

# **Reducing Memory Access Latencies using Data Compression in Sparse, Iterative Linear Solvers**

An All-College Thesis

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**Project Title:** Reducing Memory Access Latencies using Data  
Compression in Sparse, Iterative Linear Solvers  
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## **Abstract**

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
<b>2</b>	<b>Background</b>	<b>4</b>
2.1	Conjugate Gradient . . . . .	4
2.2	Multigrid Preconditioner with Gauss-Seidel Step . . . . .	7
2.3	Problem Setup of High Performance Conjugate Gradient . . . . .	7
2.4	Data Access Patterns of High Performance Conjugate Gradient . . . . .	8
2.5	Compression Strategies . . . . .	8
2.5.1	Restrictions on Compression Strategies . . . . .	9
2.5.2	Single and Mixed Precision Floating Point Numbers . . . . .	9
2.5.3	1 bit Compression . . . . .	9
2.5.4	Squeeze (SZ) Compression . . . . .	9
2.5.5	ZFP Compression . . . . .	9
2.5.6	Elias Gamma Coding and Delta Coding . . . . .	9
2.5.7	Op-Code Compression . . . . .	9
2.5.8	Huffman Coding . . . . .	9
2.5.9	Combined Compression Strategies . . . . .	9
<b>3</b>	<b>Test Results</b>	<b>9</b>
<b>4</b>	<b>Conclusions and Future Work</b>	<b>9</b>
<b>5</b>	<b>References</b>	<b>9</b>

# 1 Introduction

## 2 Background

### 2.1 Conjugate Gradient

Conjugate Gradient is the iterative solver used by HPCG [1]. Symmetric, positive definite matrices will guarantee the converge of Conjugate Gradient to the correct solution within  $n$  iterations, where  $n$  is the number of dimensions, when using exact algebra [4]. More importantly, Conjugate Gradient can be used as in iterative method, providing a solution,  $\vec{x}$ , where  $\|\mathbf{A}\vec{x} - \vec{b}\|$  is within some tolerance, after significantly fewer than  $n$  iterations, allowing it to find solutions to problems where even  $n$  iterations is infeasible [5].

To understand the Conjugate Gradient, first consider the quadratic form of  $\mathbf{A}\vec{x} = \vec{b}$ . The quadratic form is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  where

$$f(\vec{x}) = \frac{1}{2}\vec{x}^T \mathbf{A} \vec{x} - \vec{b} \cdot \vec{x} + c \quad (1)$$

for some  $c \in \mathbb{R}$ . Note that

$$\nabla f(\vec{x}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)\vec{x} - \vec{b}$$

Then, when  $\mathbf{A}$  is symmetric,

$$\nabla f(\vec{x}) = \mathbf{A}\vec{x} - \vec{b}$$

So, the solution to  $\mathbf{A}\vec{x} = \vec{b}$  is the sole critical point of  $f$  [3]. Since  $\mathbf{A}$  is the Hessian matrix of  $f$  at the point, if  $\mathbf{A}$  is positive definite, then that critical point is a minimum. Thus, if  $\mathbf{A}$  is a symmetric, positive definite matrix, then the minimum of  $f$  is the solution to  $\mathbf{A}\vec{x} = \vec{b}$  [5].

The method of Steepest Decent is useful for understanding Conjugate Gradient, because they both use a similar approach to minimize Equation 1, and thus solve  $\mathbf{A}\vec{x} = \vec{b}$ . This shared approach is to take an initial  $\vec{x}_0$  and move downwards in the steepest direction, within certain constraints, of the surface defined by Equation 1 [3]. Because the gradient at a point is the direction of maximal increase,  $\vec{x}$  should be moved in the opposite direction of the gradient. Thus, to compute the next value of  $\vec{x}$ , use

$$\vec{x}_{i+1} = \vec{x}_i + \alpha_i \vec{r}_i \quad (2)$$

for some  $\alpha_i > 0$  and where  $\vec{r}_i = -\nabla f(\vec{x}_i) = \vec{b} - \mathbf{A}\vec{x}_i$  is the residual of  $\vec{x}_i$ . Since  $\mathbf{A}\vec{x} = \vec{b}$  is the only critical point and a minimum of the quadratic function,  $f$ , the ideal value of  $\alpha_i$  is the one that minimizes  $f(\vec{x}_{i+1})$ . Thus, choose  $\alpha_i$  such that

$$\begin{aligned} 0 &= \frac{d}{d\alpha_i} f(\vec{x}_{i+1}) \\ &= \frac{d}{d\alpha_i} f(\vec{x}_i + \alpha \vec{r}_i) \\ \alpha_i &= \frac{\vec{r}_i \cdot \vec{r}_i}{\vec{r}_i \cdot \mathbf{A} \vec{r}_i} \end{aligned}$$

