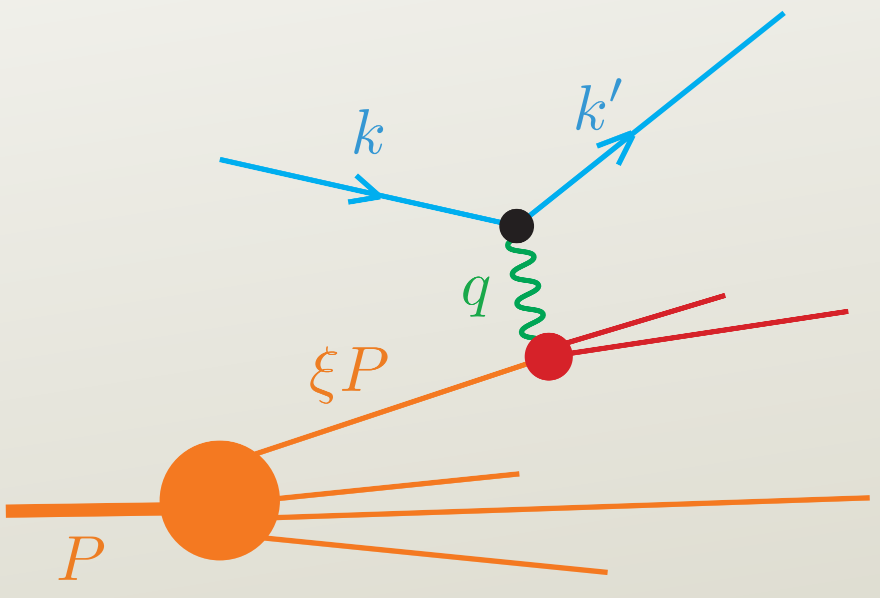
$$\Delta x^+ \sim Q/(m^2 x)$$

- so the "partons" in the hadron are effectively free as seen by the virtual photon.



Factored cross section

- This picture gives

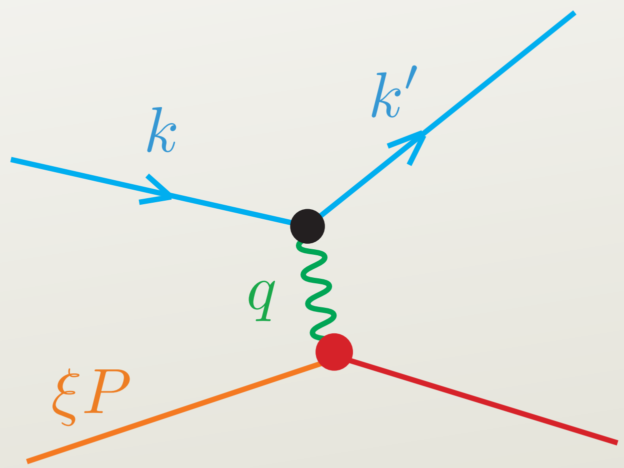


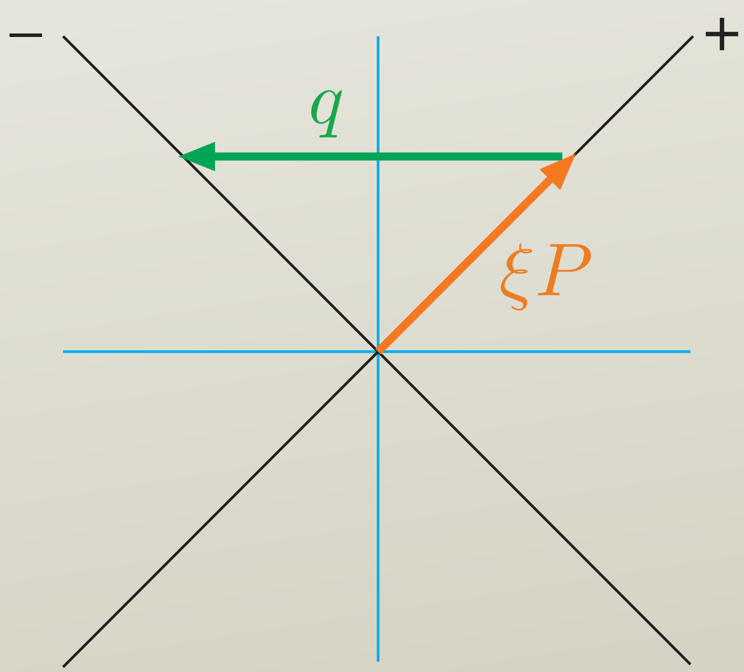
$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

$f_{a/h}(\xi,\mu)\; d\xi =$ probability to find a parton
with flavor $a = g, u, \bar{u}, d, \ldots,$
in hadron h,
carrying momentum fraction $\xi = p_i^+/p^+$.

$d\hat{\sigma}_a/dE'\,d\omega' =$ cross section for scattering that parton.

Kinematics of the leading order diagram





$$\xi P^+ + q^+ = 0$$

$$P^+ = \frac{Q}{x\sqrt{2}}$$

$$q^+ = -\frac{Q}{\sqrt{2}}$$

So

$$\boxed{\xi = x}$$

- Consequence of $\xi = x$ at lowest order:

$$F_2(x, Q^2) \sim \sum_a Q_a^2 \; x\, f_{a/h}(x, \mu) + \mathcal{O}(\alpha_s) + \mathcal{O}(m/Q)$$

- At higher orders, this is more complicated:

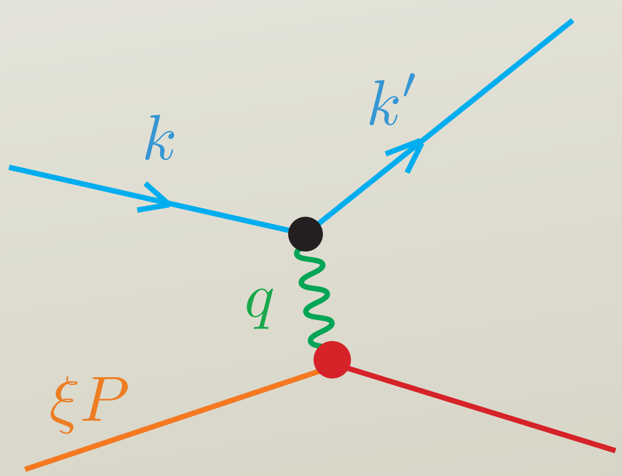
$$F_2(x.Q^2) \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi, \mu) \hat{F}_2^a(x/\xi, Q^2/\mu^2) + \mathcal{O}(m/Q)$$

- We can find out what $\hat{F}_2^a$ is by considering the cross section.

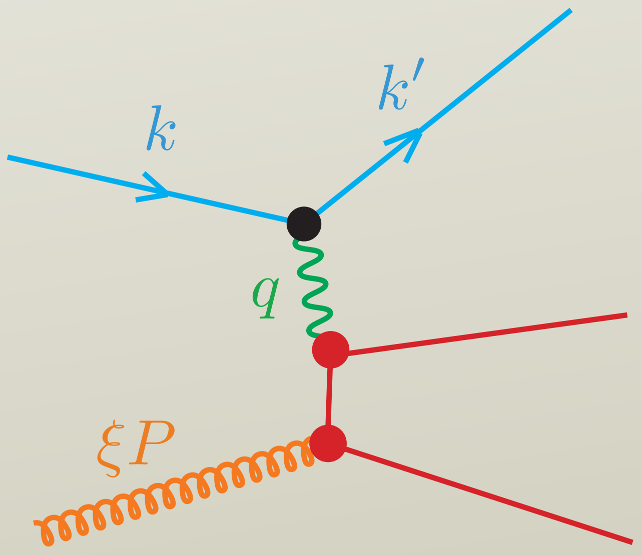
The hard scattering cross section

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

To calculate $d\hat{\sigma}_a(\mu)/dE'\,d\omega'$ use diagrams like

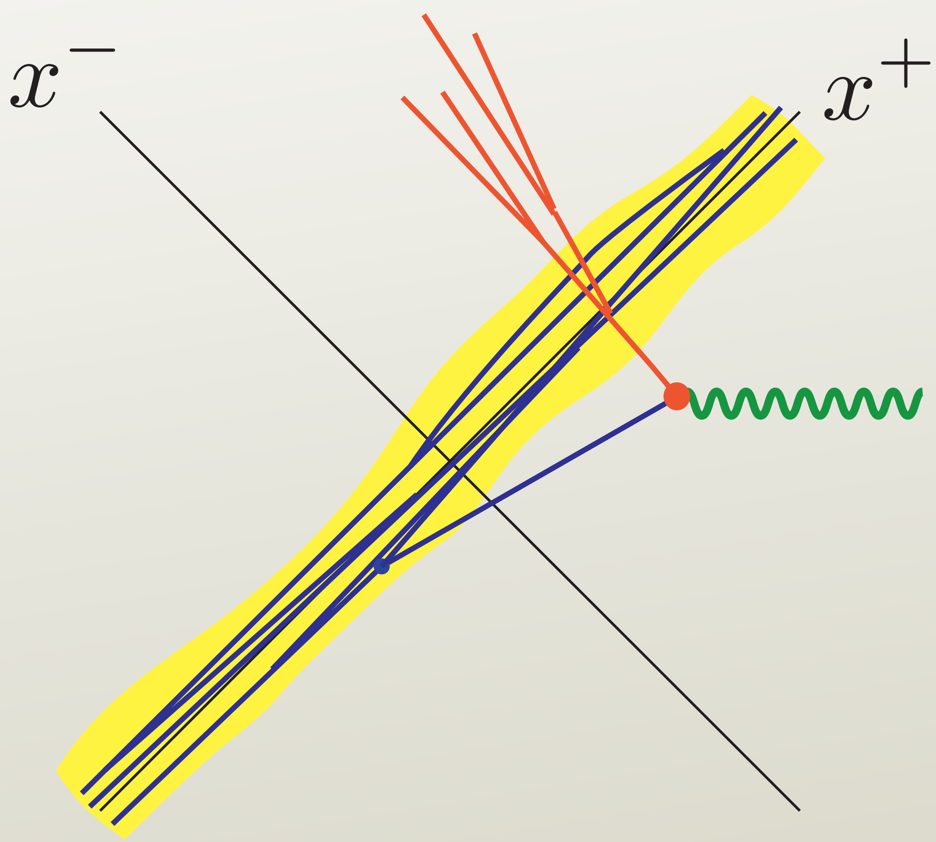


leading order

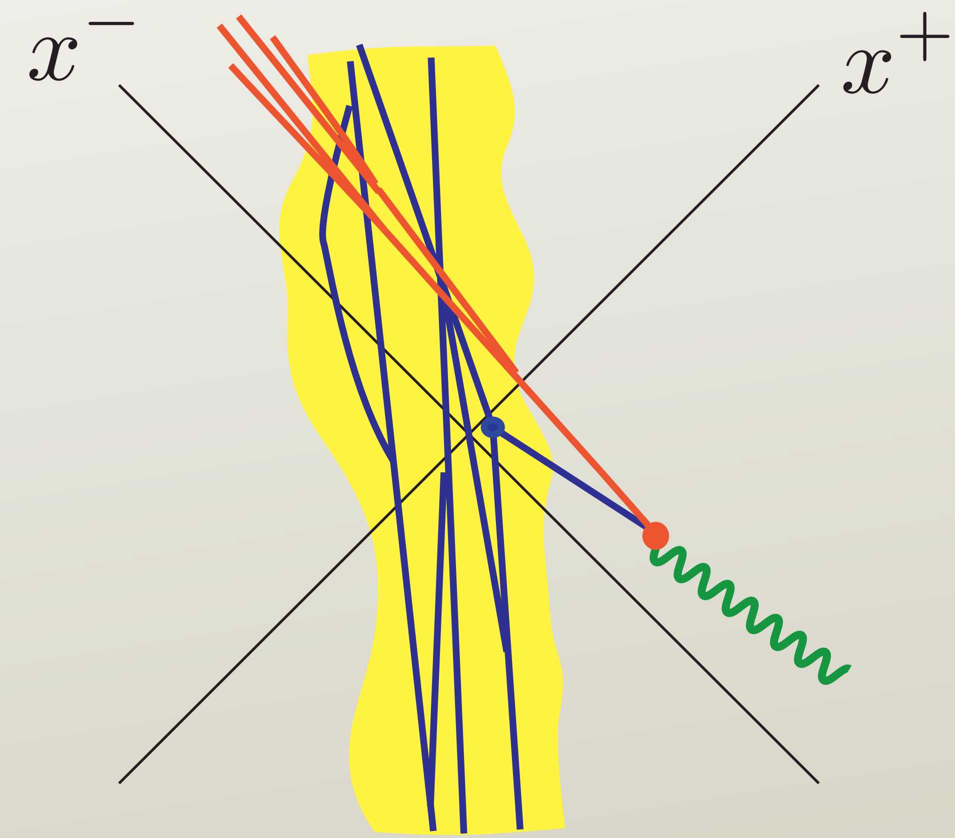


higher order

- How this looks depends on the reference frame.



Breit frame



proton rest frame

- In the proton rest frame, the photon vertex is first.
- This gives the dipole picture.

