Reducing Memory Access Latencies using Data Compression in Sparse, Iterative Linear Solvers

An All-College Thesis

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Project Title: Reducing Memory Access Latencies using Data Compression in Sparse, Iterative Linear Solvers
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Abstract

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2 Background

2.1 Conjugate Gradient

Conjugate Gradient is the iterative solver used by HPCG [1]. Symmetric, positive definite matrices will guarantee the converge of Conjugate Gradient to the correct solution within n iterations when using exact algebra [2]. As an iterative method, Conjugate Gradient can provide a solution, $\vec{x}$, where $\left\|\mathbf{A}\vec{x} - \vec{b}\right\|$ is within some tolerance, after significantly fewer than n iterations, allowing it to find solutions to problems where even n iterations is infeasible [3].

To understand the Conjugate Gradient, first consider the quadratic form of $\mathbf{A}\vec{x} = \vec{b}$. The quadratic form is a function $f : \mathbb{R}^n \to \mathbb{R}$ where

$$f(\vec{x}) = \frac{1}{2}\vec{x} \cdot (\mathbf{A}\vec{x}) - \vec{b} \cdot \vec{x} \tag{1}$$

for some $c \in \mathbb{R}$. Note that

$$\begin{aligned}
\nabla f(\vec{x}) &= \nabla \left(\frac{1}{2}\vec{x} \cdot (\mathbf{A}\vec{x}) - \vec{b} \cdot \vec{x}\right) \\
&= \frac{1}{2}\nabla\left(\vec{x} \cdot (\mathbf{A}\vec{x})\right) - \nabla\left(\vec{b} \cdot \vec{x}\right) \\
&= \frac{1}{2}\left(\mathbf{A}\vec{x} + \mathbf{A}^T\vec{x}\right) - \vec{b}
\end{aligned}$$

Then, when $\mathbf{A}$ is symmetric,

$$\nabla f(\vec{x}) = \mathbf{A}\vec{x} - \vec{b}$$

So, the solution to $\mathbf{A}\vec{x} = \vec{b}$ is the sole critical point of f. Since $\mathbf{A}$ is the Hessian matrix of f at the point, if $\mathbf{A}$ is positive definite, then that critical point is a minimum. Thus, if $\mathbf{A}$ is a symmetric, positive definite matrix, then the minimum of f is the solution to $\mathbf{A}\vec{x} = \vec{b}$ [3].

The Method of Steepest Decent is similar to Conjugate Gradient, but simpler to initially understand. The method takes an initial $\vec{x}_0$ and repeatedly computes improved points, $\vec{x}_1, \vec{x}_2, \ldots$, until reaching a point close enough to the minimum of Equation 1. Because the gradient at a point is the direction of maximal increase, $\vec{x}_{i+1} = \vec{x}_i + \alpha\vec{r}_i$ for some $\alpha > 0$ and where $\vec{r}_i = -\nabla f(\vec{x}_i) = \vec{b} - \mathbf{A}\vec{x}_i$ is the residual of $\vec{x}_i$. So, the value for α that minimizes $f\left(\vec{x}_{i+1}\right)$ occurs when

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha} f\left(\vec{x}_{i+1}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\alpha} f\left(\vec{x}_i + \alpha \vec{r}_i\right) \\
&= \nabla f\left(\vec{x}_i + \alpha \vec{r}_i\right) \cdot \vec{r}_i \\
&= \left(\mathbf{A}\left(\vec{x}_i + \alpha \vec{r}_i\right) - \vec{b}\right) \cdot \vec{r}_i \\
&= \left(\mathbf{A}\vec{x}_i - \vec{b}\right) \cdot \vec{r}_i + \alpha \mathbf{A}\vec{r}_i \cdot \vec{r}_i \\
-\alpha \mathbf{A}\vec{r}_i \cdot \vec{r}_i &= -\vec{r}_i \cdot \vec{r}_i \\
\alpha &= \frac{\vec{r}_i \cdot \vec{r}_i}{\vec{r}_i \cdot \mathbf{A}\vec{r}_i}.
\end{aligned}
$$

The resulting steps for the Method of Steepest Decent are

$$
\begin{aligned}
\vec{r}_i &= \vec{b} - \mathbf{A}\vec{x}_i \\
\alpha &= \frac{\vec{r}_i \cdot \vec{r}_i}{\vec{r}_i \cdot \mathbf{A}\vec{r}_i} \\
\vec{x}_{i+1} &= \vec{x}_i + \alpha \vec{r}_i
\end{aligned}
$$

until $\|\vec{r}_i\|$ is less than some tolerance [3].

Example 1.

3 Test Results

4 Conclusions and Future Work

5 References

[1] Jack Dongarra, Michael Heroux, and Piotr Luszczek. Hpcg benchmark: a new metric for ranking high performance computing systems. Technical Report UT-EECS-15-736, Electrical Engineering and Computer Sciente Department, Knoxville, Tennessee, November 2015.

[2] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2nd edition, 2003.

[3] Jonathan R Shewchuk. An introduction to the conjugate gradient method without the agonizing pain. Technical report, Carnegie Mellon University, Pittsburgh, PA, USA, 1994.