Note that by using Equation 2, we can derive

$$\vec{r}_{i+1} = \vec{r}_i - \alpha \mathbf{A} \vec{r}_i. \quad (3)$$

Because  $\mathbf{A} \vec{r}_i$  is already computed to find  $\alpha_i$ , using Equation 3 to compute the residual results in one less matrix-vector product per iteration. The steps for the Method of Steepest Decent are

$$\begin{aligned} \vec{r}_0 &= \vec{b} - \mathbf{A} \vec{x}_0 \\ \alpha_i &= \frac{\vec{r}_i \cdot \vec{r}_i}{\vec{r}_i \cdot \mathbf{A} \vec{r}_i} \\ \vec{x}_{i+1} &= \vec{x}_i + \alpha_i \vec{r}_i \\ \vec{r}_{i+1} &= \vec{r}_i - \alpha \mathbf{A} \vec{r}_i \end{aligned}$$

until  $\|\vec{r}_i\|$  is less than some tolerance [5].

**Example 1.** Consider the linear system

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and use  $c = 0$ . Note that the solution is

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

When starting at the origin, the iteration of Method of Steepest Decent becomes

$\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\vec{r}_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$	$\alpha_0 = 2/7$
$\vec{x}_1 = \begin{bmatrix} 10/7 \\ 10/7 \end{bmatrix}$	$\vec{r}_1 = \begin{bmatrix} 5/7 \\ -5/7 \end{bmatrix}$	$\alpha_1 = 2/3$
$\vec{x}_2 = \begin{bmatrix} 40/21 \\ 20/21 \end{bmatrix}$	$\vec{r}_2 = \begin{bmatrix} 5/21 \\ 5/21 \end{bmatrix}$	$\alpha_2 = 2/7$
$\vec{x}_3 = \begin{bmatrix} 290/147 \\ 50/49 \end{bmatrix}$	$\vec{r}_3 = \begin{bmatrix} 5/147 \\ -5/147 \end{bmatrix}$	$\alpha_3 = 2/3$
$\vdots$	$\vdots$	$\vdots$

The  $\vec{x}_i$ 's are plotted with a contour graph of the quadratic form in Figure 1.

The Conjugate Directions family of linear solvers, of which Conjugate Gradient is a member of, attempts to improve on the number of iterations needed by Steepest Decent. [5]. Note that, in Example 1, the directions of  $\vec{r}_0$  and  $\vec{r}_2$  are the same and the directions of  $\vec{r}_1$  and  $\vec{r}_3$  are the same. Thus, the same direction has to be traversed multiple times. Additionally, note that the two sets of residual directions are perpendicular to each other. Conjugate Directions attempts

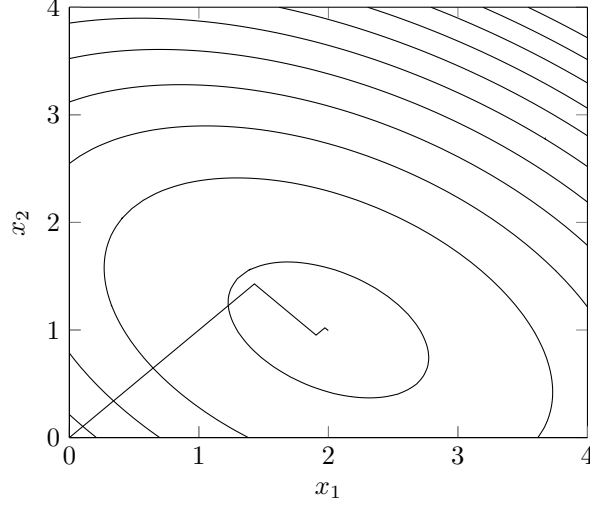


Figure 1: Contour graph of the quadratic function and the first six values of  $\vec{x}$  produced by steepest descent for Example 1