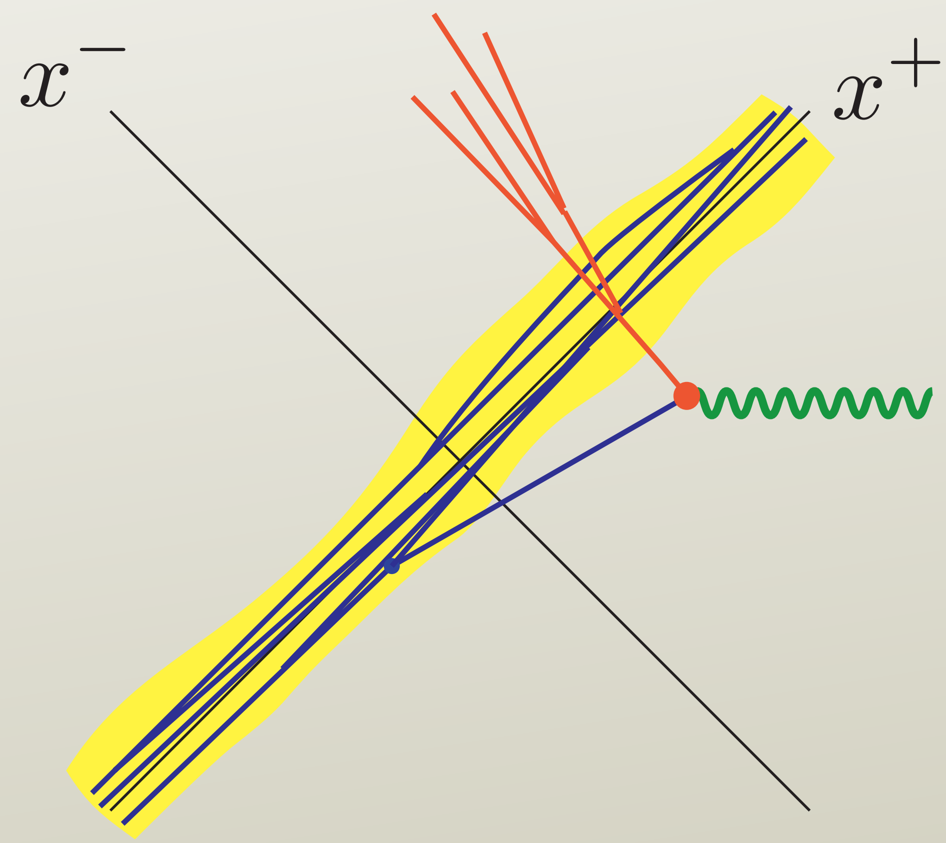
Space-time picture at small x

- Size of proton

$$\Delta y \approx \left(\frac{Q}{xm^2}, \frac{x}{Q}, \frac{1}{m}, \frac{1}{m} \right)$$

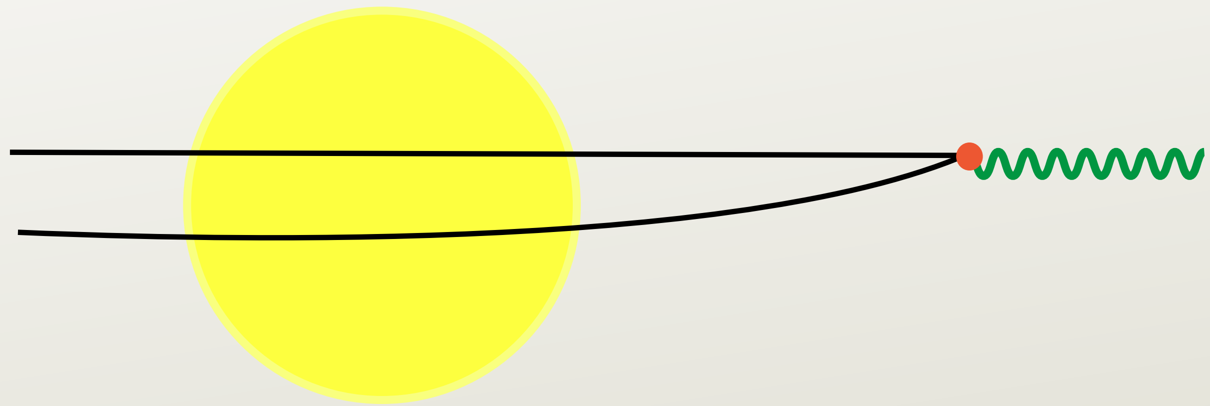
- Size of interaction point

$$\Delta \tilde{y} \approx \left(\frac{1}{Q}, \frac{1}{Q}, \infty, \infty \right)$$

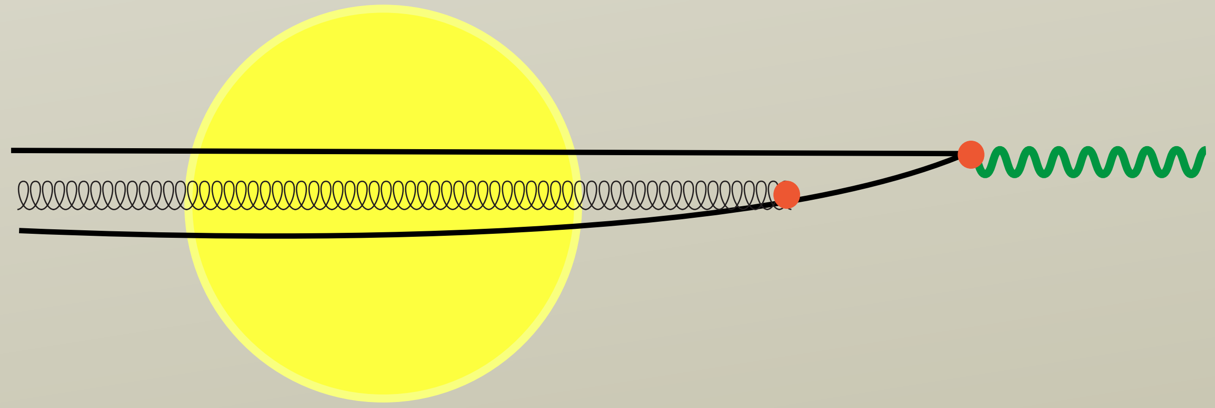


- For very small x, the point where the quark is destroyed is far outside the normal size of the proton.

The dipole picture in the proton rest frame



- The virtual photon creates a quark-antiquark dipole that shoots through the proton.
- The dipole can develop further.
- Whatever happens, it has more to do with the structure of QCD than with the structure of the proton.



Partons and renormalization

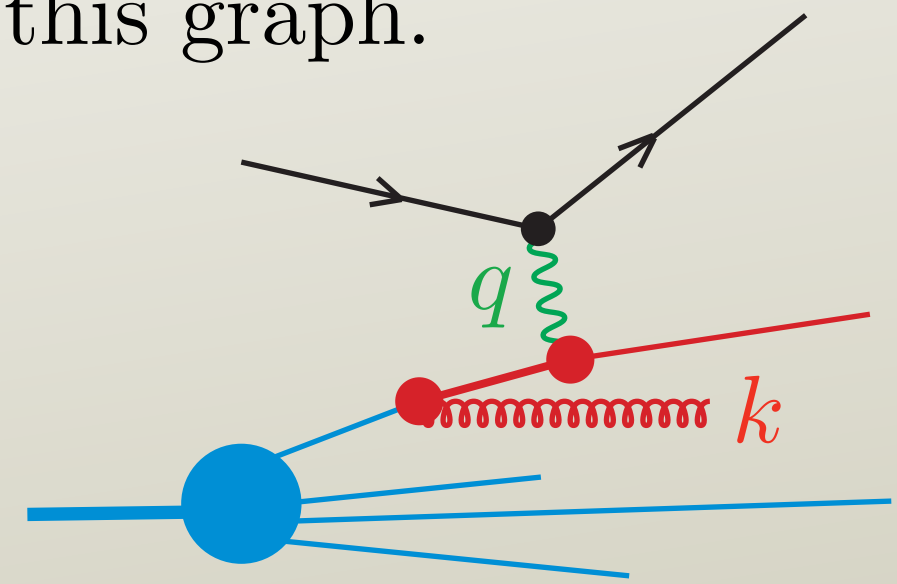
What the proton looks like depends on the resolution of your experiment

Topics

- The factorization scale.
- The definition of the parton distribution functions.
- Evolution of the parton distribution functions.
- Fitting the parton distribution functions.

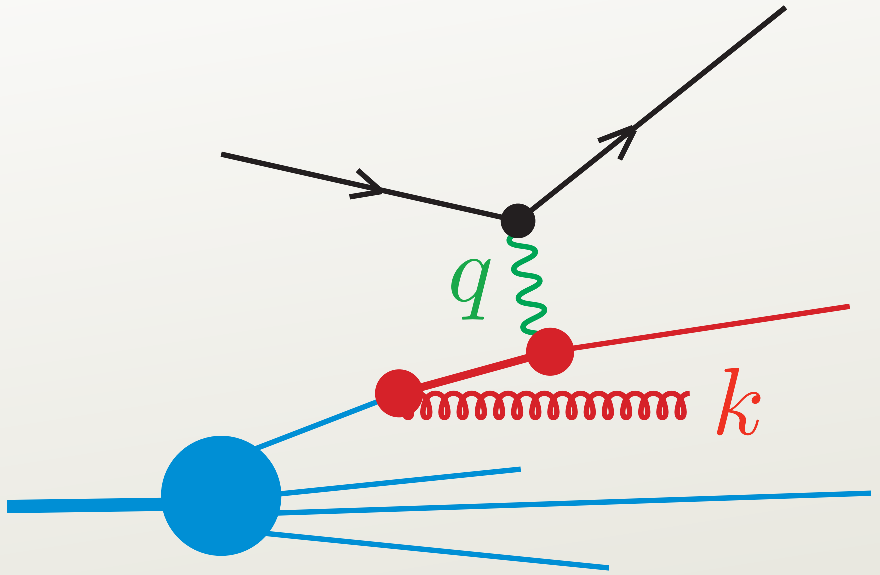
The factorization scale

- We argued that $\Delta x^+ \sim Q/(xm^2) \gg 1/Q$.
- Thus we regarded the partons as "frozen."
- But look at this graph.



- Integration over k maps to integration over Δx^+
- $1/Q \lesssim \Delta x^+ \lesssim Q/(xm^2)$.
- So we were wrong.

- Solution: divide up the integration region.



perturbative calculation **parton distribution function**

$\log(1/Q)$ $\log(1/\mu_\mathrm{F})$ $\log(\Delta x^+)$

- We call μ_F the factorization scale.

- Both $f_{a/h}(\xi, \mu_\mathrm{F})$ and $d\hat{\sigma}_a(\mu_\mathrm{F}, \mu)$ depend on μ_F.

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi, \mu_\mathrm{F}) \, \frac{d\hat{\sigma}_a(\mu_\mathrm{F}, \mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu_{\rm F})\; \frac{d\hat\sigma_a(\mu_{\rm F},\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

- Both $f_{a/h}(\xi,\mu_{\rm F})$ and $d\hat\sigma_a(\mu_{\rm F},\mu)$ depend on $\mu_{\rm F}$.
- Also, $d\hat\sigma_a(\mu_{\rm F},\mu)$ depends on the renormalization scale μ.
- The higher the order of perturbation theory that we use the smaller is the dependence on the scales.
- This applies also in hadron-hadron collisions.
- As an example, look at the one jet inclusive cross section

$$\frac{d\sigma}{dE_T\,dy}$$

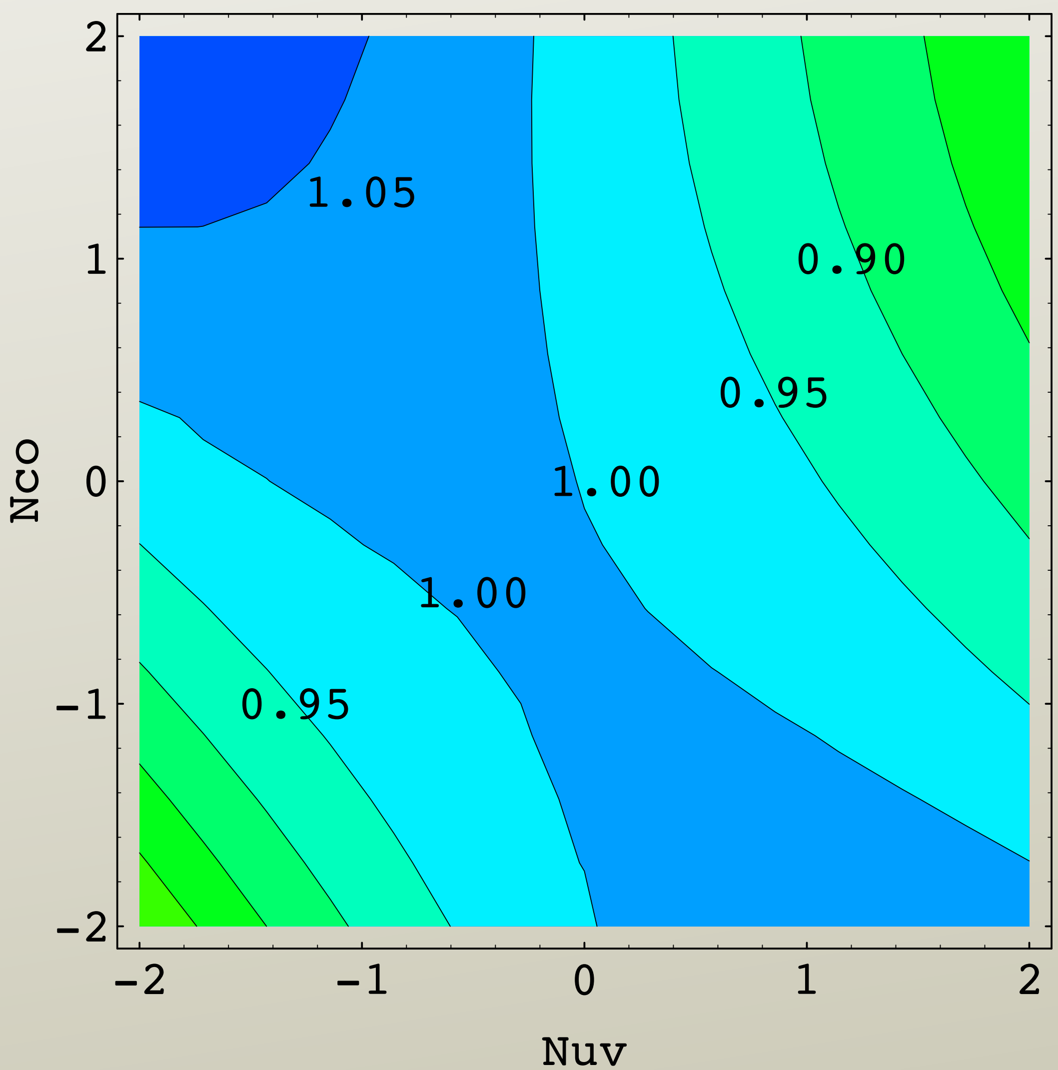
- E_T = jet transverse momentum; y = jet rapidity $= 0$.
- The example is for p-$\bar p$ collisions at $\sqrt{s} = 1800$ GeV.

