Particle lifetime, decay width, branching ratios…

1. Consider a particle P of mass M being at rest ($E=M$, $\boldsymbol{p}=0$). Its wave function Ψ_0 is:

$$\Psi_0 = e^{-iE_i t}\psi_0(x) = e^{-iMt}\,\frac{1}{\sqrt{V}}$$

2. The decay products, free particles appearing as a result of the decay of the original particle, can be described by a set of wave functions

$$\Psi_{decay\_products} = c_n(t)\cdot e^{-iE_1 t}\psi_1(x_1)\cdot e^{-iE_2 t}\psi_2(x_2)\cdot e^{-iE_3 t}\psi_3(x_3)... = c_n(t)e^{-iE_n t}\psi_n(x)$$

Note that the total energy E_n does not have to be exactly equal to the original particle's mass M. Since the original particle exists only for some finite time $\Delta t\sim\tau$ from the moment it was given to us (t=0) and due to the time-energy analog of the uncertainty principle, the final energy E_n is allowed to be different from the mass M by $\Delta E\sim 1/\tau$.

3. Then, the overall final wave function at time t is

$$\Psi = c_0(t)\Psi_0 + \sum c_n(t)\Psi_n(x), \quad \text{where } c_0(t) = e^{-\frac{\Gamma}{2}t} \quad \text{and} \quad c_n(t) << 1$$

Such defined $c_i(t)$ insures the exponential decay of the original particle: $p = \int \left|\Psi_0\right|^2 dV = e^{-\Gamma t} = e^{-t/\tau}$

4. Let us assume that the particle P decays happen due to some potential V that can be treated as a small perturbation to the free Hamiltonian. In this case, we can repeat all the argumentation of the perturbation theory of scattering, just paying attention to the effects of the redefined $c_n(t)$. This would now result in:

$\dot{c}_f(t) = -ic_0(t)\cdot\int \Psi_f^+\hat{U}\Psi_0 dV$, which transforms further to:

$$\dot{c}_f(t) = -ie^{-\frac{\Gamma}{2}t}\int e^{i(E_f-M)t}\psi_f^+\hat{U}\psi_0 dV = -i\int e^{i\left((E_f-M)+i\frac{\Gamma}{2}\right)t}\psi_f^+\hat{U}\psi_0 dV = -ie^{i\left((E_f-M)+i\frac{\Gamma}{2}\right)t}m_{f0},$$

5. Unlike the case of scatterings, we can trivially integrate this function and find the value of $c(t)$. We will set observation time to be $t=\infty$ and the initial time $t_0=0$. Integration of the above equation in these limits is trivial and gives:

$$c_f = -i\int_0^\infty e^{i\left((E_f-M)+i\frac{\Gamma}{2}\right)t}m_{f0}dt = -\frac{m_{f0}}{\left((E_f-M)+i\frac{\Gamma}{2}\right)}e^{\left(i(E_f-M)-\frac{\Gamma}{2}\right)t}\Bigg|_0^\infty = \frac{m_{f0}}{\left((E_f-M)+i\frac{\Gamma}{2}\right)}$$

6. Therefore, probability of transition from the initial state to the *dn* interval of states in the vicinity of some final *f*-state is:

$$dP_f = |c_f|^2\, dn = \frac{|m_{f0}|^2}{(E_f - M)^2 + \Gamma^2/4}\, dn = \frac{|m_{f0}|^2}{(E_f - M)^2 + \Gamma^2/4} \frac{dn}{dE_f}\, dE_f$$

7. First of all, unlike the case of scatterings, this expression does not contain a delta-function that would require that the initial energy M (mass of the decaying particle) and the final energy E_f (total energy of decay products) equal to each other. Or, by running the time back, what used to be a decay is now a creation of a particle, and the creation is possible even if the initial energy E_i is not exactly equal to M.

   All this is possible because the particle lives only a finite time. As long as the total initial ($t=-\infty$) energy E_i of particles resulting to a creation of particle P and the final energy of decay products E_f at $t=+\infty$ equal to each other, the quantum mechanics seem to allow for a short-term energy non-conservation, i.e.

   $M \neq E_f$ and, consequently, $E_i \neq M$, which means that the particle P could exist with a mass somewhat different from its nominal mass M. Of course, if $\Gamma \ll M$, the probability for a large deviation would be very small.

8. Second, although we do not find a delta-function $\delta(E_f\text{-}M)$ in the expression for the probability, it is still can be integrated over all possible E_f energies. If $\Gamma \ll M$, then the probability dies out very quickly outside of $E=M$ and one can assume that V_{f0} and $dn/dE_f=\rho_f(E_f)$ do not change too much within a few Γ-range. In such approximation, one can evaluate $|V_{f0}|$ and $\rho_f(E_f)$ at $E_f=M$ and the integral becomes:

$$P_f = \int_{-\infty}^{+\infty} \frac{|m_{f0}|^2}{(E - M)^2 + \Gamma^2/4} \rho_f(M)\, dE_f = \frac{2\pi}{\Gamma} |m_{f0}|^2 \rho_f(M)$$