to improve on this, by making the search directions,  $\vec{d}_0, \vec{d}_1, \dots$ ,  $\mathbf{A}$ -orthogonal to each other and only moving  $\vec{x}$  once in each search direction. Two vectors,  $\vec{u}, \vec{v}$  are  $\mathbf{A}$ -orthogonal, or conjugate, if  $\vec{u}^T \mathbf{A} \vec{v} = 0$ . The requirement for Conjugate Directions is to make  $\vec{e}_{i+1}$   $\mathbf{A}$ -orthogonal to  $\vec{d}_i$ , where  $\vec{e}_i = \vec{x}_i - \mathbf{A}^{-1} \vec{b}$  is the error of  $\vec{x}_i$ . The computation of  $\alpha_i$  changes to find the minimal value along  $\vec{d}_i$  instead of  $\vec{r}_i$ .

$$\alpha_i = \frac{\vec{d}_i^T \vec{r}_i}{\vec{d}_i^T \mathbf{A} \vec{d}_i}.$$

Conjugate Gradient is a form of Conjugate Directions where the residuals are made to be  $\mathbf{A}$ -orthogonal to each other. This is done using the Conjugate Gram-Schmidt Process. To do this, each search direction,  $\vec{d}_i$  is computed by taking  $\vec{r}_i$  and removing any components that are not  $\mathbf{A}$ -orthogonal to the previous  $\vec{d}$ 's. So, let  $\vec{d}_0 = \vec{r}_0$  and for  $i > 0$  let

$$\vec{d}_i = \vec{r}_i + \sum_{k=0}^{i-1} \beta_{(i,k)} \vec{d}_k$$

with  $\beta_{(i,k)}$  defined for  $i > k$ . Then, solving for  $\beta_{(i,k)}$  gives

$$\beta_{(i,k)} = -\frac{\vec{r}_i \cdot \mathbf{A} \vec{d}_k}{\vec{d}_k \cdot \mathbf{A} \vec{d}_k}.$$

Note that each residual is orthogonal to the previous search directions, and thus the previous residuals. So, it can be shown that  $\vec{r}_{i+1}$  is  $\mathbf{A}$ -orthogonal to

all previous search directions, except  $\vec{d}_i$  [5]. Then,  $\beta_{(i,k)} = 0$  for  $i - 1 \neq k$ . To simplify notation, let  $\beta_i = \beta_{(i,i-1)}$ . So, each new search direction can then be computed by

$$\vec{d}_i = \vec{r}_i + \beta_i \vec{d}_{i-1}.$$

This results in the following steps for Conjugate Gradient

$$\begin{aligned} \vec{d}_0 &= \vec{r}_0 = \vec{b} - \mathbf{A}\vec{x}_0 \\ \alpha_i &= \frac{\vec{r}_i \cdot \vec{r}_i}{\vec{d}_i \cdot \mathbf{A}\vec{d}_i} \\ \vec{x}_{i+1} &= \vec{x}_i + \alpha_i \vec{d}_i \\ \vec{r}_{i+1} &= \vec{r}_i - \alpha_i \mathbf{A}\vec{d}_i \\ \beta_{i+1} &= \frac{\vec{r}_{i+1} \cdot \vec{r}_{i+1}}{\vec{r}_i \cdot \vec{r}_i} \\ \vec{d}_{i+1} &= \vec{r}_{i+1} + \beta_{i+1} \vec{d}_i \end{aligned}$$

continuing the iteration until  $\|\vec{r}_i\|$  is less than the specified tolerance.

**Example 2.** Consider the linear system used in Example 1 where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

The result of applying Conjugate Gradient is

$$\begin{aligned} \vec{x}_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \vec{r}_0 &= \begin{bmatrix} 5 \\ 5 \end{bmatrix} & \vec{d}_0 &= \begin{bmatrix} 5 \\ 5 \end{bmatrix} & \alpha_0 &= 2/7 \\ \vec{x}_1 &= \begin{bmatrix} 10/7 \\ 10/7 \end{bmatrix} & \vec{r}_1 &= \begin{bmatrix} 5/7 \\ -5/7 \end{bmatrix} & \beta_1 &= 1/49 & \vec{d}_1 &= \begin{bmatrix} 40/49 \\ -30/49 \end{bmatrix} & \alpha_1 &= 7/10 \\ \vec{x}_2 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \vec{r}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Note that after two iterations,  $\vec{x}$  reaches the exact solution, compared to the iterations of Steepest Decent in Example 1. Figure 2 shows the values of  $\vec{x}$  with the contour graph of the quadratic function.