- Plot

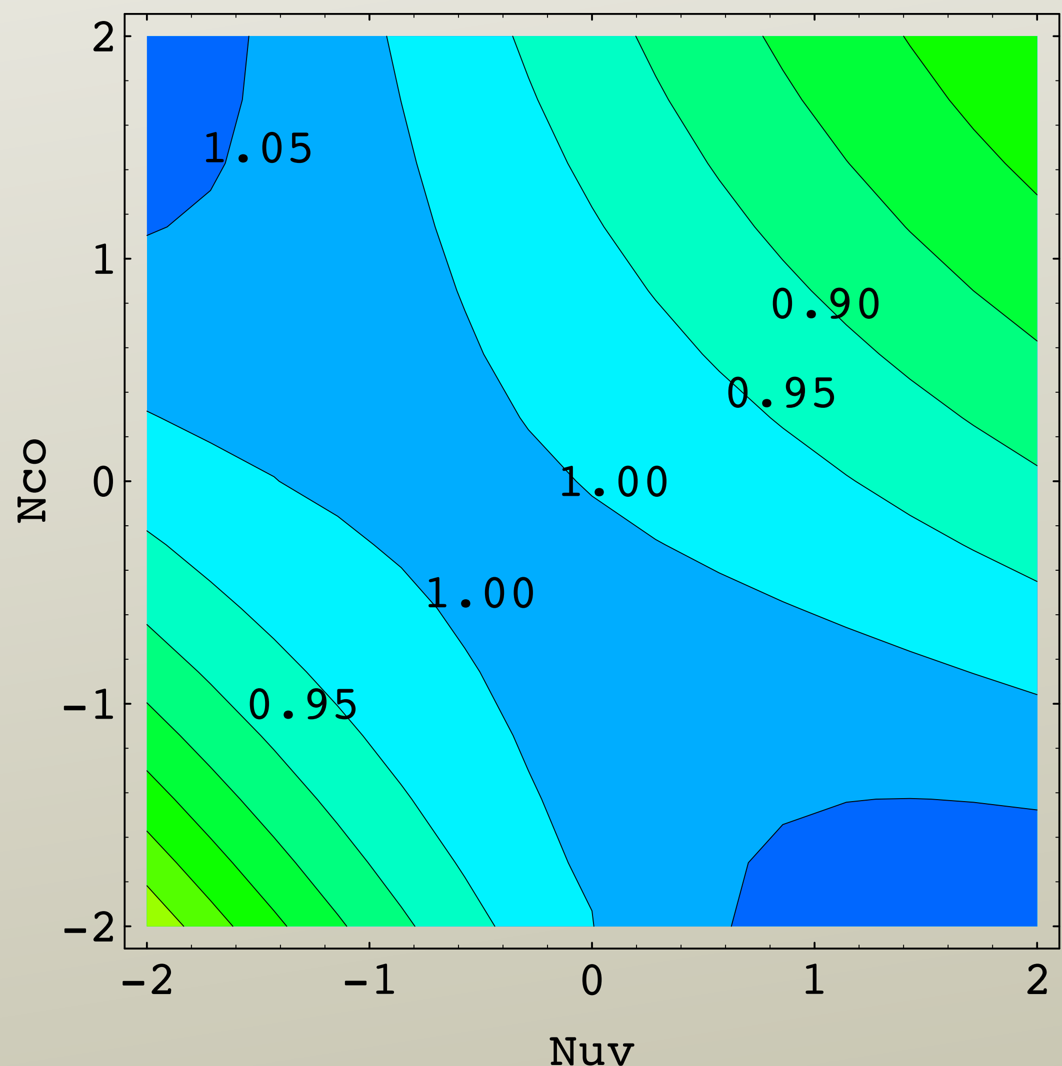
$$\frac{f(\mu, \mu_{\mathrm{F}})}{f(E_T/2, E_T/2)} \qquad f(\mu, \mu_{\mathrm{F}}) = \frac{d\sigma}{dE_T\, dy}$$

- Versus N_{uv} and N_{co}, where

$$\mu = E_T/2 \times 2^{N_{\mathrm{uv}}} \qquad \mu_{\mathrm{F}} = E_T/2 \times 2^{N_{\mathrm{co}}}$$



$E_T = 100$ GeV



$E_T = 500$ GeV

Parton distribution functions

- They are defined as proton matrix elements of a certain operator.

- For quarks,

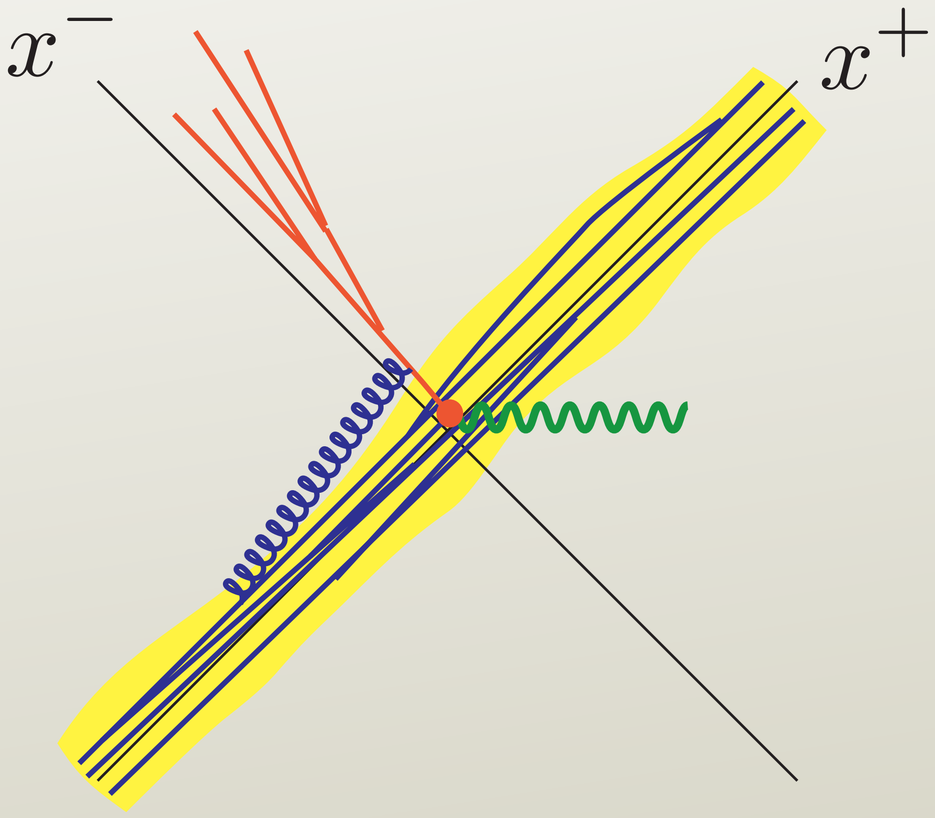
$$f_{i/h}(\xi,\mu_F) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-}\,\langle p|\bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F\psi_i(0)|p\rangle$$

$$F = \mathcal{P}\exp\left(ig\int_0^{y^-} dz^-\,A_a^+(0,z^-,\mathbf{0})\,t_a\right).$$

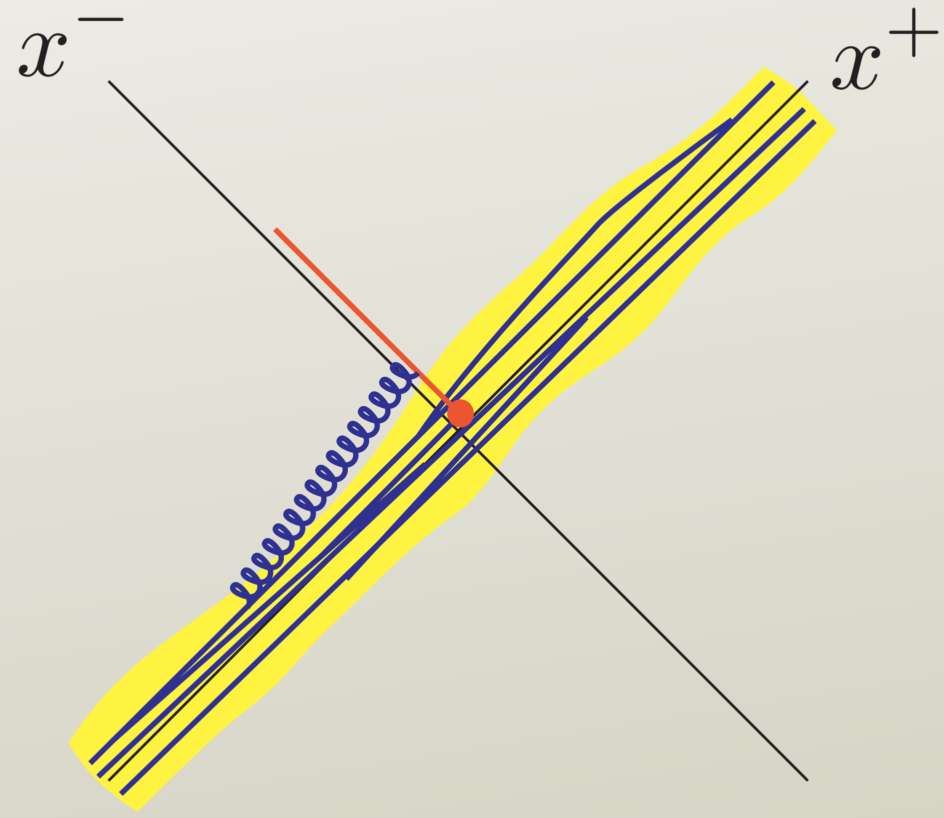
- For gluons, a similar definition.

- Renormalize with the so-called $\overline{\mathrm{MS}}$ prescription with scale μ_F.

The definition in pictures.



DIS



quark distribution function

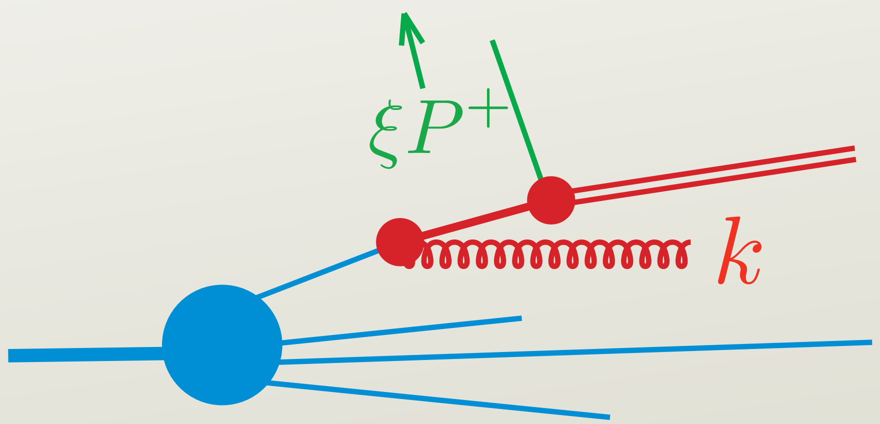
$$f_{i/h}(\xi,\mu_F) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-}\langle p|\bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F\psi_i(0)|p\rangle$$

- The definition entails certain sum rules, *eg*.

$$\int dx \; \{f_{u/p}(x,\mu) - f_{\bar{u}/p}(x,\mu)\} = 2$$

$$\sum_a \int dx \; x f_{a/p}(x,\mu) = 1$$

- Recall that we renormalize the parton distributions using the $\overline{\mathrm{MS}}$ prescription with scale μ_{F}.



- Roughly speaking, this means that we integrate over the transverse momentum of a parton that is part of the proton up to a limit,

$$\boldsymbol{k}_T^2 < \mu_F^2$$

- Thus the parton distribution function depends on μ_F.