9. By recalling the definitions of a branching ratio and partial width Γ_f, $P_f = \Gamma_f/\Gamma$, one can rewrite:

$$\Gamma_f = 2\pi |m_{f0}|^2 \rho_f(M)$$

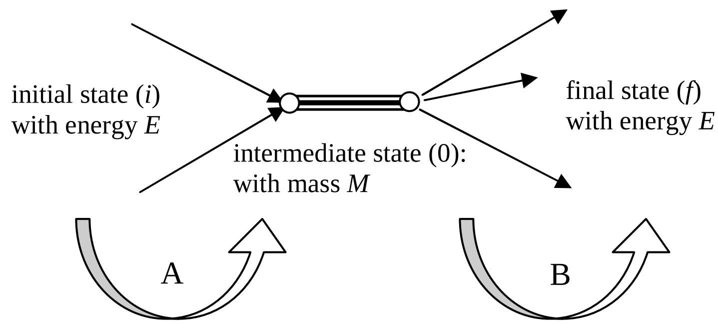
- This is how theorists calculate partial widths if they know how to calculate a matrix element responsible for the decay (as we will see, decays and scatterings look very much alike).

- Note that, in addition to the matrix element-squared, the partial widths also depend on the phase space of the decay products. This, for example, explains why particles decaying via the same interaction mechanism may live very different lifetimes. E.g., lifetimes of nuclei undergoing β-decays and those of heavy quarks/leptons (all due to weak interactions with the matrix element well approximated as a constant) range from billions of years of tiny fractions of a second.

- To calculate the total width and, consequently particle's lifetime, a theorist would calculate partial widths to all possible decay channels (at least to those that are believed to give the largest contribution). If the observed lifetime is shorter (the total width is broader) than predicted, one can tell that there are other open decay channels that have not been accounted in a theory (this method was successfully used to show that there only 3 light neutrinos in nature and, consequently, only 3 lepton-quark generations).

- Also, if a theorist can calculate relative branching ratios of any two channels and experimentally the ratio is different (e.g., much larger than expected), it may mean that there other unaccounted forces that give contributions to the matrix element. This is a very popular method for indirect searches for the new physics.

Resonant cross-section (Breit-Wigner formula)



Consider a process of two particles in state Ψ_i merging into an intermediate state/particle Ψ_0 that eventually decays into some final state Ψ_f. The energy of the collision in the center of mass frame is *E* (as was discussed above, it is not necessarily the same as the mass M of the intermediate particle)

Process A: $\dot{c}_0(t) = -ie^{i(M-E)t}m_{0i}$ (feeding the resonance state)

Process B: $\dot{c}_0(t) = -\frac{\Gamma}{2}c_0(t)$ (decaying resonance)

Combining both together, $\dot{c}_0(t) = -ie^{i(M-E)t}m_{0i} - \frac{\Gamma}{2}c_0(t)$

This can be easily solved by multiplying both sides of the equation by $e^{\frac{\Gamma}{2}t}$:

$$c_0(t) = m_{0i}\frac{e^{i(M-E)t}}{(M-E)+i\frac{\Gamma}{2}}$$

$$\dot{c}_0(t)e^{\frac{\Gamma}{2}t} + \frac{\Gamma}{2}e^{\frac{\Gamma}{2}t}c_0(t) = -ie^{\left(\frac{\Gamma}{2}+i(M-E)\right)t}m_{0i},$$
$$\frac{\partial}{\partial t}\left(e^{\frac{\Gamma}{2}t}c_0(t)\right) = -ie^{\left(\frac{\Gamma}{2}+i(M-E)\right)t}m_{0i},$$
$$\int_0^t \frac{\partial}{\partial t}\left(e^{\frac{\Gamma}{2}t}c_0(t)\right)dt = -im_{0i}\int_0^t e^{\left(\frac{\Gamma}{2}+i(M-E)\right)t}dt,$$
$$e^{\frac{\Gamma}{2}t}c_0(t) = -im_{0i}\frac{e^{\left(\frac{\Gamma}{2}+i(M-E)\right)t}-1}{\frac{\Gamma}{2}+i(M-E)},$$
$$c_0(t) = -im_{0i}\frac{e^{i(M-E)t}}{\frac{\Gamma}{2}+i(M-E)} = m_{0i}\frac{e^{i(M-E)t}}{(M-E)+i\frac{\Gamma}{2}}$$

So amount of resonance states present at any given time is

$$\left|c_0(t)\right|^2 = \frac{\left|m_{0i}\right|^2}{(M-E)^2 + \frac{\Gamma^2}{4}}$$

From $\frac{dN}{dt} = -\Gamma_f N$ for the decaying intermediate particles to the final state *f*, the rate at which the final state *f* appears to be produced is

$$Rate = \Gamma_f \cdot \left|c_0(t)\right|^2 = \frac{\Gamma_f\left|m_{0i}\right|^2}{(M-E)^2 + \frac{\Gamma^2}{4}}$$

And therefore, if the energy of collision *E* is scanned, the cross section would show a characteristic bump

$$\sigma(E) \sim \frac{1}{(M-E)^2 + \frac{\Gamma^2}{4}}$$

- This is how many new particles were discovered (Δ-resonance, J/ψ, etc.). Note if the resonance width Γ is small, one needs to scan with the energy E with very small steps not to miss the bump

- Note that the shape of the distribution (defined by its peak parameter *M* and the total width *Γ*) **does not** depend on a particular channel of decay *f*, but the distribution's amplitude (i.e. cross section) **does**.