## 2.2 Multigrid Preconditioner with Gauss-Seidel Step

## 2.3 Problem Setup of High Performance Conjugate Gradient

The problem used to create the linear system used by HPCG, and thus by this project, is a three dimensional partial differential equation (PDE) model [1]. This problem is approximating the function  $u(x, y, z)$  over the three dimensional rectangular region  $\Omega \in \mathbb{R}^3$  such that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$



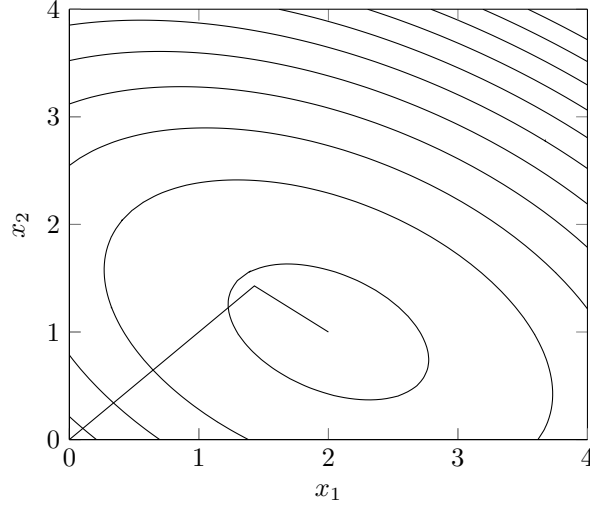


Figure 2: Contour graph of the quadratic function and the each value of  $\vec{x}$  produced by Conjugate Gradient for Example 2

with  $u(x, y, z) = 1$  along the boundaries of  $\Omega$ . Note that the solution is  $u(x, y, z) = 1$  over  $\Omega$ . The linear system is created by using the finite difference method with a 27-point stencil on the PDE over a rectangular grid with nodes of fixed distance. The matrix's diagonal consists of the value 26, and -1's fill the entries for the row's 26 grid neighbors. The right hand side of the equation has a value of 14 for corner points, 12 for edge points, 9 for side points and 0 for interior points [2]. The solution vector consists of all 1's. Finally, the zero vector is used as the initial guess.

## 2.4 Data Access Patterns of High Performance Conjugate Gradient

## 2.5 Compression Strategies

Numerous compression strategies were considered for this project. Figure 3 lists the compressions tried for each main data structure. Note that most compression methods were only used with one or two of the data types, even if able to be reasonably used within the constraints of additional data.

Strategy	Vector Values	Matrix Values	Matrix Indices
Single Precision	Yes	Yes	Not Able
Mixed Precision	Yes	Not Able	Not Able
1 Bit	Not Able	Yes	Not Able
Squeeze (SZ)	Yes	Yes	Yes
ZFP	Yes	Yes	No
Elias Gamma	Not Able	Not Able	Yes
Elias Delta	Not Able	Not Able	Yes
Huffman	Not Able	No	Yes
Op Code	Not Able	Not Able	Yes

Figure 3: Overview of Compression Strategies

- 2.5.1 Restrictions on Compression Strategies**
- 2.5.2 Single and Mixed Precision Floating Point Numbers**
- 2.5.3 1 bit Compression**
- 2.5.4 Squeeze (SZ) Compression**
- 2.5.5 ZFP Compression**
- 2.5.6 Elias Gamma Coding and Delta Coding**
- 2.5.7 Op-Code Compression**
- 2.5.8 Huffman Coding**
- 2.5.9 Combined Compression Strategies**

## 3 Test Results

## 4 Conclusions and Future Work

## 5 References

- [1] Jack Dongarra, Michael Heroux, and Piotr Luszczek. Hpcg benchmark: a new metric for ranking high performance computing systems. Technical Report UT-EECS-15-736, Electrical Engineering and Computer Science Department, Knoxville, Tennessee, November 2015.
- [2] David R. Kincaid and E. Ward Cheney. *Numerical Analysis: Mathematics of Scientific Computing*. Pure and applied undergraduate texts. American Mathematical Society, 2002.
- [3] J. Nearing. *Mathematical Tools for Physics*. Dover books on mathematics. Dover Publications, 2010.

- [4] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2nd edition, 2003.
- [5] Jonathan R Shewchuk. An introduction to the conjugate gradient method without the agonizing pain. Technical report, Carnegie Mellon University, Pittsburgh, PA, USA, 1994.