Evolution of the parton distribution functions

$$\frac{d}{d\log\mu_F} f_{a/h}(x,\mu_F) =$$

$$\sum_b \int_x^1 \frac{d\xi}{\xi}\; P_{ab}(x/\xi, \alpha_s(\mu_F))\; f_{b/h}(\xi,\mu_F)$$

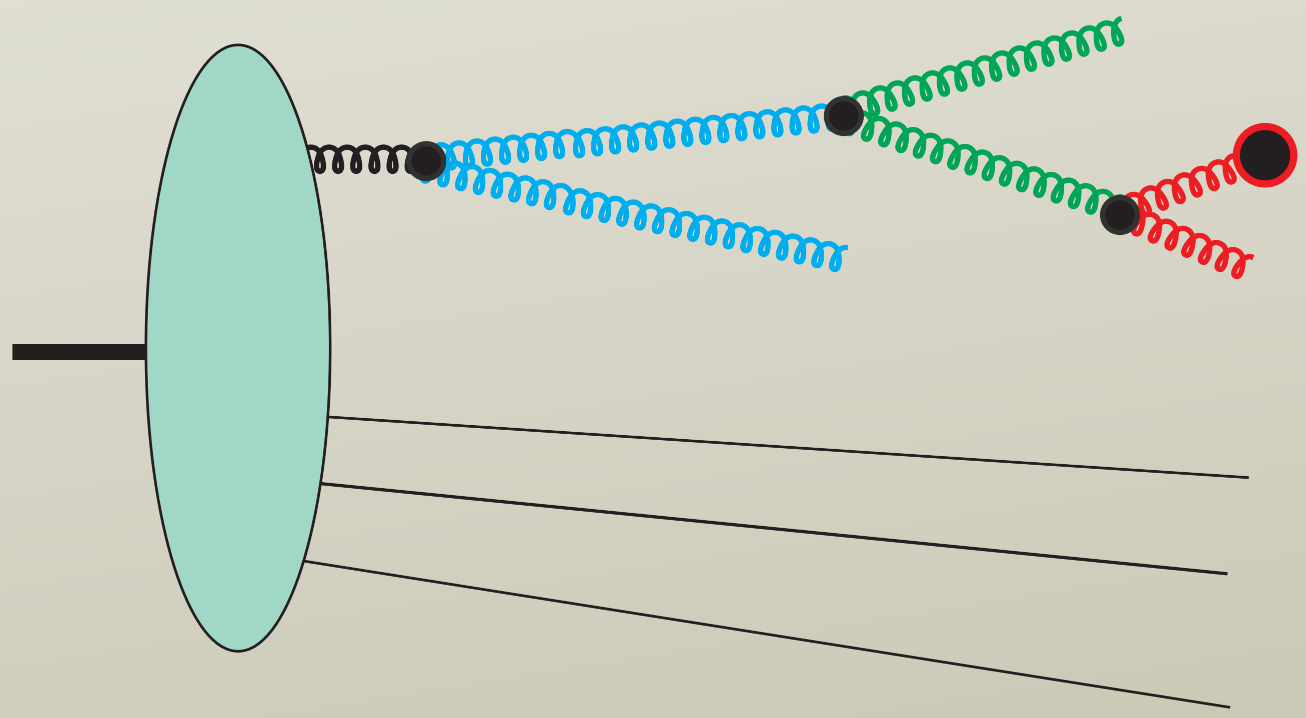
- This is called the Altarelli-Parisi equation or the DGLAP equation.

$$\begin{aligned} P_{ab}(x/\xi,\alpha_s(\mu_F)) &= P_{ab}^{(1)}(x/\xi)\;\frac{\alpha_s(\mu_F)}{\pi} \\ &\quad + P_{ab}^{(2)}(x/\xi)\;\left(\frac{\alpha_s(\mu_F)}{\pi}\right)^2 \\ &\quad + \cdots \end{aligned}$$

$$\frac{d}{d\log\mu_F} f_{a/h}(x,\mu_F) =$$

$$\sum_b \int_x^1 \frac{d\xi}{\xi}\ P_{ab}(x/\xi,\alpha_s(\mu_F))\ f_{b/h}(\xi,\mu_F)$$

- The physical effect that we account for is fluctuations within fluctuations ... as we look with a more powerful "microscope."



Fitting the parton distribution functions

- For DIS, we have

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

- For hadron-hadron collisions, we have similar formulas with two parton distribution functions.
- The parton distribution functions cannot be accurately calculated.
- But given enough data, we can fit them.

- What we need to fit is $f_a(x, \mu_0)$ at some starting scale μ_0.

- This fixes $f_a(x, \mu)$ for any other scale $\mu > \mu_0$.

- Then

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi, \mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q)$$

gives the observed cross section.

- Just adjust $f_a(x, \mu_0)$ until we get all of the observed cross sections right.

- The fact that this works means that the theory is right.

Review

- Parton distribution functions have a definition that is independent of any particular process.
- The functions obey a simple evolution equation that describes the effect of changing resolution
- The parton distribution functions appear in any short distance process with one or two hadrons in the initial state.
- They are fit to experimental results.

Hadron-hadron collisions

Initial state, hard scattering, final state

Topics

- Kinematics: rapidity.
- Drell-Yan processes.
- New particle production.
- Jets.

Rapidity

- Rapidity y (or η) is useful for hadron-hadron collisions.
- Choose c.m. frame with z-axis along the beam direction.

- Consider the production of a massive particle like a Z-boson, with momentum $q = (q^+, q^-, \boldsymbol{q}_T)$.

$$y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right)$$

$$q^\mu = \left(e^y\sqrt{(\boldsymbol{q}_T^2 + M^2)/2},\ e^{-y}\sqrt{(\boldsymbol{q}_T^2 + M^2)/2},\ \boldsymbol{q}_T\right)$$

- Property under z-boosts:

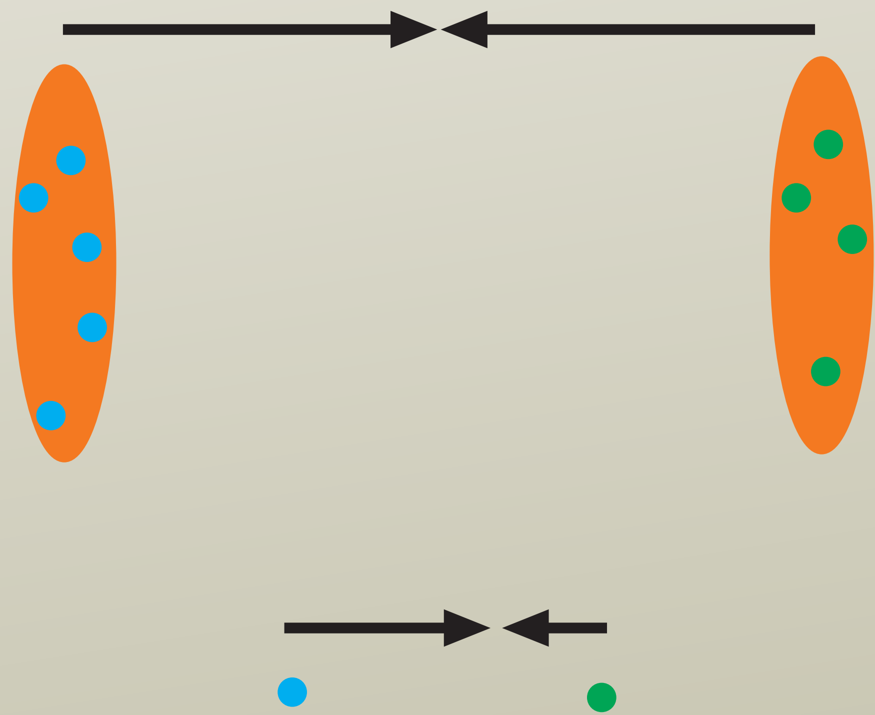
$$q^+ \to e^{\omega} q^+, \quad q^- \to e^{-\omega} q^-, \quad \boldsymbol{q}_T \to \boldsymbol{q}_T$$

$$y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right)$$

- So

$$y \to y + \omega$$

- Simple behavior under z-boosts is important because the c.m. frame is not so special.

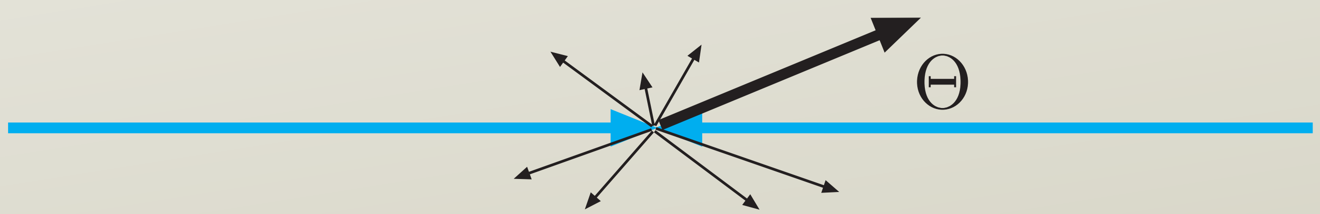


Pseudorapidity

- For a massless particle this becomes

$$y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right)$$

$$y = -\log\left(\tan(\Theta/2)\right)$$

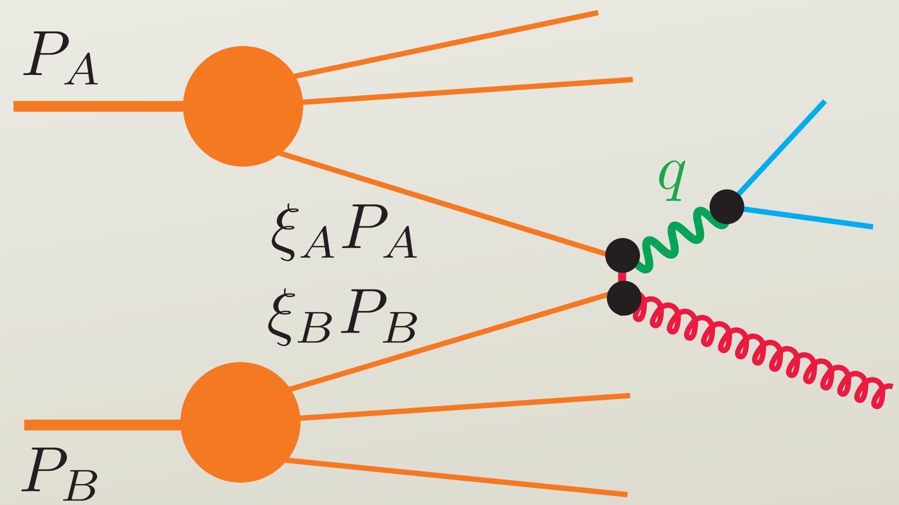


- If the particle is not quite massless, $-\log(\tan(\Theta/2))$ is called the pseudorapidity.

Virtual photon, Z, or W production

- Consider $d\sigma/dy$ for

$$A + B \to Z + X$$



- Factored form of cross section

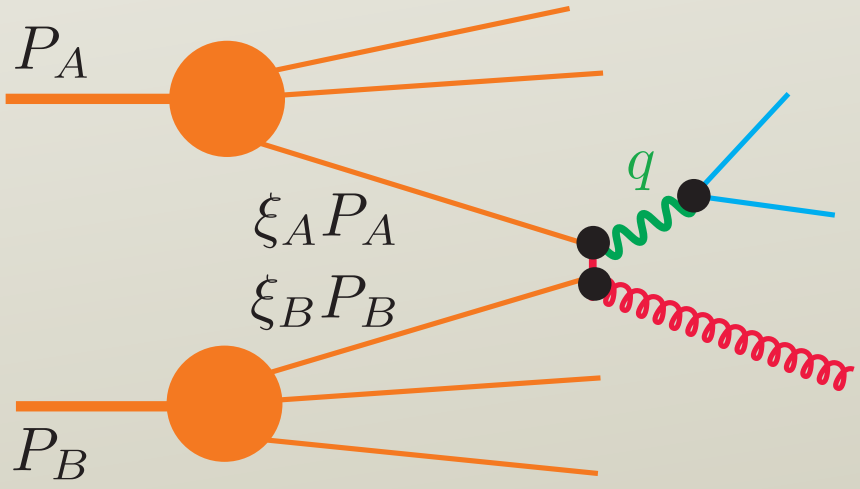
$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}_{ab}(\mu)}{dy}$$

$$+\mathcal{O}(m/M)$$

$$x_A = e^{y}\sqrt{M^2/s} \qquad x_B = e^{-y}\sqrt{M^2/s}$$

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A,\mu)\; f_{b/B}(\xi_B,\mu)\; \frac{d\hat{\sigma}_{ab}(\mu)}{dy}$$
$$+\mathcal{O}(m/M)$$

- The factored formula has power suppressed corrections.
- When $d\hat{\sigma}_{ab}/dy$ is evaluated at order α_s^n, there are also corrections of order α_s^{n+1}.

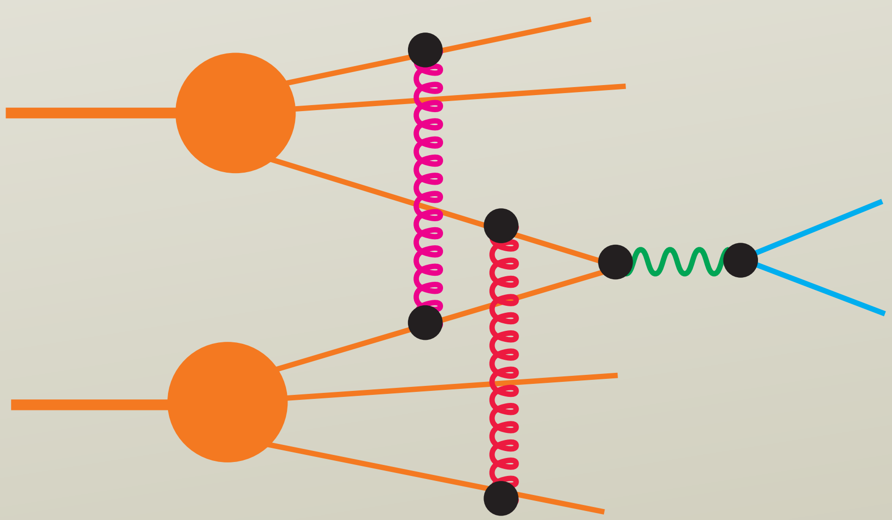


- We integrate over $\boldsymbol{q}_T$. The Z boson has mostly $\boldsymbol{q}_T^2 \lesssim M^2$.

Discussion of factorization

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}_{ab}(\mu)}{dy} + \mathcal{O}(m/M)$$

- This is not obvious.

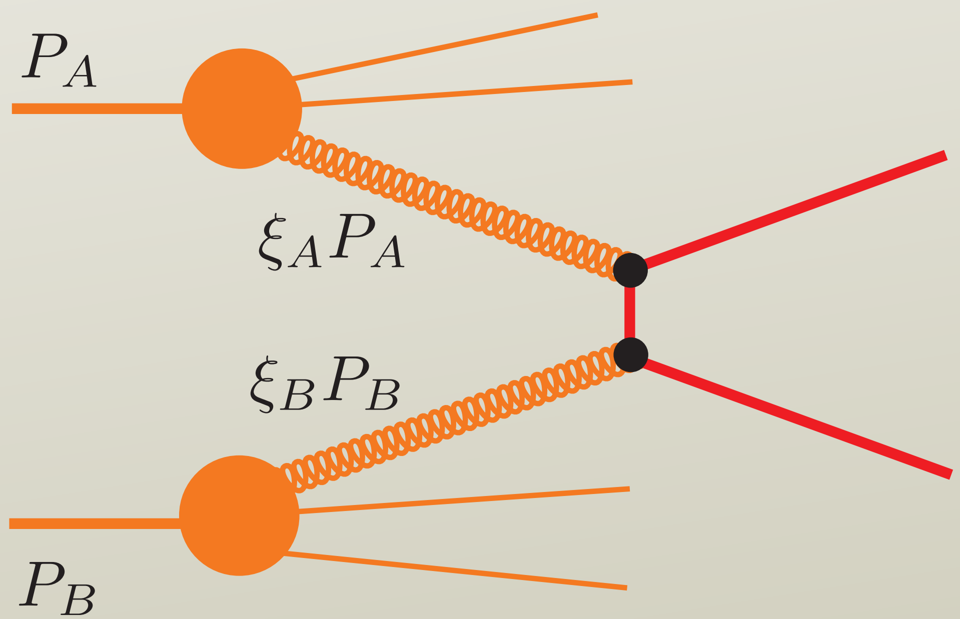


- Need unitarity, causality, gauge invariance.

Heavy particle production

- For instance to make a squark and an antisquark,

$$\sigma_T \approx \sum_{a,b} \int_0^1 d\xi_A \int_0^1 d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \hat{\sigma}_T^{ab}(\mu).$$



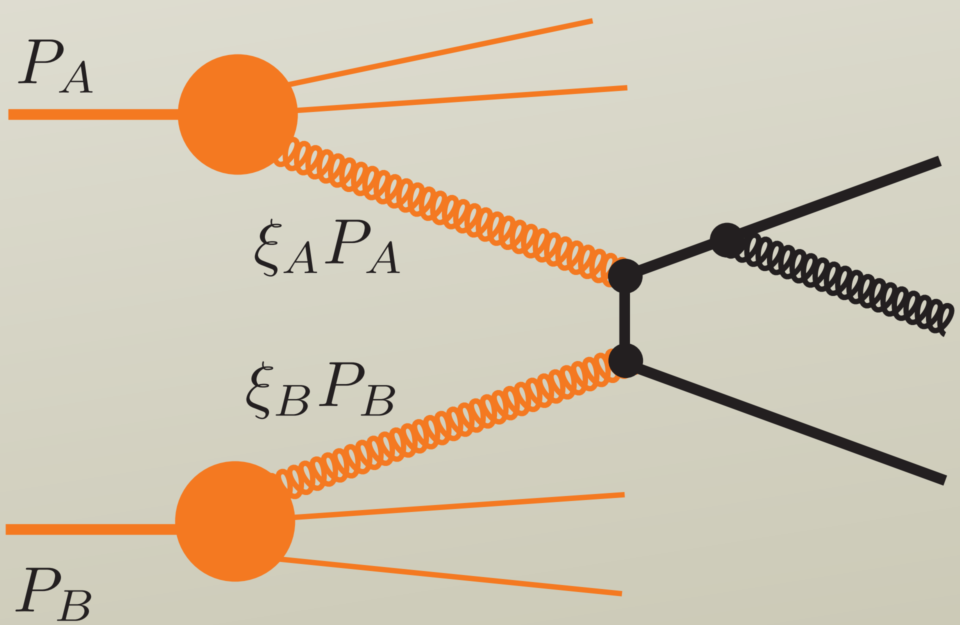
- The large scale is the squark mass, M.
- The virtuality of the exchanged squark is at least M^2.

Jet production

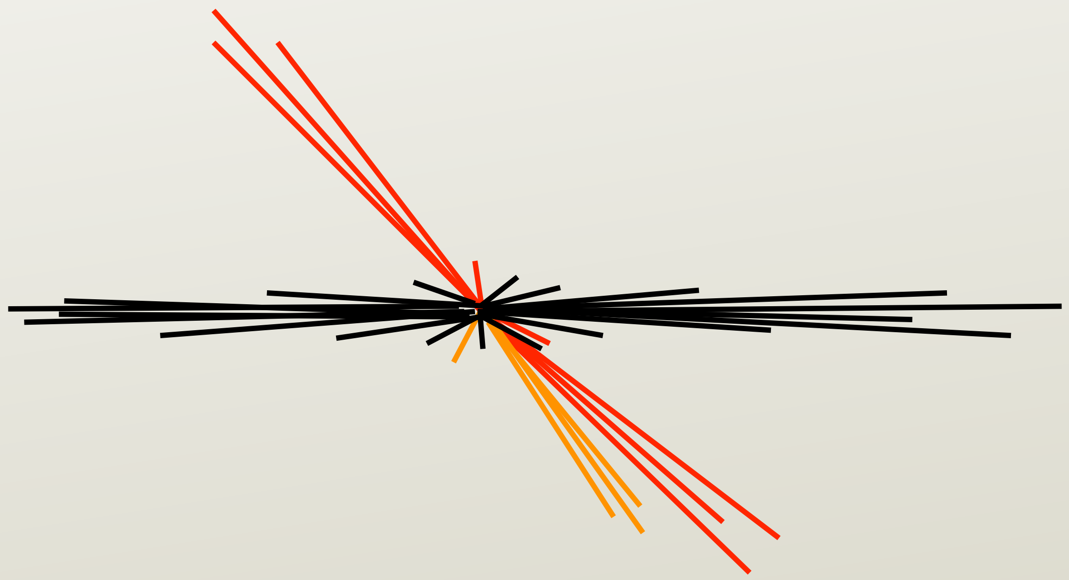
- We can measure – and calculate – a jet cross section, say the one jet inclusive cross section.

$$\frac{d\sigma}{dP_T dy} \approx$$

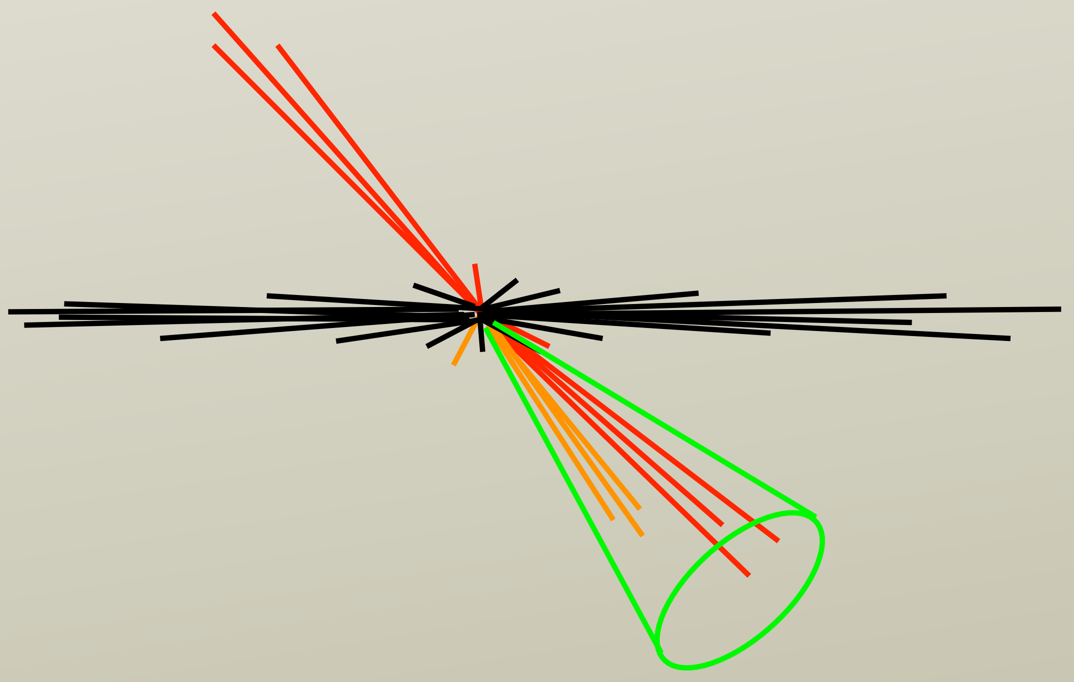
$$\sum_{a,b} \int_0^1 d\xi_A \int_0^1 d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}^{ab}(\mu)}{dP_T dy}.$$



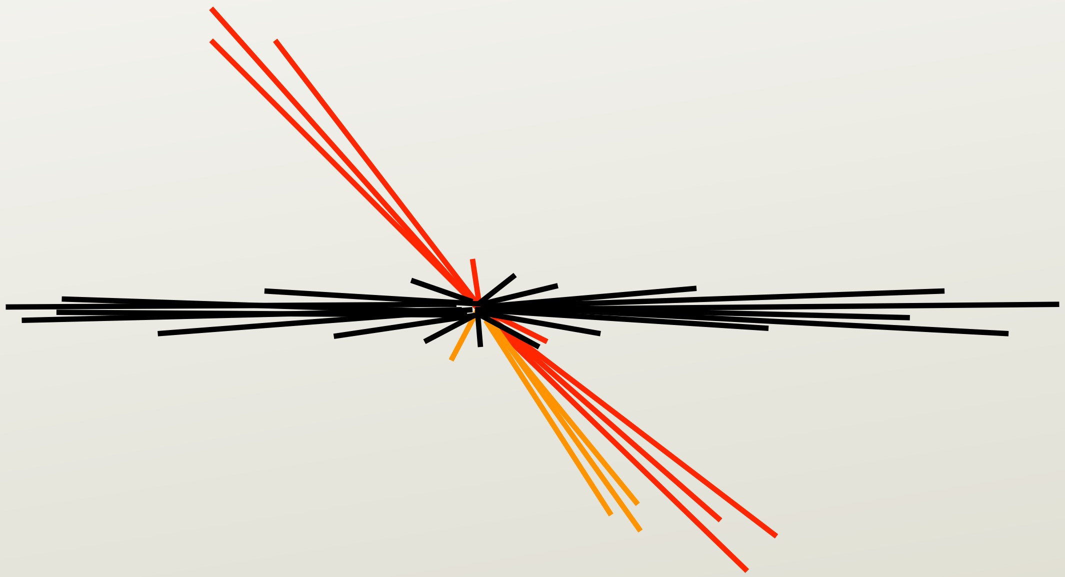
What is a jet?



- It is a spray of hadrons. But we need a more precise definition.
- The definition needs to be infrared safe.
- One traditional definition involves cones.
- The other forms jets by successive combination of hadrons.



The "k_T" jet definition.



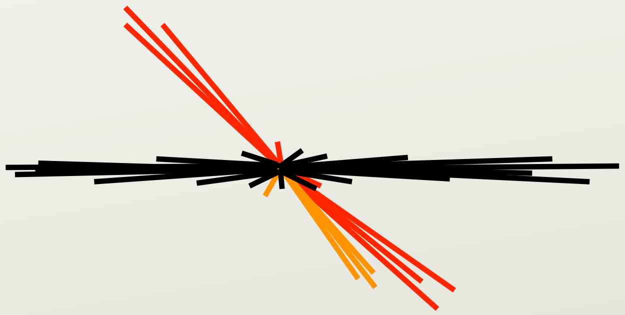
- Start with a list of protojets.
    - Each hadron could be a protojet.
- End with a list of jets.
    - Most of the "jets" will have very low p_T.
    - We will be interested in the high p_T jets.
- There is a parameter R that is similar to the cone size in a cone definition.

1. For each pair of protojets define

$$d_{ij} = \min(p_{T,i}^2, p_{T,j}^2)[(y_i - y_j)^2 + (\phi_i - \phi_j)^2]/R^2$$

For each protojet define

$$d_i = p_{T,i}^2$$



2. Find the smallest of all the d_{ij} and the d_i. Call it $d_{\rm min}$.

3. If $d_{\rm min}$ is a d_{ij}, merge protojets i and j into a new protojet k with

$$p_k^\mu = p_i^\mu + p_j^\mu$$

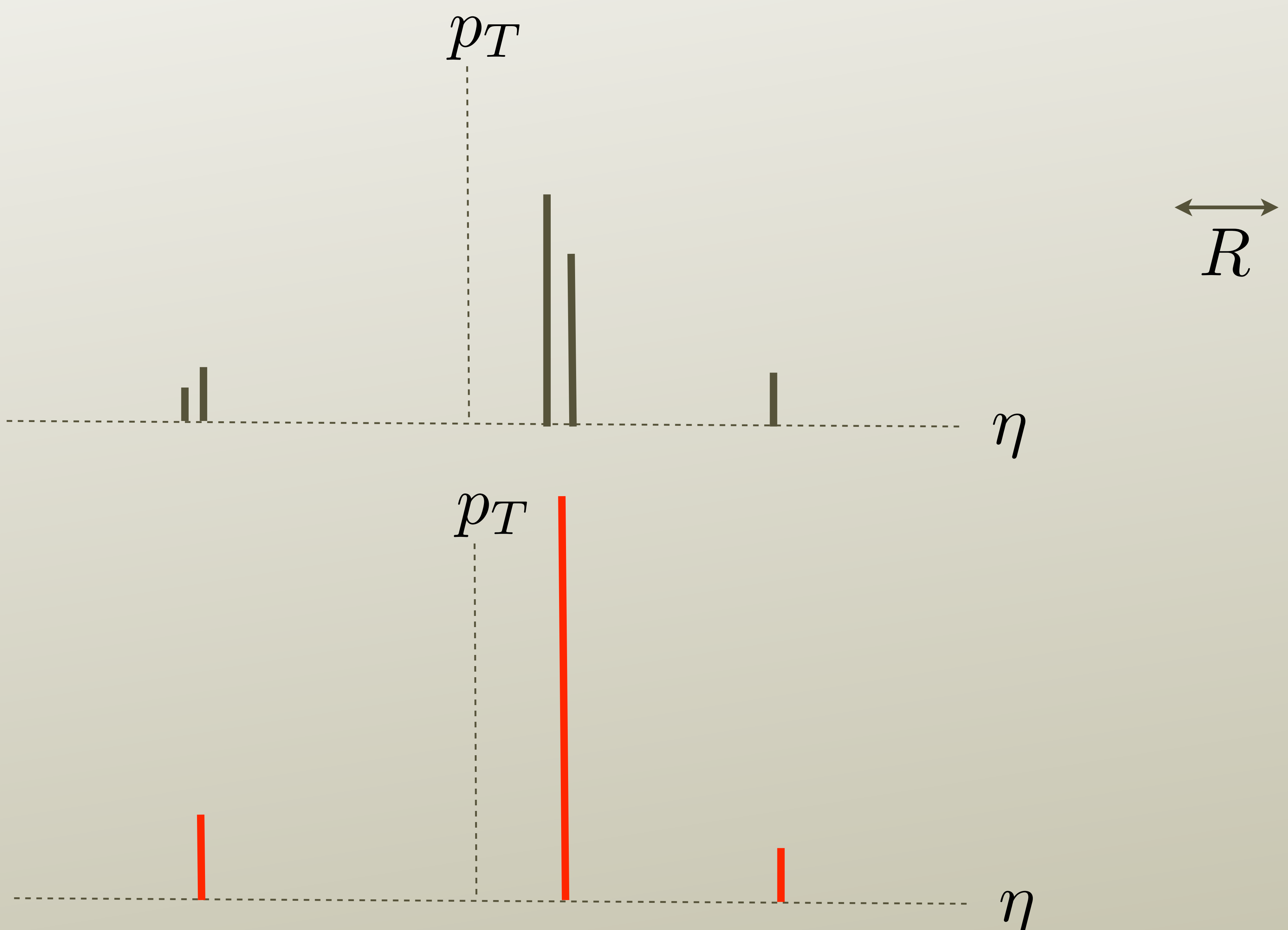
4. If $d_{\rm min}$ is a d_i, then protojet i is "not mergable." Remove it from the list of protojets and add it to the list of jets.

5. If protojets remain, go to 1.

Example

$$d_{ij} = \min(p_{T,i}^2, p_{T,j}^2)\,[(\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2]/R^2$$

$$d_i = p_{T,i}^2$$



Review

- The theory can be reliable for processes with a single short distance scale (high momentum scale).
    - Very heavy particles produced.
    - High transverse momentum particles produced.
- We need an infrared-safe observable.
    - Typically, this involves (suitably defined) jets.

- For such observables, the cross section factors:

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}_{ab}(\mu)}{dy}$$

$$+\mathcal{O}(m/M)$$

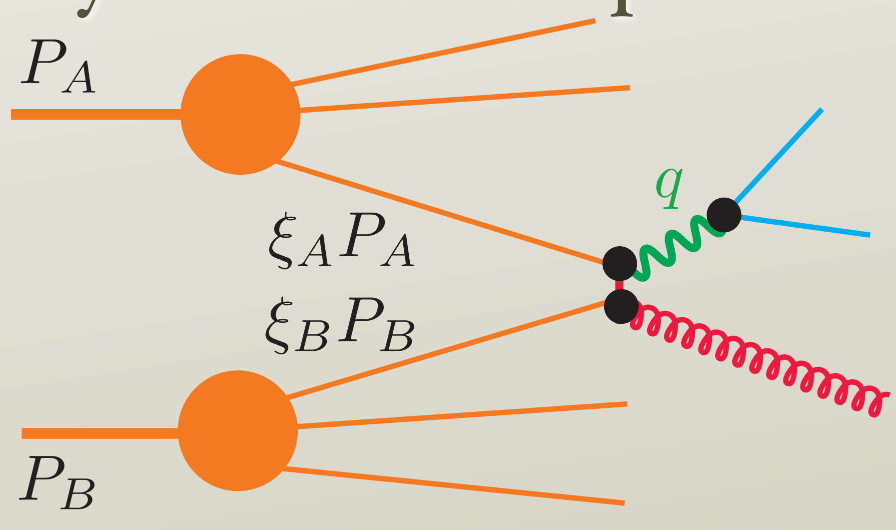
Summing large logarithms

What happens when there are two scales

Summing logs

- For an infrared-safe process with one hard scale, the theory is simple.

- If there are two hard scales, the theory is more complicated.



- Consider $A + B \to Z + X$

$$\frac{d\sigma}{dP_T\, dy}$$

- If $P_T \sim M_Z$, the theory is simple.

- If $1\ \mathrm{GeV} \ll P_T \ll M_Z$, there are two large scales.

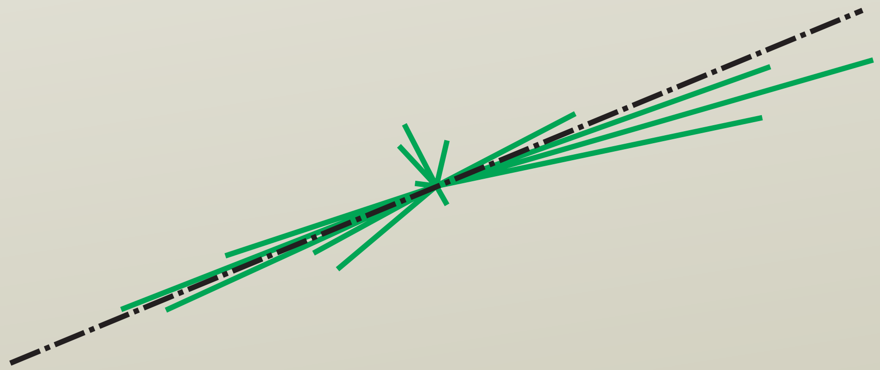
- We need to sum terms of order $\alpha_s^n \log(M_Z/P_T)^{2n-1}$.

- In many cases like this, there are known formulas for summing the logs.

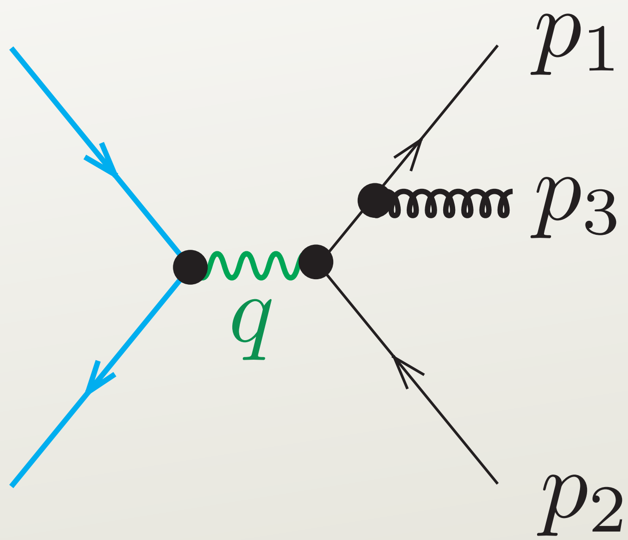
Recall the thrust distribution $d\sigma/dT$.

$$F_m(p_1^\mu,\ldots,p_m^\mu) = \delta\left(T - \mathcal{T}_m(p_1^\mu,\ldots,p_m^\mu)\right)$$

$$\mathcal{T}_m(p_1^\mu,\ldots,p_m^\mu) = \max_{\vec{u}} \frac{\sum_{i=1}^m |\vec{p}_i\cdot\vec{u}|}{\sum_{i=1}^m |\vec{p}_i|}$$



- Since the thrust distribution is infrared safe, we can calculate it in perturbation theory.



- At first order, one finds

$$\frac{d\sigma}{dT} = \left(\frac{4\pi\alpha^2}{Q^2}\sum e_f^2\right)\frac{C_{\rm F}\alpha_{\rm s}}{2\pi} \times \left[\frac{3(2-T)(2-3T)}{1-T} + \frac{4-6T(1-T)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right)\right]$$

Note the log.

- $Q^2 = q^2 = (p_1+p_2+p_3)^2$.
- $C_{\rm F} = 4/3$.
- $T \to 1$ corresponds to two narrow jets.
- $d\sigma/dT$ is singular in this limit.

Quantitative measure of infrared safety

- We can say "safe at scale $Q^2(F)$."
- For thrust, one finds

$$Q^2(F) \approx \frac{1-T}{T}\, Q^2$$

- The scale at which $d\sigma/(dT)$ is infrared safe is much smaller than Q^2 when $(1-T) \ll 1$.
- There are logs of the ratio of Q^2 to this smaller scale.
- For $(1-T) \ll 1$, one should sum the dominant terms in the perturbative expansion.

Parton showers

Factorization at multiple scales

- I will outline some ideas behind the organization of “hardness ordered” shower generators.

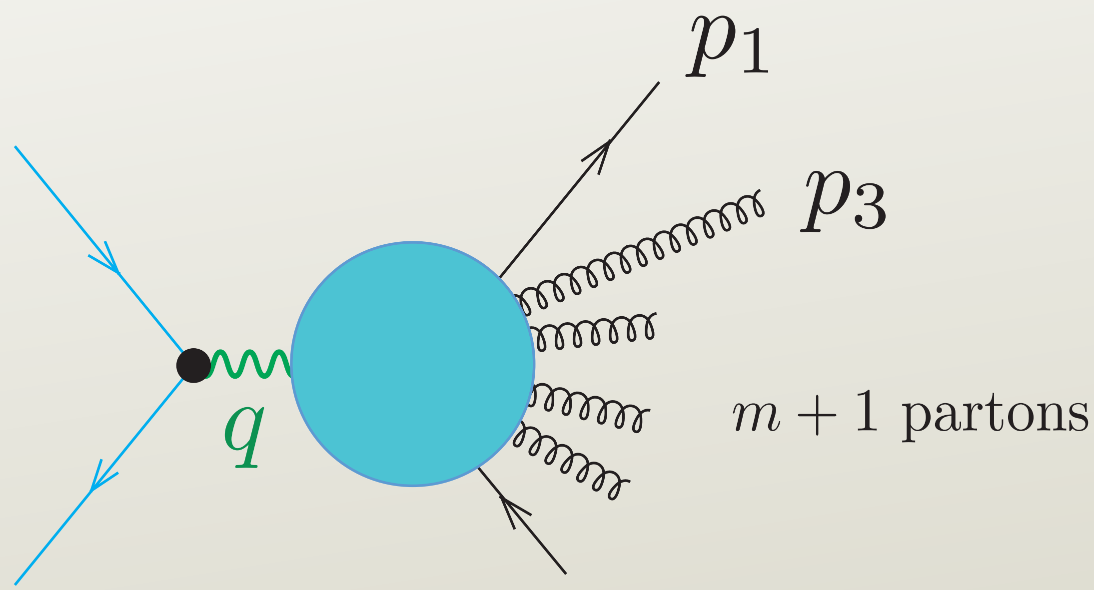
- This includes Pythia and Sherpa (as well as a new program, Deductor).

- Herwig is “angle ordered.” That is sensible, but a little more complicated to explain.

The physical basis for parton showers

- Suppose that partons 1 and 3 become collinear.

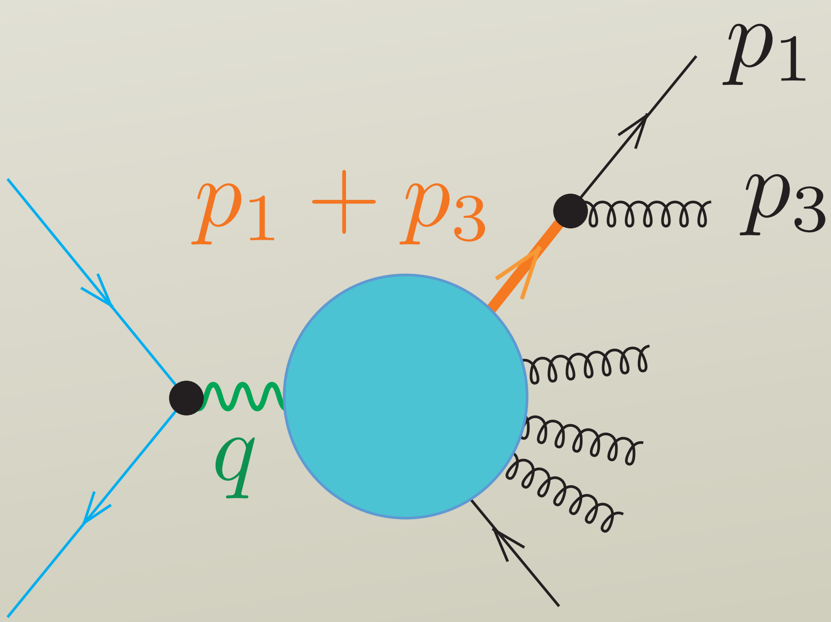
$$p_1 \to z\, p_{\widetilde{13}}$$
$$p_3 \to (1-z)\, p_{\widetilde{13}}$$
$$p_{\widetilde{13}}^2 = 0$$



- Then

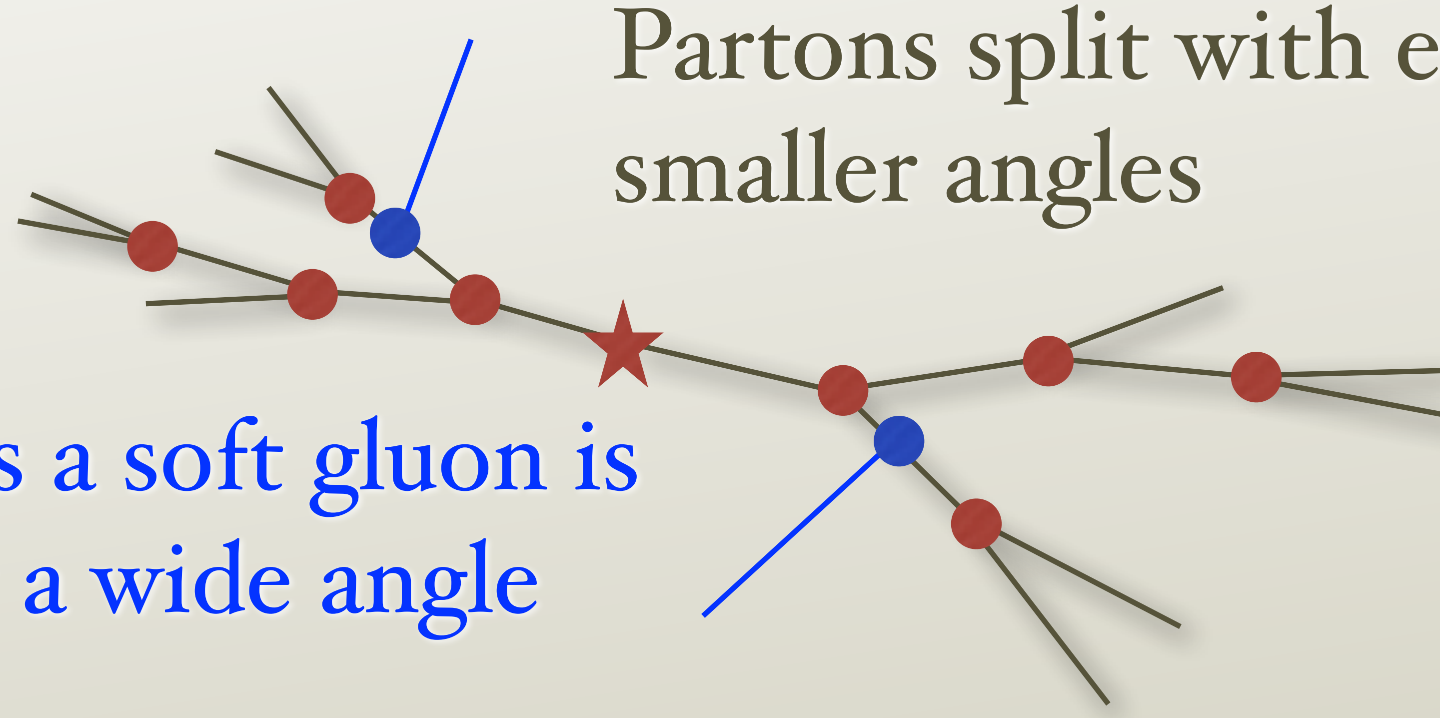
$$\mathcal{M}_{m+1} \sim [\mathcal{M}_m]_{\widetilde{13}} \frac{\text{spinors}}{(p_1+p_3)^2}$$

splitting amplitude



- This is how one starts to define a parton shower.

This suggests the following structure of events:



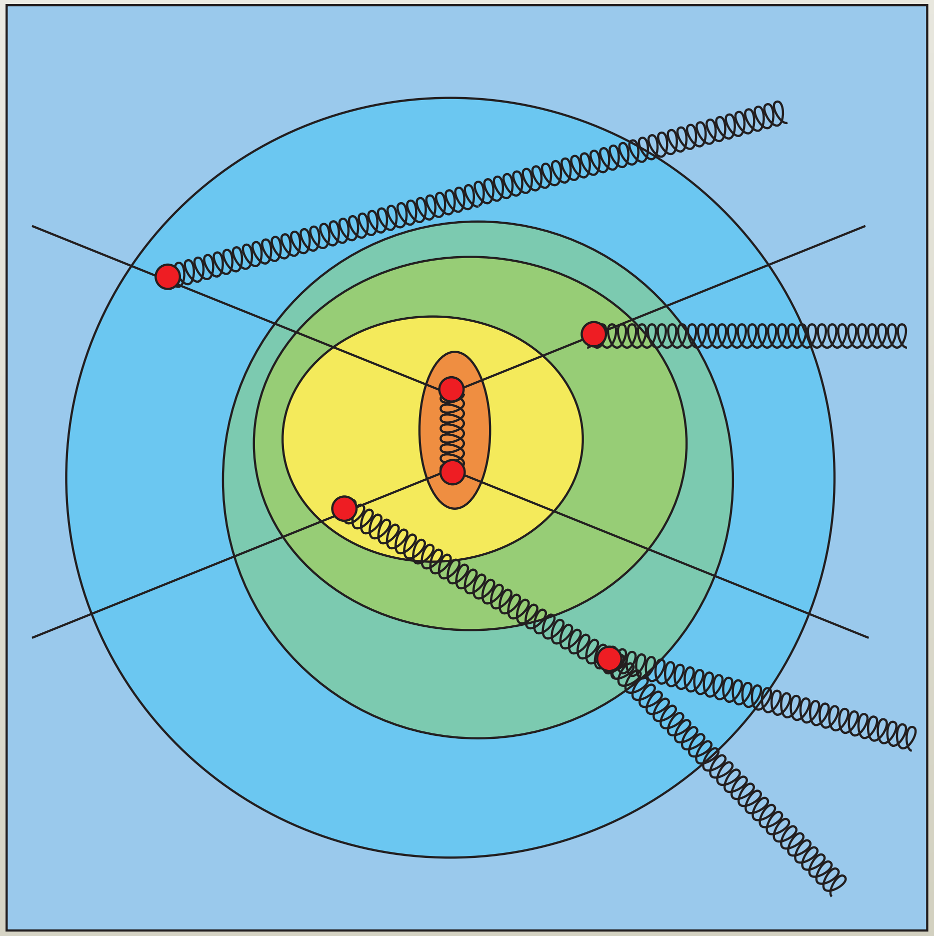
Partons split with ever smaller angles

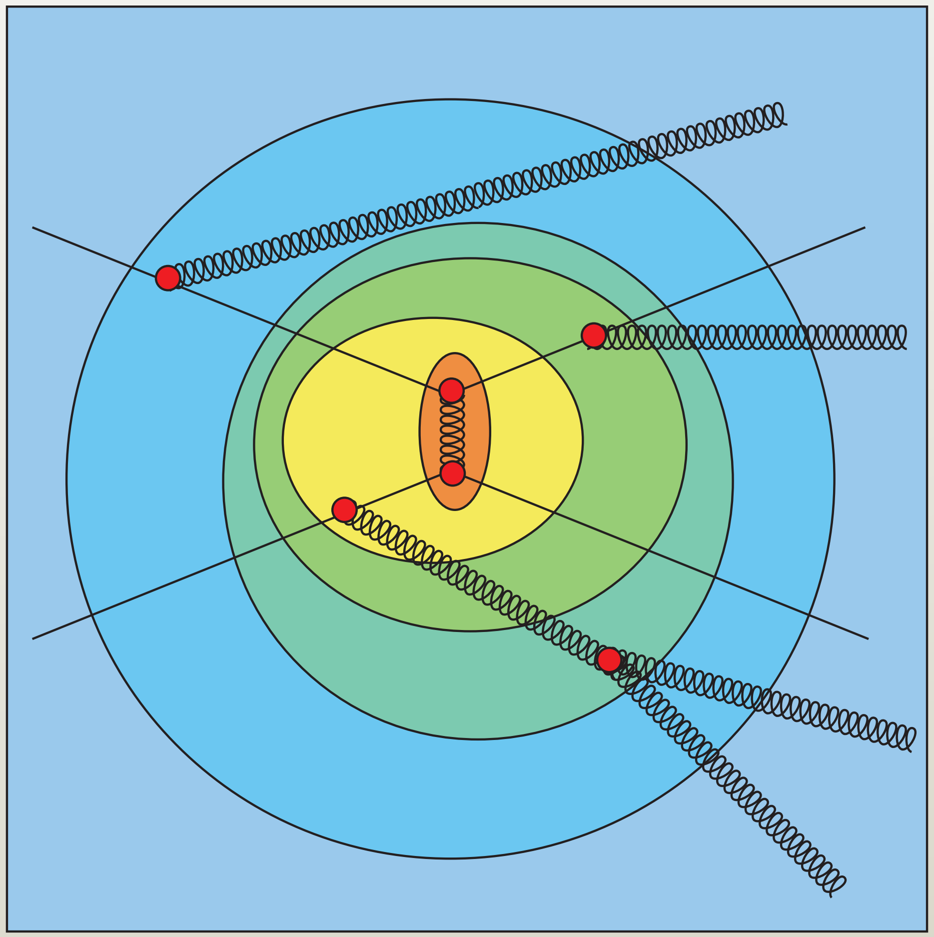
Sometimes a soft gluon is emitted at a wide angle

- Parton shower generators simulate this.

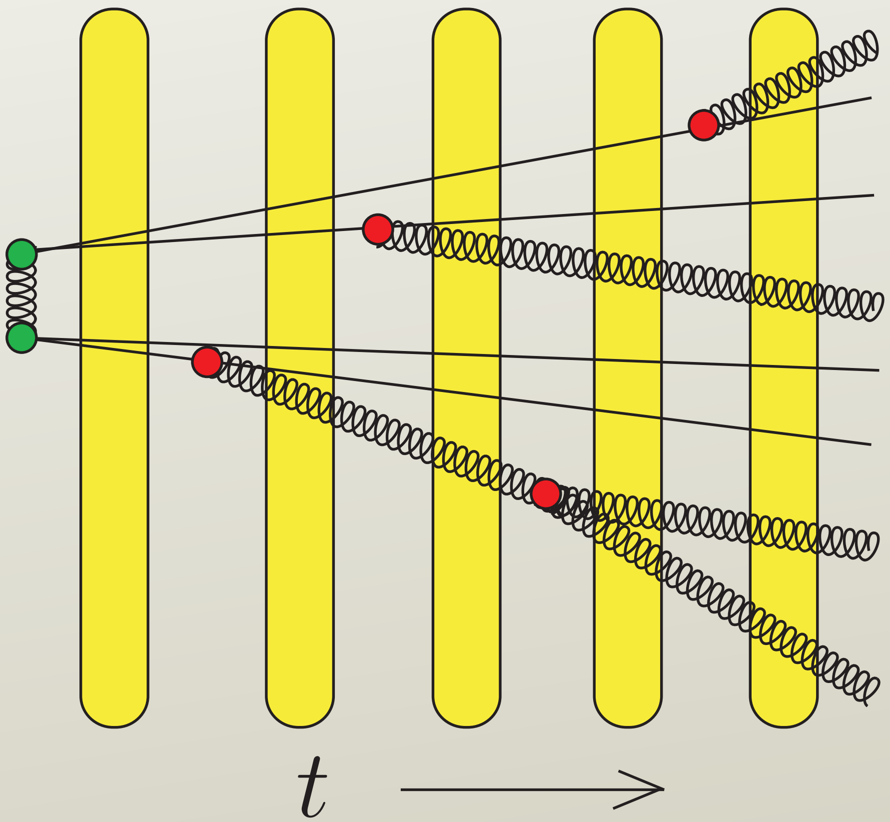
Parton showers and factorization

- First approximation is just the hard interaction.
- Softer interactions not resolved by imagined observations.
- So softer interactions are integrated out.
- Then we increase the resolution...

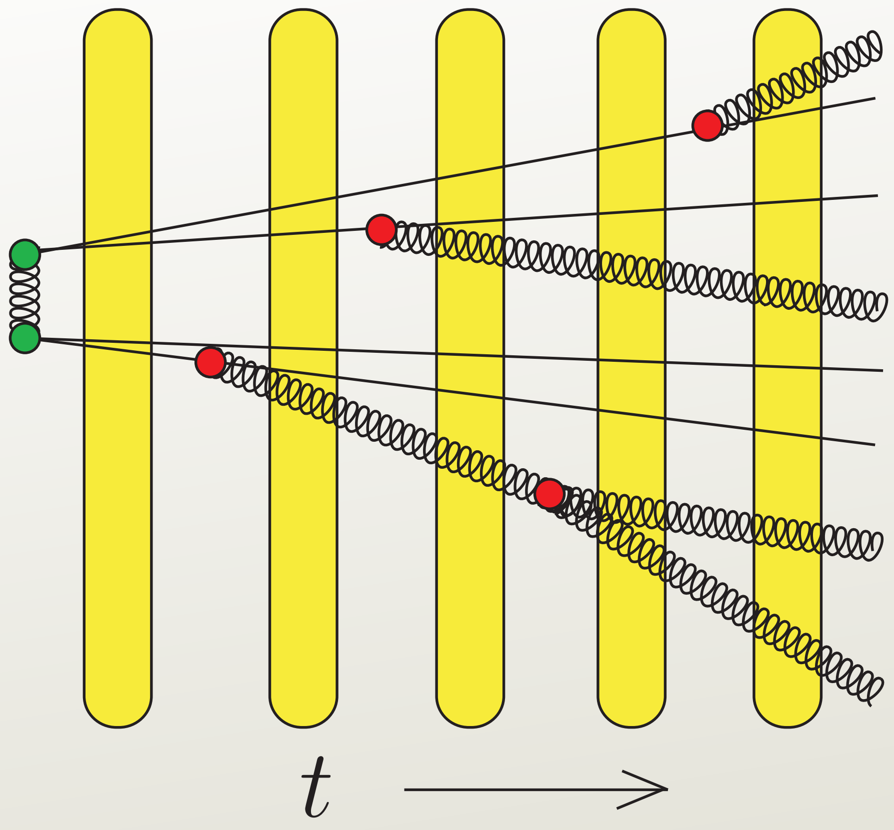




Real time picture



Shower time picture



- In the shower algorithm, the splitting vertices give the probability for a parton to split with a given sharing of the mother momentum between the daughters.

- The splitting probabilities are an approximation to the results from QCD Feynman diagrams.

- In a practical implementation, the shower is generated as a "Markov process."

- Given the parton configuration at shower time t_i, select variables for the next splitting using random numbers and the desired probability distributions.

- Analyze the shower evolution operator $\mathcal{U}(t', t)$

$$\frac{d}{dt}\mathcal{U}(t,t') = [\mathcal{H}_{\mathrm{I}}(t) - \mathcal{V}(t)]\,\mathcal{U}(t,t')$$

split don't split

- Since $\mathcal{V}(t)$ is simple, rewrite as

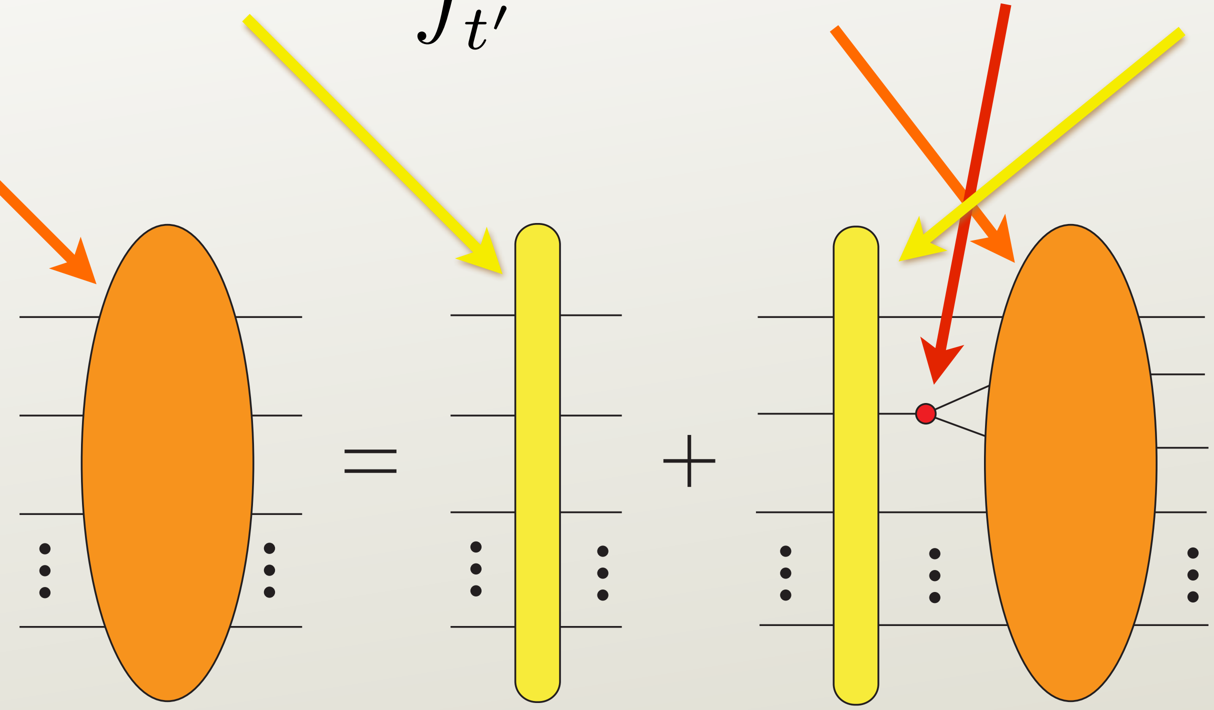
$$\mathcal{U}(t,t') = \mathcal{N}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\,\mathcal{H}_{\mathrm{I}}(\tau)\,\mathcal{N}(\tau,t')$$

exponentiate the probability of not splitting

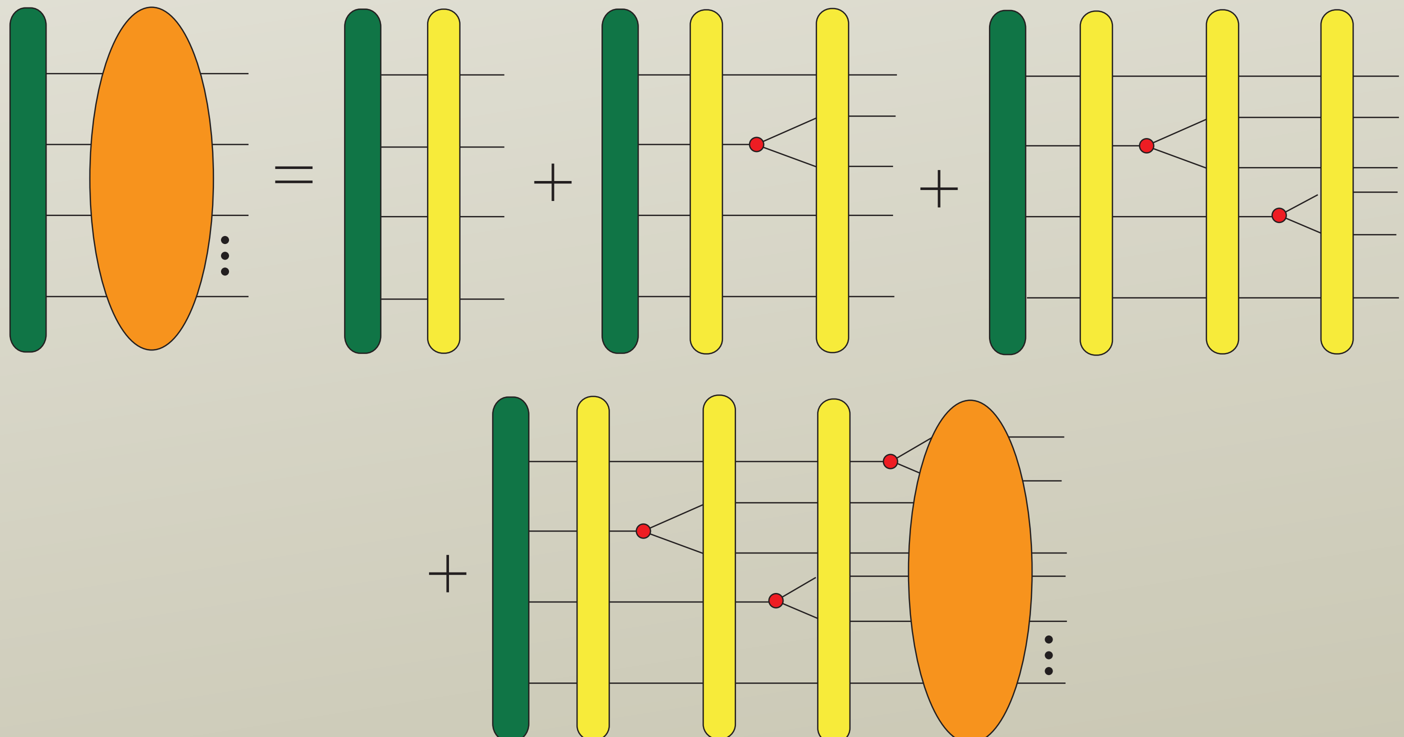
$$\mathcal{N}(t,t') = \mathbb{T}\exp\left\{-\int_{t'}^{t} d\tau\, \mathcal{V}(\tau)\right\}$$

this is the
Sudakov factor

$$\mathcal{U}(t,t') = \mathcal{N}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\, \mathcal{H}_{\mathrm{I}}(\tau)\, \mathcal{N}(\tau,t')$$

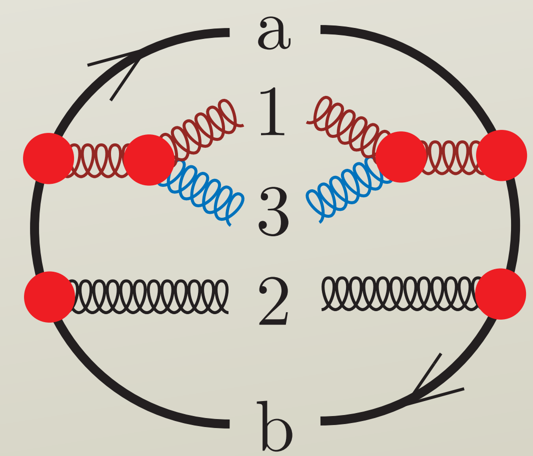


- Iterated, gives a picture of what shower evolution does...

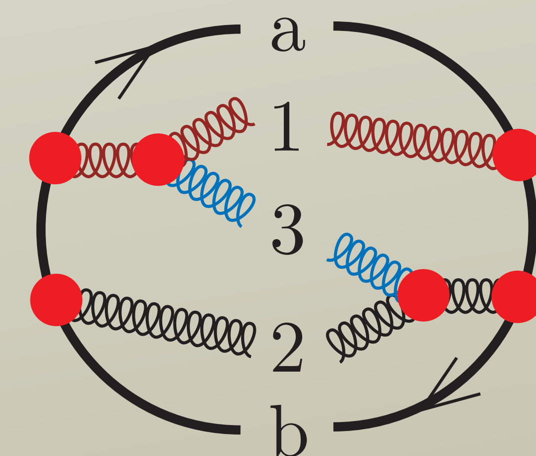


Quantum interference

- In quantum mechanics, to get a probability we sum amplitudes, then square.
- So $\mathcal{H}_{\rm I}$ has contributions of the form $\mathcal{M}_A \mathcal{M}_B^*$.
- A picture for a splitting amplitude squared:

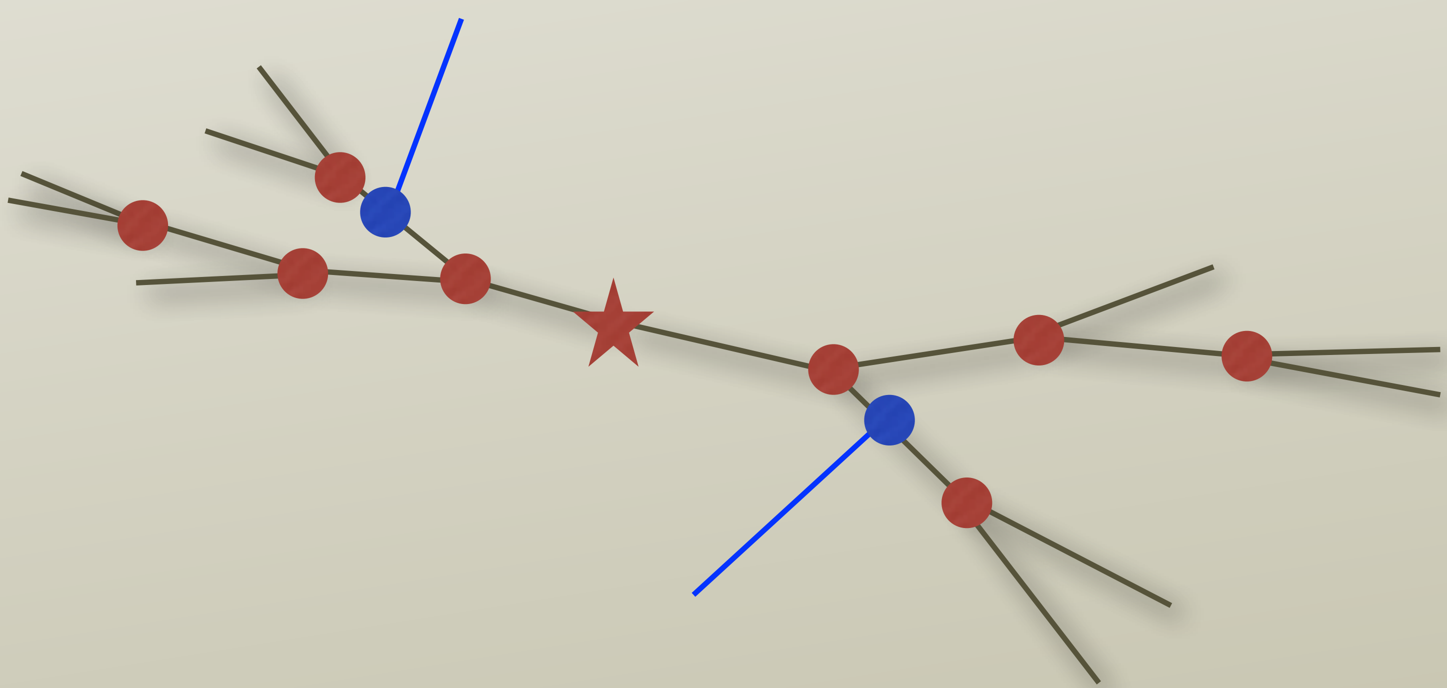


- An interference diagram:



Hadronization

- When the hardness scale reaches a GeV or so, we can't trust perturbation theory.
- So we have to turn the parton shower off.
- Pythia, Sherpa, and Herwig provide a non-perturbative model of how to make hadrons out of the partons.



Review

- Parton shower generators are based on lowest order perturbation theory, so they are not as precise as next-to-leading order (or NNLO) calculations.
- In favorable cases they can sum large logarithms.
- They produce whole events, so they are very important practically.